Chapter 9 Section 2

TOPICS
• UNDERSTAND THE DEFINITION OF A CONVERGENT INFINITE SERIES
• USE PROPERTIES OF INFINITE GEOMETRIC SERIES
• USE THE Nth TERM TEST FOR DIVERGENCE OF AN INFINITE SERIES

TEXT READING ASSIGNMENT FOR 9.2

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TEXT HOMEWORK EXCERCISES FOR 9.2

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• UNDERSTAND THE DEFINITION OF A CONVERGENT INFINITE SERIES

An ***infinite series*** $\sum_{n=1}^{\infty} a_n$ is written as a sum $a_1 + a_2 + a_3 + \cdots$. Addition is only defined for a finite number of summands, so we need an alternative definition for sum of an infinite series. Consider the following sequence defined below.

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$\vdots = \vdots$$

This infinite sequence $S_1, S_2, S_3, \ldots$ is called the ***sequence of partial sums*** for the series $\sum_{n=1}^{\infty} a_n$.

Student Exercise: Write the first five terms of the sequence of partial sums S_1, S_2, S_3, S_4, S_5 for the series $\sum_{n=1}^{\infty} \frac{3}{10^n}$.

Student Exercise Solution 9.2.1

Remark It may seem odd and maybe a little bit confusing to start with a series and then create a sequence. The reason why we do this is it gives us the definition of sum for a series that we needed.

Definition The ***sum of an infinite series*** $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$ is the limit of the sequence of partial sums $\lim_{n\to\infty} S_n$. An infinite series ***converges*** if and only if its sequence of partial sums converges. Likewise, an infinite series ***diverges*** if and only if its sequence of partial sums diverges.

Example In student Exercise 9.2.1 we showed the series $\sum_{n=1}^{\infty} \frac{3}{10^n}$ has a sequence of partial sums $.3, .33, .333, .3333, .33333, \ldots$. Clearly, the limit of this sequence is $.333\cdots = \frac{1}{3}$.

Therefore, we say the infinite series $\sum_{n=1}^{\infty} \frac{3}{10^n}$ converges to the sum $\frac{1}{3}$.

[Worked Example 9.2.2]

• USE PROPERTIES OF INFINITE GEOMETRIC SERIES

A ***geometric series*** has the form $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots$, where $a \neq 0$ and $|r| < 1$.

Remark You can see why we need r to satisfy $|r| < 1$.

If $r = 1$ the series $\sum_{n=0}^{\infty} ar^n = a + a + a + \cdots$ would have an nth partial sum $S_n = na$ which would diverge as $n \to \infty$.

If $r = -1$ the series $\sum_{n=0}^{\infty} ar^n = a - a + a - \cdots$ would have an nth partial sum $S_n = \begin{cases} 0, & n \text{ even} \\ a, & n \text{ odd} \end{cases}$ which also diverges as $n \to \infty$.

Let's find a formula for the nth partial sum S_n of a geometric series. By definition,

$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$ (notice that we stop at the exponent $n-1$ because this gives us n terms on the right)

$S_n = a + r\left(a + r + ar^2 + \cdots + ar^{n-2}\right)$ (factor out r)

$S_n = a + r\left(S_n - ar^{n-1}\right)$ (substitute)

$(1-r)S_n = a\left(1 - r^{n-1}\right)$ (distribute and regroup)

$S_n = \frac{a\left(1-r^{n-1}\right)}{(1-r)}$ (Divide).

Since $0 < |r| < 1$, $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a\left(1-r^{n-1}\right)}{(1-r)} = \frac{a}{1-r}$. Therefore, $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$.

[Worked Example 9.2.3]

Remark The series $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots$ with $|r| > 1$ clearly diverges.

Student Exercise: The decimal $.\overline{45} = .454545\cdots$ can be written as an infinite series $\frac{45}{100} + \frac{45}{100^2} + \frac{45}{100^3} + \cdots = \frac{45}{100}\left(1 + \frac{1}{100} + \frac{1}{100^2} + \cdots\right) = \sum_{n=0}^{\infty} \frac{45}{100}\left(\frac{1}{100}\right)^n$. Write this number in fraction form.

[Student Exercise Solution 9.2.4]

The following facts on series correspond directly to facts on sequences discussed in section 9.1.

If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ then 1) $\sum_{n=1}^{\infty} ca_n = cA$ 2) $\sum_{n=1}^{\infty} (a_n \pm b_n) = A \pm B$.

• USE THE Nth TERM TEST FOR DIVERGENCE OF AN INFINITE SERIES

A series $\sum_{n=1}^{\infty} a_n = L$ if and only if $\lim_{n\to\infty} S_n = L$, where S_n is the n th partial sum of the series. Suppose $\sum_{n=1}^{\infty} a_n = L$ then $a_n = S_n - S_{n-1}$ so $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = L - L = 0$.

Theorem If $\sum_{n=1}^{\infty} a_n$ is a convergent series then $\lim_{n\to\infty} a_n = 0$.

Remark Be careful how you use this theorem! It only says $\lim_{n\to\infty} a_n = 0$ is a <u>necessary</u> condition for convergence. This theorem <u>does not say</u> if $\lim_{n\to\infty} a_n = 0$ then the series $\sum_{n=1}^{\infty} a_n$ converges, this is not always true!!! The correct interpretation is that the terms of every convergent series must go to 0. If the terms in a series do not go to 0, the series itself cannot converge.

[Worked Example 9.2.5]