Chapter 9 Section 1

TOPICS
- LIST THE TERMS OF A SEQUENCE
- DETERMINE WHETHER A SEQUENCE CONVERGRES OR DIVERGES
- WRITE A FORMULA FOR THE Nth TERM OF A SEQUENCE
- USE PROPERTIES OF MONOTONIC SEQUENCES AND BOUNDED SEQUENCES

TEXT READING ASSIGNMENT FOR 9.1

PAGE 594,595596,597,598,599,600,601(OMIT PROOF OF THEOREM 9.5)

TEXT HOMEWORK EXCERCISES FOR 9.1

PAGE 602# 3,11,27,29,37,39,47,51
PAGE 603#53,55,59,73,77,85,87,89

- LIST THE TERMS OF A SEQUENCE

A ***sequence*** is simply an ordered list of numbers. For example, $\{a_n\} = \{(-2)^n\}$ is $(-2)^1, (-2)^2, (-2)^3, \ldots =$ $-2, 4, -8, \ldots$. One way to interpret the equation $\{a_n\} = \{(-2)^n\}$ is like a function $a(n) = (-2)^n$ with domain positive integers. In mathematics, we usually writes $\{a_n\} = \{(-2)^n\}$ for sequences rather than $f(x) = (-2)^x$ because the use of f and x usually implies the domain is any real number, not just positive integers.

Here are a few more examples of sequences.

a) $\{a_n\} = \left\{\dfrac{\cos\frac{\pi n}{2}}{n}\right\}$ is $\dfrac{\cos\frac{\pi}{2}}{1}, \dfrac{\cos\pi}{2}, \dfrac{\cos\frac{3\pi}{2}}{3}, \dfrac{\cos 2\pi}{4}, \ldots = -\dfrac{1}{2}, \dfrac{1}{4}, -\dfrac{1}{6}, \dfrac{1}{8}, \ldots$

b) $a_1 = 6$ and $\{a_n\} = \left\{\dfrac{a_{n-1}}{3}\right\}$ is $6, 2, \dfrac{2}{3}, \dfrac{2}{9}, \dfrac{2}{27}, \ldots$

Student Exercise: Write out the first eight terms in the Fibonacci Sequence defined by $a_1 = 1, a_2 = 1,$ and $\{a_n\} = \{a_{n-1} + a_{n-2}\}$.

Student Exercise Solution 9.1.1

Remark The sequences above all started with $n = 1$, but there are many sequences that start off with any non-negative integer value of n.

• DETERMINE WHETHER A SEQUENCE CONVERGRES OR DIVERGES

A sequence $\{a_n\}$ is said to ***converge*** to a ***limit*** L if for every $\varepsilon > 0$ there exists N such that for $n > N$, $|a_n - L| < \varepsilon$. If a sequence converges to L then we write $\lim_{n\to\infty} a_n = L$. If a sequence does not converge it is said to ***diverge***.

In other words, L is the limit of a sequence if given any small positive quantity ε, at some point a_N in the sequence all the terms become within ε of L.

Theorem Suppose $y = f(x)$ is a function and $\{a_n\} = \{f(n)\}$. If $\lim_{x\to\infty} f(x) = L$ then $\lim_{n\to\infty} a_n = L$.

Before our example let's recall from section 3.5, the limit of a rational function $\lim_{x\to\infty} \frac{p(x)}{q(x)}$ is

1) 0 if degree of p is less than the degree of q.
2) $\frac{\text{leading coefficient of } p}{\text{leading coefficient of } q}$ if degree of p is equal to degree of q.
3) non existent if degree of p is greater than the degree of q.

Worked Example 9.1.2

Here are some properties of limits of a sequence. Suppose $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = K$.

1) $\lim_{n\to\infty}(a_n \pm b_n) = L \pm K$
2) $\lim_{n\to\infty} ca_n = cL$
3) $\lim_{n\to\infty}(a_n b_n) = LK$
4) $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{L}{K}$, $b_n \neq 0$ and $K \neq 0$.

Example Find $\lim_{n\to\infty}\left\{5 - \frac{1}{n-1}\right\}$. $\lim_{n\to\infty}\{5-\} = \lim_{n\to\infty} 5 - \lim_{n\to\infty}\frac{1}{n-1} = 5 - 0 = 5$.

Squeeze theorem for sequences

Suppose $a_n \le c_n \le b_n$ for all n greater than some N. If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = L$ then $\lim_{n\to\infty} c_n = L$.

Worked Example 9.1.3

Theorem 9.4 from the texts tells us if $\lim_{n\to\infty}|a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$.

Example To determine the convergence of $\left\{-\frac{1}{n}\right\}$ we could apply the squeeze theorem. But since $\left\{\frac{1}{n}\right\}$ converges to 0, by the theorem above so does the sequence $\left\{-\frac{1}{n}\right\}$.

• WRITE A FORMULA FOR THE Nth TERM OF A SEQUENCE

Worked Example 9.1.4

Remark It is not possible to find a unique formula for an entire sequence based on only the first few terms of that sequence. So the answers in the worked example above are not unique.

The sequence $1, 1\cdot 2, 1\cdot 2\cdot 3, 1\cdot 2\cdot 3\cdot 4, \cdots$ is a divergent sequence; $a_1 = 1$ and $a_n = na_{n-1}$. The n th term is called ***n-factorial*** and is written $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$. We define $0! = 1$.

Student Exercise: Identify a formula for the n th term in each sequence.

a) $2, 5, 8, 11, \ldots$

b) $1, -\frac{3}{2}, \frac{9}{4}, -\frac{27}{8}, \ldots$

Student Exercise Solution 9.1.5

• USE PROPERTIES OF MONOTONIC SEQUENCES AND BOUNDED SEQUENCES

A sequence $\{a_n\}$ is ***monotonic*** if either its terms are

1) Non-decreasing $a_1 \geq a_2 \geq a_3 \geq \cdots$ or
2) Non-increasing $a_1 \leq a_2 \leq a_3 \leq \cdots$

There are two main ways to prove a sequence is monotonic

1) Show either the equation $a_n \geq a_{n+1}$ or $a_n \leq a_{n+1}$ is equivalent to a true statement or
2) If $\{a_n\} = \{f(n)\}$, show either $f'(x) \leq 0$ or $f'(x) > 0$ for all $x \geq 1$.

Student Exercise: Which of the following terms are monotonic? Prove your answer.

a) $\left\{\frac{n}{2^n}\right\}$

b) $\left\{\frac{n}{\ln(n+1)}\right\}$

c) $\{\sin n\}$

Student Exercise Solution 9.1.6

A sequence $\{a_n\}$ is ***bounded above*** if there exists a real number M so that $a_n \le M$ for all n.
A sequence $\{a_n\}$ is ***bounded below*** if there exists a real number M so that $a_n \ge M$ for all n.
A sequence $\{a_n\}$ is ***bounded*** if it is bounded above and bounded below.

Examples

a) $\left\{\frac{n}{2^n}\right\}$ is bounded above by $\frac{1}{2}$ and below by 0 and is therefore bounded.

b) $\left\{\frac{n}{\ln(n+1)}\right\}$ is bounded below by $\frac{1}{\ln 2}$. By L'Hopital's Rule $\lim_{x\to\infty}\frac{x}{\ln(x+1)} = \lim_{x\to\infty}\frac{1}{\frac{1}{(x+1)}} = \lim_{x\to\infty}(x+1) = \infty$. So this sequence is not bounded above.

c) $\{\sin n\}$ is bounded above by 1 and below by -1.

Theorem A bounded and monotonic sequence $\{a_n\}$ always converges.

Remark This is a very useful theorem in determining the convergence of a sequence. Notice however, this theorem does not tell us what is the limit of the sequence.

Remark Our theorem tells us only ***sufficient conditions*** for convergence. It <u>does not</u> tell us ***necessary conditions*** for convergence.

Example In the three examples $\left\{\frac{n}{2^n}\right\}$, $\left\{\frac{n}{\ln(n+1)}\right\}$, $\{\sin n\}$, only $\left\{\frac{n}{2^n}\right\}$ is both bounded and monotonic. Thus, only $\left\{\frac{n}{2^n}\right\}$ is guaranteed to converge by our theorem.

Example The sequence $\left\{-\frac{1}{n}\right\}$ does not satisfy the conditions of our theorem because it is not monotonic. However, we showed earlier using a different technique that it is still a convergent sequence.