# 12.8 Modeling: Membrane, Two-Dimensional Wave Equation

Since the modeling here will be similar to that of Sec. 12.2, you may want to take another look at Sec. 12.2.

The vibrating string in Sec. 12.2 is a basic one-dimensional vibrational problem. Equally important is its two-dimensional analog, namely, the motion of an elastic membrane, such as a drumhead, that is stretched and then fixed along its edge. Indeed, setting up the model will proceed almost as in Sec. 12.2.

## Physical Assumptions

1. The mass of the membrane per unit area is constant ("homogeneous membrane"). The membrane is perfectly flexible and offers no resistance to bending.
2. The membrane is stretched and then fixed along its entire boundary in the $xy$-plane. The tension per unit length $T$ caused by stretching the membrane is the same at all points and in all directions and does not change during the motion.
3. The deflection $u(x, y, t)$ of the membrane during the motion is small compared to the size of the membrane, and all angles of inclination are small.

Although these assumptions cannot be realized exactly, they hold relatively accurately for small transverse vibrations of a thin elastic membrane, so that we shall obtain a good model, for instance, of a drumhead.

**Derivation of the PDE of the Model ("Two-Dimensional Wave Equation") from Forces.** As in Sec. 12.2 the model will consist of a PDE and additional conditions. The PDE will be obtained by the same method as in Sec. 12.2, namely, by considering the forces acting on a small portion of the physical system, the membrane in Fig. 301 on the next page, as it is moving up and down.

Since the deflections of the membrane and the angles of inclination are small, the sides of the portion are approximately equal to $\Delta x$ and $\Delta y$. The tension $T$ is the force per unit length. Hence the forces acting on the sides of the portion are approximately $T\,\Delta x$ and $T\,\Delta y$. Since the membrane is perfectly flexible, these forces are tangent to the moving membrane at every instant.

**Horizontal Components of the Forces.** We first consider the horizontal components of the forces. These components are obtained by multiplying the forces by the cosines of the angles of inclination. Since these angles are small, their cosines are close to 1. Hence the horizontal components of the forces at opposite sides are approximately equal. Therefore, the motion of the particles of the membrane in a horizontal direction will be negligibly small. From this we conclude that we may regard the motion of the membrane as transversal; that is, each particle moves vertically.

**Vertical Components of the Forces.** These components along the right side and the left side are (Fig. 301), respectively,

$$T\,\Delta y \sin\beta \qquad \text{and} \qquad -T\,\Delta y \sin\alpha.$$

Here $\alpha$ and $\beta$ are the values of the angle of inclination (which varies slightly along the edges) in the middle of the edges, and the minus sign appears because the force on the

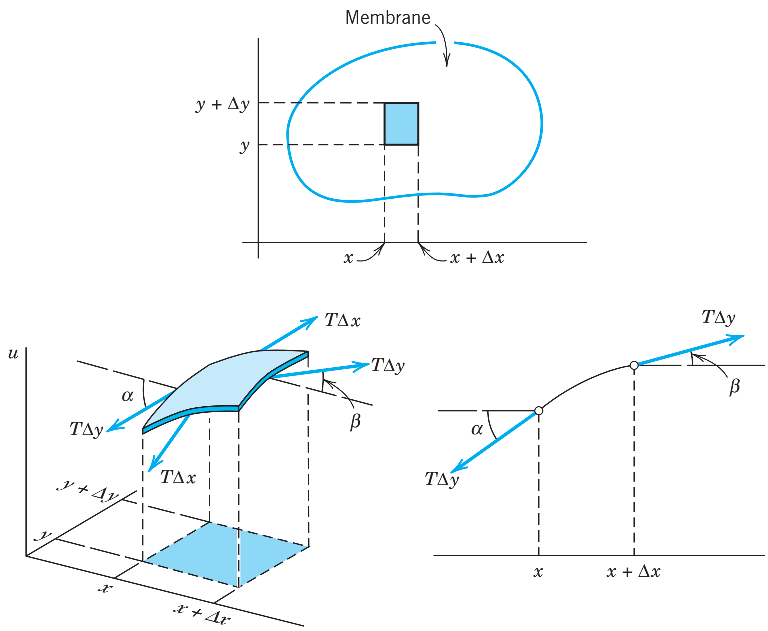


**Fig. 301.** Vibrating membrane

left side is directed downward. Since the angles are small, we may replace their sines by their tangents. Hence the resultant of those two vertical components is

$$\text{(1)} \qquad \begin{aligned} T\Delta y(\sin\beta - \sin\alpha) &\approx T\Delta y(\tan\beta - \tan\alpha) \\ &= T\Delta y[u_x(x + \Delta x, y_1) - u_x(x, y_2)] \end{aligned}$$

where subscripts $x$ denote partial derivatives and $y_1$ and $y_2$ are values between $y$ and $y + \Delta y$. Similarly, the resultant of the vertical components of the forces acting on the other two sides of the portion is

$$\text{(2)} \qquad T\Delta x[u_y(x_1, y + \Delta y) - u_y(x_2, y)]$$

where $x_1$ and $x_2$ are values between $x$ and $x + \Delta x$.

**Newton's Second Law Gives the PDE of the Model.** By Newton's second law (see Sec. 2.4) the sum of the forces given by (1) and (2) is equal to the mass $\rho\,\Delta A$ of that small portion times the acceleration $\partial^2 u/\partial t^2$; here $\rho$ is the mass of the undeflected membrane per unit area, and $\Delta A = \Delta x\,\Delta y$ is the area of that portion when it is undeflected. Thus

$$\begin{aligned} \rho\Delta x\,\Delta y\frac{\partial^2 u}{\partial t^2} &= T\Delta y[u_x(x + \Delta x, y_1) - u_x(x, y_2)] \\ &\quad + T\Delta x[u_y(x_1, y + \Delta y) - u_y(x_2, y)] \end{aligned}$$

where the derivative on the left is evaluated at some suitable point $(\tilde{x}, \tilde{y})$ corresponding to that portion. Division by $\rho\Delta x\,\Delta y$ gives

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\left[\frac{u_x(x + \Delta x, y_1) - u_x(x, y_2)}{\Delta x} + \frac{u_y(x_1, y + \Delta y) - u_y(x_2, y)}{\Delta y}\right].$$

If we let $\Delta x$ and $\Delta y$ approach zero, we obtain the PDE of the model

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad c^2 = \frac{T}{\rho}. \tag{3}$$

This PDE is called the **two-dimensional wave equation**. The expression in parentheses is the Laplacian $\Delta^2 u$ of $u$ (Sec. 10.8). Hence (3) can be written

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta^2 u. \tag{3$'$}$$

Solutions of the wave equation (3) will be obtained and discussed in the next section.

# 12.9 Rectangular Membrane. Double Fourier Series

Now we develop a solution for the PDE obtained in Sec. 12.8. Details are as follows.

The model of the vibrating membrane for obtaining the displacement $u(x, y, t)$ of a point $(x, y)$ of the membrane from rest ($u = 0$) at time $t$ is

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{1}$$

$$u = 0 \text{ on the boundary} \tag{2}$$

$$u(x, y, 0) = f(x, y) \tag{3a}$$

$$u_t(x, y, 0) = g(x, y). \tag{3b}$$

Here (1) is the **two-dimensional wave equation** with $c^2 = T/\rho$ just derived, (2) is the **boundary condition** (membrane fixed along the boundary in the $xy$-plane for all times $t \geqq 0$), and (3) are the **initial conditions** at $t = 0$, consisting of the given *initial displacement* (initial shape) $f(x, y)$ and the given *initial velocity* $g(x, y)$, where $u_t = \partial u/\partial t$. We see that these conditions are quite similar to those for the string in Sec. 12.2.

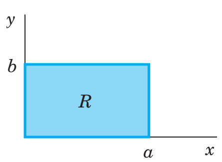


**Fig. 302.** Rectangular membrane

Let us consider the **rectangular membrane** $\boldsymbol{R}$ in Fig. 302. This is our first important model. It is much simpler than the circular drumhead, which will follow later. First we note that the boundary in equation (2) is the rectangle in Fig. 302. We shall solve this problem in three steps:

***Step 1.*** By separating variables, first setting $u(x, y, t) = F(x, y)G(t)$ and later $F(x, y) = H(x)Q(y)$ we obtain from (1) an ODE (4) for $G$ and later from a PDE (5) for $F$ two ODEs (6) and (7) for $H$ and $Q$.

***Step 2.*** From the solutions of those ODEs we determine solutions (13) of (1) ("**eigenfunctions**" $u_{mn}$) that satisfy the boundary condition (2).

***Step 3.*** We compose the $u_{mn}$ into a double series (14) solving the whole model (1), (2), (3).

## Step 1. Three ODEs From the Wave Equation (1)

To obtain ODEs from (1), we apply two successive separations of variables. In the first separation we set $u(x, y, t) = F(x, y)G(t)$. Substitution into (1) gives

$$F\ddot{G} = c^2(F_{xx}G + F_{yy}G)$$

where subscripts denote partial derivatives and dots denote derivatives with respect to $t$. To separate the variables, we divide both sides by $c^2FG$:

$$\frac{\ddot{G}}{c^2G} = \frac{1}{F}(F_{xx} + F_{yy}).$$

Since the left side depends only on $t$, whereas the right side is independent of $t$, both sides must equal a constant. By a simple investigation we see that only negative values of that constant will lead to solutions that satisfy (2) without being identically zero; this is similar to Sec. 12.3. Denoting that negative constant by $-\nu^2$, we have

$$\frac{\ddot{G}}{c^2G} = \frac{1}{F}(F_{xx} + F_{yy}) = -\nu^2.$$

This gives two equations: for the "**time function**" $G(t)$ we have the ODE

$$\textbf{(4)} \qquad \ddot{G} + \lambda^2 G = 0 \qquad \text{where } \lambda = c\nu,$$

and for the "**amplitude function**" $F(x, y)$ a PDE, called the *two-dimensional* **Helmholtz**[3] **equation**

$$\textbf{(5)} \qquad F_{xx} + F_{yy} + \nu^2 F = 0.$$

[3]HERMANN VON HELMHOLTZ (1821–1894), German physicist, known for his fundamental work in thermodynamics, fluid flow, and acoustics.

Separation of the Helmholtz equation is achieved if we set $F(x, y) = H(x)Q(y)$. By substitution of this into (5) we obtain

$$\frac{d^2H}{dx^2}Q = -\left(H\frac{d^2Q}{dy^2} + \nu^2 HQ\right).$$

To separate the variables, we divide both sides by $HQ$, finding

$$\frac{1}{H}\frac{d^2H}{dx^2} = -\frac{1}{Q}\left(\frac{d^2Q}{dy^2} + \nu^2 Q\right).$$

Both sides must equal a constant, by the usual argument. This constant must be negative, say, $-k^2$, because only negative values will lead to solutions that satisfy (2) without being identically zero. Thus

$$\frac{1}{H}\frac{d^2H}{dx^2} = -\frac{1}{Q}\left(\frac{d^2Q}{dy^2} + \nu^2 Q\right) = -k^2.$$

This yields two ODEs for $H$ and $Q$, namely,

$$\boxed{\frac{d^2H}{dx^2} + k^2 H = 0} \tag{6}$$

and

$$\boxed{\frac{d^2Q}{dy^2} + p^2 Q = 0} \qquad \text{where } p^2 = \nu^2 - k^2. \tag{7}$$

## Step 2. Satisfying the Boundary Condition

General solutions of (6) and (7) are

$$H(x) = A\cos kx + B\sin kx \qquad \text{and} \qquad Q(y) = C\cos py + D\sin py$$

with constant $A, B, C, D$. From $u = FG$ and (2) it follows that $F = HQ$ must be zero on the boundary, that is, on the edges $x = 0, x = a, y = 0, y = b$; see Fig. 302. This gives the conditions

$$H(0) = 0, \qquad H(a) = 0, \qquad Q(0) = 0, \qquad Q(b) = 0.$$

Hence $H(0) = A = 0$ and then $H(a) = B\sin ka = 0$. Here we must take $B \neq 0$ since otherwise $H(x) \equiv 0$ and $F(x, y) \equiv 0$. Hence $\sin ka = 0$ or $ka = m\pi$, that is,

$$k = \frac{m\pi}{a} \qquad (m \text{ integer}).$$

In precisely the same fashion we conclude that $C = 0$ and $p$ must be restricted to the values $p = n\pi/b$ where $n$ is an integer. We thus obtain the solutions $H = H_m$, $Q = Q_n$, where

$$H_m(x) = \sin\frac{m\pi x}{a} \qquad \text{and} \qquad Q_n(y) = \sin\frac{n\pi y}{b}, \qquad \begin{array}{l} m = 1, 2, \cdots, \\ n = 1, 2, \cdots . \end{array}$$

As in the case of the vibrating string, it is not necessary to consider $m, n = -1, -2, \cdots$ since the corresponding solutions are essentially the same as for positive $m$ and $n$, expect for a factor $-1$. Hence the functions

$$F_{mn}(x, y) = H_m(x)Q_n(y) = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \qquad \begin{array}{l} m = 1, 2, \cdots, \\ n = 1, 2, \cdots, \end{array} \tag{8}$$

are solutions of the Helmholtz equation (5) that are zero on the boundary of our membrane.

**Eigenfunctions and Eigenvalues.** Having taken care of (5), we turn to (4). Since $p^2 = \nu^2 - k^2$ in (7) and $\lambda = c\nu$ in (4), we have

$$\lambda = c\sqrt{k^2 + p^2}.$$

Hence to $k = m\pi/a$ and $p = n\pi/b$ there corresponds the value

$$\lambda = \lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}, \qquad \begin{array}{l} m = 1, 2, \cdots, \\ n = 1, 2, \cdots, \end{array} \tag{9}$$

in the ODE (4). A corresponding general solution of (4) is

$$G_{mn}(t) = B_{mn}\cos\lambda_{mn}t + B^*_{mn}\sin\lambda_{mn}t.$$

It follows that the functions $u_{mn}(x, y, t) = F_{mn}(x, y)G_{mn}(t)$, written out

$$u_{mn}(x, y, t) = (B_{mn}\cos\lambda_{mn}t + B^*_{mn}\sin\lambda_{mn}t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{10}$$

with $\lambda_{mn}$ according to (9), are solutions of the wave equation (1) that are zero on the boundary of the rectangular membrane in Fig. 302. These functions are called the **eigenfunctions** or *characteristic functions,* and the numbers $\lambda_{mn}$ are called the **eigenvalues** or *characteristic values* of the vibrating membrane. The frequency of $u_{mn}$ is $\lambda_{mn}/2\pi$.

**Discussion of Eigenfunctions.** It is very interesting that, depending on $a$ and $b$, several functions $F_{mn}$ may correspond to the same eigenvalue. Physically this means that there may exists vibrations having the same frequency but entirely different **nodal lines** (curves of points on the membrane that do not move). Let us illustrate this with the following example.

**EXAMPLE 1 Eigenvalues and Eigenfunctions of the Square Membrane**

Consider the square membrane with $a = b = 1$. From (9) we obtain its eigenvalues

$$\lambda_{mn} = c\pi\sqrt{m^2 + n^2}. \tag{11}$$

Hence $\lambda_{mn} = \lambda_{nm}$, but for $m \neq n$ the corresponding functions

$$F_{mn} = \sin m\pi x \sin n\pi y \qquad \text{and} \qquad F_{nm} = \sin n\pi x \sin m\pi y$$

are certainly different. For example, to $\lambda_{12} = \lambda_{21} = c\pi\sqrt{5}$ there correspond the two functions

$$F_{12} = \sin \pi x \sin 2\pi y \qquad \text{and} \qquad F_{21} = \sin 2\pi x \sin \pi y.$$

Hence the corresponding solutions

$$u_{12} = (B_{12} \cos c\pi\sqrt{5}t + B^*_{12} \sin c\pi\sqrt{5}t)F_{12} \qquad \text{and} \qquad u_{21} = (B_{21} \cos c\pi\sqrt{5}t + B^*_{21} \sin c\pi\sqrt{5}t)F_{21}$$

have the nodal lines $y = \frac{1}{2}$ and $x = \frac{1}{2}$, respectively (see Fig. 303). Taking $B_{12} = 1$ and $B^*_{12} = B^*_{21} = 0$, we obtain

$$u_{12} + u_{21} = \cos c\pi\sqrt{5}t\,(F_{12} + B_{21}F_{21}) \tag{12}$$

which represents another vibration corresponding to the eigenvalue $c\pi\sqrt{5}$. The nodal line of this function is the solution of the equation

$$F_{12} + B_{21}F_{21} = \sin \pi x \sin 2\pi y + B_{21} \sin 2\pi x \sin \pi y = 0$$

or, since $\sin 2\alpha = 2 \sin \alpha \cos \alpha$,

$$\sin \pi x \sin \pi y(\cos \pi y + B_{21} \cos \pi x) = 0. \tag{13}$$

This solution depends on the value of $B_{21}$ (see Fig. 304).

From (11) we see that even more than two functions may correspond to the same numerical value of $\lambda_{mn}$. For example, the four functions $F_{18}$, $F_{81}$, $F_{47}$, and $F_{74}$ correspond to the value

$$\lambda_{18} = \lambda_{81} = \lambda_{47} = \lambda_{74} = c\pi\sqrt{65}, \qquad \text{because} \qquad 1^2 + 8^2 = 4^2 + 7^2 = 65.$$

This happens because 65 can be expressed as the sum of two squares of positive integers in several ways. According to a theorem by Gauss, this is the case for every sum of two squares among whose prime factors there are at least two different ones of the form $4n + 1$ where $n$ is a positive integer. In our case we have $65 = 5 \cdot 13 = (4 + 1)(12 + 1)$. ■

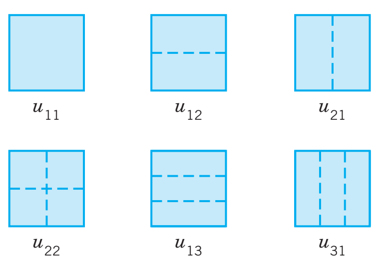


**Fig. 303.** Nodal lines of the solutions $u_{11}$, $u_{12}$, $u_{21}$, $u_{22}$, $u_{13}$, $u_{31}$ in the case of the square membrane

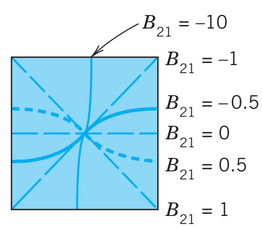


**Fig. 304.** Nodal lines of the solution (12) for some values of $B_{21}$

## Step 3. Solution of the Model (1), (2), (3). Double Fourier Series

So far we have solutions (10) satisfying (1) and (2) only. To obtain the solutions that also satisfies (3), we proceed as in Sec. 12.3. We consider the double series

$$\begin{aligned} u(x, y, t) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{mn}(x, y, t) \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (B_{mn} \cos \lambda_{mn} t + B^*_{mn} \sin \lambda_{mn} t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \end{aligned} \tag{14}$$

(without discussing convergence and uniqueness). From (14) and (3a), setting $t = 0$, we have

$$u(x, y, 0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = f(x, y). \tag{15}$$

Suppose that $f(x, y)$ can be represented by (15). (Sufficient for this is the continuity of $f$, $\partial f/\partial x$, $\partial f/\partial y$, $\partial^2 f/\partial x\, \partial y$ in $R$.) Then (15) is called the **double Fourier series** of $f(x, y)$. Its coefficients can be determined as follows. Setting

$$K_m(y) = \sum_{n=1}^{\infty} B_{mn} \sin \frac{n\pi y}{b} \tag{16}$$

we can write (15) in the form

$$f(x, y) = \sum_{m=1}^{\infty} K_m(y) \sin \frac{m\pi x}{a}.$$

For fixed $y$ this is the Fourier sine series of $f(x, y)$, considered as a function of $x$. From (4) in Sec. 11.3 we see that the coefficients of this expansion are

$$K_m(y) = \frac{2}{a} \int_0^a f(x, y) \sin \frac{m\pi x}{a}\, dx. \tag{17}$$

Furthermore, (16) is the Fourier sine series of $K_m(y)$, and from (4) in Sec. 11.3 it follows that the coefficients are

$$B_{mn} = \frac{2}{b} \int_0^b K_m(y) \sin \frac{n\pi y}{b}\, dy.$$

From this and (17) we obtain the **generalized Euler formula**

$$B_{mn} = \frac{4}{ab} \int_0^b \int_0^a f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dx\, dy \qquad \begin{aligned} m &= 1, 2, \cdots \\ n &= 1, 2, \cdots \end{aligned} \tag{18}$$

for the **Fourier coefficients** of $f(x, y)$ in the double Fourier series (15).

The $B_{mn}$ in (14) are now determined in terms of $f(x, y)$. To determine the $B^*_{mn}$, we differentiate (14) termwise with respect to $t$; using (3b), we obtain

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B^*_{mn}\,\lambda_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = g(x, y).$$

Suppose that $g(x, y)$ can be developed in this double Fourier series. Then, proceeding as before, we find that the coefficients are

$$\textbf{(19)} \qquad \boxed{B^*_{mn} = \frac{4}{ab\lambda_{mn}}\int_0^b\int_0^a g(x, y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}\,dx\,dy} \qquad \begin{array}{l} m = 1, 2, \cdots \\ n = 1, 2, \cdots . \end{array}$$

**Result.** *If $f$ and $g$ in* (3) *are such that $u$ can be represented by* (14), *then* (14) *with coefficients* (18) *and* (19) *is the solution of the model* (1), (2), (3).

**EXAMPLE 2** **Vibration of a Rectangular Membrane**

Find the vibrations of a rectangular membrane of sides $a = 4$ ft and $b = 2$ ft (Fig. 305) if the tension is 12.5 lb/ft, the density is 2.5 slugs/ft$^2$ (as for light rubber), the initial velocity is 0, and the initial displacement is

$$(20) \qquad f(x, y) = 0.1\,(4x - x^2)(2y - y^2) \text{ ft.}$$

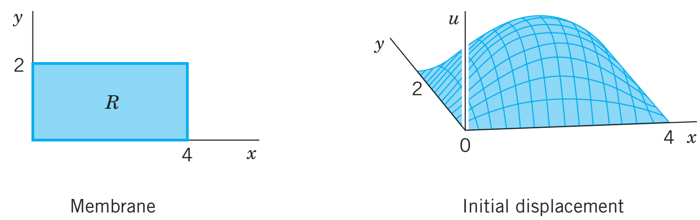


**Fig. 305.** Example 2

***Solution.*** $c^2 = T/\rho = 12.5/2.5 = 5$ [ft$^2$/sec$^2$]. Also $B^*_{mn} = 0$ from (19). From (18) and (20),

$$B_{mn} = \frac{4}{4\cdot 2}\int_0^2\int_0^4 0.1(4x - x^2)(2y - y^2)\sin\frac{m\pi x}{4}\sin\frac{n\pi y}{2}\,dx\,dy$$

$$= \frac{1}{20}\int_0^4 (4x - x^2)\sin\frac{m\pi x}{4}\,dx\int_0^2 (2y - y^2)\sin\frac{n\pi y}{2}\,dy.$$

Two integrations by parts give for the first integral on the right

$$\frac{128}{m^3\pi^3}[1 - (-1)^m] = \frac{256}{m^3\pi^3} \qquad (m \text{ odd})$$

and for the second integral

$$\frac{16}{n^3\pi^3}[1 - (-1)^n] = \frac{32}{n^3\pi^3} \qquad (n \text{ odd}).$$

For even $m$ or $n$ we get 0. Together with the factor 1/20 we thus have $B_{mn} = 0$ if $m$ or $n$ is even and

$$B_{mn} = \frac{256 \cdot 32}{20m^3n^3\pi^6} \approx \frac{0.426050}{m^3n^3} \qquad (m \text{ and } n \text{ both odd}).$$

From this, (9), and (14) we obtain the answer

$$u(x, y, t) = 0.426050 \sum_{m,n \text{ odd}} \sum \frac{1}{m^3n^3} \cos\left(\frac{\sqrt{5}\pi}{4}\sqrt{m^2 + 4n^2}\right) t \sin\frac{m\pi x}{4} \sin\frac{n\pi y}{2}$$

$$(21) \qquad = 0.426050\left(\cos\frac{\sqrt{5}\pi\sqrt{5}}{4} t \sin\frac{\pi x}{4} \sin\frac{\pi y}{2} + \frac{1}{27}\cos\frac{\sqrt{5}\pi\sqrt{37}}{4} t \sin\frac{\pi x}{4} \sin\frac{3\pi y}{2}\right.$$

$$\left. + \frac{1}{27}\cos\frac{\sqrt{5}\pi\sqrt{13}}{4} t \sin\frac{3\pi x}{4} \sin\frac{\pi y}{2} + \frac{1}{729}\cos\frac{\sqrt{5}\pi\sqrt{45}}{4} t \sin\frac{3\pi x}{4} \sin\frac{3\pi y}{2} + \cdots\right).$$

To discuss this solution, we note that the first term is very similar to the initial shape of the membrane, has no nodal lines, and is by far the dominating term because the coefficients of the next terms are much smaller. The second term has two horizontal nodal lines ($y = \frac{2}{3}, \frac{4}{3}$), the third term two vertical ones ($x = \frac{4}{3}, \frac{8}{3}$), the fourth term two horizontal and two vertical ones, and so on. ■

## PROBLEM SET 12.9

1. **Frequency.** How does the frequency of the eigenfunctions of the rectangular membrane change (a) If we double the tension? (b) If we take a membrane of half the density of the original one? (c) If we double the sides of the membrane? Give reasons.
2. **Assumptions.** Which part of Assumption 2 cannot be satisfied exactly? Why did we also assume that the angles of inclination are small?
3. Determine and sketch the nodal lines of the square membrane for $m = 1, 2, 3, 4$ and $n = 1, 2, 3, 4$.

### 4–8 DOUBLE FOURIER SERIES

Represent $f(x, y)$ by a series (15), where

4. $f(x, y) = 1, \quad a = b = 1$
5. $f(x, y) = y, \quad a = b = 1$
6. $f(x, y) = x, \quad a = b = 1$
7. $f(x, y) = xy, \quad a$ and $b$ arbitrary
8. $f(x, y) = xy(a - x)(b - y), \quad a$ and $b$ arbitrary
9. **CAS PROJECT. Double Fourier Series. (a)** Write a program that gives and graphs partial sums of (15). Apply it to Probs. 5 and 6. Do the graphs show that those partial sums satisfy the boundary condition (3a)? Explain why. Why is the convergence rapid?

   **(b)** Do the tasks in (a) for Prob. 4. Graph a portion, say, $0 < x < \frac{1}{2}, 0 < y < \frac{1}{2}$, of several partial sums on common axes, so that you can see how they differ. (See Fig. 306.)

   **(c)** Do the tasks in (b) for functions of your choice.

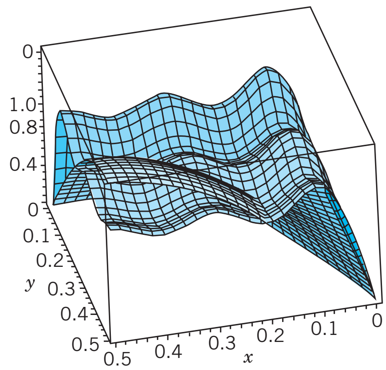


**Fig. 306.** Partial sums $S_{2,2}$ and $S_{10,10}$ in CAS Project 9b

10. **CAS EXPERIMENT. Quadruples of $F_{mn}$.** Write a program that gives you four numerically equal $\lambda_{mn}$ in Example 1, so that four different $F_{mn}$ correspond to it. Sketch the nodal lines of $F_{18}$, $F_{81}$, $F_{47}$, $F_{74}$ in Example 1 and similarly for further $F_{mn}$ that you will find.

### 11–13 SQUARE MEMBRANE

Find the deflection $u(x, y, t)$ of the square membrane of side $\pi$ and $c^2 = 1$ for initial velocity 0 and initial deflection

11. $0.1 \sin 2x \sin 4y$
12. $0.01 \sin x \sin y$
13. $0.1 xy(\pi - x)(\pi - y)$

### 14–19 RECTANGULAR MEMBRANE

**14.** Verify the discussion of (21) in Example 2.

**15.** Do Prob. 3 for the membrane with $a = 4$ and $b = 2$.

**16.** Verify $B_{mn}$ in Example 2 by integration by parts.

**17.** Find eigenvalues of the rectangular membrane of sides $a = 2$ and $b = 1$ to which there correspond two or more different (independent) eigenfunctions.

**18. Minimum property.** Show that among all rectangular membranes of the same area $A = ab$ and the same $c$ the square membrane is that for which $u_{11}$ [see (10)] has the lowest frequency.

**19. Deflection.** Find the deflection of the membrane of sides $a$ and $b$ with $c^2 = 1$ for the initial deflection

$$f(x, y) = \sin \frac{6\pi x}{a} \sin \frac{2\pi y}{b} \text{ and initial velocity 0.}$$

**20. Forced vibrations.** Show that forced vibrations of a membrane are modeled by the PDE $u_{tt} = c^2\nabla^2 u + P/\rho$, where $P(x, y, t)$ is the external force per unit area acting perpendicular to the $xy$-plane.

# 12.10 Laplacian in Polar Coordinates. Circular Membrane. Fourier–Bessel Series

It is a ***general principle*** in boundary value problems for PDEs to *choose coordinates that make the formula for the boundary as simple as possible*. Here polar coordinates are used for this purpose as follows. Since we want to discuss circular membranes (drumheads), we first transform the Laplacian in the wave equation (1), Sec. 12.9,

$$u_{tt} = c^2\nabla^2 u = c^2(u_{xx} + u_{yy}) \tag{1}$$

(subscripts denoting partial derivatives) into **polar coordinates** $r, \theta$ defined by $x = r\cos\theta$, $y = r\sin\theta$; thus,

$$r = \sqrt{x^2 + y^2}, \qquad \tan\theta = \frac{y}{x}.$$

By the chain rule (Sec. 9.6) we obtain

$$u_x = u_r r_x + u_\theta \theta_x.$$

Differentiating once more with respect to $x$ and using the product rule and then again the chain rule gives

$$\begin{aligned} u_{xx} &= (u_r r_x)_x + (u_\theta\theta_x)_x \\ &= (u_r)_x r_x + u_r r_{xx} + (u_\theta)_x\theta_x + u_\theta\theta_{xx} \\ &= (u_{rr}r_x + u_{r\theta}\theta_x)r_x + u_r r_{xx} + (u_{\theta r}r_x + u_{\theta\theta}\theta_x)\theta_x + u_\theta\theta_{xx}. \end{aligned} \tag{2}$$

Also, by differentiation of $r$ and $\theta$ we find

$$r_x = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r}, \qquad \theta_x = \frac{1}{1 + (y/x)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{r^2}.$$