TOPICS

- FIND POLYNOMIAL APPROXIMATIONS OF ELEMENTARY FUNCTIONS AND COMPARE THEM WITH THE ELEMENTARY FUNCTIONS
- FIND TAYLOR AND MACLAURIN POLYNOMIAL APPROXIMATIONS OF ELEMENTARY FUNCTIONS
- USE THE REMAINDER OF A TAYLOR POLYNOMIAL

TEXT READING ASSIGNMENT FOR 9.7

PAGE 648,649,650,651,652,653,654,655

TEXT HOMEWORK EXCERCISES FOR 9.7

PAGE 656#15,17,19,21,25,27,29
PAGE 657#45,49,51
PAGE 658#57,59

- FIND POLYNOMIAL APPROXIMATIONS OF ELEMENTARY FUNCTIONS AND COMPARE THEM WITH THE ELEMENTARY FUNCTIONS

In calculus I, much time is spent finding the equation of the tangent line $y = mx + b$ to a function $y = f(x)$ at a point $(x_0, f(x_0))$.

Student Exercise: Find the equation of the tangent line to $f(x) = \sin x$ at $(0,0)$.

Student Exercise Solution 9.7.1

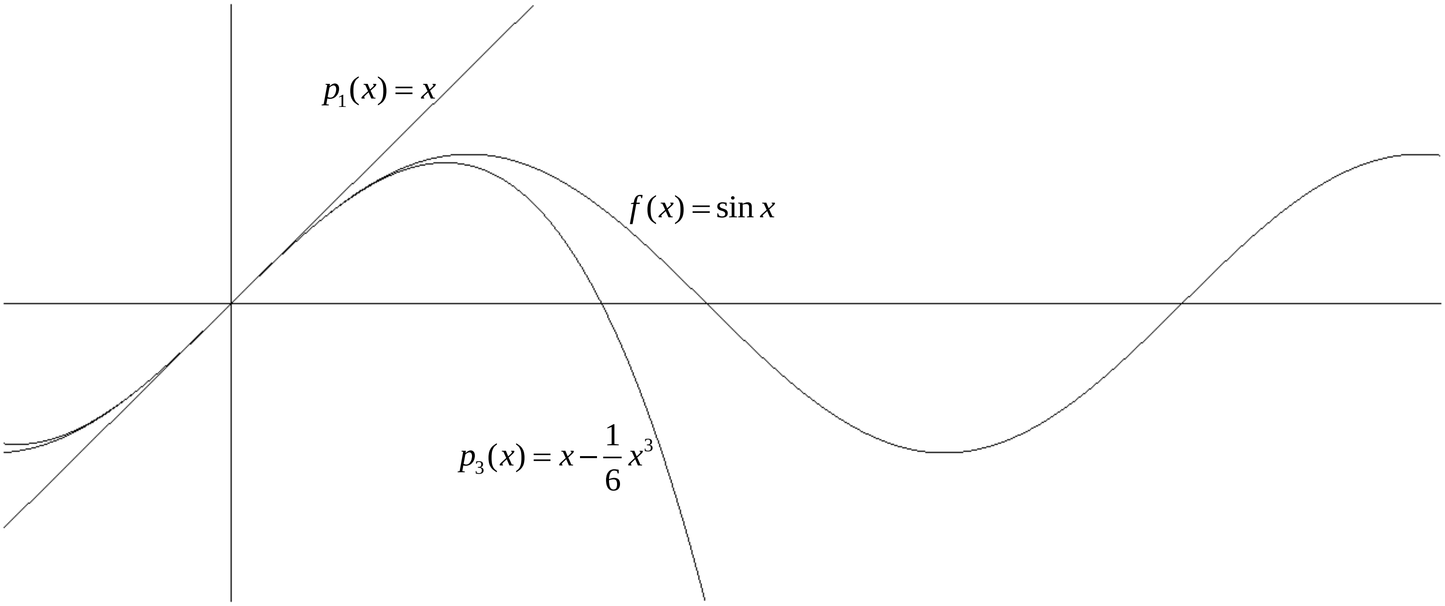
The equation of the tangent line to $y = f(x)$ at a point $(x_0, f(x_0))$ agrees with the point and slope of $y = f(x)$ at the point $(x_0, f(x_0))$. For example, in the student exercise you found the equation of the tangent line to $f(x) = \sin x$ at $(0,0)$ to be $y = x$. Both equation and tangent line satisfy $y(0) = 0$ and $y'(0) = 1$.

In this section we will study how to find a polynomial that agrees with $y = f(x)$ and its first n derivatives at a point $(x_0, f(x_0))$. This is called the n th ***Taylor Polynomial*** for $y = f(x)$ ***centered*** at x_0.

Example The first Taylor Polynomial for $f(x) = \sin x$ at $(0,0)$ is just the tangent line at $(0,0)$ $y = x$.

Worked Example 9.7.2

In the worked example above we found for $f(x) = \sin x$ at $(0,0)$, the first and third Taylor polynomials are $p_1(x) = x$ and $p_3(x) = x - \frac{1}{3}x^3$. The larger values of n correspond to better Taylor polynomial approximations to $f(x) = \sin x$ near the center $(0,0)$. This is true in general of Taylor polynomials.



• FIND TAYLOR AND MACLAURIN POLYNOMIAL APPROXIMATIONS OF ELEMENTARY FUNCTIONS

Let's find a formula for the coefficients $a_0, a_1, a_2, \ldots, a_n$ of the n th Taylor polynomial for $y = f(x)$ centered at 0. Compute n derivatives of $p_n(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ and $y = f(x)$ and evaluate at 0.

k	$p_n^k(x)$	$p_n^k(0)$	$f^k(0)$	Conclusion
0	$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$	a_0	$f(0)$	$a_0 = f(0)$
1	$a_1 + 2a_2x + \ldots + na_nx^{n-1}$	a_1	$f'(0)$	$a_1 = f'(0)$
2	$2a_2 + \ldots + n(n-1)a_nx^{n-2}$	$2a_2$	$f''(0)$	$a_2 = \frac{1}{2}f''(0)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$n!a_n$	$n!a_n$	$f^{(n)}(0)$	$a_n = \frac{1}{n!}f^{(n)}(0)$

The n th Taylor polynomial for $y = f(x)$ centered at 0 is $p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n$, provided f is n times differentiable at 0. When a Taylor polynomial is centered at 0, it is called a ***Maclaurin Polynomial***.

The formula for the n th Taylor polynomial for $y = f(x)$ centered at c is

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$, provided f is n times differentiable at c.

Worked Example 9.7.3

To find the n th Taylor polynomial for $y = f(x)$ centered at c use the steps below.

1) Compute derivatives $f(x), f'(x), f''(x), \ldots, f^{(n)}(x)$.
2) Evaluate $f(c), f'(c), f''(c), \ldots, f^{(n)}(c)$.
3) Form the sum $p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$.

Remark The k th term in the polynomial is $\frac{f^{(k)}(c)}{k!}(x-c)^k$, make sure to divide by $k!$, not just k (recall $0! = 1$ and $1! = 1$).

Student Exercise: Find the fifth Maclaurin polynomial for $f(x) = e^x$.

Student Exercise Solution 9.7.4

• USE THE REMAINDER OF A TAYLOR POLYNOMIAL

The remainder $R_n(x)$ of a function $f(x)$ estimated by $p_n(x)$ is defined by $R_n(x) = f(x) - p_n(x)$.
The error $|R_n(x)|$ of a function $f(x)$ estimated by $p_n(x)$ is defined by $|R_n(x)| = |f(x) - p_n(x)|$.

Taylor's Theorem Suppose $y = f(x)$ is an $n+1$ time differentiable function on an interval I containing c. For each fixed $x \in I$ there exists z between c and x so that

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x)$$, where $R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$.

Remark This theorem usually takes a second or third reading to really understand.

Remark The part of this theorem that seems most unusual is the factor $f^{(n+1)}(z)$ in the remainder. When estimating the error $|R_n(x)|$, the theorem tells us $|R_n(x)| = \frac{M}{(n+1)!}|x-c|^{n+1}$, where M is the maximum of $\left|f^{(n+1)}(z)\right|$ for all z between c and x.

We will make the following four uses of Taylor's theorem.

1) Estimate the error $|R_n(a)|$ created by approximating $f(a)$ with $p_n(a)$.
2) Determine n so that the approximation $p_n(a)$ will have a desired degree of accuracy.
3) Find an interval I for x so that $p_n(x)$ will approximate $f(x)$ to a desired degree of accuracy on for all x in I.
4) Show the series of Taylor polynomials $\{p_n(x)\}$ converge to the function they represent $f(x)$ (section 9.10).

Worked Example 9.7.5

Student Exercise: Determine n so that the Maclaurin polynomial approximation $p_n(.1)$ for $\cos(.1)$ will have error less than 0.000001. Hint: Since the derivatives of $\cos x$ are either $\pm\sin x$ or $\pm\cos x$, you can set the error $\frac{|f^{(n+1)}(z)|}{(n+1)!}|x-c|^{n+1} = \frac{1}{(n+1)!}|.1-0|^{n+1}$ less than 0.000001. Solve by trial and error with a calculator.

Student Exercise Solution 9.7.6

As our final example, we show how to determine the values of x so that the Maclaurin polynomial $p_3(x) = x - \frac{1}{6}x^3$ will approximate $f(x) = \sin x$ with $|R_2(x)| \le 0.001$.

Since the derivatives of $\sin x$ are either $\pm\sin x$ or $\pm\cos x$, you can set the error

$\frac{|f^{(n+1)}(z)|}{(n+1)!}|x-c|^{n+1} = \frac{1}{(3+1)!}|x-0|^{3+1} = \frac{1}{24}x^4 \le 0.001$. Solving by calculator, $-.3936 \le x \le .3936$.