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Modeling the wave motion of a guitar string

Topic: Sound theory and Fourier series

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Abstract

This investigation aims to develop a mathematical model for prescribing the behavior of an idealized vibrating string. The one-dimensional wave equation [1] is derived by physical examination of the situation and a general solution was found by the method of separation of variables. We focus on the scenario where the string is put into motion by plucking the string at a specific position. Through the implementation of the boundary and initial conditions of the pluck, a simplified special formula [14] was found. The coefficients are found using Fourier series, giving the final displacement function:

$$y(x,t) = \sum_{k=1}^{\infty} \frac{2hL^2}{\pi^2 k^2 d(L-d)} \sin\left(\frac{k\pi d}{L}\right) \sin\left(\frac{k\pi}{L}x\right) \cos\left(\frac{ck\pi}{L}t\right)$$

This formula defines the displacement of the string at any point, at any time. Due to the structure of the formula, it is understood as superposition of sine and cosine terms, with coefficients found by [16]. In chapter 10, the physical interpretation of these coefficients is found to be the amplitudes of *harmonics* of the vibrating string, forming the fundamental tone and several overtones. By this mathematical model we can calculate theoretical amplitudes for the different harmonics, based on the extent and position of the pluck.

The calculated values were compared to empirical measurements made in a previous paper by Egeland, also submitted to Unge Forskere. Due to large inconsistencies with these empiric values, a similar experiment was carried out to find the sources of deviation. After trying variations of the original experiment setup, several sources of deviations were identified. By making some adjustments and removing the error sources, results which were consistent with the mathematical model were observed.

Word count: 253

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1. Introduction

Since early childhood, I have had an interest for music and I have played the piano and the guitar for several years. At the same time I am very interested in mathematics, also taking the high level course at the IB. It was therefore natural for me to write my extended essay on either one of these topics, or a combination. Wanting to explore the mathematical properties of sound and resonance, I chose the topic known as sound theory for the essay.

During early investigation I came across a paper[1] pertaining to this research area, written by a former student at my school. The essay presented: “An investigation on the influence of the position and method of plucking a guitar string for the distribution of energy between fundamental tone and overtones”. The research was done by recording sound samples from an acoustic guitar, and numerically analyzing the data using computer software. Conclusively, the essay stated that there were more overtones produced when the string was plucked close to the edge of the guitar. Since the paper was a physics investigation, it examined the matter experimentally, presenting little to no mathematical explanations for the results.

One of the main motivations for this essay was to try and explain the discoveries, using a mathematical approach. The research focus is to try to derive a mathematical model prescribing the behavior of a vibrating string. Some simplifying assumptions will need to be applied to keep the mathematical model manageable. At the end of the essay some interpretation of the model are given, and the results are briefly compared with the empirical work of Egeland.

[1] Egeland, K. (2009)

2. The physics of a vibrating string

Wave theory

In physics, a wave is defined as:

> *A wave is a disturbance that travels in a medium transferring energy and momentum form one place to another. The direction of energy transfer is the direction of propagation of the wave.*[2]

In short, a wave is simply a disruption in the position of particles, which is transferred as a wave pulse. The distance from the equilibrium position of the particles to the most displaced position is known as the amplitude. In an ideal model, where resistance and friction is neglected, the disturbance will continue travelling as long as there is no boundary.

Although not limited to, it applies for all types of elastic matter, including, the string of a guitar. In our case, the disturbance is the displacement of the guitar string, after it is plucked. Consequently, the disruption forms what is known as a transverse wave, i.e. a wave where the disturbance is at right angles to the direction of propagation of the wave. If the there is a change in the medium in which the wave is travelling, the wave pulse is (partly) reflected.

The ideal vibrating string

In an ideal model where the medium has a physical end, all the energy of the wave will be reflected. For a guitar string, the string is fixed at both ends; hence reflection occurs at both ends of the guitar string. The two wave pulses formed produce a stationary wave, i.e. a wave which is non-travelling.

When plucking the string, it will be initially displaced from its equilibrium state, as illustrated by the bold line in figure 1. When released, the wave will vibrate from its initial shape, to its mirror shape, and back again. In reality, friction can slightly alter the wave shape, but we will consider the first few moments as the time frame.

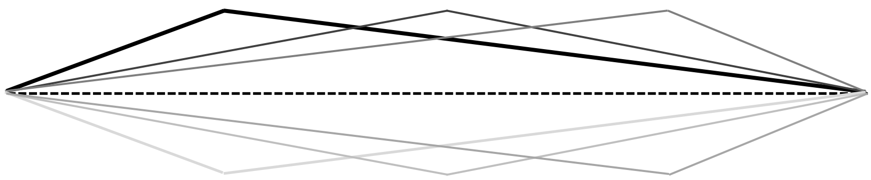


Figure 1: Initial and subsequent displacement of a plucked string.

[2] Tsokos, K. A. (2011, p.217)

Modeling a vibrating string

In this essay I will consider an ideal vibrating string and try to develop a mathematical model describing the wave formed. The purpose of this essay is to develop a function describing the shape of the wave. Let *y* denote the vertical displacement of the string from the x-axis. We need two variables to describe the displacement; the position along the string, *x*, and the time since plucking the string, *t*.

The aim of this essay is thus to develop a function *y*, so that:

$$y = y(x, t)$$

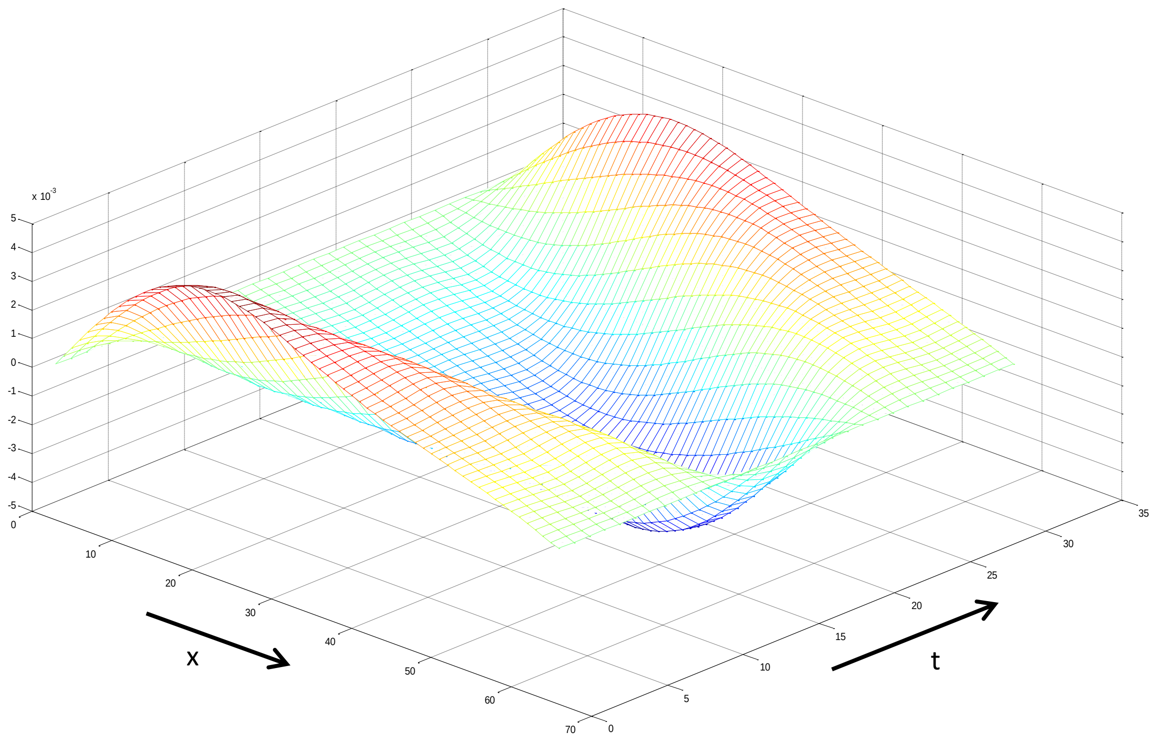


Figure 2: Illustration of the time-position-displacement function for a one-dimensional wave.

3. The wave equation

Let's consider some elastic medium stretched by a tension force T (we assume that this force is significantly larger than the string's weight). The medium now acts as a string, and has mass per unit length, μ and its equilibrium position is along the x-axis. When this string is plucked it will be forced to obtain a certain shape. In order to describe the motion caused by the plucking, we must first find the forces acting on the string. Figure 1 showed the shape of the string at some particular time. Considering an infinitesimal part of this string, dm, gives figure 3.

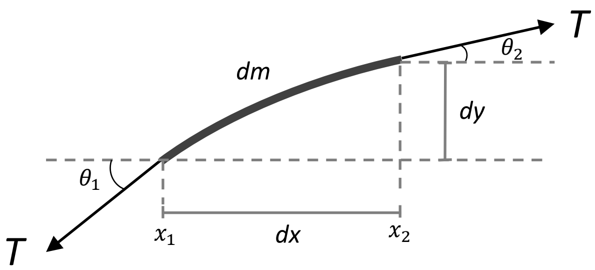


Figure 3: An infinitesimal part of a plucked string.

If we neglect gravity and all friction (with the air, as well as the friction of the string) the only force acting on the string is the tension, T. To find the y-component:

$$F_y = m \times a_y$$

At the same time, the Force in the y-direction must be equal to the sum of the forces at the points x_1 and x_2. We define upwards along the y-axis as positive, thus:

$$F_y = T \times \sin\theta_2 - T \times \sin\theta_1$$

For moderately small angles θ, $\sin\theta$ gets close to $\tan\theta$, as seen on figure 4.

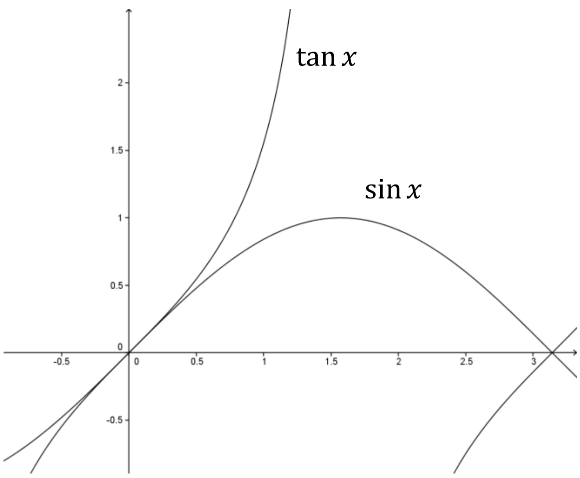


Figure 4: Graph of $tan\ (x)$ and $sin\ (x)$.

Comparing the small section of the string, *dm*, with the right-angle triangle (see fig. 3) gives:

$$\tan\theta = \frac{\partial y}{\partial x}$$

The partial derivative is applied because *y* is a function of both *x* and *t*. In this derivative, *t* is kept constant.

The second equation for the y-component of the force is then:

$$F_y = T \times \left.\frac{\partial y}{\partial x}\right|_{x\,=\,x_2} - T \times \left.\frac{\partial y}{\partial x}\right|_{x\,=\,x_1}$$

If we consider the first equation again, we can replace the acceleration with the second partial derivative of *y* with respect to the time. Again the partial derivative is applied because *x* is kept constant. The mass can also be expressed as the mass per unit length, *μ*, times the length, *dm*. However, for small values of *dx*, *dm* approaches *dx*. Thus:

$$F_y = m \times a_y = \mu \times dx \times \frac{\partial^2 y}{\partial t^2}$$

These two expressions for the y-component of the force must be equal:

$$T \times \left[\left.\frac{\partial y}{\partial x}\right|_{x\,=\,x_2} - \left.\frac{\partial y}{\partial x}\right|_{x\,=\,x_1} \right] = \mu \times dx \times \frac{\partial^2 y}{\partial t^2}$$

Rearranging gives:

$$\frac{T}{\mu} \times \frac{\left[\left.\frac{\partial y}{\partial x}\right|_{x\,=\,x_2} - \left.\frac{\partial y}{\partial x}\right|_{x\,=\,x_1} \right]}{dx} = \frac{\partial^2 y}{\partial t^2}$$

By definition, the left-hand side is the second derivative times a constant:

$$\frac{T}{\mu} \times \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$$

It is usual to introduce the variable *c* as: $\sqrt{\frac{T}{\mu}}$. This gives:

$$c^2 \times \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} \qquad [1]$$

Equation [1] is called the one-dimensional wave equation.

4. Superposition lemma

Before we start addressing the solution to the wave equation, we need to consider superpositioning, which will be used later. Mathematically, this can be described as follows:

> If $y_1(x,t), y_2(x,t), \ldots$ are solutions to the wave equation and K_i arbitrary constants, then the sum $y_s(x,t) = K_1 y_1(x,t) + K_2 y_2(x,t) + \cdots$ is another solution. [2]

To show that $y_s(x,t)$ is a valid solution, we expand $\frac{\partial^2 y_s}{\partial t^2}$, and see if it is indeed equal to $c^2 \times \frac{\partial^2 y_s}{\partial x^2}$.

Using the definition of $y_s(x,t)$:

$$\frac{\partial^2 y_s}{\partial t^2} = \frac{\partial^2}{\partial t^2}\left(K_1 y_1(x,t) + K_2 y_2(x,t) + \cdots\right)$$

Applying the sum rule of derivatives[3]:

$$= K_1 \frac{\partial^2}{\partial t^2} y_1(x,t) + K_2 \frac{\partial^2}{\partial t^2} y_2(x,t) + \cdots$$

We have assumed that $y_1(x,t), y_2(x,t), \ldots$ are solutions, hence: $c^2 \times \frac{\partial^2 y_1}{\partial x^2} = \frac{\partial^2 y_1}{\partial t^2}, c^2 \times \frac{\partial^2 y_2}{\partial x^2} = \frac{\partial^2 y_2}{\partial t^2} \ldots$
If we now substitute the right hand side with the left hand side, this gives:

$$= K_1 c^2 \times \frac{\partial^2}{\partial x^2} y_1(x,t) + K_2 c^2 \times \frac{\partial^2}{\partial x^2} y_2(x,t) + \cdots$$

$$= c^2 \times \left(K_1 \frac{\partial^2}{\partial x^2} y_1(x,t) + K_2 \frac{\partial^2}{\partial x^2} y_2(x,t) + \cdots\right)$$

Using the sum rule of derivatives again:

$$= c^2 \times \frac{\partial^2}{\partial x^2}\left(K_1 y_1(x,t) + K_2 y_2(x,t) + \cdots\right)$$

The contents of the parenthesis is simply the definition of $y_s(x,t)$:

$$= c^2 \times \frac{\partial^2}{\partial x^2} y_s(x,t)$$

Hence we have mathematically showed the principle of superposition. This can also be confirmed by physical observations for most media. From physics we know that two waves may coexist in the same medium and independent of each other. When e.g. two wave pulses meet, the amplitude will be equal to the sum of the amplitudes of each wave pulse.

[3] Wolfram Mathworld (2011), *Sum Rule*.

5. Solution of the wave equation

We would now like to solve the differential equation [1] to find a general solution for the behavior of a vibrating string. Solving the wave equation in general is very hard, but one method which seems applicable is by separation of variables[4].

This process is based on the assumption that the function for distance of position and time, $y(x,t)$, can be written as the product of two individual functions, one of positions and the other of time. $X(x)$ is only dependent on *x* and $T(t)$ is only dependent on t.

$$y(x,t) = X(x) \times T(t) \quad [3]$$

If we substitute the expression for *y* given in [3] in the wave equations [1], we get:

$$c^2 \times \frac{\partial^2}{\partial x^2}(X(x) \times T(t)) = \frac{\partial^2}{\partial t^2}(X(x) \times T(t))$$

$$c^2 \times \frac{\partial^2 X(x)}{\partial x^2} \times T(t) + \frac{\partial^2 T(t)}{\partial x^2} \times X(x) = \frac{\partial^2 X(x)}{\partial t^2} \times T(t) + \frac{\partial^2 T(t)}{\partial t^2} \times X(x)$$

$X(x)$ is independent of time, hence:

$$\frac{\partial^2 X(x)}{\partial t^2} = 0$$

$T(t)$ is independent of position, hence:

$$\frac{\partial^2 T(t)}{\partial x^2} = 0$$

The expression therefore simplifies to:

$$c^2 \times X''(x) \times T(t) + 0 \times X(x) = 0 \times T(t) + T''(t) \times X(x)$$

$$c^2 \times X''(x) \times T(t) = T''(t) \times X(x)$$

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} \times \frac{1}{c^2}$$

In this expression, the left hand side is independent of time, while the right hand side is independent of position. This means that they both must be equal to some constant, σ:

$$\frac{X''(x)}{X(x)} = \sigma \qquad \text{and} \qquad \frac{T''(t)}{T(t)} \times \frac{1}{c^2} = \sigma$$

$$X''(x) = \sigma \times X(x) \quad [4] \qquad \text{and} \qquad T''(t) = \sigma \times c^2 T(t) \quad [5]$$

This means that the functions $X(x)$ and $T(t)$ are equal to their own derivatives or in this case their second derivatives.

4 Feldman, J. (2007).

The only function[5] that fulfills this requirement is $y = ae^{rx}$, for some constants *a* and *r*.

We insert this into the equation [4] and get:

$$\frac{d^2}{dx^2}(ae^{rx}) - \sigma \times \ ae^{rx} = 0$$

$$r^2 \times ae^{rx} - \sigma \times \ ae^{rx} = 0$$

$$ae^{rx}(r^2 - \sigma) = 0$$

$$(r^2 - \sigma) = 0$$

$$r = \pm\sqrt{\sigma}$$

and into [5]:

$$\frac{d^2}{dt^2}(be^{st}) - \sigma \times \ c^2 be^{st} = 0$$

$$s^2 \times be^{st} - \sigma \times \ c^2 be^{st} = 0$$

$$be^{st}(s^2 - \sigma c^2) = 0$$

$$(s^2 - \sigma c^2) = 0$$

$$s = \pm c\sqrt{\sigma}$$

If $\sigma \neq 0$ there are two independent solutions: $a_1 e^{\sqrt{\sigma}x}$ and $a_2 e^{-\sqrt{\sigma}x}$ for $X(x)$, and two solutions: $a_3 e^{c\sqrt{\sigma}t}$ and $a_4 e^{-c\sqrt{\sigma}t}$ for $T(t)$. By the principle of superposition [2], the solution of $X(x)$ and $T(t)$ can then be written as the sum of the valid solutions:

$$X(x) = a_1 e^{\sqrt{\sigma}x} + a_2 e^{-\sqrt{\sigma}x} \qquad \text{and} \qquad T(t) = a_3 e^{c\sqrt{\sigma}t} + a_4 e^{-c\sqrt{\sigma}t}$$

If $\sigma = 0$ the equations [4] and [5] simplifies to:

$$X''(x) = 0 \qquad \text{and} \qquad T''(t) = 0$$

Any linear function will fulfill this requirement, thus we can write the solution as:

$$X(x) = a_1 x + a_2 \qquad \text{and} \qquad T(t) = a_3 t + a_4$$

We wanted to find two functions $X(x)$ and $T(t)$ by which the amplitude *y*, can be written as $y(x,t) = X(x) \times T(t)$. We have now found a large number of solution, so we can write $y(x,t)$:

$$y(x,t) = \left(a_1 e^{\sqrt{\sigma}x} + a_2 e^{-\sqrt{\sigma}x}\right)\left(a_3 e^{c\sqrt{\sigma}t} + a_4 e^{-c\sqrt{\sigma}t}\right) \quad [6] \qquad \text{for } \sigma \neq 0$$

$$y(x,t) = (a_1 x + a_2)(a_3 t + a_4) \quad [7] \qquad \text{for } \sigma = 0$$

for arbitrary constants a_1, a_2, a_3 and a_4.

[5] A sine or cosine function is also valid. However, as we will see, it is necessary to introduce complex numbers, meaning that they are essentially the same function.

6. Imposing boundary conditions

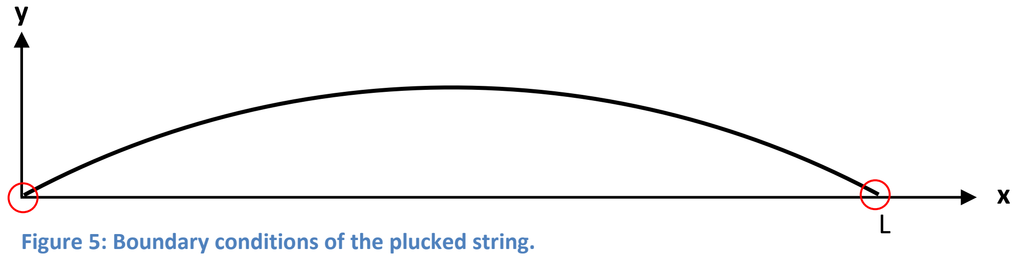


Figure 5: Boundary conditions of the plucked string.

The left end and the right end of the string are fixed at all times:

$y(0, t) = 0$ for all t > 0 [8]

$y(L, t) = 0$ for all t > 0 [9]

In the previous chapter we found a number of solutions $X_i(x)T_i(t)$. By the superpositioning principle [2] we know that $\sum_i K_i X_i(x)T_i(t)$ is also a solution for arbitrary K_i. This satisfies [8] if and only if

$\sum_i K_i X_i(0)T_i(t) = 0$ for all t > 0

If the K_i-s are nonzero and the T_i-s are independent, this implies that all $X_i(0) = 0$. Similarly, [9]

$\sum_i K_i X_i(L)T_i(t) = 0$ for all t > 0

implies that all $X_i(L) = 0$ for all *i*.

First we consider $\sigma = 0$, where $X_i(x) = a_1x + a_2$.

$X_i(0) = 0$ implies that $a_2 = 0$.

$X_i(L) = 0$ implies that $a_1L + a_2 = 0$, i.e. $a_1 = 0$. If so, $X_i(x) = a_1x + a_2 = 0$ for all *x*, so this solution is discarded.

Next, we consider $\sigma \neq 0$ where $X_i(x) = a_1e^{\sqrt{\sigma}x} + a_2e^{-\sqrt{\sigma}x}$.

$X_i(0) = 0$ implies that $a_1e^{\sqrt{\sigma}0} + a_2e^{-\sqrt{\sigma}0} = a_1 + a_2 = 0$, i.e. $a_2 = -a_1$.

$X_i(L) = 0$ implies that $a_1e^{\sqrt{\sigma}L} + a_2e^{-\sqrt{\sigma}L} = a_1(e^{\sqrt{\sigma}L} - e^{-\sqrt{\sigma}L}) = 0$. Ignoring solutions with $a_1 = 0$ (implying the trivial case $X(x) = 0$ for all *x*), this implies:

$$e^{\sqrt{\sigma}L} - e^{-\sqrt{\sigma}L} = 0$$

$$e^{\sqrt{\sigma}L} = e^{-\sqrt{\sigma}L}$$

$$e^{2\sqrt{\sigma}L} = 1.$$

The only real value of σ fulfilling $e^{2\sqrt{\sigma}L} = 1$ is $\sigma = 0$. However, this solution is discarded since we are now considering $\sigma \neq 0$.

It is therefore necessary to introduce complex numbers. Remembering $e^{2k\pi i} = 1$ for any k, this implies:

$$2\sqrt{\sigma}L = 2k\pi\mathrm{i}$$

$$\sqrt{\sigma} = \frac{k\pi}{L}i.$$

Inserting $a_2 = -a_1$and $\sqrt{\sigma} = \frac{k\pi}{L}i$ into [6] gives:

$$X_i(x)T_i(t) = \left(a_1 e^{i\frac{k\pi}{L}x} - a_1 e^{-i\frac{k\pi}{l}x}\right)\left(a_3 e^{ci\frac{k\pi}{L}t} + a_4 e^{-ci\frac{k\pi}{L}t}\right)$$

For complex numbers we can use the fact that $re^{\theta i} = r(\cos\theta + i\, sin\,\theta)$. Converting to polar form:

$$X_i(x)T_i(t) = \left[a_1\left(\cos\left(\frac{k\pi}{L}x\right) + i\sin\left(\frac{k\pi}{L}x\right)\right) - a_1\left(\cos\left(-\frac{k\pi}{L}x\right) + i\sin\left(-\frac{k\pi}{L}x\right)\right)\right]$$
$$\times\left[a_3\left(\cos\left(c\frac{k\pi}{L}t\right) + i\sin\left(c\frac{k\pi}{L}t\right)\right) + a_4\left(\cos\left(-c\frac{k\pi}{L}t\right) + i\sin\left(-c\frac{k\pi}{L}t\right)\right)\right]$$

Using the angle identities $\cos\theta = \cos(-\theta)$ and $\sin\theta = -\sin(-\theta)$, this gives:

$$X_i(x)T_i(t) = a_1\left[\cos\left(\frac{k\pi}{L}x\right) - \cos\left(\frac{k\pi}{L}x\right) + i\sin\left(\frac{k\pi}{L}x\right) + i\sin\left(\frac{k\pi}{L}x\right)\right]$$
$$\times\left[a_3\cos\left(c\frac{k\pi}{L}t\right) + a_4\cos\left(c\frac{k\pi}{L}t\right) + a_3 i\sin\left(c\frac{k\pi}{L}t\right) - a_4 i\sin\left(c\frac{k\pi}{L}t\right)\right]$$
$$= 2a_1 i\left(\sin\left(\frac{k\pi}{L}x\right)\right)\left[(a_3 + a_4)\cos\left(c\frac{k\pi}{L}t\right) + i(a_3 - a_4)\sin\left(c\frac{k\pi}{L}t\right)\right]$$
$$= \sin\left(\frac{k\pi}{L}x\right)\left[2a_1 i(a_3 + a_4)\cos\left(c\frac{k\pi}{L}t\right) - 2a_1(a_3 - a_4)\sin\left(c\frac{k\pi}{L}t\right)\right]$$
$$= \sin\left(\frac{k\pi}{L}x\right)\left(\alpha\cos\left(c\frac{k\pi}{L}t\right) + \beta\sin\left(c\frac{k\pi}{L}t\right)\right)$$

where $\alpha_k = 2a_1 i\,(a_3 + a_4)$ and $\beta_k = -2a_1\,(a_3 - a_4)$. Since a_1, a_3 and a_4 are allowed to be complex numbers, so are $\alpha_k$and β_k.

To summarize:

$$y(x,t) = \sum_i X_i(x)T_i(t) = \sum_{k=1}^{\infty}\sin\left(\frac{k\pi}{L}x\right)\left(\alpha_k\cos\left(\frac{ck\pi}{L}t\right) + \beta_k\sin\left(\frac{ck\pi}{L}t\right)\right) \qquad [10]$$

7. Imposing initial conditions

When plucking the string, it is removed by a distance *h* at position *d* from its equilibrium state. The shape of the string the moment it is plucked defines a function $f(x)$.

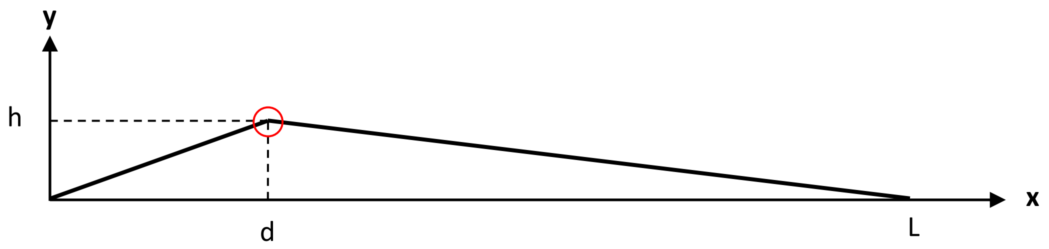


Figure 6: Initial conditions of the plucked string.

Since we are assuming that the string is motionless when released (no speed), the initial conditions are:

$$y(x,0) = f(x) \qquad \text{for all } 0 < x < L \qquad [11]$$

$$\frac{\partial}{\partial t} y(x,0) = 0 \qquad \text{for all } 0 < x < L \qquad [12]$$

Using [10] gives:

$$f(x) = y(x,0) = \sum_{k=1}^{\infty} \sin\left(\frac{k\pi}{L}x\right)\left(\alpha_k \cos\left(\frac{ck\pi}{L} \times 0\right) + \beta_k \sin\left(\frac{ck\pi}{L} \times 0\right)\right)$$

$$= \sum_{k=1}^{\infty} \alpha_k \sin\left(\frac{k\pi}{L}x\right) \qquad [13]$$

and

$$\frac{\partial}{\partial t} y(x,0) = \frac{\partial}{\partial t} \sum_{k=1}^{\infty} \sin\left(\frac{k\pi}{L}x\right)\left(\alpha_k \cos\left(\frac{ck\pi}{L}t\right) + \beta_k \sin\left(\frac{ck\pi}{L}t\right)\right)$$

$$= \sum_{k=1}^{\infty} \sin\left(\frac{k\pi}{L}x\right)\left(\alpha_k \frac{\partial}{\partial t}\cos\left(\frac{ck\pi}{L}t\right) + \beta_k \frac{\partial}{\partial t}\sin\left(\frac{ck\pi}{L}t\right)\right)$$

Using the chain rule for derivatives:

$$\frac{\partial}{\partial t}\cos\left(\frac{ck\pi}{L}t\right) = -\left(\frac{ck\pi}{L}\right)\sin\left(\frac{ck\pi}{L}t\right)$$

$$\frac{\partial}{\partial t}\sin\left(\frac{ck\pi}{L}t\right) = \left(\frac{ck\pi}{L}\right)\cos\left(\frac{ck\pi}{L}t\right)$$

At $t = 0$:

$$\frac{\partial}{\partial t}\cos\left(\frac{ck\pi}{L}t\right)\bigg|_{t\,=\,0} = -\left(\frac{ck\pi}{L}\right)\sin\left(\frac{ck\pi}{L}\times 0\right) = 0$$

$$\frac{\partial}{\partial t}\sin\left(\frac{ck\pi}{L}t\right)\bigg|_{t\,=\,0} = \left(\frac{ck\pi}{L}\right)\cos\left(\frac{ck\pi}{L}\times 0\right) = \left(\frac{ck\pi}{L}\right)$$

Giving:

$$\frac{\partial}{\partial t}y(x,0) = \sum_{k=1}^{\infty}\sin\left(\tfrac{k\pi}{L}x\right)\left(\alpha_k \times 0 + \beta_k\frac{ck\pi}{L}\,1\right) = \sum_{k=1}^{\infty}\beta_k\frac{ck\pi}{L}\,\sin\left(\tfrac{k\pi}{L}x\right)$$

From [12]:

$$\frac{\partial}{\partial t}y(x,0) = 0$$

$$\sum_{k=1}^{\infty}\beta_k\frac{ck\pi}{L}\,\sin\left(\tfrac{k\pi}{L}x\right) = 0$$

implying $\beta_k = 0$ for all *k*. This allows us to simplify [10] to:

$$y(x,t) = \sum_{k=1}^{\infty}\alpha_k\sin\left(\tfrac{k\pi}{L}x\right)\cos\left(\tfrac{ck\pi}{L}t\right) \qquad [14]$$

8. Introduction to Fourier

Fourier analysis is named after the French mathematician and physicist Joseph Fourier who initiated the investigation of Fourier series and their applications. Using Fourier series, complicated general functions can be written as the sum of simple sine and cosine functions. Fourier developed the initial ideas when studying heat propagation, but Fourier analysis is nowadays used in many scientific and engineering areas.

Decomposing a complex function into simpler pieces is called a Fourier transform[6], in which functions are decomposed into their constituent frequencies, known as a frequency spectrum. For periodic functions Fourier transform consists of computing the set of complex amplitudes called Fourier series coefficients. For instance, the transform of a musical chord made up of pure notes is a mathematical representation of the amplitudes (and phase) of the individual notes that make it up.

Fourier series

Any *smooth* function $f(x)$ has a unique representation

$$f(x) = \sum_{k=1}^{\infty} A_k \sin\left(\frac{k\pi}{L}x\right)$$

where the coefficients are computed by

$$A_k = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{k\pi x}{L}\right)dx \qquad [15]$$

[6] Wikipedia (2011), *Fourier Series*

9. Finding amplitudes by Fourier series

To find the coefficients α_k in [14] Fourier series will be used.

As noted earlier, the shape of the string at the moment it is plucked can be defined by a function $f(x)$.

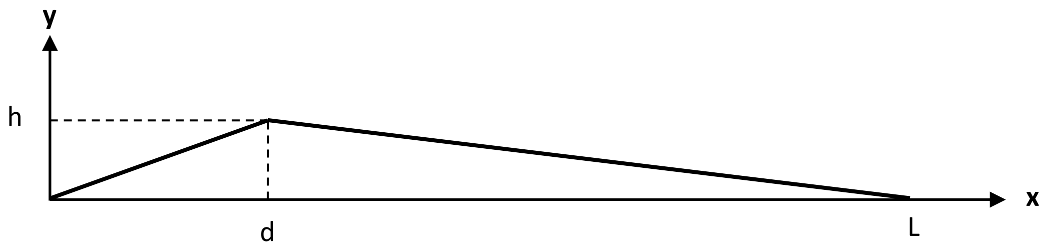


Figure 7: Initial conditions of the plucked string.

$$f(x) = \begin{cases} \frac{hx}{d}, & 0 \leq x \leq d \\ \frac{h(L-x)}{L-d}, & d < x \leq L \end{cases}$$

Equation [13] is exactly in the form of a Fourier series. The coefficients are computed by [15], computing the integral from 0 to *d* and from *d* to L separately. Solving the integral is done by using the method *integral by parts*[7].

Integration by parts (Wikipedia):

$$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$$

Coefficients in [13]:

$$A_k = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{k\pi x}{L}\right)dx = \frac{2}{L}\left[\underbrace{\int_0^d \frac{hx}{d}\sin\left(\frac{k\pi x}{L}\right)dx}_{\text{Part 1}} + \underbrace{\int_d^L \frac{h(L-x)}{L-d}\sin\left(\frac{k\pi x}{L}\right)dx}_{\text{Part 2}}\right]$$

[7] Wikipedia (2011), *Integration by parts*.

Part 1

Setting $u = \frac{hx}{d}$ and $dv = \sin\left(\frac{k\pi x}{L}\right) dx$

$$\int_0^d \frac{hx}{d}\sin\left(\frac{k\pi x}{L}\right) dx = \left[\frac{hx}{d}\left(\frac{-L}{k\pi}\right) cos\left(\frac{k\pi x}{L}\right)\right]_0^d - \int_0^d \left(\frac{-L}{k\pi}\right)\cos\left(\frac{k\pi x}{L}\right)\frac{h}{d}\, dx$$

$$= \left[\frac{-hLx}{k\pi d} cos\left(\frac{k\pi x}{L}\right)\right]_0^d - \int_0^d \frac{-hL}{k\pi d}\cos\left(\frac{k\pi x}{L}\right) dx$$

$$= \left[\frac{-hLx}{k\pi d} cos\left(\frac{k\pi x}{L}\right)\right]_0^d - \left(\frac{-hL}{k\pi d}\right)\left[\frac{L}{k\pi}\sin\left(\frac{k\pi x}{L}\right)\right]_0^d$$

$$= \frac{-hLd}{k\pi d} cos\left(\frac{k\pi d}{L}\right) - \left(\frac{-hL\times 0}{k\pi d}\right) cos\left(\frac{k\pi \times 0}{L}\right)$$

$$+\frac{hL}{k\pi d}\,\frac{L}{k\pi}\left(\sin\left(\frac{k\pi d}{L}\right) - \sin\left(\frac{k\pi\times 0}{L}\right)\right)$$

$$= \frac{-hL}{k\pi} cos\left(\frac{k\pi d}{L}\right) + \frac{hL^2}{k^2\pi^2 d}\sin\left(\frac{k\pi d}{L}\right)$$

Part 2

Setting $u = \frac{h(L-x)}{L-d}$ and $dv = \sin\left(\frac{k\pi x}{L}\right) dx$

$$\int_d^L \frac{h(L-x)}{L-d}\sin\left(\frac{k\pi x}{L}\right) dx = \left[\frac{h(L-x)}{L-d}\left(\frac{-L}{k\pi}\right) cos\left(\frac{k\pi x}{L}\right)\right]_d^L - \int_d^L \left(\frac{-L}{k\pi}\right)\cos\left(\frac{k\pi x}{L}\right)\frac{-h}{L-d}\, dx$$

$$= \left[\frac{-h(L-x)L}{(L-d)k\pi} cos\left(\frac{k\pi x}{L}\right)\right]_d^L - \int_d^L \frac{hL}{(L-d)k\pi}\cos\left(\frac{k\pi x}{L}\right) dx$$

$$= \left[\frac{-h(L-x)L}{(L-d)k\pi} cos\left(\frac{k\pi x}{L}\right)\right]_d^L - \frac{hL}{(L-d)k\pi}\left[\frac{L}{k\pi}\sin\left(\frac{k\pi x}{L}\right)\right]_d^L$$

$$= \frac{-h(L-L)L}{(L-d)k\pi} cos\left(\frac{k\pi L}{L}\right) - \frac{h(L-d)L}{(L-d)k\pi} cos\left(\frac{k\pi d}{L}\right)$$

$$-\frac{hL}{(L-d)k\pi}\,\frac{L}{k\pi}\left(\sin\left(\frac{k\pi L}{L}\right) - \sin\left(\frac{k\pi d}{L}\right)\right)$$

$$= \frac{hL}{k\pi} cos\left(\frac{k\pi d}{L}\right) + \frac{hL^2}{(L-d)k^2\pi^2}\sin\left(\frac{k\pi d}{L}\right)$$

Thus:

$$A_k = \frac{2}{L}\left[\frac{-hL}{k\pi} cos\left(\frac{k\pi d}{L}\right) + \frac{hL^2}{k^2\pi^2 d} \sin\left(\frac{k\pi d}{L}\right) + \frac{hL}{k\pi} cos\left(\frac{k\pi d}{L}\right) + \frac{hL^2}{(L-d)k^2\pi^2} \sin\left(\frac{k\pi d}{L}\right)\right]$$

$$= \frac{2}{L}\left[\frac{hL^2}{k^2\pi^2 d} \sin\left(\frac{k\pi d}{L}\right) + \frac{hL^2}{(L-d)k^2\pi^2} \sin\left(\frac{k\pi d}{L}\right)\right]$$

$$= \frac{2}{L}\left(\frac{hL^2}{k^2\pi^2 d}\frac{\times (L-d)}{\times (L-d)} + \frac{hL^2}{(L-d)k^2\pi^2}\frac{\times d}{\times d}\right) \sin\left(\frac{k\pi d}{L}\right)$$

$$= \frac{2}{L}\left(\frac{hL^3 - hL^2 d + hL^2 d}{(L-d)k^2\pi^2 d}\right) \sin\left(\frac{k\pi d}{L}\right)$$

$$= \frac{2hL^2}{(L-d)k^2\pi^2 d} \sin\left(\frac{k\pi d}{L}\right)$$

The solution:

$$A_k = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{k\pi x}{L}\right) dx = \frac{2hL^2}{\pi^2 k^2 d(L-d)} \sin\left(\frac{k\pi d}{L}\right) \qquad [16]$$

The displacement function then becomes:

$$y(x,t) = \sum_{k=1}^{\infty} \frac{2hL^2}{\pi^2 k^2 d(L-d)} \sin\left(\frac{k\pi d}{L}\right) \sin\left(\frac{k\pi}{L}x\right) \cos\left(\frac{ck\pi}{L}t\right) \qquad [17]$$

10. Interpretation

Harmonics

Equation [17] states that the displacement function is a sum of terms called *modes* or *harmonics*. Each mode represents a harmonic motion with different wavelength. For a fixed time t_1:

$$y(x,t_1)=\sum_{k=1}^{\infty}\frac{2hL^2}{\pi^2k^2d(L-d)}\sin\left(\frac{k\pi d}{L}\right)\cos\left(\frac{ck\pi}{L}t_1\right)\quad\times\sin\left(\frac{k\pi}{L}x\right)$$

implying that the each mode is a constant times $\sin\left(\frac{k\pi}{L}x\right)$. As *x* runs from 0 to L, the argument of $\sin\left(\frac{k\pi}{L}x\right)$ runs from 0 to $k\pi$, which is *k* half-periods of sin.

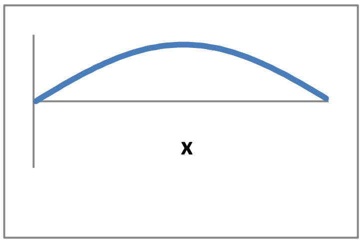


Figure 8: Amplitude for k=1.

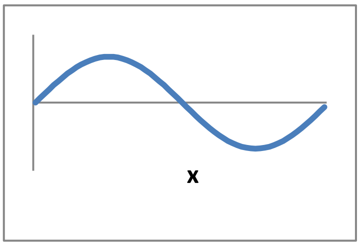


Figure 9: Amplitude for k=2.

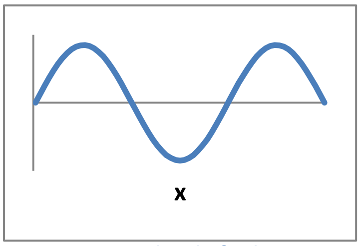


Figure 10: Amplitude for k=3.

Similarly, for any fixed position x_1:

$$y(x_1,t)=\sum_{k=1}^{\infty}\frac{2hL^2}{\pi^2k^2d(L-d)}\sin\left(\frac{k\pi d}{L}\right)\sin\left(\frac{k\pi}{L}x_1\right)\quad\times\cos\left(\frac{ck\pi}{L}t\right)$$

implying that each mode is a constant times $\cos\left(\frac{ck\pi}{L}t\right)$. As *t* increases from 0 to 1 s, the argument of $\cos\left(\frac{ck\pi}{L}t\right)$ increases by $\frac{ck\pi}{L}$, which is $\frac{ck}{2L}$ cycles. For mode $k=1$ (the fundamental tone), the frequency is $\frac{c}{2L}$ cycles per second. For mode $k=2$ (the second harmonic), the frequency is $2\frac{c}{2L}$ cycles per second. For mode $k=3$ (the third harmonic), the frequency is $3\frac{c}{2L}$ cycles per second etc.

Since $c=\sqrt{\frac{T}{\mu}}$, the frequency of oscillation of a string decrease with the density and increase with the tension or by shortening the string.

Coefficients

The coefficients in [16] found by Fourier transformation gives the amplitude of each harmonic, i.e. computed by inserting k=1, k=2, k=3, etc. By superposition [2], the total displacement is formed by the sum of the harmonics.

To illustrate the principle, we consider the following example: $L = 0.64$, $h = 0.005$ and $d = 0.16$, giving the following amplitudes according to [16]:

k	A_k
1	3.821E-03
2	1.351E-03
3	4.246E-04
4	4.138E-20
5	-1.528E-04

We compute the graphs using Excel for *t* = 0 as an example. The first harmonic:

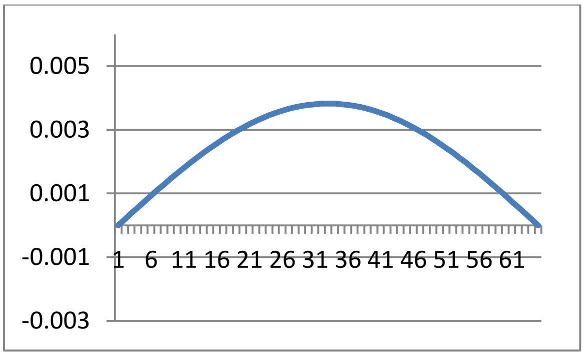


Figure 11: First harmonic at t=0.

The second harmonic and the sum of the first two harmonics:

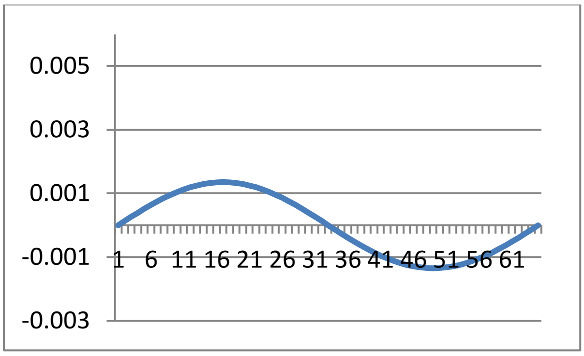


Figure 12: Second harmonic at t=0.

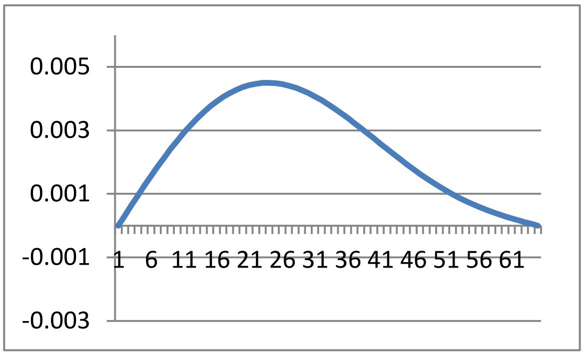


Figure 13: Sum of the two first harmonics at t=0.

The third harmonic and the sum of the first three harmonics:

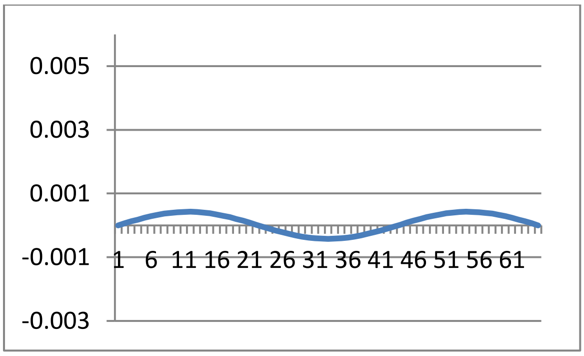


Figure 14: Third harmonic at t=0.

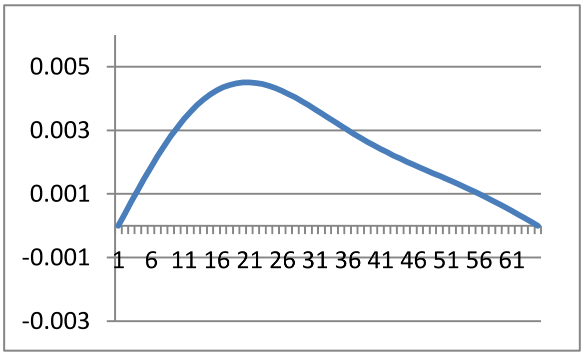


Figure 15: Sum of the first three harmonics at t=0.

As more harmonics are added, the total displacement function as the accumulated sum of harmonics becomes more and more similar to the initial displacement function *f(x)* defined in the beginning of Chapter 9.

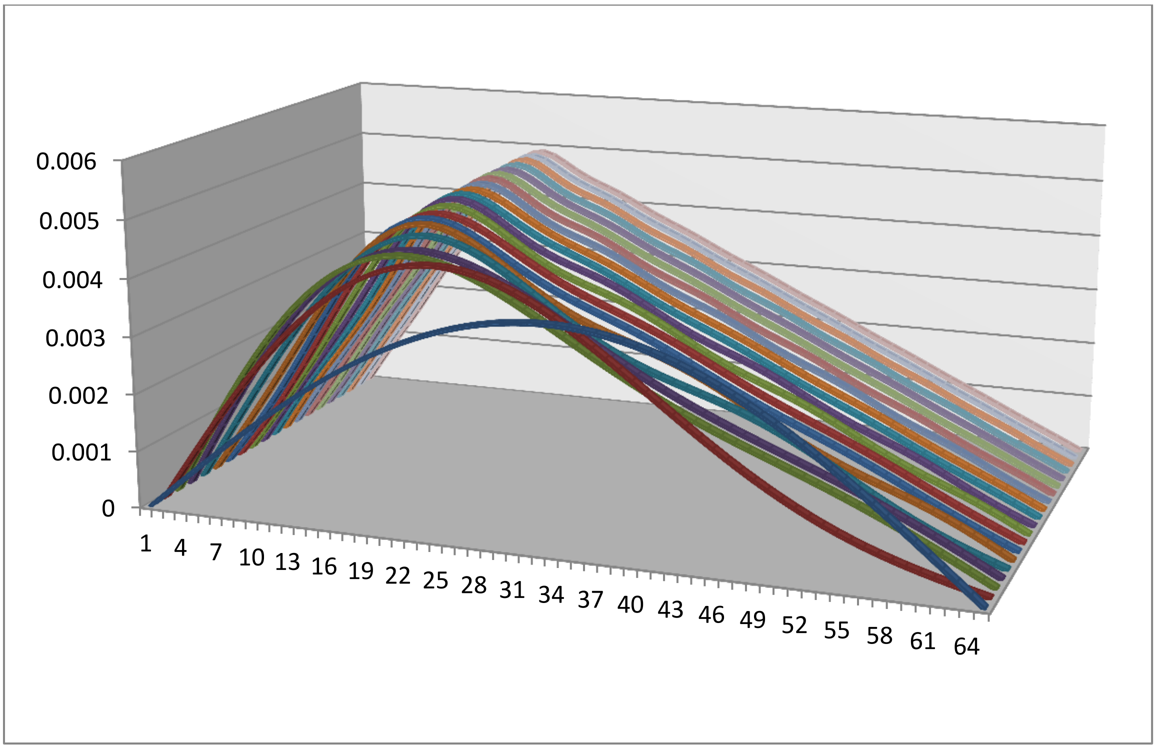


Figure 16: Accumulated sum of the first 20 harmonics for d = 0.16.

Amplitude distribution depending on position of plucking

To find the influence of the distribution of amplitudes for the harmonics based on the position of plucking, we compute amplitudes according to [16] for some example values of *d*. L and *h* are fixed ($L = 0.64$, $h = 0.005$).

d = 0.32

Plucking at the middle of the string (*d* = 0.32) should produce the following amplitude distribution:

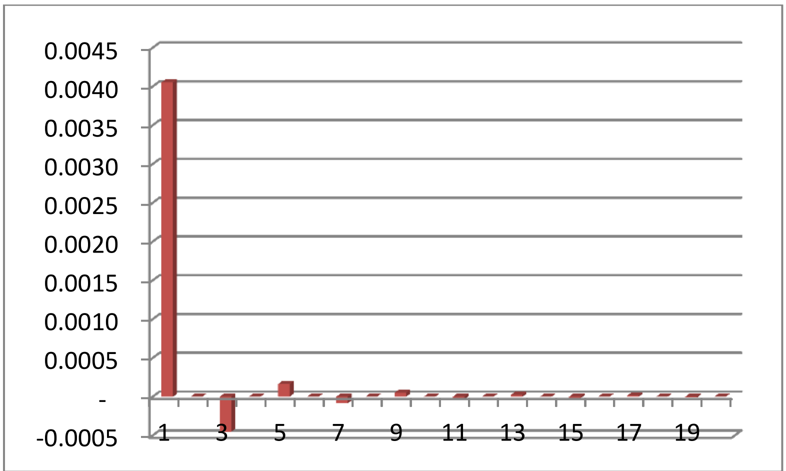


Figure 17: Amplitudes for d = 0.32.

Here the fundamental harmonic is very dominant, producing a “pure” sound with low amplitudes on overtones. Even-numbered harmonics ($k = 2$, $k = 4$, etc.) are completely missing.

$d = 0.16$

Plucking at the regular playing area of the guitar (ca $d = 0.16$) produces a slightly different amplitude distribution.

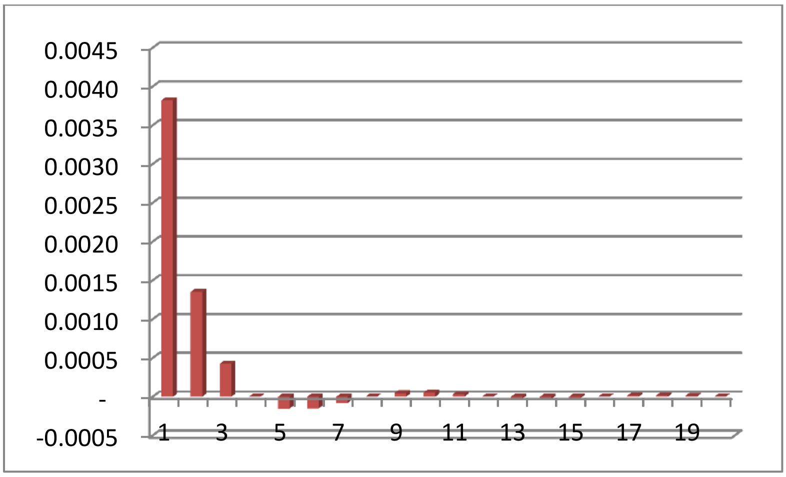


Figure 18: Amplitudes for d = 0.16.

The fundamental tone is still dominant, but the five first overtones would influence the tone *quality*.

$d = 0.05$

Plucking close to the nut or the bridge of the guitar should produce the following amplitude distribution.

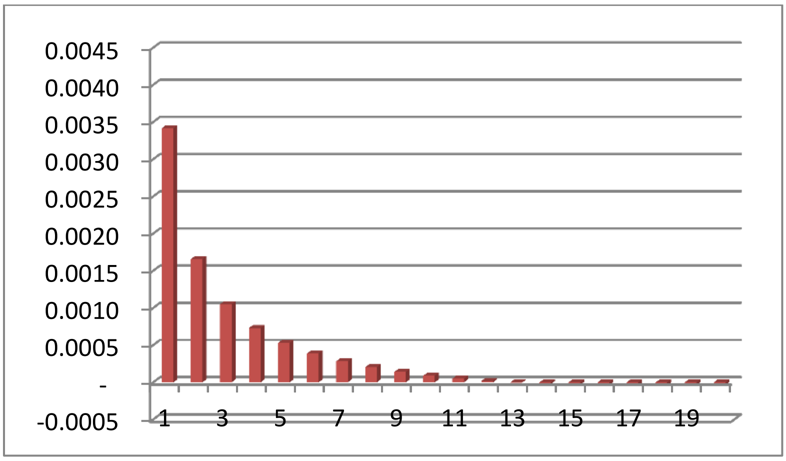


Figure 19: Amplitudes for d = 0.05.

In this case the overtones will be clearly more visible, even up to the 10^{th} overtone. This would create a ‘bright’ tone quality.

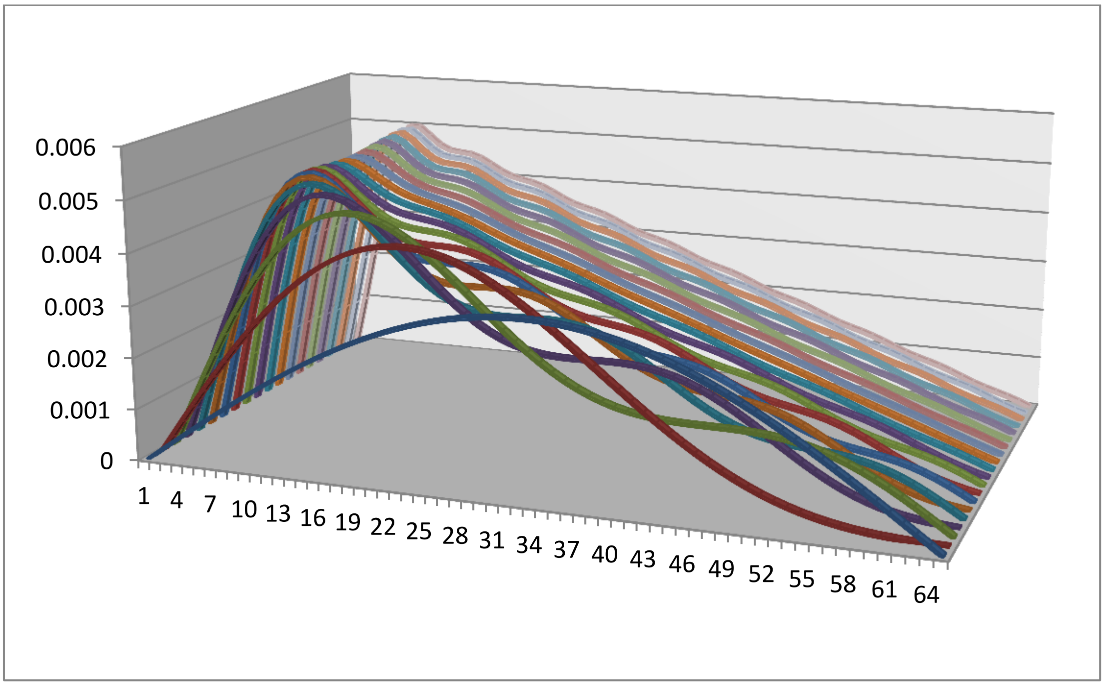


Figure 20: Accumulated sum of the first 20 harmonics for d = 0.05.

Limitations

When deriving the mathematical model for a vibrating guitar string a number of simplifying assumptions have been done. Some of the main ones are:

- No resonance from the guitar body is taken into account
- String stiffness and its effect on the vibration has been excluded
- Damping, due e.g. to friction and air resistance, has been ignored
- The string is assumed to be in an acoustic *dead* environment
- The initial shape of the string when plucking is an approximation
- Other forces, e.g. gravitational force, have been ignored

11. Validation

Egeland's essay[8] investigated the *influence of the positioning and method of plucking a guitar string for the distribution of energy between fundamental tone and overtones*. Having derived a quantitative model for a vibrating string, it was exciting to compare her empirical measurements with computed values using the idealized model.

In her experiment Egeland plucked the lower E string of an acoustic guitar using different materials (cardboard, finger, plectrum) at different positions. Using a PC tool called SpectraScope[9] she was able to get amplitude dB readouts for different frequencies. Based on these she calculated the relationship in dB between the energy in the five first overtones relative to the energy in the fundamental tone.

To my disappointment there were major discrepancies. Her measurements showed that plucking with e.g. a plastic plectrum at position $d = 0.158\ m$ produced an energy relationship $r = 14.9\ dB$, meaning significant more energy in the overtones than in the fundamental tone. Calculating this relationship using our model with $h = 0.005$ and $L = 0.63$ produces $r = -8.5\ dB$, meaning more energy in the fundamental tone than in the overtones.

Experiment setup

To find the sources and possible explanations for the discrepancies, I recreated the experiment and also tested some variations. I tried to reproduce the experiment setup performed by Egeland as closely as possible. Below a brief outline of the experiment setup is presented. For more details, consult Egeland's essay[8].

A microphone was connected to a PC running the following software tools:

- Audacity[10] for recording from the microphone
- SpectraScope for analyzing the sound files and producing frequency amplitudes
- MS Excel for examining and analyzing the output frequency tables

Using Audacity and recoding to a file enabled full control over the start and end times of the recording, avoiding transients at the beginning and distortion due to fading. Similar to Egeland, samples of 1.04 seconds were used.

SpectraScope uses an implementation of a Fast Fourier Transform (FFT) to produce a frequency plot of the amplitudes given in dB. This allowed us to derive the relative energies in the fundamental tone and overtones. SpectraScope supported export of frequency tables.

Excel was used to import and then analyze the frequency tables output form SpectraScope. To reduce sources of error, dBv values from ca 10 Hz below to ca 10 Hz above the actual frequency were summarized. Each group always contained eight values.

---

[8] Egeland, K. (2009).
[9] SpectraScope (FFT tool).
[10] Freeware sound software, available at: <http://audacity.sourceforge.net/>

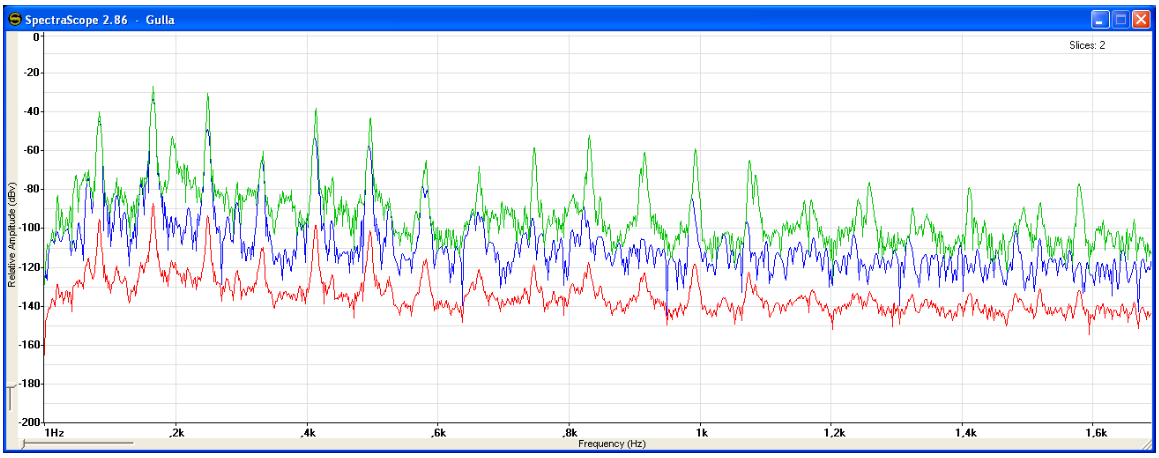
In order to get reproducible and consistent results, the position of plucking was accurately measured and tagged on the guitar. The sideways offset of 0.5 cm was also indicated.

Figure 21: Plucking the guitar string.

Measurements

Acoustic guitar, 82Hz, $d = 0.16$, plucking with plastic plectrum

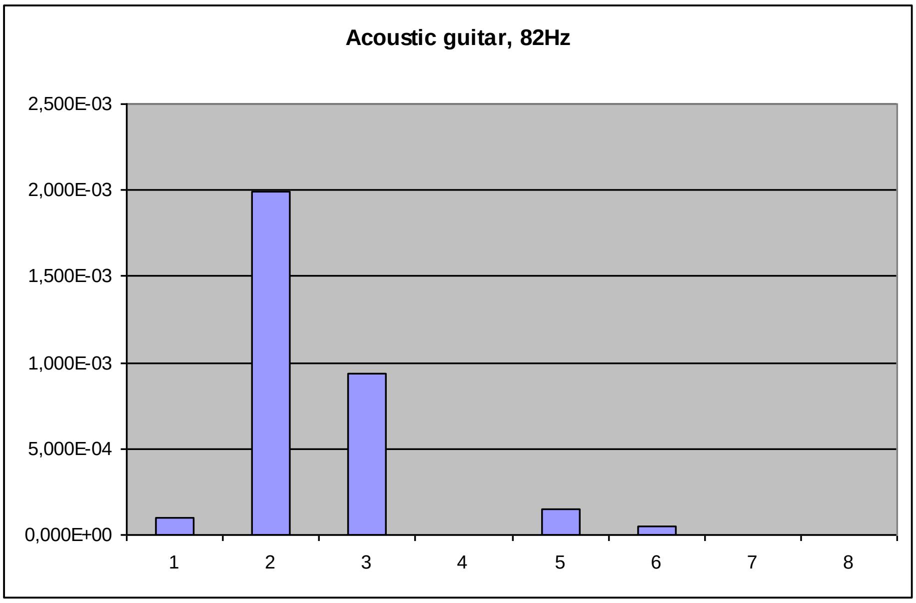
Initially I tried to closely recreate Egeland's original experiment. Using a high-quality microphone connected to a laptop, a recording of 1.04 s was done. Below the amplitude-to-frequency diagram of SpectraScope is shown.



The dBv values for different frequencies were imported into Excel for further processing.

Tone	Ideal frequency (Hz)	Actual frequency (Hz) for max amplitude	Amplitude (dBv) from SpectraScope	Amplitude
Fundamental	82	82	-39.9	1.023E-04
Overtone 1	164	164	-27.0	1.995E-03
Overtone 2	246	247	-30.3	9.333E-04

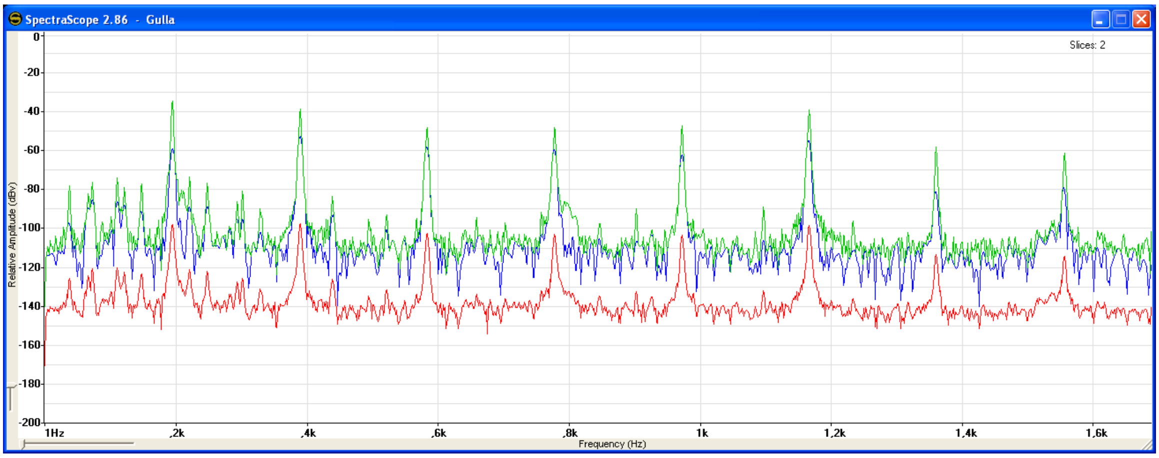
Overtone 3	328	332	-60.3	9.333E-07
Overtone 4	410	413	-38.3	1.479E-04
Overtone 5	492	496	-43.2	4.786E-05
Overtone 6	574	581	-64.7	3.388E-07
Overtone 7	656	662	-68.2	1.514E-07



As observed from the diagram above, the amplitude for the fundamental tone is quite low for 82Hz. In an acoustic guitar the sound is mainly produced by the guitar body and not the string itself[11]. A hypothesis could be that the body was not able to reproduce such low tones.

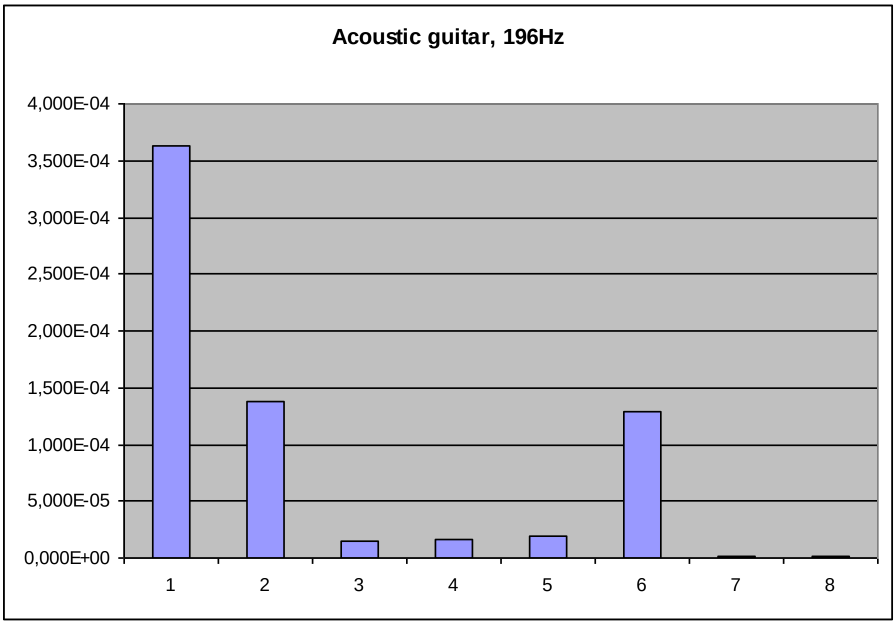
Acoustic guitar, $196Hz, d = 0.16$, plucking with plastic plectrum

Instead of plucking the low E string, the experiment was redone with the G string at approximately 196 Hz.



[11] Pelc, J. (2007).

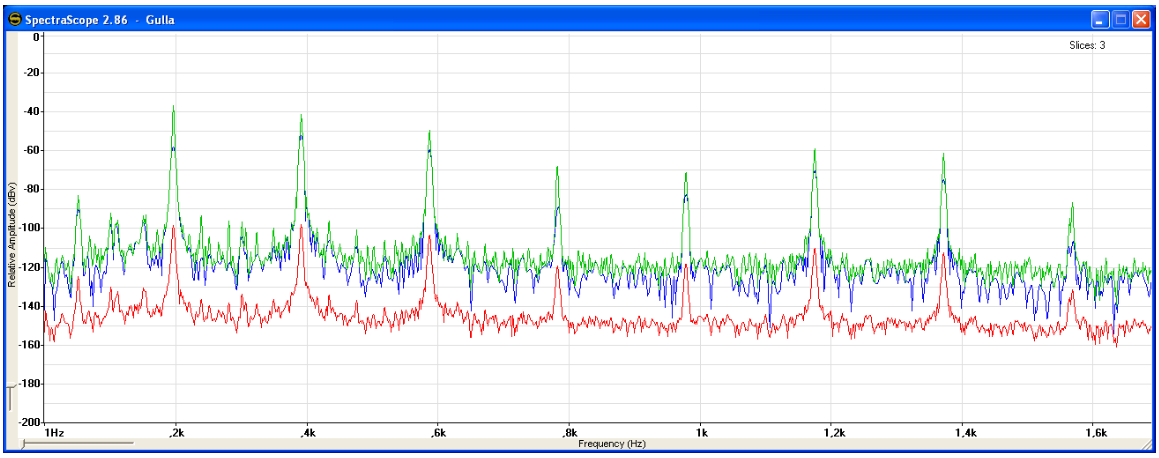
Tone	Ideal frequency (Hz)	Actual frequency (Hz) for max amplitude	Amplitude (dBv) from SpectraScope	Amplitude
Fundamental	196	193	-34.4	3.631E-04
Overtone 1	392	388	-38.6	1.380E-04
Overtone 2	588	582	-48.2	1.514E-05
Overtone 3	784	776	-48.0	1.585E-05
Overtone 4	980	971	-47.1	1.950E-05
Overtone 5	1176	1165	-38.9	1.288E-04
Overtone 6	1372	1359	-58.0	1.585E-06
Overtone 7	1568	1555	-61.1	7.762E-07



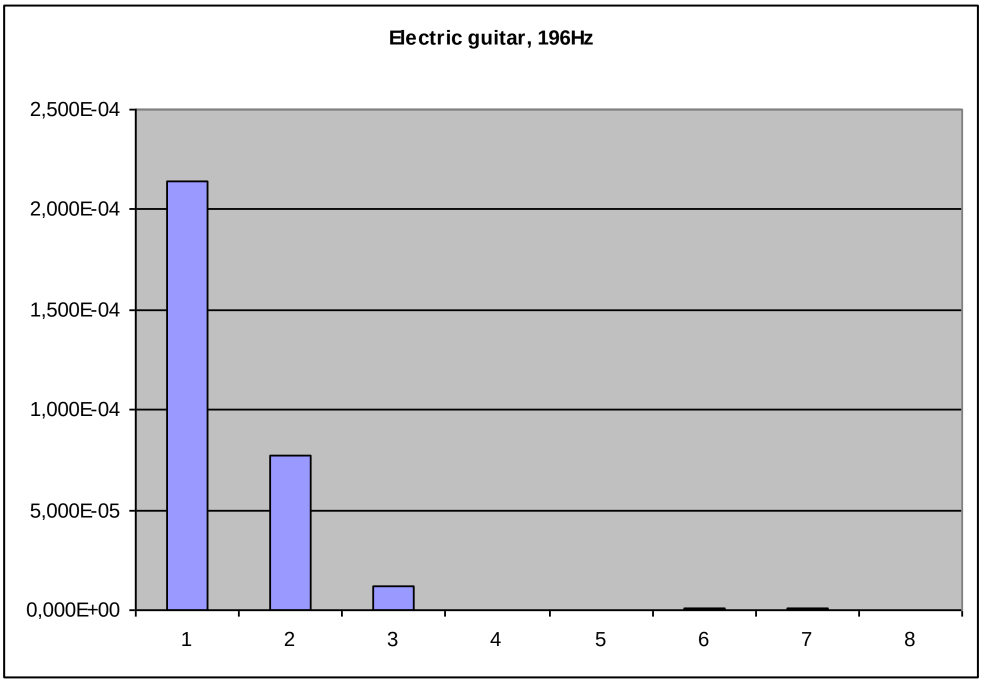
This graph matches the predicted distribution by the mathematical model much closer. Similar experiments with other mid-range tones produced similar results, indicating that the unexpected low amplitudes of the fundamental tone in Egeland’s original experiment are probably due to the acoustic guitar’s body lack of responsiveness to such low frequencies.

Electric guitar, $196Hz$, $d = 0.16$, plucking with plastic plectrum

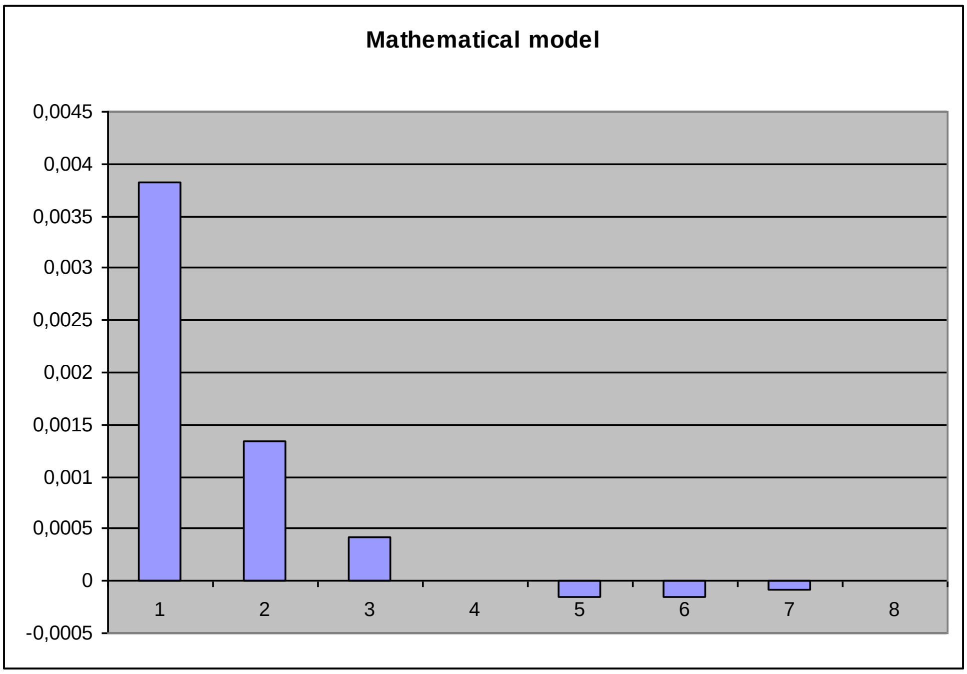
To eliminate the resonance effect of the acoustic guitar body completely, an electric guitar was used. A digital external sound was used to amplify the signal produced by the electric guitar.



Tone	Ideal frequency (Hz)	Actual frequency (Hz) for max amplitude	Amplitude (dBv) from SpectraScope	Amplitude
Fundamental	196	195	-36.7	2.138E-04
Overtone 1	392	390	-41.1	7.762E-05
Overtone 2	588	586	-49.3	1.175E-05
Overtone 3	784	781	-68.2	1.514E-07
Overtone 4	980	978	-71.0	7.943E-08
Overtone 5	1176	1174	-59.1	1.230E-06
Overtone 6	1372	1372	-61.2	7.586E-07
Overtone 7	1568	1567	-86.8	2.089E-09



For comparison the corresponding amplitudes computed by formula [16] are shown below.



As can be observed the relative distribution of amplitudes for the different harmonics for an actual electrical guitar quite closely resemble the distribution predicted by the mathematical formula.

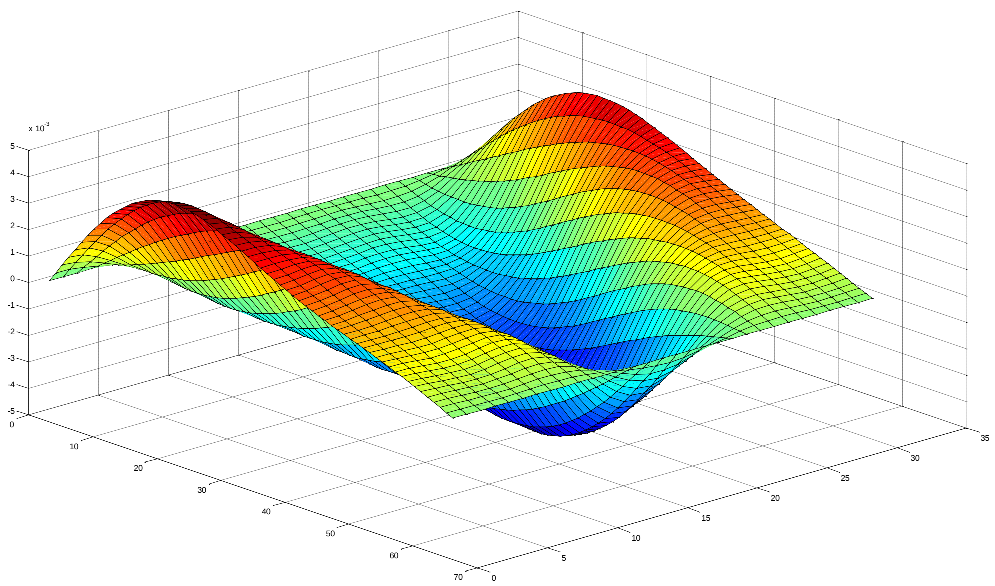


Figure 22: Displacement function of time and position for the three first harmonics.

12. Conclusion

In this study a mathematical model for an idealized vibrating string has been developed. Based on a simplified model of forces affecting a string segment, the one-dimensional wave equation was derived [1]. A general solution of the wave equation was found using the method of separation of variables. By imposing boundary and initial conditions for plucking the vibrating string at some position *d*, a simplified formula [14] was computed. By using Fourier series, expressions for the coefficient in the formula were derived.

The formula [17] defines the displacement from equilibrium for any segment of the string (denoted by position *x*) at any point of time (denoted by *t*). Due to the structure of the formula, the complex function can more easily be understood as a superposition of *modes* or *harmonics*, as shown in Chapter 10. Each harmonic defines a tone corresponding to the possible standing waves given by the string length, with frequency depending on string tension and density. The amplitudes of each harmonic [16] correspond to the coefficients derived using Fourier series.

Based on this mathematical model we are able to compute theoretical amplitudes for the fundamental tone and the overtones dependent on the extent (*h*) and position (*d*) of the plucking of the string. Since the model is based on a number of simplifying assumptions, some relevant limitations are listed at the end of Chapter 10.

When plucking close to the middle of the string, the amplitude of the fundamental tone is dominant, producing a clean, sine-like tone. When plucking closer to the edge of the string, more of the potential energy released goes into higher-pitch harmonic tones, producing a sharper sound. This also is also consistent with observations using a real guitar.

When comparing predicted values using this model with empirical measurements from a previous IB extended essay, I was initially puzzled by the large discrepancies. In an experiment using an acoustic guitar, Egeland measured the relative energies in the fundamental tone versus the overtones, depending on the position and method of plucking. However, after eliminating the major sources of deviation (Using an extreme low frequency at an acoustic guitar), results consistent with the mathematical model were observed when running similar experiments.

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SpectraScope is a FFT tool, developed by Christopher Brown. A free 10 day trial version is available from <www.spectrascope.com>