Mathematical Modeling of the Arterial Blood Flow

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Abstract. Blood flow is a study of measuring the blood pressure and finding the flow through the blood vessel. Blood flow problem has been studied for centuries where one of the motivations was to understand the conditions that contribute to high blood pressure. This occurs when the blood vessel became narrowed from its normal size. This paper presents a mathematical modeling of the arterial blood flow which was derived from the Navier-Stokes equations and some assumptions. A system of nonlinear partial differential equations for blood flow and the cross-sectional area of the artery was obtained. Finite difference method was adopted to solve the equations numerically. The result obtained is very sensitive to the values of the initial conditions and this helps to explain the condition of hypertension.

Keywords: Mathematical modeling, arterial flow, MatLab

1. Introduction

Blood flow is a study of measuring the blood pressure and finding the flow through the blood vessel. This study is important for human health. Most of the researches study the blood flow in the arteries and veins. One of the motivations to study the blood flow was to understand the conditions that may contribute to high blood pressure. Past studies indicated that one of the reasons a person having hypertension is when the blood vessel becomes narrow. This paper will focus on the diastolic hypertension. Blood is non-Newtonian fluid and to model such fluid is very complicated. In this problem, blood is assumed to be a Newtonian fluid. Even though this will make the problem much simpler, it still is valid since blood in a large vessel acting almost like a Newtonian fluid. In order to model this problem, Navier-Stokes equations will be used to derive the governing equations that represent this problem.

2. Formulation of the Governing Equations

We have adopted Yang, Zhang and Asada's [1] local arterial flow model. This includes the assumptions that the arterial vessel is rectilinear, deformable, thick shell of isotropic, incompressible material with circular section and without longitudinal movements. Meanwhile blood is considered as an incompressible Newtonian fluid, and the flow is axially symmetric. The model approach is to use the two-dimensional Navier-Stokes equations and continuity equation for a Newtonian and incompressible fluid in cylindrical coordinate (r, z, t):

$$\frac{\partial u}{\partial t}+w\frac{\partial u}{\partial r}+u\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial z}+v\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}\right), \qquad (1)$$

$$\frac{\partial w}{\partial t}+w\frac{\partial w}{\partial r}+u\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial z}+v\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}+\frac{w}{r^2}\right), \qquad (2)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rw)+\frac{\partial u}{\partial z}=0, \qquad (3)$$

where P = pressure,
ρ = density,

v = kinematic viscosity,
$u\ (r, z, t)$ = the components of velocity in axial (z) directions,
$w\ (r, z, t)$ = the components of velocity in radial (r) directions.

For convenience we define a new variable, which is the radial coordinate, η:

$$\eta = \frac{r}{R(z,t)} \tag{4}$$

where $R(z, t)$ denotes the inner radius of the vessel. Assuming that P is independent of the radial coordinate, η, then the pressure P is uniform within the cross section ($P = P(z, t)$). Hence

$$\frac{\partial^2 u}{\partial z^2} \le 1, \qquad \frac{\partial^2 w}{\partial z^2} \le 1, \qquad \frac{\partial P}{\partial r} \le 1$$

Using simple algebra to change the variable such as

$$\frac{\partial u(r,z,t)}{\partial t} = \frac{\partial u(\eta,t)}{\partial t}\cdot\frac{\partial \eta}{\partial t} + \frac{\partial u(\eta,t)}{\partial t}\cdot\frac{\partial t}{\partial t},$$
$$= -\frac{\eta}{R}\frac{\partial u(\eta,t)}{\partial t}\cdot\frac{\partial R}{\partial t} + \frac{\partial u(\eta,t)}{\partial t},$$

equations (1), (2) and (3) can be written in the new coordinate (η, z, t) as:

$$\frac{\partial u}{\partial t} + \frac{1}{R}\left(\eta\left(u\frac{\partial R}{\partial z} + \frac{\partial R}{\partial t}\right) - w\right)\frac{\partial u}{\partial \eta} + u\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \frac{v}{R^2}\left(\frac{\partial^2 u}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial u}{\partial \eta}\right), \tag{5}$$

$$\frac{\partial w}{\partial t} + \frac{1}{R}\left(\eta\left(u\frac{\partial R}{\partial z} + \frac{\partial R}{\partial t}\right) - w\right)\frac{\partial w}{\partial \eta} + u\frac{\partial w}{\partial z} = \frac{v}{R^2}\left(\frac{\partial^2 w}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial w}{\partial \eta} + \frac{w}{\eta^2}\right), \tag{6}$$

$$\frac{1}{R}\frac{\partial w}{\partial \eta} + \frac{w}{\eta R} + \frac{\partial u}{\partial z} - \frac{\eta}{R}\frac{\partial R}{\partial z}\frac{\partial u}{\partial \eta} = 0\cdot \tag{7}$$

The system of equations above is a hemodynamic type of model. [1] stated that according to Belardinelli and Cavalcanti in 1991, the velocity profile in the axial direction, $u(\eta,z,t)$, is assumed to have the expression in the polynomial form below

$$u(\eta,z,t) = \sum_{k=1}^{N} q_k\left(\eta^{2k} - 1\right)\cdot \tag{8}$$

While the velocity profile in the radial direction is

$$w(\eta,z,t) = \frac{\partial R}{\partial z}\eta w + \frac{\partial R}{\partial t}\eta - \frac{\partial R}{\partial t}\frac{1}{N}\eta\sum_{k=1}^{N}\frac{1}{k}\left(\eta^{2k} - 1\right)\cdot \tag{9}$$

[1] chose $N = 1$ to simplify (8) and (9), so that

$$u(\eta,z,t) = q(z,t)\left(\eta^2 - 1\right), \tag{10}$$

$$w(\eta,z,t) = \frac{\partial R}{\partial z}\eta w + \frac{\partial R}{\partial t}\eta - \frac{\partial R}{\partial t}\eta\left(\eta^2 - 1\right)\cdot \tag{11}$$

Then, when equations (10) and (11) are substituted into equations (5) and (7), we will get the dynamic equations of $q(z, t)$ and $R(z, t)$, which are

$$\frac{\partial q}{\partial t} - \frac{4q}{R}\frac{\partial R}{\partial t} - \frac{2q^2}{R}\frac{\partial R}{\partial z} + \frac{4v}{R^2}q + \frac{1}{\rho}\frac{\partial P}{\partial z} = 0, \tag{12}$$

$$2R\frac{\partial R}{\partial t} + \frac{R^2}{2}\frac{\partial q}{\partial z} + q\frac{\partial R}{\partial z} = 0\cdot \tag{13}$$

Now, the cross-sectional area $S(z, t)$ and blood flow $Q(z, t)$ are defined as

$$S = \pi R^2, \qquad Q = \iint_S u\,\partial\eta = \frac{1}{2}\pi q R^2\cdot$$

We can use these definitions to express equations (12) and (13) in terms of $Q(z,t)$ and $S(z,t)$:

$$\frac{\partial Q}{\partial t}-\frac{3Q}{S}\frac{\partial S}{\partial t}-\frac{2Q^2}{S^2}\frac{\partial S}{\partial z}+\frac{4\pi v}{S}Q+\frac{S}{2\rho}\frac{\partial P}{\partial z}=0, \tag{14}$$

$$\frac{\partial S}{\partial t}+\frac{\partial Q}{\partial z}=0. \tag{15}$$

The solutions for the cross-sectional area of the artery and its corresponding blood flow can now be obtained by solving the governing equations (14) and (15). The system of equations (14)-(15) is nonlinear partial differential equations. Finite difference method is used to solve such problem. First, the equations will be discretized using the following difference formula in first order accuracy:

$$\frac{\partial Q_i}{\partial z}=\frac{Q_i-Q_{i-1}}{\Delta z} \text{ and } \frac{\partial S_i}{\partial z}=\frac{S_{i+1}-S_i}{\Delta z}$$

where $\Delta z = L/(N-1)$, so that the equations becomes difference equations:

$$\frac{\partial Q_i}{\partial t}+\frac{3Q_i}{S_i}\frac{Q_i-Q_{i-1}}{\Delta z}-\frac{2Q_i^2}{S_i^2}\frac{S_{i+1}-S_i}{\Delta z}+\frac{4\pi v}{S_i}Q_i+\frac{S_i}{2\rho}\frac{\partial P}{\partial z}=0, \tag{16}$$

$$\frac{\partial S_i}{\partial t}=-\frac{Q_i-Q_{i-1}}{\Delta z}, \tag{17}$$

where $i=1,2,\dots N$. Here, the pressure gradient $\frac{\partial P}{\partial z}$ is kept constant and the value is prescribed. The discretization of the artery model is shown in Figure 1 below.

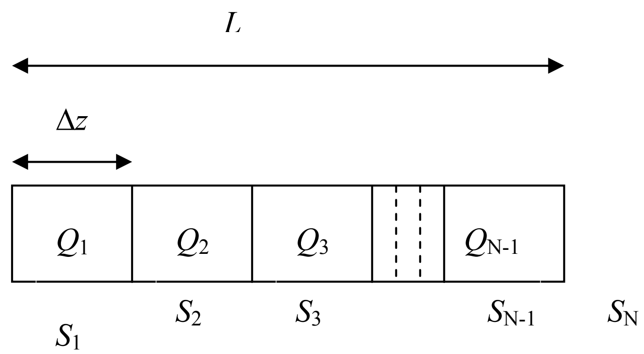

Figure 1: Discretization of the arterial model

Since we are considering local arterial segment, we can simplify the governing equations by linearizing equation (16):

$$\frac{\partial Q_i}{\partial t}+\frac{4\pi v}{S_0}Q_i+\frac{S_0}{2\rho}\frac{\partial P}{\partial z}+\frac{S_i}{2\rho}\frac{\partial P}{\partial z}=0. \tag{18}$$

The system of equations that needs to be solved is now equations (17) - (18) where the numerical solutions are discussed in the following section.

3 Numerical Method

We notice that the difference equations (17) – (18) can be written in the form $\frac{\partial y}{\partial t}=f(y)$

where

$$y = [Q_1\ Q_2\ \cdots Q_N\ S_1\ S_2 \cdots S_N]^T \text{ and}$$

$$f(y) = \begin{pmatrix} -\left(\frac{4\pi v}{S_0} y(1) + \frac{S_0}{2\rho}\frac{\partial P}{\partial z} + \frac{y(N+1)}{2\rho}\frac{\partial P}{\partial z}\right) \\ -\left(\frac{4\pi v}{S_0} y(2) + \frac{S_0}{2\rho}\frac{\partial P}{\partial z} + \frac{y(2+N)}{2\rho}\frac{\partial P}{\partial z}\right) \\ \vdots \\ \vdots \\ -\left(\frac{4\pi v}{S_0} y(N-1) + \frac{S_0}{2\rho}\frac{\partial P}{\partial z} + \frac{y(2N-1)}{2\rho}\frac{\partial P}{\partial z}\right) \\ -\left(\frac{4\pi v}{S_0} y(N) + \frac{S_0}{2\rho}\frac{\partial P}{\partial z} + \frac{y(2N)}{2\rho}\frac{\partial P}{\partial z}\right) \\ -\frac{y(1) - Q_0}{\Delta z} \\ -\frac{y(2) - y(0)}{\Delta z} \\ \vdots \\ \vdots \\ -\frac{y(N-1) - y(N-2)}{\Delta z} \\ -\frac{y(N) - y(N-1)}{\Delta z} \end{pmatrix}.$$

The simplest and fast way to solve such problem is by using MatLab built-in function ODE45, which is based on Runge-Kutta method. The values of parameters that are required are the initial value of the blood flow, Q_0, the initial cross-sectional area, S_0, the axial pressure gradient $\frac{\partial P}{\partial z}$, the kinematic viscosity v and density ρ for blood. The required values in normal condition can be obtained from past works in the field such as:

Initial value of Q and Q_0 = 1 to 5.4 liter/minute [2]
Initial value of S and S_0 = 1.5 to 2.0 cm^3 [3]
$\frac{\partial P}{\partial z}$ = 100 to 40 mmHg [4]
v = 0.035 cm^2/s [5]
ρ = 1.05 g/cm^3 [5].

4. Result Analysis

In order to simulate how the cross-sectional area of the artery affects the blood flow within the artery, the values of parameters mentioned in the previous section are chosen to be: ρ = 1.05 g/cm^3, v = 0.035 cm^2/sec, Q_0 = 16.7 cm^3/sec and S_0 = 1.5 cm^2. For simplicity, we chose the length of the artery model, L = 15 cm and the number of nodes of the system, N = 3. Since we consider arteries in a diastole condition only, the chosen time span is 0.2 seconds.

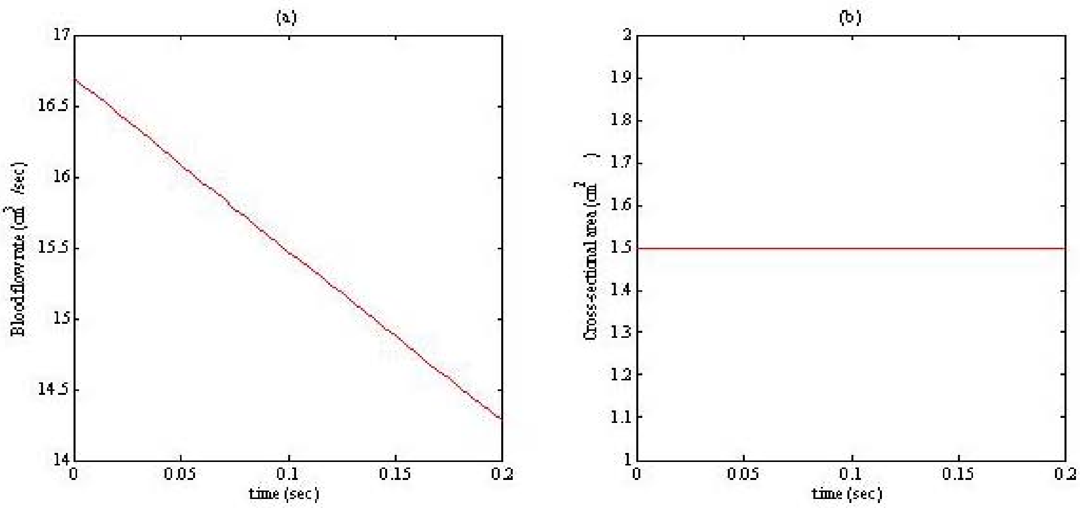

Figure 2: (a) is the blood flow rate against time and (b) is the cross-sectional area against time.

Figure 2 shows the blood flow rate and cross-sectional areas for each node. It is observed that the results for Q_1, Q_2 and Q_3 are almost the same as depicted in Figure 2(a). Similarly, the values of S_1, S_2 and S_3 in Figure 2(b) are very close and it is almost a constant. This shows that the values of the blood flow rate and the cross-sectional area are almost the same through out in the small section of arteries. This could be due to the absence of viscoelastic effect in the model. Now since there is not much difference in the blood flow rate between the sections, we will consider only one section which is S_2 to make the comparison of the different values of the cross-sectional area. As we can see, the value for the blood flow is decreasing from its initial value. This is also the case for the cross-sectional area, although it decreases in smaller range as shown in Figure 3.

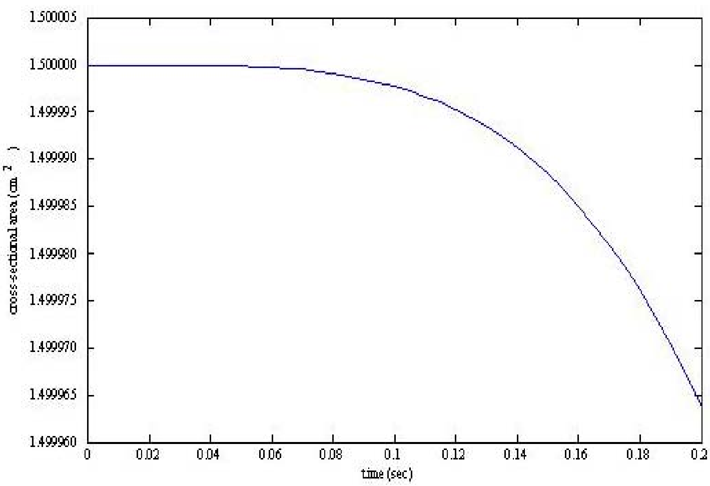

Figure 3: The cross-sectional area against time

This shows that without changing the value of the pressure gradient and the cross-sectional area of the arteries, the blood flow rate through the arteries is decreasing significantly as time increasing. It also shows that the blood flow is linearly decreasing. We assume this condition is valid in the diastole condition only. Next, we compare this result with smaller cross-sectional area. From Figure 4, we can see that if the value of cross-sectional area is smaller, the blood flow rate is decreasing slower than the blood flow rate at a normal condition which implies that when the cross-sectional area is decreased, the blood flow is increased. This condition occurs when the value of cross-sectional area is in the range between 1.5 cm^2 to 0.9 cm^2. Due to the fact that larger amount of blood flows through the arteries in a smaller cross-sectional area, may cause the increasing of pressure in the artery's wall. Thus, blood pressure increases and contributes to high blood pressure.

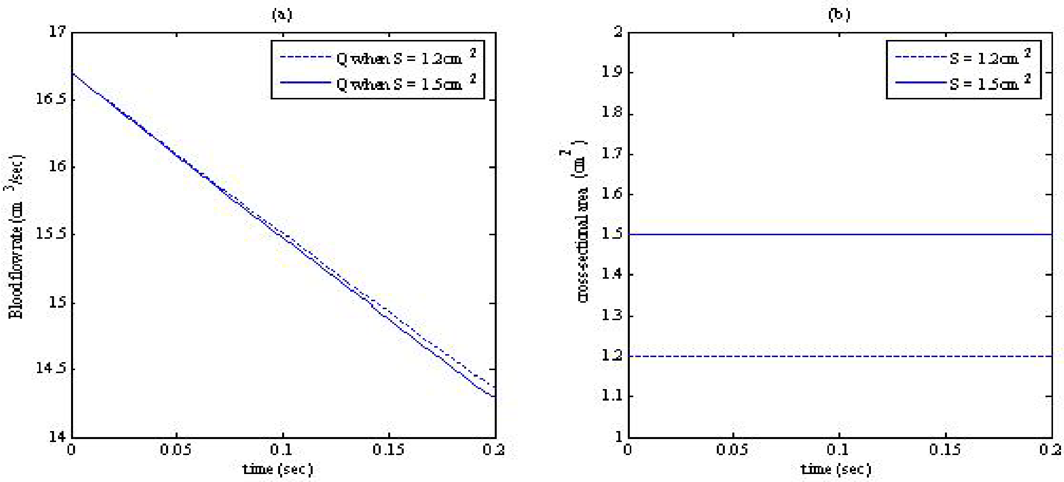

Figure 4: Comparison graph for blood flow with different value of cross-sectional area.

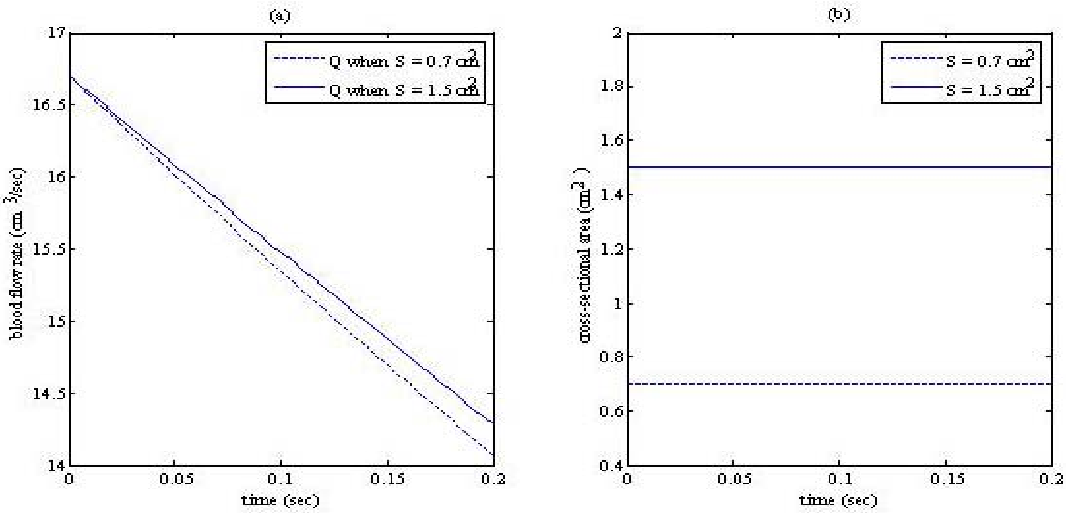

Figure 5: Comparison of *Q* at normal cross-sectional area and much smaller cross-sectional area

As shown in Figure 5 above, when the value of cross-sectional area is below 0.8 cm^2, the blood flow rate decreases faster that the normal rate. Figure 6 shows the value of blood flow rate when the cross-sectional area is in range of between 0.1 cm^2 to 0.8cm^2. Clearly, if the cross-sectional area continues to decrease below 0.8 cm^2, the blood flow rate also decreased drastically. From this observation, we can say that this condition occur because the cross-sectional area is too small for the blood to get through it. This is also a dangerous condition for human.

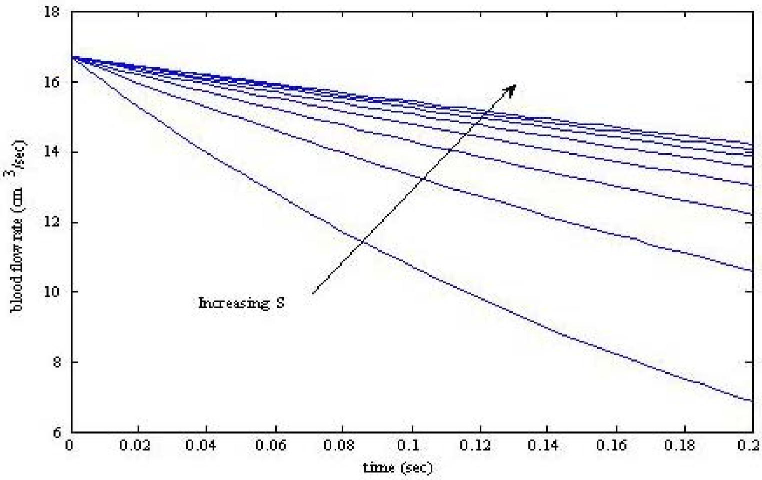

Figure 6: Q when cross-sectional area is in range between 0.1 cm^2 to 0.8cm^2

From the results obtained, we can conclude that cross-sectional area plays an important part in order for the blood to flow smoothly through the blood vessel. A small change in the value for the cross-sectional area may affect the amount of blood flow rate through the arteries which also may affect the blood pressure. In other words, smaller cross-sectional area from normal size may contribute to hypertension or high blood pressure. When a large amount of fluid flows through a small vessel, it may cause the pressure in the vessel to increase.

5. Conclusion

In this paper, we have derived a simple mathematical model that can represent the blood flow in the arteries. Even though the model does not include viscoelastic effect, the results obtained is considered valid since we are able to make a conclusion that from this model, we observe that the size of the blood vessel does influence the blood flow. A little change on the cross-sectional value makes vast change on the blood flow rate.

6. References

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