TOPICS
• USE THE ALTERNATING SERIES TEST TO DETERMINE WHETHER AN INFINITE SERIES CONVERGES
• USE THE ALTERNATING SERIES REMAINDER TO APPROXIMATE THE SUM OF AN ALTERNATING SERIES
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TEXT READING ASSIGNMENT FOR 9.5

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TEXT HOMEWORK EXCERCISES FOR 9.5

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• USE THE ALTERNATING SERIES TEST TO DETERMINE WHETHER AN INFINITE SERIES CONVERGES

In this section we will focus on series whose terms alternate between positive and negative. We define an ***alternating series*** as one whose terms are of the form $(-1)^n a_n$ or $(-1)^{n+1} a_n$, where $a_n > 0$ for all n.

The ***Alternating Series Test*** states that either alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ or $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$, where $a_n > 0$ converge if the two conditions below are met.

1) $\lim_{n\to\infty} a_n = 0$
2) $a_{n+1} \le a_n$ for all n

Remark Again it is imperative that you demonstrate all the conditions are met when applying the Alternating Series Test.

Remark We can apply the Alternating Series Test to any series satisfying the conditions above but starts at $n \ne 1$. Any test for convergence can be applied to a series starting at $n \ne 1$. This is because a finite number of terms does not affect the convergence of the series.

Remark The Alternating Series Test gives only <u>sufficient conditions</u> for an alternating series to converge. It is possible for an alternating series to not satisfy a condition but still converge. Make sure you understand this thoroughly.

Worked Example 9.5.1

To apply the Alternating Series Test use the steps below.

1) Verify $a_{n+1} \le a_n$. (Remember to $a_n > 0$; do not include the alternating term $(-1)^n$ in steps1 or 2)

2) Verify $\lim_{n\to\infty} a_n = 0$.

3) Any alternating series satisfying 1 and 2 converges. If either or both conditions fail, we do not know if the convergence of the series. A different test is required.

Rather than give a proof, lets make the an argument as to why this theorem is true.

Worked Example 9.5.2

• USE THE ALTERNATING SERIES REMAINDER TO APPROXIMATE THE SUM OF AN ALTERNATING SERIES

As with the integral test, there is a theorem estimating the remainder of an alternating series $R_N = S - S_N$, where S is the sum and S_N is the N th partial sum of a convergent series. For the integral test, $0 \le R_N \le \int_N^\infty f(x)dx$. For the Alternating Series Test, $|R_N| \le a_{N+1}$.

Remark The exprseeion S_N can sometimes cause confusion. For example, in $\sum_{n=3}^{\infty} \frac{1}{n}$ the fifth partial sum is $S_5 = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$. Occasionally, students will write $S_5 = \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$ by mistake. Remember S_N always means the partial sum of adding the first N terms, it does not mean you add all terms up to $n = N$.

Worked Example 9.5.3

Student Exercise: Show the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges. Find N so that $|R_N| < 0.001$.

Student Exercise Solution 9.5.4

• CLASSIFY A CONVERGENT SERIES AS ABSOLUTELY OR CONDITIONALLY CONVERGENT

Theorem If the series $\sum_{n=1}^{\infty} |a_n|$ converges then so does the series $\sum_{n=1}^{\infty} a_n$. For a proof see page 634.

Of course this is true if $a_n > 0$ for all n. So this theorem can help to show a series with negative terms converges. The terms in the series need not even alternate!

Example Consider the series $\sum_{n=2}^{\infty} a_n = \frac{1}{4}+\frac{1}{9}-\frac{1}{16}-\frac{1}{25}+\frac{1}{36}+\frac{1}{49}-\cdots$. This series is not alternating and so we cannot apply the Alternating Series Test. Since $\sum_{n=2}^{\infty} |a_n| = \frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\frac{1}{49}+\cdots$ is the convergent *p*-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, our theorem above tells us $\sum_{n=2}^{\infty} a_n = \frac{1}{4}+\frac{1}{9}-\frac{1}{16}-\frac{1}{25}+\frac{1}{36}+\frac{1}{49}-\cdots$ also converges.

The series $\sum_{n=1}^{\infty} a_n$ is ***absolutely convergent*** if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

The series $\sum_{n=1}^{\infty} a_n$ is ***conditionally convergent*** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Example The series $\sum_{n=2}^{\infty} a_n = \frac{1}{4}+\frac{1}{9}-\frac{1}{16}-\frac{1}{25}+\frac{1}{36}+\frac{1}{49}-\cdots$ converges absolutely because $\sum_{n=2}^{\infty} |a_n| = \frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\frac{1}{49}+\cdots$ converges.

Example The alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges conditionally because $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges but $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

[Worked Example 9.5.5]

To test whether a series $\sum_{n=1}^{\infty} a_n$ converges conditionally, converges absolutely or diverges, use the steps below.

1) Test for divergence; if $\lim_{n\to\infty} a_n \neq 0$ the series diverges. If $\lim_{n\to\infty} a_n = 0$ continue to step 2.

2) Test for absolute convergence; if $\sum_{n=1}^{\infty} |a_n|$ the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. If $\sum_{n=1}^{\infty} |a_n|$ does not converge continue to step 3.

3) Test for conditional convergence; if $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} a_n$ is said to converge conditionally. Otherwise, $\sum_{n=1}^{\infty} a_n$ diverges.

• REARRANGE AN INFINITE SERIES TO OBTAIN A DIFFERENT SUM

A pretty interesting fact is that if a series converges conditionally than you can rearrange the terms to make a sum of any real number you like! However, if a series converges absolutely then you can rearrange the terms of the series in any way and you still will come up with the same sum!!! See page 636 for an example with the alternating harmonic series.