
GPU-ACCELERATED COUNTERFACTUAL REGRET MINIMIZATION

Juho Kim

Faculty of Applied Science and Engineering
University of Toronto
Toronto, Ontario, Canada
`juho.kim@mail.utoronto.ca`

ABSTRACT

Counterfactual regret minimization (CFR) is a family of algorithms of no-regret learning dynamics capable of solving large-scale imperfect information games. There has been a notable lack of work on making CFR more computationally efficient. We propose implementing this algorithm as a series of dense and sparse matrix and vector operations, thereby making it highly parallelizable for a graphical processing unit. Our experiments show that our implementation performs up to about 352.5 times faster than OpenSpiel’s Python implementation and up to about 22.2 times faster than OpenSpiel’s C++ implementation and the speedup becomes more pronounced as the size of the game being solved grows.

1 INTRODUCTION

Counterfactual regret minimization (CFR) is a family of algorithms of no-regret learning dynamics capable of solving large-scale imperfect information games. Since Zinkevich et al. (2007) proposed a “vanilla” version of CFR, various variants have been proposed: CFR+ by Tammelin (2014) (optionally) eliminates the averaging step while improving the convergence rate; Sampling variants (Lanctot et al., 2009) makes a complete recursive tree traversal unnecessary; Burch et al. (2014) proposes CFR-D in which games are decomposed into subgames; Brown & Sandholm (2019) explores modifying CFR such as to explore alternate weighted averaging (and discounting) schemes; Xu et al. (2024) learns a discounting technique from smaller games to be used in larger games.

While many prior works have focused on making CFR converge faster, there has been a notable lack of work focusing on making CFR more computationally efficient. In this paper, we propose implementing this algorithm as a series of dense and sparse matrix and vector operations, thereby making it highly parallelizable for a graphical processing unit (GPU). We then experiment by analyzing the runtimes of our implementation both with a computer processing unit (CPU) backend and a graphical processing units (GPU) backend and comparing them to DeepMind’s OpenSpiel (Lanctot et al., 2020) implementations (Python and C++) on 8 games of differing sizes, from small to large.

Our experiments show that, compared to DeepMind OpenSpiel’s (Lanctot et al., 2020) Python implementation, ours with a GPU performs worse (about 2.7 times slower) for small games but is up to about 352.5 times faster for large games. When compared against their C++ implementation, our performance with a GPU ranges from 75.7 times slower to 22.2 times faster. While our main focus was running CFR on a GPU, we found that, even without a GPU (i.e. with a CPU backend), our implementation shows speedups compared to the OpenSpiel baselines (from about 1.7 to 56.4 times faster than their Python implementation and from 17.0 times slower to 5.6 times faster than theirs in C++). In general, we see that the speedup becomes more pronounced as the size of the game grows.

2 BACKGROUND

2.1 FINITE EXTENSIVE-FORM GAMES

Definition 1 Formally, a **finite extensive-form game** (Osborne & Rubinstein, 1994) is a structure $\mathcal{G} = \langle \mathcal{T}, \mathbb{H}, f_h, \mathbb{A}, f_a, \mathbb{I}, f_i, \sigma_0, u \rangle$ where:

-
- $\mathcal{T} = \langle \mathbb{V}, v_0, \mathbb{T}, f_{Pa} \rangle$ is a **finite game tree** with a **finite set of nodes** (i.e. vertices) \mathbb{V} , a **unique initial node** (i.e. a root) $v_0 \in \mathbb{V}$, a **finite set of terminal nodes** (i.e. leaves) $\mathbb{T} \subseteq \mathbb{V}$, and a **parent function** $f_{Pa} : \mathbb{V}_+ \rightarrow \mathbb{D}$ that maps a non-initial node (i.e. a non-root) $v_+ \in \mathbb{V}_+$ to an immediate predecessor (i.e. a parent) $d \in \mathbb{D}$, with $\mathbb{V}_+ = \mathbb{V} \setminus \{v_0\}$ the finite set of non-initial nodes (i.e. non-roots) and $\mathbb{D} = \mathbb{V} \setminus \mathbb{T}$ the finite set of decision nodes (i.e. internal vertices),
 - \mathbb{H} is a **finite set of information sets**, $f_h : \mathbb{D} \rightarrow \mathbb{H}$ is an **information partition** of \mathbb{D} associating each decision node $d \in \mathbb{D}$ to an information set $h \in \mathbb{H}$,
 - \mathbb{A} is a **finite set of actions**, $f_a : \mathbb{V}_+ \rightarrow \mathbb{A}$ is an **action partition** of \mathbb{V}_+ associating each non-initial node $v_+ \in \mathbb{V}_+$ to an action $a \in \mathbb{A}$ such that $\forall d \in \mathbb{D}$ the restriction $f_{a,d} : S(d) \rightarrow A(f_h(d))$ is a bijection, with $S(d \in \mathbb{D}) = \{v_+ \in \mathbb{V}_+ : f_{Pa}(v_+) = d\}$ the finite set of immediate successors (i.e. children) of a node $d \in \mathbb{D}$ and $A(h \in \mathbb{H}) = \{a \in \mathbb{A} : [\exists v_+ \in \mathbb{V}_+] (f_h(f_{Pa}(v_+)) = h \wedge f_a(v_+) = a)\}$ the finite set of available actions at an information set $h \in \mathbb{H}$,
 - \mathbb{I} is a **finite set of (rational) players and, optionally, the nature** (i.e. chance) $i_0 \in \mathbb{I}$, $f_i : \mathbb{H} \rightarrow \mathbb{I}$ is a **player partition** of \mathbb{H} associating each information set $h \in \mathbb{H}$ to a player $i \in \mathbb{I}$,
 - $\sigma_0 : \mathbb{Q}_0 \rightarrow [0, 1]$ is a **chance probabilities function** that associates each pair of a nature information set and an available action $(h_0, a) \in \mathbb{Q}_0$ to an independent probability value, with $\mathbb{Q}_j = \{(h, a) \in \mathbb{Q} : h \in \mathbb{H}_j\}$ the finite set of pairs of a player information set $h_j \in \mathbb{H}_j$ and an available action $a \in A(h_j)$, $\mathbb{Q} = \{(h, a) \in \mathbb{H} \times \mathbb{A} : a \in A(h)\}$ the finite set of pairs of an information set $h \in \mathbb{H}$ and an available action $a \in A(h)$, and $\mathbb{H}_j = \{h \in \mathbb{H} : f_i(h) = i_j\}$ the finite set of information sets associated with a player $i_j \in \mathbb{I}$, and
 - $u : \mathbb{T} \times \mathbb{I}_+ \rightarrow \mathbb{R}$ is a **utility function** that associates each pair of a terminal node $t \in \mathbb{T}$ and a (rational) player $i_+ \in \mathbb{I}_+$ to a real payoff value, with $\mathbb{I}_+ = \mathbb{I} \setminus \{i_0\}$ the finite set of (rational) players.

2.2 NASH EQUILIBRIUM

Each player $i_j \in \mathbb{I}$ selects a **player strategy** $\sigma_j : \mathbb{Q}_j \rightarrow [0, 1]$ from a **set of player strategies** Σ_j . A player strategy $\sigma_j \in \Sigma_j$ associates, for each player information set $h_j \in \mathbb{H}_j$, a probability distribution over a finite set of available actions $A(h_j)$. A **strategy profile** $\sigma : \mathbb{Q} \rightarrow [0, 1]$ is a direct sum of the strategies of each player $\sigma = \bigoplus_{i_j \in \mathbb{I}} \sigma_j$ which, for each information set $h \in \mathbb{H}$, gives a probability distribution over a finite set of available actions $A(h)$. Σ is a set of strategy profiles. $\sigma_{-j} = \bigoplus_{i_k \in \mathbb{I} \setminus \{i_j\}} \sigma_k$ is used to denote a direct sum of all player strategies in σ except σ_j (i.e. that of player $i_j \in \mathbb{I}$).

Define $\pi : \Sigma \times \mathbb{V} \rightarrow \mathbb{R}$ to be a probability of reaching a vertex $v \in \mathbb{V}$ following a strategy profile $\sigma \in \Sigma$.

$$\pi(\sigma \in \Sigma, v \in \mathbb{V}) = \begin{cases} \sigma(f_h(f_{Pa}(v)), f_a(v))\pi(\sigma, f_{Pa}(v)) & v \in \mathbb{V}_+ \\ 1 & v = v_0 \end{cases}$$

Then, let $\hat{u} : \Sigma \times \mathbb{I} \rightarrow \mathbb{R}$ be an expected payoff of a (rational) player $i_+ \in \mathbb{I}_+$, following a strategy profile $\sigma \in \Sigma$.

$$\hat{u}(\sigma \in \Sigma, i_+ \in \mathbb{I}_+) = \sum_{t \in \mathbb{T}} \pi(\sigma, t) u(t, i_+)$$

A strategy profile $\sigma^* \in \Sigma$ is a **Nash equilibrium** if

$$\forall i_{+,j} \in \mathbb{I}_+ \quad \hat{u}(\sigma^*, i_{+,j}) \geq \max_{\sigma'_j \in \Sigma_j} \hat{u}(\sigma'_j \oplus \sigma_{-j}^*, i_{+,j})$$

A strategy profile that approximates a Nash equilibrium σ^* is an **ϵ -Nash equilibrium** $\sigma^{*,\epsilon} \in \Sigma$ if

$$\forall i_{+,j} \in \mathbb{I}_+ \quad \hat{u}(\sigma^{*,\epsilon}, i_{+,j}) + \epsilon \geq \max_{\sigma'_j \in \Sigma_j} \hat{u}(\sigma'_j \oplus \sigma_{-j}^{*,\epsilon}, i_{+,j})$$

2.3 COUNTERFACTUAL REGRET MINIMIZATION

Define $\check{u} : \Sigma \times \mathbb{V} \times \mathbb{I}_+ \rightarrow \mathbb{R}$ as an expected payoff of a (rational) player $i_+ \in \mathbb{I}_+$ at a node $v \in \mathbb{V}$, following a strategy profile $\sigma \in \Sigma$.

$$\check{u}(\sigma \in \Sigma, v \in \mathbb{V}, i_+ \in \mathbb{I}_+) = \begin{cases} \sum_{s \in S(v)} \sigma(f_h(v), f_a(s)) \check{u}(\sigma, s, i_+) & v \in \mathbb{D} \\ u(v, i_+) & v \in \mathbb{T} \end{cases} \quad (1)$$

Let $\check{\pi} : \Sigma \times \mathbb{V} \times \mathbb{I} \rightarrow \mathbb{R}$ be a probability of reaching a vertex $v \in \mathbb{V}$ following a strategy profile $\sigma \in \Sigma$ while ignoring a strategy of a player $i \in \mathbb{I}$.

$$\check{\pi}(\sigma \in \Sigma, v \in \mathbb{V}, i \in \mathbb{I}) = \begin{cases} \check{\pi}(\sigma, f_{Pa}(v), i) & \begin{cases} \sigma(f_h(f_{Pa}(v)), f_a(v)) & f_i(f_h(f_{Pa}(v))) \neq i \\ 1 & f_i(f_h(f_{Pa}(v))) = i \end{cases} \\ 1 & v = v_0 \end{cases} \quad (2)$$

Below definition shows a “counterfactual” reach probability $\tilde{\pi} : \Sigma \times \mathbb{H} \rightarrow \mathbb{R}$.

$$\tilde{\pi}(\sigma \in \Sigma, h \in \mathbb{H}) = \sum_{d \in \mathbb{D}: f_h(d) = h} \check{\pi}(\sigma, d, f_i(h)) \quad (3)$$

Now, let $\tilde{u} : \Sigma \times \mathbb{H}_+ \rightarrow \mathbb{R}$ be a counterfactual utility, with $\mathbb{H}_+ = \mathbb{H} \setminus \mathbb{H}_0$ the finite set of information sets associated with (rational) players.

$$\tilde{u}(\sigma \in \Sigma, h_+ \in \mathbb{H}_+) = \frac{\sum_{d \in \mathbb{D}: f_h(d) = h_+} \check{\pi}(\sigma, d, f_i(h_+)) \check{u}(\sigma, d, f_i(h_+))}{\check{\pi}(\sigma, h_+)} \quad (4)$$

$\sigma|_{h \rightarrow a} \in \Sigma$ is used to denote an overridden strategy profile of σ where an action $a \in A(h)$ is always taken at an information set $h \in \mathbb{H}$.

$$\sigma|_{h \rightarrow a}((h', a') \in \mathbb{Q}) = \begin{cases} \mathbf{1}_{a=a'} & h = h' \\ \sigma(h', a') & h \neq h' \end{cases}$$

$\tilde{r} : \Sigma \times \mathbb{Q}_+ \rightarrow \mathbb{R}$ is the instantaneous counterfactual regret, with $\mathbb{Q}_+ = \mathbb{Q} \setminus \mathbb{Q}_0$ the finite set of pairs of a (rational) player information set $h_+ \in \mathbb{H}_+$ and an available action $a \in A(h_+)$.

$$\tilde{r}(\sigma \in \Sigma, (h_+, a) \in \mathbb{Q}_+) = \tilde{\pi}(\sigma, h_+) (\tilde{u}(\sigma|_{h_+ \rightarrow a}, h_+) - \tilde{u}(\sigma, h_+)) \quad (5)$$

$\bar{r}^{(T)} : \mathbb{Q}_+ \rightarrow \mathbb{R}$ is the average counterfactual regret at an iteration T . $\sigma^{(\tau)} \in \Sigma$ is the strategy played at an iteration τ .

$$\bar{r}^{(T)}(q_+ \in \mathbb{Q}_+) = \frac{1}{T} \sum_{\tau=1}^T \tilde{r}(\sigma^{(\tau)}, q_+) \quad (6)$$

The strategy profile for the next iteration $T + 1$ is $\sigma^{(T+1)} \in \Sigma$.

$$\sigma^{(T+1)}((h, a) \in \mathbb{Q}) = \begin{cases} \begin{cases} \frac{(\bar{r}^{(T)}(h, a))^+}{\sum_{a' \in A(h)} (\bar{r}^{(T)}(h, a'))^+} & \sum_{a' \in A(h)} (\bar{r}^{(T)}(h, a'))^+ > 0 \\ \frac{1}{|A(h)|} & \sum_{a' \in A(h)} (\bar{r}^{(T)}(h, a'))^+ = 0 \\ \sigma_0(h, a) & \end{cases} & (h, a) \in \mathbb{Q}_+ \\ & (h, a) \in \mathbb{Q}_0 \end{cases} \quad (7)$$

Counterfactual regret minimization (Zinkevich et al., 2007) is an algorithm that iteratively approximates a coarse correlated equilibrium $\bar{\sigma}^{(T)} : \mathbb{Q} \rightarrow \mathbb{R}$ (Hart & Mas-Colell, 2000).

$$\bar{\sigma}^{(T)}((h, a) \in \mathbb{Q}) = \frac{\sum_{\tau=1}^T \tilde{\pi}(\sigma^{(\tau)}, h) \sigma^{(\tau)}(h, a)}{\sum_{\tau=1}^T \tilde{\pi}(\sigma^{(\tau)}, h)} \quad (8)$$

Define $r^{(T)} : \mathbb{I}_+ \rightarrow \mathbb{R}$ as the average overall regret of a (rational) player $i_{+,j} \in \mathbb{I}_+$ at an iteration T .

$$r^{(T)}(i_{+,j} \in \mathbb{I}_+) = \frac{1}{T} \max_{\sigma'_j \in \Sigma_j} \sum_{\tau=1}^T (\hat{u}(\sigma'_j \oplus \sigma_{-j}^{(\tau)}, i_{+,j}) - \hat{u}(\sigma^{(\tau)}, i_{+,j}))$$

In 2-player ($|\mathbb{I}_+| = 2$) zero-sum games, if $\forall i_+ \in \mathbb{I}_+ r^{(T)}(i_+) \leq \epsilon$, the average strategy $\bar{\sigma}^{(T)}$ (at an iteration T) is also a 2ϵ -Nash equilibrium $\sigma^{*,2\epsilon} \in \Sigma$ (Zinkevich et al., 2007).

3 IMPLEMENTATION

In order to highly parallelize the execution of the counterfactual regret minimization algorithm, we implement the algorithm as a series of dense and sparse matrix operations and avoid recursive game tree traversal.

3.1 SETUP

The calculations of expected payoffs of (rational) players $\check{u} : \Sigma \times \mathbb{V} \times \mathbb{I}_+ \rightarrow \mathbb{R}$ in Equation 1 and “excepted” reach probability $\check{\pi} : \Sigma \times \mathbb{V} \times \mathbb{I} \rightarrow \mathbb{R}$ in Equation 2 are classical problems of dynamic programming on trees. To calculate these values efficiently with matrix operations, we represent the game tree \mathcal{T} as an adjacency matrix $\mathbf{G} \in \mathbb{R}^{\mathbb{V}^2}$ and the level graphs of the game tree \mathcal{T} as adjacency matrices $\mathbf{L}^{(1)}, \mathbf{L}^{(2)}, \dots, \mathbf{L}^{(D)} \in \mathbb{R}^{\mathbb{V}^2}$, with $D = \max_{t \in \mathbb{T}} d_{\mathcal{T}}(t)$ the maximum depth of any (terminal) node in the game tree \mathcal{T} and $d_{\mathcal{T}} : \mathbb{V} \rightarrow \mathbb{Z}$ the depth of a vertex $v \in \mathbb{V}$ in the game tree \mathcal{T} with respect to the root v_0 . Level graphs are illustrated in Figure 1.

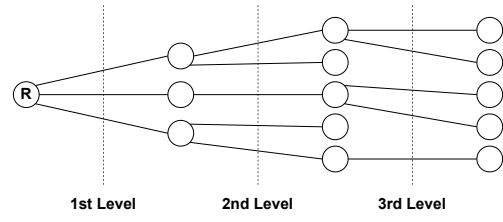


Figure 1: An example tree with 3 levels. Each level graph is composed of edges connecting nodes from one depth to another. The root node is denoted with a character ‘R’.

$$d_{\mathcal{T}}(v \in \mathbb{V}) = \begin{cases} 1 + d_{\mathcal{T}}(f_{Pa}(v)) & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases}$$

$$\mathbf{G} = \left(\begin{cases} \mathbf{1}_{v=f_{Pa}(v')} & v \in \mathbb{D} \wedge v' \in \mathbb{V}_+ \\ 0 & v \in \mathbb{T} \vee v' = v_0 \end{cases} \right)_{(v,v') \in \mathbb{V}^2} \quad (9)$$

$$\forall l \in [1, D] \cap \mathbb{Z} \quad \mathbf{L}^{(l)} = \left(\begin{cases} \mathbf{1}_{v=f_{Pa}(v') \wedge d_{\mathcal{T}}(v')=l} & v \in \mathbb{D} \wedge v' \in \mathbb{V}_+ \\ 0 & v \in \mathbb{T} \vee v' = v_0 \end{cases} \right)_{(v,v') \in \mathbb{V}^2} \quad (10)$$

Game (in OpenSpiel)	Sparsities (%)			
	$\mathbf{M}^{(Q_+, V)}$	$\mathbf{M}^{(H_+, Q_+)}$	$\mathbf{L}^{(l)}$ (Average)	\mathbf{G}
first_sealed_auction	99.54200	95.00000	99.99765	99.98591
kuhn_poker	96.55172	91.66667	99.71760	98.30559
kuhn_poker(players=3)	99.02755	97.91667	99.98202	99.83819
leduc_poker	99.95730	99.89316	99.99912	99.98943
liars_dice	99.99797	99.99593	99.99998	99.99966
tic.tac.toe	99.99982	99.99966	99.99998	99.99982
tiny_bridge_2p	99.98600	99.97210	99.99992	99.99907
tiny_hanabi	96.36364	87.50000	99.64298	98.21488

Table 1: The sparsities of sparse matrix constants in our implementation. The entries in the leftmost column correspond exactly to the game name in Deepmind’s OpenSpiel (Lanctot et al., 2020) library. CUDA’s (Nickolls et al., 2008) cuSPARSE “library targets matrices with sparsity ratios in the range between 70%-99.9%” (cuS). Our values fall under this recommended range. We project that the matrices for games not tested in our work will typically have similar sparsity values as those we test.

We also define matrices $\mathbf{M}^{(Q_+, V)} \in \mathbb{R}^{\mathbb{Q}_+ \times \mathbb{V}}$, $\mathbf{M}^{(H_+, Q_+)} \in \mathbb{R}^{\mathbb{H}_+ \times \mathbb{Q}_+}$, $\mathbf{M}^{(V, I_+)} \in \mathbb{R}^{\mathbb{V} \times \mathbb{I}_+}$ to represent the game \mathcal{G} . Matrix $\mathbf{M}^{(Q_+, V)}$ describes whether a node $v \in \mathbb{V}$ is a result of an action from a (rational) player information set $(h_+, a) \in \mathbb{Q}_+$. Matrix $\mathbf{M}^{(H_+, Q_+)}$ describes whether a (rational) player information set $h_+ \in \mathbb{H}_+$ is the first element of the corresponding (rational) player information set-action pair $(h_+, a) \in \mathbb{Q}_+$. Finally, matrix $\mathbf{M}^{(V, I_+)}$ describes whether a node $v \in \mathbb{V}$ has a parent whose associated information set’s associated player is $i_+ \in \mathbb{I}_+$ (i.e. which player $i_+ \in \mathbb{I}_+$ acted to reach a node $v \in \mathbb{V}$). Note that we omit the nature player i_0 and related information sets \mathbb{H}_0 and information set-action pairs \mathbb{Q}_0 as only the strategies of (rational) players are updated by the algorithm.

$$\mathbf{M}^{(Q_+, V)} = \left(\begin{array}{ll} \mathbf{1}_{q_+ = (f_h(f_{Pa}(v)), f_a(v))} & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{array} \right)_{(q_+, v) \in \mathbb{Q}_+ \times \mathbb{V}} \quad (11)$$

$$\mathbf{M}^{(H_+, Q_+)} = \left(\mathbf{1}_{h_+ = h'_+} \right)_{(h_+, (h'_+, a)) \in \mathbb{H}_+ \times \mathbb{Q}_+} \quad (12)$$

$$\mathbf{M}^{(V, I_+)} = \left(\begin{array}{ll} \mathbf{1}_{f_i(f_h(f_{Pa}(v))) = i_+} & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{array} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \quad (13)$$

The matrices $\mathbf{G}, \mathbf{L}^{(1)}, \mathbf{L}^{(2)}, \dots, \mathbf{L}^{(D)}, \mathbf{M}^{(Q_+, V)}, \mathbf{M}^{(H_+, Q_+)}, \mathbf{M}^{(V, I_+)}$ are constant matrices. In the games we test later, all aforesaid matrices except $\mathbf{M}^{(V, I_+)}$ are highly sparse (as demonstrated in Table 1) and therefore we implement them as sparse matrices in a compressed sparse row (CSR) format. Matrix $\mathbf{M}^{(V, I_+)}$ and all other matrices and vectors defined below throughout our description are implemented as dense.

We also define a dense vector $\mathbf{s}^{(\sigma_0)}$ representing the probabilities of nature information set-action pairs \mathbb{Q}_0 .

$$\mathbf{s}^{(\sigma_0)} = \left(\begin{array}{ll} \sigma_0(f_h(f_{Pa}(v)), f_a(v)) & f_h(f_{Pa}(v)) \in \mathbb{H}_0 \\ 0 & f_h(f_{Pa}(v)) \in \mathbb{H}_+ \\ 0 & v = v_0 \end{array} \right)_{v \in \mathbb{V}} \quad (14)$$

Let a dense vector $\boldsymbol{\sigma} \in \mathbb{R}^{\mathbb{Q}_+}$ represent the strategy related to (rational) player information set-action pairs \mathbb{Q}_+ at an iteration T .

$$\boldsymbol{\sigma} = \left(\sigma^{(T)}(q_+) \right)_{q_+ \in \mathbb{Q}_+} \quad (15)$$

A dense vector $\sigma^{(T=1)} \in \mathbb{R}^{\mathbb{Q}_+}$ representing the initial strategy profile (i.e. at $T = 1$) is shown below.

$$\sigma^{(T=1)} = (\sigma^{(1)}(q_+))_{q_+ \in \mathbb{Q}_+} = \left(\frac{1}{|A(h_+)|} \right)_{(h_+, a) \in \mathbb{Q}_+} = \mathbf{1}_{|\mathbb{Q}_+|} \oslash \left((\mathbf{M}^{(H_+, Q_+)})^\top ((\mathbf{M}^{(H_+, Q_+)}) \mathbf{1}_{|\mathbb{Q}_+|}) \right) \quad (16)$$

On each iteration, the strategy at the next iteration $\sigma' = (\sigma^{(T+1)}(q_+))_{q_+ \in \mathbb{Q}_+}$ is to be calculated using σ .

3.2 ITERATION

3.2.1 TREE TRAVERSAL

Let a dense vector $s \in \mathbb{R}^{\mathbb{V}}$ represent the probabilities of taking an action that reaches a node $v \in \mathbb{V}$ at an iteration T . This value is irrelevant for the unique initial node v_0 .

$$s = \begin{pmatrix} \sigma^{(T)}(f_h(f_{Pa}(v)), f_a(v)) & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{pmatrix}_{v \in \mathbb{V}} = (\mathbf{M}^{(Q_+, V)})^\top \sigma + s^{(\sigma_0)} \quad (17)$$

For later use, we also broadcast the vector s to be a matrix $S \in \mathbb{R}^{\mathbb{V}^2}$. Note that we define this for notational convenience and, in our implementation, we don't actually store this matrix in memory as this matrix can be quite large for larger games.

$$S = (s_{v'})_{(v, v') \in \mathbb{V}^2} \quad (18)$$

To represent the expected payoffs of (rational) players $\check{u} : \Sigma \times \mathbb{V} \times \mathbb{I}_+ \rightarrow \mathbb{R}$, as defined in Equation 1, we express the recurrence relations with matrices. For this purpose, we define dense matrices $\check{U}^{(1)}, \check{U}^{(2)}, \dots, \check{U}^{(D+1)} \in \mathbb{R}^{\mathbb{V} \times \mathbb{I}_+}$.

$$\forall l \in [1, D+1] \cap \mathbb{Z} \quad \check{U}^{(l)} = \begin{pmatrix} \check{u}(\sigma^{(T)}, v, i_+) & d_{\mathcal{T}}(v) \geq l-1 \vee v \in \mathbb{T} \\ 0 & d_{\mathcal{T}}(v) < l-1 \wedge v \in \mathbb{D} \end{pmatrix}_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \quad (19)$$

$$\check{U}^{(D+1)} = \begin{pmatrix} u(v, i_+) & v \in \mathbb{T} \\ 0 & v \in \mathbb{D} \end{pmatrix}_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \quad (20)$$

$$\forall l \in [1, D] \cap \mathbb{Z} \quad \check{U}^{(l)} = (\mathbf{L}^{(l)} \odot S) \check{U}^{(l+1)} + \check{U}^{(l+1)} \quad (21)$$

Let a dense matrix $\check{U} \in \mathbb{R}^{\mathbb{V} \times \mathbb{I}_+}$ represent $\check{u} : \Sigma \times \mathbb{V} \times \mathbb{I}_+ \rightarrow \mathbb{R}$.

$$\check{U} = (\check{u}(\sigma^{(T)}, v, i_+))_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} = \check{U}^{(1)} \quad (22)$$

Let $\check{S} \in \mathbb{R}^{\mathbb{V} \times \mathbb{I}_+}$ be a dense matrix to be used in a later calculation.

$$\check{S} = \begin{pmatrix} s_v & (\mathbf{M}^{(V, I_+)})_{v, i_+} = 0 \\ 1 & (\mathbf{M}^{(V, I_+)})_{v, i_+} = 1 \end{pmatrix}_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \quad (23)$$

In order to represent a restriction (ignoring nature) of the “excepted” reach probabilities (defined in Equation 2) $\check{\pi} : \Sigma \times \mathbb{V} \times \mathbb{I} \rightarrow \mathbb{R}$ with matrices, we, again, express the recurrence relations with matrices. We therefore define the following dense matrices: $\check{\Pi}^{(0)}, \check{\Pi}^{(1)}, \check{\Pi}^{(2)}, \dots, \check{\Pi}^{(D)} \in \mathbb{R}^{\mathbb{V} \times \mathbb{I}_+}$.

$$\forall l \in [0, D] \cap \mathbb{Z} \quad \check{\mathbf{\Pi}}^{(l)} = \begin{pmatrix} \check{\pi}(\sigma^{(T)}, v, i_+) & d_{\mathcal{T}}(v) \leq l \\ 0 & d_{\mathcal{T}}(v) > l \end{pmatrix}_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \quad (24)$$

$$\check{\mathbf{\Pi}}^{(0)} = (\mathbf{1}_{v=v_0})_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \quad (25)$$

$$\forall l \in [1, D] \cap \mathbb{Z} \quad \check{\mathbf{\Pi}}^{(l)} = \left(\left(\mathbf{L}^{(l)} \right)^{\top} \check{\mathbf{\Pi}}^{(l-1)} \right) \odot \check{\mathbf{S}} + \check{\mathbf{\Pi}}^{(l-1)} \quad (26)$$

Then, let a dense vector $\check{\boldsymbol{\pi}} \in \mathbb{R}^{\mathbb{V}}$ be the terms in Equation 3 for “counterfactual” reach probabilities $\tilde{\pi} : \Sigma \times \mathbb{H} \rightarrow \mathbb{R}$.

$$\check{\boldsymbol{\pi}} = \left(\begin{array}{lll} \left\{ \begin{array}{ll} \tilde{\pi}(\sigma^{(T)}, v, f_i(f_h(f_{Pa}(v)))) & f_h(f_{Pa}(v)) \in \mathbb{H}_+ \\ 0 & f_h(f_{Pa}(v)) \in \mathbb{H}_0 \end{array} \right. & v \in \mathbb{V}_+ \\ \left. \begin{array}{l} v = v_0 \\ 0 \end{array} \right. & v \in \mathbb{V} \end{array} \right)_{v \in \mathbb{V}} = \left(\mathbf{M}^{(V, I_+)} \odot \check{\mathbf{\Pi}}^{(D)} \right) \mathbf{1}_{|\mathbb{I}_+|} \quad (27)$$

3.2.2 AVERAGE STRATEGY PROFILE

The average strategy profile $\bar{\sigma}^{(T)} : \mathbb{Q} \rightarrow \mathbb{R}$ at the current iteration T , formulated in Equation 8 and represented as a dense vector $\bar{\boldsymbol{\sigma}} \in \mathbb{R}^{\mathbb{Q}_+}$ can be updated, from the previous iteration’s $\bar{\sigma}^{(T-1)} : \mathbb{Q} \rightarrow \mathbb{R}$, represented as a dense vector $\bar{\boldsymbol{\sigma}}' \in \mathbb{R}^{\mathbb{Q}_+}$. For this, the “counterfactual” reach probabilities $\tilde{\pi} : \Sigma \times \mathbb{H} \rightarrow \mathbb{R}$ (Equation 3), a restriction of which is represented by a dense vector $\tilde{\boldsymbol{\pi}} \in \mathbb{R}^{\mathbb{H}_+}$, and their running sums, a restriction of which is represented by a dense vector $\tilde{\boldsymbol{\pi}}^{(\Sigma)} \in \mathbb{R}^{\mathbb{H}_+}$, must also be calculated. The previous running sum of “counterfactual” reach probabilities is denoted as a dense vector $\tilde{\boldsymbol{\pi}}^{(\Sigma)'} \in \mathbb{R}^{\mathbb{H}_+}$.

$$\tilde{\boldsymbol{\pi}} = \left(\tilde{\pi}(\sigma^{(T)}, h_+) \right)_{h_+ \in \mathbb{H}_+} = \left(\left(\mathbf{M}^{(H_+, Q_+)} \right) \left(\mathbf{M}^{(Q_+, V)} \right) \tilde{\boldsymbol{\pi}} \right) \oslash \left(\left(\mathbf{M}^{(H_+, Q_+)} \right) \mathbf{1}_{|\mathbb{Q}_+|} \right) \quad (28)$$

$$\tilde{\boldsymbol{\pi}}^{(\Sigma)} = \left(\sum_{\tau=1}^T \tilde{\pi}(\sigma^{(\tau)}, h_+) \right)_{h_+ \in \mathbb{H}_+} = \tilde{\boldsymbol{\pi}}^{(\Sigma)'} + \tilde{\boldsymbol{\pi}} \quad (29)$$

$$\bar{\boldsymbol{\sigma}} = \left(\bar{\sigma}^{(T)}(q_+) \right)_{q_+ \in \mathbb{Q}_+} = \bar{\boldsymbol{\sigma}}' + \left(\left(\mathbf{M}^{(H_+, Q_+)} \right)^{\top} \left(\tilde{\boldsymbol{\pi}} \oslash \tilde{\boldsymbol{\pi}}^{(\Sigma)} \right) \right) \odot \left(\boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}}' \right) \quad (30)$$

3.2.3 NEXT STRATEGY PROFILE

Let a dense vector $\tilde{\mathbf{r}} \in \mathbb{R}^{\mathbb{Q}_+}$ represent instantaneous counterfactual regrets $\tilde{r} : \Sigma \times \mathbb{Q}_+ \rightarrow \mathbb{R}$, defined in Equation 5, for strategy profile $\sigma^{(T)}$.

$$\tilde{\mathbf{r}} = \left(\tilde{r}(\sigma^{(T)}, q_+) \right)_{q_+ \in \mathbb{Q}_+} = \left(\mathbf{M}^{(Q_+, V)} \right) \left(\tilde{\boldsymbol{\pi}} \odot \left(\left(\mathbf{M}^{(V, I_+)} \right) \odot \left(\check{\mathbf{U}} - \mathbf{G}^{\top} \check{\mathbf{U}} \right) \right) \mathbf{1}_{|\mathbb{I}_+|} \right) \quad (31)$$

Then, average counterfactual regrets $\bar{r}^{(T)} : \mathbb{Q}_+ \rightarrow \mathbb{R}$, in Equation 6, can be represented with a dense vector $\bar{\mathbf{r}} \in \mathbb{R}^{\mathbb{Q}_+}$, with a dense vector $\bar{\mathbf{r}}' \in \mathbb{R}^{\mathbb{Q}_+}$ the average counterfactual regrets at previous iteration $\bar{r}^{(T-1)} : \mathbb{Q}_+ \rightarrow \mathbb{R}$.

$$\bar{\mathbf{r}} = \left(\bar{r}^{(T)}(q_+) \right)_{q_+ \in \mathbb{Q}_+} = \bar{\mathbf{r}}' + \frac{1}{T} (\tilde{\mathbf{r}} - \bar{\mathbf{r}}') \quad (32)$$

Game (in OpenSpiel)	# Nodes	# Terminals	# Infosets	# Actions	# Players
first_sealed_auction	7096	3410	3056	11	2
kuhn_poker	58	30	16	3	2
kuhn_poker(players=3)	617	312	65	4	3
leduc_poker	9457	5520	1093	6	2
liars_dice	294883	147420	24583	13	2
tic_tac_toe	549946	255168	294778	9	2
tiny_bridge_2p	107129	53340	3613	28	2
tiny_hanabi	55	36	11	3	2

Table 2: The 8 games tested in our benchmark and relevant statistics: number of nodes, terminal nodes, information sets, actions, and (rational) players. The names in the first column correspond exactly to the game name in Deepmind’s OpenSpiel (Lanctot et al., 2020) library.

Game (in OpenSpiel)	Average CFR Iteration Runtime (seconds)			
	OpenSpiel		Ours	
	Python	C++	CPU	GPU
first_sealed_auction	0.08123	0.00370	0.00144	0.00283
kuhn_poker	0.00101	0.00004	0.00061	0.00271
kuhn_poker(players=3)	0.01522	0.00072	0.00102	0.00383
leduc_poker	0.15373	0.01544	0.00277	0.00467
liars_dice	1.35128	0.09811	0.07802	0.00766
tic_tac_toe	2.62992	0.16539	0.11971	0.00746
tiny_bridge_2p	0.64078	0.03752	0.01936	0.00480
tiny_hanabi	0.00085	0.00003	0.00051	0.00227

Table 3: The average per-iteration runtimes of CFR implementations: reference OpenSpiel’s (Lanctot et al., 2020) and ours (with a CPU or a GPU). The performances of the fastest implementation for each game are bolded. The names in the first column correspond exactly to the game name in Deepmind’s OpenSpiel (Lanctot et al., 2020) library.

Then, we normalize the clipped regrets to obtain a restriction of the next strategy profile $\sigma^{(T+1)} : \mathbb{Q}_+ \rightarrow \mathbb{R}$ from Equation 7 for (rational) player information set-action pairs, represented as a dense vector $\boldsymbol{\sigma}'$.

$$\bar{\mathbf{r}}^{(+,\Sigma)} = \left(\sum_{a' \in A(h_+)} \left(\bar{r}^{(T)}(h_+, a') \right)^+ \right)_{(h_+, a) \in \mathbb{Q}_+} = \left(\mathbf{M}^{(H_+, Q_+)} \right)^\top \left(\left(\mathbf{M}^{(H_+, Q_+)} \right) \bar{\mathbf{r}}^+ \right) \quad (33)$$

$$\boldsymbol{\sigma}' = \left(\sigma^{(T+1)}(q_+) \right)_{q_+ \in \mathbb{Q}_+} = \left(\begin{cases} (\bar{\mathbf{r}}^+ \oslash \bar{\mathbf{r}}^{(+,\Sigma)})_{q_+} & (\bar{\mathbf{r}}^{(+,\Sigma)})_{q_+} > 0 \\ (\boldsymbol{\sigma}^{(T=1)})_{q_+} & (\bar{\mathbf{r}}^{(+,\Sigma)})_{q_+} = 0 \end{cases} \right)_{q_+ \in \mathbb{Q}_+} \quad (34)$$

4 BENCHMARKS

We run 1,000 CFR iterations on 8 games of varying sizes implemented in DeepMind’s OpenSpiel (Lanctot et al., 2020) using their Python and C++ CFR implementations and our implementation (in Python). The statistics of each game tested are tabulated in Table 2. The games represent a diverse range of sizes from small (Kuhn poker and tiny bridge (2-player)), medium (first sealed auction, Kuhn poker (3-player), and Leduc poker), to large (liar’s dice, tic-tac-toe, and tiny bridge (2-player)). In our implementation, we use CuPy (Okuta et al., 2017) for GPU-accelerated matrix and vector operations. We also disable our GPU and simply run our implementation with NumPy (Harris et al., 2020) and SciPy (Virtanen et al., 2020) with a CPU backend. Our testbench computer contains an AMD Ryzen 9 3900X 12-core, 24-thread desktop processor, 128 GB memory, and Nvidia GeForce RTX 4090 24 GB VRAM graphics card.

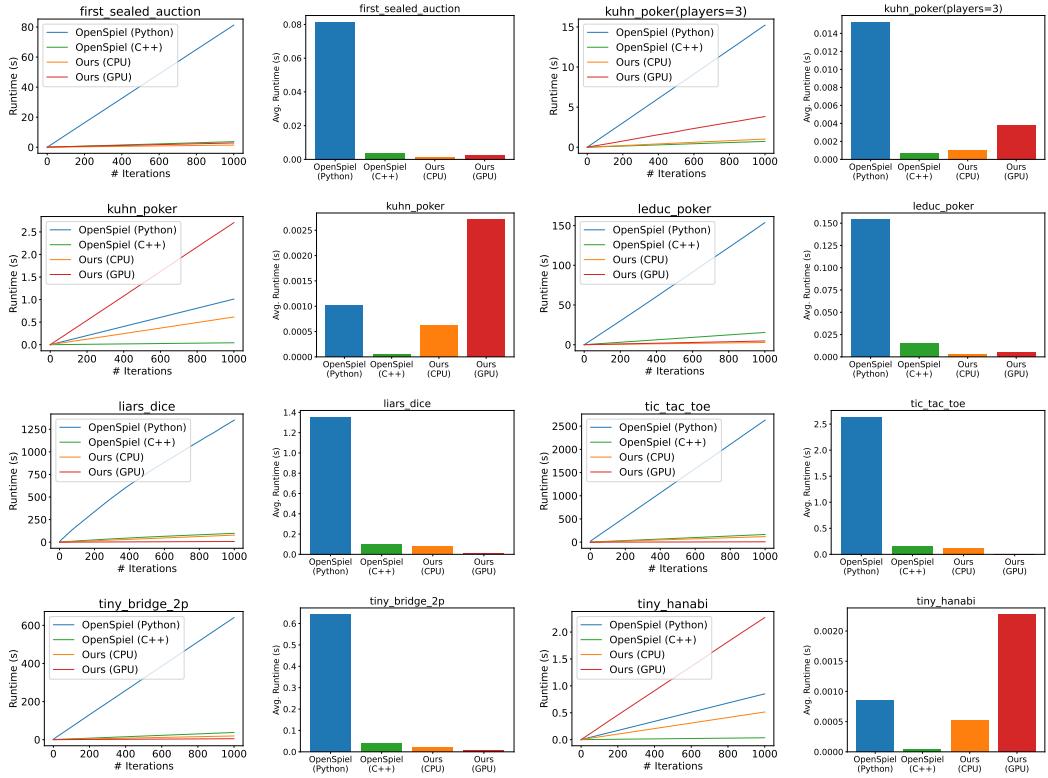


Figure 2: Pairs of plots for each game tested showing the runtimes for up to 1,000 iterations and a bar graph showing the average runtimes per iteration for four implementations of counterfactual regret minimization: Deepmind’s OpenSpiel (Lanctot et al., 2020) CFR implementation in Python and C++ and our implementation with a CPU and a GPU backend.

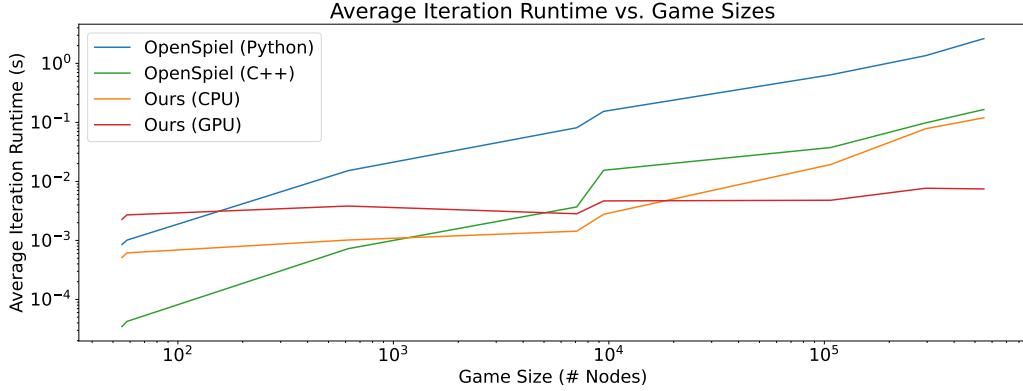


Figure 3: A log-log graph showing the average CFR iteration runtime with respect to the game size. The four line shows the runtimes of Deepmind’s OpenSpiel (Lanctot et al., 2020) CFR implementation in Python and C++ and our implementation with a CPU and a GPU backend.

The benchmark runtime results are tabulated in Table 3 and plotted in Figure 2. The results vary depending on the size of the game being played. The relationship between the game sizes and the runtimes of each implementation is shown more clearly in the log-log graph in Figure 3. Note that our GPU implementation clearly scales better than both OpenSpiel’s (Lanctot et al., 2020) and our CPU implementation.

4.1 SMALL GAMES: KUHN POKER AND TINY HANABI

In small games like Kuhn poker (58 nodes) and tiny Hanabi (55 nodes), our CPU implementation shows modest gains over the OpenSpiel’s (Lanctot et al., 2020) Python baseline (about 1.7 times faster for both). However, our GPU implementation is actually about 2.7 times slower for both. OpenSpiel’s C++ baseline vastly outperforms others by at least an order of magnitude. This suggests the overheads from GPU and Python make our implementation impractical for games of small sizes.

4.2 MEDIUM GAMES: FIRST SEALED AUCTION, KUHN POKER (3P), AND LEDUC POKER

In medium-sized games like first sealed auction (7096 nodes), Kuhn poker (3-player) (617 nodes), and Leduc poker (9457 nodes), noticeable performance gains can be observed for both our CPU implementation (about 56.4, 14.9, and 55.5 times faster, respectively) and GPU implementation (about 28.7, 4.0, and 32.9 times faster, respectively) compared to OpenSpiel’s (Lanctot et al., 2020) Python implementation. Comparisons with OpenSpiel’s C++ implementation is mixed. For the Kuhn poker (3-player), OpenSpiel’s C++ implementation is about 1.4 times faster than ours with a CPU backend and 5.3 times faster than ours with a GPU backend. But, for first sealed auction and Leduc poker, our CPU implementation is about 2.6 and 5.6 times faster, respectively, and our GPU implementation is about 1.3 and 3.3 times faster, respectively. The GPU overhead seems to make using it less preferable than not for medium-sized games.

4.3 LARGE GAMES: LIAR’S DICE, TIC-TAC-TOE, AND TINY BRIDGE (2P)

In larger games like liar’s dice (294883 nodes), tic-tac-toe (549946 nodes), and tiny bridge (2-player) (107129 nodes), noticeable performance gains over OpenSpiel’s (Lanctot et al., 2020) Python implementation can be observed for both our CPU implementation (about 17.3, 22.0, and 33.1 times faster, respectively) and GPU implementation (about 176.4, 352.5, and 133.5 times faster, respectively). The same can be said for OpenSpiel’s C++ implementation to a lesser degree: our CPU implementation is about 1.3, 1.4, and 1.9 times faster, respectively, and our GPU implementation is about 12.8, 22.2, and 7.8 times faster, respectively. We predict that the performance differences will be even more pronounced for games of larger sizes than the ones explored.

5 LIMITATIONS

In this work, we only explore parallelizing the vanilla counterfactual regret minimization algorithm, as proposed by Zinkevich et al. (2007). Later variants of CFR show improvements, namely in convergence speeds, which modify various aspects of the algorithm. The discounting techniques proposed by Brown and Sandholm can trivially be applied by altering Equation 32 and Equation 30. However, pruning techniques (Brown & Sandholm, 2015) and alternating player updates (Burch et al., 2019) would require non-trivial manipulations on the game-related matrices – possibly between iterations – problematic since updating certain types of sparse matrices like the CSR format we use is computationally expensive. In addition, unlike sampling variants of CFR (Lanctot et al., 2009), on each iteration, our implementation deals with the entire game tree and stores and updates values for every node – impractical for extremely large games.

6 CONCLUSION

We introduce our implementation of counterfactual regret minimization, designed to be highly parallelized by computing each iteration as dense and sparse matrix and vector operations and eliminating costly recursive tree traversal. While our goal was to run the algorithm on a GPU, the tight nature of our code also allows for a vastly more efficient computation when a GPU is not leveraged. Our experiments on solving 8 games of differing sizes show that, in larger games, our implementation achieves orders of magnitude performance improvements over DeepMind’s OpenSpiel (Lanctot et al., 2020) baselines in Python and C++, and predict that the performance benefit will be even more pronounced for games of sizes larger than those we tested. Incorporating the non-vanilla variants of counterfactual regret minimization with our implementation scheme remains a promising avenue for future research.

REFERENCES

- URL <https://docs.nvidia.com/cuda/cusparse/index.html>.
- Noam Brown and Tuomas Sandholm. Regret-based pruning in extensive-form games. In C. Cortes, N. Lawrence, D. Lee, M. Sugiyama, and R. Garnett (eds.), *Advances in Neural Information Processing Systems*, volume 28. Curran Associates, Inc., 2015. URL https://proceedings.neurips.cc/paper_files/paper/2015/file/c54e7837e0cd0ced286cb5995327d1ab-Paper.pdf.
- Noam Brown and Tuomas Sandholm. Solving imperfect-information games via discounted regret minimization. In *Proceedings of the Thirty-Third AAAI Conference on Artificial Intelligence and Thirty-First Innovative Applications of Artificial Intelligence Conference and Ninth AAAI Symposium on Educational Advances in Artificial Intelligence*, AAAI’19/IAAI’19/EAAI’19. AAAI Press, 2019. ISBN 978-1-57735-809-1. doi: 10.1609/aaai.v33i01.33011829. URL <https://doi.org/10.1609/aaai.v33i01.33011829>.
- Neil Burch, Michael Johanson, and Michael Bowling. Solving imperfect information games using decomposition. *Proceedings of the AAAI Conference on Artificial Intelligence*, 28(1), Jun. 2014. doi: 10.1609/aaai.v28i1.8810. URL <https://ojs.aaai.org/index.php/AAAI/article/view/8810>.
- Neil Burch, Matej Moravcik, and Martin Schmid. Revisiting cfr+ and alternating updates. *J. Artif. Int. Res.*, 64(1):429–443, jan 2019. ISSN 1076-9757. doi: 10.1613/jair.1.11370. URL <https://doi.org/10.1613/jair.1.11370>.
- Charles R. Harris, K. Jarrod Millman, Stéfan J. van der Walt, Ralf Gommers, Pauli Virtanen, David Cournapeau, Eric Wieser, Julian Taylor, Sebastian Berg, Nathaniel J. Smith, Robert Kern, Matti Picus, Stephan Hoyer, Marten H. van Kerkwijk, Matthew Brett, Allan Haldane, Jaime Fernández del Río, Mark Wiebe, Pearu Peterson, Pierre Gérard-Marchant, Kevin Sheppard, Tyler Reddy, Warren Weckesser, Hameer Abbasi, Christoph Gohlke, and Travis E. Oliphant. Array programming with NumPy. *Nature*, 585(7825):357–362, September 2020. doi: 10.1038/s41586-020-2649-2. URL <https://doi.org/10.1038/s41586-020-2649-2>.
- Sergiu Hart and Andreu Mas-Colell. A simple adaptive procedure leading to correlated equilibrium. *Econometrica*, 68(5):1127–1150, 2000. ISSN 00129682, 14680262. URL <http://www.jstor.org/stable/2999445>.
- Marc Lanctot, Kevin Waugh, Martin Zinkevich, and Michael Bowling. Monte carlo sampling for regret minimization in extensive games. In Y. Bengio, D. Schuurmans, J. Lafferty, C. Williams, and A. Culotta (eds.), *Advances in Neural Information Processing Systems*, volume 22. Curran Associates, Inc., 2009. URL https://proceedings.neurips.cc/paper_files/paper/2009/file/00411460f7c92d2124a67ea0f4cb5f85-Paper.pdf.
- Marc Lanctot, Edward Lockhart, Jean-Baptiste Lespiau, Vinicius Zambaldi, Satyaki Upadhyay, Julien Pérolat, Sriram Srinivasan, Finbarr Timbers, Karl Tuyls, Shayegan Omidshafiei, Daniel Hennes, Dustin Morrill, Paul Muller, Timo Ewalds, Ryan Faulkner, János Kramár, Bart De Vylder, Brennan Saeta, James Bradbury, David Ding, Sebastian Borgeaud, Matthew Lai, Julian Schrittwieser, Thomas Anthony, Edward Hughes, Ivo Danihelka, and Jonah Ryan-Davis. Open-spiel: A framework for reinforcement learning in games, 2020. URL <https://arxiv.org/abs/1908.09453>.
- John Nickolls, Ian Buck, Michael Garland, and Kevin Skadron. Scalable parallel programming with cuda: Is cuda the parallel programming model that application developers have been waiting for? *Queue*, 6(2):40–53, mar 2008. ISSN 1542-7730. doi: 10.1145/1365490.1365500. URL <https://doi.org/10.1145/1365490.1365500>.
- Ryosuke Okuta, Yuya Unno, Daisuke Nishino, Shohei Hido, and Crissman Loomis. Cupy: A numpy-compatible library for nvidia gpu calculations. In *Proceedings of Workshop on Machine Learning Systems (LearningSys) in The Thirty-first Annual Conference on Neural Information Processing Systems (NIPS)*, 2017. URL http://learningsys.org/nips17/assets/papers/paper_16.pdf.

M.J. Osborne and A. Rubinstein. *A Course in Game Theory*. MIT Press, 1994. ISBN 9780262650403. URL <https://books.google.ca/books?id=PuSMEAAAQBAJ>.

Oskari Tammelin. Solving large imperfect information games using cfr+, 2014. URL <https://arxiv.org/abs/1407.5042>.

Pauli Virtanen, Ralf Gommers, Travis E. Oliphant, Matt Haberland, Tyler Reddy, David Cournapeau, Evgeni Burovski, Pearu Peterson, Warren Weckesser, Jonathan Bright, Stéfan J. van der Walt, Matthew Brett, Joshua Wilson, K. Jarrod Millman, Nikolay Mayorov, Andrew R. J. Nelson, Eric Jones, Robert Kern, Eric Larson, C J Carey, İlhan Polat, Yu Feng, Eric W. Moore, Jake VanderPlas, Denis Laxalde, Josef Perktold, Robert Cimrman, Ian Henriksen, E. A. Quintero, Charles R. Harris, Anne M. Archibald, Antônio H. Ribeiro, Fabian Pedregosa, Paul van Mulbregt, and SciPy 1.0 Contributors. SciPy 1.0: Fundamental Algorithms for Scientific Computing in Python. *Nature Methods*, 17:261–272, 2020. doi: 10.1038/s41592-019-0686-2.

Hang Xu, Kai Li, Haobo Fu, QIANG FU, Junliang Xing, and Jian Cheng. Dynamic discounted counterfactual regret minimization. In *The Twelfth International Conference on Learning Representations*, 2024. URL <https://openreview.net/forum?id=6PbvbLyqT6>.

Martin Zinkevich, Michael Johanson, Michael Bowling, and Carmelo Piccione. Regret minimization in games with incomplete information. In J. Platt, D. Koller, Y. Singer, and S. Roweis (eds.), *Advances in Neural Information Processing Systems*, volume 20. Curran Associates, Inc., 2007. URL https://proceedings.neurips.cc/paper_files/paper/2007/file/08d98638c6fcfd194a4b1e6992063e944-Paper.pdf.

A INITIAL STRATEGY PROFILE

An expanded form of Equation 16 is shown below.

$$\begin{aligned}
\boldsymbol{\sigma}^{(T=1)} &= \left(\sigma^{(1)}(q_+) \right)_{q_+ \in \mathbb{Q}_+} \\
&= \left(\frac{1}{|A(h_+)|} \right)_{(h_+, a) \in \mathbb{Q}_+} \\
&= \mathbf{1}_{|\mathbb{Q}_+|} \oslash (|A(h_+)|)_{(h_+, a) \in \mathbb{Q}_+} \\
&= \mathbf{1}_{|\mathbb{Q}_+|} \oslash \left(\sum_{h'_+ \in \mathbb{H}_+} \mathbf{1}_{h_+ = h'_+} |A(h'_+)| \right)_{(h_+, a) \in \mathbb{Q}_+} \\
&= \mathbf{1}_{|\mathbb{Q}_+|} \oslash \left(\left(\mathbf{1}_{h_+ = h'_+} \right)_{((h_+, a), h'_+) \in \mathbb{Q}_+ \times \mathbb{H}_+} (|A(h_+)|)_{h_+ \in \mathbb{H}_+} \right) \\
&= \mathbf{1}_{|\mathbb{Q}_+|} \oslash \left(\left(\mathbf{1}_{h_+ = h'_+} \right)_{(h_+, (h'_+, a)) \in \mathbb{H}_+ \times \mathbb{Q}_+} \right)^\top (|A(h_+)|)_{h_+ \in \mathbb{H}_+}
\end{aligned}$$

Using Equation 12

$$\begin{aligned}
&= \mathbf{1}_{|\mathbb{Q}_+|} \oslash \left(\left(\mathbf{M}^{(H_+, Q_+)} \right)^\top (|A(h_+)|)_{h_+ \in \mathbb{H}_+} \right) \\
&= \mathbf{1}_{|\mathbb{Q}_+|} \oslash \left(\left(\mathbf{M}^{(H_+, Q_+)} \right)^\top \left(\sum_{(h'_+, a) \in \mathbb{Q}_+} \mathbf{1}_{h_+ = h'_+} \right)_{h_+ \in \mathbb{H}_+} \right) \\
&= \mathbf{1}_{|\mathbb{Q}_+|} \oslash \left(\left(\mathbf{M}^{(H_+, Q_+)} \right)^\top \left(\left(\mathbf{1}_{h_+ = h'_+} \right)_{(h_+, (h'_+, a)) \in \mathbb{H}_+ \times \mathbb{Q}_+} \right) \mathbf{1}_{|\mathbb{Q}_+|} \right)
\end{aligned}$$

Using Equation 12

$$= \mathbf{1}_{|\mathbb{Q}_+|} \oslash \left(\left(\mathbf{M}^{(H_+, Q_+)} \right)^\top \left(\mathbf{M}^{(H_+, Q_+)} \right) \mathbf{1}_{|\mathbb{Q}_+|} \right)$$

To take advantage of the sparsity of $\mathbf{M}^{(H_+, Q_+)}$ (see Table 1)

$$= \mathbf{1}_{|\mathbb{Q}_+|} \oslash \left(\left(\mathbf{M}^{(H_+, Q_+)} \right)^\top \left(\left(\mathbf{M}^{(H_+, Q_+)} \right) \mathbf{1}_{|\mathbb{Q}_+|} \right) \right)$$

B STRATEGIES

An expanded form of Equation 17 is shown below.

$$\begin{aligned} \mathbf{s} &= \left(\begin{cases} \sigma^{(T)}(f_h(f_{Pa}(v)), f_a(v)) & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right)_{v \in \mathbb{V}} \\ &= \left(\begin{cases} \begin{cases} \sigma^{(T)}(f_h(f_{Pa}(v)), f_a(v)) & f_h(f_{Pa}(v)) \in \mathbb{H}_+ \\ 0 & f_h(f_{Pa}(v)) \in \mathbb{H}_0 \end{cases} & v \in \mathbb{V}_+ \\ \begin{cases} 0 & v = v_0 \end{cases} & v = v_0 \end{cases} \right)_{v \in \mathbb{V}} \\ &\quad + \left(\begin{cases} \begin{cases} \sigma_0(f_h(f_{Pa}(v)), f_a(v)) & f_h(f_{Pa}(v)) \in \mathbb{H}_0 \\ 0 & f_h(f_{Pa}(v)) \in \mathbb{H}_+ \end{cases} & v \in \mathbb{V}_+ \\ \begin{cases} 0 & v = v_0 \end{cases} & v = v_0 \end{cases} \right)_{v \in \mathbb{V}} \end{aligned}$$

Using Equation 14

$$\begin{aligned} &= \left(\begin{cases} \begin{cases} \sigma^{(T)}(f_h(f_{Pa}(v)), f_a(v)) & f_h(f_{Pa}(v)) \in \mathbb{H}_+ \\ 0 & f_h(f_{Pa}(v)) \in \mathbb{H}_0 \end{cases} & v \in \mathbb{V}_+ \\ \begin{cases} 0 & v = v_0 \end{cases} & v = v_0 \end{cases} \right)_{v \in \mathbb{V}} + \mathbf{s}^{(\sigma_0)} \\ &= \left(\begin{cases} \sum_{h_+ \in \mathbb{H}_+} (\mathbf{1}_{h_+ = f_h(f_{Pa}(v))}) \sigma^{(T)}(h_+, f_a(v)) & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right)_{v \in \mathbb{V}} + \mathbf{s}^{(\sigma_0)} \\ &= \left(\begin{cases} \sum_{q_+ \in \mathbb{Q}_+} (\mathbf{1}_{q_+ = (f_h(f_{Pa}(v)), f_a(v))}) \sigma^{(T)}(q_+) & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right)_{v \in \mathbb{V}} + \mathbf{s}^{(\sigma_0)} \\ &= \left(\left(\begin{cases} \mathbf{1}_{q_+ = (f_h(f_{Pa}(v)), f_a(v))} & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right)_{(v, q_+) \in \mathbb{V} \times \mathbb{Q}_+} \right) \left(\sigma^{(T)}(q_+) \right)_{q_+ \in \mathbb{Q}_+} + \mathbf{s}^{(\sigma_0)} \\ &= \left(\left(\begin{cases} \mathbf{1}_{q_+ = (f_h(f_{Pa}(v)), f_a(v))} & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right)_{(q_+, v) \in \mathbb{Q}_+ \times \mathbb{V}} \right)^\top \left(\sigma^{(T)}(q_+) \right)_{q_+ \in \mathbb{Q}_+} + \mathbf{s}^{(\sigma_0)} \end{aligned}$$

Using Equation 11 and Equation 15

$$= (\mathbf{M}^{(Q_+, V)})^\top \boldsymbol{\sigma} + \mathbf{s}^{(\sigma_0)}$$

C EXPECTED PAYOFFS

C.1 INITIAL CONDITION

An expanded form of Equation 20 is shown below.

$$\begin{aligned}\check{\mathbf{U}}^{(D+1)} &= \left(\begin{cases} \check{u}(\sigma^{(T)}, v, i_+) & d_{\mathcal{T}}(v) \geq l - 1 \vee v \in \mathbb{T} \\ 0 & d_{\mathcal{T}}(v) < l - 1 \wedge v \in \mathbb{D} \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \\ &= \left(\begin{cases} \check{u}(\sigma^{(T)}, v, i_+) & d_{\mathcal{T}}(v) \geq D \vee v \in \mathbb{T} \\ 0 & d_{\mathcal{T}}(v) < D \wedge v \in \mathbb{D} \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+}\end{aligned}$$

Since $\forall d \in \mathbb{D} \quad d_{\mathcal{T}}(d) < D = \max_{t \in \mathbb{T}} d_{\mathcal{T}}(t)$

$$= \left(\begin{cases} \check{u}(\sigma^{(T)}, v, i_+) & v \in \mathbb{T} \\ 0 & v \in \mathbb{D} \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+}$$

Using Equation 1

$$\begin{aligned}&= \left(\begin{cases} \left(\begin{cases} \sum_{s \in S(v)} \sigma^{(T)}(f_h(v), f_a(s)) \check{u}(\sigma^{(T)}, s, i_+) & v \in \mathbb{D} \\ u(v, i_+) & v \in \mathbb{T} \end{cases} \right) & v \in \mathbb{D} \\ 0 & v \in \mathbb{T} \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \\ &= \left(\begin{cases} u(v, i_+) & v \in \mathbb{T} \\ 0 & v \in \mathbb{D} \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+}\end{aligned}$$

C.2 RECURRENCE

An expanded form of Equation 21 is shown below.

$\forall l \in [1, D] \cap \mathbb{Z}$

$$\begin{aligned}\check{\mathbf{U}}^{(l)} &= \left(\begin{cases} \check{u}(\sigma^{(T)}, v, i_+) & d_{\mathcal{T}}(v) \geq l - 1 \vee v \in \mathbb{T} \\ 0 & d_{\mathcal{T}}(v) < l - 1 \wedge v \in \mathbb{D} \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \\ &= \left(\begin{cases} \check{u}(\sigma^{(T)}, v, i_+) & d_{\mathcal{T}}(v) = l - 1 \wedge v \in \mathbb{D} \\ 0 & d_{\mathcal{T}}(v) \neq l - 1 \vee v \in \mathbb{T} \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \\ &\quad + \left(\begin{cases} \check{u}(\sigma^{(T)}, v, i_+) & (d_{\mathcal{T}}(v) \neq l - 1 \vee v \in \mathbb{T}) \wedge (d_{\mathcal{T}}(v) \geq l - 1 \vee v \in \mathbb{T}) \\ 0 & (d_{\mathcal{T}}(v) = l - 1 \wedge v \in \mathbb{D}) \vee (d_{\mathcal{T}}(v) < l - 1 \wedge v \in \mathbb{D}) \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \\ &= \left(\begin{cases} \check{u}(\sigma^{(T)}, v, i_+) & d_{\mathcal{T}}(v) = l - 1 \wedge v \in \mathbb{D} \\ 0 & d_{\mathcal{T}}(v) \neq l - 1 \vee v \in \mathbb{T} \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \\ &\quad + \left(\begin{cases} \check{u}(\sigma^{(T)}, v, i_+) & d_{\mathcal{T}}(v) \geq l \vee v \in \mathbb{T} \\ 0 & d_{\mathcal{T}}(v) < l \wedge v \in \mathbb{D} \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+}\end{aligned}$$

Using Equation 19

$$= \left(\begin{cases} \check{u}(\sigma^{(T)}, v, i_+) & d_{\mathcal{T}}(v) = l - 1 \wedge v \in \mathbb{D} \\ 0 & d_{\mathcal{T}}(v) \neq l - 1 \vee v \in \mathbb{T} \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} + \check{\mathbf{U}}^{(l+1)}$$

Using Equation 1

$$\begin{aligned} &= \left(\begin{cases} \left(\begin{cases} \sum_{s \in S(v)} \sigma^{(T)}(f_h(v), f_a(s)) \check{u}(\sigma^{(T)}, s, i_+) & v \in \mathbb{D} \\ u(v, i_+) & v \in \mathbb{T} \end{cases} \right) & d_{\mathcal{T}}(v) = l - 1 \wedge v \in \mathbb{D} \\ 0 & d_{\mathcal{T}}(v) \neq l - 1 \vee v \in \mathbb{T} \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} + \check{\mathbf{U}}^{(l+1)} \\ &= \left(\begin{cases} \left(\begin{cases} \sum_{s \in S(v)} \sigma^{(T)}(f_h(v), f_a(s)) \check{u}(\sigma^{(T)}, s, i_+) & d_{\mathcal{T}}(v) = l - 1 \wedge v \in \mathbb{D} \\ 0 & d_{\mathcal{T}}(v) \neq l - 1 \vee v \in \mathbb{T} \end{cases} \right) & (v, i_+) \in \mathbb{V} \times \mathbb{I}_+ \\ + \check{\mathbf{U}}^{(l+1)} & \end{cases} \right) \\ &= \left(\begin{cases} \left(\begin{cases} \sum_{s \in S(v)} (\mathbf{1}_{d_{\mathcal{T}}(s)=l}) \sigma^{(T)}(f_h(v), f_a(s)) \check{u}(\sigma^{(T)}, s, i_+) & v \in \mathbb{D} \\ 0 & v \in \mathbb{T} \end{cases} \right) & (v, i_+) \in \mathbb{V} \times \mathbb{I}_+ \\ + \check{\mathbf{U}}^{(l+1)} & \end{cases} \right) \\ &= \left(\begin{cases} \left(\begin{cases} \sum_{v' \in \mathbb{V}} (\mathbf{1}_{v=f_{Pa}(v') \wedge d_{\mathcal{T}}(v')=l}) \sigma^{(T)}(f_h(v), f_a(v')) \check{u}(\sigma^{(T)}, v', i_+) & v \in \mathbb{D} \wedge v' \in \mathbb{V}_+ \\ 0 & v \in \mathbb{T} \vee v' = v_0 \end{cases} \right) & (v, i_+) \in \mathbb{V} \times \mathbb{I}_+ \\ + \check{\mathbf{U}}^{(l+1)} & \end{cases} \right) \\ &= \left(\sum_{v' \in \mathbb{V}} \left(\begin{cases} \left(\begin{cases} (\mathbf{1}_{v=f_{Pa}(v') \wedge d_{\mathcal{T}}(v')=l}) \sigma^{(T)}(f_h(v), f_a(v')) & v \in \mathbb{D} \wedge v' \in \mathbb{V}_+ \\ 0 & v \in \mathbb{T} \vee v' = v_0 \end{cases} \right) & (v, i_+) \in \mathbb{V} \times \mathbb{I}_+ \\ + \check{\mathbf{U}}^{(l+1)} & \end{cases} \right) \right. \\ &\quad \left. \left(\begin{cases} \check{u}(\sigma^{(T)}, v', i_+) & d_{\mathcal{T}}(v') \geq l \vee v' \in \mathbb{T} \\ 0 & d_{\mathcal{T}}(v') < l \wedge v' \in \mathbb{D} \end{cases} \right) \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \\ &= \left(\begin{cases} \left(\begin{cases} (\mathbf{1}_{v=f_{Pa}(v') \wedge d_{\mathcal{T}}(v')=l}) \sigma^{(T)}(f_h(v), f_a(v')) & v \in \mathbb{D} \wedge v' \in \mathbb{V}_+ \\ 0 & v \in \mathbb{T} \vee v' = v_0 \end{cases} \right) & (v, v') \in \mathbb{V}^2 \\ + \check{\mathbf{U}}^{(l+1)} & \end{cases} \right) \\ \left(\begin{cases} \check{u}(\sigma^{(T)}, v, i_+) & d_{\mathcal{T}}(v) \geq l \vee v \in \mathbb{T} \\ 0 & d_{\mathcal{T}}(v) < l \wedge v \in \mathbb{D} \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \end{aligned}$$

Using Equation 19

$$\begin{aligned} &= \left(\left(\begin{cases} \left(\begin{cases} (\mathbf{1}_{v=f_{Pa}(v') \wedge d_{\mathcal{T}}(v')=l}) \sigma^{(T)}(f_h(v), f_a(v')) & v \in \mathbb{D} \wedge v' \in \mathbb{V}_+ \\ 0 & v \in \mathbb{T} \vee v' = v_0 \end{cases} \right) & (v, v') \in \mathbb{V}^2 \\ + \check{\mathbf{U}}^{(l+1)} & \end{cases} \right) \right) \check{\mathbf{U}}^{(l+1)} + \check{\mathbf{U}}^{(l+1)} \\ &= \left(\left(\begin{cases} \left(\begin{cases} (\mathbf{1}_{v=f_{Pa}(v') \wedge d_{\mathcal{T}}(v')=l}) \sigma^{(T)}(f_h(f_{Pa}(v')), f_a(v')) & v \in \mathbb{D} \wedge v' \in \mathbb{V}_+ \\ 0 & v \in \mathbb{T} \vee v' = v_0 \end{cases} \right) & (v, v') \in \mathbb{V}^2 \\ + \check{\mathbf{U}}^{(l+1)} & \end{cases} \right) \right) \check{\mathbf{U}}^{(l+1)} \\ &\quad + \check{\mathbf{U}}^{(l+1)} \end{aligned}$$

Using Equation 17

$$\begin{aligned}
&= \left(\left(\begin{cases} (\mathbf{1}_{v=f_{Pa}(v') \wedge d_{\mathcal{T}}(v')=l}) \mathbf{s}_{v'} & v \in \mathbb{D} \wedge v' \in \mathbb{V}_+ \\ 0 & v \in \mathbb{T} \vee v' = v_0 \end{cases} \right)_{(v,v') \in \mathbb{V}^2} \right) \check{\mathbf{U}}^{(l+1)} + \check{\mathbf{U}}^{(l+1)} \\
&= \left(\left(\left(\begin{cases} (\mathbf{1}_{v=f_{Pa}(v') \wedge d_{\mathcal{T}}(v')=l}) & v \in \mathbb{D} \wedge v' \in \mathbb{V}_+ \\ 0 & v \in \mathbb{T} \vee v' = v_0 \end{cases} \right) \mathbf{s}_{v'} \right)_{(v,v') \in \mathbb{V}^2} \right) \check{\mathbf{U}}^{(l+1)} + \check{\mathbf{U}}^{(l+1)} \\
&= \left(\left(\left(\begin{cases} (\mathbf{1}_{v=f_{Pa}(v') \wedge d_{\mathcal{T}}(v')=l}) & v \in \mathbb{D} \wedge v' \in \mathbb{V}_+ \\ 0 & v \in \mathbb{T} \vee v' = v_0 \end{cases} \right)_{(v,v') \in \mathbb{V}^2} \right) \odot (\mathbf{s}_{v'})_{(v,v') \in \mathbb{V}^2} \right) \check{\mathbf{U}}^{(l+1)} + \check{\mathbf{U}}^{(l+1)}
\end{aligned}$$

Using Equation 10 and Equation 18

$$= (\mathbf{L}^{(l)} \odot \mathbf{S}) \check{\mathbf{U}}^{(l+1)} + \check{\mathbf{U}}^{(l+1)}$$

D “EXCEPTED” REACH PROBABILITIES

D.1 INITIAL CONDITION

An expanded form of Equation 25 is shown below.

$$\begin{aligned}
\check{\Pi}^{(0)} &= \left(\begin{cases} \check{\pi}(\sigma^{(T)}, v, i_+) & d_{\mathcal{T}}(v) \leq l \\ 0 & d_{\mathcal{T}}(v) > l \end{cases} \right)_{(v,i_+) \in \mathbb{V} \times \mathbb{I}_+} \\
&= \left(\begin{cases} \check{\pi}(\sigma^{(T)}, v, i_+) & d_{\mathcal{T}}(v) \leq 0 \\ 0 & d_{\mathcal{T}}(v) > 0 \end{cases} \right)_{(v,i_+) \in \mathbb{V} \times \mathbb{I}_+} \\
&= \left(\begin{cases} \check{\pi}(\sigma^{(T)}, v, i_+) & v = v_0 \\ 0 & v \in \mathbb{V}_+ \end{cases} \right)_{(v,i_+) \in \mathbb{V} \times \mathbb{I}_+}
\end{aligned}$$

Using Equation 2

$$\begin{aligned}
&= \left(\begin{cases} \dots & v \in \mathbb{V}_+ \\ 1 & v = v_0 \\ 0 & v \in \mathbb{V}_+ \end{cases} \right)_{(v,i_+) \in \mathbb{V} \times \mathbb{I}_+} \\
&= \left(\begin{cases} 1 & v = v_0 \\ 0 & v \in \mathbb{V}_+ \end{cases} \right)_{(v,i_+) \in \mathbb{V} \times \mathbb{I}_+} \\
&= (\mathbf{1}_{v=v_0})_{(v,i_+) \in \mathbb{V} \times \mathbb{I}_+}
\end{aligned}$$

D.2 RECURRENCE

An expanded form of Equation 26 is shown below.

$$\forall l \in [1, D] \cap \mathbb{Z}$$

$$\begin{aligned}\widetilde{\Pi}^{(l)} &= \left(\begin{cases} \tilde{\pi}(\sigma^{(T)}, v, i_+) & d_{\mathcal{T}}(v) \leq l \\ 0 & d_{\mathcal{T}}(v) > l \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \\ &= \left(\begin{cases} \tilde{\pi}(\sigma^{(T)}, v, i_+) & d_{\mathcal{T}}(v) = l \\ 0 & d_{\mathcal{T}}(v) \neq l \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} + \left(\begin{cases} \tilde{\pi}(\sigma^{(T)}, v, i_+) & d_{\mathcal{T}}(v) \leq l - 1 \\ 0 & d_{\mathcal{T}}(v) > l - 1 \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+}\end{aligned}$$

Using Equation 24

$$= \left(\begin{cases} \tilde{\pi}(\sigma^{(T)}, v, i_+) & d_{\mathcal{T}}(v) = l \\ 0 & d_{\mathcal{T}}(v) \neq l \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} + \widetilde{\Pi}^{(l-1)}$$

Using Equation 2

$$\begin{aligned}&= \left(\begin{cases} \left(\begin{cases} \tilde{\pi}(\sigma^{(T)}, f_{Pa}(v), i_+) & f_h(f_{Pa}(v)) = f_a(v) \\ 1 & f_i(f_h(f_{Pa}(v))) \neq i_+ \\ 0 & f_i(f_h(f_{Pa}(v))) = i_+ \end{cases} \right) & v \in \mathbb{V}_+ \\ \left(\begin{cases} \tilde{\pi}(\sigma^{(T)}, f_{Pa}(v), i_+) & f_i(f_h(f_{Pa}(v))) \neq i_+ \\ 1 & f_i(f_h(f_{Pa}(v))) = i_+ \end{cases} \right) & v = v_0 \\ 0 & d_{\mathcal{T}}(v) \neq l \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \\ &\quad + \widetilde{\Pi}^{(l-1)} \\ &= \left(\begin{cases} \left(\begin{cases} \tilde{\pi}(\sigma^{(T)}, f_{Pa}(v), i_+) & f_i(f_h(f_{Pa}(v))) \neq i_+ \\ 1 & f_i(f_h(f_{Pa}(v))) = i_+ \end{cases} \right) & d_{\mathcal{T}}(v) = l \\ 0 & d_{\mathcal{T}}(v) \neq l \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} + \widetilde{\Pi}^{(l-1)} \\ &= \left(\left(\begin{cases} \left(\begin{cases} \tilde{\pi}(\sigma^{(T)}, f_{Pa}(v), i_+) & d_{\mathcal{T}}(v) = l \\ 0 & d_{\mathcal{T}}(v) \neq l \end{cases} \right) & \left(\begin{cases} \left(\begin{cases} \tilde{\pi}(\sigma^{(T)}, f_{Pa}(v), i_+) & f_i(f_h(f_{Pa}(v))) \neq i_+ \\ 1 & f_i(f_h(f_{Pa}(v))) = i_+ \end{cases} \right) & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right) \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \\ &\quad + \widetilde{\Pi}^{(l-1)} \\ &= \left(\begin{cases} \left(\begin{cases} \tilde{\pi}(\sigma^{(T)}, f_{Pa}(v), i_+) & d_{\mathcal{T}}(v) = l \\ 0 & d_{\mathcal{T}}(v) \neq l \end{cases} \right) & \left(\begin{cases} \left(\begin{cases} \tilde{\pi}(\sigma^{(T)}, f_{Pa}(v), i_+) & f_i(f_h(f_{Pa}(v))) \neq i_+ \\ 1 & f_i(f_h(f_{Pa}(v))) = i_+ \end{cases} \right) & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right) \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \\ &\quad \odot \left(\begin{cases} \left(\begin{cases} \tilde{\pi}(\sigma^{(T)}, f_{Pa}(v), i_+) & d_{\mathcal{T}}(v) = l \\ 0 & d_{\mathcal{T}}(v) \neq l \end{cases} \right) & \left(\begin{cases} \left(\begin{cases} \tilde{\pi}(\sigma^{(T)}, f_{Pa}(v), i_+) & f_i(f_h(f_{Pa}(v))) \neq i_+ \\ 1 & f_i(f_h(f_{Pa}(v))) = i_+ \end{cases} \right) & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right) \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} + \widetilde{\Pi}^{(l-1)}\end{aligned}$$

Using Equation 17 and Equation 13

$$\begin{aligned}&= \left(\left(\begin{cases} \tilde{\pi}(\sigma^{(T)}, f_{Pa}(v), i_+) & d_{\mathcal{T}}(v) = l \\ 0 & d_{\mathcal{T}}(v) \neq l \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \right) \odot \left(\begin{cases} \left(\begin{cases} \mathbf{s}_v & (\mathbf{M}^{(V, I_+)})_{v, i_+} = 0 \\ 1 & (\mathbf{M}^{(V, I_+)})_{v, i_+} = 1 \end{cases} \right) & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \\ &\quad + \widetilde{\Pi}^{(l-1)}\end{aligned}$$

Using Equation 17

$$\begin{aligned}&= \left(\left(\begin{cases} \tilde{\pi}(\sigma^{(T)}, f_{Pa}(v), i_+) & d_{\mathcal{T}}(v) = l \\ 0 & d_{\mathcal{T}}(v) \neq l \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \right) \odot \left(\begin{cases} \left(\begin{cases} \mathbf{s}_v & (\mathbf{M}^{(V, I_+)})_{v, i_+} = 0 \\ 1 & (\mathbf{M}^{(V, I_+)})_{v, i_+} = 1 \end{cases} \right) & v \in \mathbb{V}_+ \\ \mathbf{s}_v & v = v_0 \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \\ &\quad + \widetilde{\Pi}^{(l-1)}\end{aligned}$$

Using Equation 13

$$= \left(\left(\begin{cases} \tilde{\pi}(\sigma^{(T)}, f_{Pa}(v), i_+) & d_{\mathcal{T}}(v) = l \\ 0 & d_{\mathcal{T}}(v) \neq l \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \right) \odot \left(\begin{cases} \left(\begin{cases} \mathbf{s}_v & (\mathbf{M}^{(V, I_+)})_{v, i_+} = 0 \\ 1 & (\mathbf{M}^{(V, I_+)})_{v, i_+} = 1 \end{cases} \right) & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} + \widetilde{\Pi}^{(l-1)}$$

Using Equation 23

$$\begin{aligned}
&= \left(\left(\begin{cases} \pi(\sigma^{(T)}, f_{Pa}(v), i_+) & d_{\mathcal{T}}(v) = l \\ 0 & d_{\mathcal{T}}(v) \neq l \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \right) \odot \check{\mathbf{s}} + \check{\mathbf{\Pi}}^{(l-1)} \\
&= \left(\left(\begin{cases} \mathbf{1}_{d_{\mathcal{T}}(v)=l} \pi(\sigma^{(T)}, f_{Pa}(v), i_+) & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \right) \odot \check{\mathbf{s}} + \check{\mathbf{\Pi}}^{(l-1)} \\
&= \left(\left(\sum_{v' \in \mathbb{V}} \begin{cases} \mathbf{1}_{v'=f_{Pa}(v) \wedge d_{\mathcal{T}}(v)=l} \pi(\sigma^{(T)}, v', i_+) & v' \in \mathbb{D} \wedge v \in \mathbb{V}_+ \\ 0 & v' \in \mathbb{T} \vee v = v_0 \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \right) \odot \check{\mathbf{s}} + \check{\mathbf{\Pi}}^{(l-1)} \\
&= \left(\left(\sum_{v' \in \mathbb{V}} \left(\begin{cases} \mathbf{1}_{v'=f_{Pa}(v) \wedge d_{\mathcal{T}}(v)=l} & v' \in \mathbb{D} \wedge v \in \mathbb{V}_+ \\ 0 & v' \in \mathbb{T} \vee v = v_0 \end{cases} \right) \left(\begin{cases} \pi(\sigma^{(T)}, v', i_+) & d_{\mathcal{T}}(v') \leq l-1 \\ 0 & d_{\mathcal{T}}(v') > l-1 \end{cases} \right) \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \right) \odot \check{\mathbf{s}} \\
&\quad + \check{\mathbf{\Pi}}^{(l-1)} \\
&= \left(\left(\begin{cases} \mathbf{1}_{v'=f_{Pa}(v) \wedge d_{\mathcal{T}}(v)=l} & v' \in \mathbb{D} \wedge v \in \mathbb{V}_+ \\ 0 & v' \in \mathbb{T} \vee v = v_0 \end{cases} \right)_{(v, v') \in \mathbb{V}^2} \right) \left(\begin{cases} \pi(\sigma^{(T)}, v, i_+) & d_{\mathcal{T}}(v) \leq l-1 \\ 0 & d_{\mathcal{T}}(v) > l-1 \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \right) \odot \check{\mathbf{s}} \\
&\quad + \check{\mathbf{\Pi}}^{(l-1)}
\end{aligned}$$

Using Equation 24

$$\begin{aligned}
&= \left(\left(\begin{cases} \mathbf{1}_{v'=f_{Pa}(v) \wedge d_{\mathcal{T}}(v)=l} & v' \in \mathbb{D} \wedge v \in \mathbb{V}_+ \\ 0 & v' \in \mathbb{T} \vee v = v_0 \end{cases} \right)_{(v, v') \in \mathbb{V}^2} \right) \check{\mathbf{\Pi}}^{(l-1)} \odot \check{\mathbf{s}} + \check{\mathbf{\Pi}}^{(l-1)} \\
&= \left(\left(\begin{cases} \mathbf{1}_{v=f_{Pa}(v') \wedge d_{\mathcal{T}}(v')=l} & v \in \mathbb{D} \wedge v' \in \mathbb{V}_+ \\ 0 & v \in \mathbb{T} \vee v' = v_0 \end{cases} \right)_{(v, v') \in \mathbb{V}^2} \right)^{\top} \check{\mathbf{\Pi}}^{(l-1)} \odot \check{\mathbf{s}} + \check{\mathbf{\Pi}}^{(l-1)}
\end{aligned}$$

Using Equation 10

$$= \left((\mathbf{L}^{(l)})^{\top} \check{\mathbf{\Pi}}^{(l-1)} \right) \odot \check{\mathbf{s}} + \check{\mathbf{\Pi}}^{(l-1)}$$

E “COUNTERFACTUAL” REACH PROBABILITY TERMS

An expanded form of Equation 27 is shown below.

$$\begin{aligned}
\check{\pi} &= \left(\begin{cases} \begin{cases} \check{\pi}(\sigma^{(T)}, v, f_i(f_h(f_{Pa}(v)))) & f_h(f_{Pa}(v)) \in \mathbb{H}_+ \\ 0 & f_h(f_{Pa}(v)) \in \mathbb{H}_0 \end{cases} & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right)_{v \in \mathbb{V}} \\
&= \left(\begin{cases} \begin{cases} \sum_{i_+ \in \mathbb{I}_+} (\mathbf{1}_{f_i(f_h(f_{Pa}(v)))=i_+}) \check{\pi}(\sigma^{(T)}, v, i_+) & f_h(f_{Pa}(v)) \in \mathbb{H}_+ \\ 0 & f_h(f_{Pa}(v)) \in \mathbb{H}_0 \end{cases} & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right)_{v \in \mathbb{V}} \\
&= \left(\begin{cases} \begin{cases} (\mathbf{1}_{f_i(f_h(f_{Pa}(v)))=i_+}) \check{\pi}(\sigma^{(T)}, v, i_+) & f_h(f_{Pa}(v)) \in \mathbb{H}_+ \\ 0 & f_h(f_{Pa}(v)) \in \mathbb{H}_0 \end{cases} & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \mathbf{1}_{|\mathbb{I}_+|} \\
&= \left(\begin{cases} (\mathbf{1}_{f_i(f_h(f_{Pa}(v)))=i_+}) \check{\pi}(\sigma^{(T)}, v, i_+) & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \mathbf{1}_{|\mathbb{I}_+|} \\
&= \left(\left(\begin{cases} \mathbf{1}_{f_i(f_h(f_{Pa}(v)))=i_+} & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right) \check{\pi}(\sigma^{(T)}, v, i_+) \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \mathbf{1}_{|\mathbb{I}_+|} \\
&= \left(\left(\left(\begin{cases} \mathbf{1}_{f_i(f_h(f_{Pa}(v)))=i_+} & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \right) \odot \left(\check{\pi}(\sigma^{(T)}, v, i_+) \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \right) \mathbf{1}_{|\mathbb{I}_+|}
\end{aligned}$$

Using Equation 13

$$\begin{aligned}
&= \left(\left(\mathbf{M}^{(V, I_+)} \right) \odot \left(\check{\pi}(\sigma^{(T)}, v, i_+) \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \right) \mathbf{1}_{|\mathbb{I}_+|} \\
&= \left(\left(\mathbf{M}^{(V, I_+)} \right) \odot \left(\begin{cases} \check{\pi}(\sigma^{(T)}, v, i_+) & d_{\mathcal{T}}(v) \leq D \\ 0 & d_{\mathcal{T}}(v) > D \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \right) \mathbf{1}_{|\mathbb{I}_+|}
\end{aligned}$$

Using Equation 24

$$= \left(\left(\mathbf{M}^{(V, I_+)} \right) \odot \check{\Pi}^{(D)} \right) \mathbf{1}_{|\mathbb{I}_+|}$$

F “COUNTERFACTUAL” REACH PROBABILITIES

An expanded form of Equation 28 is shown below.

$$\tilde{\pi} = \left(\tilde{\pi}(\sigma^{(T)}, h_+) \right)_{h_+ \in \mathbb{H}_+}$$

Using Equation 3

$$\begin{aligned}
&= \left(\sum_{d \in \mathbb{D}: f_h(d) = h_+} \tilde{\pi}(\sigma^{(T)}, d, f_i(h_+)) \right)_{h_+ \in \mathbb{H}_+} \\
&= \left(\frac{|A(h_+)|}{|A(h_+)|} \sum_{d \in \mathbb{D}: f_h(d) = h_+} \tilde{\pi}(\sigma^{(T)}, d, f_i(h_+)) \right)_{h_+ \in \mathbb{H}_+} \\
&= \left(\left(\sum_{d \in \mathbb{D}: f_h(d) = h_+} |A(h_+)| \tilde{\pi}(\sigma^{(T)}, d, f_i(h_+)) \right)_{h_+ \in \mathbb{H}_+} \right) \oslash (|A(h_+)|)_{h_+ \in \mathbb{H}_+} \\
&= \left(\sum_{v \in \mathbb{V}} \begin{cases} (\mathbf{1}_{h_+ = f_h(f_{Pa}(v))}) \tilde{\pi}(\sigma^{(T)}, f_{Pa}(v), f_i(f_h(f_{Pa}(v)))) & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right)_{h_+ \in \mathbb{H}_+} \\
&\quad \oslash \left(\sum_{(h'_+, a) \in \mathbb{Q}_+} \mathbf{1}_{h_+ = h'_+} \right)_{h_+ \in \mathbb{H}_+}
\end{aligned}$$

Using Equation 2

$$\begin{aligned}
&= \left(\sum_{v \in \mathbb{V}} \begin{cases} (\mathbf{1}_{h_+ = f_h(f_{Pa}(v))}) \tilde{\pi}(\sigma^{(T)}, v, f_i(f_h(f_{Pa}(v)))) & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right)_{h_+ \in \mathbb{H}_+} \\
&\quad \oslash \left(\sum_{(h'_+, a) \in \mathbb{Q}_+} \mathbf{1}_{h_+ = h'_+} \right)_{h_+ \in \mathbb{H}_+} \\
&= \left(\begin{cases} \mathbf{1}_{h_+ = f_h(f_{Pa}(v))} & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right)_{(h_+, v) \in \mathbb{H}_+ \times \mathbb{V}} \\
&\quad \left(\begin{cases} \tilde{\pi}(\sigma^{(T)}, v, f_i(f_h(f_{Pa}(v)))) & f_h(f_{Pa}(v)) \in \mathbb{H}_+ \\ 0 & f_h(f_{Pa}(v)) \in \mathbb{H}_0 \end{cases} \right)_{v \in \mathbb{V}} \\
&\quad \oslash \left(\left(\mathbf{1}_{h_+ = h'_+} \right)_{(h_+, (h'_+, a)) \in \mathbb{H}_+ \times \mathbb{Q}_+} \right) \mathbf{1}_{|\mathbb{Q}_+|}
\end{aligned}$$

Using Equation 12

$$\begin{aligned}
&= \left(\begin{cases} \mathbf{1}_{h_+ = f_h(f_{Pa}(v))} & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right)_{(h_+, v) \in \mathbb{H}_+ \times \mathbb{V}} \\
&\quad \left(\begin{cases} \tilde{\pi}(\sigma^{(T)}, v, f_i(f_h(f_{Pa}(v)))) & f_h(f_{Pa}(v)) \in \mathbb{H}_+ \\ 0 & f_h(f_{Pa}(v)) \in \mathbb{H}_0 \end{cases} \right)_{v \in \mathbb{V}} \\
&\quad \oslash \left(\left(\mathbf{M}^{(H_+, Q_+)} \right) \mathbf{1}_{|\mathbb{Q}_+|} \right)
\end{aligned}$$

Using Equation 27

$$\begin{aligned}
&= \left(\left(\left(\begin{array}{ll} \mathbf{1}_{h_+ = f_h(f_{Pa}(v))} & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{array} \right)_{(h_+, v) \in \mathbb{H}_+ \times \mathbb{V}} \right) \check{\pi} \right) \oslash \left(\left(M^{(H_+, Q_+)} \right) \mathbf{1}_{|\mathbb{Q}_+|} \right) \\
&= \left(\left(\left(\begin{array}{ll} \mathbf{1}_{(h_+, f_a(v)) = (f_h(f_{Pa}(v)), f_a(v))} & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{array} \right)_{(h_+, v) \in \mathbb{H}_+ \times \mathbb{V}} \right) \check{\pi} \right) \oslash \left(\left(M^{(H_+, Q_+)} \right) \mathbf{1}_{|\mathbb{Q}_+|} \right) \\
&= \left(\left(\left(\sum_{(h'_+, a) \in \mathbb{Q}_+} \mathbf{1}_{h_+ = h'_+} \begin{cases} \mathbf{1}_{(h'_+, a) = (f_h(f_{Pa}(v)), f_a(v))} & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right)_{(h_+, v) \in \mathbb{H}_+ \times \mathbb{V}} \right) \check{\pi} \right) \\
&\quad \oslash \left(\left(M^{(H_+, Q_+)} \right) \mathbf{1}_{|\mathbb{Q}_+|} \right) \\
&= \left(\left(\mathbf{1}_{h_+ = h'_+} \right)_{(h_+, (h'_+, a)) \in \mathbb{H}_+ \times \mathbb{Q}_+} \right) \left(\left(\begin{array}{ll} \mathbf{1}_{q_+ = (f_h(f_{Pa}(v)), f_a(v))} & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{array} \right)_{(q_+, v) \in \mathbb{Q}_+ \times \mathbb{V}} \right) \check{\pi} \right) \\
&\quad \oslash \left(\left(M^{(H_+, Q_+)} \right) \mathbf{1}_{|\mathbb{Q}_+|} \right)
\end{aligned}$$

Using Equation 12 and Equation 11

$$= \left(\left(M^{(H_+, Q_+)} \right) \left(M^{(Q_+, V)} \right) \check{\pi} \right) \oslash \left(\left(M^{(H_+, Q_+)} \right) \mathbf{1}_{|\mathbb{Q}_+|} \right)$$

G COUNTERFACTUAL REACH PROBABILITY SUMS

An expanded form of Equation 29 is shown below.

$$\begin{aligned}
\tilde{\pi}^{(\Sigma)} &= \left(\sum_{\tau=1}^T \tilde{\pi}(\sigma^{(\tau)}, h_+) \right)_{h_+ \in \mathbb{H}_+} \\
&= \left(\left(\sum_{\tau=1}^{T-1} \tilde{\pi}(\sigma^{(\tau)}, h_+) \right) + \tilde{\pi}(\sigma^{(T)}, h_+) \right)_{h_+ \in \mathbb{H}_+} \\
&= \left(\sum_{\tau=1}^{T-1} \tilde{\pi}(\sigma^{(\tau)}, h_+) \right)_{h_+ \in \mathbb{H}_+} + \left(\tilde{\pi}(\sigma^{(T)}, h_+) \right)_{h_+ \in \mathbb{H}_+}
\end{aligned}$$

Using Equation 29 and Equation 28

$$= \tilde{\pi}^{(\Sigma)'} + \tilde{\pi}$$

H AVERAGE STRATEGY PROFILE

An expanded form of Equation 30 is shown below.

$$\bar{\sigma} = \left(\bar{\sigma}^{(T)}(q_+) \right)_{q_+ \in \mathbb{Q}_+}$$

Using Equation 8

$$\begin{aligned}
&= \left(\frac{\sum_{\tau=1}^T \tilde{\pi}(\sigma^{(\tau)}, h_+) \sigma^{(\tau)}(h_+, a)}{\sum_{\tau=1}^T \tilde{\pi}(\sigma^{(\tau)}, h_+)} \right)_{(h_+, a) \in \mathbb{Q}_+} \\
&= \left(\frac{\sum_{\tau=1}^{T-1} \tilde{\pi}(\sigma^{(\tau)}, h_+) \sigma^{(\tau)}(h_+, a)}{\sum_{\tau=1}^T \tilde{\pi}(\sigma^{(\tau)}, h_+)} + \frac{\tilde{\pi}(\sigma^{(T)}, h_+) \sigma^{(T)}(h_+, a)}{\sum_{\tau=1}^T \tilde{\pi}(\sigma^{(\tau)}, h_+)} \right)_{(h_+, a) \in \mathbb{Q}_+} \\
&= \left(\left(\frac{\sum_{\tau=1}^{T-1} \tilde{\pi}(\sigma^{(\tau)}, h_+)}{\sum_{\tau=1}^T \tilde{\pi}(\sigma^{(\tau)}, h_+)} \right) \left(\frac{\sum_{\tau=1}^{T-1} \tilde{\pi}(\sigma^{(\tau)}, h_+) \sigma^{(\tau)}(h_+, a)}{\sum_{\tau=1}^{T-1} \tilde{\pi}(\sigma^{(\tau)}, h_+)} \right) + \right. \\
&\quad \left. + \left(\frac{\tilde{\pi}(\sigma^{(T)}, h_+)}{\sum_{\tau=1}^T \tilde{\pi}(\sigma^{(\tau)}, h_+)} \right) \sigma^{(T)}(h_+, a) \right)_{(h_+, a) \in \mathbb{Q}_+}
\end{aligned}$$

Using Equation 8

$$\begin{aligned}
&= \left(\left(\frac{\sum_{\tau=1}^{T-1} \tilde{\pi}(\sigma^{(\tau)}, h_+)}{\sum_{\tau=1}^T \tilde{\pi}(\sigma^{(\tau)}, h_+)} \right) \bar{\sigma}^{(T-1)}(h_+, a) + \left(\frac{\tilde{\pi}(\sigma^{(T)}, h_+)}{\sum_{\tau=1}^T \tilde{\pi}(\sigma^{(\tau)}, h_+)} \right) \sigma^{(T)}(h_+, a) \right)_{(h_+, a) \in \mathbb{Q}_+} \\
&= \left(\left(1 - \frac{\tilde{\pi}(\sigma^{(T)}, h_+)}{\sum_{\tau=1}^T \tilde{\pi}(\sigma^{(\tau)}, h_+)} \right) \bar{\sigma}^{(T-1)}(h_+, a) + \right. \\
&\quad \left. + \left(\frac{\tilde{\pi}(\sigma^{(T)}, h_+)}{\sum_{\tau=1}^T \tilde{\pi}(\sigma^{(\tau)}, h_+)} \right) \sigma^{(T)}(h_+, a) \right)_{(h_+, a) \in \mathbb{Q}_+} \\
&= \left(\bar{\sigma}^{(T-1)}(h_+, a) + \left(\frac{\tilde{\pi}(\sigma^{(T)}, h_+)}{\sum_{\tau=1}^T \tilde{\pi}(\sigma^{(\tau)}, h_+)} \right) (\sigma^{(T)}(h_+, a) - \bar{\sigma}^{(T-1)}(h_+, a)) \right)_{(h_+, a) \in \mathbb{Q}_+} \\
&= \left(\bar{\sigma}^{(T-1)}(q_+) \right)_{q_+ \in \mathbb{Q}_+} \\
&\quad + \left(\frac{\tilde{\pi}(\sigma^{(T)}, h_+)}{\sum_{\tau=1}^T \tilde{\pi}(\sigma^{(\tau)}, h_+)} \right)_{(h_+, a) \in \mathbb{Q}_+} \\
&\quad \odot \left(\left(\sigma^{(T)}(q_+) \right)_{q_+ \in \mathbb{Q}_+} - \left(\bar{\sigma}^{(T-1)}(q_+) \right)_{q_+ \in \mathbb{Q}_+} \right)
\end{aligned}$$

Using Equation 30 and Equation 15

$$\begin{aligned}
&= \bar{\sigma}' + \left(\left(\frac{\tilde{\pi}(\sigma^{(T)}, h_+)}{\sum_{\tau=1}^T \tilde{\pi}(\sigma^{(\tau)}, h_+)} \right)_{(h_+, a) \in \mathbb{Q}_+} \right) \odot (\sigma - \bar{\sigma}') \\
&= \bar{\sigma}' + \left(\left(\sum_{h'_+ \in \mathbb{H}_+} (\mathbf{1}_{h_+ = h'_+}) \frac{\tilde{\pi}(\sigma^{(T)}, h'_+)}{\sum_{\tau=1}^T \tilde{\pi}(\sigma^{(\tau)}, h'_+)} \right)_{(h_+, a) \in \mathbb{Q}_+} \right) \odot (\sigma - \bar{\sigma}') \\
&= \bar{\sigma}' + \left(\left((\mathbf{1}_{h_+ = h'_+})_{((h_+, a), h'_+) \in \mathbb{Q}_+ \times \mathbb{H}_+} \right) \left(\frac{\tilde{\pi}(\sigma^{(T)}, h_+)}{\sum_{\tau=1}^T \tilde{\pi}(\sigma^{(\tau)}, h_+)} \right)_{h_+ \in \mathbb{H}_+} \right) \odot (\sigma - \bar{\sigma}') \\
&= \bar{\sigma}' \\
&\quad + \left(\left((\mathbf{1}_{h_+ = h'_+})_{(h_+, (h'_+, a)) \in \mathbb{H}_+ \times \mathbb{Q}_+} \right)^\top \right. \\
&\quad \left. \left(\left((\tilde{\pi}(\sigma^{(T)}, h_+))_{h_+ \in \mathbb{H}_+} \right) \oslash \left(\sum_{\tau=1}^T \tilde{\pi}(\sigma^{(\tau)}, h_+) \right)_{h_+ \in \mathbb{H}_+} \right) \right) \odot (\sigma - \bar{\sigma}')
\end{aligned}$$

Using Equation 12, Equation 28, and Equation 29

$$= \bar{\sigma}' + \left(\left(\mathbf{M}^{(H_+, Q_+)} \right)^\top \left(\tilde{\pi} \oslash \tilde{\pi}^{(\Sigma)} \right) \right) \odot (\sigma - \bar{\sigma}')$$

I INSTANTANEOUS COUNTERFACTUAL REGRETS

An expanded form of Equation 31 is shown below.

First, define a dense vector $\rho \in \mathbb{R}^V$ for intermediate values.

$$\begin{aligned}
\rho &= \left(\begin{cases} \begin{cases} \tilde{u}(\sigma^{(T)}, v, f_i(f_h(f_{Pa}(v)))) - \tilde{u}(\sigma^{(T)}, f_{Pa}(v), f_i(f_h(f_{Pa}(v)))) & f_h(f_{Pa}(v)) \in \mathbb{H}_+ \\ 0 & f_h(f_{Pa}(v)) \in \mathbb{H}_0 \end{cases} & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right)_{v \in \mathbb{V}} \quad (35) \\
&= \left(\left(\mathbf{M}^{(V, I_+)} \right) \odot \left(\tilde{\mathbf{U}} - \mathbf{G}^\top \tilde{\mathbf{U}} \right) \right) \mathbf{1}_{|\mathbb{I}_+|}
\end{aligned}$$

Then,

$$\hat{\pi} = \left(\tilde{r}(\sigma^{(T)}, q_+) \right)_{q_+ \in \mathbb{Q}_+}$$

Using Equation 5

$$= \left(\tilde{\pi}(\sigma^{(T)}, h_+) (\tilde{u}(\sigma^{(T)}|_{h_+ \rightarrow a}, h_+) - \tilde{u}(\sigma^{(T)}, h_+)) \right)_{(h_+, a) \in \mathbb{Q}_+}$$

Using Equation 4

$$\begin{aligned}
&= \left(\tilde{\pi}(\sigma^{(T)}, h_+) \right. \\
&\quad \left(\frac{\sum_{d \in \mathbb{D}: f_h(d) = h_+} \tilde{\pi}(\sigma^{(T)}|_{h_+ \rightarrow a}, d, f_i(h_+)) \tilde{u}(\sigma^{(T)}|_{h_+ \rightarrow a}, d, f_i(h_+))}{\tilde{\pi}(\sigma^{(T)}|_{h_+ \rightarrow a}, h_+)} \right. \\
&\quad \left. - \frac{\sum_{d \in \mathbb{D}: f_h(d) = h_+} \tilde{\pi}(\sigma^{(T)}, d, f_i(h_+)) \tilde{u}(\sigma^{(T)}, d, f_i(h_+))}{\tilde{\pi}(\sigma^{(T)}, h_+)} \right) \Big)_{(h_+, a) \in \mathbb{Q}_+} \\
&= \left(\tilde{\pi}(\sigma^{(T)}, h_+) \right. \\
&\quad \left(\frac{\sum_{d \in \mathbb{D}: f_h(d) = h_+} \tilde{\pi}(\sigma^{(T)}, d, f_i(h_+)) \tilde{u}(\sigma^{(T)}|_{h_+ \rightarrow a}, d, f_i(h_+))}{\tilde{\pi}(\sigma^{(T)}, h_+)} \right. \\
&\quad \left. - \frac{\sum_{d \in \mathbb{D}: f_h(d) = h_+} \tilde{\pi}(\sigma^{(T)}, d, f_i(h_+)) \tilde{u}(\sigma^{(T)}, d, f_i(h_+))}{\tilde{\pi}(\sigma^{(T)}, h_+)} \right) \Big)_{(h_+, a) \in \mathbb{Q}_+} \\
&= \left(\tilde{\pi}(\sigma^{(T)}, d, f_i(h_+)) \tilde{u}(\sigma^{(T)}|_{h_+ \rightarrow a}, d, f_i(h_+)) - \right. \\
&\quad \left. - \sum_{d \in \mathbb{D}: f_h(d) = h_+} \tilde{\pi}(\sigma^{(T)}, d, f_i(h_+)) \tilde{u}(\sigma^{(T)}, d, f_i(h_+)) \right) \Big)_{(h_+, a) \in \mathbb{Q}_+} \\
&= \left(\tilde{\pi}(\sigma^{(T)}, d, f_i(h_+)) (\tilde{u}(\sigma^{(T)}|_{h_+ \rightarrow a}, d, f_i(h_+)) - \tilde{u}(\sigma^{(T)}, d, f_i(h_+))) \right) \Big)_{(h_+, a) \in \mathbb{Q}_+} \\
&= \left(\sum_{v \in \mathbb{V}} \left(\begin{cases} \mathbf{1}_{q_+ = (f_h(f_{Pa}(v)), f_a(v))} & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right) \right. \\
&\quad \left(\begin{cases} \begin{cases} \tilde{\pi}(\sigma^{(T)}, f_{Pa}(v), f_i(f_h(f_{Pa}(v)))) & f_h(f_{Pa}(v)) \in \mathbb{H}_+ \\ 0 & f_h(f_{Pa}(v)) \in \mathbb{H}_0 \end{cases} & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right) \\
&\quad \left(\begin{cases} \begin{cases} \begin{cases} \tilde{u}(\sigma^{(T)}|_{f_h(f_{Pa}(v)) \rightarrow a}, f_{Pa}(v), f_i(f_h(f_{Pa}(v)))) & f_h(f_{Pa}(v)) \in \mathbb{H}_+ \\ -\tilde{u}(\sigma^{(T)}, f_{Pa}(v), f_i(f_h(f_{Pa}(v)))) & f_h(f_{Pa}(v)) \in \mathbb{H}_0 \end{cases} & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \end{cases} \right) \Big)_{q_+ \in \mathbb{Q}_+} \\
&= \left(\sum_{v \in \mathbb{V}} \left(\begin{cases} \mathbf{1}_{q_+ = (f_h(f_{Pa}(v)), f_a(v))} & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right) \right. \\
&\quad \left(\begin{cases} \begin{cases} \tilde{\pi}(\sigma^{(T)}, v, f_i(f_h(f_{Pa}(v)))) & f_h(f_{Pa}(v)) \in \mathbb{H}_+ \\ 0 & f_h(f_{Pa}(v)) \in \mathbb{H}_0 \end{cases} & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right) \\
&\quad \left(\begin{cases} \begin{cases} \begin{cases} \tilde{u}(\sigma^{(T)}, v, f_i(f_h(f_{Pa}(v)))) - \tilde{u}(\sigma^{(T)}, f_{Pa}(v), f_i(f_h(f_{Pa}(v)))) & f_h(f_{Pa}(v)) \in \mathbb{H}_+ \\ f_h(f_{Pa}(v)) \in \mathbb{H}_0 & f_h(f_{Pa}(v)) \in \mathbb{H}_0 \end{cases} & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \end{cases} \right) \Big)_{q_+ \in \mathbb{Q}_+} \\
&= \left(\begin{cases} \mathbf{1}_{q_+ = (f_h(f_{Pa}(v)), f_a(v))} & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right)_{(q_+, v) \in \mathbb{Q}_+ \times \mathbb{V}} \\
&\quad \left(\left(\begin{cases} \begin{cases} \tilde{\pi}(\sigma^{(T)}, v, f_i(f_h(f_{Pa}(v)))) & f_h(f_{Pa}(v)) \in \mathbb{H}_+ \\ 0 & f_h(f_{Pa}(v)) \in \mathbb{H}_0 \end{cases} & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right)_{v \in \mathbb{V}} \right. \\
&\quad \left. \odot \left(\begin{cases} \begin{cases} \tilde{u}(\sigma^{(T)}, v, f_i(f_h(f_{Pa}(v)))) - \tilde{u}(\sigma^{(T)}, f_{Pa}(v), f_i(f_h(f_{Pa}(v)))) & f_h(f_{Pa}(v)) \in \mathbb{H}_+ \\ f_h(f_{Pa}(v)) \in \mathbb{H}_0 & f_h(f_{Pa}(v)) \in \mathbb{H}_0 \end{cases} & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right)_{v \in \mathbb{V}} \right)
\end{aligned}$$

Using Equation 11, Equation 27, and Equation 35

$$= (M^{(Q_+, V)}) (\check{\pi} \odot \rho)$$

Using Equation 35

$$= (\mathbf{M}^{(Q+, V)}) (\check{\pi} \odot (((\mathbf{M}^{(V, I_+)}) \odot (\check{\mathbf{U}} - \mathbf{G}^\top \check{\mathbf{U}})) \mathbf{1}_{|\mathbb{I}_+|}))$$

I.1 INTERMEDIATE VALUES

An expanded form of Equation 35 is shown below.

$$\begin{aligned} \rho &= \left(\begin{cases} \begin{cases} \check{u}(\sigma^{(T)}, v, f_i(f_h(f_{Pa}(v)))) - \check{u}(\sigma^{(T)}, f_{Pa}(v), f_i(f_h(f_{Pa}(v)))) & f_h(f_{Pa}(v)) \in \mathbb{H}_+ \\ 0 & f_h(f_{Pa}(v)) \in \mathbb{H}_0 \\ 0 & v = v_0 \end{cases} & v \in \mathbb{V}_+ \\ \begin{cases} \sum_{i_+ \in \mathbb{I}_+} \begin{cases} \mathbf{1}_{i_+ = f_i(f_h(f_{Pa}(v)))} \begin{cases} \check{u}(\sigma^{(T)}, v, i_+) - \check{u}(\sigma^{(T)}, f_{Pa}(v), i_+) & f_h(f_{Pa}(v)) \in \mathbb{H}_+ \\ 0 & f_h(f_{Pa}(v)) \in \mathbb{H}_0 \\ 0 & v = v_0 \end{cases} & v \in \mathbb{V}_+ \\ 0 & v \in \mathbb{V}_+ \end{cases} & v \in \mathbb{V}_+ \end{cases} \\ \begin{cases} \begin{cases} \mathbf{1}_{i_+ = f_i(f_h(f_{Pa}(v)))} \begin{cases} \check{u}(\sigma^{(T)}, v, i_+) - \check{u}(\sigma^{(T)}, f_{Pa}(v), i_+) & f_h(f_{Pa}(v)) \in \mathbb{H}_+ \\ 0 & f_h(f_{Pa}(v)) \in \mathbb{H}_0 \\ 0 & v = v_0 \end{cases} & v \in \mathbb{V}_+ \\ 0 & v \in \mathbb{V}_+ \end{cases} & v \in \mathbb{V}_+ \end{cases} \\ \begin{cases} \begin{cases} \begin{cases} \mathbf{1}_{i_+ = f_i(f_h(f_{Pa}(v)))} \begin{cases} \check{u}(\sigma^{(T)}, v, i_+) - \check{u}(\sigma^{(T)}, f_{Pa}(v), i_+) & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} & (v, i_+) \in \mathbb{V} \times \mathbb{I}_+ \\ 0 & (v, i_+) \in \mathbb{V} \times \mathbb{I}_+ \end{cases} & \mathbf{1}_{|\mathbb{I}_+|} \end{cases} \\ \begin{cases} \begin{cases} \begin{cases} \begin{cases} \mathbf{1}_{f_i(f_h(f_{Pa}(v))) = i_+} \begin{cases} v \in \mathbb{V}_+ \\ v = v_0 \end{cases} & (v, i_+) \in \mathbb{V} \times \mathbb{I}_+ \\ 0 & (v, i_+) \in \mathbb{V} \times \mathbb{I}_+ \end{cases} & \mathbf{1}_{|\mathbb{I}_+|} \end{cases} \\ \begin{cases} \begin{cases} \begin{cases} \mathbf{1}_{f_i(f_h(f_{Pa}(v))) = i_+} \begin{cases} v \in \mathbb{V}_+ \\ v = v_0 \end{cases} & (v, i_+) \in \mathbb{V} \times \mathbb{I}_+ \\ 0 & (v, i_+) \in \mathbb{V} \times \mathbb{I}_+ \end{cases} & \mathbf{1}_{|\mathbb{I}_+|} \end{cases} \\ \odot \begin{cases} \begin{cases} \begin{cases} \check{u}(\sigma^{(T)}, v, i_+) - \check{u}(\sigma^{(T)}, f_{Pa}(v), i_+) & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} & (v, i_+) \in \mathbb{V} \times \mathbb{I}_+ \\ 0 & (v, i_+) \in \mathbb{V} \times \mathbb{I}_+ \end{cases} & \mathbf{1}_{|\mathbb{I}_+|} \end{cases} \\ \begin{cases} \begin{cases} \begin{cases} \mathbf{1}_{f_i(f_h(f_{Pa}(v))) = i_+} \begin{cases} v \in \mathbb{V}_+ \\ v = v_0 \end{cases} & (v, i_+) \in \mathbb{V} \times \mathbb{I}_+ \\ 0 & (v, i_+) \in \mathbb{V} \times \mathbb{I}_+ \end{cases} & \mathbf{1}_{|\mathbb{I}_+|} \end{cases} \\ \odot \begin{cases} \begin{cases} \begin{cases} \check{u}(\sigma^{(T)}, v, i_+) - \check{u}(\sigma^{(T)}, f_{Pa}(v), i_+) & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} & (v, i_+) \in \mathbb{V} \times \mathbb{I}_+ \\ 0 & (v, i_+) \in \mathbb{V} \times \mathbb{I}_+ \end{cases} & \mathbf{1}_{|\mathbb{I}_+|} \end{cases} \end{cases} \end{aligned}$$

Using Equation 13

$$\begin{aligned} &= \left((\mathbf{M}^{(V, I_+)}) \odot \left(\check{u}(\sigma^{(T)}, v, i_+) - \begin{cases} \check{u}(\sigma^{(T)}, f_{Pa}(v), i_+) & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \right) \mathbf{1}_{|\mathbb{I}_+|} \\ &= \left((\mathbf{M}^{(V, I_+)}) \odot \left(\left(\check{u}(\sigma^{(T)}, v, i_+) \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} - \left(\begin{cases} \check{u}(\sigma^{(T)}, f_{Pa}(v), i_+) & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \right) \right) \mathbf{1}_{|\mathbb{I}_+|} \end{aligned}$$

Using Equation 22

$$\begin{aligned} &= \left((\mathbf{M}^{(V, I_+)}) \odot \left(\check{\mathbf{U}} - \left(\begin{cases} \check{u}(\sigma^{(T)}, f_{Pa}(v), i_+) & v \in \mathbb{V}_+ \\ 0 & v = v_0 \end{cases} \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \right) \right) \mathbf{1}_{|\mathbb{I}_+|} \\ &= \left((\mathbf{M}^{(V, I_+)}) \odot \left(\check{\mathbf{U}} - \left(\sum_{v' \in \mathbb{V}} \left(\begin{cases} \mathbf{1}_{v' = f_{Pa}(v)} & v' \in \mathbb{D} \wedge v \in \mathbb{V}_+ \\ 0 & v' \in \mathbb{T} \vee v = v_0 \end{cases} \right) \check{u}(\sigma^{(T)}, v', i_+) \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \right) \right) \mathbf{1}_{|\mathbb{I}_+|} \\ &= \left((\mathbf{M}^{(V, I_+)}) \odot \left(\check{\mathbf{U}} - \left(\left(\begin{cases} \mathbf{1}_{v' = f_{Pa}(v)} & v' \in \mathbb{D} \wedge v \in \mathbb{V}_+ \\ 0 & v' \in \mathbb{T} \vee v = v_0 \end{cases} \right)_{(v, v') \in \mathbb{V}^2} \right) \left(\check{u}(\sigma^{(T)}, v, i_+) \right)_{(v, i_+) \in \mathbb{V} \times \mathbb{I}_+} \right) \right) \mathbf{1}_{|\mathbb{I}_+|} \end{aligned}$$

Using Equation 22

$$\begin{aligned} &= \left((\mathbf{M}^{(V, I_+)}) \odot \left(\check{\mathbf{U}} - \left(\left(\begin{cases} \mathbf{1}_{v' = f_{Pa}(v)} & v' \in \mathbb{D} \wedge v \in \mathbb{V}_+ \\ 0 & v' \in \mathbb{T} \vee v = v_0 \end{cases} \right)_{(v, v') \in \mathbb{V}^2} \right) \check{\mathbf{U}} \right) \right) \mathbf{1}_{|\mathbb{I}_+|} \\ &= \left((\mathbf{M}^{(V, I_+)}) \odot \left(\check{\mathbf{U}} - \left(\left(\begin{cases} \mathbf{1}_{v = f_{Pa}(v')} & v \in \mathbb{D} \wedge v' \in \mathbb{V}_+ \\ 0 & v \in \mathbb{T} \vee v' = v_0 \end{cases} \right)_{(v, v') \in \mathbb{V}^2} \right)^\top \check{\mathbf{U}} \right) \right) \mathbf{1}_{|\mathbb{I}_+|} \end{aligned}$$

Using Equation 9

$$= \left((\mathbf{M}^{(V, I_+)}) \odot (\mathbf{\tilde{U}} - \mathbf{G}^\top \mathbf{\tilde{U}}) \right) \mathbf{1}_{|I_+|}$$

J AVERAGE COUNTERFACTUAL REGRETS

An expanded form of Equation 32 is shown below.

$$\bar{\mathbf{r}} = \left(\tilde{r}^{(T)}(q_+) \right)_{q_+ \in \mathbb{Q}_+}$$

Using Equation 6

$$\begin{aligned} &= \left(\frac{1}{T} \sum_{\tau=1}^T \tilde{r}(\sigma^{(\tau)}, q_+) \right)_{q_+ \in \mathbb{Q}_+} \\ &= \left(\frac{1}{T} \left(\left(\sum_{\tau=1}^{T-1} \tilde{r}(\sigma^{(\tau)}, q_+) \right) + \tilde{r}(\sigma^{(T)}, q_+) \right) \right)_{q_+ \in \mathbb{Q}_+} \\ &= \left(\left(\frac{1}{T} \sum_{\tau=1}^{T-1} \tilde{r}(\sigma^{(\tau)}, q_+) \right) + \left(\frac{1}{T} \right) \tilde{r}(\sigma^{(T)}, q_+) \right)_{q_+ \in \mathbb{Q}_+} \\ &= \left(\left(\frac{(T-1)}{T} \right) \left(\frac{1}{(T-1)} \sum_{\tau=1}^{T-1} \tilde{r}(\sigma^{(\tau)}, q_+) \right) + \left(\frac{1}{T} \right) \tilde{r}(\sigma^{(T)}, q_+) \right)_{q_+ \in \mathbb{Q}_+} \end{aligned}$$

Using Equation 6

$$\begin{aligned} &= \left(\left(\frac{(T-1)}{T} \right) \bar{r}^{(T-1)}(q_+) + \left(\frac{1}{T} \right) \tilde{r}(\sigma^{(T)}, q_+) \right)_{q_+ \in \mathbb{Q}_+} \\ &= \left(\left(1 - \frac{1}{T} \right) \bar{r}^{(T-1)}(q_+) + \left(\frac{1}{T} \right) \tilde{r}(\sigma^{(T)}, q_+) \right)_{q_+ \in \mathbb{Q}_+} \\ &= \left(\bar{r}^{(T-1)}(q_+) + \frac{1}{T} \left(\tilde{r}(\sigma^{(T)}, q_+) - \bar{r}^{(T-1)}(q_+) \right) \right)_{q_+ \in \mathbb{Q}_+} \\ &= \left(\bar{r}^{(T-1)}(q_+) \right)_{q_+ \in \mathbb{Q}_+} + \frac{1}{T} \left(\left(\tilde{r}(\sigma^{(T)}, q_+) \right)_{q_+ \in \mathbb{Q}_+} - \left(\bar{r}^{(T-1)}(q_+) \right)_{q_+ \in \mathbb{Q}_+} \right) \end{aligned}$$

Using Equation 32 and Equation 31

$$= \bar{\mathbf{r}}' + \frac{1}{T} (\tilde{\mathbf{r}} - \bar{\mathbf{r}}')$$

K REGRET NORMALIZERS

An expanded form of Equation 33 is shown below.

$$\begin{aligned}
\bar{\mathbf{r}}^{(+,\Sigma)} &= \left(\sum_{a' \in A(h_+)} \left(\bar{r}^{(T)}(h_+, a') \right)^+ \right)_{(h_+, a) \in \mathbb{Q}_+} \\
&= \left(\sum_{(h'_+, a') \in \mathbb{Q}_+} \mathbf{1}_{h_+ = h'_+} \left(\bar{r}^{(T)}(h'_+, a') \right)^+ \right)_{(h_+, a) \in \mathbb{Q}_+} \\
&= \left(\left(\mathbf{1}_{h_+ = h'_+} \right)_{((h_+, a), (h'_+, a')) \in \mathbb{Q}_+^2} \right) \left(\left(\bar{r}^{(T)}(q_+) \right)^+ \right)_{q_+ \in \mathbb{Q}_+} \\
&= \left(\left(\sum_{h''_+ \in \mathbb{H}_+} \mathbf{1}_{h_+ = h''_+} \mathbf{1}_{h'_+ = h''_+} \right)_{((h_+, a), (h'_+, a')) \in \mathbb{Q}_+^2} \right) \left(\left(\bar{r}^{(T)}(q_+) \right)_{q_+ \in \mathbb{Q}_+} \right)^+
\end{aligned}$$

Using Equation 32

$$\begin{aligned}
&= \left(\left(\sum_{h''_+ \in \mathbb{H}_+} \mathbf{1}_{h_+ = h''_+} \mathbf{1}_{h'_+ = h''_+} \right)_{((h_+, a), (h'_+, a')) \in \mathbb{Q}_+^2} \right) \bar{\mathbf{r}}^+ \\
&= \left(\left(\mathbf{1}_{h_+ = h'_+} \right)_{((h_+, a), (h'_+, a')) \in \mathbb{Q}_+ \times \mathbb{H}_+} \right) \left(\left(\mathbf{1}_{h_+ = h'_+} \right)_{(h_+, (h'_+, a)) \in \mathbb{H}_+ \times \mathbb{Q}_+} \right) \bar{\mathbf{r}}^+ \\
&= \left(\left(\mathbf{1}_{h_+ = h'_+} \right)_{(h_+, (h'_+, a)) \in \mathbb{H}_+ \times \mathbb{Q}_+} \right)^\top \left(\left(\mathbf{1}_{h_+ = h'_+} \right)_{(h_+, (h'_+, a)) \in \mathbb{H}_+ \times \mathbb{Q}_+} \right) \bar{\mathbf{r}}^+
\end{aligned}$$

Using Equation 12

$$= \left(\mathbf{M}^{(H_+, Q_+)} \right)^\top \left(\mathbf{M}^{(H_+, Q_+)} \right) \bar{\mathbf{r}}^+$$

To take advantage of the sparsity of $\mathbf{M}^{(H_+, Q_+)}$ (see Table 1)

$$= \left(\mathbf{M}^{(H_+, Q_+)} \right)^\top \left(\left(\mathbf{M}^{(H_+, Q_+)} \right) \bar{\mathbf{r}}^+ \right)$$

L NEXT STRATEGY PROFILE

An expanded form of Equation 34 is shown below.

$$\boldsymbol{\sigma}' = \left(\sigma^{(T+1)}(q_+) \right)_{q_+ \in \mathbb{Q}_+}$$

Using Equation 7

$$\begin{aligned}
&= \left(\begin{cases} \begin{cases} \frac{(\bar{r}^{(T)}(h_+, a))^+}{\sum_{a' \in A(h_+)} (\bar{r}^{(T)}(h_+, a'))^+} & \sum_{a' \in A(h_+)} (\bar{r}^{(T)}(h_+, a'))^+ > 0 \\ \frac{1}{|A(h_+)|} & \sum_{a' \in A(h_+)} (\bar{r}^{(T)}(h_+, a'))^+ = 0 \end{cases} & (h_+, a) \in \mathbb{Q}_+ \\ \sigma_0(h_+, a) & (h_+, a) \in \mathbb{Q}_0 \end{cases} \right)_{(h_+, a) \in \mathbb{Q}_+} \\
&= \left(\begin{cases} \begin{cases} \frac{(\bar{r}^{(T)}(h_+, a))^+}{\sum_{a' \in A(h_+)} (\bar{r}^{(T)}(h_+, a'))^+} & \sum_{a' \in A(h_+)} (\bar{r}^{(T)}(h_+, a'))^+ > 0 \\ \frac{1}{|A(h_+)|} & \sum_{a' \in A(h_+)} (\bar{r}^{(T)}(h_+, a'))^+ = 0 \end{cases} & (h_+, a) \in \mathbb{Q}_+ \end{cases} \right)_{(h_+, a) \in \mathbb{Q}_+}
\end{aligned}$$

Using Equation 32, Equation 33, and Equation 16

$$= \left(\begin{cases} \begin{cases} (\bar{\mathbf{r}}^+ \oslash \bar{\mathbf{r}}^{(+,\Sigma)})_{q_+} & (\bar{\mathbf{r}}^{(+,\Sigma)})_{q_+} > 0 \\ (\boldsymbol{\sigma}^{(T=1)})_{q_+} & (\bar{\mathbf{r}}^{(+,\Sigma)})_{q_+} = 0 \end{cases} & q_+ \in \mathbb{Q}_+ \end{cases} \right)_{q_+ \in \mathbb{Q}_+}$$