TOPICS
• EVALUATE AN IMPROPER INTEGRAL THAT HAS AN INFINITE LIMIT OF INTEGRATION
• EVALUATE AN IMPROPER INTEGRAL THAT HAS AN INFINITE DISCONTINUITY

TEXT READING ASSIGNMENT FOR 8.8

PAGE 578,579,580,581(OMIT EXAMPLE 5),582,583,584(OMIT SURFACE AREA ONLY)

TEXT HOMEWORK EXCERCISES FOR 8.8

PAGE 585# 5,7,11,15,17,19,23,25,27,29,33,35,39

• EVALUATE AN IMPROPER INTEGRAL THAT HAS AN INFINITE LIMIT OF INTEGRATION

In section 4.3, the definite integral of $y = f(x)$ was defined on closed intervals $[a,b]$. In section 4.4, the fundamental theorem of calculus allowed us to evaluate the definite integral of a continuous function f by simply computing the difference of an antiderivative of f at b and a, respectively.

If a definite integral $\int_a^b f(x)dx$ is either not defined on a closed interval $[a,b]$ or not continuous on $[a,b]$ or both then we say $\int_a^b f(x)dx$ is an ***improper integral***. If $\int_a^b f(x)dx$ is finite then we say the improper integral ***converges***, otherwise we say the improper integral ***diverges***.

Each of the following are examples of improper integrals;
Here is an example of how to solve an improper integral.

Worked Example 8.8.1

Improper integrals with infinite limits of integration are defined as follows;

1) $\int_a^{\infty} f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$, provided f is continuous on $[a,\infty)$.

2) $\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx$, provided f is continuous on $(-\infty,b]$.

3) $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_c^{\infty} f(x)dx$ for any real number c, provided f is continuous on $(-\infty,\infty)$.

Student Exercise: Evaluate the improper integral $\int_1^{\infty} \frac{dx}{\sqrt[4]{x}}$.

Student Exercise Solution 8.8.2

Let's take a look at a doubly improper integral: $\int_{-\infty}^{\infty} \frac{2xdx}{x^4+1}$.

According to the definition above we must split this up into two separate improper integrals $\int_{-\infty}^{c} \frac{2xdx}{x^4+1} + \int_{c}^{\infty} \frac{2xdx}{x^4+1}$.

We can choose any c, so let's make things easy and put $c=0$. We need an antiderivative for $\frac{2x}{x^4+1}$. Let $u=2x$ so that $du=2xdx$ and our first integrals becomes,

$$\lim_{a\to-\infty} \int_{x=a}^{x=0} \frac{du}{u^2+1} = \lim_{a\to-\infty} \tan^{-1} u\Big|_{x=a}^{x=0} = \lim_{a\to-\infty} \tan^{-1}(x^2)\Big|_a^0 = \lim_{a\to-\infty} \left[\tan^{-1} 0 - \tan^{-1}(a^2)\right] = -\frac{\pi}{2}.$$

Our second integral is done in a similar way,

$$\lim_{b\to\infty} \int_{x=0}^{x=b} \frac{du}{u^2+1} = \lim_{b\to\infty} \tan^{-1} u\Big|_{x=0}^{x=b} = \lim_{b\to\infty} \tan^{-1}(x^2)\Big|_0^b = \lim_{b\to\infty} \left[\tan^{-1}(b^2) - \tan^{-1} 0\right] = \frac{\pi}{2}.$$

Therefore, $\int_{-\infty}^{\infty} \frac{2xdx}{x^4+1} = 0$.

• EVALUATE AN IMPROPER INTEGRAL THAT HAS AN INFINITE DISCONTINUITY

Improper integrals that have infinite discontinuities (vertical asymptotes) are defined as follows;

1) $\int_a^b f(x)dx = \lim_{c\to b^-} \int_a^c f(x)dx$, when f is continuous on $[a,b)$,but has an infinite discontinuity at b .

2) $\int_a^b f(x)dx = \lim_{c\to a^+} \int_c^b f(x)dx$, when f is continuous on $(a,b]$,but has an infinite discontinuity at a .

3) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, when f is continuous on $[a,b]$, except at the infinite discontinuity c in (a,b).

Worked Example 8.8.3

Student Exercise: Evaluate $\int_0^1 x\ln x dx$. Hint: you will need integration by parts and L'Hopital's Rule.

Student Exercise Solution 8.8.4