Business Cycles and Labor Market Search

Replication of D. Andolfatto's paper

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1 Introduction

The paper of Andolfatto that studied here presents a Real Business Cycle model that embeds a Search and Matching framework to characterize the functioning of the labor market. The objective of such an approach is to enhance the performances of RBC models to replicate the properties of the business cycles. To understand and identify properly these objectives, one can first recall the main failures of RBC models and the direction of the field's research agenda in the mid-nineteens.
Although the first RBC models were able to replicate some key features of the business cycles, such as an important part of the volatility of the cyclical component of output, they suffer several shortcomings.

- Without extensive margin of employment, the first RBC models could not generate the high relative volatility of total hours to real wages, which is observed in the data, with consistent value of the Frisch elasticity. The low values suggested by micro evidences lead to a weak Lucas-Rapping effect. Thus, important changes in wages are required to make households adjust their hours worked over time. Important volatility in hours comes with strong volatility in wages, at odds with the data.

- With the competitive labor market and technological shock as the only source of volatility, households are always on their labor supply curve, so that real wages are predicted to be strongly positively correlated with hours. Again this is at odd with the data.

- As noted by Cogley and Nason, the workhorse RBC models did not embed a significant internal propagation mechanism. The ability of these models to match the observed output persistence is only due to the persistence of the impulse. Moreover, they were not able to generate amplifications in the response to the impulse, while we observe such shapes in the data.

Introducing search and matching framework in the workhorse RBC model aims at solving those limitations. Indeed there were *a priori* justifications to hope that such an approach would enhance those dimensions. First, the introduction of extensive margin has been shown by Hansen, with indivisible hours, to reconciliate a low Frisch elasticity with a high relative volatility of total hours with respect to real wages. As the search framework also introduces an extensive margin, one can hope for similar success here. Moreover, by introducing a more sophisticated wage setting rule one could expect a weaker correlation between real wages and total hours. The search environment also offers some rationales for internal propagation and amplification mechanisms, as the matching process delays the responses of employment.
So far, three key objectives for the search economy has emerged. These are:

- Propagation and amplification mechanisms

- High relative volatility of total hours to real wages

- Weak correlation of real wages to labor input

As the model gives important attention to the labor market, its performance relies on its ability to replicate other key properties of the cyclical components of the related variables. The objective of the search economy is thus to generate the following empirical regularities:

- Large fluctuations in total hours, mainly driven by the volatility of extensive margin

- Volatile and persistent unemployment

- High volatility of vacancies
- Beveridge curve: negative correlation between unemployment and vacancies
- Asymmetric pattern in cross correlations of total hours with labor productivity and with real wages

This document is organized as follows. Section 2 presents the derivation of the decentralized equilibrium of the search economy. The following section proposes a resolution of the model and replicates the results of Andolfatto. In section 4, I discuss the quantitative performance of the search economy. I consider some extensions in 5 and section 6 concludes.

2 Decentralized equilibrium

In the workhorse RBC model, assuming complete markets yields a full insurance for the agents and thus ensure that the first theorem of welfare holds, that is, the Pareto optimal allocation can be decentralized. Incorporating a search and matching framework modifies the functioning of the labor market as compared to standard RBC and leads to a different wage setting mechanism. Hence, to ensure that the first theorem of welfare still holds, one needs to assume that the outcome of this mechanism will be efficient. I detail this assumption in what follows. Consequently the Pareto allocation can be decentralized and the planner first-order condition gives the first-order condition of the decentralized equilibrium, with the exception of the wage setting rule which is determined by the search and the wage bargaining process. Therefore, I focus in a first time on the planner objective and then detail the wage setting mechanisms.
All along this part, I will directly use the calibration described by Andolfatto later in the paper. I also make some slight notation changes as compared to Andolfatto papers. Variables without time subscripts refer to the steady state value of the variable. Hat variables refer to the percentage point deviation from steady state. As an example, one can read

- c_t as the consumption in period t.
- c as the steady state level of consumption.
- $\hat{c}_t$ as the percentage deviation of consumption from its steady state value.

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2.1 Decentralization of the planner allocation

In its recursive formulation, the objective of the planner is to maximize

$$W_t = \max_{\{c_t, l_t, k_{t+1}, n_{t+1}, v_t\}} \left\{ \log(c_t) + n_t\phi_1 \frac{(1-l_t)^{1-\eta}}{1-\eta} + (1-n_t)\phi_2 \frac{(1-e)^{1-\eta}}{1-\eta} + \beta \mathrm{E_t} W_{t+1}) \right\} \tag{1}$$

The first term is the utility of consumption, the second is the utility of labor for employed households (a share n_t of households are employed in t, and they supply l_t individual hours, thus the total hours worked in the economy are $n_t l_t$). The third term is the utility from leisure of non-employed agents. The expectation operator is conditional on G, the transition function describing the stochastic behavior of the technological process z_t. This maximization is subject to two constraints (equations 2 and 3). The two associated multipliers are denoted λ_t and μ_t.

$$y_t + (1-\delta)k_t = c_t + k_{t+1} + \kappa v_t \tag{2}$$

This first equation corresponds to the resources constraints of the economy, where output is defined by $y_t = e^{z_t}\zeta k_t^{\theta}(n_t l_t)^{1-\theta}$. The only modification induced by the search and matching framework is that the firm has to pay a hiring cost to open new vacancies: κv_t.
The planner also takes into account the law of motion of employment. At each period, a fraction σ of matches are destroyed, and m_t new matches are formed. The law of motion of extensive margin is thus given by:

$$n_{t+1} = (1-\sigma)n_t + m_t \tag{3}$$

Where the matching function is defined by:

$$m_t = \chi v_t^{\alpha}((1-n_t)e)^{1-\alpha}$$

Where e is the individual search effort, assumed to be constant and $(1-n_t)e$ the total search effort provided by non-employed households. One can now derive the first order conditions, starting with the Euler equation:

$$\frac{1}{c_t} = \beta \mathrm{E_t}\left[\theta\frac{y_{t+1}}{k_{t+1}} + 1 - \delta\right]\frac{1}{c_{t+1}} \tag{4}$$

That gives the optimal consumption path over time. Then the intra temporal condition for labor leisure arbitrage is given by:

$$\phi_1(1-l_t)^{-\eta} = (1-\theta)\frac{y_t}{n_t l_t}\frac{1}{c_t} \tag{5}$$

Households supply individual hours in such a way that their marginal utility of labor equates their marginal productivity, valued in terms of marginal utility of consumption. Then, substituting the value of Lagrange multipliers in the first order condition for vacancies yields:

$$\frac{\kappa}{c_t} = \mu_t \alpha \frac{m_t}{v_t} \tag{6}$$

This equation indicates that the marginal cost of opening a new vacancy, valued in terms of marginal utility of consumption terms must equates the marginal benefit of a vacancy. Indeed, $q_t = \frac{m_t}{v_t}$ is the probability to fill a vacancy, so αq_t is the marginal increase in matches due to the opening of a vacancy, valued by μ_t.
Then, the first order condition associated with n_{t+1} and the envelope theorem give the dynamic first order condition for employment:

$$\mu_t = \beta \mathrm{E_t}\left[\phi_1\frac{(1-l_{t+1})^{1-\eta}}{1-\eta} - \phi_2\frac{(1-e)^{1-\eta}}{1-\eta} + \frac{1}{c_{t+1}}(1-\theta)\frac{y_{t+1}}{n_{t+1}} + \mu_{t+1}\left(1-\sigma-(1-\alpha)p_{t+1}\right)\right] \tag{7}$$

This equation ensures the inter temporal optimality of the employment allocation, as it equates the current value of a new match to the discounted expected surplus this match will generate. Indeed, one can read the right hand side as the total surplus generated by a new match at next period. The term $\phi_1\frac{(1-l_{t+1})^{1-\eta}}{1-\eta} - \phi_2\frac{(1-e)^{1-\eta}}{1-\eta}$ is the change in utility of a matched agent as she was non-employed this period and will be employed next period. The following term is the marginal productivity of labor ($(1-\theta)\frac{y_{t+1}}{n_{t+1}l_{t+1}}$) generated by a new match, that will supply l_{t+1} hours, valued in terms of marginal utility of consumption $\frac{1}{c_{t+1}}$. The last term represents next period value of matches. But in $t+1$, a fraction $p_{t+1} = \frac{m_{t+1}}{1-n_{t+1}}$ of non employed will be matched, reducing so the pool of job seekers. As $(1-\alpha)$ is the elasticity of matching function to the total search effort, the number of new matches at next period will reduce by $(1-\alpha)p_{t+1}$. To represent this search externality on the labor market, the above equation depreciates μ_{t+1} consequently. Moreover, as a share σ of new matches will be exogenously destroyed, future value of matches is also depreciated by this term.

The two last first order conditions are the constraints (2) and (3) described above. For a purpose of clarity and in perspective of the log linearization of the model, I use the variables y_t, m_t and p_t in the above equations. The set of equilibrium conditions should thus be completed by the definitions of these variables, that is:

$$y_t = e^{z_t}\zeta k_t^{\theta}(n_t l_t)^{1-\theta} \tag{8}$$

$$m_t = \chi v_t^{\alpha}((1-n_t)e)^{1-\alpha} \tag{9}$$

$$p_t = \frac{m_t}{1-n_t} \tag{10}$$

I also define explicitly in equilibrium conditions the stochastic process of z_t

$$z_{t+1} = \rho z_t + \varepsilon_t \tag{11}$$

2.2 Wage setting in the search economy

The remaining equilibrium condition is the wage setting rule. Wage setting rule is the result of the vacancy opening behavior of the firm and the wage bargaining process.The value of a filled match for a firm is given by:

$$J_t = \pi_t + (1-\sigma)\mathrm{E}_t\Delta_t J_{t+1}$$

Where J_t is the value of an active job for the firm. This value equates the induced dividend, that is the profit π_t plus the discounted expected change in the value of an active job, taking into account that this match will be destroyed with a probability σ. As complete financial market are assumed, the decentralization requires to use the stochastic discount factor $\Delta_t = \beta\frac{c_t}{c_{t+1}}$. It means that firms discount their future profit by the future change in marginal utility. One can note that, due to the constant return to scale of production function and using the equilibrium condition of capital market, profits can be written $\pi_t = l_t\left((1-\theta)\frac{y_t}{n_t l_t} - w_t\right)$. Opening a new vacancy yields the value:

$$Q_t = -\kappa + \mathrm{E}_t\Delta_t\left(q_t J_{t+1} + (1-q_t)Q_{t+1}\right)$$

The interpretation is quite straightforward. Opening a new vacancy entails a cost κ. The next period expected return is J_{t+1} in the case the vacancy is filled, with probability q_t and Q_{t+1} otherwise. The expected return is discounted using the stochastic discount factor.
The search and matching framework generates externalities on the labor market. Equation (7) made clear that the search activity of the non employed has an external effect through the job finding rate. Here one can see that posting new vacancies also has an external effect, by decreasing the job filling rate q_t it decreases the return of posting vacancies.
As there is free entry for firms in opening vacancy, at equilibrium, we must that $Q_t = 0$. This free-entry requirement leads to the job creation condition:

$$\kappa = q_t\mathrm{E}_t\Delta_t J_{t+1}$$

Consider now that workers and firms engage in a Nash bargaining process. Denoting ξ the bargaining power of the firms we have that firms get a share ξ of the total surplus generated by a match, expressed in terms of marginal utility. Denoting this total surplus $W_{2,t}$ one can write $J_t = \xi c_t W_{2,t}$ or alternatively $W_{2,t+1} = \frac{J_{t+1}}{\xi c_{t+1}}$. Using now the equation (7), which writes with this notation as $\mu_t = \beta\mathrm{E}_t W_{2,t+1} = \frac{1}{\xi c_t}\mathrm{E}_t\Delta_{t+1}J_{t+1}$. Substituting in the free-entry condition gives:

$$\kappa = q_t\xi c_t\mu_t \Leftrightarrow \frac{\kappa v_t}{c_t} = \xi\mu_t m_t$$

So the assumption that the outcome of the bargaining process is efficient reduces to $\alpha = \xi$, so that the decentralized allocation is compatible with the one implied by the first order condition for the vacancies of the planner program (equation (6)). Now, replacing the job creation condition and the surplus sharing rule in the equation for the value of an active job yields:

$$\alpha c_t W_{2,t} = \pi_t + (1-\sigma)\frac{\kappa}{q_t}$$

Deducing the value of $W_{2,t}$ from equation (7) one obtain the wage setting rule:

$$w_t = (1-\alpha)(1-\theta)\frac{y_t}{n_t l_t} + \frac{\alpha c_t}{l_t}\left[\phi_2\frac{(1-e)^{1-\eta}}{1-\eta} - \phi_1\frac{(1-l_t)^{1-\eta}}{1-\eta} + (1-\alpha)p_t\mu_t\right] \tag{12}$$

Real wages are a weighted average of the labor marginal productivity and the return from the search activity. Indeed, as being non employed delivers return from search activity, the outside option of workers constitute a credible threat in the bargaining process. Thus, the workers can extract a share of the rent generated by an active job, that is a share of the hiring costs that the firm saves.

In this equation α shouldn't be viewed as the the elasticity of matching function but as the bargaining power of firm (and so $1-\alpha$ as the bargaining power of workers). The return from non employed search is composed by $\phi_2\frac{(1-e)^{1-\eta}}{1-\eta} - \phi_1\frac{(1-l_t)^{1-\eta}}{1-\eta}$, the differential in utility when being non employed and enjoying leisure rather than working and by $(1-\alpha)p_t\mu_t = p_t(1-\xi)\beta \mathrm{E}_t W_{2,t+1}$ is the expected share of total surplus extracted by the job seeker if he is matched. This return from search is expressed in terms of marginal utility, and per hours supplied.

That completes the characterization of the decentralized equilibrium of the search economy, which is defined by the equations (2) to (12).

3 Quantitative version of the model

The Real Business Cycles approach is a quantitative and applied exercise that aims at evaluating how a theoretical economy can replicates the stylized facts of business cycles. The spirit is thus to simulate the behavior of the economy over a given time period, and to compare the predicted properties of cyclical components of key variables to their empirical counterpart. Such a simulation requires a quantitative version of the model. To implement it, I solve the dynamics by using Matlab, which requires beforehand to have a linearized version of the model. For this purpose I first log linearise equilibrium equations around the steady state.

3.1 Log linearization of the model

I present here the log linearized equations for the decentralized equilibrium of the search economy. The detail of the computation is provided in appendix.

As in Andolfatto paper, the decentralized equilibrium is characterized by 7 equations. For sake of clarity, I use output, matches and job finding probability variables rather than directly substituting for their values. Thus, one needs to consider the equations that define those variables in addition. Taking into account the equation for technological shock, we are left with 11 equations to characterize the equilibrium.

This system is composed by 6 static equations and 5 dynamic ones. Consequently, there are 6 control variables, namely the output, vacancies, hours per worker, job finding probability, matches and real wages. The 5 states variables are consumption, capital, the marginal value of a new match, the level of employment and the technological shock. Among them, only k_t,

n_t and z_t have initial conditions and thus are backward-looking variables. The two remaining c_t and μ_t are forward-looking variables.

$$\hat{y}_t - \hat{l}_t \left(1 + \eta \frac{l}{1-l}\right) = \hat{c}_t + \hat{n}_t \tag{13}$$

$$\hat{v}_t - \hat{m}_t = \hat{c}_t + \hat{\mu}_t \tag{14}$$

$$\hat{w}_t = \left(\frac{y(1-\theta)(1-\alpha)}{wnl}\right)(\hat{y}_t - \hat{n}_t - \hat{c}_t) + \hat{c}_t + \hat{l}_t \left(\frac{\alpha c(1-l)^{-\eta}}{w} - 1\right) + (\hat{p}_t + \hat{\mu}_t)\left(\frac{(1-\alpha)\kappa v}{wl(1-n)}\right) \tag{15}$$

$$\hat{y}_t = \hat{z}_t + \theta \hat{k}_t + (1-\theta)(\hat{l}_t + \hat{n}_t) \tag{16}$$

$$\hat{m}_t = -(1-\alpha)\frac{n}{1-n}\hat{n}_t + \alpha \hat{v}_t \tag{17}$$

$$\hat{p}_t = \hat{m}_t + \hat{n}_t \frac{n}{1-n} \tag{18}$$

$$\hat{c}_{t+1} = \hat{c}_t + \hat{y}_{t+1}\beta\theta\frac{y}{k} - \hat{k}_{t+1}\beta\theta\frac{y}{k} \tag{19}$$

$$\frac{\hat{\mu}_t}{\beta} = \frac{\phi_1 l(1-l)^{-\eta}}{\mu}\left(-\hat{l}_{t+1} - \hat{c}_{t+1} + \hat{y}_{t+1} - \hat{n}_{t+1}\right) + \hat{\mu}_{t+1}(1-\sigma-(1-\alpha)p) - \hat{p}_{t+1}(1-\alpha)p \tag{20}$$

$$\hat{k}_{t+1} + \hat{c}_t \frac{c}{k} + \hat{v}_t \kappa \frac{v}{k} = \hat{y}_t \frac{y}{k} + (1-\delta)\hat{k}_t \tag{21}$$

$$\hat{n}_{t+1} = (1-\sigma)\hat{n}_t + \hat{m}_t \frac{m}{n} \tag{22}$$

$$\hat{z}_{t+1} = \rho \hat{z}_t + \varepsilon_t \tag{23}$$

The first set of equations, from (13) to (18) captures static equilibrium relations. This system of equations can be written in a matrix form:

$$M_1 \begin{pmatrix} \hat{y}_t \\ \hat{v}_t \\ \hat{l}_t \\ \hat{p}_t \\ \hat{m}_t \\ \hat{w}_t \end{pmatrix} = M_2 \begin{pmatrix} \hat{c}_t \\ \hat{k}_t \\ \hat{\mu}_t \\ \hat{n}_t \\ \hat{z}_t \end{pmatrix} \tag{24}$$

The second set of equations, from (19) to (23) captures dynamic relations. This system of equations can be written in a matrix form:

$$M3I \begin{pmatrix} \hat{c}_{t+1} \\ \hat{k}_{t+1} \\ \hat{\mu}_{t+1} \\ \hat{n}_{t+1} \\ \hat{z}_{t+1} \end{pmatrix} + M3L \begin{pmatrix} \hat{c}_t \\ \hat{k}_t \\ \hat{\mu}_t \\ \hat{n}_t \\ \hat{z}_t \end{pmatrix} = M4I \begin{pmatrix} \hat{y}_{t+1} \\ \hat{v}_{t+1} \\ \hat{l}_{t+1} \\ \hat{p}_{t+1} \\ \hat{m}_{t+1} \\ \hat{w}_{t+1} \end{pmatrix} + M4L \begin{pmatrix} \hat{y}_t \\ \hat{v}_t \\ \hat{l}_t \\ \hat{p}_t \\ \hat{m}_t \\ \hat{w}_t \end{pmatrix} + M5 \begin{pmatrix} \hat{\varepsilon}^c_{t+1} \\ \hat{\varepsilon}^k_{t+1} \\ \hat{\varepsilon}^\mu_{t+1} \\ \hat{\varepsilon}^n_{t+1} \\ \hat{\varepsilon}^z_{t+1} \end{pmatrix} \tag{25}$$

3.2 Solving the dynamics

I first simplify the dynamic system (25) by substituting the values of control variables, using (24) and obtain

$$\hat{S}_{t+1} = W\hat{S}_t + Q\hat{\varepsilon}_{t+1} \tag{26}$$

Where $\hat{S}_{t+1} = (\hat{c}_{t+1}\ \hat{k}_{t+1}\ \hat{\mu}_{t+1}\ \hat{n}_{t+1}\ \hat{z}_{t+1})'$. I then diagonalize W and order the matrix of eigenvectors P in such a way that the eigenvalues in $D = P^{-1}WP$ are in increasing order. Denoting $P1I = P^{-1}$ one can define

$$\tilde{S}_t = \begin{pmatrix} \tilde{k}_t \\ \tilde{\mu}_t \\ \tilde{z}_t \\ \tilde{n}_t \\ \tilde{c}_t \end{pmatrix} = P1I.\hat{S}_t = P1I \begin{pmatrix} \hat{c}_t \\ \hat{k}_t \\ \hat{\mu}_t \\ \hat{n}_t \\ \hat{z}_t \end{pmatrix} \tag{27}$$

The order of variables in $\tilde{S}_t$ reflect the increasing order of eigenvalues. The dynamic system can now be written as:

$$\tilde{S}_{t+1} = D\tilde{S}_t + R\hat{\varepsilon}_{t+1}$$

Where D is diagonal. The two last eigenvalues are explosive: $\lambda_4 = 1.06$ and $\lambda_5 = 2.05$. As there are two forward looking variable c_t and μ_t, the condition of Blanchard and Kahn is fulfilled. One can impose saddle-path condition on the two last equations of the system, that is:

$$\begin{pmatrix} \tilde{n}_t \\ \tilde{c}_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} P_{41} & P_{42} & P_{43} & P_{44} & P_{45} \\ P_{51} & P_{52} & P_{53} & P_{54} & P_{55} \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \hat{k}_t \\ \hat{\mu}_t \\ \hat{n}_t \\ \hat{z}_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{28}$$

Where P_{ij} are the coefficients of $P1I$. This system can be used to express the forward looking variables in terms of the backward looking. To do so, one can write the system (28) as

$$P1IF \begin{pmatrix} \hat{c}_t \\ \hat{\mu}_t \end{pmatrix} + P1IB \begin{pmatrix} \hat{k}_t \\ \hat{n}_t \\ \hat{z}_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{29}$$

As $P1IF = \begin{pmatrix} P_{41} & P_{43} \\ P_{51} & P_{53} \end{pmatrix}$ is invertible, one can solve (29) as $\begin{pmatrix} \hat{c}_t \\ \hat{\mu}_t \end{pmatrix} = SP \begin{pmatrix} \hat{k}_t \\ \hat{n}_t \\ \hat{z}_t \end{pmatrix}$ Where $SP = -P1IF^{-1}.P1IB$. Substituting then these values of $\hat{c}_t$ and $\hat{\mu}_t$ in (26) and gives the decision rule for state variables

$$\begin{pmatrix} \hat{k}_{t+1} \\ \hat{n}_{t+1} \\ \hat{z}_{t+1} \end{pmatrix} = PIB \begin{pmatrix} \hat{k}_t \\ \hat{n}_t \\ \hat{z}_t \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \hat{\varepsilon}_{t+1} \end{pmatrix} \tag{30}$$

And using (24) again gives the decision rule for control variables.

$$\begin{pmatrix} \hat{y}_t \\ \hat{v}_t \\ \hat{l}_t \\ \hat{p}_t \\ \hat{m}_t \\ \hat{w}_t \end{pmatrix} = PIC \begin{pmatrix} \hat{k}_t \\ \hat{\mu}_t \\ \hat{z}_t \end{pmatrix} \tag{31}$$

3.3 Calibration

A crucial step to obtain a quantitative version of the model is the calibration exercise. To avoid the shortcomings of the reduced form model highlighted by the Lucas critique, RBCs models ground on microfoundations and thus on deep structural parameters. To set the value of those parameters, I calibrate the deep parameters in such a way that long-run relations predicted by the model are consistent with their empirical counterparts. As a consequence, I will exploit steady state relations to calibrate these parameters. As I aim at replicating the results of Andolfatto, I replicate here his calibration.
In a first time, I set the targets for the long-term values of key economic variables, where the output is normalized to unity. Those objectives are:

y	n	l	r
1	0.57	0.33	1.1

Then, using reasonable assumptions or exogenous information one can directly set the value of deep parameters or the long run value of key economic variables.

e	q	θ	α	δ	ψ	σ	η	ρ	σ_ε
0.33/2	0.90	0.36	0.60	0.025	0.01	0.15	2	0.95	0.007

Different justifications come with those values. Some of these are set consistently with existing research in the literature, such as the separation rate σ, the capital intensity of output θ, the match elasticity to vacancies α or the inverse of Frisch elasticity η. Other parameter such as δ or e are set so that their values seems reasonable and realistic.
One can now exploit the steady state relations to calibrate the remaining parameters or steady state values. I report here the relations used and the calibrated values.

$\beta = \frac{1}{r}$	k $= \frac{\theta}{r+\delta-1}$	$c = 1 - \delta k - \psi$	$m = \sigma n$	$v = \frac{m}{q}$	$\kappa = \frac{\psi}{v}$	$\mu = \frac{\psi}{\alpha c m}$
$\beta = 0.99$	$k = 10.2$	$c = 0.73$	$m = 0.08$	$v = 0.095$	$\kappa = 0.105$	$\mu = 0.26$

$\phi_1 = \frac{y(1-l)^\eta(1-\theta)}{nlc}$	$\phi_2 = \frac{(1-\eta)}{(1-e)^{1-\eta}}\left(\frac{\phi_1(1-l)^{1-\eta}}{1-\eta} + \frac{(1-\theta)y}{nc} + \mu(1 - \frac{\mu}{\beta} - \sigma - (1-\alpha)p)\right);$
$\phi_1 = 2.08$	$\phi_2 = 1.54$

$w = \frac{1}{nl}\left((1-\theta) - \psi\frac{1-\beta(1-\sigma)}{\beta\sigma}\right)$
$w = 3.34$

4 Evaluation of quantitative performance

The objective of this section is twofold. I present my replications of Andolfatto figures and discuss the performance of the search economy regarding the objective described in the introduction. It is worthwhile noting that these figures replicate quite accurately the ones reported by Andolfatto.

4.1 Business cycles properties

First, I aim at evaluating the ability of search economy to generate the main stylized properties of business cycles. To do so, I have replicated the third part of Table 1 in Andolfatto paper, by performing 1000 stochastic simulations of the search economy, with a 500-length time period. For each simulations, I have stored the standard deviation, the correlation with output, and the autocorrelation of cyclical components[1] of the 10 key variables listed in the table 1 of the paper[2]. Here, I present the average simulations obtained. Row (1) expresses the standard deviation of each variable relative to the output standard deviation. Row (2) is the contemporaneous correlation with output. Row (3) is the autocorrelation with one lag of each variable.

Table 1: Business cycles properties

	Output	Consumption	Investment	Total hours	Employment
(1)	1.00	0.31	3.01	0.61	0.56
(2)	1.00	0.91	0.99	0.96	0.82
(3)	0.81	0.81	0.83	0.89	0.85
	Hours per worker	Wage Bill	labor share	Productivity	Real wages
(1)	0.22	0.92	0.16	0.45	0.34
(2)	0.60	0.99	-0.57	0.92	0.93
(3)	0.38	0.86	0.36	0.57	0.76

One can now compare those measures with the actual business cycles properties of the U.S. economy reported by Andolfatto to evaluate the achievements of the search economy. Several achievements could be noted.

- The search economy generates a volatility of output which is closer to the observed one that standard RBC (I obtain a value of 1.43 for the standard deviation of output, close to the 1.45 obtained by Andolfatto).
- By introducing the extensive margin, search economy delivers high relative volatility of total hours to real wages (the value of the ratio is 2.11 in the data reported by Andolfatto, 1.51 in his simulations and 1.79 here).
- The extensive margin of employment is much more volatile than the intensive, indicating that much of variations in total hours comes from the extensive margin, in accordance with the data.
- The labor intensity of output is countercyclical while it was constant by construction in standard RBC.

However some points remain disappointing.

- The volatility in total hours generated by the search economy is still insufficient to account for the important volatility observed in the data.
- The wage bill displays a strong contemporaneous correlation with output, that is one cannot obtain the lag of the wage bill on output that we observe in the data.

[1]Cyclical components were extracted by using a HP filter, with a smoother parameter set at 1600 as the time length considered is the quarter.

[2]Log-linearized equations of investment, wage bill, labor share and productivity are provided in appendix.

4.2 Total hours, productivity and real wages

As search and matching mechanisms soften the link between real wages and productivity, by augmenting real wages by the return of search activity, there is some hope that the model mimics more accurately the cyclical behavior of those variables. One can first use the table (1) to note some achievements of the search economy. The real wage volatility generated is smaller than the variance of the productivity, which is in accordance with the data. One can also see that both those variables have a smaller variance than total hours, unlike in standard RBC models. However, the model cannot generates the leading behavior of productivity and real wages. Indeed, both are predicted to be very strongly correlated with the contemporaneous value of output. Let's focus now one the cross-correlation of total hours with both productivity and real wages. To do so, I have replicated the table 2 of Andolfatto paper.

Table 2: Cross correlations of total hours with

Lags	-4	-3	-2	-1	0	1	2	3	4
Productivity	0.10	0.34	0.60	0.81	0.78	0.51	0.36	0.25	0.17
Real wages	-0.06	0.19	0.48	0.75	0.83	0.69	0.57	0.46	0.36

Like in the data, one can see that productivity leads hours. However, the model clearly underestimate this lead, as here the largest correlations is with 1 lag, against 4 in the data. But worse, the search economy is unable to deliver the asymmetric pattern in correlations over time. It always predicts a positive correlation between total hours and productivity or between total hours and real wages, while in the data, total hours are negatively correlated with the future values of productivity and the past values of real wages. Ultimately, real wages still behave too much like productivity. Even if the contemporaneous correlation between total hours has decreased as compared to standard RBC, it remains far too high as compared to the data, where the correlation is negative.

4.3 Unemployment and vacancies

A great interest of integrating the search and matching environment in RBC model is that as a consequence, it naturally embeds a decreasing relation between unemployment and vacancies, the so-called Beveridge curve. As this is a well known empirical regularity, the model's ability to replicate the stylized facts associated with the Beveridge curve is a good way to assess the quantitative performance of the search economy.

For this purpose, I replicate the third table of Andolfatto paper. Before focusing on the relation between unemployment and vacancies, one can look at the cross-correlations of employment. The search economy achieves to generate the important persistence of unemployment over time that we observe in the data.

On the relation between unemployment and vacancies, there is indeed a negative contemporaneous correlation between vacancies and unemployment. However this correlation is far weaker than the one observed in the data. The search economy also delivers the leading behavior of vacancies on unemployment (by 2 quarters). However in the data, unemployment is negatively correlated with both past and future values of vacancies, while the search economy predicts that future values of vacancies to be positively related to unemployment. Another limit of the search economy is the weak magnitude of the correlations between unemployment and vacancies that it predicts and in a more striking way, the too weak volatility of vacancies it generates. I compute that the relative volatility of vacancies to output is 3.27, while the one observed in the data reported by Andolfatto is about 9.

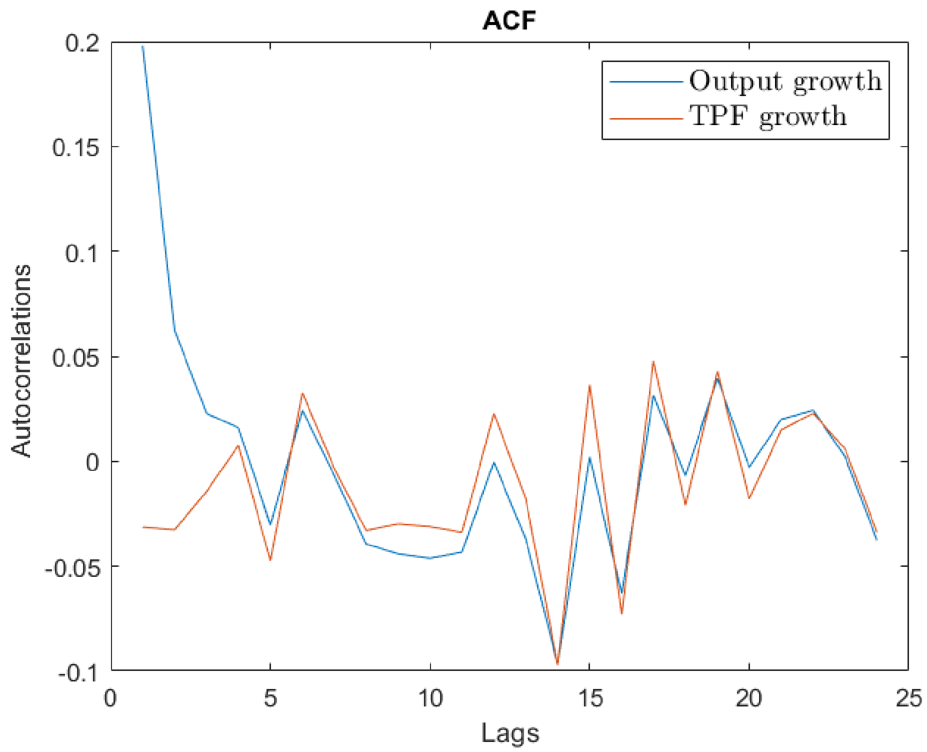
Table 3: Cross correlations of unemployment with

Lags	-4	-3	-2	-1	0	1	2	3	4
Unemployment	0.17	0.37	0.61	0.85	1.00	0.85	0.61	0.37	0.17
Vacancies	-0.46	-0.61	-0.71	-0.65	-0.16	0.07	0.18	0.23	0.24

4.4 Propagation mechanisms

A last key point of the evaluation is to assess the ability of the search economy to amplify the response to the impulse, that is, whether the model embeds an internal propagation mechanisms. Standard RBC models didn't succeed to generate such mechanisms, all the persistence was due to external propagation mechanisms, that is to the technological process. As a consequence, the persistence of output was identical to the one of the technical process. On the contrary, one can see of the figure (1) where is replicated the second panel of figure 1 of Andolfatto paper, that for the first lags, the persistence of output growth significantly differs from the one of the technological process: there is a positive autocorrelation on the short run. The search economy features a propagation mechanisms that increase the persistence of a shock on output.

Figure 1: Autocorrelation functions of output and TPF growth



4.5 Impulse Response function

The investigation of the impulse response functions confirms the results that the search economy embeds a significant propagation mechanisms that amplify the impulse. The technological shock propagates into the economy with a delay, which delivers hump-shaped IRFs, such as the response of the output characterized by Cogley and Nason.

In response to the shift in the labor productivity, and to the subsequent increase in the expected return of new matches, firms want to use more labor input. Thus they open new vacancies.

However, as forming new matches takes time, the answer of employment is delayed, the immediate answer of labor input is through the intensive margin. Indeed, households are willing to supply more individual hours due to the higher wages. After a small delay, new matches are formed and the employment increases. However, this delay in the employment answer generates an amplification in the response of output, leading to the hump shape that we observe. After this delay, as productivity falls, wages and total hours return to their steady state value.
In parallel, as the response of total hours and output is delayed, the rental rate of capital does not decrease immediately after its drop at the time of the shock, it continues to increase for some periods. Thus, investment continues to increase and consequently the capital stock response is also hump-shaped.

Figure 2:

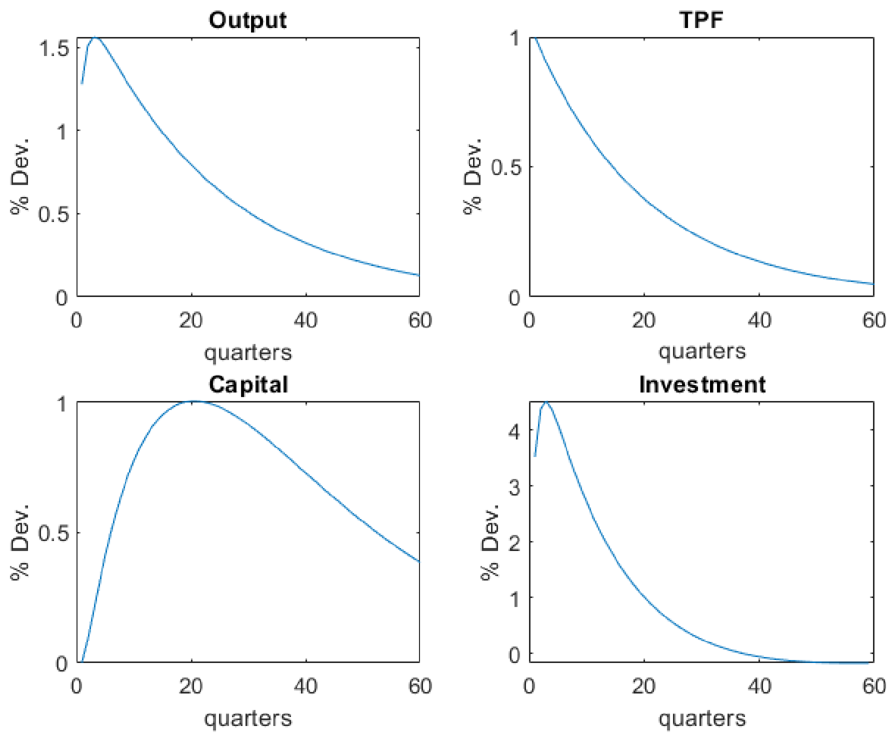


Figure 3:

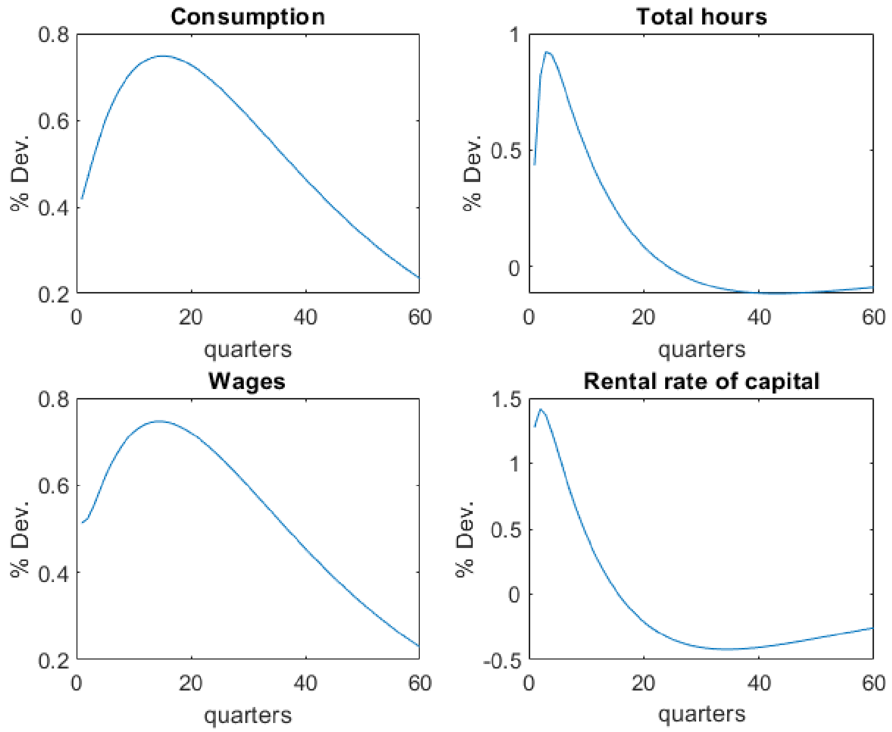
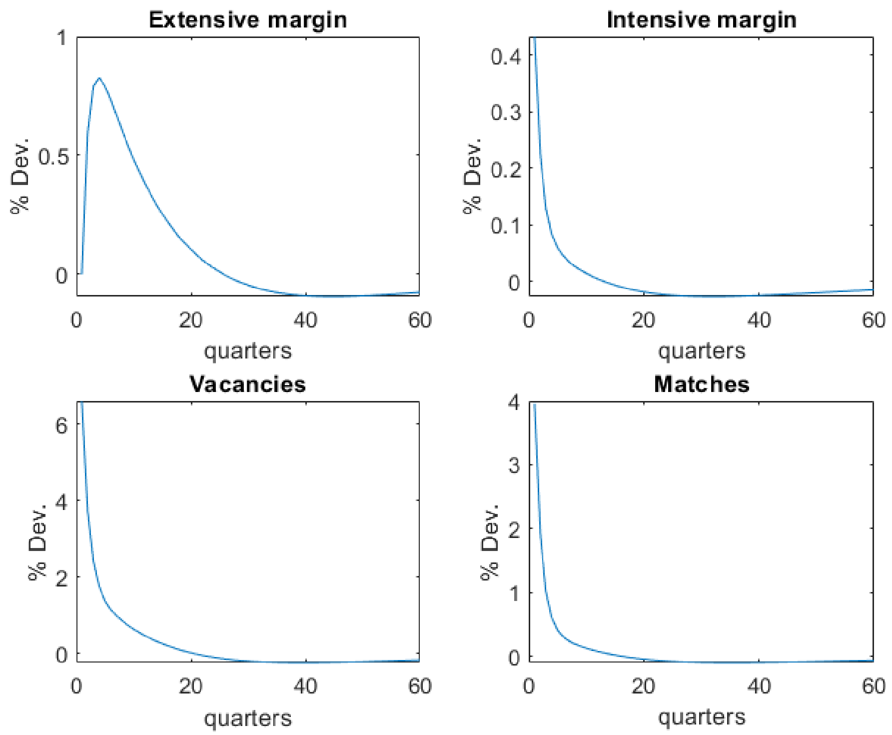


Figure 4:



5 Extensions

5.1 Allocative shocks

A first extension that could be added to the search economy is already discussed by Andolfatto. He proposes to introduce shocks in the matching function that capture shifts in the Beveridge curve. Such shifts represent changes in the structure of the labor market. Andolfatto proposes an example of a sectoral shift. One can also think to a shock of information technology that would reduce the time involved in the search process and so increases the number of matches for a given level of input in the matching function. Another example of allocative shock could be the introduction of a benevolent matching agency, such as Pole Emploi in France, whose aim is to enhance the efficiency of the matching process.

The search economy offers a natural framework to evaluate whether allocative shocks affect the business cycles properties. Given the functional form assumed for the matching function, the natural way to introduce those shocks is to consider that the scale parameter χ_t follows a stochastic process. In his paper Andolfatto assumes that technological and allocative shocks follow a VAR(1) process. However, one shall note a limitation of his analysis, as the search effort e is assumed to be constant, changes in χ_t can capture both structural disturbances or changes in the search effort. To identify properly the allocative shocks, one must endogenized the search effort.

The quantitative results obtained by Andolfatto shed some doubts on the ability of allocative mechanisms to enhance the performance of the search economy. When considering three different level of variance for allocative shocks, in all cases the volatility of output exceeds the one observed in the data. One can note a higher volatility in total hours, closer to the value observed in the data, but this is at the cost of undesirable properties for both intensive and extensive margin. The relative standard deviation of employment exceeds the one of the total hours, and the contemporaneous correlation of individual hours with output is either too weak or negative. One shall also note a theoretical failure of allocative mechanisms. As such shocks increase employment, the fall in labor productivity provides incentives to firms to use the intensive margin, which yields a negative relation between employment and individual hours.

5.2 Shimer puzzle

Since the work of Andolfatto, other studies have embedded search and matching framework in RBC models. Among those researches, the one of Shimer in 2005 have highlighted an inherent failure of the search economy. Due to the bargaining process and the consequent ability of workers to obtain a share of the return of the search activity in addition to their productivity. Consequently, wages are strongly procyclical. When a positive technological shock hits the economy the incentives for firms to open new vacancies due to the increase in labor productivity is offset by the increase of real wages. Therefore, productivity shocks affect labor market tightness to a very small extent and as the only source of volatility in the search economy is the technology, it cannot generates enough volatility in unemployment. I compute in table (4) the standard deviation for the three key labor market variables[3].

Table 4: Volatility of labor market variables

Vacancies	Unemployment	Tightness
0.05	0.01	0.05

[3]The values reported are not expressed in relative to output standard deviation for this table. This is to the purpose of comparing with the table provided by Shimer.

These values that are clearly comparable to the order of magnitude obtained by Shimer, confirming that the search economy fails to generate a consistent unemployment volatility.
It could be interesting to note that those results are obtained for a value of elasticity of matches to vacancies far above the one considered by Shimer. Indeed, this is obtained with value set by Andolfatto, that is $\alpha = 0.60$, while Shimer calibrated α at 0.28. As 0.60 value is already above the range of plausible values reported by Petrongolo and Pissarides, one cannot try to modify the calibration to increase the elasticity of tightness to productivity. Thus, solving the Shimer puzzle in Andolfatto paper will requires others rationales, such as wages rigidities for instance. It remains worthwhile noting that all the elements highlighted by Shimer were already present in Andolfatto analysis, almost 10 years in advance.

6 Conclusion

Along several margins, the incorporation of search and matching framework into the workhorse RBC model has significantly enhanced the performance of the model. As shown in section 4.4, the search economy embeds a significant propagation and amplification mechanism. Moreover, the extensive margin accounts for the most volatility of total labor input, whose volatility is increased compared to standard RBC. The model also generates a high relative volatility of total hours with respect to the real wage. The volatility of real wages is now smaller than the one of productivity. Concerning unemployment, the model replicates its persistancy and its negative correlation with vacancies.
Interestingly, Andolfatto presents the ability of the search economy to reduce the contemporaneous correlation between total hours and productivity as a success. However, one should note that the correlation decreases to a very small extent compared to standard RBC and remains far too high with respect to the data, where we observe a negative correlation. In the same way, the correlation between total hours and real wages predicted by the search economy is far too high. Real wages are also too strongly procyclical in the model, while almost a-cyclical in the data. Furthermore, the asymmetric pattern in dynamic correlation between total hours with productivity and real wages is not generated by the model.
The search economy presents other limitations. It cannot generate the volatility of the key variables introduced by the search and matching framework, that is, unemployment and vacancies and underestimates the correlation between those two variables (that is the strength of the Beveridge curve).
The contribution of Andolfatto to literature remains no less substantial. It has paved the road for a better understanding of the labor market within the real business cycle framework. Further research had grounded on his work, such as the model of den Haan, Ramey and Watson (2000), which endogenize separation in a model highly comparable to the one of Andolfatto.
Moreover, by calling for other rationales for unemployment fluctuations, one can consider that the research of Andolfatto have pushed the research agenda in a fruitful direction, that is, the introduction of wage rigidity and the broader development of DSGE analysis.

7 Appendix

7.1 Log linearizations of decentralized equilibrium

For the 11 equations of the decentralized equilibrium, one can use the following procedure in order to obtain log-linearized equations :

- take the log of the equation of the decentralized equilibrium ;
- make a first order Taylor development of all logarithms that have appeared, around the steady state ;
- simplify the equation obtained by using the steady state expression of the decentralize equilibrium (which permits to delete steady state terms on each side of the equation).

Here, I detail two equations for the example, for the Euler equation and the intratemporal first order condition.
Taking the log of the decentralized equilibrium Euler equation gives:

$$-log(c_t) = log(\beta) - log(c_{t+1}) + log(\theta \frac{y_{t+1}}{k_{t+1}} + 1 - \delta) \tag{32}$$

Then, using a Taylor development at order 1 around the steady state yields:

$$-\hat{c}_t = log(\beta) - \hat{c}_{t+1} + \left[log(\theta \frac{y}{k} + 1 - \delta) + (y_{t+1} - y) \frac{\frac{\theta}{k}}{\theta \frac{y}{k} + 1 - \delta} + (k_{t+1} - k) \frac{\frac{-\theta y}{k^2}}{\theta \frac{y}{k} + 1 - \delta} \right] \tag{33}$$

As far as one consider the decentralized equilibrium Euler equation at the steady state $\frac{1}{\beta} = \theta \frac{y}{k} + 1 - \delta$, one can simplify the equation above and obtain :

$$-\hat{c}_t = -\hat{c}_{t+1} + \frac{(y_{t+1} - y)}{y} \beta \theta \frac{y}{k} - \frac{(k_{t+1} - k)}{k} \beta \theta \frac{y}{k} \tag{34}$$

Eventually that leads to the log linearized Euler equation :

$$\hat{c}_{t+1} = \hat{c}_t + \hat{y}_{t+1} \beta \theta \frac{y}{k} - \hat{k}_{t+1} \beta \theta \frac{y}{k} \tag{35}$$

Taking the log of the intratemporal first order condition and Taylor developing yields:

$$log(\phi_1) - \eta \left[log(1 - l) - (l_t - l) \frac{1}{1 - l} \right] = log(1-\theta) + log(y) + \hat{y}_t - log(n) - \hat{n}_t - log(l) - \hat{l}_t - log(c) - \hat{c}_t \tag{36}$$

One can simplify using the expression of the intratemporal first order condition at the steady state, and obtain the linearized intratemporal first order condition :

$$\hat{y}_t - \hat{l}_t \left(1 + \eta \frac{l}{1 - l} \right) = \hat{c}_t + \hat{n}_t \tag{37}$$

7.2 Matricial form of the dynamic system

Static equilibrium relations are captured by the system

$$M_1 \begin{pmatrix} \hat{y}_t \\ \hat{v}_t \\ \hat{l}_t \\ \hat{p}_t \\ \hat{m}_t \\ \hat{w}_t \end{pmatrix} = M_2 \begin{pmatrix} \hat{c}_t \\ \hat{k}_t \\ \hat{\mu}_t \\ \hat{n}_t \\ \hat{z}_t \end{pmatrix} \tag{38}$$

Where

$$M1 = \begin{pmatrix} 1 & 0 & -(1+\eta\frac{l}{1-l}) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ \frac{y(1-\alpha)(1-\theta)}{wnl} & 0 & \frac{\alpha c(1-l)^{-\eta}}{w} - 1 & \frac{(1-\alpha)\kappa v}{wl(1-n)} & 0 & -1 \\ 1 & 0 & -(1-\theta) & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix} \tag{39}$$

$$M2 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ -1 + \frac{y(1-\theta)(1-\alpha)}{wnl} & 0 & -\frac{(1-\alpha)\kappa v}{wl(1-n)} & \frac{y(1-\theta)(1-\alpha)}{wnl} & 0 \\ 0 & \theta & 0 & (1-\theta) & 1 \\ 0 & 0 & 0 & -(1-\alpha)\frac{n}{1-n} & 0 \\ 0 & 0 & 0 & \frac{n}{1-n} & 0 \end{pmatrix} \tag{40}$$

Dynamics relation are captured by

$$M3I \begin{pmatrix} \hat{c}_{t+1} \\ \hat{k}_{t+1} \\ \hat{\mu}_{t+1} \\ \hat{n}_{t+1} \\ \hat{z}_{t+1} \end{pmatrix} + M3L \begin{pmatrix} \hat{c}_t \\ \hat{k}_t \\ \hat{\mu}_t \\ \hat{n}_t \\ \hat{z}_t \end{pmatrix} = M4I \begin{pmatrix} \hat{y}_{t+1} \\ \hat{v}_{t+1} \\ \hat{l}_{t+1} \\ \hat{p}_{t+1} \\ \hat{m}_{t+1} \\ \hat{w}_{t+1} \end{pmatrix} + M4L \begin{pmatrix} \hat{y}_t \\ \hat{v}_t \\ \hat{l}_t \\ \hat{p}_t \\ \hat{m}_t \\ \hat{w}_t \end{pmatrix} + M5 \begin{pmatrix} \hat{\varepsilon}^c_{t+1} \\ \hat{\varepsilon}^k_{t+1} \\ \hat{\varepsilon}^\mu_{t+1} \\ \hat{\varepsilon}^n_{t+1} \\ \hat{\varepsilon}^z_{t+1} \end{pmatrix} \tag{41}$$

Where

$$M3I = \begin{pmatrix} 1 & \beta\theta\frac{y}{k} & 0 & 0 & 0 \\ \frac{\phi_1 l}{(1-l)^\eta \mu} & 0 & -(1-\sigma-(1-\alpha)p) & \frac{\phi_1 l}{(1-l)^\eta \mu} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{42}$$

$$M3L = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\beta} & 0 & 0 \\ \frac{c}{k} & -(1-\delta) & 0 & 0 & 0 \\ 0 & 0 & 0 & -(1-\sigma) & 0 \\ 0 & 0 & 0 & 0 & -\rho \end{pmatrix} \tag{43}$$

$$M4I = \begin{pmatrix} \beta\theta\frac{y}{k} & 0 & 0 & 0 & 0 & 0 \\ \frac{\phi_1 l}{(1-l)^\eta \mu} & 0 & -\frac{\phi_1 l}{(1-l)^\eta \mu} & -(1-\alpha)p & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{44}$$

$$M4L = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{y}{k} & -\kappa\frac{v}{k} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{m}{n} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{45}$$

$$M5 = (0\ 0\ 0\ 0\ 1)' \tag{46}$$

7.3 Additional log linearizations

Throughout the quantitative evaluation of the model, I use additional variables. I give here their log linearized expression as they are used in the Matlab code.

$$\text{Investment}_t = \frac{1}{\delta} k_{t+1} - \frac{1-\delta}{\delta} \hat{k}_t \tag{47}$$

$$\text{Total hours}_t = \hat{l}_t + \hat{n}_t \tag{48}$$

$$\text{Wage bill}_t = \hat{w}_t + \hat{l}_t + \hat{n}_t \tag{49}$$

$$\text{labor productivity}_t = \hat{y}_t - \hat{n}_t - \hat{l}_t \tag{50}$$

$$\text{Unemployment}_t = -\frac{n}{1-n} \hat{n}_t \tag{51}$$

$$\text{labor market thightness}_t = \hat{v}_t + \frac{n}{1-n} \hat{n}_t \tag{52}$$