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Groups, radical rings and the Yang–Baxter equation

A combinatorial approach to solutions

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preliminaries

1.1 Groups

A *semigroup* is a set S with an associative operation

$$*\colon S\times S \longrightarrow S, \quad (x,y)\mapsto x*y,$$

that is $(x*y)*z = x*(y*z)$ for all $x,y,z\in S$.

If there exists an element $e\in S$ such that $e*x = x*e = x$ for all $x\in S$, then the semigroup $(S,*)$ is called a *monoid* and e is the *neutral element* for $*$. The neutral element in a monoid $(S,*)$ is unique.

A *group* is a monoid $(G,*)$ such that for every $x\in G$ there exists $x'\in G$ satisfying

$$x*x' = x'*x = e,$$

where $e\in G$ is the neutral element for $*$. The element x' is called the *symmetric element* of x for $*$ and it is unique.

The standard notation for the operation of a general semigroup is the multiplicative terminology. Thus if $(S,\cdot)$ is a semigroup, then we write the multiplication $x\cdot y = xy$ and we say that xy is the product of x and y. If $(S,\cdot)$ is a monoid, then 1 denotes its neutral element and it is called the *unit-element* of S. If $(S,\cdot)$ is a group, then the symmetric element x' of $x\in S$ is called the *inverse* of x and it is denoted by x^{-1}.

In some chapters, we also will use the additive terminology for general semigroups, monoids and groups. Thus if $(S,+)$ is a semigroup the addition of elements $x+y$ is called the sum of x and y. If $(S,+)$ is a monoid, then 0 denotes its neutral element and it is called the *zero* of S. If $(S,\cdot)$ is a group, then the symmetric element x' of $x\in S$ is called the *opposite* of x and it is denoted by $-x$.

In the remainder of this chapter we shall use the multiplicative terminology for general semigroups, monoids and groups.

A semigroup S is said to be *commutative* if $xy = yx$ for all $x,y\in S$. A commutative group is also called an *abelian group*.

Example 1.1.1. $(\mathbb{Z},+)$, $(\mathbb{Q},+)$, $(\mathbb{R},+)$, $(\mathbb{C},+)$, $(\mathbb{Q}\setminus\{0\},\cdot)$, $(\mathbb{R}\setminus\{0\},\cdot)$, $(\mathbb{C}\setminus\{0\},\cdot)$ are abelian groups.

Example 1.1.2. The *symmetric group* on a set X is the set

$$\mathbb{S}_X = \{f\colon X \to X : f \text{ is bijective}\}$$

with the composition of maps. For every positive integer n the symmetric group of degree n is $\mathbb{S}_n = \mathbb{S}_{\{1,2,\ldots,n\}}$. Note that $\mathbb{S}_n$ is not abelian for $n > 2$.

A *subgroup* of a group G is a non-empty subset H of G such that $xy^{-1} \in H$ for all $x, y \in H$. Note that every subgroup H of G also is a group with the operation of G restricted to H. The notation $H \leq G$ will mean that H is a subgroup of G.

intersection

Proposition 1.1.3. *Let $(H_i)_{i\in I}$ be a non-empty family of subgroups of a group G. Then*

$$\bigcap_{i\in I} H_i$$

is a subgroup of G. □

Let S be a subset of a group G. The *subgroup of G generated by S* is

$$\langle S\rangle = \bigcap_{S\subseteq H\leq G} H.$$

Note that if S is a non-empty subset of G, then

$$\langle S\rangle = \{x_1^{\varepsilon_1}\cdots x_n^{\varepsilon_n} : n \geq 0,\ \varepsilon_i = \pm 1,\ x_i \in S,\ 1 \leq i \leq n\}.$$

We say that G is *finitely generated* if there exists a finite subset $F = \{x_1,\ldots,x_n\}$ of G such that $G = \langle F\rangle$. In this case we also write $G = \langle x_1,\ldots,x_n\rangle$. We say that G is *cyclic* if there exists $x \in G$ such that $G = \langle x\rangle$.

Let H be a subgroup of a group G. The *left cosets* of H in G are the subsets of G of the form

$$xH = \{xh : h \in H\},$$

for $x \in G$. A subset of G containing just one element from each left coset of H in G is called a *left transversal* of H in G. Note that $xH = yH$ if and only if $x^{-1}y \in H$. Right cosets and right transversals are defined similarly. We say that H is a *normal subgroup* of G if $xH = Hx$ for all $x \in G$. In this case the set $G/H = \{xH : x \in G\}$, with the operation defined by the rule $(xH)\cdot(yH) = (xy)H$ for all $x, y \in G$, is a group called the *quotient group* of G by the normal subgroup H. The notation $H \trianglelefteq G$ will mean that H is a normal subgroup of G. Note that every subgroup of an abelian group is normal.

Example 1.1.4. The subgroups of the group $(\mathbb{Z},+)$ are of the form

$$n\mathbb{Z} = \{nz : z \in \mathbb{Z}\},$$

for a non-negative integer n. Note that $\mathbb{Z}$ is a cyclic group.

Example 1.1.5. The group $\mathbb{Z}/n\mathbb{Z} = \{0, 1, \ldots, n-1\}$ of integers modulo n is a cyclic group.

Example 1.1.6. Let G be a group and let H be a subgroup of G. The *normalizer* of H in G is the set

$$N_G(H) = \{x \in G : xH = Hx\}.$$

Note that $H \trianglelefteq N_G(H) \leq G$.

Let G be a group. If G is finite, then the number of elements of G is called the *order* of G and it is denoted by $|G|$. Let H be a subgroup of G. If the number of left cosets of H in G is finite then it coincides with the number of right cosets of H in G, and this number is called the *index* of H in G and it is denoted by $(G : H)$.

Theorem 1.1.7 (Lagrange's theorem). *Let G be a finite group and let H be a subgroup of G. Then $|G| = (G : H)|H|$.* □

A *homomorphism of groups* is a map $f : G_1 \to G_2$, where G_1 and G_2 are groups, and such that $f(xy) = f(x)f(y)$ for all $x, y \in G_1$. A bijective homomorphism is called an *isomorphism*. Two groups G_1 and G_2 are isomorphic if there exists an isomorphism $f : G_1 \to G_2$. The notation $G_1 \cong G_2$ will mean that G_1 and G_2 are isomorphic. An *automorphism* of a group G is an isomorphism $G \to G$. The set $\mathrm{Aut}(G)$ of all automorphisms of a group G is a group with the usual composition of maps; it is called the *automorphism group* of G.

Example 1.1.8. Let G be a group and let H be a subgroup of G. The inclusion mapping $\iota : H \to G$, $h \mapsto h$, is an injective homomorphism of groups.

Example 1.1.9. Let G be a group. For every $x \in G$, the map $\varphi_x : G \to G$ defined by $\varphi_x(y) = xyx^{-1}$, for all $y \in G$, is an automorphism of G. The map $\varphi : G \to \mathrm{Aut}(G)$ defined by $\varphi(x) = \varphi_x$, for all $x \in G$, is a homomorphism of groups. The group $\mathrm{Inn}(G)$ of *inner automorphisms* of G is defined as the image of φ.

The *kernel* of a homomorphism of groups $f : G_1 \to G_2$ is the set

$$\ker(f) = \{x \in G_1 : f(x) = 1\}.$$

The kernel $\ker(f)$ of f is a normal subgroup of G_1. The image $\mathrm{im}(f) = \{f(g) : g \in G_1\}$ of f is a subgroup of G_2.

Example 1.1.10. Let G be a group and N be a normal subgroup of G. The natural map $\pi : G \to G/N$, $x \mapsto xN$, is a surjective homomorphism of groups with kernel equal to N.

Theorem 1.1.11 (First isomorphism theorem). *For any homomorphism of groups $f : G_1 \to G_2$ there exists a unique isomorphism $\tilde{f} : G_1/\ker(f) \to \mathrm{im}(f)$ such that the diagram*

$$\begin{array}{ccc} G_1 & \xrightarrow{f} & G_2 \\ {\scriptstyle\pi}\downarrow & & \uparrow{\scriptstyle\iota} \\ G_1/\ker(f) & \xrightarrow{\tilde{f}} & \operatorname{im}(f) \end{array}$$

is commutative, that is $f = \iota\tilde{f}\pi$, where ι is the inclusion mapping and π is the natural homomorphism. □

Let H and K be subgroups of a group G. If $H \leq N_G(K)$, then

$$KH = \{xy : x \in K,\ y \in H\}$$

is a subgroup of G. Furthermore $K \trianglelefteq KH$.

Theorem 1.1.12 (Second isomorphism theorem). *Let G be a group and H and K be subgroups of G such that $H \leq N_G(K)$. Then $K \cap H \trianglelefteq H$ and*

$$H/(K \cap H) \cong (KH)/K. \quad \square$$

Theorem 1.1.13 (Third isomorphism theorem). *Given a group G and $N \trianglelefteq G$, the map $H \mapsto H/N$ yields a bijective correspondence between (normal) subgroups of G containing N and (normal) subgroups of G/N. Furthermore, if $N \leq H \trianglelefteq G$, then*

$$(G/N)/(H/N) \cong G/H. \quad \square$$

Let n be a positive integer. The elements of $\mathbb{S}_n$ are called permutations. We denote each element $\sigma \in \mathbb{S}_n$ by

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix}.$$

A permutation $\sigma \in \mathbb{S}_n$ is a *cycle* of length r if there exist r distinct elements $a_1, \dots, a_r \in \{1, \dots, n\}$ such that $\sigma(a_i) = a_{i+1}$ for $1 \leq i \leq r-1$, $\sigma(a_n) = a_1$ and $\sigma(b) = b$ for all $b \in \{1, \dots, n\} \setminus \{a_1, \dots, a_r\}$. In this case, we will denote σ by

$$\sigma = (a_1 \dots a_r).$$

A *transposition* is a cycle of length two. We say that the cycles $(a_1 \dots a_r)$ and $(b_1 \dots b_s)$ are disjoint if

$$\{a_1, \dots, a_r\} \cap \{b_1, \dots, b_s\} = \emptyset.$$

Note that if σ and τ are disjoint cycles, then $\sigma\tau = \tau\sigma$.

Theorem 1.1.14. *Every permutation $\sigma \in \mathbb{S}_n$ can be written as a product of pairwise disjoint cycles. This factorization is unique except for the order in which the factors occur.* □

Theorem 1.1.15. *Let $n > 1$ be an integer. There exists a unique surjective group homomorphism* $\operatorname{sign}\colon \mathbb{S}_n \to (\{1,-1\},\cdot)$ *such that* $\operatorname{sign}(\tau) = -1$ *for every transposition* $\tau \in \mathbb{S}_n$. □

The group homomorphism sign is known as the *sign homomorphism* of $\mathbb{S}_n$. The kernel of sign is the *alternating group* $\mathbb{A}_n$. We say that the elements of $\mathbb{A}_n$ are the *even* permutations of $\mathbb{S}_n$ and the elements of $\mathbb{S}_n \setminus \mathbb{A}_n$ are the *odd* permutations of $\mathbb{S}_n$.

Theorem 1.1.16. *Let $n > 1$ be an integer. Then the following statements hold:*

1) $\mathbb{S}_n = \langle (ij) : 1 \le i < j \le n \rangle$.
2) $\mathbb{S}_n = \langle (12), (13), \ldots, (1n) \rangle$.
3) $\mathbb{S}_n = \langle (12), (23), \ldots, (n-1\,n) \rangle$.
4) $\mathbb{S}_n = \langle (12), (12\cdots n) \rangle$.
5) $\mathbb{A}_n = \langle (ijk) : 1 \le i < j < k \le n \rangle$.
6) $\mathbb{A}_n = \langle (12k) : 3 \le k \le n \rangle$. □

Theorem 1.1.17 (Cayley's theorem). *Let G be a finite group of order n. Then G is isomorphic to a subgroup of $\mathbb{S}_n$.* □

Let p be a prime. A *p-group* is a group of order a power of p. Let G be a finite group of order $n = p^\alpha m$, where $\gcd(p,m) = 1$. Any subgroup of G of order p^α is called a *Sylow p-subgroup* of G.

Let $\operatorname{Syl}_p(G)$ be the set of Sylow p-subgroups of G.

Theorem 1.1.18 (Sylow). *Let G be a finite group and p a prime. The following statements hold:*

1) $\operatorname{Syl}_p(G) \neq \emptyset$.
2) *Every p-subgroup of G is contained in a Sylow p-subgroup of G.*
3) *If $P_1, P_2 \in \operatorname{Syl}_p(G)$, then there exists $x \in G$ such that $xP_1x^{-1} = P_2$.*
4) $|\operatorname{Syl}_p(G)| \equiv 1 \bmod p$. □

Theorem 1.1.19. *Let p be a prime. Any finite non-trivial p-group has a non-trivial center.* □

1.2 Rings

A *ring* is a set R with two operations

$$R \times R \to R, \quad (x,y) \mapsto xy, \qquad R \times R \to R, \quad (x,y) \mapsto x+y,$$

satisfying the following properties:

1) $(R,+)$ is an abelian group.
2) $(ab)c = a(bc)$ for all $a,b,c \in R$.
3) $a(b+c) = ab+ac$ and $(a+b)c = ac+bc$ for all $a,b,c \in R$.

If $(R,\cdot)$ has an identity element, then this is denoted by 1 and it is said that R is a unitary ring.

Convention 1.2.1. From now on, a ring will mean a unitary ring, unless otherwise specified.

A *commutative ring* is a ring R such that $ab = ba$ for all $a, b \in R$.

Let R be a ring. We say that $a \in R$ is a *zero-divisor* if there exists $b \in R \setminus \{0\}$ such that $ab = 0$ or $ba = 0$. We say that $a \in R$ is an *idempotent* if $a^2 = a$. We say that $a \in R$ is *nilpotent* if there exists a positive integer n such that $a^n = 0$. An idempotent $a \in R$ is *non-trivial* if $a \notin \{0, 1\}$.

An *integral domain* is a ring R with $0 \neq 1$ and without non-zero zero-divisors. A *division ring* is a ring R such that $(R \setminus \{0\}, \cdot)$ is a group. A *field* is a commutative division ring. Any division ring is an integral domain.

Example 1.2.2. The zero ring is $\{0\}$ with $0+0 = 0 = 0 \cdot 0$. This is the only ring such that $0 = 1$.

Example 1.2.3. On the one hand, $\mathbb{Z}$ is a commutative integral domain which is not a field. On the other hand, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are fields.

Example 1.2.4. If $\{R_i\}_{i\in I}$ is a non-empty family of rings, then $\prod_{i\in I} R_i$, with the addition and the multiplication defined component-wise, is a ring. If $|I| \geq 2$ and the rings R_i are non-zero, then $\prod_{i\in I} R_i$ has non-trivial idempotents. For example,

$$e_i = (\delta_{ij})_{j\in I} \quad \text{where} \quad \delta_{ij} = \begin{cases} 0 & \text{if } j \neq i, \\ 1 & \text{if } j = i. \end{cases}$$

Example 1.2.5. Let R be a ring. The *power series ring* over R is the set

$$R[[X]] = \left\{ \sum_{i=0}^{\infty} a_i X^i : a_i \in R \right\}$$

with the addition and the multiplication defined by the rules

$$\sum_{i=0}^{\infty} a_i X^i + \sum_{i=0}^{\infty} b_i X^i = \sum_{i=0}^{\infty} (a_i + b_i) X^i,$$

and

$$\left(\sum_{i=0}^{\infty} a_i X^i \right) \left(\sum_{i=0}^{\infty} b_i X^i \right) = \sum_{i=0}^{\infty} c_i X^i,$$

where $c_i = \sum_{j=0}^{i} a_j b_{i-j}$.

Example 1.2.6. Let R be a ring and let n be a positive integer. The full $n \times n$ *matrix ring* is the set

$$M_n(R) = \left\{ \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{pmatrix} : a_{i,j} \in R \right\}$$

with the addition defined component-wise and the multiplication defined by the rule

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,n} \\ b_{2,1} & b_{2,2} & \dots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \dots & b_{n,n} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} & \dots & c_{1,n} \\ c_{2,1} & c_{2,2} & \dots & c_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n,1} & c_{n,2} & \dots & c_{n,n} \end{pmatrix},$$

where $c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}$.

A *subring* of a ring R is a subset S of R such that $a - b \in S$ and $ab \in S$ for all $a, b \in S$ and $1 \in S$.

Example 1.2.7. $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$ is a chain of subrings.

Example 1.2.8. Let R be a ring. The *polynomial ring* over R is the following subring of the power series ring over R:

$$R[X] = \left\{ \sum_{i=0}^{\infty} a_i X^i \in R[[X]] : a_i \neq 0 \text{ for finitely many non-negative integers } i \right\}.$$

The elements of $R[X]$ are usually written as finite sums of the form $\sum_{i=0}^{n} a_i X^i$ and are called polynomials. Note also that R is a subring of $R[X]$.

Let R be a ring. A *right ideal* of R is an additive subgroup I of R such that $xa \in I$ for all $x \in I$ and $a \in R$. Similarly one defines a *left ideal* of R. An ideal of R is, by definition, an additive subgroup of R that is both a left and right ideal of R.

If I and J are left (right) ideals of a ring R, then

$$IJ = \left\{ \sum_{i=1}^{n} a_i b_i : n \in \mathbb{Z}_{\geq 0},\ a_i \in I,\ b_i \in J \right\}$$

also is a left (right) ideal of R.

Example 1.2.9. Let R be a ring. Then $\{0\}$ and R are ideals of R. The ideal $\{0\}$ is called the zero ideal and sometimes it is written as 0.

Any ideal of R different from R is said to be *proper*.

Example 1.2.10. The ideals of the ring $\mathbb{Z}$ are of the form $n\mathbb{Z}$ for some integer $n \geq 0$.

Example 1.2.11. Let $\{I_j\}_{j \in J}$ be a non-empty family of ideals of a ring R. Then $\cap_{j \in J} I_j$ is an ideal of R.

If S is a subset of a ring R, then the ideal of R generated by S is the intersection of all ideal I of R containing S. If $\{I_j\}_{j\in J}$ is a non-empty family of ideals of a ring R, then its sum $\sum_{j\in J} I_j$ is the ideal of R generated by $\cup_{j\in J} I_j$.

Let I be an ideal of a ring R. The quotient ring of R by the ideal I is the additive quotient group R/I with the multiplication of left cosets defined by

$$(a+I)(b+I) = (ab)+I$$

for $a,b \in R$. Since I is an ideal of R, the multiplication is well-defined. It follows that R/I with the addition and the multiplication of left cosets is a ring.

Let R and S be rings. A map $f\colon R \to S$ is said to be a *homomorphism of rings* if $f(a+b) = f(a)+f(b)$ and $f(ab) = f(a)f(b)$ for all $a,b \in R$ and $f(1) = 1$. The image $\mathrm{im}(f)$ of f is a subring of S and the kernel $\ker(f) = \{r \in R : f(r) = 0\}$ of f is an ideal of R.

Example 1.2.12. Let R be a ring. The map $f\colon \mathbb{Z} \to R$ defined by

$$f(z) = \begin{cases} 1+\cdots+1 \ (z \text{ times}) & \text{if } z > 0, \\ 0 & \text{if } z = 0, \\ (-1)+\cdots+(-1) \ (-z \text{ times}) & \text{if } z < 0, \end{cases}$$

is a homomorphism of rings. There exists a unique non-negative integer n such that $\ker(f) = n\mathbb{Z}$. This integer n is said to be the *characteristic* of R.

Example 1.2.13. Let R be a ring and let S be a subring of R. The inclusion $\iota\colon S \to R$, $a \mapsto a$, is an injective homomorphism of rings. Let I be an ideal of R. The natural map $\pi\colon R \to R/I$, $a \mapsto a+I$, is a surjective homomorphism of rings with kernel equal to I.

An *isomorphism* of rings is a bijective homomorphism. Two rings R,S are isomorphic if there exists an isomorphism $f\colon R \to S$. The notation $R \cong S$ will mean that R and S are isomorphic. An *automorphism* of a ring R is an isomorphism from R to itself.

Theorem 1.2.14 (First isomorphism theorem). *For any homomorphism of rings $f\colon R \to S$ there exists a unique isomorphism $\tilde{f}\colon R/\ker(f) \to \mathrm{im}(f)$ such that the diagram*

$$\begin{array}{ccc} R & \xrightarrow{\ f\ } & S \\ {\scriptstyle\pi}\downarrow & & \uparrow{\scriptstyle\iota} \\ R/\ker(f) & \xrightarrow{\ \tilde{f}\ } & \mathrm{im}(f) \end{array}$$

is commutative, that is $f = \iota\tilde{f}\pi$, where ι is the inclusion mapping and π is the natural homomorphism. □

Theorem 1.2.15 (Second isomorphism theorem). *Let R be a ring, S a subring and I an ideal of R. Then $I \cap S$ is an ideal of S and*

$$S/(I \cap S) \cong (S+I)/I. \quad \square$$

Theorem 1.2.16 (Third isomorphism theorem). *Let R be a ring and I an ideal of R. Then there is a natural bijection between the subrings (respectively ideals) of R containing I and the subrings (respectively ideals) of R/I. Furthermore, if J is an ideal of R containing I, then*

$$(R/I)/(J/I) \cong R/J. \quad \square$$

Let $(A, \leq)$ be a *partially order set*, this means that A is a set together with a reflexive, transitive and anti-symmetric binary relation R on $A \times A$, where $a \leq b$ if and only if $(a,b) \in R$. Recall that the relation is reflexive if $a \leq a$ for all $a \in A$, the relation is transitive if $a \leq b$ and $b \leq c$ imply that $a \leq c$ and the relation is anti-symmetric if $a \leq b$ and $b \leq a$ imply $a = b$.

The elements $a, b \in A$ are said to be *comparable* if $a \leq b$ or $b \leq a$. An element $a \in A$ is said to be *maximal* if $c \leq a$ for all $c \in A$ that is comparable with a. An *upper bound* for a non-empty subset $B \subseteq A$ is an element $d \in A$ such that $b \leq d$ for all $b \in B$. A *chain* in A is a subset B such that every pair of elements of B are comparable. *Zorn's lemma* states the following property:

If A is a non-empty partially ordered set such that every chain in A has an upper bound in A, then A contains a maximal element.

1.3 Modules

Let R be a ring. A (left) R-module is an abelian group M with a multiplication by elements of R

$$R \times M \to M, \quad (a,m) \mapsto am,$$

satisfying the following properties:

1) $(a+b)m = am + bm$ for all $m \in M$ and $a, b \in R$.
2) $a(m_1 + m_2) = am_1 + am_2$ for all $m_1, m_2 \in M$ and $a \in R$.
3) $(ab)m = a(bm)$ for all $m \in M$ and $a, b \in R$.
4) $1m = m$ for all $m \in M$.

Right R-modules are defined similarly.

Convention 1.3.1. From now on, a module will mean a left module, unless otherwise specified.

Example 1.3.2. Let K be a field. The K-modules are the K-vector spaces.

Example 1.3.3. The $\mathbb{Z}$-modules are the abelian groups.

Let M be an R-module. A *submodule* of M is a subgroup N of M such that $am \in N$ for all $m \in N$ and $a \in R$. Let N be a submodule of M. The *quotient module*

of M by the submodule N is the quotient additive group M/N with the multiplication by elements of R defined by

$$a(m+N) = am+N$$

for all $m \in M$ and $a \in R$. This multiplication is well-defined and, moreover, M/N is an R-module. If $(m_i)_{i\in I}$ is a family of elements of M, the submodule generated by this family is $\sum_{i\in I} Rm_i$.

Let M and N be R-modules. A map $f\colon M \to N$ is said to be a *homomorphism* of R-modules if

$$f(a_1m_1 + a_2m_2) = a_1f(m_1) + a_2f(m_2)$$

for all $m_1, m_2 \in M$ and $a_1, a_2 \in R$. If f is injective, surjective or bijective, we say that f is a monomorphism, an epimorphism or an isomorphism, respectively. If $M = N$, we say that f is a endomorphism of M. A bijective endomorphism is an automorphism. We say that M and N are isomorphic if there exists an isomorphism from M to N. The notation $M \cong N$ will mean that M and N are isomorphic.

Let $f\colon M \to N$ be a homomorphism of R-modules. Then $\mathrm{im}(f)$ is a submodule of N and $\ker(f) = \{m \in M : f(m) = 0\}$ is a submodule of M.

Theorem 1.3.4 (First isomorphism theorem). *Let R be a ring. For any homomorphism of R-modules $f\colon M \to N$ there exists a unique R-modules isomorphism $\tilde{f}\colon M/\ker(f) \to \mathrm{im}(f)$ such that the diagram*

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & N \\ {\scriptstyle\pi}\downarrow & & \uparrow{\scriptstyle\iota} \\ M/\ker(f) & \xrightarrow{\quad \tilde{f} \quad} & \mathrm{im}(f) \end{array}$$

is commutative, that is $f = \iota\tilde{f}\pi$, where ι is the inclusion mapping and π is the natural homomorphism. □

Theorem 1.3.5 (Second isomorphism theorem). *Let R be a ring. Let M be an R-module. If N and L are submodules of M, then*

$$N/(N \cap L) \cong (N+L)/L. \quad \square$$

Theorem 1.3.6 (Third isomorphism theorem). *Let R be a ring. Given an R-module M and a submodule N of M, there is a natural bijection between the submodules of M containing N and the submodules of M/N. Furthermore, if H is a submodule of M containing N, then*

$$(M/N)/(H/N) \cong M/H. \quad \square$$

1.4 Free groups and free monoids

Let F be a group and X a set. It is said that F is a *free group* on X if there exists a function $\sigma\colon X \to F$ such that for every function $\alpha\colon X \to G$, where G is a group, there exists a unique group homomorphism $f\colon F \to G$ such that $f\sigma = \alpha$. Note that, in this case, σ is injective and F also is a free group on $\sigma(X)$ with respect the inclusion map $\sigma(X) \to F$. A group is free if it is free on some set.

Theorem 1.4.1. *Let X be a set. Then there exists a free group F on X.* □

Proposition 1.4.2. *Let G be a group and X a subset of G. If every element $g \in G$ can be written uniquely in the form*

$$g = x_1^{n_1} \cdots x_s^{n_s}, \tag{1.1}$$

lnormalform

for some $s \geq 0$, where $x_1, \ldots, x_s \in X$, $n_1, \ldots, n_s \in \mathbb{Z} \setminus \{0\}$ and $x_i \neq x_{i+1}$ for all $1 \leq i < s$, then G is free on X. □

If G is a free group on a subset $X \subseteq G$, then we say that X is a *basis* of G, and the right-hand side of the equality (1.1) is called the *normal form* of $g \in G$ with respect the basis X. All the bases of G have the same cardinal, which is then called the *rank* of the free group G.

A *free presentation* of a group G is a surjective homomorphism $\pi\colon F \to G$, where F is a free group. The kernel of π is the subgroup of the *relators* of the presentation. Let Y be a basis of F and let S be a subset of $\ker(\pi)$ such that $\ker(\pi)$ is the smallest normal subgroup of F containing S, i.e. $\ker(\pi)$ is the intersection of all normal subgroups containing S. Every $s \in S$ has a normal form $w(s)$ with respect to Y. Then we write

$$G = \mathrm{gr}(Y : w(s) = 1 \text{ for all } s \in S)$$

and say that the right-hand side of this equality is a presentation of G with set of generators Y and defining relations $w(s) = 1$ for all $s \in S$. If $Y = \{y_1, \ldots, y_n\}$, then we also write

$$G = \mathrm{gr}(y_1, \ldots, y_n : w(s) = 1 \text{ for all } s \in S).$$

If, for example, $y_1 y_2 y_1^{-1} y_2^{-1} \in S$, then the defining relation $y_1 y_2 y_1^{-1} y_2^{-1} = 1$ can also be written in the form $y_1 y_2 = y_2 y_1$.

Example 1.4.3. Let C_n be a cyclic group of order n. Then $C_n = \mathrm{gr}(x : x^n = 1)$ is a presentation of C_n.

prop:presentation

Proposition 1.4.4. *Let $G = \mathrm{gr}(Y : w(s) = 1 \text{ for all } s \in S)$. Let H be a group and let $\alpha\colon Y \to H$ be a function such that for every $s \in S$, if $w(s) = y_1^{n_1} \cdots y_k^{n_k}$, then $\alpha(y_1)^{n_1} \cdots \alpha(y_k)^{n_k} = 1$. Then there exists a unique group homomorphism $f\colon G \to H$ such that $f(y) = \alpha(y)$ for all $y \in Y$.* □

A *homomorphism of monoids* is a map $f\colon M_1 \to M_2$, where M_1 and M_2 are monoids, and such that $f(xy) = f(x)f(y)$ for all $x, y \in M_1$ and $f(1) = 1$. A bijective homomorphism is called an *isomorphism*. Two monoids M_1 and M_2 are isomorphic if there exists an isomorphism $f\colon M_1 \to M_2$. The notation $M_1 \cong M_2$ will mean that M_1 and M_2 are isomorphic.

Let X be a set. We denote by $FM(X)$ the set of all the words $x_1 \cdots x_n$ with letters $x_1, \ldots, x_n \in X$. The empty word is denoted by 1. We define an operation on $FM(X)$ by the rule

$$(x_1 \cdots x_n) \cdot (y_1 \cdots y_m) = x_1 \cdots x_n y_1 \cdots y_m$$

for all $x_1, \ldots, x_n, y_1, \ldots, y_m \in X$. It is clear that $FM(X)$ with this operation is a monoid. This is the *free monoid* on X.

Proposition 1.4.5. *Let X be a set. Let M be a monoid and let $\alpha\colon X \to M$ be a function. Then there exists a unique homomorphism of monoids $f\colon FM(X) \to M$ such that $f(x) = \alpha(x)$ for all $x \in X$.* □

Let M be a monoid. A congruence on M is an equivalence relation $\sim$ on M such that, for $a, b \in M$,

$$a \sim b \Rightarrow \forall c, d \in M, cad \sim cbd.$$

Let $\bar{a} = \{b \in M : b \sim a\}$, where $\sim$ is as congruence on the monoid M. We define an operation on $M/\!\sim = \{\bar{a} : a \in M\}$ by the rule $\bar{a} \cdot \bar{b} = \overline{ab}$ for all $a, b \in M$. It is easy to check that this operation is well-defined and that $M/\!\sim$ with this operation is a monoid. The natural map $\pi\colon M \to M/\!\sim$, $a \mapsto \bar{a}$, is a surjective homomorphism of monoids.

Example 1.4.6. Let $f\colon M_1 \to M_2$ be a homomorphism of monoids. Let ρ be the binary relation on M_1 defined by $a \rho b$ if and only if $f(a) = f(b)$. Then ρ is a congruence on M_1 and the map $\tilde{f}\colon M_1/\rho \to M_2$, defined by $\tilde{f}(\bar{a}) = f(a)$ for all $a \in M_1$, is an injective homomorphism of monoids.

Example 1.4.7. Let X be a set. Let $X' = \{x' : x \in X\}$ be a copy of X and let Y be the disjoint union of X and X'. Let $G = \mathrm{gr}(X)$ be the free group on X. Let $\alpha\colon Y \to G$ be the function defined by $\alpha(x) = x$ and $\alpha(x') = x^{-1}$ for all $x \in X$. Let $f\colon FM(Y) \to G$ be the unique homomorphism of monoids such that $f(y) = \alpha(y)$ for all $y \in Y$. Let $\sim$ be the congruence on $FM(Y)$ such that $a \sim b$ if and only if $f(a) = f(b)$. Then the homomorphism $\tilde{f}\colon FM(Y)/\!\sim \to G$, defined by $\tilde{f}(\bar{a}) = f(a)$ for all $a \in FM(Y)$, is an isomorphism of groups. Note that $\sim$ is the smallest congruence on $FM(Y)$ such that $xx' \sim 1$ and $x'x \sim 1$ for all $x \in X$.

Let X be a set. Let $W_1 = (w_i)_{i \in I}$ and $W_2 = (u_i)_{i \in I}$ be two families of elements in $FM(X)$. The monoid M presented with set of generators X and set of relations $\{(w_i, u_i) : i \in I\}$ is $FM(X)/\!\sim$, where $\sim$ is the smallest congruence on $FM(X)$ such that $w_i \sim u_i$ for all $i \in I$. We will denote M by

$$M = \langle X : w_i = u_i \text{ for all } i \in I \rangle.$$

1.5 Free modules and abelian groups

In this section we use the additive notation for abelian groups.

Let R be a ring. An R-module F is *free* if there exists a family $(x_i)_{i\in I}$ of elements of F such that for every element $x \in F$ there exist unique elements $r_i \in R$, only finitely many of the r_i are nonzero, satisfying

$$x = \sum_{i\in I} r_i x_i.$$

In this case, we say that $(x_i)_{i\in I}$ is a *basis* of the free module F.

Example 1.5.1. Let R be a ring. Let I be a set. For every $i \in I$, let $R_i = R$. Let $M = \prod_{i\in I} R_i$ be the Cartesian product of the R_i. Then M with the sum and the multiplication by elements of R defined componentwise is an R-module. Let $e_i = (\delta_{ij})_{j\in I} \in M$, where

$$\delta_{ij} = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Then the submodule

$$F = \bigoplus_{i\in I} R_i = \sum_{i\in I} Re_i$$

of M is a free R-module and $(e_i)_{i\in I}$ is a basis of F. F is the *direct sum* of the R_i.

Proposition 1.5.2. *Every R-module is an homomorphic image of a free R-module.* □

A *free abelian group* is a free $\mathbb{Z}$-module. An abelian group G is *torsion-free* if G has no nonzero elements of finite order.

Proposition 1.5.3. *A finitely generated abelian group G is torsion-free if and only if it is a free abelian group.* □

Theorem 1.5.4. *Let G be a finitely generated abelian group. Then there exist non-negative integers n, m and integers $d_i > 1$ for $1 \le i \le m$ such that*

$$G \cong \mathbb{Z}^n \oplus \bigoplus_{i=1}^{m} \mathbb{Z}/d_i\mathbb{Z}$$

and $d_{j+1} \| d_j$ for all $1 \le j < m$. Furthermore, the integers n, m and d_i are unique satisfying these conditions. the integer n is called the rank *of G, and $d_1, \ldots, d_m$ are the* invarian factors *of G.* □

1.6 Exercises

1.6.1. Prove all the results stated in this chapter.

1.7 Notes

The material in this chapter is basic and standard, see for example [33].

Chapter 2
The Jacobson radical

radical

2.1 Primitive rings

Let M be an R-module and S a subset of M. The *annihilator* of S is

$$\mathrm{Ann}_R(S) = \{a \in R : as = 0 \text{ for all } s \in S\}.$$

If $S = \{m\}$, then we will write $\mathrm{Ann}_R(m) = \mathrm{Ann}_R(\{m\})$.

It is an exercise to check that $\mathrm{Ann}_R(S)$ is a left ideal of R. Furthermore, if S is a submodule of M, then $\mathrm{Ann}_R(S)$ is an ideal of R.

Let T be a subset of a right R-module, then the annihilator of T is

$$\mathrm{Ann}^r_R(T) = \{a \in R : ta = 0 \text{ for all } t \in T\}.$$

Definition 2.1.1. A *faithful* R-module is an R-module M such that $\mathrm{Ann}_R(M) = \{0\}$.

Similarly one defines faithful right R-modules.

Example 2.1.2. Let K be a field and $R = M_2(K)$. Then

$$I = \begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in K \right\}$$

is a right ideal of R which is not a left ideal. It is an exercise to show that

$$\mathrm{Ann}_R(I) = \begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix} \quad \text{and} \quad \mathrm{Ann}^r_R(I) = \{0\}.$$

Hence I is a faithful right R-module.

Lema 1.1.1

Lemma 2.1.3. *Let M be an R-module. Then M is a faithful $(R/\mathrm{Ann}_R(M))$-module.*

Proof. We know that $\mathrm{Ann}_R(M)$ is an ideal of R, and thus $R/\mathrm{Ann}_R(M)$ is a ring. We define on M the multiplication by elements of $R/\mathrm{Ann}_R(M)$ by

$$(a+\mathrm{Ann}_R(M))m = am$$

for all $m \in M$ and $a \in R$. This multiplication is well-defined. Let $m \in M$ and $a, a' \in R$ be such that $a+\mathrm{Ann}_R(M) = a'+\mathrm{Ann}_R(M)$. Since $a-a' \in \mathrm{Ann}_R(M)$, $(a-a')m = 0$, that is $am = a'm'$. A straightforward calculation shows that M with the addition and this multiplication is a faithful $R/\mathrm{Ann}_R(M)$-module. □

Definition 2.1.4. A module M is *simple* if $M \neq \{0\}$ and $\{0\}$ and M are the only submodules of M.

Lema 1.1.2

Lemma 2.1.5. *An R-module M is simple if and only if there exists a maximal left ideal I of R such that $M \cong R/I$.*

Proof. Suppose first that M is simple. Let $m \in M \setminus \{0\}$. Then $M = Rm$. Let $f : R \to M$ be the map defined by $f(a) = am$. It is clear that f is an epimorphism of R-modules. By the first isomorphism theorem, $M \cong R/\ker(f)$. Furthermore, if J is a left ideal of R such that $\ker(f) \subseteq J \subseteq R$, then $J/\ker(f)$ is isomorphic to $\{0\}$ or M. Hence J is equal to $\ker(f)$ or R, and thus $\ker(f)$ is a maximal left ideal of R.

The converse is clear. □

Note that Zorn's lemma implies that every unitary ring has simple modules, see Exercise 2.6.1.

Definition 2.1.6. Let $(M_i)_{i\in I}$ be a family of submodules of an R-module M. We say that the sum $\sum_{i\in I} M_i$ is a *direct sum* if for every $j \in I$, $M_i \cap \sum_{i\in I\setminus\{j\}} M_i = \{0\}$. We will write $\bigoplus_{i\in I} M_i$ to denote the direct sum of the submodules M_i of M.

Definition 2.1.7. An R-module is *semisimple* if it is the sum of its simple submodules.

prop:semisimple

Proposition 2.1.8. *Let M be a semisimple module. Let N be a submodule of M. Then*

1) *There exists a submodule L of M such that $M = N \oplus L$,*
2) *N and M/N are semisimple,*
3) *M is a direct sum of simple submodules.*

Proof. Let $I = \{S : S \text{ is a simple submodule of } M\}$. Let

$$C = \{J : J \subseteq I, \text{ such that } N + \sum_{S\in J} S \text{ is a direct sum }\}.$$

Note that $\emptyset \in C$. Then, applying Zorn's lemma to C ordered by the inclusion, we get that there exists a maximal element $J \in C$. Let

$$M_1 = N \oplus \bigoplus_{S \in J} S.$$

We shall see that $M_1 = M$. Suppose that $M_1 \neq M$. Then there exists a simple submodule $S' \in I$ such that $S' \nsubseteq M_1$. Since S' is simple, we have that $S' \cap M_1 = \{0\}$ and thus $J \cup \{S'\} \in C$, in contradiction with the maximality of J. Hence $M_1 = M$ and (1) follows. Note that for $N = \{0\}$, we have that $M = \bigoplus_{S \in J} S$ is a dirct sum of simple submodules, and this proves (3).

Let $I' = \{S : S \text{ is a simple submodule of } N\}$. $N_1 = \sum_{S \in I'} S$. By (1), there exists a submodule L of M such that $M = N_1 \oplus L$. Hence $N = N_1 \oplus (N \cap L)$. Suppose that $N_1 \neq N$. Let $a \in N \cap L$. By Zorn's lemma there exists a submodule N_2 of $N \cap L$ maximal with the property that $a \notin N_2$. By (1), there exists a submodule L_1 of M such that $M = N_2 \oplus L_1$. Hence $N \cap L = N_2 \oplus (N \cap L \cap L_1)$. Since $N \cap L \cap L_1$ is not simple, there exist a nonzero proper submodule N_3 of $N \cap L \cap L_1$. Then, by the maximality of N_2, $a \in N_2 \oplus N_3$. By (1) there exists a submodule L_2 of M such that $M = N_3 \oplus L_2$. Hence $N \cap L \cap L_1 = N_3 \oplus (N \cap L \cap L_1 \cap L_2)$. By the maximality of N_2, $a \in N_2 \oplus (N \cap L \cap L_1 \cap L_2)$. Hence there exist $a_1, a_1' \in N_2$, $a_2 \in N_3$ and $a_3 \in N \cap L \cap L_1 \cap L_2$ such that

$$a = a_1 + a_2 = a_1' + a_3.$$

Since $N \cap L = N_2 \oplus N_3 \oplus (N \cap L \cap L_1 \cap L_2)$, we have that $a_2 = a_3 = 0$ and $a = a_1 = a_1' \in N_2$, a contradiction. Therefore $N_1 = N$ is semisimple. Since $M = N \oplus L$, we have that $M/N \cong L$ also is semisimple, and this proves (2). □

Convention 2.1.9. From now on, a ring will mean a non-zero ring, unless otherwise specified.

Definition 2.1.10. A ring R is *left primitive* if there exists a faithful simple R-module.

Similarly one defines *right primitive* rings.

Definition 2.1.11. A ring R is *simple* if $\{0\}$ and R are the only ideals of R.

Note that every simple ring is (left and right) primitive.

Example 2.1.12. Let K be a field. Then $M_n(K)$ is a simple ring for all positive integer n. Furthermore,

$$\begin{pmatrix} K & K & \dots & K \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

is a minimal right ideal of $M_n(K)$ and thus it is a faithful simple right $M_n(K)$-module. Similarly

$$\begin{pmatrix} K & 0 & \dots & 0 \\ K & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ K & 0 & \dots & 0 \end{pmatrix}$$

is a minimal left ideal of $M_n(K)$ and thus it is a faithful simple $M_n(K)$-module.

Prop1.2.1 **Proposition 2.1.13.** *Let R be a commutative ring. Then the following statements are equivalent:*

1) *R is primitive.*
2) *R is simple.*
3) *R is a field.*

Proof. The implications 3) $\implies$ 2) $\implies$ 1) are trivial. Let us prove that 1) $\implies$ 3). Suppose that R is primitive. Let V be a faithful simple R-module. By Lemma 2.1.5, $V \cong R/I$ for some maximal ideal I of R. Now $\{0\} = \mathrm{Ann}_R(V) = \mathrm{Ann}_R(R/I) = I$, and thus $\{0\}$ is a maximal ideal of R. If $r \in R \setminus \{0\}$, then $rR = Rr = R$. Hence R is a field. □

The next result shows an example of a primitive ring which is not simple.

Prop1.2.2 **Proposition 2.1.14.** *Let K be a field and V an infinite dimensional K-vector space. Then $\mathrm{End}_K(V)$ is a primitive ring which is not simple.*

Proof. Let $R = \mathrm{End}_K(V)$. Define on V a multiplication by elements of R by the rule $f \cdot v = f(v)$ for $f \in R$ and $v \in V$. Then V is an R-module. It is clear that for every $v \in V \setminus \{0\}$ and $w \in V$ there exists $f \in R$ such that $f(v) = w$. Hence V is a simple R-module. Clearly, $\mathrm{Ann}_R(V) = \{0\}$ and thus V is a faithful R-module. Hence R is left primitive.

Let $V^* = \mathrm{Hom}_K(V, K)$ the dual space of V. Define on V^* a multiplication by elements of R by the rule $\omega \cdot f = \omega f$ for $f \in R$ and $\omega \in V^*$. Then V^* is a faithful simple right R-module. Hence R is right primitive.

Note that $I = \{f \in R : \dim_K(\mathrm{im}(f)) < \infty\}$ is a non-zero proper ideal of R. Hence R is not simple. □

Bergman constructed in [14, 15] the first example of a right primitive ring which is not left primitive.

Prop1.2.3 **Proposition 2.1.15.** *Let R be a left primitive ring. Then $M_n(R)$ is left primitive for all positive integer n.*

Proof. Let V be a faithful simple R-module. Then $M_{n\times 1}(V)$ is a faithful simple $M_n(R)$-module. □

Note that every division ring is a simple ring. The following result gives us a method to construct division rings.

Lemma 2.1.16 (Schur). *Let V be a simple R-module. Then $\mathrm{End}_R(V)$ is a division ring.*

Proof. Let $f \in \mathrm{End}_R(V) \setminus \{0\}$. Since V is simple and $\mathrm{im}(f) \neq \{0\}$, it follows that $\mathrm{im}(f) = V$. Since $\ker(f) \neq V$, $\ker(f) = \{0\}$. Hence f is an automorphism of V and therefore $\mathrm{End}_R(V)$ is a division ring. □

The following theorem goes back to Chevalley and Jacobson.

Theorem 2.1.17 (Density theorem). *Let V be a simple R-module and $D = \mathrm{End}_R(V)$. Then V is a D-vector space. Furthermore, if $x_1, \ldots, x_n \in V$ are D-linearly independent and $y_1, \ldots, y_n \in V$, then there exists $r \in R$ such that $rx_i = y_i$ for all $i = 1, \ldots, n$.*

Proof. By Schur's lemma, D is a division ring. We define on V a multiplication by elements of D by

$$f \cdot v = f(v)$$

for $f \in D$ and $v \in V$. Then V with the addition and this multiplication is a D-vector space.

We claim that there exists $r \in R$ such that $rx_i = y_i$ for all $i = 1, \ldots, n$. We proceed by induction on n. For $n = 1$, since V is a simple R-module, $Rx_1 = V$ and thus there exists $r \in R$ such that $rx_1 = y_1$.

Suppose that $n > 1$ and that the result is true for $n-1$. By the induction hypothesis, $V^{n-1} = R(x_1, \ldots, x_{n-1})$. We shall show that there exists $r_n \in R$ such that $r_n x_n \neq 0$ and $r_n x_i = 0$ for all $i = 1, \ldots, n-1$. Suppose that there is no such an element. In this case, the map

$$\varphi \colon R(x_1, \ldots, x_{n-1}) \to V, \quad r(x_1, \ldots, x_{n-1}) \mapsto rx_n,$$

is well-defined. Since $\mathrm{Hom}_R(V^{n-1}, V) \cong (\mathrm{End}_R(V))^{n-1} = D^{n-1}$, we may identify $\mathrm{Hom}_R(V^{n-1}, V)$ with D^{n-1}, and then there exist $d_1, \ldots, d_{n-1} \in D$ such that $\varphi = (d_1, \ldots, d_{n-1})$. Thus $x_n = \varphi(x_1, \ldots, x_{n-1}) = \sum_{i=1}^{n-1} d_i x_i$, a contradiction because $x_1, \ldots, x_n$ are D-linearly independent. Hence there exists $r_n \in R$ such that $r_n x_n \neq 0$ and $r_n x_i = 0$ for all $i = 1, \ldots, n-1$. Similarly one can see that for every $j \in \{1, \ldots, n\}$, there exists $r_j \in R$ such that $r_j x_j \neq 0$ and $r_j x_i = 0$ for all $i \neq j$. Since V is a simple R-module, there exists $s_j \in R$ such that $s_j r_j x_j = y_j$. Let $r = \sum_{j=1}^{n} s_j r_j$. One checks that $rx_i = y_i$ for all $i = 1, \ldots, n$. Therefore the result follows by induction. □

Teorema 1.2.4

Theorem 2.1.18. *Let R be a left primitive ring. Then one of the following conditions holds:*

1) *$R \cong M_n(D)$ for some division ring D and some positive integer n.*
2) *There exists a division ring D such that for every positive integer m there exists a subring S_m of R and an surjective homomorphismm $\varphi_m \colon S_m \to M_m(D)$.*

Proof. Let V be a faithful simple R-module and $D = \mathrm{End}_R(V)$. By Schur's lemma, D is a division ring. Moreover, V is a D-vector space. Let $\psi \colon R \to \mathrm{End}_D(V)$, $\psi(r)(v) = rv$ for $r \in R$ and $v \in V$. It is clear that ψ is a homomorphism of rings.

Suppose that $\dim_D(V) < \infty$ and let $v_1, \ldots, v_n$ be a D-basis of V. Let $f \in \mathrm{End}_D(V)$. By the density theorem, there exists $r \in R$ such that $f(v_i) = rv_i$ for all $i = 1, \ldots, n$. Hence $\psi(r) = f$ and thus ψ is surjective. Since V is a faithful R-module, ψ is injective. Therefore $R \cong \mathrm{End}_D(V) \cong M_n(D)$.

Suppose now that V is an infinite dimensional D-vector space. Let $(v_i)_{i \geq 1}$ be a family of D-linearly independent vectors of V. Let

$$V_m = Dv_1 + \cdots + Dv_m \quad \text{and} \quad S_m = \{r \in R \mid rV_m \subseteq V_m\}.$$

It is clear that S_m is a subring of R. By the density theorem, the restriction map $\psi|_{S_m} \colon S_m \to \mathrm{End}_D(V_m)$ is surjective. Hence there exists an surjective homomorphism from S_m to $M_m(D)$ and the result follows. □

2.2 Radicals

Definition 2.2.1. Let R be a ring. The *Jacobson radical* of R is

$$J(R) = \bigcap_{V \text{ simple } R\text{−module}} \mathrm{Ann}_R(V).$$

Since each $\mathrm{Ann}_R(V)$ is an ideal of R, $J(R)$ also is an ideal of R.

Teorema 1.3.1

Theorem 2.2.2 (Jacobson). *Let R be a ring and $a \in R$. Then the following conditions are equivalent.*

(i) $a \in J(R)$.
(ii) $a \in M$, *for every maximal left ideal M of R.*
(iii) $R(1-xa) = R$ *for all* $x \in R$.
(iv) $1-xay$ *is invertible for all* $x, y \in R$.
(i^*) -(iv^*) the dual left-right conditions of $(i)-(iv)$.

Proof. $(i) \Rightarrow (ii)$. Let M be a maximal left ideal of R and suppose that $a \in J(R)$. By Lemma 2.1.5, $V = R/M$ is a simple R-module. Thus $a \in \mathrm{Ann}_R(V) \subseteq M$.

$(ii) \Rightarrow (iii)$. Let $x \in R$ and suppose that $a \in M$, for every maximal left ideal M of R. Suppose that $R(1-xa) \neq R$. By Zorn's lemma, there exists a maximal left ideal M of R such that $R(1-xa) \subseteq M$. But $a \in M$, hence $1 = 1-xa+xa \in M$, a contradiction. Therefore $R(1-xa) = R$.

$(iii) \Rightarrow (iv)$. Let $x, y \in R$ and suppose that $R(1-xa) = R$ for all $x \in R$. Suppose that $R(1-xay) \neq R$. By the implication $(i) \Rightarrow (iii)$, there exists a simple R-module V such that $ayV \neq 0$. Hence $aV \neq 0$. Let $v \in V$ such that $av \neq 0$. Since V is simple, there exists $z \in R$ such that $zav = v$. Thus $(1-za)v = 0$ and, since $v \neq 0$, we have that $R(1-za) \neq R$, a contradiction. Therefore $R(1-xay) = R$. Hence there exists $b \in R$ such that $(1-b)(1-xay) = 1$. Thus $b = bxay - xay = (bx-x)ay$ and, then we get that $R(1-b) = R$. Hence $1-b$ is invertible and $(1-b)^{-1} = 1-xay$.

$(iv) \Rightarrow (i)$. Let V be a simple R-module and suppose that $1-xay$ is invertible for all $x, y \in R$. Suppose that $aV \neq 0$. Hence as above there exist $v \in V \setminus \{0\}$ and $z \in R$ such that $(1-za)v = 0$. But $1-za$ is invertible, a contradiction, therefore $a \in \mathrm{Ann}_R(V)$ for all simple R-module V, that is $a \in J(R)$.

Since condition (iv) is left-right symmetric, the result follows. □

Definition 2.2.3. Let $(R_i)_{i \in I}$ be a non-empty family of rings. Let

$$\pi_j \colon \prod_{i \in I} R_i \to R_j \quad (j \in I),$$

be the natural maps. A subring S of $\prod_{i \in I} R_i$ is said to be a *subdirect product* of the rings R_i if for every $j \in I$ the restiction $\pi_j|_S$ of π_j to S is surjective.

Let R be a ring and $I = \{\mathrm{Ann}_R(V) : V \text{ is a simple } R\text{-module }\}$. Define

$$\begin{aligned} \varphi \colon \ R &\to \textstyle\prod_{A \in I} R/A \\ r &\mapsto (r + A)_{A \in I} \end{aligned}$$

It is clear that φ is a homomorphism of rings and $\ker(\varphi) = J(R)$. Hence, by the first isomorphism theorem there is an injective homomorphism

$$\begin{aligned} \overline{\varphi} \colon \ R/J(R) &\to \textstyle\prod_{A \in I} R/A \\ r + J(R) &\mapsto (r + A)_{A \in I} \end{aligned}$$

Thus $R/J(R) \cong \varphi(R)$, which is a subdirect product of the rings R/A, and these rings are left primitive rings by Lemma 2.1.3.

Definition 2.2.4. A ring is said to be *semiprimitive* if it is isomorphic to a subdirect product of left primitive rings.

Prop1.3.2 **Proposition 2.2.5.** *Let R be a ring. Then the following conditions are equivalent.*

(i) R is semiprimitive.
(ii) $J(R) = 0$.
(iii) R is isomorphic to a subdirect product of right primitive rings.

Proof. Since (ii) is left-right symmetric, it is enough to prove that (i) and (ii) are equivalent. Above we have seen that (ii) implies (i).

We may assume that R is a subdirect product of the rings $\{R_i\}_{i \in I}$ and that R_j is left primitive for all $j \in I$. Let V_j be a faithful simple R_j-module. Let $\pi_j \colon \prod_{i \in I} R_i \to R_j$ be the natural map. Then V_j is an R-module via π_j, that is

$$r \cdot v := \pi_j(r)v,$$

for all $v \in V_j$ and all $r \in R$. It is clear that V_j is a simple R-module. Since V_j is a faithful R_j-module, we have that $\mathrm{Ann}_R(V_j) = \ker(\pi_j|_R)$. Since R is a subring of $\prod_{i \in I} R_i$, we have that $\bigcap_{j \in I} \mathrm{Ann}_R(V_j) = \{0\}$. Since $J(R) \subseteq \bigcap_{j \in I} \mathrm{Ann}_R(V_j)$, we get that $J(R) = \{0\}$, and the result follows. □

Note that every left (or right) primitive ring is semiprimitive. A trivial consequence of Proposition 2.2.5 is the following result.

Cor1.3.3 **Corollary 2.2.6.** *If R is a ring, then $J(R/J(R)) = \{0\}$.* □

Proposition 2.2.7 (Nakayama's lemma). *Let R be a ring and $S \subseteq J(R)$. Let M be a finitely generated R-module. If $SM = M$, then $M = 0$.*

Proof. Let $m_1 = 0, m_2, \ldots, m_n \in M$ such that $M = Rm_1 + Rm_2 + \cdots + Rm_n$. We shall prove that $M = \{0\}$ by induction on n. For $n = 1$, it is clear. Suppose that $n > 1$ and the result is true for $n-1$. We know that there exist $s_1, s_2, \ldots, s_n \in S$ such that

$$m_n = s_1 m_1 + s_2 m_2 + \cdots + s_n m_n.$$

Hence $(1 - s_n)m_n = s_1 m_1 + \cdots + s_{n-1} m_{n-1}$ and, since $1 - s_n$ is invertible, we get that $m_n = (1 - s_n)^{-1} s_1 m_1 + \cdots + (1 - s_n)^{-1} s_{n-1} m_{n-1}$. Hence $M = Rm_1 + \cdots + Rm_{n-1}$ and by the induction hypothesis, $M = \{0\}$. Therefore the result follows by induction. □

Note that if a is a nilpotent element of a ring R, then there exists a positive integer n such that $a^n = 0$, and then

$$(1 - a)(1 + a + \cdots + a^{n-1}) = 1.$$

Definition 2.2.8. A *nil left (right) ideal* of a ring R is a left (right) ideal I of R such that every $a \in I$ is nilpotent.

By Theorem 2.2.2 and the above remark, it is clear that every nil left (right) ideal of R is contained in $J(R)$.

Definition 2.2.9. A left (right) ideal I of R is *nilpotent* if there exists a positive integer n such that $I^n = 0$.

Definition 2.2.10. An ideal J of R is said to be a *nilradical* if it is a nil ideal and R/J has no nonzero nilpotent ideal.

Prop1.3.4

Proposition 2.2.11. *Let R be a ring. Then there exists a unique maximal nil ideal $\mathcal{U}(R)$ of R. Furthermore, $\mathcal{U}(R)$ is a nilradical and it is called the upper nilradical of R.*

Proof. First we shall see that the sum of two nil ideals is a nil ideal. Let I_1, I_2 be nil ideals of R. Let $x \in I_1$ and $y \in I_2$. There exists a positive integer n such that $x^n = 0$. Hence $(x + y)^n \in I_2$. Thus $x + y$ is nilpotent and $I_1 + I_2$ is a nil ideal. Let $C = \{I : I \text{ is a nil ideal of } R\}$. Let $\mathcal{U}(R) = \sum_{I \in C} I$. It is clear that $\mathcal{U}(R)$ is the unique maximal nil ideal of R. Let I be a nilpotent ideal of $R/\mathcal{U}(R)$ and let $\pi : R \to R/\mathcal{U}(R)$ be the natural map. Then there exists a positive integer m such that $I^m = \{0\}$. Hence, $\pi^{-1}(I)^m = \mathcal{U}(R)$. Thus $\pi^{-1}(I)$ is a nil ideal and therefore $\pi^{-1}(I) = \mathcal{U}(R)$. Hence $I = \{0\}$, and the result follows. □

Definition 2.2.12. A ring R is said to be *prime* if for ideals I and J of R, $IJ = \{0\}$ implies that $I = \{0\}$ or $J = \{0\}$.

Note that the previous definition is equivalent to $aRb \neq \{0\}$ for all $a, b \in R \setminus \{0\}$.

Example 2.2.13. The commutative prime rings are precisely the commutative integral domains.

Prop1.3.5

Proposition 2.2.14. *Every left primitive ring is prime.*

Proof. Let V be a faithful simple R-module. Let I, J be nonzero ideals of R. We shall see that $IJ \neq \{0\}$. Since V is faithful and simple, $IV = V$ and $JV = V$. Hence $IJV = IV = V$ and thus $IJ \neq \{0\}$. □

Definition 2.2.15. An ideal P of a ring R is said to be *prime* if R/P is a prime ring.

Let $C = \{P : P \text{ prime ideal of } R\}$. Define

$$\varphi\colon R \to \prod_{P \in C} R/P$$

by $\varphi(r) = (r + P)_{P \in C}$ for all $r \in R$. It is clear thate φ is a homomorphism of rings and $\ker(\varphi) = \bigcap_{P \in C} P$. The *Baer-McCoy radical* or *(Baer's) lower nilradical* of R is

$$\mathcal{B}(R) = \bigcap_{P \in C} P.$$

$\mathcal{B}(R)$ is also called the *prime radical* of R. A ring R is said to be *semiprime* if it is isomorphic to a subdirect product of prime rings. Note that $\mathcal{B}(R) = \ker(\varphi)$. Hence, if $\mathcal{B}(R) = \{0\}$, then R is semiprime. In fact we have the following result.

Teorema 1.3.6

Theorem 2.2.16. *Let R be a ring. Then the following conditions are equivalent.*

(i) R is semiprime.
(ii) $\mathcal{B}(R) = \{0\}$.
(iii) For an ideal I of R, $I^2 = \{0\}$ implies $I = \{0\}$.

Proof. $(i) \Rightarrow (ii)$ We may assume that R is a subdirect product of prime rings $(R_i)_{i \in I}$. Let $\pi_j\colon \prod_{i \in I} R_i \to R_j$ be the natural map for all $j \in I$. Then $P_j = \ker(\pi_j|_R)$ is a prime ideal of R because $R/P_j \cong R_j$. Hence $\bigcap_{i \in I} P_i = \{0\}$. Since $\mathcal{B}(R) \subseteq \bigcap_{i \in I} P_i$, we have that $\mathcal{B}(R) = \{0\}$.

$(ii) \Rightarrow (i)$ We have seen this above.

$(ii) \Rightarrow (iii)$ Suppose that $\mathcal{B}(R) = \{0\}$. Let I be an ideal of R such that $I^2 = \{0\}$. Let P be a prime ideal of R. Since $I^2 \subseteq P$, we have that $I \subseteq P$. Hence $I \subseteq \mathcal{B}(R) = \{0\}$.

$(iii) \Rightarrow (ii)$ Let $a \in R \setminus \{0\}$. We shall prove that there exists a prime ideal P of R such that $a \notin P$ and thus $\mathcal{B}(R) = \{0\}$.

By (iii), we have that $aRa \neq 0$. We shall construct inductively a sequence $a_0 = a, a_1, \ldots, a_n, \ldots$ of nonzero elements of R such that $a_n \in a_{n-1}Ra_{n-1}$ for all positive integer n. Suppose we have constructed $a_0, a_1, \ldots, a_m$ for some $m \geq 0$. Then by (iii), $a_mRa_m \neq 0$. Let $a_{m+1} \in a_mRa_m \setminus \{0\}$. Hence we get $a_0, a_1, \ldots, a_n, \ldots$ nonzero elements in R such that $a_n \in a_{n-1}Ra_{n-1}$. Let $S = \{a_n \mid n \geq 0\}$. Let

$$C = \{I : I \text{ ideal of } R \text{ such that } I \cap S = \emptyset\}.$$

It is clear that $\{0\} \in C$ and that any chain in C (ordered by inclusion) has an upper bound in C. By Zorn's lemma, there exists a maximal element P in C. We shall see that P is prime. Let I, J be ideals of R such that $I \not\subseteq P$ and $J \not\subseteq P$. Since P is maximal in C, there exist non-negative integers m, n such that $a_m \in I + P$ and $a_n \in J + P$. We

may assume that $m \leq n$. Hence $a_n \in (I+P) \cap (J+P)$. Since $a_{n+1} \in a_n R a_n$, we have that $a_{n+1} \in (I+P)(J+P) \subseteq IJ + P$. Since $P \cap S = \emptyset$, $IJ \nsubseteq P$. Therefore the result follows. □

Prop1.3.7

Proposition 2.2.17. *Let R be a ring. Then $\mathcal{B}(R)$ is a nilradical of R, and if I is a nilradical of R, then $\mathcal{B}(R) \subseteq I$.*

Proof. Let I be a nilradical of R. By definition R/I has no nonzero nilpotent ideals. By Theorem 2.2.16, R/I is semiprime. Hence I is the intersection of the prime ideals of R containing I, and thus $\mathcal{B}(R) \subseteq I$. By Proposition 2.2.11, R has nilradicals, and thus $\mathcal{B}(R)$ is a nil ideal of R. By Theorem 2.2.16, $R/\mathcal{B}(R)$ has no nonzero nilpotent ideals. Hence $\mathcal{B}(R)$ is a nilradical of R, and the result follows. □

Let R be a ring. Let $\mathcal{N} = \{I : I \text{ nilpotent ideal of } R\}$. The *nilpotent radical* of R is $N(R) = \sum_{I \in \mathcal{N}} I$. It is clear that $N(R) \subseteq \mathcal{B}(R)$, but in general this inclusion is not an equality, that is, in general, $N(R)$ is not a nilradical of R. Furthermore, $N(R)$, in general, is not a nilpotent ideal.

Note that if R is a commutative ring, then

$$N(R) = \mathcal{B}(R) = \mathcal{U}(R) = \{x \in R \mid x \text{ is nilpotent}\}$$

is the unique nilradical of R, and thus, in this case, $N(R)$ is said to be the nilradical of R.

2.3 Artinian rings

An R-module M is said to be *Artinian* if every descending chain of submodules of M is stationary, that is, if

$$M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \ldots$$

is a descending chain of submodules of M, there exists a positive integer n_0 such that $M_n = M_{n_0}$ for all $n \geq n_0$.

The proof of the next result is easy and we left it to the reader.

Prop1.4.1

Proposition 2.3.1. *An R-module M is Artinian if and only if every non-empty set of submodules of M has a minimal element.* □

A ring R is said to be *right Artinian* if it is Artinian as right R-module. Similarly one defines left Artinian ring. We say that the ring R is Artinian if it is right and left Artinian.

Example 2.3.2. Let R be a ring. Suppose that R has a subring which is a division ring D. Then every R-module also is a D-vector space. Hence every R-module finitely dimensional as D-vector space is an Artinian R-module. For example, if D is a division ring, then $M_n(D)$ is an Artinian ring for all positive integer n.

Note that if e is an idempotent element in a ring R, then eRe is a ring (possibly zero) and e is its unit-element.

Lema 1.4.2

Lemma 2.3.3. *Let R be a ring and $e \in R$ idempotent. Then $eRe \cong \mathrm{End}_R(eR)$.*

Proof. Consider the map $\varphi\colon eRe \to \mathrm{End}_R(eR)$ defined by $\varphi(ere)(es) = eres$ for all $r, s \in R$. It is easy to check that φ is an injective homomorphism of rings. Let $f \in \mathrm{End}_R(eR)$. Note that $f(e) = f(e^2) = f(e)e \in eRe$. One can verify that $\varphi(f(e)) = f$, and thus φ is an isomorphism and the result follows. □

Teorema 1.4.3

Theorem 2.3.4 (Artin–Wedderburn). *The following conditions are equivalent.*

(i) R is an Artinian simple ring.
(ii) R is a left Artinian simple ring.
(iii) R is a left Artinian and a left primitive ring.
(iv) $R \cong M_n(D)$, for some division ring D and some positive integer n.
(i^*) -(iv^*) the left-right dual condition of (i)-(iv).

Furthermore, in (iv), n is uniquely determined by R and D is determined up to isomorphism.

Proof. It is clear that $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$.

$(iii) \Rightarrow (iv)$. Suppose that R is a left Artinian and left primitive ring. Let V be a faithful simple R-module and $D = \mathrm{End}_R(V)$. By Schur's lemma, D is a division ring. We know that V is a D-vector space. Suppose that V is an infinite dimensional D-vector space. Let $(v_n)_{n\geq 1}$ be a family of D-linearly independent vectors of V. Let $A_m = \mathrm{Ann}_R(\{v_1, \ldots, v_m\})$. It is clear that

$$A_1 \supseteq A_2 \supseteq \cdots \supseteq A_m \supseteq \ldots$$

By the Density theorem these inclusions are strict, but this is a contradiction because R is left Artinian. Hence V is finite dimensional D-vector space. Let $\dim_D(V) = n$. By the proof of Theorem 2.1.18, $R \cong M_n(D)$.

$(iv) \Rightarrow (i)$ It is easy to check that $M_n(D)$ is an Artinian simple ring.

Let D, D' be division rings and n, m positive integers such that $M_n(D) \cong M_m(D')$. We shall prove that $n = m$ and $D \cong D'$. Let $\varphi\colon M_n(D) \to M_m(D')$ be an isomorphism. Let $E_{1,1} \in M_n(D)$ be the matrix with 1 in the $(1,1)$-entry and zero in the other entries. Note that $E_{1,1}$ is an idempotent of $M_n(D)$, and it is easy to see that $E_{1,1}M_n(D)$ is a simple right $M_n(D)$-module and $\dim_D(E_{1,1}M_n(D)) = n$. By Lemma 2.3.3, we have:

$$\begin{aligned} D \cong & E_{1,1}M_n(D)E_{1,1} \cong \mathrm{End}_{M_n(D)}(E_{11}M_n(D)) \\ & \cong \mathrm{End}_{M_m(D')}(\varphi(E_{1,1})M_m(D')) \cong \varphi(E_{1,1})M_m(D')\varphi(E_{1,1}). \end{aligned}$$

Since $\varphi(E_{1,1})$ is an idempotent, using an automorphism of $M_m(D')$, we may assume that

$$\varphi(E_{1,1}) = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \in M_m(D'),$$

where I_r is the $r \times r$ identity matrix. Since $\varphi(E_{1,1})M_m(D')$ is a simple $M_m(D')$-module, it is clear that $r = 1$. Hence $D \cong \varphi(E_{1,1})M_m(D')\varphi(E_{1,1}) \cong D'$. Furthermore,

$$n = \dim_D(E_{1,1}M_n(D)) = \dim_{D'}(\varphi(E_{1,1})M_m(D')) = m,$$

and the result follows. □

Teorema 1.4.4

Theorem 2.3.5. *Let R be a right Artinian ring. Then $J(R)$ is nilpotent.*

Proof. Let $J = J(R)$ and consider the chain

$$J \supseteq J^2 \supseteq \cdots \supseteq J^n \supseteq \ldots$$

Since R is right Artinian, there exists a positive integer n such that $J^n = J^m$ for all $m \geq n$. Let $I = J^n$. Suppose that $I \neq 0$. Since R is right Artinian and $I = I^2$, there exists a right ideal M which is minimal with the property $MI \neq 0$. Since $MI^2 = MI \neq 0$, we have that $MI = M$. It is clear that M is generated by one element, hence, by Nakayasma's lemma, $M = \{0\}$, in contradiction with $MI \neq 0$. Therefore, $I = J^n = 0$ and the result follows. □

Theorem 2.3.6 (Wedderburn-Artin theorem). *Let R be a ring. Then R is right (left) Artinian and semiprimitive if and only if*

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r),$$

where the D_i are division rings. Furthermore, r is determined by R, and the pairs (D_i, n_i) are determined up to a permutation.

Proof. Let $C = \{A : A = Ann_R^r(S)$ where S is a simple right R-module $\}$. Suppose that R is right Artinian and semiprimitive. By Theorem 2.2.2, $J(R) = \bigcap_{A \in C} A = 0$. Let $\mathcal{A} = \{\bigcap_{A \in \mathcal{F}} A : \mathcal{F}$ is a finite subset of $C\}$. Since R is right Artinian, there exist $A_1, \ldots, A_r \in C$ such that $\bigcap_{i=1}^r A_i$ is a minimal element of $\mathcal{A}$. Suppose that $\bigcap_{i=1}^r A_i \neq 0$ and let $a \in \bigcap_{i=1}^r A_i \setminus \{0\}$. Since $J(R) = 0$, there exists $A \in C$ such that $a \notin A$. Hence $A \cap (\bigcap_{i=1}^r A_i)$ is strictly contained in $\bigcap_{i=1}^r A_i$, in contradiction with the minimality of $\bigcap_{i=1}^r A_i$. Hence $\bigcap_{i=1}^r A_i = 0$.

We may assume that for every $j \in \{1, \ldots, r\}$

$$\bigcap_{1 \leq i \leq r,\, i \neq j} A_i \neq \{0\}.$$

Let $\varphi\colon R \to \prod_{i=1}^r R/A_i$ be the map defined by $\varphi(x) = (x + A_1, \ldots, x + A_r)$. It is clear that φ is an injective homomorphism of rings. We shall see that it is surjective. By Theorem 2.3.4, R/A_i is simple. Hence A_i is a maximal ideal of R. Thus $A_1 + \bigcap_{i=2}^r A_i = R$. Hence there exist $a_1 \in A_1$ and $b_1 \in \bigcap_{i=2}^r A_i$ such that $1 = a_1 + b_1$, and thus $\varphi(1 - a_1) = (1 + A_1, A_2, \ldots, A_r)$. Similarly one can see that there exists $a_j \in A_j$ such that $\varphi(1 - a_j) = (A_1, \ldots, A_{j-1}, 1 + A_j, A_{j+1}, \ldots, A_r)$. Now it is easy to see that φ is surjective. Therefore φ is an isomorphism. By Theorem 2.3.4,

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r),$$

where the D_i are division rings.

The converse is a consequence of the fact that every finite product of right Artinian rings is right Artinian. Note that r is the number of minimal ideals of R and that these minimal ideals are rings isomorphic to $M_{n_i}(D_i)$. By Theorem 2.3.4, the result follows. □

2.4 Noetherian rings

An R-module M is said to be *Noetherian* if every ascending chain of submodules of M is stationary, that is, if

$$M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq \ldots$$

is an ascending chain of submodules of M, there exists a positive integer n_0 such that $M_n = M_{n_0}$ for all $n \geq n_0$.

The proof of the next result is easy and we left it to the reader.

prop: Noetherian1

Proposition 2.4.1. *An R-module M is Noetherian if and only if every non-empty set of submodules of M has a maximal element.* □

prop:Noetherian2

Proposition 2.4.2. *An R-module M is Noetherian if and only if every submodule of M is finitely generated.*

Proof. Suppose that M is a Noetherian R-module. Suppose that there exists a submodule N of M which is not finitely generated. By induction we shall construct a sequence $x_1, x_2, \ldots, x_n, \cdots \in N$ such that $x_{n+1} \notin Rx_1 + \cdots + Rx_n$ for all positive integer n. Let $x_1 \in N$. Since $N \neq Rx_1$, there exists $x_2 \in N \setminus Rx_1$. Suppose that $n \geq 2$ and $x_1, \ldots, x_n \in N$ satisfy that

$$x_{i+1} \notin Rx_1 + \cdots + Rx_i$$

for all $1 \leq i < n$. Since $N \neq Rx_1 + \cdots + Rx_n$, there exists $x_{n+1} \in N \setminus (Rx_1 + \cdots + Rx_n)$. Hence by induction there exist $x_1, x_2, \ldots, x_n, \cdots \in N$ such that $x_{n+1} \notin Rx_1 + \cdots + Rx_n$ for all positive integer n. Let $N_k = \sum_{i=1}^{k} Rx_i$. We have that the ascending chain

$$N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n \subseteq \ldots$$

is not stationary, a contradiction. Hence every submodule of M is finitely generated.

Conversely, suppose that every submodule of M is finitely generated. Let

$$M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq \ldots$$

be an ascending chain of submodules of M. Let $N = \bigcup_{n=1}^{\infty} M_n$. Since N is a submodule of M, there exist $x_1, \ldots, x_k \in N$ such that $N = \sum_{i=1}^{k} Rx_i$. Now there exists a positive

integer n_0 such that $x_1, \dots, x_k \in M_{n_0}$. Hence $M_n = M_{n_0} = N$ for all $n \geq n_0$. Therefore M is Noetherian. □

cor:Noetherian3

Corollary 2.4.3. *Let M be an R-module and let N be a submodule of M. Then M is Noetherian if and only if N and M/N are Noetherian.*

Proof. Suppose that M is Noetherian. By Proposition 2.4.2, every submodule of M is finitely generated. In particular, every submodule of N is finitely generated, and thus N is Noetherian. Since the submodules of M/N are of the form L/N, where L is a submodule of M containing N, it is clear that the submodules of M/N are finitely generated, and thus M/N is Noetherian.

Conversely, suppose that N and M/N are Noetherian. Let K be a submodule of M. By Proposition 2.4.2, $K \cap N$ is finitely generated and $K/(K \cap N) \cong (K+N)/N$ also is finitely generated. Let $x_1, \dots, x_s, y_1, \dots, y_t \in K$ such that

$$K \cap N = Rx_1 + \cdots + Rx_s \quad \text{and} \quad K/(K \cap N) = R(y_1 + (K \cap N)) + \cdots + R(y_t + (K \cap N)).$$

It is easy to see that

$$K = Rx_1 + \cdots + Rx_s + Ry_1 + \cdots + Ry_t.$$

By Proposition 2.4.2, it follows that M is Noetherian. □

A ring R is said to be *right Noetherian* if it is Noetherian as right R-module. Similarly one defines left Noetherian rings. We say that the ring R is Noetherian if it is right and left Noetherian.

Example 2.4.4. $\mathbb{Z}$ is a Noetherian ring.

Example 2.4.5. Every division ring is Noetherian.

thm:Noetherian4

Theorem 2.4.6. *Let R be a right Noetherian ring. Then the polynomial ring $R[X]$ is right Noetherian.*

Proof. Let I be a right ideal of $R[X]$. Let

$$J = \{a \in R : \text{ there exists } a_0 + a_1X + \cdots + a_nX^n \in I \text{ such that } a_n = a\}.$$

It is an easy exercise to check that J is a right ideal of R. Since R is right Noetherian, there exists $c_1, \dots, c_m \in J$ such that

$$J = c_1R + \cdots + c_mR.$$

Let $p_i(X) = a_{i,0} + a_{i,1}X + \cdots + a_{i,n_i}X^{n_i} \in I$ such that $a_{i,n_i} = c_i$ for all $1 \leq i \leq m$. Let n be a positive integer such that $n \geq n_i$ for all $1 \leq i \leq m$. Since R is right Noetherian, by Corollary 2.4.3, it is easy to see by induction on n that R^{n+1} is a Noetherian right R-module. Consider the epimorphism of right R-modules $\pi\colon R^{n+1} \to R + RX + \cdots + RX^n$, defined by $\pi(a_0, a_1, \dots, a_n) = a_0 + a_1X + \cdots + a_nX^n$. By Corollary 2.4.3, $R + RX + \cdots + RX^n$ is a Noetherian right R-module and $I \cap (R + RX + \cdots + RX^n)$ also

is a Noetherian right R-module. Hence there exist $g_1(X), \ldots, g_s(X) \in I \cap (R + RX + \cdots + RX^n)$ such that

$$I \cap (R + RX + \cdots + RX^n) = \sum_{j=1}^{s} g_j(X)R.$$

We shall prove by induction on the degree of $p(X) \in I$ that

$$p(X) \in \sum_{j=1}^{s} g_j(X)R[X] + \sum_{i=1}^{m} p_i(X)R[X].$$

We may assume that $\deg(p(X)) = t > n$ and that for every $q(X) \in I$ of degree $< t$,

$$q(X) \in \sum_{j=1}^{s} g_j(X)R[X] + \sum_{i=1}^{m} p_i(X)R[X].$$

Let $a_0, a_1, \ldots, a_t \in R$ shuch that $p(X) = a_0 + a_1X + \cdots + a_tX^t$. Since $a_t \in J$, there exist $r_1, \ldots, r_m \in R$ such that $a_t = \sum_{i=1}^{m} c_i r_i$. Hence

$$q(X) = p(X) - \sum_{i=1}^{m} p_i(X)X^{t-n_i} r_i \in I$$

and $\deg q(X) < t$. By the inductive hypothesis,

$$q(X) \in \sum_{j=1}^{s} g_j(X)R[X] + \sum_{i=1}^{m} p_i(X)R[X].$$

Hence

$$p(X) \in \sum_{j=1}^{s} g_j(X)R[X] + \sum_{i=1}^{m} p_i(X)R[X].$$

By induction, I is finitely generated and therefore $R[X]$ is right Noetherian. □

Theorem 2.4.7 (Hopkins' theorem). *Let R be a ring. Then R is right Artinian if and only if R is right Noetherian, $J(R)$ is nilpotent and $R/J(R)$ is a semisimple right R-module.*

Proof. Suppose that R is right Artinian. By Theorem 2.3.5, $J(R)$ is nilpotent. Since $R/J(R)$ is right Artinian and semiprimitive, by Wedderburn-Artin theorem, it is easy to see that $R/J(R)$ is a semisimple R-module. Let n be a positive integer such that $J(R)^n = 0$. Consider the descending chain

$$R \supseteq J(R) \supseteq \cdots \supseteq J(R)^n = 0.$$

Note that $J(R)^i/J(R)^{i+1}$ is a right $R/J(R)$-module for all $1 \leq i < n$. It is an exercise to prove that every direct sum of semisimple right modules is semisimple. Since

$J(R)^i/J(R)^{i+1}$ is an epimorphic image of a direct sum of copies of $R/J(R)$, we have that $J(R)^i/J(R)^{i+1}$ is a semisimple right R-module. Since R is right Artinian, $J(R)^i/J(R)^{i+1}$ is an Artinian right module. By Proposition **??**, $J(R)^i/J(R)^{i+1}$ and its right submodules are finite direct sums of simple right R-modules. In particular, every right submodule of $J(R)^i/J(R)^{i+1}$ is finitely generated. Hence $J(R)^i/J(R)^{i+1}$ is a Noetherian R-module. Hence R also is right Noetherian.

Conversely, assume that R is right Noetherian, $J(R)$ is nilpotent and $R/J(R)$ is a semisimple right R-module. There exists a positive integer n such that

$$R \supseteq J(R) \supseteq \cdots \supseteq J(R)^n = 0.$$

As above $J(R)^i/J(R)^{i+1}$ is a semisimple right R-module. Since R is right Noetherian, by Proposition 2.1.8, $J(R)^i/J(R)^{i+1}$ and its right submodules are finite direct sums of simple right R-modules. Now it is clear that $J(R)^i/J(R)^{i+1}$ is an Artinian R-module and therefore R is right Artinian. □

2.5 Rings without unity

Let S be a ring without unity. Consider $S^1 = \mathbb{Z} \times S$ with the addition defined componentwise and the multiplication defined by the rule

$$(z_1, a) \cdot (z_2, b) = (z_1 z_2, z_1 b + z_2 a + ab),$$

for all $z_1, z_2 \in \mathbb{Z}$ and all $a, b \in S$. It is easy to check that S^1 is a ring and that $(1,0)$ is its unit-element. Furthermore, $\{0\} \times S$ is an ideal of S^1, which is naturally isomorphic to S (as rings without unity).

We say that a ring S without unity is a *(Jacobson) radical ring* if it is isomorphic to the Jacobson radical of a ring (with unity). Note that if S is a radical ring and R is a ring (with unity) such that $S \cong J(R)$, we have that for every $a \in S$ there exists a unique $b \in S$ such that $a + b + ab = a + b + ba = 0$. This can be proved as follows. Let $\psi : S \to R$ be an injective homomorphism of rings without uniy such that $\mathrm{im}(\psi) = J(R)$, this exists because $S \cong J(R)$. Then, for $a \in S$, $1 + \psi(a) \in R$ is invertible. Thus there exists $c \in R$ such that $(1+\psi(a))(1+c) = (1+c)(1+\psi(a)) = 1$. Note that this implies that $c = -\psi(a)c - \psi(a) \in J(R)$. Hence there exists $b \in S$ such that $\psi(b) = c$. Therefore

$$a + b + ab = a + b + ba = 0.$$

The uniqueness of b is clear. We shall study the maximal left ideals of $S^1 = \mathbb{Z} \times S$. Let M be a maximal left ideal of S^1. Since for all $a \in S$ there exists $b \in S$ such that $a + b + ab = a + b + ba = 0$, we have that $(1,a)(1,b) = (1,b)(1,a) = 1$. Note that if $(0,a) \notin M$, then there exist $(z_1, c) \in S^1$ and $(z_2, d) \in M$ such that

$$(z_1, c)(0,a) + (z_2, d) = (1,0),$$

that is $z_2 = 1$ and $z_1 a + ca + d = 0$, thus $(1, d) \in M$, a contradiction because $(1, d)$ is invertible in S^1. Hence $\{0\} \times S \subseteq M$. Now it is easy to see that $M = I \times S$, where I is a maximal ideal of $\mathbb{Z}$. Therefore $J(S^1) = \{0\} \times S \cong S$.

Thus we have proved the following result.

Prop5.1

Proposition 2.5.1. *Let S be a ring without unity. Then the following conditions are equivalent.*

(i) S is a radical ring.
(ii) For all $a \in S$ there exists a unique $b \in S$ such that $a + b + ab = a + b + ba = 0$.
(iii) $S \cong J(S^1)$. □

Let S be a ring without unity (or with unity). Define on S the operation $\circ$ by

$$a \circ b = a + b + ab,$$

for all $a, b \in S$. It is easy to see that $(S, \circ)$ is a monoid with neutral element 0. Note that S is a radical ring if and only if $(S, \circ)$ is a group. If $a \in S$ is invertible in the monoid $(S, \circ)$, we shall denote by a' its inverse.

A *nil ring* is a ring S without unity such that every element of S is nilpotent. Note that every nil ring is a radical ring.

Example 2.5.2. Let K be a field. Then $XK[[X]]$ is a radical ring and it is not a nil ring.

2.6 Exercises

prob:Zorn_simple

2.6.1. Prove that unitary rings have simple modules.

2.6.2. Let M be an R-module. Prove that if for every submodule N of M there exists a submodule L of M such that $M = N \oplus L$, then M is semisimple.

prob:Rump

2.6.3. Prove Proposition 3.3.5.

2.6.4. If X is a cycle set, then $x \cdot (y \cdot y) = ((y * x) \cdot y) \cdot ((y * x) \cdot y)$, where $y * x = z$ if and only if $y \cdot z = x$.

prob:CS

2.6.5. Prove Theorem 3.3.9.

prob:Herstein

2.6.6. Let D be the ring of rationals with odd denominators. Let $R = \begin{pmatrix} D & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$. Prove that R is right noetherian and $J(R) = \begin{pmatrix} J(D) & \mathbb{Q} \\ 0 & 0 \end{pmatrix}$. Prove that $J(R)^n \supseteq \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{pmatrix}$ and hence $\bigcap_n J(R)^n$ is non-zero.

2.7 Open problems

prob:Kaplansky

Open problem 2.7.1 (Kaplansky). If a ring R is right primitive and Noetherian, is R left primitive?

prob:Jacobson

Open problem 2.7.2 (Jacobson). Let R be a noetherian ring. Is then

$$\bigcap_{n\geq 1} J(R)^n = \{0\}?$$

prob:Koethe

Open problem 2.7.3 (Köthe). Let R be a ring. Is the sum of two arbitrary nil left ideals of R is nil?

Open problem 2.7.4. Construct and enumerate involutive solutions of size 11.

Open problem 2.7.5. Estimate the number of solutions of size n for $n \to \infty$.

Notes

The material on non-commutative ring theory is standard, see for example [17] **Ferran: quizás deberíamos citar el libro Algebra, vol 2, de P. M. Cohn**. Radical rings were introduced by Jacobson in [49]. Nil rings were used by Zelmanov in his solution to Burnside's problem, see for example [79].

In 1957 [**?**] Kaplansky formulated the question of whether right primitive rings are left primitive. After the example of Bergman of a right primitive ring which is not left primitive [14, 15] and the examples of Jategaonkar [**?**] with the additional property of being principal left ideal domains, Kaplansky formulated Open problem 2.7.1 in [**?**].

Open problem 2.7.2 was originally formulated by Jacobson in 1956 [**?**] for one-sided noetherian rings. In 1965 Herstein [**?**] found a counterexample in the case of one-sided noetherian rings (see Exercise 2.6.6) and reformulated the conjecture as it appears here.

Open problem 2.7.3 is the well-known Köthe's conjecture. The conjecture was first formulated in 1930, see [53]. It is known to be true in several cases. In full generality, the problem is still open. In [54] Krempa proved that the following statements are equivalent:

1) Köthe's conjecture is true.
2) If R is a nil ring, then $R[X]$ is a radical ring.
3) If R is a nil ring, then $M_2(R)$ is a nil ring.
4) Let $n \geq 2$. If R is a nil ring, then $M_n(R)$ is a nil ring.

In 1956 Amitsur formulated the following conjecture, see for example [4]: If R is a nil ring, then $R[X]$ is a nil ring. In [67] Smoktunowicz found a counterexample to Amitsur's conjecture. This counterexample suggests that Köthe's conjecture might

be false. A simplification of Smoktunowicz's example appears in [59]. See [68, 69] for more information on Köthe's conjecture and related topics.

Rump introduced cycle sets in [64]. The bijective correspondence of Theorem 3.3.9 was also proved by Rump in [64]. A similar result can be found in [39, Proposition 2.2].

The numbers of Table 3.3 were computed in [2] using a combination of [41] and constraint programming techniques. The algorithm is based on an idea of Plemmons [61], originally conceived to construct non-isomorphic semigroups.

Chapter 3
The Yang–Baxter equation

YB

3.1 Set-theoretic solutions

In [36], Drinfeld briefly discuss set-theoretic solutions to the YBE. He not only observed that it makes sense to consider the YBE in the category of sets but also that "maybe it would be interesting to study set-theoretical solutions".

Definition 3.1.1. A *set-theoretic solution* to the Yang–Baxter equation (YBE) is a pair (X,r), where X is a non-empty set and $r\colon X\times X\to X\times X$ is a bijective map that satisfies

$$(r\times\mathrm{id})(\mathrm{id}\times r)(r\times\mathrm{id}) = (\mathrm{id}\times r)(r\times\mathrm{id})(\mathrm{id}\times r),$$

where, if $r(x,y) = (\sigma_x(y),\tau_y(x))$, then

$$\begin{aligned} r\times\mathrm{id}\colon X\times X\times X\to X\times X\times X, &\qquad (r\times\mathrm{id})(x,y,z) = (\sigma_x(y),\tau_y(x),z),\\ \mathrm{id}\times r\colon X\times X\times X\to X\times X\times X, &\qquad (\mathrm{id}\times r)(x,y,z) = (x,\sigma_y(z),\tau_z(y)). \end{aligned}$$

The solution (X,r) is said to be *finite* if X is a finite set.

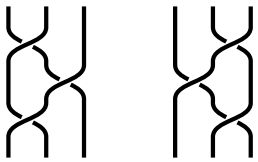


fig:braid

Figure 3.1: The Yang–Baxter equation.

Example 3.1.2. Let X be a non-empty set. Then (X,id) is a set-theoretic solution to the YBE.

Example 3.1.3. Let X be a non-empty set. Then (X,r), where $r(x,y)=(y,x)$, is a set-theoretic solution to the YBE. This solution is known as the *trivial solution* over the set X.

Convention 3.1.4. If (X,r) is a set-theoretic solution to the YBE, we always write

$$r(x,y)=(\sigma_x(y),\tau_y(x)).$$

By definition, the map $r\colon X\times X\to X\times X$ is invertible. By convention, we always write

$$r^{-1}(x,y)=(\widehat{\sigma}_x(y),\widehat{\tau}_y(x)).$$

It follows that

$$x=\widehat{\sigma}_{\sigma_x(y)}\tau_y(x),\quad y=\widehat{\tau}_{\tau_y(x)}\sigma_x(y).$$

lem:YB **Lemma 3.1.5.** *Let X be a non-empty set and $r\colon X\times X\to X\times X$ be a bijective map. Then (X,r) is a set-theoretic solution to the YBE if and only if*

$$\sigma_x\sigma_y=\sigma_{\sigma_x(y)}\sigma_{\tau_y(x)},\quad \sigma_{\tau_{\sigma_y(z)}(x)}\tau_z(y)=\tau_{\sigma_{\tau_y(x)}(z)}\sigma_x(y),\quad \tau_z\tau_y=\tau_{\tau_z(y)}\tau_{\sigma_y(z)}$$

for all $x,y,z\in X$.

Proof. We write $r_1=r\times\mathrm{id}$ and $r_2=\mathrm{id}\times r$. We first compute

$$\begin{aligned}r_1r_2r_1(x,y,z)&=r_1r_2(\sigma_x(y),\tau_y(x),z)=r_1(\sigma_x(y),\sigma_{\tau_y(x)}(z),\tau_z\tau_y(x))\\&=\left(\sigma_{\sigma_x(y)}\sigma_{\tau_y(x)}(z),\tau_{\sigma_{\tau_y(x)}(z)}\sigma_x(y),\tau_z\tau_y(x)\right).\end{aligned}$$

Then we compute

$$\begin{aligned}r_2r_1r_2(x,y,z)&=r_2r_1(x,\sigma_y(z),\tau_z(y))=r_2(\sigma_x\sigma_y(z),\tau_{\sigma_y(z)}(x),\tau_z(y))\\&=\left(\sigma_x\sigma_y(z),\sigma_{\tau_{\sigma_y(z)}(x)}\tau_z(y),\tau_{\tau_z(y)}\tau_{\sigma_y(z)}(x)\right)\end{aligned}$$

and the first claim follows. □

Example 3.1.6. If (X,r) is a set-theoretic solution to the YBE, then so is (X,r^{-1}). By Lemma 3.1.5, the following formulas hold:

$$\widehat{\tau}_y\widehat{\tau}_x=\widehat{\tau}_{\tau_y(x)}\widehat{\tau}_{\sigma_x(y)},\quad \widehat{\sigma}_x\widehat{\sigma_y}=\widehat{\sigma}_{\sigma_x(y)}\widehat{\sigma}_{\tau_y(x)}.$$

Example 3.1.7. Let $X=\{1,2,3,4\}$ and $r(x,y)=(\sigma_x(y),\tau_y(x))$, where

$$\begin{aligned}&\sigma_1=(132),&&\sigma_2=(124),&&\sigma_3=(143),&&\sigma_4=(234),\\&\tau_1=(12)(34),&&\tau_2=(12)(34),&&\tau_3=(12)(34),&&\tau_4=(12)(34).\end{aligned}$$

Then r is invertible with $r^{-1}(x,y)=(\widehat{\sigma}_x(y),\widehat{\tau}_y(x))$ given by

$$\begin{aligned}&\widehat{\sigma}_1=(12)(34),&&\widehat{\sigma}_2=(12)(34),&&\widehat{\sigma}_3=(12)(34),&&\widehat{\sigma}_4=(12)(34),\\&\widehat{\tau}_1=(142),&&\widehat{\tau}_2=(123),&&\widehat{\tau}_3=(243),&&\widehat{\tau}_4=(134).\end{aligned}$$

Definition 3.1.8. A *homomorphism* between the set-theoretic solutions (X,r) and (Y,s) is a map $f\colon X \to Y$ such that the diagram

$$\begin{array}{ccc} X\times X & \xrightarrow{r} & X\times X \\ {\scriptstyle f\times f}\downarrow & & \downarrow{\scriptstyle f\times f} \\ Y\times Y & \xrightarrow[s]{} & Y\times Y \end{array}$$

is commutative, that is $s(f\times f) = (f\times f)r$. An *isomorphism* of solutions is a bijective homomorphism of solutions.

Since we are interested in studying the combinatorics behind set-theoretic solutions to the YBE, it makes sense to study the following family of solutions.

Definition 3.1.9. We say that a set-theoretic solution (X,r) to the YBE is *non-degenerate* if all the maps σ_x and τ_x are permutations of X.

Convention 3.1.10. A *solution* we will always mean a non-degenerate set-theoretic solution to the YBE.

Lemma 3.1.11. *Let (X,r) be a solution.*

1) *Given $x,u \in X$, there exist unique $y,v \in X$ such that $r(x,y) = (u,v)$.*
2) *Given $y,v \in X$, there exist unique $x,u \in X$ such that $r(x,y) = (u,v)$.*

Proof. For the first claim take $y = \sigma_x^{-1}(u)$ and $v = \tau_y(x)$. For the second, $x = \tau_y^{-1}(v)$ and $u = \sigma_x(y)$. □

The bijectivity of r means that any row determines the whole square. Lemma **??** means that any column also determines the whole square, see Figure 3.2.

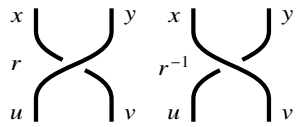


fig:braid **Figure 3.2:** Any row or column determines the whole square.

Example 3.1.12. If the map $(x,y) \mapsto (\sigma_x(y), \tau_y(x))$ satisfies the YBE, then so does $(x,y) \mapsto (\tau_x(y), \sigma_y(x))$.

exa:Lyubashenko **Example 3.1.13.** Let X be a non-empty set and σ and τ be bijections on X such that $\sigma\tau = \tau\sigma$. Then (X,r), where $r(x,y) = (\sigma(y), \tau(x))$, is a solution. This is known as the *permutation solution* associated with permutations σ and τ.

exa:Venkov **Example 3.1.14.** Let G be a group. Then (G,r), where $r(x,y) = (xyx^{-1}, x)$, is a solution.

Now we will prove the main theorem of this chapter. The result shows an intriguing connection between group actions and non-degenerate solutions. It was proved by Lu, Yan and Zhu.

thm:LYZ **Theorem 3.1.15.** *Let G be a group and let $\xi\colon G\times G\to G$, $\xi(x,y)=x\triangleright y$, be a left action of the group G on itself as a set, and let $\eta\colon G\times G\to G$, $\eta(x,y)=x\triangleleft y$, be a right action of the group G on itself as a set. If the compatibility condition*

$$uv=(u\triangleright v)(u\triangleleft v)$$

holds for all $u,v\in G$, then the pair (G,r), where

$$r\colon G\times G\to G\times G,\quad r(u,v)=(u\triangleright v,u\triangleleft v)$$

is a solution.

Proof. We write $r_1=r\times\mathrm{id}$ and $r_2=\mathrm{id}\times r$. Let

$$r_1r_2r_1(u,v,w)=(u_1,v_1,w_1),\quad r_2r_1r_2(u,v,w)=(u_2,v_2,w_2).$$

The compatibility condition implies that $u_1v_1w_1=u_2v_2w_2$. So we need to prove that $u_1=u_2$ and $w_1=w_2$. We note that

$$\begin{aligned} u_1&=(u\triangleright v)\triangleright((u\triangleleft v)\triangleright w), & w_1&=(u\triangleleft v)\triangleleft w,\\ u_2&=u\triangleright(v\triangleright w), & w_2&=(u\triangleleft(v\triangleright w))\triangleleft(v\triangleleft w).\end{aligned}$$

Using the compatibility condition and the fact that ξ is a left action,

$$u_1=((u\triangleright v)(u\triangleleft v))\triangleright w=(uv)\triangleright w=u\triangleright(v\triangleright w)=u_2.$$

Similarly, since η is a right action,

$$w_2=u\triangleleft((v\triangleright w)(v\triangleleft w))=u\triangleleft(vw)=(u\triangleleft v)\triangleleft w=w_1.$$

To prove that r is invertible we proceed as follows. Write $r(u,v)=(x,y)$, thus $u\triangleright v=x$, $u\triangleleft v=y$ and $uv=xy$. Since

$$(y\triangleright v^{-1})u=(y\triangleright v^{-1})(y\triangleleft v^{-1})=yv^{-1}=x^{-1}u,$$

it follows that $y\triangleright v^{-1}=x^{-1}$, i.e. $v^{-1}=y^{-1}\triangleright x^{-1}$. Similarly,

$$v(u^{-1}\triangleleft x)=(u^{-1}\triangleright x)(u^{-1}\triangleleft x)=u^{-1}x=vy^{-1}$$

implies that $u^{-1}=y^{-1}\triangleleft x^{-1}$. Clearly $r^{-1}=\zeta(i\times i)r(i\times i)\zeta$, is the inverse of r, where $\zeta(x,y)=(y,x)$ and $i(x)=x^{-1}$. □

prop:LYZ **Proposition 3.1.16.** *Under the assumptions of Theorem 3.1.15, if $r(x,y)=(u,v)$, then*

$$r(v^{-1},u^{-1}) = (y^{-1},x^{-1}), \quad r(x^{-1},u) = (y,v^{-1}), \quad r(v,y^{-1}) = (u^{-1},x).$$

Proof. In the proof of Theorem 3.1.15 we found that the inverse of the map r is given by $r^{-1} = \zeta(i\times i)r(i\times i)\zeta$, where $\zeta(x,y) = (y,x)$ and $i(x) = x^{-1}$. Hence

$$r^{-1}(y^{-1},x^{-1}) = \zeta(i\times i)r(i\times i)\zeta(y^{-1},x^{-1}) = \zeta(i\times i)r(x,y) = (v^{-1},u^{-1}).$$

It follows that $r(v^{-1},u^{-1}) = (y^{-1},x^{-1})$. To prove the equality $r(x^{-1},u) = (y,v^{-1})$ we proceed as follows. Since $r(x,y) = (u,v)$, it follows that $x \triangleright y = u$. Then $x^{-1} \triangleright u = y$ and hence $r(x^{-1},u) = (y,z)$ for some $z \in G$. Since $xy = uv$ and $x^{-1}u = yz$, it immediately follows that $yz = yv^{-1}$. Then $z = v^{-1}$. Similarly one proves $r(v,y^{-1}) = (u^{-1},x)$. □

3.2 Racks

defn:rack

Definition 3.2.1. A *rack* is a pair $(X,\triangleleft)$, where X is a non-empty set and $X\times X \to X$, $(x,y) \mapsto x \triangleleft y$, is a binary operation on X such that the maps $\rho_y\colon X \to X$, $x \mapsto x \triangleleft y$, are bijective for all $y \in X$, and

$$(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z) \tag{3.1}$$

eq:rack

for all $x,y,z \in X$.

Racks are used in low-dimensional topology [37], singularities [18] and in the classification of finite-dimensional pointed Hopf algebras [5].

Example 3.2.2. Let X be a set. Then $x \triangleleft y = x$ turns X into a rack. This is the *trivial rack* on X.

Example 3.2.3. Let $X = \mathbb{Z}/n$. Then $x \triangleleft y = 2y - x$ turns X into a rack. This is the *dihedral rack* of size n.

Example 3.2.4. Let A be an abelian group and $f \in \mathrm{Aut}(A)$. Then

$$x \triangleleft y = (\mathrm{id} - f)(y) + f(x)$$

turns A into a rack. These racks are known as the *Alexander racks*.

Definition 3.2.5. Let X and Z be racks. A *rack homomorphism* between the racks X and Z is a map $f\colon X \to Z$ such that $f(x \triangleleft y) = f(x) \triangleleft f(y)$ for all $x,y \in X$. An *isomorphism* of racks is a bijective rack homomorphism.

For $n \in \mathbb{N}$, let $r(n)$ be the number of isomorphism classes of racks of size n. Some values of $r(n)$ appear in Table 3.1, see for example [76].

Table 3.1: Enumeration of non-isomorphic racks.

tab:racks

n	2	3	4	5	6	7	8	9	10	11	12	13
$r(n)$	2	6	19	74	353	2080	16023	159526	2093244	36265070	836395102	25794670618

pro:Venkov **Proposition 3.2.6.** *Let X be a non-empty set and $X \times X \to X$, $(x,y) \mapsto x \triangleleft y$ be a binary operation on X. Then $r(x,y) = (y, x \triangleleft y)$ is a solution if and only if $(X, \triangleleft)$ is a rack.*

Proof. The map r satisfies $(r \times \mathrm{id})(\mathrm{id} \times r)(r \times \mathrm{id}) = (\mathrm{id} \times r)(r \times \mathrm{id})(\mathrm{id} \times r)$ if and only if (3.1) holds for all $x, y, z \in X$. The solution (X, r) is non-degenerate if the maps $X \to X$, $x \mapsto x \triangleleft y$, are bijective. □

The connection between racks and solutions goes deeper than the phenomenon appearing in Proposition 3.2.6.

pro:derived **Proposition 3.2.7.** *Let (X, r) be a solution. Then*

$$x \triangleleft y = \sigma_y \tau_{\sigma_x^{-1}(y)}(x) = \sigma_y \widehat{\sigma}_y^{-1}(x) \tag{3.2}$$ eq:derived

turns X into a rack and each σ_x is a rack homomorphism. Moreover, (X, r) is involutive if and only if the rack $(X, \triangleleft)$ is trivial.

Proof. Since $r(x, \sigma_x^{-1}(y)) = (y, \tau_{\sigma_x^{-1}(y)}(x))$, it follows that $\widehat{\sigma}_y^{-1}(x) = \tau_{\sigma_x^{-1}(y)}(x)$ for all $x, y \in X$. Hence the second equality of (3.2) holds and the maps $X \to X$, $x \mapsto x \triangleleft y$ are bijective.

Now we show that

$$\sigma_x(y) \triangleleft \sigma_x \sigma_y(z) = \sigma_x(y \triangleleft \sigma_y(z)) \tag{3.3}$$ eq:rackhom

for all $x, y \in X$. Write $r(x,y) = (u, v)$. On the one hand, by Lemma 3.1.5,

$$\begin{aligned} \sigma_x(y) \triangleleft \sigma_x \sigma_y(z) &= u \triangleleft \sigma_u \sigma_v(z) \\ &= \sigma_{\sigma_u \sigma_v(z)} \tau_{\sigma_{\tau_y(x)}(z)} \sigma_x(y) = \sigma_{\sigma_x \sigma_y(z)} \sigma_{\tau_{\sigma_y(z)}(x)} \tau_z(y). \end{aligned}$$

On the other hand,

$$\sigma_x(y \triangleleft \sigma_y(z)) = \sigma_x \sigma_{\sigma_y(z)} \tau_z(y) = \sigma_{\sigma_x \sigma_y(z)} \sigma_{\tau_{\sigma_y(z)}(x)} \tau_z(y).$$

Therefore (3.3) follows.

By Proposition 3.2.6, to prove that $(X, \triangleleft)$ is a rack it is enough to show that $s(x,y) = (y, x \triangleleft y)$ satisfies the YBE. For that purpose, we demonstrate that the map $J \colon X^3 \to X^3$, $J(x,y,z) = (x, \sigma_x(y), \sigma_x \sigma_y(z))$ is invertible and satisfies

$$(\mathrm{id} \times s) J = J(\mathrm{id} \times r), \quad (s \times \mathrm{id}) J = J(r \times \mathrm{id}).$$

One checks that the map $(x,y,z) \mapsto (x, \sigma_x^{-1}(y), \sigma^{-1}_{\sigma_x^{-1}(y)} \sigma_x^{-1}(z))$ is the inverse of J.

By (3.3),

$$\sigma_x(y) \triangleleft \sigma_x \sigma_y(z) = \sigma_x(y \triangleleft \sigma_y(z)) = \sigma_x \sigma_{\sigma_y(z)} \tau_{\sigma_y^{-1} \sigma_y(z)}(y) = \sigma_x \sigma_{\sigma_y(z)} \tau_z(y)$$

Then it follows that

$$\begin{aligned}(\mathrm{id}\times s)J(x,y,z) &= (\mathrm{id}\times s)(x,\sigma_x(y),\sigma_x\sigma_y(z))\\ &= (x,\sigma_x\sigma_y(z),\sigma_x(y)\triangleleft\sigma_x\sigma_y(z))\\ &= (x,\sigma_x\sigma_y(z),\sigma_x\sigma_{\sigma_y(z)}\tau_z(y))\\ &= J(x,\sigma_y(z),\tau_z(y))\\ &= J(\mathrm{id}\times r)(x,y,z).\end{aligned}$$

Similarly one proves that $(s\times\mathrm{id})J = J(r\times\mathrm{id})$. This implies that (X,s) is a solution and hence $(X,\triangleleft)$ is a rack by Proposition 3.2.6.

If (X,r) is involutive, then $x\triangleleft\sigma_x(y) = \sigma_{\sigma_x(y)}\tau_y(x) = x$ by (3.7). Conversely, if $x\triangleleft y = x$ for all $x,y\in X$, then r is involutive, as

$$r^2(x,\sigma_x^{-1}(y)) = r(y,\sigma_y^{-1}(x)) = (x,\sigma_x^{-1}(y)).$$ □

Definition 3.2.8. The rack constructed in Proposition 3.2.7 is known as the *derived rack* of (X,r).

There is a dual version of the derived rack:

pro:derived_dual

Proposition 3.2.9. *Let (X,r) be a solution. Then*

$$x\blacktriangleleft y = \tau_y\sigma_{\tau_x^{-1}(y)}(x) = \tau_y\widehat{\tau}_y^{-1}(x)$$

turns X into a rack and each τ_x is a rack homomorphism.

Proof. Since (X,r) is a solution, then so is (X,r_0), where $r_0(x,y) = (\tau_x(y),\sigma_y(x))$. Then the claim follows from Proposition 3.2.7 applied to the solution (X,r_0). □

Definition 3.2.10. The rack constructed in Proposition 3.2.9 is known as the *dual derived rack* of (X,r).

In general, the racks constructed in Propositions 3.2.7 and 3.2.9 are different:

Example 3.2.11. Let $X = \{1,\dots,5\}$ and (X,r) be the solution given by

$$\begin{array}{lllll}\sigma_1=\mathrm{id}, & \sigma_2=\mathrm{id}, & \sigma_3=\mathrm{id}, & \sigma_4=(13)(45), & \sigma_5=(12)(45),\\ \tau_1=\mathrm{id}, & \tau_2=\mathrm{id}, & \tau_3=\mathrm{id}, & \tau_4=(23)(45), & \tau_5=(23)(45).\end{array}$$

On the one hand the derived rack of (X,r) is given by the permutations

$$\sigma_1\widehat{\sigma}_1^{-1} = \sigma_2\widehat{\sigma}_2^{-1} = \sigma_3\widehat{\sigma}_3^{-1} = \mathrm{id}, \qquad \sigma_4\widehat{\sigma}_4^{-1} = (132), \qquad \sigma_5\widehat{\sigma}_5^{-1} = (123).$$

On the other hand, the dual derived rack by

$$\tau_1\widehat{\tau}_1^{-1} = \tau_2\widehat{\tau}_2^{-1} = \tau_3\widehat{\tau}_3^{-1} = \mathrm{id}, \qquad \tau_4\widehat{\tau}_4^{-1} = (123), \qquad \tau_5\widehat{\tau}_5^{-1} = (132).$$

We now prove that the racks of Propositions 3.2.7 and 3.2.9 are isomorphic. We shall need a lemma.

lem:T_invertible

Lemma 3.2.12. *Let (X,r) be a solution. The map $T\colon X \to X$, $x \mapsto \sigma_x^{-1}(x)$, is invertible with inverse $U\colon X \to X$, $x \mapsto \tau_x^{-1}(x \blacktriangleleft x)$.*

Proof. Let $x \in X$ and $y = U(x) = \tau_x^{-1}(x \blacktriangleleft x)$. Then $\tau_x(y) = x \blacktriangleleft x = \tau_x\widehat{\tau}_x^{-1}(x)$ and hence $y = \widehat{\tau}_x^{-1}(x)$. Then $\widehat{\tau}_x(y) = x$ and

$$r^{-1}(y,x) = (\widehat{\sigma}_y(x),x) = (z,x),$$

where $z \in X$ is such that $\sigma_z(x) = y$. By Lemma 3.1.5, $\sigma_y = \sigma_z$. Then it follows that $x = \sigma_y^{-1}(y) = T(y)$. Therefore $y = U(x) = U(T(y))$.

To prove that $T(U(x)) = x$, first note that

$$r(\tau_x^{-1}(x),x) = (\sigma_{\tau_x^{-1}(x)}(x),x)$$

and Lemma 3.1.5 imply that $\sigma_{\tau_x^{-1}(x)} = \sigma_{\sigma_{\tau_x^{-1}(x)}(x)}$. Now

$$\begin{aligned} T(U(x)) &= T(\tau_x^{-1}(x \blacktriangleleft x)) = T(\sigma_{\tau_x^{-1}(x)}(x)) \\ &= \sigma^{-1}_{\sigma_{\tau_x^{-1}(x)}(x)}\sigma_{\tau_x^{-1}(x)}(x) = \sigma^{-1}_{\tau_x^{-1}(x)}\sigma_{\tau_x^{-1}(x)}(x) = x. \end{aligned}$$ □

There is a version of Proposition 3.3.6 for arbitrary solutions. A similar result appears in Exercise 3.4.8.

Proposition 3.2.13. *Let (X,r) be a solution. Then $T\colon X \to X$, $x \mapsto \sigma_x^{-1}(x)$, is a bijective map such that*

$$T\tau_y = \widehat{\sigma}_y^{-1}T, \quad T\widehat{\tau}_y = \sigma_y^{-1}T$$

and $T(x \blacktriangleleft y) = T(x) \blacktriangleleft T(y)$ for all $x, y \in X$.

Proof. Lemma 3.2.12 proves that T is bijective. We now compute

$$\begin{aligned} T\tau_y(x) &= \sigma^{-1}_{\tau_y(x)}\tau_y(x) = \sigma^{-1}_{\tau_y(x)}\sigma^{-1}_{\sigma_x(y)}\sigma_{\sigma_x(y)}\tau_y(x) \\ &= \sigma_y^{-1}\sigma_x^{-1}\sigma_{\sigma_x(y)}\tau_y(x) = \sigma_y^{-1}\sigma_x^{-1}(x \blacktriangleleft \sigma_x(y)) = \sigma_y^{-1}(T(x) \blacktriangleleft y) = \widehat{\sigma}_y^{-1}T(x). \end{aligned}$$

Since $\widehat{\tau}_y(x) = \sigma^{-1}_{\widehat{\sigma}_x(y)}(x)$, Lemma 3.1.5 implies that

$$T\widehat{\tau}_y(x) = \sigma^{-1}_{\widehat{\tau}_y(x)}\widehat{\tau}_y(x) = \sigma^{-1}_{\widehat{\tau}_y(x)}\sigma^{-1}_{\widehat{\sigma}_x(y)}(x) = \sigma_y^{-1}\sigma_x^{-1}(x) = \sigma_y^{-1}T(x).$$

These formulas imply that

$$T\tau_y\widehat{\tau}_y^{-1} = \widehat{\sigma}_y^{-1}T\widehat{\tau}_y^{-1} = \widehat{\sigma}_y^{-1}\sigma_y T. \tag{3.4}$$ eq:T_rack

We evaluate Equality (3.4) on X. On the one hand, $T(x \blacktriangleleft y) = T\tau_y\widehat{\tau}_y^{-1}(x)$. On the other hand,

$$\widehat{\sigma}_y^{-1}\sigma_y T(x) = \sigma_y^{-1}\sigma_y\widehat{\sigma}_y^{-1}\sigma_y T(x) = \sigma_y^{-1}(\sigma_y T(x) \blacktriangleleft y) = T(x) \blacktriangleleft T(y).$$ □

As it happens in the involutive case, there is a nice combinatorial structure that describes a solution.

defn:skewCS

Definition 3.2.14. A *skew cycle set* is a triple $(X, \triangleleft, \cdot)$, where X is a non-empty set, $(X, \triangleleft)$ is a rack and $X \times X \to X$, $(x, y) \mapsto x \cdot y$, is a binary operation on X such that the maps $X \to X$, $y \mapsto x \cdot y$, are rack isomorphisms, and

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot (x \triangleleft y)) \cdot (y \cdot z) \tag{3.5}$$

eq:skew_CS

for all $x, y, z \in X$. A skew cycle set $(X, \triangleleft, \cdot)$ is said to be *non-degenerate* if the map $X \to X$, $x \mapsto x \cdot x$, is bijective.

If $(X, \triangleleft, \cdot)$ is a skew cycle set and $(X, \triangleleft)$ is a trivial rack, then $(X, \cdot)$ is a cycle set.

Definition 3.2.15. Let X and Z be skew cycle sets. A *homomorphism* between the skew cycle sets X and Z is a map $f \colon X \to Z$ such that $f(x \cdot y) = f(x) \cdot f(y)$ and $f(x \triangleleft y) = f(x) \triangleleft f(y)$ for all $x, y \in X$. An *isomorphism* of skew cycle sets is a bijective homomorphism of skew cycle sets.

Theorem 3.3.9 can be generalized to arbitrary solutions.

thm:skewCS

Theorem 3.2.16. *There exists a bijective correspondence between solutions and non-degenerate skew cycle sets.*

Proof. Let (X, r) be a solution and $(X, \triangleleft)$ be its derived rack. We will prove that the operation $x \cdot y = \sigma_x^{-1}(y)$ turns $(X, \triangleleft)$ into a skew cycle set. By Proposition 3.2.7, the maps $X \to X$, $y \mapsto x \cdot y$, are bijective rack homomorphisms.

On the one hand, since $r(x, \sigma_x^{-1}(y)) = (y, \tau_{\sigma_x^{-1}(y)}(x))$,

$$\begin{aligned}(x \cdot y) \cdot (x \cdot z) &= \sigma_x^{-1}(y) \cdot \sigma_x^{-1}(z) = \sigma^{-1}_{\sigma_x^{-1}(y)} \sigma_x^{-1}(z) \\ &= \left(\sigma_x \sigma_{\sigma_x^{-1}(y)}\right)^{-1}(z) = \left(\sigma_y \sigma_{\tau_{\sigma_x^{-1}(y)}(x)}\right)^{-1}(z).\end{aligned}$$

On the other hand,

$$\begin{aligned}(y \cdot (x \triangleleft y)) \cdot (y \cdot z) &= \sigma_y^{-1}(\sigma_y \tau_{\sigma_x^{-1}(y)}(x)) \sigma_y^{-1}(z) \\ &= \sigma^{-1}_{\tau_{\sigma_x^{-1}(y)}(x)} \sigma_y^{-1}(z) = \left(\sigma_y \sigma_{\tau_{\sigma_x^{-1}(y)}(x)}\right)^{-1}(z).\end{aligned}$$

Therefore $(X, \triangleleft, \cdot)$ is a skew cycle set. Furthermore, by Lemma 3.2.12, this skew cycle set is non-degenerate.

Now we prove the converse statement. Let $(X, \triangleleft, \cdot)$ be a non-degenerate skew cycle set. For $x, y \in X$ let

$$\sigma_x(y) = x * y, \quad \tau_y(x) = \sigma^{-1}_{\sigma_x(y)}(x \triangleleft \sigma_x(y)),$$

where $x * y = z$ if and only if $x \cdot z = y$. Since X is a skew cycle set, each σ_x is bijective. Let us prove that the τ_x are bijective. Equality (3.5) with $y = \sigma_x(z)$ implies that

$$\begin{aligned}\sigma_z^{-1}\sigma_x^{-1} &= \sigma^{-1}_{\sigma_x^{-1}(y)}\sigma_x^{-1}\\ &= \sigma^{-1}_{\sigma_y^{-1}(x\triangleleft y)}\sigma_y^{-1} = \sigma^{-1}_{\sigma^{-1}_{\sigma_x(z)}(x\triangleleft\sigma_x(z))}\sigma^{-1}_{\sigma_x(z)} = \sigma^{-1}_{\tau_z(x)}\sigma^{-1}_{\sigma_x(z)}\end{aligned} \tag{3.6}$$

eq:solskew

for all $x, z \in X$. Since each σ_x is a rack homomorphism and $\sigma^{-1}_{\sigma_x(y)}\sigma_x = \sigma_{\tau_y(x)}\sigma_y^{-1}$ holds for all $x, y \in X$, it follows that

$$\tau_y(x) = \sigma^{-1}_{\sigma_x(y)}(x\triangleleft\sigma_x(y)) = \sigma^{-1}_{\sigma_x(y)}\sigma_x(\sigma_x^{-1}(x)\triangleleft y) = \sigma_{\tau_y(x)}\sigma_y^{-1}(\sigma_x^{-1}(x)\triangleleft y).$$

holds for all $x, y \in X$. Therefore $T\tau_y = \sigma_y^{-1}\rho_y T$, where $T\colon X \to X$, $T(x) = x\cdot x$ and $\rho_y\colon X \to X$, $\rho_y(x) = x\triangleleft y$ are bijective maps. In particular, τ_y is bijective for all $y \in X$.

Let $r\colon X\times X \to X\times X$ be the map defined by $r(x,y) = (\sigma_x(y), \tau_y(x))$, for all $x, y \in X$. Now we prove that (X, r) is a solution. Let $s\colon X\times X \to X\times X$ be the map defined by $s(x,y) = (y, x\triangleleft y)$. By Proposition 3.2.6, (X, s) is a solution. As in the proof of Proposition 3.2.7, The map $J\colon X^3 \to X^3$, $J(x,y,z) = (x, \sigma_x(y), \sigma_x\sigma_y(z))$ is invertible and satisfies that

$$(\mathrm{id}\times s)J = J(\mathrm{id}\times r),$$

because the σ_x are rack homomorphisms. Furthermore, by (3.6) we have that

$$\begin{aligned}(s\times\mathrm{id})J(x,y,z) &= (s\times\mathrm{id})(x, \sigma_x(y), \sigma_x\sigma_y(z))\\ &= (\sigma_x(y), x\triangleleft\sigma_x(y), \sigma_x\sigma_y(z))\\ &= (\sigma_x(y), \sigma_{\sigma_x(y)}\tau_y(x), \sigma_{\sigma_x(y)}\sigma_{\tau_y(x)}(z))\\ &= J(\sigma_x(y), \tau_y(x), z)\\ &= J(r\times\mathrm{id})(x,y,z).\end{aligned}$$

Therefore (X, r) is a solution.

Let $(X, \triangleleft, \cdot)$ be a non-degenerate skew cycle set. Let $G(X, \triangleleft, \cdot) = (X, r)$, where $r(x,y) = (x * y, (x * y)\cdot(x\triangleleft(x * y)))$, where $x * y = z$ if and only if $y = x\cdot z$. We have seen that (X, r) is a solution.

For every solution (X, r) we define $F(X, r) = (X, \triangleleft, \cdot)$, where $x\triangleleft y = \sigma_y\tau_{\sigma_x^{-1}(y)}(x)$ and $x\cdot y = \sigma_x^{-1}(y)$. We have seen that $(X, \triangleleft, \cdot)$ is a non-degenerate skew cycle set.

A direct calculation shows that $F(G(X, \triangleleft, \cdot)) = (X, \triangleleft, \cdot)$ for every non-degenerate skew cycle set $(X, \triangleleft, \cdot)$ and $G(F(X, r)) = (X, r)$ for every solution (X, r). □

Theorem 3.2.16 can be used to construct small solutions, see Table 3.2.

Table 3.2: Enumeration of non-involutive solutions.

tab:non_involutive

n	2	3	4	5	6	7	8
$s(n)$	2	21	253	3519	100071	4602720	422449480

3.3 Involutive solutions

We now go back to study solutions to the YBE and discuss the intriguing interplay between radical rings and involutive solutions.

Definition 3.3.1. A solution (X,r) is said to be *involutive* if $r^2 = \mathrm{id}$.

For $n \geq 2$, the *symmetric group* $\mathbb{S}_n$ can be presented as the group with generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$\begin{aligned} &\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} && \text{if } 1 \leq i \leq n-2, \\ &\sigma_i\sigma_j = \sigma_j\sigma_i && \text{if } |i-j| > 1, \\ &\sigma_i^2 = 1 && \text{for all } i \in \{1, \ldots, n-1\}. \end{aligned}$$

Let (X,r) be an involutive solution. Then the map $\sigma_i \mapsto r_{i,i+1} = \mathrm{id}_{X^{i-1}} \times r \times \mathrm{id}_{X^{n-i-1}}$ extends to an action of $\mathbb{S}_n$ on X^n.

Example 3.3.2. Let X be a non-empty set and σ be a bijection on X. Then (X,r), where $r(x,y) = (\sigma(y), \sigma^{-1}(x))$, is an involutive solution.

We now present a very important family of involutive solutions. These examples show an intriguing connection between the YBE and the theory of non-commutative rings.

Example 3.3.3. Let p be a prime and let $A = \mathbb{Z}/(p^2) = \mathbb{Z}/p^2\mathbb{Z}$ be the cylic additive group of order p^2. Then A with a new multiplication $*$ defined by $a * b = pab$ is a radical ring. In this case, $x \circ y = x + y + pxy$, and $x' = -x + px^2$.

Example 3.3.4. Let n be an integer such that $n > 1$. Let $A = \left\{ \frac{nx}{ny+1} : x, y \in \mathbb{Z} \right\} \subseteq \mathbb{Q}$. Note that A is a subring without unity of the field $\mathbb{Q}$. In fact A is a radical ring. A straightforward computation shows that

$$\left(\frac{nx}{ny+1}\right)' = \frac{-nx}{n(x+y)+1}.$$

The following fundamental family of solutions appears in [65]. It turns out to be fundamental in the study of set-theoretic solutions to the YBE.

pro:Rump **Proposition 3.3.5.** *Let R be a radical ring. Then (R,r), where*

$$r(x,y) = (-x + x \circ y, (-x + x \circ y)' \circ x \circ y)$$

is an involutive solution.

The proposition can be demonstrated using Theorem 3.1.15, see Exercise 2.6.3. We will prove a stronger result in Theorem 6.1.23.

Note that if (X,r) is an involutive solution, then

$$(x,y) = r^2(x,y) = r(\sigma_x(y),\tau_y(x)) = (\sigma_{\sigma_x(y)}\tau_y(x),\tau_{\tau_y(x)}\sigma_x(y)).$$

Hence

$$\tau_y(x) = \sigma^{-1}_{\sigma_x(y)}(x), \quad \sigma_x(y) = \tau^{-1}_{\tau_y(x)}(y) \tag{3.7}$$

eq:involutive

for all $x,y \in X$. Thus for involutive solutions it is enough to know $\{\sigma_x : x \in X\}$, as from this we obtain the set $\{\tau_x : x \in X\}$.

pro:T

Proposition 3.3.6. *Let (X,r) be an involutive solution. Then the map $T\colon X \to X$, $x \mapsto \sigma_x^{-1}(x)$, is invertible with inverse $T^{-1}(y) = \tau_y^{-1}(y)$ and*

$$T^{-1}\sigma_x^{-1}T = \tau_x$$

for all $x \in X$.

Proof. Let $U(x) = \tau_x^{-1}(x)$. Since r is involutive,

$$(U(x),x) = r^2(U(x),x) = r(\sigma_{U(x)}(x),x) = (\sigma_{\sigma_{U(x)}(x)}(x),\tau_x\sigma_{U(x)}(x)).$$

In particular, $x = \tau_x\sigma_{U(x)}(x)$ and hence $U(x) = \sigma_{U(x)}(x)$. This implies that

$$T(U(x)) = \sigma^{-1}_{U(x)}(U(x)) = x.$$

Similarly one obtains $U(T(x)) = x$.

Since (X,r) is a solution, Lemma 3.1.5 implies that $\sigma_x\sigma_y = \sigma_{\sigma_x(y)}\sigma_{\tau_y(x)}$ holds for all $x,y \in X$. Then

$$\sigma_y^{-1}T(x) = \sigma_y^{-1}\sigma_x^{-1}(x) = \sigma^{-1}_{\tau_y(x)}\sigma^{-1}_{\sigma_x(y)}(x) = \sigma^{-1}_{\tau_y(x)}\tau_y(x) = T\tau_y(x)$$

for all $y \in X$, by Equality (3.7). □

Definition 3.3.7. A *cycle set* is a pair $(X,\cdot)$, where X is a non-empty set provided with a binary operation $X \times X \to X$, $(x,y) \mapsto x \cdot y$, such that

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z) \tag{3.8}$$

eq:cycle_set

holds for all $x,y,z \in X$ and each map $\varphi_x : X \to X$, $y \mapsto x \cdot y$, is bijective. A cycle set $(X,\cdot)$ is said to be *non-degenerate* if the map $X \to X$, $x \mapsto x \cdot x$, is bijective.

Definition 3.3.8. Let X and Z be cycle sets. A *homomorphism* between the cycle sets X and Z is a map $f : X \to Z$ such that $f(x \cdot y) = f(x) \cdot f(y)$ for all $x,y \in X$. An *isomorphism* of cycle sets is a bijective homomorphism of cycle sets.

thm:CS

Theorem 3.3.9. *There exists a bijective correspondence between non-isomorphic involutive solutions and non-isomorphic non-degenerate cycle sets.*

For the readers who are not familiar with the above-mentioned result, the bijective correspondence is given by

$$r(x,y) = (x * y, (x * y) \cdot x),$$

$x * y = \varphi_x^{-1}(y)$. We leave the proof for the reader, see Exercise 2.6.5. However, we will prove a more general result in Theorem 3.2.16.

Theorem 3.3.9 can be used to construct and enumerate small involutive solutions [2]. Table 3.3 shows the number of non-isomorphic involutive solutions of size ≤ 10. For size ≤ 7 the numbers of Table 3.3 coincide with those in [39] but differ by two for $n = 8$, as two solutions of size eight are missing in [39].

Table 3.3: Involutive solutions of size ≤ 10.

tab:IYB

n	2	3	4	5	6	7	8	9	10
solutions	2	5	23	88	595	3456	34530	321931	4895272

3.4 Exercises

3.4.1. For $n \geq 2$, the *braid group* $\mathbb{B}_n$ is defined as the group with generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$\begin{aligned} &\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} && \text{if } 1 \leq i \leq n-2, \\ &\sigma_i\sigma_j = \sigma_j\sigma_i && \text{if } |i-j| > 1. \end{aligned}$$

Let (X,r) be a set-theoretic solution to the YBE. Write $X^n = X \times \cdots \times X$ (n-times). For $i < n$ let $r_{i,i+1} = \mathrm{id}_{X^{i-1}} \times r \times \mathrm{id}_{X^{n-i-1}} : X^n \to X^n$. Then the map $\sigma_i \mapsto r_{i,i+1}$ extends to an action of $\mathbb{B}_n$ on X^n.

prob:Wada

3.4.2. Let G be a group. Prove that the following maps satisfy the set-theoretic YBE:

a) $r(x,y) = (y, x^{-1})$.
b) $r(x,y) = (y^{-1}, x^{-1})$.
c) $r(x,y) = (xyx, x^{-1})$.
d) $r(x,y) = (x^2y, y^{-1}x^{-1}y)$.

prob:Wada_racks

3.4.3. Let G be a group. Prove that the following maps satisfy the set-theoreic YBE:

a) $r(x,y) = (x^m y x^{-m}, x)$ for every integer m.
b) $r(x,y) = (xy^{-1}x, x)$.

prob:D_n

3.4.4. Let $n \geq 2$ and $X = \mathbb{Z}/(n)$ be the ring of integers modulo n. Prove that the map $r(x,y) = (2x - y, x)$ satisfies the the set-theoretic YBE.

3.4.5. Let G be a group and $f \in \mathrm{Aut}(G)$. Prove that the map

$$r(x,y) = (f(y), f(y)^{-1}xy)$$

satisfies the set-theoretic YBE.

prob:xx **3.4.6.** Let (X,r) be a solution. Let $(X,\triangleleft)$ and $(X,\blacktriangleleft)$ be the derived rack and the dual derived rack of (X,r), respectively. Prove that $x \triangleleft x = x \blacktriangleleft x$ for all $x \in X$.

prob:tau_hat **3.4.7.** Let (X,r) be a solution and $(X,\triangleleft)$ be its derived rack. Prove that

$$\widehat{\tau}_x(y \triangleleft z) = \widehat{\tau}_x(y) \triangleleft \widehat{\tau}_x(z)$$

for all $x,y,z \in X$.

prob:variationT **3.4.8.** Let (X,r) be a solution and let $(X,\triangleright)$ be its derived rack. Prove that

$$T\sigma_x(y) = \tau_x^{-1}(x \triangleright T(y))$$

for all $x \in X$, where $T\colon X \to X$, $T(y) = \tau_y^{-1}(y)$.

prob:guitar **3.4.9.** Let (X,r) be a solution **and** (X,s) **its derived solution**. Let $J_2(x,y) = (x,\sigma_x(y))$ and $J_{n+1} = Q_{n+1} \circ (\mathrm{id} \times J_n)$ for $n \geq 2$, where

$$Q_{n+1}(x_1,\dots,x_{n+1}) = (x_1,\sigma_{x_1}(x_2),\dots,\sigma_{x_1}(x_{n+1})).$$

Prove that $J_n \circ r_{i,i+1} = s_{i,i+1} \circ J_n$ for all $n \geq 2$ and $i \in \{1,\dots,n-1\}$, where $r_{i,i+1} = \mathrm{id}^{\times i-1} \times r \times \mathrm{id}^{\times n-i-1}$ and $s_{i,i+1} = \mathrm{id}^{\times i-1} \times s \times \mathrm{id}^{\times n-i-1}$.

3.5 Open problems

problem:racks14 **Open problem 3.5.1.** Enumerate isomorphism classes of racks of size 14.

problem:non_involutive9 **Open problem 3.5.2.** Enumerate non-involutive solutions of size ≥ 9.

problem:involutive11 **Open problem 3.5.3.** Enumerate involutive solutions of size ≥ 11.

3.6 Notes

The first papers where set-theoretic solutions are studied are those of Etingof, Schedler and Soloviev [39] and Gateva–Ivanova and Van den Bergh [44]. Both papers deal with non-degenerate involutive solutions, i.e. solutions (X,r) where $r^2 = \mathrm{id}$.

In [36], Drinfeld attributes Example 3.1.13 to Lyubashenko.

Theorem 3.1.15 goes back to Lu, Yan and Zhu, see [57]. Similar results can be found in the work of Etingof, Schedler and Soloviev [39] for involutive solutions and in Soloviev's paper [73].

Exercises 3.4.2 and 3.4.3 appear in the work of Wada [77] on representations of braid groups.

In [36], Drinfeld attributes Proposition 3.2.6 to Venkov.

A particular family of racks turns out to be useful in combinatorial **knot** theory. A quandle is a rack $(X, \triangleleft)$ such that $x \triangleleft x = x$ for all $x \in X$.

There are several papers on the enumeration of isomorphic classes of finite racks [6, 16, 46]. Estimations on the number of finite racks of size n appear in [16].

The numbers of Table 3.2 were computed using Theorem 3.2.16 essentially with the same technique used to construct involutive solutions [2]. The construction of non-involutive solutions of size 9 seems to be feasible with these methods. However, it should be noted that a huge number of solutions is expected.

Exercises 3.4.6 and 3.4.7 appear in [56].

The map J_n of Exercise 3.4.9 is known as the *guitar map*. It was first considered by Etingof, Schedler and Soloviev in [39] for involutive solutions. The construction was extended to non-involutive solutions by Soloviev in [73] and Lu, Yan and Zhu in [57]. In [35] Dehornoy used the inverse of the guitar map to develop his right-cyclic calculus and to obtain short proofs for results on the structure group of involutive solutions. In [5] Andruskiewitsch and Graña use the guitar map to study certain isomorphisms of Nichols algebras. A particular case of the guitar map also appears in the work of Przytycki [63].

The derived rack of a solution was first defined in the work of Soloviev [73]. Most of the properties of the derived racks mentioned in this chapter were proved in [56].

Problem 3.5.1 appears in [76].

Problems 3.5.2 and 3.5.3 appear in [2].

Chapter 4
Nilpotent groups

nilpotent

4.1 Central series

Let G be a group. The *center* of G is

$$Z(G) = \{x \in G \mid xy = yx, \text{ for all } y \in G\}.$$

A subgroup H of G is *central* in G if $H \subseteq Z(G)$ and H is said to be *characteristic* if $f(H) = H$ for all $f \in \mathrm{Aut}(G)$. The center $Z(G)$ of G is a characteristic subgroup of G. Every characteristic subgroup of G is normal in G. If H is a characteristic subgroup of K and K is normal in G, then H is normal in G.

Let X be a non-empty subset of G. The *centralizer* of X in G is

$$C_G(X) = \{y \in G \mid xy = yx, \text{ for all } x \in X\}.$$

Note that $C_G(X)$ is a subgroup of G. In particular, $C_G(G) = Z(G)$.

Let H be a normal subgroup of G. Let $\varphi\colon G \to \mathrm{Aut}(H)$ be the map defined by $\varphi(x) = \varphi_x$ and $\varphi_x(h) = xhx^{-1}$ for all $x \in G$ and $h \in H$. It is easy to check that φ is a homomorphism of groups and $\ker(\varphi) = C_G(H)$. Hence $C_G(H)$ is a normal subgroup of G. We also say that φ is a left action of G on H by conjugation.

If G is a group and $x, y, z \in G$, then the conjugation (as a left action) will be denoted by ${}^x y = xyx^{-1}$. The *commutator* between x and y is then

$$[x, y] = xyx^{-1}y^{-1} = ({}^x y)y^{-1}.$$

We also write $[x, y, z] = [x, [y, z]]$. If X, Y and Z are subgroups of G, we write

$$[X, Y] = \langle [x, y] : x \in X, y \in Y \rangle$$

and $[X, Y, Z] = [X, [Y, Z]]$. Note that $[X, Y, Z] = [X, Z, Y]$, as $[X, Y] = [Y, X]$. The *commutator subgroup* $[G, G]$ of G is a characteristic subgroup of G.

The proof of the following lemma is an easy exercise.

xca:HallWitt

Lemma 4.1.1 (Hall–Witt's identity). *Let G be a group and $x, y, z \in G$. Then*

$$\left({}^{y}[x, y^{-1}, z]\right)\left({}^{z}[y, z^{-1}, x]\right)\left({}^{x}[z, x^{-1}, y]\right) = 1. \tag{4.1}$$

eq:HallWitt

Note that if G is such that $[G, G]$ is central, then Hall–Witt's identity turns out to be Jacobi's identity, i.e.

$$[x, y, z][y, z, x][z, x, y] = 1.$$

lemma:3subgrupos_general

Lemma 4.1.2 (three subgroups lemma). *Let N be a normal subgroup of G and let X, Y and Z be subgroups of G. If $[X, Y, Z] \subseteq N$ and $[Y, Z, X] \subseteq N$, then $[Z, X, Y] \subseteq N$.*

Proof. We first assume that $N = \{1\}$. Then $[Y, Z] \subseteq C_G(X)$ and $[Z, X] \subseteq C_G(Y)$. It is enough to prove that $[z, x^{-1}, y] = 1$ for all $x \in X$, $y \in Y$ and $z \in Z$. Let $x \in X$, $y \in Y$ and $z \in Z$. Since $[x, y^{-1}, z] = 1$, it follows that ${}^{y}[x, y^{-1}, z] = 1$. Similarly, ${}^{z}[y, z^{-1}, x] = 1$. Hence the Hall–Witt identity yields $[z, x^{-1}, y] = 1$.

We now demonstrate the general case. Let N be a normal subgroup of G and $\pi : G \to G/N$ be the canonical map. Since $[X, Y, Z] \subseteq N$,

$$\begin{aligned}\{1\} &= \pi([X, Y, Z]) = \pi([X, [Y, Z]]) \\ &= [\pi(X), \pi([Y, Z])] = [\pi(X), [\pi(Y), \pi(Z)]] = [\pi(X), \pi(Y), \pi(Z)].\end{aligned}$$

Similarly one proves that $[\pi(Y), \pi(Z), \pi(X)] = \{1\}$. By the previous paragraph, $[\pi(Z), \pi(X), \pi(Y)] = \{1\}$, so $[Z, X, Y] \subseteq N$. □

The *lower central series* of a group G is the sequence $\gamma_k(G)$, for positive integers k, defined recursively as

$$\gamma_1(G) = G, \quad \gamma_{i+1}(G) = [G, \gamma_i(G)] \quad i \geq 1.$$

A group G is said to be *nilpotent* if there exists a non-negative integer c such that $\gamma_{c+1}(G) = \{1\}$. The smallest c such that $\gamma_{c+1}(G) = \{1\}$ is the *nilpotency index* (or *nilpotency class*) of G.

Example 4.1.3. A group is nilpotent of class one if and only if it is non-trivial and abelian.

Example 4.1.4. The group $G = \mathbb{A}_4$ is not nilpotent, as

$$\gamma_1(G) = G, \quad \gamma_j(G) = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\} \simeq C_2 \times C_2$$

for all $j \geq 2$.

xca:gamma

Lemma 4.1.5. *Let G be group. Then the following conditions hold.*

1) *Each $\gamma_i(G)$ is a characteristic subgroup of G.*
2) *$\gamma_i(G) \supseteq \gamma_{i+1}(G)$ for all $i \geq 1$.*

3) *If $f : G \to H$ is a surjective group homomorphism, then $f(\gamma_i(G)) = \gamma_i(H)$ for all $i \geq 1$.*

Proof. We shall prove the first statement by induction on i. Let $h \in \mathrm{Aut}(G)$. For $i = 1$, the result is clear. Suppose that the result holds for some $i \geq 1$. By the inductive hypothesis,

$$h(\gamma_{i+1}(G)) = h([G, \gamma_i(G)]) = [h(G), h(\gamma_i(G))] = [G, \gamma_i(G)] = \gamma_{i+1}(G).$$

Therefore, the first statement follows by induction.

The second and the third statements can be easily proved by induction on i. □

theorem:nilpotent

Theorem 4.1.6. *Let G be a nilpotent group.*

1) *If H is a subgroup of G, then H is nilpotent.*
2) *If $f : G \to H$ is a surjective group homomorphism, then H is nilpotent.*

Proof. For the first statement note that $\gamma_i(H) \subseteq \gamma_i(G)$ for all $i \geq 1$. Let us prove the second claim, if there exists c such that $\gamma_{c+1}(G) = \{1\}$, then

$$\gamma_{c+1}(H) = f(\gamma_{c+1}(G)) = f(\{1\}) = \{1\}.$$ □

There exist a non-nilpotent group G with a normal subgroup K such that K and G/K are both nilpotent. For example, take $G = \mathbb{S}_3$ and $K = \mathbb{A}_3$.

theorem:gamma

Theorem 4.1.7. *Let G be a group. Then $[\gamma_i(G), \gamma_j(G)] \subseteq \gamma_{i+j}(G)$ for all $i, j \geq 1$.*

Proof. We proceed by induction on i. The case where $i = 1$ is trivial, as by definition one has $[G, \gamma_j(G)] = \gamma_{j+1}(G)$. Assume now that the result holds for some $i \geq 1$ and all $j \geq 1$. We first note that

$$[G, \gamma_i(G), \gamma_j(G)] \subseteq [G, \gamma_{i+j}(G)] = \gamma_{i+j+1}(G)$$

by the inductive hypothesis. Moreover, again using the inductive hypothesis,

$$[\gamma_i(G), \gamma_j(G), G] = [\gamma_i(G), G, \gamma_j(G)] = [\gamma_i(G), \gamma_{j+1}(G)] \subseteq \gamma_{i+j+1}(G).$$

Lemma 4.1.2 implies that $[\gamma_j(G), G, \gamma_i(G)] \subseteq \gamma_{i+j+1}(G)$. Thus

$$[\gamma_{i+1}(G), \gamma_j(G)] = [[G, \gamma_i(G)], \gamma_j(G)] = [\gamma_j(G), G, \gamma_i(G)] \subseteq \gamma_{i+j+1}(G).$$ □

Certainly we can consider other type of arbitrary commutators, say for example $[[G, G], G]$ and $[G, G, G] = [G, [G, G]]$. This naturally suggest the notion of the weight of a commutator. For example, $[[G, G], G]$ and $[G, G, G] = [G, [G, G]]$ are both commutators of weight three.

Corollary 4.1.8. *Every commutator of weight n is contained in $\gamma_n(G)$.*

Proof. We proceed by induction on n. The case $n = 1$ is trivial, so assume that $n \geq 1$ and the result holds for all m such that $n \geq m \geq 1$. Let $[A, B]$ be a commutator, where A is a commutator of weight k, B is a commutator of weight l and $n+1 = k+l$. Since $k \leq n$ and $l \leq n$, the inductive hypothesis implies that $A \subseteq \gamma_k(G)$ y $B \subseteq \gamma_l(G)$. Thus

$$[A, B] \subseteq [\gamma_k(G), \gamma_l(G)] \subseteq \gamma_{k+l}(G)$$

by the previous theorem. Therefore, the result follows by induction. □

A group G satisfies the *normalizer condition* if each proper subgroup H is properly contained in its normalizer $N_G(H)$, that is $H \subsetneq N_G(H)$.

lem:normalizer

Lemma 4.1.9 (normalizer condition). *Let G be a nilpotent group. If H is a proper subgroup of G, then $H \subsetneq N_G(H)$.*

Proof. Since G is nilpotent, there a positive integer c such that

$$G = \gamma_1(G) \supseteq \cdots \supseteq \gamma_{c+1}(G) = \{1\}.$$

Since $\{1\} = \gamma_{c+1}(G) \subseteq H$ and $\gamma_1(G) \nsubseteq H$, let k be the smallest positive integer such that $\gamma_k(G) \subseteq H$. Since

$$[H, \gamma_{k-1}(G)] \subseteq [G, \gamma_{k-1}(G)] = \gamma_k(G) \subseteq H,$$

it follows that $xHx^{-1} \subseteq H$ for all $x \in \gamma_{k-1}(G)$, so $\gamma_{k-1}(G) \subseteq N_G(H)$. If $N_G(H) = H$, then $\gamma_{k-1}(G) \subseteq H$, in contradiction with the minimality of k. Therefore H is a proper subgroup of $N_G(H)$. □

For a group G we define the sequence $\zeta_0(G), \zeta_1(G), \ldots$ recursively as

$$\zeta_0(G) = \{1\}, \quad \zeta_{i+1}(G) = \{g \in G : [x, g] \in \zeta_i(G) \text{ for all } x \in G\}, \quad i \geq 0.$$

In particular, $\zeta_1(G) = Z(G)$.

lem:central_ascendente

Lemma 4.1.10. *Let G be a group. Each $\zeta_i(G)$ is a normal subgroup of G.*

Proof. We proceed by induction on i. The case $i = 0$ is trivial, as $\zeta_0(G) = \{1\}$. Assume that the result holds for some $i \geq 0$. Thus $\zeta_i(G)$ is a normal subgroup of G. Let $\pi\colon G \to G/\zeta_i(G)$ be the canonical map. Note that

$$\zeta_{i+1}(G) = \pi^{-1}(Z(G/\zeta_i(g))).$$

Therefore $\zeta_{i+1}(G)$ is a normal subgroup of G, and the result follows by induction. □

For a group G the ***upper central series*** of G is the sequence

$$\{1\} = \zeta_0(G) \subseteq \zeta_1(G) \subseteq \zeta_2(G) \subseteq \cdots$$

A subgroup K of G *normalizes* a subgroup H if $K \subseteq N_G(H)$. A subgroup K of G *centralizes* a subgroup H if $K \subseteq C_G(H)$, that is if and only if $[H, K] = \{1\}$.

lem:gamma_zeta

Lemma 4.1.11. *Let G be a group. There exists c such that $\zeta_c(G) = G$ if and only if $\gamma_{c+1}(G) = \{1\}$. In this case,*

$$\gamma_{i+1}(G) \subseteq \zeta_{c-i}(G)$$

for all $i \in \{0, 1, \ldots, c\}$.

Proof. For $c = 0$, the result is trivial. Suppose that $c \geq 1$. Assume first that $\zeta_c(G) = G$. To prove that $\gamma_{i+1}(G) \subseteq \zeta_{c-i}(G)$ we proceed by induction. The case where $i = 0$ is trivial, so assume the result holds for some $i \geq 0$. Let $x \in G$ and $g \in \gamma_{i+1}(G)$. By the inductive hypothesis $g \in \zeta_{c-i}(G)$. Hence $[x, g] \in \zeta_{c-i-1}(G)$ and $\gamma_{i+2}(G) \subseteq \zeta_{c-i-1}$. By induction, this claim follows. In particular, $\gamma_{c+1}(G) \subseteq \zeta_0(G) = \{1\}$.

Assume now that $\gamma_{c+1}(G) = \{1\}$. We prove that $\gamma_{i+1}(G) \subseteq \zeta_{c-i}(G)$ for all i. We proceed by backward induction on i. The case $i = c$ is trivial, so assume the result holds for some $i + 1 \leq c$. Let $g \in \gamma_i(G)$. By the inductive hypothesis,

$$[x, g] \in [G, \gamma_i(G)] = \gamma_{i+1}(G) \subseteq \zeta_{c-i}(G),$$

for all $x \in G$. Thus $g \in \zeta_{c-i+1}(G)$ by definition. Hence $\gamma_i(G) \subseteq \zeta_{c-i+1}(G)$, and the result follows by induction. □

Example 4.1.12. If $G = \mathbb{S}_3$, then $\zeta_j(G) = \{1\}$ for all $j \geq 0$.

A *central series* of a group G is a sequence

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \{1\}$$

of normal subgroups of G such that for each $i \in \{1, \ldots, n\}$, $\pi_i(G_{i-1})$ is a subgroup of $Z(G/G_i)$, where $\pi_i \colon G \to G/G_i$ is the canonical map.

pro:serie_central

Proposition 4.1.13. *Let G be a group and let $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \{1\}$ be a central series of G. Then $\gamma_{i+1}(G) \subseteq G_i \subseteq \zeta_{n-i}(G)$ for all i.*

Proof. We shall prove that $\gamma_{i+1}(G) \subseteq G_i$ by induction on i. The case $i = 0$ is trivial, so we assume that the result holds for some $i \geq 0$. Let $\pi_{i+1} \colon G \to G/G_{i+1}$ is the canonical map. Since $\pi_{i+1}(G_i) \subseteq Z(G/G_{i+1})$, it follows that

$$\pi_{i+1}([G, G_i]) = [\pi_{i+1}(G), \pi_{i+1}(G_i)] = \{1\}.$$

This implies that $[G, G_i] \subseteq \ker \pi_{i+1} = G_{i+1}$. Hence, by the inductive hypothesis,

$$\gamma_{i+2}(G) = [G, \gamma_{i+1}(G)] \subseteq [G, G_i] \subseteq G_{i+1}.$$

Thus the first inclusion follows by induction.

We now prove that $G_i \subseteq \zeta_{n-i}(G)$. We proceed by induction on $n - i$. If $n - i = 0$, then $G_n = \{1\} = \zeta_0(G)$. Suppose that the inclusion we want to prove holds for some $n - i \geq 0$. Since $G_i \subseteq \zeta_{n-i}(G)$, there exists a surjective group homomorphism $f \colon G/G_i \to G/\zeta_{n-i}(G)$, such that $f(gG_i) = g\zeta_{n-i}(G)$ for all $g \in G$. Hence

$f(Z(G/G_i)) \subseteq Z(G/\zeta_{n-i}(G))$. Since $f\pi_i \colon G \to G/\zeta_{n-i}(G)$ is the canonical map and $f(\pi_i(G_{i-1})) \subseteq f(Z(G/G_i)) \subseteq Z(G/\zeta_{n-i}(G))$,

$$G_{i-1} \subseteq (f\pi_i)^{-1}(Z(G/\zeta_{n-i}(G))) = \zeta_{n-i+1}(G).$$

Therefore the result follows. □

thm:Z(nilpotent)

Theorem 4.1.14 (Hirsch). *Let G be a non-trivial nilpotent group. If H is a non-trivial normal subgroup of G, then $H \cap Z(G) \neq \{1\}$. In particular, $Z(G) \neq \{1\}$.*

Proof. Since $\zeta_0(G) = \{1\}$ and there exists $c \geq 1$ such that $\zeta_c(G) = G$, there exists

$$m = \min\{k : H \cap \zeta_k(G) \neq \{1\}\}.$$

Since H is normal,

$$[H \cap \zeta_m(G), G] \subseteq H \cap [\zeta_m(G), G] \subseteq H \cap \zeta_{m-1}(G) = \{1\}.$$

Thus $\{1\} \neq H \cap \zeta_m(G) \subseteq H \cap Z(G)$. If $H = G$, then $Z(G) \neq \{1\}$. □

A subgroup M of G is *maximal normal* if it is maximal among all normal proper subgroups of G.

Corollary 4.1.15. *Let G be a non-abelian nilpotent group and A be an abelian maximal normal subgroup of G. Then $A = C_G(A)$.*

Proof. Since A is abelian, $A \subseteq C_G(A)$. Assume that $A \neq C_G(A)$. We know that the centralizer of a normal subgroup is normal. Thus the centralizer $C_G(A)$ is normal in G. Let $\pi \colon G \to G/A$ be the canonical map. Then $\pi(C_G(A))$ is a non-trivial normal subgroup of $\pi(G)$. Since G is nilpotent, $\pi(G)$ is nilpotent. By Hirsch's theorem, $\pi(C_G(A)) \cap Z(\pi(G)) \neq \{1\}$. Let $x \in C_G(A) \setminus A$ be such that $\pi(x)$ is central in $\pi(G)$. Note that if $g \in G$, then $gxg^{-1} \in xA$. Hence $\langle A, x\rangle$ is an abelian normal subgroup of G such that $A \subsetneq \langle A, x\rangle \subseteq G$, a contradiction. Therefore $A = C_G(A)$. □

A subgroup M of a group G is said to be *minimal normal* if $M \neq \{1\}$, M is normal in G and the unique normal subgroup of G strictly contained in M is the trivial subgroup. Every finite group contains a minimal normal subgroup.

Example 4.1.16. If a non-trivial normal subgroup M is minimal (with respect to the inclusion), then it is minimal normal. The converse statement is not true. The subgroup of $\mathbb{A}_4$ generated by $(12)(34)$, $(13)(24)$ and $(14)(23)$ is minimal normal in $\mathbb{A}_4$ but it is not minimal **non-trivial**.

Example 4.1.17. Let $G = \mathbb{D}_6 = \langle r, s : r^6 = s^2 = 1, srs = r^{-1}\rangle$ be the dihedral group of size twelve. The subgroups $S = \langle r^2\rangle$ and $T = \langle r^3\rangle$ are minimal normal subgroups.

theorem:minmax_nilpotent

Theorem 4.1.18. *Let G be a non-trivial nilpotent group.*

1) *Every minimal normal subgroup of G has prime order and it is central.*

2) *Every maximal subgroup of G is normal, has prime index and contains $[G,G]$.*

Proof. We first prove (1). Let N be a minimal normal subgroup of G. Since G is nilpotent, $N \cap Z(G) \neq \{1\}$ by Hirsch's theorem. It follows that $N \cap Z(G)$ is a normal subgroup of G contained in N. Thus $N = N \cap Z(G) \subseteq Z(G)$ by the minimality of N. In particular, N is abelian. Since every subgroup of N is normal in G, N should be cyclic of prime order.

We now prove (2). If M is a maximal subgroup, then M is normal in G by the normalizer condition. The maximality of M implies that G/M contains no non-trivial proper subgroups. It follows that G/M is cyclic of prime order. In particular, G/M is abelian and thus $[G,G] \subseteq M$. □

The theorem does not prove the existence of maximal subgroups, see for example what happens with the additive group of rational numbers.

pro:g^n **Proposition 4.1.19.** *Let G be a nilpotent group and H be a subgroup of G of index n. If $g \in G$, then $g^n \in H$.*

Proof. We proceed by induction on n. The case $n = 1$ is trivial. Assume that $n > 1$ and the result holds for all subgroups of index $< n$. If H is a normal subgroup of index n, then clearly $g^n \in H$ for all $g \in G$. Hence we may assume that H is a non-normal subgroup of G of index n. By the normalizer condition, $H \subsetneq N_G(H)$. Hence $(G : N_G(H)), (N_G(H) : H) < n$. By the inductive hypothesis, $g^{(G:N_G(H))} \in N_G(H)$ and thus

$$g^n = g^{(G:H)} = g^{(G:N_G(H))(N_G(H):H)} \in H,$$

for all $g \in G$. The result follows. □

Example 4.1.20. The nilpotency of G is needed in the previous proposition. If $G = \mathbb{S}_3$ and $H = \{\mathrm{id}, (12)\}$, then $(G : H) = 3$. If $g = (13)$, then $g^3 = (13) \notin H$.

The following tool is useful to perform induction on nilpotent groups.

lem:a[GG] **Lemma 4.1.21.** *Let G be a nilpotent group of class $c \geq 2$. If $x \in G$, then the subgroup $\langle x, [G,G]\rangle$ is nilpotent of class $< c$.*

Proof. Let $H = \langle x, [G,G]\rangle$. Note that

$$H = \{x^n c : n \in \mathbb{Z}, c \in [G,G]\},$$

as $[G,G]$ is normal in G.

It is enough to show that $[H,H] \subseteq \gamma_3(G)$. Let $h = x^n c, k = x^m d \in H$ where $c, d \in [G,G]$. Since

$$[h,x^m] = [x^n,[c,x^m]][c,x^m] \in \gamma_4(G)\gamma_3(G) \subseteq \gamma_3(G),$$

it follows that

$$\begin{aligned}[h,k] &= [h,x^m][x^m,[h,d]][h,d] \\ &= [x^n,[c,x^m]][c,x^m][x^m,[h,d]][h,d] \in \gamma_3(G).\end{aligned}$$

□

Example 4.1.22. Let $G = \mathbb{D}_8 = \langle r, s : r^8 = s^2 = 1, srs = r^{-1} \rangle$ be the dihedral group of order 16. Then G is nilpotent of class three and $[G,G] = \{1, r^2, r^4, r^6\} \simeq C_4$. The subgroup $\langle s, [G,G] \rangle \simeq \mathbb{D}_4$ is nilpotent of class two.

Now an application of Lemma 4.1.21.

thm:T(nilpotent)

Theorem 4.1.23. *If G is a nilpotent group, then*

$$T(G) = \{g \in G : g^n = 1 \text{ for some positive integer } n\}$$

is a subgroup of G.

Proof. We show the result by induction on the nilpotency class c of G. If G is abelian, then the result is clear. Suppose that $c \geq 2$ and that the result holds for all nilpotent groups of class $< c$. Let $a, b \in T(G)$ and let $A = \langle a, [G,G] \rangle$ and $B = \langle b, [G,G] \rangle$. Since A and B are both nilpotent of class $< c$ by the previous lemma, the inductive hypothesis implies that $T(A)$ is a subgroup of A and $T(B)$ is a subgroup of B. Since $T(A)$ is characteristic in A and A is normal in G, it follows that $T(A)$ is normal in G. Similarly, $T(B)$ is normal in G. We claim that every element of $T(A)T(B)$ has finite order. Indeed, if $x \in T(A)T(B)$, say $x = a_1 b_1$ with $a_1 \in T(A)$ of order m and $b_1 \in T(B)$, then x has finite order since

$$x^m = (a_1 b_1)^m = (a_1 b_1 a_1^{-1})(a_1^2 b_1 a_1^{-2}) \cdots (a_1^{m-1} b_1 a_1^{-m+1}) b_1 \in T(B).$$

In particular, ab has finite order. It follows that $T(G)$ is a subgroup of G. □

Another application of Lemma 4.1.21.

thm:a=b

Theorem 4.1.24. *Let G be a torsion-free nilpotent group and let $a, b \in G$. If $a^n = b^n$ for some $n \neq 0$, then $a = b$.*

Proof. We proceed by induction on the nilpotency class c of G. The claim holds if G is abelian. Assume that G is nilpotent of class $c \geq 2$. Since $\langle a, [G,G] \rangle$ is a nilpotent subgroup of G of nilpotency class $< c$ and $bab^{-1} = [b,a]a \in \langle a, [G,G] \rangle$, the inductive hypothesis implies that $ba = ab$, as $a^n = (bab^{-1})^n = b^n$. Thus $(ab^{-1})^n = a^n b^{-n} = 1$. Since G is torsion-free, it follows that $a = b$. □

Corollary 4.1.25. *Let G be a torsion-free nilpotent group. If $x, y \in G$ are such that $x^n y^m = y^m x^n$ for some $n, m \neq 0$, then $xy = yx$.*

Proof. Let $a = x$ and $b = y^m x y^{-m}$. Since $a^n = b^n$, Theorem 4.1.24 implies that $a = b$. Thus $xy^m = y^m x$. By using Theorem 4.1.24 now with $a = y$ and $b = xyx^{-1}$, we conclude that $xy = yx$. □

We need the following lemma.

lem:fg

Lemma 4.1.26. *Let G be a finitely generated group and H be a finite-index subgroup of G. Then H is finitely generated.*

Proof. Assume that $G = \langle g_1, \dots, g_m \rangle$. Without loss of generality we may assume that for each i there exists k such that $g_i^{-1} = g_k$. Let $\{1 = t_1, \dots, t_n\}$ be a right transversal of H in G. For $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$ we write

$$t_i g_j = h(i,j) t_{k(i,j)},$$

where $h(i,j) \in H$. We claim that H is generated by the $h(i,j)$. If $x \in H$, then

$$\begin{aligned} x &= g_{i_1} \cdots g_{i_s} \\ &= (t_1 g_{i_1}) g_{i_2} \cdots g_{i_s} \\ &= h(1,i_1) t_{k_1} g_{i_2} \cdots g_{i_s} \\ &= h(1,i_1) h(k_1,i_2) t_{k_2} g_{i_3} \cdots g_{i_s} \\ &= h(1,i_1) h(k_1,i_2) \cdots h(k_{s-1},i_s) t_{k_s}, \end{aligned}$$

where $k_1, \dots, k_{s-1} \in \{1, \dots, n\}$. Thus $t_{k_s} \in H$ and hence $t_{k_s} = 1$, which implies the claim. □

Let G be a group. The *order* of an element $x \in G$ is the order of the group $\langle x \rangle$, and it is denoted by $|x|$. The group G is said to be *torsion* or *periodic* if every element of G has finite order.

thm:T(G)finito

Theorem 4.1.27. *Let G be a finitely generated torsion nilpotent group. Then G is finite.*

Proof. We proceed by induction on the nilpotency class c. The case $c = 1$ is true since G is abelian. Assume that $c \geq 2$ and the result holds groups of nilpotency class $< c$. Since $G/[G,G]$ is an abelian finitely generated torsion group, it is finite. By Lemma 4.1.26, $[G,G]$ also is finitely generated. Since $[G,G]$ is a finitely generated torsion nilpotent group of class $< c$, the inductibe hypothesis implies that $[G,G]$ is finite. Thus G is finite. Therefore the result follows by induction. □

4.2 Finite nilpotent groups

Let G be a finite group and p a prime. We denote by $\mathrm{Syl}_p(G)$ the set of the Sylow p-subgroups of G.

lemma:N_G(H)=H

Lemma 4.2.1. *Let G be a finite group and let $P \in \mathrm{Syl}_p(G)$. If H is a subgroup of G such that $N_G(P) \subseteq H$, then $N_G(H) = H$.*

Proof. Let $x \in N_G(H)$. Since $P \in \mathrm{Syl}_p(H)$ and $xPx^{-1} \subseteq xHx^{-1} = H$, we have that there exists $h \in H$ such that $(hx)P(hx)^{-1} = P$. Hence $hx \in N_G(P) \subseteq H$, and thus $x \in H$. This proves the result. □

pgroup

Lemma 4.2.2. *Let p be a prime number. Then p-groups are nilpotent.*

Proof. We know that every non-trivial p-group has non-trivial center. Let G be a p-group. Suppose that $|G| = p^n$. We shall prove the result by induction on n. For $n = 1$, G is cyclic of order p and the result is trivial. Suppose that $n > 1$ and that the result holds for p-groups of order $< p^n$. We may assume that G is non-abelian. Since $Z(G)$ is a non-trivial proper subgroup of G, by the inductive hypothesis, $Z(G)$ and $G/Z(G)$ are nilpotent. Let $\pi\colon G \to G/Z(G)$ be the canonical map. By Lemma 4.1.11, there exists c such that $\zeta_c(G/Z(G)) = G/Z(G)$. Note that

$$\pi^{-1}(\zeta_i(G/Z(G))) = \zeta_{i+1}(G).$$

Hence $\zeta_{c+1}(G) = G$ is nilpotent by Lemma 4.1.11. Therefore the result follows by induction. □

We say that a group G is the *direct product* of its subgroups $H_1, \ldots, H_n$ if every subgroup H_i is normal in G, $G = H_1 \cdots H_n$ and $H_{k+1} \cap H_1 \cdots H_k = \{1\}$ for all $k = 1, \ldots, n-1$. Note that, for $i < j$, $h_i \in H_i$ and $h_j \in H_j$, we have that $[h_i, h_j] \in H_i \cap H_j = \{1\}$. Note that, in this case the map $f\colon H_1 \times \cdots \times H_n \to G$, defined by $f(h_1, \ldots, h_n) = h_1 \cdots h_n$ for all $h_i \in H_i$, is an isomorphism of groups. It is clear that f is surjective. Let $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in H_1 \times \cdots \times H_n$. We have that

$$\begin{aligned} f((a_1, \ldots, a_n)(b_1, \ldots, b_n)) &= f(a_1 b_1, \ldots, a_n b_n) \\ &= a_1 b_1 \cdots a_n b_n \\ &= a_1 \cdots a_n b_1 \cdots b_n \\ &= f(a_1, \ldots, a_n) f(b_1, \ldots, b_n). \end{aligned}$$

Thus f is a surjective homomorphism of groups. One proves that $\ker(f) = \{1\}$.

thm:nilpotente:eq

Theorem 4.2.3. *Let G be a finite group. The following conditions are equivalent:*

1) *G is nilpotent.*
2) *Every Sylow subgroup of G is normal.*
3) *G is the direct product of its Sylow subgroups.*

Proof. We prove that 1) $\implies$ 2). Suppose that G is nilpotent. Let $P \in \mathrm{Syl}_p(G)$. By Lemma 4.2.1, $N_G(N_G(P)) = N_G(P)$. Now $N_G(P) = G$ by the normalizer condition. Hence P is normal in G.

Now we prove that 2) $\implies$ 3). Let $p_1, \ldots, p_n$ be the distinct prime divisors of $|G|$. For every $i \in \{1, \ldots, n\}$, let $P_i \in \mathrm{Syl}_{p_i}(G)$. By hypothesis, each P_j is normal in G. Since $|P_1 \cdots P_n| = |G|$, we have that $G = P_1 \cdots P_n$. Clearly $|P_{k+1}|, |P_1 \cdots P_k|$ are coprime for all $k = 1, \ldots, n-1$. Hence $P_{k+1} \cap P_1 \cdots P_k = \{1\}$ for all $k = 1, \ldots, n-1$. Thus G is the direct product of its Sylow subgroups.

Finally, to prove that 3) $\implies$ 1) we note that every p-group is nilpotent and that a finite direct product of nilpotent groups is nilpotent. □

Theorem 4.2.4 (Baumslag–Wiegold). *Let G be a finite group such that $|xy| = |x||y|$ whenever $x, y \in G$ have coprime order. Then G is nilpotent.*

Proof. Let $p_1, \dots, p_n$ be the distinct prime divisors of the order of G. For each $i \in \{1, \dots, n\}$ let $P_i \in \mathrm{Syl}_{p_i}(G)$. We claim that $G = P_1 \cdots P_n$. The non-trivial inclusion is equivalent to show that the map

$$\psi \colon P_1 \times \cdots \times P_n \to G, \quad (x_1, \dots, x_n) \mapsto x_1 \cdots x_n$$

is surjective. We first show that ψ is injective. If $\psi(x_1, \dots, x_n) = \psi(y_1, \dots, y_n)$, then

$$x_1 \cdots x_n = y_1 \cdots y_n.$$

If $y_n \neq x_n$, then $x_1 \cdots x_{n-1} = (y_1 \cdots y_{n-1}) y_n x_n^{-1}$. But $x_1 \cdots x_{n-1}$ has order coprime with p_n and $y_1 \cdots y_{n-1} y_n x_n^{-1}$ has order divisible by p_n, a contradiction. Thus $x_n = y_n$ and hence the same argument proves that ψ is injective. Since $|P_1 \times \cdots \times P_n| = |G|$, we conclude that ψ is bijective. In particular, ψ is surjective and hence $G = P_1 \cdots P_n$.

We now show that each P_j is normal in G. Let $j \in \{1, \dots, n\}$ and $x_j \in P_j$. Let $g \in G$ and $y_j = g x_j g^{-1}$. Since $y_j \in G$, we write $y_j = z_1 \cdots z_n$ with $z_k \in P_k$ for all k. Since the order of y_j is a power of p_j, it follows that $z_1 \cdots z_n$ has order a power of p_j. Thus $z_k = 1$ for all $k \neq j$ and $y_j = z_j \in P_j$. Since each Sylow subgroup is normal, G is nilpotent, by Theorem 4.2.3. □

4.3 Fratini subgroup

Let G be a group. If G has maximal subgroups, the *Frattini subgroup* $\Phi(G)$ of G is defined as the intersection of all maximal subgroups of G. Otherwise, $\Phi(G) = G$.

Note that $\Phi(G)$ is a characteristic subgroup of G.

Example 4.3.1. Let $G = \mathbb{S}_3$. The maximal subgroups of G are

$$M_1 = \langle (123) \rangle, \quad M_2 = \langle (12) \rangle, \quad M_3 = \langle (23) \rangle, \quad M_4 = \langle (13) \rangle.$$

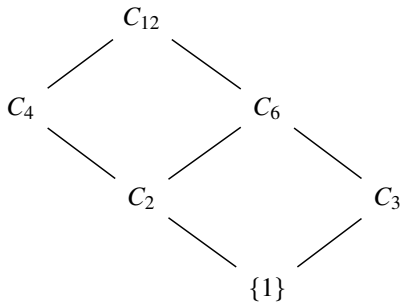
Thus $\Phi(G) = \{1\}$.

Example 4.3.2. Let $G = \langle g \rangle \simeq C_{12}$. The subgroups of G are

$$\{1\}, \quad \langle g^6 \rangle \simeq C_2, \quad \langle g^4 \rangle \simeq C_3, \quad \langle g^3 \rangle \simeq C_4, \quad \langle g^2 \rangle \simeq C_6, \quad G.$$

The maximal subgroups are $\langle g^3 \rangle \simeq C_4$ and $\langle g^2 \rangle \simeq C_6$. Hence

$$\Phi(G) = \langle g^3 \rangle \cap \langle g^2 \rangle = \langle g^6 \rangle \simeq C_2.$$



lem:Dedekind

Lemma 4.3.3 (Dedekind). *Let H, K and L be subgroups of G such that $H \subseteq L \subseteq G$. Then $HK \cap L = H(K \cap L)$.*

Proof. We only need to prove that $HK \cap L \subseteq H(K \cap L)$, as the other inclusion is trivial. If $x = hk \in HK \cap L$, where $x \in L$, $h \in H$ and $k \in K$, then $k = h^{-1}x \in L \cap K$ since $H \subseteq L$. Thus $x = hk \in H(L \cap K)$. □

lem:G=HPhi(G)

Lemma 4.3.4. *Let G be a finite group. If H is a subgroup of G such that $G = H\Phi(G)$, then $H = G$.*

Proof. Assume that $H \neq G$. Let M be a maximal subgroup of G such that $H \subseteq M$. Since $\Phi(G) \subseteq M$, it follows that $G = H\Phi(G) \subseteq M$, a contradiction □

pro:phi(N)phi(G)

Proposition 4.3.5. *Let G be a finite group and N be a normal subgroup of G. Then $\Phi(N) \subseteq \Phi(G)$.*

Proof. Since $\Phi(N)$ is characteristic in N and N is normal in G, it follows that $\Phi(N)$ is normal in G. Let M be a maximal subgroup of G such that $\Phi(N) \nsubseteq M$. Then $\Phi(N)M = G$, as $M = \Phi(N)M \supseteq \Phi(N)$ otherwise. By applying Lemma 4.3.3 with $H = \Phi(N)$, $K = M$ and $L = N$,

$$N = G \cap N = (\Phi(N)M) \cap N = \Phi(N)(M \cap N).$$

By Lemma 4.3.4 $N = M \cap N \subseteq M$, a contradiction. Hence every maximal subgroup of G contains $\Phi(N)$ and therefore $\Phi(G) \supseteq \Phi(N)$. □

The Frattini subgroup can be characterized in terms of non-generators.

lemma:nongenerators

Lemma 4.3.6 (non-generators). *Let G be a finite group. Then*

$$\Phi(G) = \{x \in G : \text{if } G = \langle x, Y\rangle \text{ for some } Y \subseteq G, \text{ then } G = \langle Y\rangle\}.$$

Proof. Let $x \in \Phi(G)$ be such that $G = \langle x, Y\rangle$ for some subset Y of G. If $G \neq \langle Y\rangle$, then there exists a maximal subgroup M such that $\langle Y\rangle \subseteq M$. Since $x \in M$, $G = \langle x, Y\rangle \subseteq M$, a contradiction.

Conversely, let $x \in G$ and M be a maximal subgroup of G. If $x \notin M$, then, since $G = \langle x, M\rangle$, it follows that $G = \langle M\rangle = M$, a contradiction. Thus $x \in M$ for all maximal subgroup M. Hence $x \in \Phi(G)$. □

Lemma 4.3.7 (Frattini's argument). *Let G be a finite group, H a normal subgroup of G and p a prime. Then, for every $P \in \mathrm{Syl}_p(H)$, $G = HN_G(P)$.*

Proof. Let $P \in \mathrm{Syl}_p(H)$ and $g \in G$. We have that $gPg^{-1} \subseteq gHg^{-1} = H$. Hence $gPg^{-1} \in \mathrm{Syl}_p(H)$. Therefore, there exists $h \in H$ such that $hPh^{-1} = gPg^{-1}$. Thus $h^{-1}g \in N_G(P)$. Now we have that $g = hh^{-1}g \in HN_G(P)$, and the result follows. □

theorem:Frattini

Theorem 4.3.8 (Frattini). *If G is a finite group, then $\Phi(G)$ is nilpotent.*

Proof. Let $P \in \mathrm{Syl}_p(\Phi(G))$ for some prime p. Since $\Phi(G)$ is normal in G, Frattini's argument implies that $G = \Phi(G)N_G(P)$. By Lemma 4.3.4, $G = N_G(P)$. Since Sylow subgroups of $\Phi(G)$ are normal in G, $\Phi(G)$ is nilpotent, by Theorem 4.2.3. □

thm:Gaschutz

Theorem 4.3.9 (Gaschütz). *If G is a finite group, then $[G,G] \cap Z(G) \subseteq \Phi(G)$.*

Proof. Let $D = [G,G] \cap Z(G)$. Assume that D is not contained in $\Phi(G)$. Hence there exists a maximal subgroup M such that D is not contained in M. Thus $G = MD$. Since $D \subseteq Z(G)$, it is clear that M is normal in G. Since $G/M \cong D/(D \cap M)$ is abelian, $[G,G] \subseteq M$ and hence $D \subseteq [G,G] \subseteq M$, a contradiction. Therefore, the result follows. □

theorem:Wielandt

Theorem 4.3.10 (Wielandt). *Let G be a finite group. Then G is nilpotent if and only if $[G,G] \subseteq \Phi(G)$.*

Proof. Assume first that $[G,G] \subseteq \Phi(G)$. Let $P \in \mathrm{Syl}_p(G)$. If $N_G(P) \neq G$, then $N_G(P) \subseteq M$ for some maximal subgroup M of G. Let $g \in G$ and $m \in M$. Since

$$gmg^{-1}m^{-1} = [g,m] \in [G,G] \subseteq \Phi(G) \subseteq M,$$

we get that $gmg^{-1} \in M$. Hence M is normal in G, that is $G = N_G(M)$. But, since $N_G(P) \subseteq M$, by lemma 4.2.1, $N_G(M) = M$, a contradiction. Therefore P is normal in G, and thus G is nilpotent, by Theorem 4.2.3.

The converse follows by Theorem 4.1.18. □

theorem:G/phi(G)

Theorem 4.3.11. *Let G be a finite group. Then G is nilpotent if and only if $G/\Phi(G)$ is nilpotent.*

Proof. If G is nilpotent, then clearly $G/\Phi(G)$ also is nilpotent. Suppose that $G/\Phi(G)$ is nilpotent. Let $P \in \mathrm{Syl}_p(G)$. Since $\Phi(G)P/\Phi(G) \in \mathrm{Syl}_p(G/\Phi(G))$ and $G/\Phi(G)$ is nilpotent, $\Phi(G)P/\Phi(G)$ is a normal subgroup of $G/\Phi(G)$. Hence, $\Phi(G)P$ is normal in G. By Frattini's argument,

$$G = \Phi(G)PN_G(P) = \Phi(G)N_G(P).$$

By Lemma 4.3.4), $G = N_G(P)$, and thus P is normal in G. By Theorem 4.2.3, G is nilpotent. □

theorem:Hall_nilpotente

Theorem 4.3.12 (Hall). *Let G be a finite group and N a normal subgroup of G. If N and $G/[N,N]$ are nilpotent, then G is nilpotent.*

Proof. Since N is nilpotent, by Theorem 4.3.10, $[N,N] \subseteq \Phi(N)$. By Proposition 4.3.5, $[N,N] \subseteq \Phi(N) \subseteq \Phi(G)$. Hence, there exists a surjective homomorphism $G/[N,N] \to G/\Phi(G)$. Since $G/[N,N]$ is nilpotent, we get that $G/\Phi(G)$ is nilpotent. Hence G is nilpotent by Theorem 4.3.11. □

Definition 4.3.13. A *minimal generating set* of a group G is a subset X of G such that $G = \langle X \rangle$ and $G \neq \langle Y \rangle$ for every proper subset Y of X.

Note that the minimal number of generators of a group G can be less than the number of element of a minimal generating set of G.

Example 4.3.14. Let $G = \langle g \mid g^6 = 1 \rangle$. If $a = g^2$ and $b = g^3$, then $\{a, b\}$ is a minimal generating set of G, but the minimal number of generators of G is 1.

lemma:Burnside:minimal

Lemma 4.3.15. *Let p be a prime and G a p-group. Then $G/\Phi(G)$ is a vector space over the field $\mathbb{F}_p$ of p elements.*

Proof. Let K be a maximal subgroup of G. Since G is nilpotent, by Theorem 4.1.18, K is normal in G of index p. Hence G/K is a cyclic group of order p. Let $K_1, \ldots, K_m$ be the distinct maximal subgroups of G. Let $f\colon G \to G/K_1 \times \cdots \times G/K_m$ be the map defined by $f(g) = (gK_1, \ldots, gK_m)$ for all $g \in G$. It is clear that f is a homomorphism of groups and $\ker(f) = \Phi(G)$. Hence $G/\Phi(G)$ is isomorphic to $\mathrm{im}(f)$, a subgroup of the $\mathbb{F}_p$-vector space $G/K_1 \times \cdots \times G/K_m$. Therefore, the result follows. □

theorem:Burnside:basis

Theorem 4.3.16 (Burnside). *Let p be a prime and G a p-group. If X is a minimal generating set of G, then $|X| = \dim_{\mathbb{F}_p}(G/\Phi(G))$.*

Proof. By the previous lemma, we know that $G/\Phi(G)$ is an $\mathbb{F}_p$-vector space. Let $\pi\colon G \to G/\Phi(G)$ be the canonical map and $\{x_1, \ldots, x_n\}$ a minimal generating set of G, with $|\{x_1, \ldots, x_n\}| = n$. We shall see that $\pi(x_1), \ldots, \pi(x_n)$ are linearly independent in $G/\Phi(G)$. Suppose that $\pi(x_1), \ldots, \pi(x_n)$ are not linearly independent. Without loss of generality, we may assume that $\pi(x_1) \in \langle \pi(x_2), \ldots, \pi(x_n) \rangle$. Then there exists $y \in \langle x_2, \ldots, x_n \rangle$ such that $x_1 y^{-1} \in \Phi(G)$. Since $G = \langle x_1 y^{-1}, x_2, \ldots, x_n \rangle$ and $x_1 y^{-1} \in \Phi(G)$, by lemma 4.3.6, we have that $G = \langle x_2, \ldots, x_n \rangle$, a contradiction. Therefore $n = \dim_{\mathbb{F}_p}(G/\Phi(G))$. □

4.4 Fitting subgroup

Definition 4.4.1. Let G be a finite group and p a prime. The *p-radical* of G is the subgroup

$$O_p(G) = \bigcap_{P \in \mathrm{Syl}_p(G)} P.$$

The following result is an easy consequence of Sylow's theorem.

lemma:core:Op(G)

Lemma 4.4.2. *Let G be a finite group and p a prime.*

1) *$O_p(G)$ is normal in G.*
2) *If N is a normal p-subgroup of G, then $N \subseteq O_p(G)$.*

Definition 4.4.3. Let G be a finite group and let $p_1, \ldots, p_k$ be the distinct prime divisors of $|G|$. The *Fitting's subgroup* of G is the subgroup

$$F(G) = O_{p_1}(G) \cdots O_{p_k}(G)$$

Note that $F(G)$ is a characteristic subgroup of G.

Example 4.4.4. Let $G = \mathbb{S}_3$. It is easy to see that $O_2(G) = \{1\}$ and $O_3(G) = \langle(123)\rangle$. Thus $F(G) = \langle(123)\rangle$.

theorem:Fitting

Theorem 4.4.5 (Fitting). *Let G be a finite group. The Fitting subgroup $F(G)$ is nilpotent and it is normal in G. Furthermore $F(G)$ contains all normal nilpotent subgroups of G.*

Proof. Note that for every prime p, $O_p(G)$ is a Sylow p-subgroup of $F(G)$. Since $O_p(G)$ is normal in G, also is normal in $F(G)$. Hence by Theorem 4.2.3, $F(G)$ is nilpotent.

Let N be a normal nilpotent subgroup of G and let $P \in \mathrm{Syl}_p(N)$. Since N is nilpotent, P is normal in N. Then P is the unique Sylow p-subgrup of N. Hence P is a characteristic subgroup of N and thus P is normal in G. By Lemma 4.4.2, $P \subseteq O_p(G)$. By Theorem 4.2.3, N is the direct product of its Sylow subgroups. Hence $N \subseteq F(G)$. □

corollary:Z(G)subsetF(G)

Corollary 4.4.6. *Let G be a finite group. Then $Z(G) \subseteq F(G)$.*

Proof. Since $Z(G)$ is a normal nilpotent subgroup of G, by Fitting's theorem, $Z(G) \subseteq F(G)$. □

corollary:Fitting

Corollary 4.4.7 (Fitting). *Let K and L be normal nilpotent subgroups of a finite group G. Then KL is nilpotent.*

Proof. By Fitting's theorem, K and L are subgroups of $F(G)$. Hence KL also is a subgroup of $F(G)$. By Fitting's theorem, $F(G)$ is nilpotent. Therefore KL also is nilpotent. □

corollary:McapF(G)

Corollary 4.4.8. *Let G be a finite group and N a normal subgroup of G. Then $N \cap F(G) = F(N)$.*

Proof. Since $F(N)$ is a characteristic subgroup of N, $F(N)$ is normal in G. By Fitting's theorem, $F(N)$ is nilpotent and $F(N) \subseteq N \cap F(G)$. On the other hand, since $F(G)$ is normal in G, we have that $F(G) \cap N$ is normal in N. By Fitting's theorem $F(G)$ is nilpotent and thus $F(G) \cap N$ also is nilpotent. Again by Fitting's theorem, $F(G) \cap N \subseteq F(N)$. Therefore the result follows. □

Theorem 4.4.9. *Let G be a finite group. Then $\Phi(G) \subseteq F(G)$ and $F(G)/\Phi(G) \cong F(G/\Phi(G))$.*

Proof. By Theorem 4.3.8, $\Phi(G)$ is a normal nilpotent subgroup of G. Hence, by Fitting's theorem, $\Phi(G) \subseteq F(G)$.

Let $\pi\colon G \to G/\Phi(G)$ be the canonical map. By Fitting's theorem $F(G)$ is nilpotent, thus $\pi(F(G))$ also is nilpotent. Since $\Phi(G) \subseteq F(G)$ and $F(G)$ is normal in G, we have that $\pi(F(G)) = F(G)/\Phi(G)$ is normal in $G/\Phi(G)$. By Fitting's theorem,

$$\pi(F(G)) \subseteq F(G/\Phi(G)).$$

Let $H = \pi^{-1}(F(G/\Phi(G)))$. We know that H i a normal subgroup of G such that $\Phi(G) \subseteq F(G) \subseteq H$. Let $P \in \mathrm{Syl}_p(H)$. Then $\pi(P) = (P\Phi(G))/\Phi(G) \cong P/P \cap \Phi(G)$ is a p-subgroup of $\pi(H) = F(G/\Phi(G))$. Since

$$(\pi(H) : \pi(P)) = \frac{|\pi(H)|}{|\pi(P)|} = \frac{|H/\Phi(G)|}{|P/P \cap \Phi(G)|} = \frac{(H : P)}{(\Phi(G) : P \cap \Phi(G))}$$

is a divisor of $(H : P)$, we have that $\pi(P) \in \mathrm{Syl}_p(\pi(H))$. By Fitting's theorem $\pi(H)$ is nilpotent. Hence $\pi(P)$ is a characteristic subgroup of $\pi(H)$, and thus $P\Phi(G) = \pi^{-1}(\pi(P))$ is normal in G. Since $P \in \mathrm{Syl}_p(P\Phi(G))$, by Frattini's argument, $G = P\Phi(G)N_G(P) = \Phi(G)N_G(P)$. By Lemma 4.3.4, $G = N_G(P)$, and thus P is normal in G. Hence P is normal in H. By Theorem 4.2.3, H is nilpotent. By Fitting's theorem, $H \subseteq F(G)$. Hence $F(G/\Phi(G)) = \pi(H) \subseteq \pi(F(G))$, and the result follows. □

Exercises

4.4.1. Prove that a group is nilpotent if and only if it admits a central series.

xca:nilpotente_central

4.4.2. Let G be a group. Prove that if K is a central subgroup of G such that G/K is nilpotent, then G is nilpotent.

nilpotente_minimalnormal

4.4.3. Let G be a nilpotent group and M be a minimal normal subgroup of G. Prove that $M \subseteq Z(G)$.

xca:normalizadora

4.4.4. Let G be a finite group. Prove that the following statements are equivalent:

1) G is nilpotent.
2) If $H \subsetneq G$ is a subgroup, then $H \subsetneq N_G(H)$.
3) Every maximal subgroup of G is normal in G.

4.4.5. Let G be a finite nilpotent group. Prove that if p is a prime dividing the order of G, then there exists a minimal normal subgroup of order p and there exists a maximal subgroup of index p.

xca:pgrupos

4.4.6. Let p be a prime and let G be a non-trivial group of order p^n. Prove the following statements.

1) G has a normal subgroup of order p.
2) For each $j \in \{0, \ldots, n\}$ there exists a normal subgroup of order p^j.

:nilpotente_equivalencia

4.4.7. Let G be a finite group. Prove that the following conditions are equivalent.

1) G is nilpotent.
2) Any two elements of coprime order commute.
3) Every non-trivial quotient of G has non-trivial center.
4) If d divides $|G|$, there exists a normal subgroup of G of order d.

4.4.8. Let G be a group and $x, y \in G$. Prove the following statements.

1) If $[x,y] \in C_G(x) \cap C_G(y)$, then $[x,y]^n = [x^n, y] = [x, y^n]$ for all $n \in \mathbb{Z}$.
2) If G is nilpotent of class two, then $(xy)^n = [y,x]^{n(n-1)/2} x^n y^n$ for all positive integer n.

4.4.9. Let p be an odd prime number and let P be a p-group of nilpotency class ≤ 2. Prove the following statements.

1) If $[y,x]^p = 1$ for all $x, y \in P$, then $P \to [P,P]$, $x \mapsto x^p$, is a group homomorphism.
2) $\{x \in P : x^p = 1\}$ is a subgroup of P.

Notes

Chapter 5
Solvable groups

solvable

Derived series

For a group G we define

$$G^{(0)} = G, \quad G^{(i+1)} = [G^{(i)}, G^{(i)}] \quad i \geq 0.$$

The *derived series* of G is the sequence

$$G = G^{(0)} \supseteq G^{(1)} \supseteq G^{(2)} \supseteq \cdots$$

Each $G^{(i)}$ is a characteristic subgroup of G. We say that G is *solvable* if $G^{(n)} = \{1\}$ for some $n \in \mathbb{N}$. Clearly every abelian group is solvable. A non-trivial group G is said to be simple if $\{1\}$ and G are the only normal subgroups of G. Note that a non-abelian simple group cannot be solvable. Nilpotent groups are solvable.

Let p be a prime number. An *elementary abelian* p-group is a p-group P such that $x^p = 1$ for all $x \in P$.

lem:minimal_normal

Lemma 5.0.1. *Let G be a non-trivial group. Let M be a minimal normal subgroup of G. If M is solvable and finite, then M is an elementary abelian p-group for some prime number p.*

Proof. Since M is solvable, $[M, M] \subsetneq M$. Moreover, $[M, M]$ is normal in G, as $[M, M]$ is characteristic in M and M is normal in G. Since M is minimal normal, it follows that $[M, M] = \{1\}$ and hence M is abelian. Now, **since** M is finite, there exists a prime number p such that $P = \{x \in M : x^p = 1\}$ is a non-trivial subgroup of M. Since P is characteristic in M, the subgroup P is normal in G. Thus $P = M$. □

theorem:resoluble

Theorem 5.0.2. *Let G be a group.*

1) *If G is solvable, then each subgroup H of G is solvable.*
2) *Let K be a normal subgroup of G. Then G is solvable if and only if K and G/K are both solvable.*

Proof. By induction one proves that $H^{(i)} \subseteq G^{(i)}$ for all $i \geq 0$. Let us prove the second claim. Let $Q = G/K$ and let $\pi\colon G \to Q$ be the canonical map. By induction we prove that $\pi(G^{(i)}) = Q^{(i)}$ for all $i \geq 0$. The case $i = 0$ is trivial, as π is surjective. Now assume that the result holds for some $i \geq 0$. Then

$$\pi(G^{(i+1)}) = \pi([G^{(i)}, G^{(i)}]) = [\pi(G^{(i)}), \pi(G^{(i)})] = [Q^{(i)}, Q^{(i)}] = Q^{(i+1)}.$$

Assume that Q and K are both solvable. Since Q is solvable, there exists n such that $Q^{(n)} = \{1\}$. Since $\pi(G^{(n)}) = Q^{(n)} = \{1\}$, it follows that $G^{(n)} \subseteq K$. Since K is solvable, there exists m such that

$$G^{(n+m)} = (G^{(n)})^{(m)} \subseteq K^{(m)} = \{1\},$$

and hence G is solvable.

Let us now assume that G is solvable. There exists $n \in \mathbb{N}$ such that $G^{(n)} = \{1\}$. Thus Q is solvable, as $Q^n = \pi(G^{(n)}) = \pi(\{1\}) = \{1\}$. The group K also is solvable, as it is a subgroup of G. □

theorem:F(G)centraliza

Theorem 5.0.3. *Let G be a finite non-trivial solvable group and let N be a minimal normal subgroup of G. Then $N \subseteq Z(F(G))$ and thus $F(G) \subseteq C_G(N)$.*

Proof. Let N be a minimal normal subgroup of G. By Theorem 5.0.2, N is solvable. By Lemma 5.0.1, N is an elementary abelian p-group for some prime number p. In particular, N is a normal nilpotent subgroup of G. By Fitting's theorem, $N \subseteq F(G)$ and $F(G)$ is a normal nilpotent subgroup of G. By Hirsch's theorem $N \cap Z(F(G)) \neq \{1\}$. Since $Z(F(G))$ is a characteristic subgroup of $F(G)$ and $F(G)$ is normal in G, we have that $Z(F(G))$ is a normal subgroup of G. Hence $N \cap Z(F(G))$ also is a normal subgroup of G. Since N is a minimal normal subgroup of G and $N \cap Z(F(G))$ is non-trivial, we have that $N = N \cap Z(F(G))$. Therefore the result follows. □

Burnside's theorem

In this section we shall prove an important theorem of Burnside, which says that every finite group of order $p^n q^m$, for some primes p, q and positive integers m, n, is solvable. For its proof, we shall introduce some results of representation theory and integral extensions.

Let K be a field. A *K-algebra* is a K-vector space R joint with a multiplication on R such that $(R, +, \cdot)$ is a ring and $\alpha(a \cdot b) = (\alpha a) \cdot b = a \cdot (\alpha b)$, for all $\alpha \in K$ and $a, b \in R$.

Let G be a group and K a field. The *group algebra* $K[G]$ of the group G over the field K is a K-vector space with basis G with a multiplication on $K[G]$ defined by

$$\left(\sum_{i=1}^{n} a_i x_i\right)\left(\sum_{j=1}^{m} b_j y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} (a_i b_j)(x_i y_j),$$

for all $a_1, \ldots, a_n, b_1, \ldots, b_m \in K$ and $x_1, \ldots, x_n, y_1, \ldots, y_m \in G$. We will write the elements of $a \in K[G]$ in the form

$$a = \sum_{x \in G} a_x x,$$

with $a_x \in K$, assuming that $\{x \in G \mid a_x \neq 0\}$ is finite. With this notation, we have that

$$\left(\sum_{x \in G} a_x x\right)\left(\sum_{x \in G} b_x x\right) = \sum_{x \in G} c_x x,$$

where

$$c_x = \sum_{y \in G} a_y b_{y^{-1}x},$$

for all $x \in G$. One can easily check that the K-vector space $K[G]$ with this multiplication is a K-algebra. Note that the unit-element 1_G of G is the unit-element of the ring $(K[G], +, \cdot)$. The map $K \to K[G] : \alpha \mapsto \alpha 1_G$ is an injective ring homomorphism, and we will identify the elements $\alpha \in K$ with $\alpha 1_G \in K[G]$, thus $K \subseteq K[G]$. We will denote the unit-element of $K[G]$ by 1.

Definition 5.0.4. A *(linear) representation* of a group G over a field K is a group homomorphism

$$\rho \colon G \to \mathrm{Aut}(V),$$

Where V is a K-vector space. If $\dim(V) = n < \infty$, then we say that ρ has degree n. It is said that ρ is faithful if $\ker(\rho) = \{1\}$.

Let $\rho \colon G \to \mathrm{Aut}(V)$ be a representation of a group G over a field K of finite degree n. Then, fixing a basis $\mathcal{B}$ of V, we have a group isomorphism $\psi_{\mathcal{B}} \colon \mathrm{Aut}(V) \to \mathbf{GL}_n(K)$, where $\mathbf{GL}_n(K) = \{A \in M_n(K) \mid A \text{ is invertible}\}$, defined by $\psi_{\mathcal{B}}(f) = M(f, \mathcal{B})$, the matrix of f with respect the basis $\mathcal{B}$, for all $f \in \mathrm{Aut}(V)$. We say that $\psi_{\mathcal{B}} \circ \rho \colon G \to \mathbf{GL}_n(K)$ is a matricial representation of G over K. The map $\chi \colon G \to K$ defined by $\chi(x) = \mathrm{tr}(\psi_{\mathcal{B}}(\rho(x)))$, the trace of the matrix $\psi_{\mathcal{B}}(\rho(x))$, for all $x \in G$, is called the *character* of ρ. It is easy to see that χ is independent of the choice of the basis $\mathcal{B}$ of V.

Example 5.0.5. Let G be a finite group of order n. Let K be a field. Then $K[G]$ is a K-vector space of dimension n. The map

$$\rho \colon G \to \mathrm{Aut}_K(K[G]),$$

defined by $\rho(x)(a) = xa$, for all $x \in G$ and $a \in K[G]$, is a representation of G over K of degree n, and it is called the regular representation of G over K. If χ is the character of ρ, then $\chi(1) = n$ and $\chi(x) = 0$, for all $x \in G \setminus \{1\}$.

Given a representation $\rho\colon G \to \mathrm{Aut}(V)$ of a group G over a field K, we define

$$\widehat{\rho}\colon K[G] \to \mathrm{End}_K(V),$$

by $\widehat{\rho}\left(\sum_{x\in G} a_x x\right) = \sum_{x\in G} a_x \rho(x)$, for all $\sum_{x\in G} a_x x \in K[G]$. Note that $\widehat{\rho}$ is a ring homomorphism. We define on V a multiplication by elements of $K[G]$ by

$$\left(\sum_{x\in G} a_x x\right)\cdot v := \sum_{x\in G} a_x \rho(x)(v),$$

for all $\sum_{x\in G} a_x x \in K[G]$ and $v \in V$. One can check that V with the sum and this product by elements of $K[G]$ is a $K[G]$-module. This is the $K[G]$-module corresponding to the representation ρ.

Conversely, if W is a $K[G]$-module, then W also is a K-vector space, and the map $\rho\colon G \to \mathrm{Aut}_K(W)$, defined by $\rho(x)(w) = xw$, for all $x \in G$ and $w \in W$, is a representation of G over K. We say that ρ is the representation of G over K corresponding to the $K[G]$-module W.

One can easily check that this gives a bijective correspondence between the representations of G over K and the $K[G]$-modules.

equivrep

Lemma 5.0.6. *Let G be a group and K a field. Let M_1, M_2 be isomorphic $K[G]$-modules of finite dimension over K. Let ρ_1, ρ_2 be the representations over K corresponding to M_1, M_2 respectively. Let χ_1, χ_2 be the characters of ρ_1, ρ_2 respectively. Then $\chi_1(g) = \chi_2(g)$, for all $g \in G$.*

Proof. Let $f : M_1 \to M_2$ be an isomorphism of $K[G]$-modules. Then

$$\rho_2(g)(f(m)) = gf(m) = f(gm) = f(\rho_1(g)(m)),$$

for all $g \in G$ and $m \in M_1$. Hence

$$\rho_2(g) = f \circ \rho_1(g) \circ f^{-1}.$$

Therefore $\chi_2(g) = \chi_1(g)$, and the result follows. □

Definition 5.0.7. A representation of a group G over a field K is said to be irreducible if its corresponding $K[G]$-module is simple.

Definition 5.0.8. Let R be a ring. An R-module is *semisimple* if it is a direct sum of simple submodules. We say that R is (left) semisimple is R is semisimple as (left) R-module, that is if R is the direct sum of minimal nonzero left ideals.

Example 5.0.9. Let D be a division ring. For every positive integer n, $M_n(D)$ is a semisimple ring. In fact, if $e_{i,j} \in M_n(D)$ is the matrix with 1 in the (i,j)-entry and 0 in the other entries, then $M_n(D)e_{i,j}$ is a minimal nonzero left ideal of $M_n(D)$, and

$$M_n(D) = \bigoplus_{j=1}^{n} M_n(D)e_{j,j}.$$

semisimple

Example 5.0.10. Let R be an Artinian semiprimitive ring. The by Wedderburn-Artin theorem, there exist positive integers $n_1, \ldots, n_r$ and division rings $D_1, \ldots, D_r$ such that

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r).$$

Since each ring $M_{n_i}(D_i)$ is semisimple, it is easy to see that $M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$ also is semisimple, and thus R is semisimple.

Theorem 5.0.11 (Maschke's theorem). *Let G be a finite group. Let K be a field of characteristic $p \geq 0$. If p is not a divisor of $|G|$, then $J(K[G]) = \{0\}$ and therefore $K[G]$ is semisimple.*

Proof. Since $K[G]$ is finite dimensional, it is Artinian. By Example 5.0.10, it is enough to show that $J(K[G]) = \{0\}$. Let W be a vector subspace of $K[G]$ such that $K[G] = J(K[G]) \oplus W$. Let $\pi\colon K[G] \to K[G]$ be the map defined by $\pi(a+b) = a$, for all $a \in J(K[G])$ and $b \in W$. It is clear that π is a K-linear map. We define $\pi^*\colon K[G] \to K[G]$ by

$$\pi^*(c) = \frac{1}{|G|} \sum_{x \in G} x^{-1}\pi(xc),$$

for all $c \in K[G]$. Note that π^* is a K-linear map. Let $c \in K[G]$ and $y \in G$. We have that

$$\begin{aligned} \pi^*(yc) =& \frac{1}{|G|} \sum_{x \in G} x^{-1}\pi(xyc) \\ =& \frac{y}{|G|} \sum_{x \in G} y^{-1}x^{-1}\pi(xyc) \\ =& y\pi^*(c). \end{aligned}$$

Hence π^* is a homomorphism of $K[G]$-modules. Furthermore, $\pi^*(K[G]) \subseteq J(K[G])$. Let $a \in J(K[G])$. Since $\pi(xa) = xa$, for all $x \in G$, we have that $\pi^*(a) = a$. Hence $\pi^*(K[G]) = J(K[G])$ and $(\pi^*)^2 = \pi^*$. Thus $\ker(\pi^*) = \mathrm{im}(\mathrm{id} - \pi^*)$ and $K[G] = J(K[G]) \oplus \ker(\pi^*)$. Now there exist $a \in J(K[G])$ and $b \in \ker(\pi^*)$ such that $1 = a + b$. Hence $a = a^2 + ab$. Since $ab \in J(K[G]) \cap \ker(\pi^*) = \{0\}$, we get that $a = a^2$. Hence $(1-a)a = 0$. Since $1 - a$ is invertible, we have that $a = 0$. Hence $1 = b \in \ker(\pi^*)$, and thus $\ker(\pi^*) = K[G]$. Therefore $J(K[G]) = \{0\}$, and the result follows. □

The *center* of a ring R is $Z(R) = \{a \in R \mid ab = ba, \text{ for all } b \in R\}$.

Let G be a group. The *conjugate class* of an element $x \in G$ is $C_x = \{yxy^{-1} \mid y \in G\}$. Note that if C_x is finite, then $|C_x| = (G : C_G(x))$. In fact, if T is a left transversal of $C_G(x)$ in G, then $C_x = \{yxy^{-1} \mid y \in T\}$, and if $yxy^{-1} = zxz^{-1}$, for $y, z \in T$, then $z^{-1}y \in C_G(x)$, and thus $y = z$.

basiscenter

Lemma 5.0.12. *Let G be a finite group. Let K be a field. Let $C_1, \ldots, C_r$ be the distinct conjugacy classes of G. Let*

$$\alpha_i = \sum_{x \in C_i} x.$$

Then $\alpha_1, \ldots, \alpha_r$ is a K-basis of $Z(K[G])$.

Proof. It is clear that $\alpha_1, \ldots, \alpha_r$ are K-linearly independent. Since

$$y\alpha_i y^{-1} = \sum_{x \in C_i} yxy^{-1} = \sum_{x \in C_i} x = \alpha_i,$$

for all $y \in G$, we have that $\alpha_i \in Z(K[G])$. Let $b = \sum_{x \in G} b_x x \in Z(K[G])$. Then, for every $y \in G$,

$$b = yby^{-1} = \sum_{x \in X} b_x yxy^{-1}.$$

Hence $b_x = b_{yxy^{-1}}$, for all $x, y \in G$. For every $i = 1, \ldots, r$, choose an element $x_i \in C_i$. Now we have that

$$b = \sum_{i=1}^{r} b_{x_i} \alpha_i,$$

and the result follows. □

Definition 5.0.13. Let R be a subring of a commutative ring S. An element $a \in S$ is said to be *integral* over R if it is a root of a monic polynomial

$$a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + x^n \in R[x],$$

for some positive integer n.

integralelement

Proposition 5.0.14. *Let S be a commutative integral domain. Let R be a subring of S. For any $a \in S$, the following conditions are equivalent.*

1) *a is integral over R.*
2) *$R[a]$ is a finitelly generated R-module.*
3) *There is a non-zero finitely generated R-submodule M of S such that $aM \subseteq M$.*

Proof. 1) $\implies$ 2). Suppose that a is integral over R. Then there exist a positive integer n and $a_0, \ldots, a_{n-1} \in R$ such that

$$a_0 + a_1 a + \cdots + a_{n-1} a^{n-1} + a^n = 0.$$

We shall prove by induction on m that $a^m \in R + Ra + \cdots + a^{n-1}R$. If $m < n$, then it is clear that $a^m \in R + Ra + \cdots + a^{n-1}R$. Assume that $m \geq n$ and that $a^k \in R + Ra + \cdots + a^{n-1}R$, for all $k < m$. We have that

$$a^m = -a_0 a^{m-n} - a_1 a^{m-n+1} - \cdots - a_{n-1} a^{m-1}.$$

Hence, by the inductive hypothesis, $a^m \in R + Ra + \cdots + a^{n-1}R$. Thus, by induction, $R[a] = R + Ra + \cdots + Ra^{n-1}$.

2) $\implies$ 3). Suppose that $R[a]$ is a finitely generated R-module. Note that $aR[a] \subseteq R[a]$. Thus (3) follows.

3) $\implies$ 1). Suppose that there exists a non-zero finitely generated R-submodule M of S such that $aM \subseteq M$. Let $s_1, \ldots, s_m \in S$ be elements such that $M = Rs_1 + \cdots + Rs_m$. Hence, there exist elements $a_{i,j} \in R$ such that

$$as_i = \sum_{j=1}^{m} a_{i,j} s_j,$$

that is

$$\sum_{j=1}^{m} (a\delta_{i,j} - a_{i,j}) s_j = 0,$$

for all $i = 1, \ldots, m$. Since $M \neq \{0\}$, some $s_j \neq 0$. Therefore

$$\det \begin{pmatrix} a - a_{1,1} & -a_{1,2} & \ldots & -a_{1,m} \\ -a_{2,1} & a - a_{2,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & -a_{m-1,m} \\ -a_{m,1} & \ldots & -a_{m,m-1} & a - a_{m,m} \end{pmatrix} = 0.$$

Therefore a is integral over R. □

integralclousure

Corollary 5.0.15. *Let R be a subring of a commutative integral domain S. Then*

$$\{a \in S \mid a \text{ is integral over } R\}$$

is a subring of S containing R.

Proof. Let $T = \{a \in S \mid a \text{ is integral over } R\}$. It is clear that $R \subseteq T$. Let $a, b \in T$. Since a is integral over R, by Proposition 5.0.14, $R[a]$ is a finitely generated R-submodule of S. Since b is integral over R, b also is integral over $R[a]$. Hence, by Proposition 5.0.14, $R[a,b]$ is a finitely generated $R[a]$-submodule of S. Since $R[a]$ is finitely generated as R-module, we have that $R[a,b]$ also is finitely generated as R-module. Since $(a-b)R[a,b] \subseteq R[a,b]$ and $abR[a,b] \subseteq R[a,b]$, by Proposition 5.0.14, $a - b, ab \in T$. Therefore, the result follows. □

Zintclosed

Example 5.0.16. We shall see that $\mathbb{Z} = \{a \in \mathbb{Q} \mid a \text{ is integral over } \mathbb{Z}\}$. Let $a \in \mathbb{Q}$ be integral over $\mathbb{Z}$. We shall see that $a \in \mathbb{Z}$. We may assume that $a = b/c$, where b, c are nonzero coprime integers. Since a is integer over $\mathbb{Z}$, there exist a positive integer n and $a_0, \ldots, a_{n-1} \in \mathbb{Z}$ such that

$$a_0 + a_1 \frac{b}{c} + \cdots + a_{n-1} \frac{b^{n-1}}{c^{n-1}} + \frac{b^n}{c^n} = 0.$$

Hence

$$-b^n = c(a_0 c^{n-1} + \cdots + a_{n-1} b^{n-1}).$$

Since b, c are coprime, we get that $c = \pm 1$, and thus $a \in \mathbb{Z}$.

A field K is said to be *algebraically closed* if every polynomial $p(x) \in K[x]$ of positive degree has a root in K.

algclosedfield

Proposition 5.0.17. *Let K be an algebraically closed field. Let R be a semiprimitive finite dimensional K-algebra. Then*

$$R \cong M_{n_1}(K) \times \cdots \times M_{n_r}(K),$$

for some positive integers $n_1, \ldots, n_r$.

Proof. Since $\dim_K(R) < \infty$, R is Artinian. By Wedderburn-Artin theorem there exist positive integers $n_1, \ldots, n_r$ and division rings $D_1, \ldots, D_r$ such that

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r).$$

Furthermore, each D_i is a finite dimensional K-algebra. We may assume that $K \subseteq D_i$. Let $a \in D_i$. Since $\dim_K(D_i) < \infty$, a is algebraic over K. Hence $K[a]$ is an algebraic field extension of K. Since K is algebraically closed, $K[a] = K$. Therefore $D_i = K$, and the result follows. □

integraloverZ

Proposition 5.0.18. *Let K be an algebraically closed field of characteristic zero. Let G be a finite group. Let ρ be an irreducible representation of G over K of finite degree n. Let χ be the character of ρ. If $g \in G$ has l conjugate elements in G, then*

$$\frac{l\chi(g)}{n}$$

is integral over $\mathbb{Z}$.

Proof. Let $C_1, \ldots, C_r$ be the distinct conjugacy classes of G. We may assume that $C_1 = \{1\}$. Let $\alpha_i = \sum_{x \in C_i} x$. By Lemma 5.0.12, $\alpha_1, \ldots, \alpha_r$ is a K-basis of $Z(K[G])$. For each i, fix an element $x_i \in C_i$. Since $\alpha_i \alpha_j \in Z(K[G])$,

$$\alpha_i \alpha_j = \sum_{k=1}^{r} m_{i,j,k} \alpha_k, \tag{5.1}$$

charintegral

where

$$m_{i,j,k} = |\{(x, y) \in C_i \times C_j \mid xy = x_k\}|.$$

Let M be the $K[G]$-module corresponding to ρ. Since ρ is irreducible, M is simple. By Schur's lemma, $\mathrm{End}_{K[G]}(M)$ is a division ring. Since $\dim_K(M) = n$, we have that $\dim_K(\mathrm{End}_{K[G]}(M)) \leq n^2$. Thus $\mathrm{End}_{K[G]}(M)$ is a finite dimensional K-algebra. As we have seen in the proof of Proposition 5.0.17, we have that $\mathrm{End}_{K[G]}(M) \cong K$. Since $\alpha_i \in Z(K[G])$,

$$\widehat{\rho}(\alpha_i) = \sum_{x \in C_i} \rho(x) \in \mathrm{End}_{K[G]}(M).$$

Hence there exists $\lambda_i \in K$ such that

$$\widehat{\rho}(\alpha_i)(m) = \lambda_i m,$$

for all $m \in M$. Now we have that

$$|C_i|\chi(x_i) = \sum_{x \in C_i} \chi(x) = \operatorname{tr}(\widehat{\rho}(\alpha_i)) = n\lambda_i.$$

Hence

$$\lambda_i = \frac{|C_i|\chi(x_i)}{n}.$$

Applying $\widehat{\rho}$ to (5.1), we get $\lambda_i\lambda_j = \sum_{k=1}^{r} m_{i,j,k}\lambda_k$. Hence

$$\sum_{k=1}^{r} (\lambda_i\delta_{i,j} - m_{i,j,k})\lambda_k = 0,$$

for all $j = 1, \ldots, r$. Since $\alpha_1 = 1$, we have that $\lambda_1 = 1 \neq 0$. Hence

$$\det\begin{pmatrix} \lambda_i - m_{i,1,1} & -m_{i,1,2} & \cdots & -m_{i,1,r} \\ -m_{i,2,1} & \lambda_i - m_{i,2,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & -m_{i,r-1,r} \\ -m_{i,r,1} & \cdots & -m_{i,r,r-1} & \lambda_i - m_{i,r,r} \end{pmatrix} = 0.$$

Therefore λ_i is integral over $\mathbb{Z}$. Note that if $g \in C_i$, then there exists $y \in G$ such that $g = yx_iy^{-1}$, and thus

$$\lambda_i = \frac{|C_i|\chi(x_i)}{n} = \frac{l\chi(g)}{n}.$$

Therefore, the result follows. □

Note that if X is a non-empty set and V is a vector space over a field F, then the set $\operatorname{Map}(X, V)$ of all maps $f\colon X \to V$ with the sum and the product by elements of F defined by

$$(f+g)(x) = f(x) + g(x), \quad (\lambda f)(x) = \lambda f(x),$$

for all $f, g \in \operatorname{Map}(X, V)$, $\lambda \in F$ and $x \in X$, is an F-vector space.

Lemma 5.0.19 (Dedekind's lemma). *Any family of distinct homomorphisms of a field K into another field F is F-linearly independent.*

Proof. Suppose that the result is not true. Then there exist a smallest integer $n > 1$, distinct homomorphisms $f_1, \ldots, f_n$ of K into F and nonzero $a_1, \ldots, a_n \in F$, such that

$$a_1f_1 + \cdots + a_nf_n = 0.$$

Hence

$$a_1f_1(x) + \cdots + a_nf_n(x) = 0, \tag{5.2}$$

Dedekind1

for all $x \in K$. Let $x, y \in K$. We have

$$\begin{aligned}0 =& a_1 f_1(yx) + \cdots + a_n f_n(yx)\\ =& a_1 f_1(y) f_1(x) + \cdots + a_n f_n(y) f_n(x).\end{aligned}$$

Multiplying (5.2) by $f_1(y)$, we get

$$0 = a_1 f_1(y) f_1(x) + \cdots + a_n f_1(y) f_n(x).$$

Hence, substracting the last two equalities, we get

$$0 = a_2(f_2(y) - f_1(y)) f_2(x) + \cdots + a_n(f_n(y) - f_1(y)) f_n(x).$$

Therefore

$$a_2(f_2(y) - f_1(y)) f_2 + \cdots + a_n(f_n(y) - f_1(y)) f_n = 0,$$

for all $y \in K$. By the minimality of n, we have that $f_2, \ldots, f_n$ are F-linearly independent. Hence $a_2(f_2(y) - f_1(y)) = 0$ and, since $a_2 \neq 0$, we get that $f_2(y) = f_1(y)$, for all $y \in K$, a contradiction, because $f_1 \neq f_2$. Therefore the result follows. □

fixedfield

Lemma 5.0.20. *Let $\xi_1, \ldots, \xi_n \in \mathbb{C}$ be roots of unity. Let $F = \mathbb{Q}(\xi_1, \ldots, \xi_n)$ be the subfield of $\mathbb{C}$ generated by $\xi_1, \ldots, \xi_n$. Let G be the group of automorphisms of F as $\mathbb{Q}$-algebra. Then*

$$\mathbb{Q} = \{a \in F \mid \alpha(a) = a \text{ for all } \alpha \in G\}.$$

Proof. Note that the multiplicative subgroup of $\mathbb{C} \setminus \{0\}$ generated by $\xi_1, \ldots, \xi_n$ is a finite cyclic group generated by some root of unity ξ. Hence $F = \mathbb{Q}[\xi]$. Let $p(x) \in \mathbb{Q}[x]$ be a monic irreducible polynomial such that $p(\xi) = 0$. Let r be the degree of $p(x)$. Note that if the multiplicative order of ξ is m, then $p(x)$ is a divisor of $x^m - 1$. Hence all roots of $p(x)$ are roots of $x^m - 1$, and thus they belong to F. Since $p(x)$ is irreducible over $\mathbb{Q}$, all its roots are simple. Let $\eta_1, \ldots \eta_r \in F$ be the distinct roots of $p(x)$. Every automorphism $\alpha \in G$ is determined by $\alpha(\xi)$. Note that $\alpha(\xi) \in \{\eta_1, \ldots, \eta_r\}$. Furthermore, for $i = 1, \ldots, r$, the map $\alpha_i \colon F \to F$, defined by $\alpha_i(q(\xi)) = q(\eta_i)$, for all $q(x) \in \mathbb{Q}[x]$ of degree $< r$, is an automorphism of $F \cong \mathbb{Q}[x]/(p(x))$. Hence $|G| = r$. Let $L = \{a \in F \mid \alpha(a) = a \text{ for all } \alpha \in G\}$. It is easy to see that L is a subfield of F and $\mathbb{Q} \subseteq L \subseteq F$. We have that $\dim_{\mathbb{Q}} F = r$. Suppose that $\dim_L F = s < r$. Let $a_1, \ldots a_s \in F$ be an L-basis of F. Since $s < r$, the system of s equations in r unknowns x_i:

$$\sum_{i=1}^{r} \alpha_i(a_j) x_i = 0 \quad (j = 1, \ldots, s),$$

has a nonzero solution $x_i = b_i$ in F. Let $a \in F$. There exist $c_1, \ldots, c_s \in L$ such that $a = \sum_{j=1}^{s} c_j a_j$. Hence

$$\sum_{i=1}^{r} \alpha_i(a) b_i = \sum_{i=1}^{r} \sum_{j=1}^{s} c_j \alpha_i(a_j) b_i = \sum_{j=1}^{s} c_j \sum_{i=1}^{r} \alpha_i(a_j) b_i = 0.$$

Thus

$$\sum_{i=1}^{r} b_i \alpha_i = 0,$$

in contradiction with Dedekind's lemma. Therefore $\dim_L F = r$ and thus $L = \mathbb{Q}$. □

Burnsidekey

Lemma 5.0.21. *Let ρ be an irreducible representation of a finite group G over $\mathbb{C}$ of finite degree n. Let χ be the character of ρ. Assume that $g \in G$ has l conjugate elements in G and that l and n are coprime. Then, either $\chi(g) = 0$ or $\rho(g)$ is a scalar.*

Proof. By Proposition 5.0.18, $\frac{l\chi(g)}{n}$ is integral over $\mathbb{Z}$. Since l and n are coprime, there exist integers r, s such that $1 = rl + sn$. Hence

$$\frac{\chi(g)}{n} = \frac{rl\chi(g)}{n} + s\chi(g).$$

Since G is finite, there exists a positive integer m such that $g^m = 1$. Hence $\chi(g)$ is a sum of roots of unity in $\mathbb{C}$. Hence, by Corollary 5.0.15, $\frac{\chi(g)}{n}$ is integral over $\mathbb{Z}$. Let $\xi_1, \ldots, \xi_n \in \mathbb{C}$ roots of unity such that $\chi(g) = \sum_{i=1}^{n} \xi_i$. Hence

$$\left|\frac{\chi(g)}{n}\right| = \frac{\left|\sum_{i=1}^{n} \xi_i\right|}{n} \leq 1.$$

If $\xi_1 = \cdots = \xi_n$, then we may assume that there exists a $\mathbb{C}$-basis $\mathcal{B}$ of M such that the matrix of $\rho(g)$ with respect to $\mathcal{B}$ is

$$A = \begin{pmatrix} \xi_1 & 0 & \ldots & \ldots & 0 \\ \varepsilon_1 & \xi_1 & \ddots & & \vdots \\ 0 & \varepsilon_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \varepsilon_{n-1} & \xi_1 \end{pmatrix},$$

where $\varepsilon_i \in \{0, 1\}$. Since $g^m = 1$, we have that $\rho(g)^m = \mathrm{id}$. Let $I \in M_n(\mathbb{C})$ be the identity matrix and $N = A - \xi_1 I$. Note that N is nilpotent, in fact $N^n = 0$. We have

$$I = A^m = (\xi_1 I + N)^m = \xi_1^m I + \sum_{i=1}^{m} \xi_1^{m-i} N^i.$$

Hence $\xi^m = 1$ and

$$\begin{aligned} 0 = \sum_{i=1}^{m} \xi_1^{m-i} N^i &= \xi_1^{m-1} N \left(\sum_{i=1}^{m} \xi_1^{1-i} N^{i-1} \right) \\ &= \xi_1^{m-1} N \left(I + \sum_{i=2}^{m} \xi_1^{1-i} N^{i-1} \right). \end{aligned}$$

Since $I + \sum_{i=2}^{m} \xi_1^{1-i} N^{i-1}$ is invertible, we get that $N = 0$. Therefore $A = \xi_1 I$, and thus $\rho(g)$ is a scalar in this case.

Suppose that $\xi_1, \ldots, \xi_n$ are not equal. In this case

$$\frac{|\chi(g)|}{n} < 1.$$

Let $F = \mathbb{Q}(\xi_1, \ldots, \xi_n)$ be the subfield of $\mathbb{C}$ generated by $\xi_1, \ldots, \xi_n$. Let H be the group of automorphisms of F. Then, for every $\alpha \in H$, $\alpha(\xi_i)$ is a root of unity. Hence

$$\frac{|\alpha(\chi(g))|}{n} = \frac{|\alpha(\xi_1) + \cdots + \alpha(\xi_n)|}{n} < 1.$$

By Lemma 5.0.20,

$$\prod_{\alpha \in H} \alpha\left(\frac{\chi(g)}{n}\right) \in \mathbb{Q}.$$

Since $\frac{\chi(g)}{n}$ is integral over $\mathbb{Z}$, $\alpha\left(\frac{\chi(g)}{n}\right)$ also is integral over $\mathbb{Z}$. By Corollary 5.0.15,

$$\prod_{\alpha \in H} \alpha\left(\frac{\chi(g)}{n}\right)$$

is integral over $\mathbb{Z}$. By Example 5.0.16,

$$\prod_{\alpha \in H} \alpha\left(\frac{\chi(g)}{n}\right) \in \mathbb{Z}.$$

Since

$$\left|\prod_{\alpha \in H} \alpha\left(\frac{\chi(g)}{n}\right)\right| < 1,$$

we have that $\chi(g) = 0$ and the result follows. □

Burnside

Theorem 5.0.22 (Burnside). *Let G be a finite group with a conjugacy class with p^m elements, for some prime p and positive integer m. Then G is not simple.*

Proof. Suppose that G is simple. It is clear that G is not abelian. Let $g \in G$ be an element with p^m conjugate elements in G. Let ρ be an irreducible non-trivial representation of G over $\mathbb{C}$ of degree n. Let χ be the character of ρ. Suppose that $\chi(g) \neq 0$ and that p is not a divisor of n. By Lemma 5.0.21, $\rho(g)$ is scalar and thus it is central in $\rho(G)$. Since G is simple and ρ is non-trivial, we have that $\ker(\rho) = \{1\}$ and $\rho(G) \cong G$ is simple. Hence $\rho(g) = \mathrm{id}$, and thus $g = 1$, in contradiction with the fact that g has p^m conjugate elements. Therefore, if p is not a divisor of n, then $\chi(g) = 0$.

Let σ be the regular representation of G over $\mathbb{C}$, that is, $\sigma\colon G \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}[G])$ is defined by $\sigma(h)(a) = ha$, for all $h \in G$ and $a \in \mathbb{C}[G]$. Let ψ be the character of σ. By Maschke's theorem $J(\mathbb{C}[G]) = \{0\}$. Since $\mathbb{C}[G]$ is Artinian, by Proposition 5.0.17,

there exist positive integers $n_1, \ldots, n_s$ such that $\mathbb{C}[G] \cong M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_s}(\mathbb{C})$. Let $f : \mathbb{C}[G] \to M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_s}(\mathbb{C})$ be an isomorphism. Let

$$I_{i,j} = f^{-1}(\{0\} \times \cdots \times \{0\} \times M_{n_i}(\mathbb{C}) e_{j,j} \times \{0\} \times \cdots \times \{0\}),$$

where $e_{j,j} \in M_{n_i}(\mathbb{C})$ is the matrix with 1 in the (j,j) entry and 0 elsewhere. Note that $I_{i,1}, \ldots, I_{i,n_i}$ are minimal isomorphic left ideals of $\mathbb{C}[G]$ of dimension n_i over $\mathbb{C}$. Furthermore

$$\mathbb{C}[G] = \bigoplus_{i=1}^{s} \bigoplus_{j=1}^{n_i} I_{i,j}.$$

Let $\sigma_{i,j} : G \to \mathrm{Aut}_{\mathbb{C}}(I_{i,j})$ be the map defined by $\sigma_{i,j}(h)(a) = ha$, for all $h \in G$ and $a \in I_{i,j}$. Let $\psi_{i,j}$ be the character of the representation $\sigma_{i,j}$. Then it is easy to see that $\psi(h) = \sum_{i=1}^{s} \sum_{j=1}^{n_i} \psi_{i,j}(h)$, for all $h \in G$. By Lemma 5.0.6, $\psi_{i,1}(h) = \psi_{i,j}(h)$, for all i and all $j = 1, \ldots, n_i$. Hence

$$\psi(h) = \sum_{i=1}^{s} n_i \psi_{i,1}(h),$$

for all $h \in G$. We may assume that $n_1 = 1$ and

$$I_{1,1} = \mathbb{C}[G] \frac{1}{|G|} \sum_{x \in G} x = \mathbb{C} \frac{1}{|G|} \sum_{x \in G} x,$$

which corresponds to the trivial representation $\sigma_{1,1}$ of degree 1. Note that $\sigma_{i,1}$ is irreducible and non-trivial, for all $i > 1$. Since n_i is the degree of $\sigma_{i,1}$ and p^m is the number of conjugate elements of g, we have seen that if p is not a divisor of n_i and $i > 1$, then $\psi_{i,1}(g) = 0$. Hence

$$0 = \psi(g) = \psi_{1,1}(g) + \sum_{i=2}^{s} n_i \psi_{i,1}(g) = 1 + p \sum_{i=2}^{s} r_i \psi_{i,1}(g),$$

where

$$r_i = \begin{cases} \frac{n_i}{p} & \text{if } p \text{ is a divisor of } n_i \\ 0 & \text{otherwise} \end{cases}$$

Hence, by Corollary 5.0.15

$$-\frac{1}{p} = \sum_{i=2}^{s} r_i \psi_{i,1}(g)$$

is integral over $\mathbb{Z}$, because $\psi_{i,1}$ is a sum of roots of unity, in contradiction with Example 5.0.16. Therefore the result follows. □

Theorem 5.0.23 (Burnside's p-q-theorem). *Let G be a finite group of order $p^n q^m$, for some prime numbers p, q and non-negative integers m, n. Then G is solvable.*

Proof. Let G be a counterexample of minimal order, that is G is a finite non-solvable group, $|G| = p^n q^m$ and every group H of order $|H| = p_1^r q_1^s < |G|$, for some prime numbers p_1, q_1 and non-negative integers r, s, is solvable. Let S be a Sylow q-subgroup of G. Since finite p-groups are nilpotent, and thus solvable, we have that S is non-trivial. Since S is a non-trivial q-subgroup, we know that $Z(S)$ is non-trivial. Let $g \in Z(S) \setminus \{1\}$. Then the number of conjugate elements of g in G is $(G : C_G(g))$. Since $S \subseteq C_G(g)$ and $p^n = (G : S) = (G : C_G(g))(C_g(g) : S)$, we have that either $g \in Z(G)$ or g has $p^r > 1$ conjugate elements. By Theorem 5.0.22, in both cases G is not simple. Let N be a non-trivial proper normal subgroup of G. Since $|N|, |G/N| < |G|$ and $|N|, |G/N|$ are divisors of $|G|$, we have that N and G/N are solvable. By Theorem 5.0.2, G is solvable, a contradiction. Therefore the result follows. □

A non-character-theoretic proof of this theorem is known but it is harder, see [47]. Burnside's theorem has several generalizations.

thm:KegelWielandt

Theorem 5.0.24 (Kegel–Wielandt). *Let G be a finite group such that $G = AB$ for nilpotent subgroups A and B of G. Then G is solvable.*

For the proof we refer to [3, Theorem 2.4.3].

thm:FeitThompson

Theorem 5.0.25 (Feit–Thompson). *Every finite group of odd order is solvable.*

The proof of the theorem is extremely hard. It occupies a full volume of *Pacific Journal of Mathematics*, see [40].

Hall subgroups

In this section we shall prove a characterization of finite solvable groups due to P. Hall. For this we will use Burnside's p-q-theorem and Wielandt's theorem. To prove Wielandt's theorem on solvable groups we need the following lemma.

lemma:4Wielandt

Lemma 5.0.26. *Let G be a finite group. If H and K are subgroups of G of coprime indices, then $G = HK$ and $(H : H \cap K) = (G : K)$.*

Proof. Let $D = H \cap K$. Since

$$(G : D) = \frac{|G|}{|H \cap K|} = (G : H)(H : H \cap K),$$

$(G : H)$ divides $(G : D)$. Similarly, $(G : K)$ divides $(G : D)$. Since $(G : H)$ and $(G : K)$ are coprime, $(G : H)(G : K)$ divides $(G : D)$. In particular,

$$\frac{|G|}{|H|}\frac{|G|}{|K|} = (G : H)(G : K) \leq (G : D) = \frac{|G|}{|H \cap K|}$$

and hence $|G| = |HK|$. Since

$$|G| = |HK| = |H||K|/|H \cap K|,$$

it follows that $(G : K) = (H : H \cap K)$. □

The *normal closure* H^G of a subgroup H of G is the subgroup

$$H^G = \langle xHx^{-1} : x \in G \rangle$$

generated by all conjugates of H. The subgroup H^G is the smallest normal subgroup of G containing H.

Example 5.0.27. Let $G = \mathbb{A}_4$ and $H = \{\mathrm{id}, (12)(34)\}$. Then

$$H^G = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\} \simeq C_2 \times C_2.$$

heorem:Wielandt:solvable

Theorem 5.0.28 (Wielandt). *Let G be a finite group and H, K and L be subgroups of G with pair-wise coprime indices. If H, K and L are solvable, then G is solvable.*

Proof. Assume the theorem is not valid and let G be a minimal counterexample. Then G is not trivial. Let N be a minimal normal subgroup of G and $\pi\colon G \to G/N$, $g \mapsto gN$, be the canonical map. Since by definition N is non-trivial, it follows that $|G/N| < |G|$. The subgroups $\pi(H) = \pi(HN)$, $\pi(K) = \pi(KN)$ and $\pi(L) = \pi(LN)$ of $\pi(G) = G/N$ are solvable. The correspondence theorem implies that the indices of $\pi(H)$, $\pi(K)$ and $\pi(L)$ in $\pi(G)$ are pair-wise coprime. By the minimality of G, the group $\pi(G)$ is solvable. If $H = \{1\}$, then $|G| = (G : H)$ is coprime with $(G : K)$ and hence $G = K$ is solvable. So we may assume that $H \neq \{1\}$. Let M be a minimal normal subgroup of H. By Lemma 5.0.1, M is a p-group for some prime number p. We may assume that p does not divide $(G : K)$ (if p divides $(G : K)$, then p does no divide $(G : L)$ and hence it is enough to replace K by L). There exists $P \in \mathrm{Syl}_p(G)$ such that $P \subseteq K$. By Sylow's theorem, there exists $g \in G$ such that $M \subseteq gKg^{-1}$. Since $(G : gKg^{-1}) = (G : K)$ and $(G : H)$ are coprime, Lemma 5.0.26 implies that $G = (gKg^{-1})H$.

We claim that all conjugates of M are included in gKg^{-1}. If $x \in G$, then $x = uv$ for some $u \in gKg^{-1}$ and $v \in H$. Since M is normal in H,

$$xMx^{-1} = (uv)M(uv)^{-1} = uMu^{-1} \subseteq gKg^{-1}.$$

In particular, $\{1\} \neq M^G \subseteq gKg^{-1}$ is solvable, as gKg^{-1} is solvable. The minimality of G implies that G/M^G is solvable. Hence G is solvable by Theorem 5.0.2. □

Let G be a finite group of order $p^\alpha m$ with p a prime number coprime with m. A subgroup H of G is a *p-complement* if $|H| = m$.

Example 5.0.29. Let $G = \mathbb{S}_3$. Then $H = \langle (123) \rangle$ is a 2-complement and $K = \langle (12) \rangle$ is a 3-complement.

theorem:Hall:solvable

Theorem 5.0.30 (Hall). *Let G be a finite group that admits a p-complement for all primes p dividing the order of G. Then G is solvable.*

Proof. Let $|G| = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ with the p_j being distinct primes. We proceed by induction on k. If $k = 1$, then G is a p_1-group and the result is clear. If $k = 2$, then Burnside's p-q-theorem implies the claim. Assume now that $k \geq 3$. For each $j \in \{1,2,3\}$ let H_j be p_j-complement in G. Since $|H_j| = |G|/p_j^{\alpha_j}$, the subgroups H_j have pair-wise coprime indices.

We claim that H_1 is solvable. Note that $|H_1| = p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. Let p be a prime number that divides $|H_1|$ and let Q be a p-complement in G. Since $(G : H_1)$ and $(G : Q)$ are coprime, Lemma 5.0.26 implies that

$$(H_1 : H_1 \cap Q) = (G : Q).$$

Thus $H_1 \cap Q$ is a p-complement in H_1. Hence H_1 is solvable by the inductive hypothesis. Similarly, H_2 and H_3 are both solvable.

Since H_1, H_2 and H_3 are solvable and have pair-wise coprime indices, by Wielandt's theorem, G also is solvable. Therefore the result follows by induction. □

Definition 5.0.31. Let π be a non-empty set of primes. A finite group G is a *π-group* if every prime divisor p of $|G|$ belongs to π.

Definition 5.0.32. Let π be a non-empty set of primes. A π-subgroup H of a finite group G is a *Hall π-subgroup* of G if $(G : H)$ and p are coprime, for all $p \in \pi$.

thm:Hall_pi_subgroup

Theorem 5.0.33 (Hall). *Let G be a finite solvable group. Then the following conditions hold.*

1) *For every non-empty set π of primes, there exists a Hall π-subgroup of G.*
2) *Let π be a non-empty set of primes. Let H be a Hall π-subgroup of G. Then for every π-subgroup L of G there exists $g \in G$ such that $L \subseteq gHg^{-1}$.*

Proof. Let G be a counterexample to 1) of smallest order. Hence G is a finite solvable group and there exists a non-empty set of prime divisors of $|G|$ such that G has no Hall π-subgroups. Furthermore, every finite solvable group K such that $|K| < |G|$ has Hall τ-subgroups for all non-empty set of primes τ. Let N be a minimal normal subgroup of G. By Theorem 5.0.2, N and G/N are solvable. By Lemma 5.0.1, N is an elementary abelian p-subgroup of G, for some prime p. Suppose that $p \in \pi$. In this case, since $|G/N| < |G|$, there exists a subgroup H of G such that $N \subseteq H$ and H/N is a Hall π-subgroup of G/N. Since $|H| = |H/N| \cdot |N|$, we have that H is a π-subgroup of G. Since $(G : H) = (G/N : H/N)$, we get that H is a Hall π-subgroup of G, a contradiction. Therefore, $p \notin \pi$. Since $|G/N| < |G|$, there exists a subgroup H of G such that $N \subseteq H$ and H/N is a Hall π-subgroup of G/N. Since $|H| = |H/N| \cdot |N|$, we have that H is a $\pi \cap \{p\}$-subgroup of G. Note that every Hall π-subgroup of H also is a Hall π-subgroup of G. Hence $H = G$. Note that N is a Sylow p-subgroup of G. Let M be a normal subgroup of G such that $N \subseteq M$ and M/N is a minimal normal

subgroup of G/N. Since M/N is solvable, by Lemma 5.0.1, M/N is an elementary abelian q-group, for some prime q. Since N is a Sylow p-subgroup of G, we have that $q \neq p$. Hence $q \in \pi$. Let S be a Sylow q-subgroup of M. By Frattini's argument, $G = MN_G(S) = NSN_G(S) = NN_G(S)$. Note that, by the above argument, S is not normal in G. Hence $|N_G(S)| < |G|$. Since $N_G(S)$ is solvable, there exists a Hall π-subgroup H_1 of $N_G(S)$. Since $(G : H_1) = (G : N_G(S))(N_G(S) : H_1)$ is a divisor of $|N|(N_G(S) : H_1)$, we have that H_1 also is a Hall π-subgroup of G, a contradiction. Therefore 1) is proved.

Assume now that 2) does not hold. Let G be a counterexample to 2) of smallest order. Hence G is a finite solvable group and there exist a non-empty set of primes π, a Hall π-subgroup H of G and a π-subgroup L of G such that, for every $g \in G$, $L \not\subseteq gHg^{-1}$. Furthermore, if K is a finite solvable group such that $|K| < |G|$, then for every non-empty set of primes τ, every Hall τ-subgroup U and every τ-subgroup V of K, there exists $x \in K$ such that $V \subseteq xUx^{-1}$. Let N be a minimal normal subgroup of G. As above, we know that N is an elementary abelian p-group. Then NH/N and NL/N are π-subgroups of G/N. Since H is a Hall π-subgroup of G, we have that NH/N is a Hall π-subgroup of G/N. Suppose that $p \in \pi$. In this case, NH is a π-subgroup. Since H is a Hall π-subgroup of G, we have that $H = NH$. Since $|G/N| < |G|$, there exists $x \in G$ such that $NL \subseteq (xN)NH(x^{-1}N) = xNHx^{-1} = xHx^{-1}$, and thus $L \subseteq xHx^{-1}$, a contradiction. Hence $p \notin \pi$, and $L \subseteq xNHx^{-1}$. If $|xNHx^{-1}| < |G|$, then, since xHx^{-1} is a Hall π-subgroup of $xNHx^{-1}$, there exists $y \in xNHx^{-1}$ such that $L \subseteq yxHx^{-1}y^{-1}$, a contradiction. Hence $|NH| = |xNHx^{-1}| = |G|$, and thus $G = NH$. Suppose that $|L| < |H|$. In this case, $|NL| = |N| \cdot |L| < |N| \cdot |H| = |G|$. Note that L is a Hall π-subgroup of NL and

$$|(NL) \cap H| = \frac{|NL| \cdot |H|}{|(NL)H|} = \frac{|N| \cdot |L| \cdot |H|}{|NH|} = \frac{|N| \cdot |L| \cdot |H|}{|N| \cdot |H|} = |L|.$$

Hence there exists $z \in NL$ such that $L \subseteq z((NL) \cap H)z^{-1} \subseteq zHz^{-1}$, a contradiction. Hence $|L| = |H|$. Let M be a normal subgroup of G such that $N \subseteq M$ and M/N is a minimal normal subgroup of G/N. As above, we know that M/N is an elementary abelian q-subgroup, for some prime $q \in \pi$. Note that

$$|H \cap M| = \frac{|H| \cdot |M|}{|MH|} = \frac{|H| \cdot |M|}{|NH|} = \frac{|M|}{|N|}.$$

Hence $H \cap M$ is a Sylow q-subgroup of M. Since $L \cap M$ is a q-subgroup of M, by Sylow's theorem, there exists $z \in M$ such that $L \cap M \subseteq z(H \cap M)z^{-1}$. Since $|L| = |H|$, we have that $G = NH = NL = MH = ML$ and

$$|L \cap M| = \frac{|L| \cdot |M|}{|ML|} = \frac{|H| \cdot |M|}{|MH|} = |H \cap M|.$$

Hence $L \cap M = z(H \cap M)z^{-1}$ and thus

$$L \subseteq N_G(L \cap M) = N_G(z(H \cap M)z^{-1}) = zN_G(H \cap M)z^{-1}.$$

By the above argument, we have that $G \neq N_G(H \cap M)$. Hence $|zN_G(H \cap M)z^{-1}| = |N_G(H \cap M)| < |G|$. Since zHz^{-1} is a Hall π-subgroup of $zN_G(H \cap M)z^{-1}$, there exists $t \in zN_G(H \cap M)z^{-1}$ such that $L \subseteq tzHz^{-1}t^{-1}$, a contradiction. Therefore 2) follows. □

Definition 5.0.34. Let G be a finite group. Let $p_1, \ldots, p_r$ be the distinct prime divisors of $|G|$. A *Sylow system* of G is a set $\{S_1, \ldots, S_r\}$, where S_i is a Sylow p_i-subgroup of G, such that $S_iS_j = S_jS_i$, for all i, j.

Theorem 5.0.35 (Hall). *Let G be a finite group. Then G is solvable if and only if G has a Sylow system.*

Proof. Suppose that G has a Sylow system $\{S_1, \ldots, S_r\}$, where S_i is a Sylow p_i-subgroup of G. Since $S_iS_j = S_jS_i$, for all i, j, $C_i = S_1 \cdots S_{i-1}S_{i+1} \cdots S_r$ is a subgroup of G. Furthermore

$$|C_i| = |S_1| \cdots |S_{i-1}| \cdot |S_{i+1}| \cdots |S_r|$$

and$(G : C_i) = |S_i|$. Hence C_i is a p_i-complement of G. By Theorem 5.0.30, G is solvable.

Conversely, suppose that G is solvable. By Theorem 5.0.33, for every prime divisor p_i of $|G|$, G has a p_i-complement H_i. Let $p_1, \ldots, p_r$ be the distinct prime divisors of $|G|$. Clearly, to prove that G has a Sylow system, we may assume that $r > 2$. Let I be a non-empty subset of $\{1, \ldots, r\}$. Let $H_I = \bigcap_{i \in I} H_i$. We shall show that

$$(G : H_I) = \prod_{i \in I} (G : H_i) \tag{5.3}$$ eq:indexH_I

by induction on $|I|$. For $|I| = 1$, it is clear. Assume that $|I| > 1$ and that (5.3) holds for every non-empty subset J of $\{1, \ldots, r\}$ such that $|J| < |I|$. Let $i \in I$ and $I_1 = I \setminus \{i\}$. By the inductive hypothesis, we have that

$$(G : H_{I_1}) = \prod_{k \in I_1} (G : H_k).$$

Hence $(G : H_i)$ and $(G : H_{I_1})$ are coprime. By Lemma 5.0.26, $G = H_iH_{I_1}$. Now we have that

$$\begin{aligned} \prod_{k \in I} (G : H_k) =& (G : H_i)(G : H_{I_1}) = \frac{|G|^2}{|H_i| \cdot |H_{I_1}|} \\ =& \frac{|G| \cdot |H_iH_{I_1}|}{|H_i| \cdot |H_{I_1}|} = \frac{|G|}{|H_i \cap H_{I_1}|} \\ =& (G : H_i \cap H_{I_1}) = (G : H_I) \end{aligned}$$

Hence (5.3) follows by induction. Let I, J be non-empty subsets of $\{1, \ldots, r\}$. We shall prove that

$$H_IH_J = H_JH_I \tag{5.4}$$ eq:H_IH_J

by induction on $|I| \cdot |J|$. For $|I| = |J| = 1$, (5.4) follows from Lemma 5.0.26. Suppose that $|I| \cdot |J| > 1$ and that (5.4) holds for all non-empty subsets I_1, J_1 of $\{1, \ldots r\}$, such

that $|I_1| \cdot |J_1| < |I| \cdot |J|$. If $I \subseteq J$, then $H_J \subseteq H_I$ and $H_I H_J = H_I = H_J H_I$. Hence we may assume that there exist $i \in I \setminus J$. Let $I_1 = I \setminus \{i\}$. By the inductive hypothesis, $H_{I_1} H_J = H_J H_{I_1}$ is a subgroup of G. Note that H_I, H_J are subgroups of the group $H_{I_1} H_J$. Since $H_I = H_{I_1} \cap H_i$,

$$|H_I H_J| = \frac{|H_I| \cdot |H_J|}{|H_I \cap H_J|} = \frac{|H_{I_1}| \cdot |H_i| \cdot |H_J|}{|H_{I_1} H_i| \cdot |H_{I_1} \cap H_i \cap H_J|} = \frac{|H_{I_1} H_J| \cdot |H_i| \cdot |H_{I_1} \cap H_J|}{|H_{I_1} H_i| \cdot |H_{I_1} \cap H_i \cap H_J|}.$$

Note that $(G : H_i)$ and $(G : H_{I_1})$ are coprime. Similarly, $(G : H_i)$ and $(G : H_J)$ are coprime. By Lemma 5.0.26, $G = H_i H_{I_1} = H_i H_J = H_i (H_{I_1} \cap H_J)$. Hence

$$|H_I H_J| = \frac{|H_{I_1} H_J| \cdot |H_i| \cdot |H_{I_1} \cap H_J|}{|H_i H_{I_1}| \cdot |H_{I_1} \cap H_i \cap H_J|} = \frac{|H_{I_1} H_J| \cdot |H_i (H_{I-1} \cap H_J)|}{|H_i H_{I_1}|} = |H_{I_1} H_J|.$$

Hence $H_I H_J = H_{I_1} H_J = H_J H_I$. Therefore (5.4) follows by induction. Let $i \in \{1, \ldots, r\}$, $J_i = \{1, \ldots, r\} \setminus \{i\}$ and $S_i = H_{J_i}$. By (5.3), S_i is a Sylow p_i-subgroup of G. By (5.4), $S_i S_j = S_j S_i$. Hence $\{S_1, \ldots, S_r\}$ is a Sylow system of G, and the result follows. □

Perfect groups

A group G is said to be *perfect* if $[G, G] = G$. Note that G is perfect if and only if $G/[G, G]$ is trivial. Every non-abelian simple group is perfect.

Let K be a field. the *special linear* group of degree n over K is

$$\mathbf{SL}_n(K) = \{A \in \mathbf{GL}_n(K) \mid \det(A) = 1\}.$$

If $K = \mathbb{F}_q$ is the finite field of q elements, then we will write $\mathbf{SL}_n(q) = \mathbf{SL}_n(\mathbb{F}_q)$.

Example 5.0.36. The groups $\mathbf{SL}_2(2)$ and $\mathbf{SL}_2(3)$ are not perfect.

Let p be a prime number and $q = p^m$, for some positive integer m. The groups $\mathbf{SL}_n(q)$ are perfect except the cases $\mathbf{SL}_2(2)$ and $\mathbf{SL}_2(3)$. As an example, let us prove that $\mathbf{SL}_2(q)$ is perfect if $q > 3$. We first claim that $\mathbf{SL}_2(q)$ is generated by matrices $X_{ij}(\lambda) = I + \lambda E_{ij}$, where I denotes the identity matrix, E_{ij} is the matrix with a one at position (i, j) and zero in all other entries, $i, j \in \{1, 2\}$ are distinct and $\lambda \in \mathbb{F}_q \setminus \{0\}$. First note that a matrix $\begin{pmatrix} 1 & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(q)$ is a product of some of the $X_{ij}(\lambda)$, as

$$\begin{pmatrix} 1 & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = X_{21}(c) X_{12}(b).$$

This implies that a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(q)$, with $c \neq 0$, also is a product of some $X_{ij}(\lambda)$. Indeed, if λ is such that $a = 1 - \lambda c$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b+\lambda d \\ c & d \end{pmatrix} = X_{12}(-\lambda)X_{21}(c)X_{12}(b+\lambda d).$$

Finally,

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ a & b+a^{-1} \end{pmatrix}$$

and therefore $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ is a product of some $X_{ij}(\lambda)$ since $\begin{pmatrix} a & b \\ a & b+a^{-1} \end{pmatrix}$ is a product of some $X_{ij}(\lambda)$. To prove that $[\mathbf{SL}_2(q), \mathbf{SL}_2(q)] = \mathbf{SL}_2(q)$ we first note that

$$\left[\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} 1 & (a^2-1)b \\ 0 & 1 \end{pmatrix}.$$

Since $q > 3$, given $\lambda \in \mathbb{F}_q$ and $a \in \mathbb{F}_q \setminus \{-1, 0, 1\}$, there exists $b \in \mathbb{F}_q$ such that $\lambda = (a^2-1)b$. This implies that each $X_{ij}(\lambda)$ belongs to the commutator subgroup of $\mathbf{SL}_2(q)$.

theorem:Grun

Theorem 5.0.37 (Grün). *If G is a perfect group, then $Z(G/Z(G)) = \{1\}$.*

Proof. The three subgroups lemma with $X = Y = G$, $Z = \zeta_2(G)$ and $N = \{1\}$ yields

$$\{1\} = [\zeta_2(G), G, G] = [\zeta_2(G), [G, G]] = [\zeta_2(G), G].$$

Thus $\zeta_2(G) \subseteq Z(G)$. Now we prove that $Z(G/Z(G))$ is trivial. Let $\pi\colon G \to G/Z(G)$ be the canonical map and $x \in G$ be such that $\pi(x)$ is a central element. Then

$$[\pi(x), \pi(y)] = \pi([x, y]) = 1$$

for all $y \in G$. In particular, $[x, y] \in Z(G) = \zeta_1(G)$ for all $y \in G$. This means that $x \in \zeta_2(G) \subseteq Z(G)$. □

Exercises

5.0.1. Prove that the group $\mathbf{SL}_2(3)$ is solvable.

5.0.2. Compute the Frattini subgroup of $G = \mathbf{SL}_2(3)$.

5.0.3. Let p be a prime. Prove that if G is an elementary abelian p-group, then $\Phi(G) = \{1\}$.

5.0.4. Let $G = \mathbf{SL}_2(3)$. The unique minimal normal subgroup of G is $Z(\mathbf{SL}_2(3)) \simeq C_2$:

5.0.5. Let $q \geq 5$. Prove that $\mathbf{SL}_n(q)$ is perfect.

5.0.6. Let G be a perfect group and N be a normal subgroup of G. Then G/N is perfect.

Notes

Solvable groups...

Chapter 6
Skew braces

braces

6.1 Basic definitions

By convention, an additive group A will be a (not necessarily abelian) group with a binary operation $(a,b) \mapsto a+b$. The identity of A will be denoted by 0 and the inverse of an element a will be denoted by $-a$.

def:brace

Definition 6.1.1. A *skew left brace* is a triple $(A,+,\circ)$, where $(A,+)$ and $(A,\circ)$ are (not necessarily abelian) groups and

$$a \circ (b+c) = (a \circ b) - a + (a \circ c) \tag{6.1}$$

eq:compatibility

holds for all $a,b,c \in A$. The groups $(A,+)$ and $(A,\circ)$ are respectively the *additive* and *multiplicative* group of the skew left brace A.

As usual, multiplication is granted a higher precedence than addition. This means that one can write (6.1) simply as $a \circ (b+c) = a \circ b - a + a \circ c$. We write a' to denote the inverse of a with respect to the circle operation $\circ$.

Skew right braces are defined similarly, one needs to replace (6.1) by

$$(a+b) \circ c = a \circ c - c + b \circ c.$$

There is a bijective correspondence between skew left braces and skew right braces, see Exercise 6.3.1. For that reason, a skew brace will always mean a skew left brace.

Definition 6.1.2. Let $\mathcal{X}$ be a family of groups. A skew brace A is said to be of $\mathcal{X}$-type if its additive group belongs to $\mathcal{X}$.

One particularly interesting family of skew braces is the family of *skew braces of abelian type*, that is skew braces with abelian additive group. Skew braces of abelian type were introduced by Rump in [65] to study involutive solutions to the Yang–Baxter equation. In the literature, skew braces of abelian type are called *left braces*.

exa:trivial **Example 6.1.3.** Let A be an additive group. Then A is a skew brace with $a \circ b = a + b$ for all $a, b \in A$. A skew brace $(A, +, \circ)$ such that $a \circ b = a + b$ for all $a, b \in A$ is said to be *trivial*. Similarly, the operation $a \circ b = b + a$ turns A into a skew brace.

exa:times **Example 6.1.4.** Let A and B be skew braces. Then $A \times B$ with

$$(a, b) + (a_1, b_1) = (a + a_1, b + b_1), \quad (a, b) \circ (a_1, b_1) = (a \circ a_1, b \circ b_1),$$

is a skew brace. This is the *direct product* of the skew braces A and B.

exa:sd **Example 6.1.5.** Let A and M be additive groups and let $\alpha \colon A \to \mathrm{Aut}(M)$ be a group homomorphism. Then $M \times A$ with

$$(x, a) + (y, b) = (x + y, a + b), \quad (x, a) \circ (y, b) = (x + \alpha_a(y), a + b)$$

is a skew brace. Similarly, $M \times A$ with

$$(x, a) + (y, b) = (x + \alpha_a(y), a + b), \quad (x, a) \circ (y, b) = (x + y, b + a)$$

is a skew brace.

exa:s3c6 **Example 6.1.6.** Let A be the symmetric group $\mathbb{S}_3$ written additively. For example, $(12) + (13) = (132)$. Let $\lambda \colon A \to \mathrm{Aut}(A)$ be the map given by

$$\begin{aligned} &\lambda_{\mathrm{id}} = \lambda_{(123)} = \lambda_{(132)} = \mathrm{id}, \\ &\lambda_{(12)} = \lambda_{(23)} = \lambda_{(13)} = \text{conjugation by } (23). \end{aligned}$$

The operation $a \circ b = a + \lambda_a(b)$ turns A into a skew brace with additive group $\mathbb{S}_3$ and multiplicative group cyclic of order six. The multiplicative group of A is generated by (12).

exa:WX **Example 6.1.7.** Let A be an additive group and B and C be subgroups of A such that $B \cap C = \{0\}$ and $A = B + C$. In this case, one says that A admits an *exact factorization* through the subgroups B and C. Thus each $a \in A$ can be written in a unique way as $a = b + c$, for some $b \in B$ and $c \in C$. The map

$$f \colon B \times C^{\mathrm{op}} \to A, \quad (b, c) \mapsto b + c,$$

is bijective. Since f is bijective, A is a group with the operation

$$a \circ a_1 = f\left(f^{-1}(a) f^{-1}(a_1)\right). \tag{6.2}$$

eq:factorization

To compute (6.2) write $a = b + c$ and $a_1 = b_1 + c_1$ for $b, b_1 \in B$ and $c, c_1 \in C$. Then

$$\begin{aligned} a \circ a_1 &= f\left((b, c)(b_1, c_1)\right) \\ &= f\left((b + b_1, c_1 + c)\right) = b + (b_1 + c_1) + c = b + a_1 + c. \end{aligned}$$

It follows that $(A, \circ)$ is a group isomorphic to $B \times C^{\mathrm{op}} \simeq B \times C$. Moreover, if $x, y \in A$, then

$$a \circ x - a + a \circ y = b + x + c - (b + c) + b + y + c = b + x + y + c = a \circ (x + y),$$

and therefore $(A, +, \circ)$ is a skew brace.

We now give concrete some examples of the previous construction.

exa:QR **Example 6.1.8.** Let n be a positive integer. For

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix}$$

let

$$A^* = \bar{A}^t = \begin{pmatrix} \bar{a}_{1,1} & \bar{a}_{2,1} & \cdots & \bar{a}_{n,1} \\ \bar{a}_{1,2} & \bar{a}_{2,2} & \cdots & \bar{a}_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{n,1} & \bar{a}_{2,n} & \cdots & \bar{a}_{n,n} \end{pmatrix},$$

where $\overline{a + bi} = a - bi$ for all $a, b \in \mathbb{R}$. The group $\mathbf{GL}_n(\mathbb{C})$ admits an exact factorization through the subgroups $U(n)$ and $T(n)$, where $U(n) = \{A \in \mathbf{GL}_n(\mathbb{C}) : AA^* = I\}$ is the unitary group and $T(n)$ is the group of upper triangular matrices with positive diagonal entries. Therefore there exists a skew brace with additive group isomorphic to $\mathbf{GL}_n(\mathbb{C})$ and multiplicative group isomorphic to $U(n) \times T(n)$.

The following examples appeared in the theory of Hopf–Galois structures, see [20, Corollary 1.1].

exa:a5a4c5 **Example 6.1.9.** The alternating simple group $\mathbb{A}_5$ admits an exact factorization through the subgroups $A = \langle (123), (12)(34) \rangle \simeq \mathbb{A}_4$ and $B = \langle (12345) \rangle \simeq C_5$. There exists a skew brace with additive group isomorphic to $\mathbb{A}_5$ and multiplicative group isomorphic to $\mathbb{A}_4 \times C_5$.

exa:PSL27S4C7 **Example 6.1.10.** The simple group $\mathbf{PSL}_2(7) = \mathbf{SL}_2(7)/Z(\mathbf{SL}_2(7))$ admits an exact factorization through the subgroups

$$A = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} Z(\mathbf{SL}_2(7)), \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix} Z(\mathbf{SL}_2(7)) \right\rangle \simeq \mathbb{S}_4$$

and

$$B = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} Z(\mathbf{SL}_2(7)) \right\rangle \simeq C_7.$$

Hence there exists a skew brace with additive group isomorphic to $\mathbf{PSL}_2(7)$ and multiplicative group isomorphic to $\mathbb{S}_4 \times C_7$.

lem:basic **Lemma 6.1.11.** *Let A be a skew brace. Then the following properties hold:*

1) $0 = 1$.

2) $a \circ (-b+c) = a-(a\circ b)+(a\circ c)$*, for all* $a,b,c \in A$.
3) $a \circ (b-c) = (a\circ b)-(a\circ c)+a$*, for all* $a,b,c \in A$.

Proof. The first claim follows from the compatibility condition (6.1) with $c = 1$. To prove the second claim let $d = b+c$. Then (6.1) becomes

$$a \circ d = a \circ b - a + a \circ (-b+d)$$

and the claim follows. The third claim is proved similarly. □

pro:lambda **Proposition 6.1.12.** *Let A be a skew brace. For each $a \in A$, the map*

$$\lambda_a \colon A \to A, \quad b \mapsto -a + (a \circ b),$$

is an automorphism of $(A,+)$. Moreover, the map $\lambda \colon (A,\circ) \to \mathrm{Aut}(A,+)$, $a \mapsto \lambda_a$, is a group homomorphism.

Proof. The inverse of λ_a is given by $\lambda_a^{-1} \colon A \to A$, $b \mapsto a' \circ (a+b)$. To prove that $\lambda_a \in \mathrm{Aut}(A,+)$ we note that

$$\lambda_a(b+c) = -a + a \circ (b+c) = -a + a \circ b - a + a \circ c = \lambda_a(b) + \lambda_a(c).$$

Note that $\lambda_a(b) = -a + a \circ b = a \circ (a'+b)$, for all $a,b \in A$. Hence

$$\begin{aligned}\lambda_a(\lambda_b(c)) &= a \circ (a' + b \circ (b'+c)) = -a + a \circ b \circ (b'+c) \\ &= -a + a - a \circ b + a \circ b \circ c = -a \circ b + a \circ b \circ c = \lambda_{a\circ b}(c).\end{aligned}$$ □

If A is a skew brace, then the map λ in the previous proposition yields a left action from $(A,\circ)$ on $(A,+)$ by automorphisms.

The following formulas

$$a \circ b = a + \lambda_a(b), \qquad a + b = a \circ \lambda_a^{-1}(b), \qquad \lambda_a(a') = -a \tag{6.3}$$ eq:formulas

hold for $a,b \in A$. Moreover, if

$$a * b = \lambda_a(b) - b = -a + a \circ b - b,$$

then the following identities are easily verified:

$$a * (b+c) = a * b + b + a * c - b, \tag{6.4}$$ eq:commutator1

$$(a \circ b) * c = (a * (b * c)) + b * c + a * c. \tag{6.5}$$ eq:commutator2

These last two identities are similar to the usual *commutator identities*.

There is also a right action of $(A,\circ)$ on the set A.

pro:mu **Proposition 6.1.13.** *Let A be a skew brace. For each $a \in A$, the map*

$$\mu_a \colon A \to A, \quad b \mapsto \lambda_b(a)' \circ b \circ a,$$

is bijective. Moreover, the map $\mu\colon (A,\circ) \to \mathbb{S}_A$, $a \mapsto \mu_a$, *satisfies* $\mu_b \circ \mu_a = \mu_{a\circ b}$, *for all* $a, b \in A$.

Proof. Note that

$$\mu_a(b) = \lambda_b(a)' \circ b \circ a = (b \circ (b'+a))' \circ b \circ a = (b'+a)' \circ a,$$

for all $a, b \in A$. Hence μ_a is bijective and

$$\mu_a^{-1}(b) = ((b \circ a')' - a)' = (a \circ b' - a)' = (b'+a')' \circ a',$$

for all $a, b \in A$. Now we have

$$\begin{aligned}\mu_b(\mu_a(c)) &= \mu_b((c'+a)' \circ a) = (a' \circ (c'+a) + b)' \circ b \\ &= (a' \circ c' - a' + b)' \circ b = (a' \circ (c'+a \circ b))' \circ b \\ &= (c'+a\circ b)' \circ a \circ b = \mu_{a\circ b}(c),\end{aligned}$$

for all $a, b, c \in A$. Therefore the result follows. □

Definition 6.1.14. A map $f\colon A \to B$ between two skew braces A and B is a *homomorphism of skew braces* if $f(x \circ y) = f(x) \circ f(y)$ and $f(x+y) = f(x) + f(y)$, for all $x, y \in A$. The *kernel* of f is

$$\ker f = \{a \in A : f(a) = 0\}.$$

We write $\mathrm{Hom}_b(A, B)$ to denote the set of skew brace homomorphisms $A \to B$. A bijective homomorphism of skew braces is an isomorphism. An automorphism of a skew brace A is an isomorphism from the skew brace A to it self. We write $\mathrm{Aut}_b(A)$ to denote the group of skew brace automorphisms of A. Two skew braces A and B are isomorphic if there exist an isomorphism $f\colon A \to B$. We write $A \cong B$ to denote that the skew braces A and B are isomorphic.

prop:semidirect

Proposition 6.1.15. *Let* A, B *be skew braces. Let* $\alpha\colon (B,\circ) \to \mathrm{Aut}_b(A)$ *be a homomorphism of groups. Define two operations on* $A \times B$ *by*

$$(a_1,b_1)+(a_2,b_2) = (a_1+a_2, b_1+b_2) \text{ and } (a_1,b_1)\circ(a_2,b_2) = (a_1 \circ \alpha_{b_1}(a_2), b_1 \circ b_2),$$

where $\alpha_{b_1} = \alpha(b_1)$, *for all* $a_1, a_2 \in A$ *and* $b_1, b_2 \in B$. *Then* $(A \times B, +, \circ)$ *is a skew brace. This skew brace is the* semidirect product *of the skew brace* A *by* B *via* α, *and it is denoted by* $A \rtimes_\alpha B$.

Proof. Note that $(A \times B, +)$ is the direct product of the groups $(A,+)$ and $(B,+)$, and $(A \times B, \circ)$ is the semidirect product of the group $(A,\circ)$ by $(B,\circ)$ via α. Hence it is enough to check the compatibility condition. Let $a_1, a_2, a_3 \in A$ and $b_1, b_2, b_3 \in B$. We have

$$\begin{aligned}
&(a_1,b_1)\circ((a_2,b_2)+(a_3,b_3))\\
&\quad=(a_1,b_1)\circ(a_2+a_3,b_2+b_3)\\
&\quad=(a_1\circ\alpha_{b_1}(a_2+a_3),b_1\circ(b_2+b_3))\\
&\quad=(a_1\circ(\alpha_{b_1}(a_2)+\alpha_{b_1}(a_3)),b_1\circ b_2-b_1+b_1\circ b_3)\\
&\quad=(a_1\circ\alpha_{b_1}(a_2)-a_1+a_1\circ\alpha_{b_1}(a_3)),b_1\circ b_2-b_1+b_1\circ b_3)\\
&\quad=(a_1\circ\alpha_{b_1}(a_2),b_1\circ b_2)-(a_1,b_1)+(a_1\circ\alpha_{b_1}(a_3)),b_1\circ b_3)\\
&\quad=(a_1,b_1)\circ(a_2,b_2)-(a_1,b_1)+(a_1,b_1)\circ(a_2,b_3).
\end{aligned}$$

Therefore the result follows. □

lem:homlambda

Lemma 6.1.16. *Let A and B be skew braces. A map $f\colon A\to B$ is a homomorphism of skew braces if and only if $f(a\circ b)=f(a)\circ f(b)$ and $f(\lambda_a(b))=\lambda_{f(a)}(f(b))$, for all $a,b\in A$.*

Proof. Let $a,b\in A$. Suppose that f is a homomorphism of skew braces. Then

$$f(\lambda_a(b))=f(-a+a\circ b)=-f(a)+f(a)\circ f(b)=\lambda_{f(a)}(f(b)).$$

Conversely, suppose that $f(x\circ y)=f(x)\circ f(y)$ and $f(\lambda_x(y))=\lambda_{f(x)}(f(y))$, for all $x,y\in A$. We have that

$$f(a+b)=f(a\circ\lambda_{a'}(b))=f(a)\circ f(\lambda_{a'}(b))=f(a)\circ\lambda_{f(a)'}(f(b))=f(a)+f(b).$$

Therefore the result follows. □

Definition 6.1.17. A skew brace A is said to be a *skew two-sided brace* if

$$(a+b)\circ c=a\circ c-c+b\circ c \tag{6.6}$$

eq:right_compatibility

holds for all $a,b,c\in A$.

If A is a skew two-sided brace, then

$$a\circ(-b)=a-a\circ b+a,\qquad (-a)\circ b=b-a\circ b+b \tag{6.7}$$

eq:2sided

hold for all $a,b\in A$. The first equality holds for every skew brace and follows from Lemma 6.1.11. The second equality follows from (6.6).

Example 6.1.18. Any skew brace with abelian multiplicative group is two-sided.

Example 6.1.19. Let $n\in\mathbb{N}$ be such that $n=p_1^{a_1}\cdots p_k^{a_k}$, where the p_j are distinct primes, all $a_j\in\{1,2\}$ and $p_i^m\not\equiv 1 \bmod p_j$ for all i,j,m with $1\le m\le a_i$. Then every brace of size n is a skew two-sided brace of abelian type, since every group of order n is abelian, see for example [60].

thm:radical

Theorem 6.1.20. *A skew brace of abelian type $(A,+,\circ)$ is two-sided if and only if $(A,+,*)$ it is a radical ring.*

Proof. Assume first that A is a skew two-sided brace of abelian type. Then $(A,+)$ is an abelian group. Let us prove that the operation

$$a * b = -a + a \circ b - b$$

turns A into a radical ring. Left distributivity follows from the compatibility condition and the fact that the addition is commutative:

$$a * (b+c) = -a + a \circ (b+c) - (b+c) = -a + a \circ b - a + a \circ c - c - b = a * b + a * c.$$

Similarly, since A is two-sided, one proves $(a+b) * c = a * c + b * c$. It remains to show that the operation $*$ is associative. On the one hand, using the first equality of (6.7) and the compatibility condition, we write

$$\begin{aligned} a * (b * c) &= a * (-b + b \circ c - c) \\ &= -a + a \circ (-b + b \circ c - c) - (-b + b \circ c - c) \\ &= -a + a \circ (-b) - a + a \circ (b \circ c) - a + a \circ (-c) + c - b \circ c + b \\ &= a \circ (b \circ c) - a \circ b - a \circ c - b \circ c + a + b + c, \end{aligned}$$

since the group $(A,+)$ is abelian. On the other hand, the second equality of (6.7) and Equality (6.6) imply that

$$\begin{aligned} (a * b) * c &= (-a + a \circ b - b) * c = -(-a + a \circ b - b) + (-a + a \circ b - b) \circ c - c \\ &= b - a \circ b + a + (-a) \circ c - c + (a \circ b) \circ c - c + (-b) \circ c - c \\ &= (a \circ b) \circ c - a \circ b - a \circ c - b \circ c + a + b + c. \end{aligned}$$

It then follows that the operation $*$ is associative.

Conversely, if A is a radical ring, say with ring multiplication $(a,b) \mapsto ab$, then $a \circ b = a + ab + b$ turns A into a skew two-sided brace of abelian type. In fact, since A is a radical ring, then $(A,+)$ is an abelian group and $(A,\circ)$ is a group. Moreover,

$$a \circ (b+c) = a + a(b+c) + (b+c) = a + ab + ac + b + c = a \circ b - a + a \circ c.$$

Similarly ones proves $(a+b) \circ c = a \circ c - c + b \circ c$. □

A skew brace is said to be *associative* if the operation $(x,y) \mapsto x * y = \lambda_x(y) - y$ is associative. In [25, Question 2.1(2)], Cedó, Gateva–Ivanova and Smoktunowicz asked whether associative skew braces of abelian type are always radical rings. To answer this question, we need some lemmas.

lem:CGIS **Lemma 6.1.21.** *If A is an associative skew brace of abelian type, then*

$$(-a) * b = -(a * b)$$

holds for all $a,b \in A$. In particular, $(-a) \circ b = 2b - (a \circ b)$ for all $a,b \in A$.

Proof. Since A is of abelian type, Identity (6.5) implies that

$$\begin{aligned}(a*(-a))*b &= (a*(-a)+a+(-a))*b\\ &= (a\circ(-a))*b\\ &= a*((-a)*b)+(-a)*b+a*b.\end{aligned}$$

The associativity of A then implies that $(-a)*b=-(a*b)$. □

If A is a skew brace of abelian type, then one proves by induction that

$$a\circ\left(\sum_{i=1}^{n}b_i-\sum_{j=1}^{m}c_j\right)=\sum_{i=1}^{n}a\circ b_i-\sum_{j=1}^{m}a\circ c_j+(m-n+1)a \tag{6.8}$$

eq:Lau

holds for all $a,b,c\in A$, see Exercise 6.3.8.

thm:Lau

Theorem 6.1.22. *If $(A,+,\circ)$ is an associative skew brace of abelian type, then $(A,+,*)$ is a radical ring.*

Proof. We need to prove that the right compatibility condition holds. Since A is associative, $(a*b)*c=a*(b*c)$ for all $a,b,c\in A$. Write the associativity condition between $a,b,c\in A$ as

$$(a\circ b-a-b)\circ c-(a\circ b-a-b)-c=a\circ(b\circ c-b-c)-a-(b\circ c-b-c),$$

which is equivalent to

$$a'\circ((a\circ b-a-b)\circ c-a\circ b)=a'\circ(a\circ(b\circ c-b-c)-a-a-b\circ c+2c).$$

By Lemma 6.1.21, this is equivalent to

$$a'\circ((a\circ b-a-b)\circ c-a\circ b)=a'\circ(a\circ(b\circ c-b-c)-a-a+(-b)\circ c).$$

By using the formula (6.8), this is equivalent to

$$(b+a'\circ(-b))\circ c-b+a'=b\circ c-b-c+a'\circ(-b)\circ c+a'$$

Hence the associativity of $(A,*)$ is equivalent to

$$(b+a'\circ(-b))\circ c+c=b\circ c+a'\circ(-b)\circ c. \tag{6.9}$$

eq:asociatividad

Let $b,c\in A$. If $d\in A$, then there exists $a\in A$ such that $d=a'\circ(-b)$. Equality (6.9) implies that

$$(b+d)\circ c+c=b\circ c+d\circ c.$$

□

In the database of [45] one finds associative skew braces (of size 16) that are not two-sided, see for example [52].

In Proposition 3.3.5 we used radical rings to produce examples of solutions of the YBE. A natural question arises: Does one need radical rings? Surprisingly, radical rings are just the tip of the iceberg.

thm:YB **Theorem 6.1.23.** *Let A be a skew brace. Then (A, r), where*

$$r\colon A\times A \to A\times A, \quad r(x,y) = (-x + x\circ y, (-x+x\circ y)'\circ x\circ y),$$

is a solution to the YBE.

Proof. By Theorem 3.1.15, since $x\circ y = (-x+x\circ y)\circ((-x+x\circ y)'\circ x\circ y)$ for all $x, y\in A$, we only need to check that $x\triangleright y = \lambda_x(y) = -x+x\circ y$ is a left action of $(A,\circ)$ on the set A and that $x\triangleleft y = \mu_y(x) = (-x+x\circ y)'\circ x\circ y$ is a right action of $(A,\circ)$ on the set A. For the left action we use Proposition 6.1.12 and for the right action we use Proposition 6.1.13. □

In Theorem 6.1.23 it is possible to prove that the solution is involutive if and only if the additive group of the brace is abelian. The next result generalizes this fact. We shall need a lemma.

lem:|r| **Lemma 6.1.24.** *Let A be a skew brace and r be its associated solution. Then*

$$\begin{aligned} r^{2n}(a,b) &= (-n(a\circ b) + a + n(a\circ b),\\ &\qquad (-n(a\circ b)+a+n(a\circ b))'\circ a\circ b), \end{aligned} \tag{6.10}$$ eq:r^2n

$$\begin{aligned} r^{2n+1}(a,b) &= (-n(a\circ b) - a + (n+1)(a\circ b),\\ &\qquad (-n(a\circ b)-a+(n+1)(a\circ b))'\circ a\circ b), \end{aligned} \tag{6.11}$$ eq:r^2n+1

for all $n\geq 0$. Moreover, the following statements hold:

1) *$r^{2n} = \mathrm{id}$ if and only if $a+nb = nb+a$ for all $a, b\in A$.*
2) *$r^{2n+1} = \mathrm{id}$ if and only if $\lambda_a(b) = n(a\circ b)+a-n(a\circ b)$ for all $a, b\in A$.*

Proof. First we shall prove (6.10) and (6.11) by induction on n. The case $n=0$ is trivial for (6.10) and (6.11). Assume that the claim holds for some $n\geq 0$. By applying the map r to Equation (6.11) we obtain that

$$\begin{aligned} r^{2(n+1)}(a,b) &= r\,(-n(a\circ b) - a + (n+1)(a\circ b),\\ &\qquad (-n(a\circ b)-a+(n+1)(a\circ b))'\circ a\circ b)\\ &= (-(n+1)(a\circ b) + a + (n+1)(a\circ b),\\ &\qquad (-(n+1)(a\circ b)+a+(n+1)(a\circ b))'\circ a\circ b)\,. \end{aligned}$$

By applying r again to this equality, we get

$$\begin{aligned} r^{2(n+1)+1}(a,b) &= r\,(-(n+1)(a\circ b) + a + (n+1)(a\circ b),\\ &\qquad (-(n+1)(a\circ b)+a+(n+1)(a\circ b))'\circ a\circ b)\\ &= (-(n+1)(a\circ b) - a + (n+2)(a\circ b),\\ &\qquad (-(n+1)(a\circ b)-a+(n+2)(a\circ b))'\circ a\circ b)\,. \end{aligned}$$

Thus Equations (6.10) and (6.11) hold by induction. The other claims follow easily from Equations (6.10) and (6.11). □

Recall that the (minimal) *exponent* $\exp(G)$ of a finite group G is the least positive integer n such that $g^n = 1$ for all $g \in G$.

thm:|r|

Theorem 6.1.25. *Let A be a finite skew brace with more than one element and let G be the additive group of A. If r is the solution associated with A, then r has order $2\exp(G/Z(G))$.*

Proof. Assume that r has order $2n+1$. Since $r^{2n+1} = \mathrm{id}$, by applying Lemma 6.1.24, one obtains that $-a + (n+1)(a \circ b) = n(a \circ b) + a$ for all $a, b \in A$. In particular, for $b = 0$, we get $a = 0$, for all $a \in A$, a contradiction. Therefore we may assume that the order of the permutation r is $2n$, where

$$n = \min\{k \in \mathbb{Z} : k > 0 \text{ and } kb + a = a + kb \text{ for all } a, b \in A\}.$$

Now one computes

$$\begin{aligned} n &= \min\{k \in \mathbb{Z} : k > 0 \text{ and } kb \in Z(G) \text{ for all } b \in A\} \\ &= \min\{k \in \mathbb{Z} : k > 0 \text{ and } k(b + Z(G)) = Z(G) \text{ for all } b \in A\} = \exp(G/Z(G)). \end{aligned}$$ □

An inmmediate consequence is the following result.

Corollary 6.1.26. *Let A be a finite skew brace and r be its associated solution. Then r is involutive if and only if A is of abelian type.*

6.2 Subbraces and ideals

Definition 6.2.1. Let A be a skew brace. A *subbrace* of A is a subset B of A such that $(B, +)$ is a subgroup of $(A, +)$ and $(B, \circ)$ is a subgroup of $(A, \circ)$.

Definition 6.2.2. Let A be a skew brace. A *left ideal* of A is a subgroup $(I, +)$ of $(A, +)$ such that $\lambda_a(I) \subseteq I$ for all $a \in A$, i.e. $\lambda_a(x) \in I$ for all $a \in A$ and $x \in I$. A *strong left ideal* of A is a left ideal I of A such that $(I, +)$ is a normal subgroup of $(A, +)$.

Proposition 6.2.3. *A left ideal I of a skew brace A is a subbrace of A.*

Proof. We need to prove that $(I, \circ)$ is a subgroup of $(A, \circ)$. Clearly I is non-empty, as it is an additive subgroup of A. If $x, y \in I$, then $x \circ y = x + \lambda_x(y) \in I + I \subseteq I$ and $x' = -\lambda_{x'}(x) \in I$. □

Example 6.2.4. Let A be a skew brace. Then

$$\mathrm{Fix}(A) = \{a \in A : \lambda_x(a) = a \text{ for all } x \in A\}$$

is a left ideal of A.

Definition 6.2.5. An *ideal* of a skew brace A is a strong left ideal I of A such that $(I,\circ)$ is a normal subgroup of $(A,\circ)$.

In general, left ideals, strong left ideals and ideals are different notions, see Exercise 6.3.13.

Example 6.2.6. Consider the semidirect product $A=\mathbb{Z}/(3)\rtimes\mathbb{Z}/(2)$ of the trivial skew braces $\mathbb{Z}/(3)$ and $\mathbb{Z}/(2)$ via the non-trivial action of $\mathbb{Z}/(2)$ over $\mathbb{Z}/(3)$. Then

$$\lambda_{(x,y)}(a,b)=-(x,y)+(x,y)\circ(a,b)=-(x,y)+(x+(-1)^y a,y+b)=((-1)^y a,b)$$

and hence $\mathrm{Fix}(A)=\{(0,0),(0,1)\}$ is not a normal subgroup of $(A,\circ)$. In particular, $\mathrm{Fix}(A)$ is not an ideal of A.

Example 6.2.7. Let $f\colon A\to B$ be a homomorphism of skew braces. Then $\ker f$ is an ideal of A.

If X and Y are subsets of a brace A, $X*Y$ is defined as the subgroup of $(A,+)$ generated by elements of the form $x*y$, $x\in X$ and $y\in Y$, i.e.

$$X*Y=\langle x*y : x\in X, y\in Y\rangle_+.$$

pro:A*I **Proposition 6.2.8.** *Let A be a skew brace. A subgroup I of $(A,+)$ is a left ideal of A if and only if $A*I\subseteq I$.*

Proof. Let $a\in A$ and $x\in I$. If I is a left ideal, then $a*x=\lambda_a(x)-x\in I$. Conversely, if $A*I\subseteq I$, then $\lambda_a(x)=a*x+x\in I$. □

pro:I*A **Proposition 6.2.9.** *Let A be a skew brace. A normal subgroup I of $(A,+)$ is an ideal of A if and only $\lambda_a(I)\subseteq I$, for all $a\in A$, and $I*A\subseteq I$.*

Proof. Let $x\in I$ and $a\in A$. Assume first that I is invariant under the action of λ and that $I*A\subseteq I$. Then

$$\begin{aligned} a\circ x\circ a' &= a+\lambda_a(x\circ a')\\ &= a+\lambda_a(x+\lambda_x(a'))=a+\lambda_a(x)+\lambda_a\lambda_x(a')+a-a\\ &= a+\lambda_a(x+\lambda_x(a')-a')-a=a+\lambda_a(x+x*a')-a\in I, \end{aligned} \tag{6.12}$$

eq:trick:I*A

and hence I is an ideal.

Conversely, assume that I is an ideal. Then $I*A\subseteq I$ since

$$\begin{aligned} x*a &= -x+x\circ a-a\\ &= -x+a\circ(a'\circ x\circ a)-a=-x+a+\lambda_a(a'\circ x\circ a)-a\in I \end{aligned}$$

for all $x\in I$ and $a\in A$. □

Let I and J be ideals of a skew brace A. Then $I\cap J$ is an ideal of A, see Exercise 6.3.11. The sum $I+J$ of I and J is defined as the additive subgroup of A generated by all the elements of the form $u+v$, $u\in I$ and $v\in J$.

Proposition 6.2.10. *Let A be a skew brace and let I and J be ideals of A. Then $I+J$ is an ideal of A.*

Proof. Since I and J are normal subgroups of $(A,+)$, we have that

$$I+J=\{u+v \mid u\in I,\ v\in J\}$$

is a normal subgroup of $(A,+)$. Let $a\in A$, $u\in I$ and $v\in J$. Then

$$\lambda_a(u+v)=\lambda_a(u)+\lambda_a(v)\in I+J$$

and hence it follows that $\lambda_a(I+J)\subseteq I+J$. Moreover, by Proposition 6.2.9,

$$(u+v)*a=(u\circ\lambda_u^{-1}(v))*a=u*(\lambda_u^{-1}(v)*a)+\lambda_u^{-1}(v)*a+u*a\in I+J.$$

Hence $(I+J)*A\subseteq I+J$. Therefore the result follows by Proposition 6.2.9. □

Definition 6.2.11. Let A be a skew brace. The subset $\mathrm{Soc}(A)=\ker\lambda\cap Z(A,+)$ is the *socle* of A.

lem:socle **Lemma 6.2.12.** *Let A be a skew brace and $x\in\mathrm{Soc}(A)$. Then*

$$a+a\circ x=a\circ x+a \quad \textit{and} \quad \lambda_a(x)=a\circ x\circ a'$$

hold for all $a\in A$. In particular, $x\circ a=x+a$ for all $x\in\mathrm{Soc}(A)$ and $a\in A$, and $x'=-x$ for all $x\in\mathrm{Soc}(A)$.

Proof. Let $a\in A$ and $x\in\mathrm{Soc}(A)$. Since

$$a'\circ(a\circ x+a)=x-a' \quad \text{and} \quad a'\circ(a+a\circ x)=-a'+x,$$

the first claim follows since $x\in Z(A,+)$. Now we prove the second claim:

$$a\circ x\circ a'=a\circ(x\circ a')=a\circ(x+a')=a\circ x-a=-a+a\circ x=\lambda_a(x).$$ □

pro:socle **Proposition 6.2.13.** *Let A be a skew brace. Then $\mathrm{Soc}(A)$ is an ideal of A.*

Proof. We first prove that $(\mathrm{Soc}(A),+)$ is a normal subgroup of $(A,+)$. Clearly $0\in\mathrm{Soc}(A)$, since λ is a group homomorphism. Let $x,y\in\mathrm{Soc}(A)$ and $a\in A$. By Lemma 6.2.12, $x-y=x\circ(-y)=x\circ y'$ and hence

$$\lambda_{x-y}(a)=\lambda_{x\circ y'}(a)=\lambda_x\lambda_y^{-1}(a)=a.$$

Thus $x-y\in\mathrm{Soc}(A)$ and therefore the claim follows, as $\mathrm{Soc}(A)$ is central in $(A,+)$.

We now prove that $\lambda_a(x)\in\mathrm{Soc}(A)$ for all $a\in A$ and $x\in\mathrm{Soc}(A)$. First we note that $\lambda_a(x)$ is central, as

$$\begin{aligned}b+\lambda_a(x)&=b-a+a\circ x=a\circ(a'\circ b+x)\\&=a\circ(x+a'\circ b)=a\circ x-a+b=-a+a\circ x+b=\lambda_a(x)+b\end{aligned}$$

for all $b \in A$, by Lemma 6.2.12. Moreover, by Lemma 6.2.12,

$$\lambda_{\lambda_a(x)}(b) = \lambda_{a\circ x\circ a'}(b) = \lambda_a\lambda_x\lambda_a^{-1}(b) = b$$

for all $b \in A$. Since $x * a = -x + x \circ a - a = -x + x + a - a = 0 \in \mathrm{Soc}(A)$, the claim follows from Proposition 6.2.9. □

As a corollary we obtain that the socle of a skew brace A is a trivial skew brace of abelian type.

pro:soc_kernels

Proposition 6.2.14. *Let A be a skew brace. Then* $\mathrm{Soc}(A) = \ker\lambda \cap \ker\mu$.

Proof. Let $a \in \mathrm{Soc}(A)$ and $b \in A$. Then $\lambda_a = \mathrm{id}$ and $a \in Z(A,+)$. By Lemma 6.2.12,

$$\mu_a(b) = \lambda_b(a)' \circ b \circ a = (b \circ a \circ b')' \circ b \circ a = b.$$

Thus $a \in \ker\lambda \cap \ker\mu$.

Conversely, let $a \in \ker\lambda \cap \ker\mu$ and $b \in A$. Then $b' = \mu_a(b') = \lambda_{b'}(a)' \circ b' \circ a$, so $\lambda_{b'}(a) = b' \circ a \circ b$. By Lemma 6.2.12 and (6.3),

$$b + a = b \circ \lambda_b^{-1}(a) = b \circ \lambda_{b'}(a) = b \circ b' \circ a \circ b = a \circ b = a + \lambda_a(b) = a + b.$$

Hence $a \in \mathrm{Soc}(A)$. □

Definition 6.2.15. Let A be a skew brace. The *annihilator* of A is defined as the set $\mathrm{Ann}(A) = \mathrm{Soc}(A) \cap Z(A,\circ)$.

Note that $\mathrm{Ann}(A) \subseteq \mathrm{Fix}(A)$.

Proposition 6.2.16. *The annihilator of a skew brace A is an ideal of A.*

Proof. Let $x, y \in \mathrm{Ann}(A)$. Note that $x - y = x \circ y' \in Z(A,\circ)$. Hence $\mathrm{Ann}(A)$ is a subbrace of A. Since $\mathrm{Ann}(A) \subseteq Z(A,+) \cap Z(A,\circ)$, we only need to note that $\lambda_a(x) = x \in \mathrm{Ann}(A)$, for all $a \in A$. □

If A is a skew brace and I is an ideal of A, then $a + I = a \circ I$ for all $a \in A$. Indeed, $a \circ x = a + \lambda_a(x) \in a + I$ and $a + x = a \circ \lambda_a^{-1}(x) = a \circ \lambda_{a'}(x) \in a \circ I$ for all $a \in A$ and $x \in I$. This allows us to prove that there exists a unique skew brace structure over A/I such that the map

$$\pi\colon A \to A/I, \quad a \mapsto a + I = a \circ I,$$

is a homomorphism of skew braces. The brace A/I is the *quotient brace* of A modulo I. It is possible to prove the isomorphism theorems for skew braces, see Exercises 6.3.16, 6.3.17, 6.3.18 and 6.3.19.

Finite skew braces

We now use Hall's theorem to obtain information related to the structure of finite skew braces of nilpotent type.

thm:add_nilpotent

Theorem 6.2.17. *Let A be a finite skew brace of nilpotent type. Then the multiplicative group of A is solvable.*

Proof. Let K be the additive group of A and G be the multiplicative group of A. Assume that $|A| = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$ for distinct primes $p_1, \ldots, p_n$ and positive integers $\alpha_1, \ldots, \alpha_n$. Since K is nilpotent and finite, each $K_i \in \mathrm{Syl}_{p_j}(K)$ is normal in K, so each K_i is a left ideal of A. It follows that for each $i \in \{1, \ldots, n\}$ both K_i and $\prod_{j \neq i} K_j$ are subbraces of coprime order. In particular, for each $i \in \{1, \ldots, n\}$ there exists a p_i-complement of G. Then G is solvable by Hall's theorem. □

6.3 Exercises

prob:left_right

6.3.1. Prove that there exists a bijective correspondence between skew left braces and skew right braces.

6.3.2. Let p be a prime number. Prove that $\mathbb{Z}/(p^2)$ is a skew brace of abelian type with the operation $x \circ y = x + y + pxy$.

6.3.3. Let A be a skew brace. Prove that

$$\mu_b(a) = \lambda^{-1}_{\lambda_a(b)}(-a \circ b + a + a \circ b).$$

prob:star

6.3.4. Let A be an additive (not necessarily abelian) group. Prove that a structure of skew brace over A is equivalent to an operation $A \times A \to A$, $(a, b) \mapsto a * b$, such that

$$a * (b + c) = a * b + b + a * c - b$$

holds for all $a, b, c \in A$, and the operation $a \circ b = a + a * b + b$ turns A into a group.

prob:equivalences

6.3.5. Let $(A, +, \circ)$ be a triple, where $(A, +)$ and $(A, \circ)$ are groups, and let $\lambda \colon A \to \mathbb{S}_A$, $a \mapsto \lambda_a$, $\lambda_a(b) = -a + a \circ b$. Prove that the following statements are equivalent:

1) A is a skew brace.
2) $\lambda_{a \circ b}(c) = \lambda_a \lambda_b(c)$ for all $a, b, c \in A$.
3) $\lambda_a(b + c) = \lambda_a(b) + \lambda_a(c)$ for all $a, b, c \in A$.

6.3.6. Let A and B be skew braces. Prove that $f \colon A \to B$ is a homomorphism of skew braces if and only if $f(a + b) = f(a) + f(b)$ and $f(\lambda_a(b)) = \lambda_{f(a)}(f(b))$, for all $a, b \in A$.

prob:2sided

6.3.7. Let A be a skew brace such that $\lambda_a(a) = a$ for all $a \in A$. Prove that A is two-sided. (Hint: Prove first that $a' = -a$ for all $a \in A$.)

prob:Lau **6.3.8.** Prove Equality (6.8).

prob:Landau **6.3.9.** Let $k,n \in \mathbb{N}$ and let A be a finite skew brace of size n and k orbits. Then $|k| \geq \frac{\log\log n}{\log 4}$.

prob:radical **6.3.10.** Recall that skew two-sided braces of abelian type are equivalent to radical rings. Prove that under this equivalence, (left) ideals of the radical ring correspond to (left) ideals of the associated brace.

prob:sum_ideals **6.3.11.** Prove that the intersection of ideals of a skew brace A is an ideal of A.

6.3.12. Let A be a skew brace and I be a characteristic subgroup of the additive group of A. Prove that I is a left ideal of A.

rob:exleftidealnotstrong **6.3.13.** Give an example of a left ideal which is not a strong left ideal of a skew brace. Give an example of a strong left ideal which is not an ideal of a skew brace.

prob:Bachiller1 **6.3.14.** Prove that the socle of a skew brace A is the kernel of the group homomorphism $(A,\circ) \to \mathrm{Aut}(A,+)\times\mathbb{S}_A$, $a \mapsto (\lambda_a,\mu_a^{-1})$.

prob:Bachiller2 **6.3.15.** Prove that the socle of a skew brace A is the kernel of the group homomorphism $(A,\circ) \to \mathrm{Aut}(A,+)\times\mathrm{Aut}(A,+)$, $a \mapsto (\lambda_a,\xi_a)$, where $\xi_a(b) = a+\lambda_a(b)-a$.

prob:iso1 **6.3.16.** Let $f\colon A \to B$ be a homomorphism of skew braces. Prove that $A/\ker f \simeq f(A)$.

prob:iso2 **6.3.17.** Let A be a skew brace and let B be a subbrace of A. Prove that if I is an ideal of A, then $B\circ I$ is a subbrace of A, $B\cap I$ is an ideal of B and $(B\circ I)/I \simeq B/(B\cap I)$.

prob:iso3 **6.3.18.** Let A be a skew brace and I and J be ideals of A. Prove that if $I \subseteq J$, then $A/J \simeq (A/I)/(J/I)$.

prob:correspondence **6.3.19.** Let A be a skew brace and let I be an ideal of A. Prove that there is a bijective correspondence between (left) ideals of A containing I and (left) ideals of A/I.

prob:G(X,r)solvable **6.3.20.** Let (X,r) be a finite involutive solution. Prove that $G(X,r)$ is solvable.

6.4 Notes

Skew braces of abelian type (i.e. left braces) were introduced by Rump in [65] for studying involutive solutions to the YBE. Rump's definition was reformulated by Cedó, Jespers and Okniński in [27]. With this definition at hand, Guarnieri and Vendramin introduced arbitrary skew braces in [45].

Example is motivated by the paper [78].

Theorem 6.1.20 was proved by Rump in [65].

Theorem 6.1.22 was proved by Lau [55] and independently by Kinyon (unpublished). It answers a question of Cedó, Gateva–Ivanova and Smoktunowicz, see [25].

Theorem 6.1.23 for skew braces of abelian type appears implicit in the work [65] of Rump, see also [27]. The general case was proved by Guarnieri and Vendramin in [45].

Theorem 6.1.25 was proved by Smoktunowicz and Vendramin in [72].

The socle was defined by Rump in [65]. The annihilator first appeared in the work [22] of Catino, Colazzo and Stefanelli.

Theorem 6.2.17 was proved by Byott in the context of Hopf-Galois extensions in [20].

Exercise 6.3.5 combines results of Bachiller, Rump [65] and Gateva–Ivanova [43]. Exercise 6.3.7 for skew braces of abelian type comes from [27]. Exercises 6.3.14 and 6.3.15 appear in [10].

Exercise 6.3.9 is a direct translation into the context of braces (or more precisely, the context of groups acting by automorphisms on groups) of Newmann's result based on Landau's method.

Chapter 7
Complements

cocycles

7.1 Extensions and 1-cocycles

Let K, Q be groups. An *extension* of K by Q is a short exact sequence of group homomorphisms

$$1 \longrightarrow K \xrightarrow{f} G \xrightarrow{g} Q \longrightarrow 1$$

This means that f is injective, g is surjective and $\ker g = \operatorname{im} f$. Note that in this case, K is isomorphic to $f(K)$, which is a normal subgroup of G and $G/f(K) \simeq Q$. We also say that G is an extension of K by Q.

Example 7.1.1. C_6 and $\mathbb{S}_3$ are both extensions of C_3 by C_2.

Example 7.1.2. C_6 is an extension of C_2 by C_3.

Example 7.1.3. The direct product $K \times Q$ of the groups K and Q is an extension of K by Q and an extension of Q by K.

Example 7.1.4. Let G be an extension of K by Q. If L is a subgroup of G containing K, then L is an extension of K by L/K.

Let $E: \ 1 \longrightarrow K \longrightarrow G \xrightarrow{p} Q \longrightarrow 1$ be an extension of groups. A *lifting* of E is a map $\ell\colon Q \to G$ such that $p(\ell(x)) = x$, for all $x \in Q$.

An extension E *splits* if there is a lifting of E that it is a group homomorphism.

Let Q and K be groups. Assume that Q acts by automorphisms on K, that is there is a group homomorphism $\alpha\colon Q \to \operatorname{Aut}(k)$. We write $\alpha_x = \alpha(x)$, for all $x \in Q$. A map $\varphi\colon Q \to K$ is said to be a 1-*cocycle* (or a derivation) if

$$\varphi(xy) = \varphi(x)\alpha_x(\varphi(y)),$$

for all $x, y \in Q$. The set of 1-cocycles $Q \to K$ is defined as

$$\operatorname{Der}(Q,K) = Z^1(Q,K) = \{\delta\colon Q \to K : \delta \text{ is 1-cocycle}\}.$$

Example 7.1.5. Let Q, K be groups. Let $\alpha\colon Q \to \mathrm{Aut}(K)$ be a group homomorphism. For each $k \in K$, the map $\delta_k\colon Q \to K, x \mapsto \delta_k(x) = [k,x] = k\alpha_x(k)^{-1}$, is a derivation.

A subgroup K of a group G admits a *complement* Q if G admits an exact factorization through K and Q, i.e. $G = KQ$ with $K \cap Q = \{1\}$. A classical example is the (inner) semidirect product $G = K \rtimes Q$, where K is a normal subgroup of G and Q is a subgroup of G such that $K \cap Q = \{1\}$.

Let K, Q be groups. Let $\alpha\colon Q \to \mathrm{Aut}(K), x \mapsto \alpha_x$ be a group homomorphism. Consider the set $K \times Q$. We define a multiplication on this set by

$$(k_1, q_1)(k_2, q_2) = (k_1\alpha_{q_1}(k_2), q_1q_2),$$

for all $k_1, k_2 \in K$ and $q_1, q_2 \in Q$. Then $K \times Q$ with this multiplication is a group, called the semidirect product of K by Q via α, and denoted by $K \rtimes_\alpha Q$. Note that $K \times \{1\}$ is a normal subgroup of $K \rtimes_\alpha Q$, and $\{1\} \times Q$ is a subgroup of $K \rtimes_\alpha Q$. Thus $K \rtimes_\alpha Q$ is the inner semidirect product of $K \times \{1\}$ by $\{1\} \times Q$. Since $K \cong K \times \{1\}$ and $Q \cong \{1\} \times Q$, we identify these groups, i. e. $k = (k, 1)$ and $q = (1, q)$, for all $k \in K$ and $q \in Q$. Note that

$$qkq^{-1} = (1,q)(k,1)(1,q^{-1}) = (\alpha_q(k),q)(1,q^{-1}) = (\alpha_q(k),1) = \alpha_q(k),$$

for all $k \in K$ and $q \in Q$.

thm:complements

Theorem 7.1.6. *Let Q, K be groups and let $\alpha\colon Q \to \mathrm{Aut}(K)$ be a group homomorphism. Then there exists a bijective correspondence between the set C of complements of K in $K \rtimes Q$ and the set $\mathrm{Der}(Q, K)$ of 1-cocycles $Q \to K$.*

Proof. Since Q acts by conjugation on K, it follows that $\delta \in \mathrm{Der}(Q, K)$ if and only if $\delta(xy) = \delta(x)x\delta(y)x^{-1}$ for all $x, y \in Q$. In this case, one obtains that $\delta(1) = 1$ and $\delta(x^{-1}) = x^{-1}\delta(x)^{-1}x$.

Let $C \in \mathcal{C}$. If $x \in Q$, then there exist unique elements $k \in K$ and $c \in C$ such that $x = k^{-1}c$. Hence the map $\delta_C\colon Q \to K, x \mapsto k$, is well-defined. Moreover, $\delta_C(x)x = c \in C$.

We claim that $\delta_C \in \mathrm{Der}(Q, K)$. If $x, x_1 \in Q$, we write $x = k^{-1}c$ and $x_1 = k_1^{-1}c_1$ for $k, k_1 \in K$ and $c, c_1 \in C$. Since K is a normal subgroup of the semidirect product $K \rtimes Q$, we can write xx_1 as $xx_1 = k_2c_2$, where $k_2 = k^{-1}(ck_1^{-1}c^{-1}) \in K$, $c_2 = cc_1 \in C$. Thus $\delta_C(xx_1)xx_1 = cc_1 = \delta_C(x)x\delta_C(x_1)x_1$ implies that $\delta_C(xx_1) = \delta_C(x)x\delta_C(x_1)x^{-1}$. So there is a map $F\colon \mathcal{C} \to \mathrm{Der}(Q, K)$, $F(C) = \delta_C$.

We now construct a map $G\colon \mathrm{Der}(Q, K) \to \mathcal{C}$. For each $\delta \in \mathrm{Der}(Q, K)$ we find a complement Δ of K in $K \rtimes Q$. Let $\Delta = \{\delta(x)x : x \in Q\}$. We claim that Δ is a subgroup of $K \rtimes Q$. Since $\delta(1) = 1$, $1 \in \Delta$. If $x, y \in Q$, then $\delta(x)x\delta(y)y = \delta(x)x\delta(y)x^{-1}xy = \delta(xy)xy \in \Delta$. Finally, if $x \in Q$, then

$$(\delta(x)x)^{-1} = x^{-1}\delta(x)^{-1}xx^{-1} = \delta(x^{-1})x^{-1} \in \Delta.$$

Thus Δ is a subgroup of $K \rtimes Q$. We claim that $\Delta \cap K = \{1\}$. If $x \in Q$ is such that $\delta(x)x \in K$, then since $\delta(x) \in K$, it follows that $x \in K \cap Q = \{1\}$. If $g \in G$, then there

are unique $k \in K$ and $x \in Q$ such that $g = kx$. We write $g = k\delta(x)^{-1}\delta(x)x$. Since $k\delta(x)^{-1} \in K$ and $\delta(x)x \in \Delta$, we conclude that $G = K\Delta$. Thus there is a well-defined map $G\colon \operatorname{Der}(Q,K) \to C$, $G(\delta) = \Delta$.

We claim that $G \circ F = \mathrm{id}_C$. Let $C \in C$. Then

$$G(F(C)) = G(\delta_C) = \{\delta_C(x)x : x \in Q\} = C,$$

by construction. (We know that $\delta_C(x)x \in C$. Conversely, if $c \in C$, we write $c = kx$ for unique elements $k \in K$ and $x \in Q$. Thus $x = k^{-1}c$ and hence $c = \delta_c(x)x$.)

Finally, we prove that $F \circ G = \mathrm{id}_{\operatorname{Der}(Q,K)}$. Let $\delta \in \operatorname{Der}(Q,K)$. Then

$$F(G(\delta)) = F(\Delta) = \delta_\Delta.$$

Finally, we need to show that $\delta_\Delta = \delta$. Let $x \in Q$. There exists $\delta(y)y \in \Delta$ for some $y \in Q$ such that $x = k^{-1}\delta(y)y$. Thus $\delta_\Delta(x)x = \delta(y)y$ and hence $x = y$ and $\delta_\Delta(x) = \delta(y)$ by the uniqueness. Therefore, $\delta_\Delta = \delta$, and the result follows. □

Let K, Q be groups and let $\alpha\colon Q \to \operatorname{Aut}(K)$ be a group homomorphism. A derivation $\delta \in \operatorname{Der}(Q,K)$ is said to be *inner* if there exists $k \in K$ such that $\delta(x) = [k,x]$ for all $x \in Q$. The set of *inner derivations* will be denoted by

$$\operatorname{Inn}(Q,K) = B^1(Q,K) = \{\delta \in \operatorname{Der}(Q,K) : \delta \text{ is inner}\}.$$

An inner derivation is also called a 1-*coboundary*.

theorem:Sysak

Theorem 7.1.7 (Sysak). *Let K, Q be groups and let $\alpha\colon Q \to \operatorname{Aut}(K)$ be a group homomorphism. Let $\delta \in \operatorname{Der}(Q,K)$.*

1) *$\Delta = \{\delta(x)x : x \in Q\}$ is a complement of K in $K \rtimes Q$.*
2) *$\delta \in \operatorname{Inn}(Q,K)$ if and only if $\Delta = kQk^{-1}$ for some $k \in K$.*
3) *$\ker\delta = Q \cap \Delta$.*
4) *δ is surjective if and only if $K \rtimes Q = \Delta Q$.*

Proof. In the proof of Theorem 7.1.6 we found that Δ is a complement of K in $K \rtimes Q$.

Let us prove the second statement. If δ is inner, then there exists $k \in K$ such that $\delta(x) = [k,x] = kxk^{-1}x^{-1}$ for all $x \in Q$. Since $\delta(x)x = kxk^{-1}$ for all $x \in Q$, $\Delta = kQk^{-1}$. Conversely, if there exists $k \in K$ such that $\Delta = kQk^{-1}$, for each $x \in Q$ there exists $y \in Q$ such that $\delta(x)x = kyk^{-1}$. Since $[k,y] = kyk^{-1}y^{-1} \in K$, $\delta(x) \in K$ and $\delta(x)x = [k,y]y \in KQ$, we conclude that $x = y$ and hence $\delta(x) = [k,x]$.

Let us prove the third statement. If $x \in Q$ is such that $\delta(x)x = y \in Q$, then

$$\delta(x) = yx^{-1} \in K \cap Q = \{1\}.$$

Conversely, if $x \in Q$ is such that $\delta(x) = 1$, then $x = \delta(x)x \in Q \cap \Delta$.

Finally we prove the fourth statement. If δ is surjective, then for each $k \in K$ there exits $y \in Q$ such that $\delta(y) = k$. Thus $K \rtimes Q \subseteq \Delta Q$, as

$$kx = \delta(y)x = (\delta(y)y)y^{-1}x \in \Delta Q.$$

Since Δ and Q are subgroups of $K \rtimes Q$, we have that $\Delta Q \subseteq K \rtimes Q$, and therefore $\Delta Q = K \rtimes Q$. Conversely, if $k \in K$ and $x \in Q$ there exist $y, z \in Q$ such that $kx = \delta(y)yz$. Then it follows that $k = \delta(y)$. □

A group G admits a *triple factorization* if there are subgroups A, B and M such that $G = MA = MB = AB$ and $A \cap M = B \cap M = \{1\}$. The following result is an immediate consequence of Sysak's theorem.

Corollary 7.1.8. *If the group Q acts by automorphisms on K and $\delta \in \mathrm{Der}(Q,K)$ is surjective, then $G = K \rtimes Q$ admits a triple factorization.*

7.2 Schur–Zassenhaus' theorem

In this section we shall prove Schur–Zassenhaus' theorem about complements of a normal subgroup N of a finite group G.

lem:1cocycle

Lemma 7.2.1. *Let G be a group and N be a normal subgroup of G. If G acts on N by conjugation and $\varphi\colon G \to N$ is a 1-cocycle with kernel K, then $\varphi(x) = \varphi(y)$ if and only if $xK = yK$. In particular, $(G : K) = |\varphi(G)|$.*

Proof. If $\varphi(x) = \varphi(y)$, then, since

$$\varphi(x^{-1}y) = \varphi(x^{-1})x^{-1}\varphi(y)x = \varphi(x^{-1})x^{-1}\varphi(x)x = \varphi(x^{-1}x) = \varphi(1) = 1,$$

we obtain that $xK = yK$. Conversely, if $x^{-1}y \in K$, then, since

$$\varphi(x^{-1})x^{-1}\varphi(x)x = \varphi(x^{-1}x) = 1 = \varphi(x^{-1}y) = \varphi(x^{-1})x^{-1}\varphi(y)x,$$

we conclude that $\varphi(x) = \varphi(y)$.

The second claim is now trivial, as φ is constant in each coset of K and there are $(G : K)$ different possible values. □

lem:d

Lemma 7.2.2. *Let G be a finite group, N be a normal abelian subgroup of G and S, T and U be transversals of N in G. Let*

$$d(S,T) = \prod st^{-1} \in N,$$

where the product is taken over all $s \in S$ and $t \in T$ such that $sN = tN$. The following statements hold:

1) *$d(S,T)d(T,U) = d(S,U)$.*
2) *$d(gS,gT) = gd(S,T)g^{-1}$ for all $g \in G$.*
3) *$d(nS,S) = n^{(G:N)}$ for all $n \in N$.*

Proof. If $s \in S$, $t \in T$ and $u \in U$ are such that $sN = tN = uN$, then, since N is abelian and $(st^{-1})(tu^{-1}) = su^{-1}$,

$$d(S,T)d(T,U) = \prod (st^{-1})(tu^{-1}) = \prod su^{-1} = d(S,U).$$

Since $sN = tN$ if and only if $gsN = gtN$ for all $g \in G$,

$$g\left(\prod st^{-1}\right)g^{-1} = \prod gst^{-1}g^{-1} = \prod (gs)(gt)^{-1} = d(gS,gT).$$

Finally, since N is normal, $nsN = sN$ for all $n \in N$. Thus

$$d(nS,S) = \prod (ns)s^{-1} = n^{(G:N)}. \qquad \square$$

We now prove the first version of Schur–Zassenhaus' theorem.

:SchurZassenhaus:abelian

Theorem 7.2.3 (Schur–Zassenhaus). *Let G be a finite group and let N be an abelian normal subgroup of G. If $|N|$ and $(G : N)$ are coprime, then N admits a complement in G. In this case, all complements of N are conjugate in G.*

Proof. Let T be a transversal of N in G. Let $\theta\colon G \to N$, $\theta(g) = d(gT,T)$. Since N is abelian, Lemma 7.2.2 implies that θ is a 1-cocycle, where G acts on N by conjugation:

$$\begin{aligned}\theta(xy) &= d(xyT,T) = d(xyT,xT)d(xT,T)\\ &= (xd(yT,T)x^{-1})d(xT,T) = (x\theta(y)x^{-1})\theta(x).\end{aligned}$$

Claim. $\theta|_N : N \to N$ is surjective.

If $n \in N$, then $\theta(n) = d(nT,T) = n^{(G:N)}$ by Lemma 7.2.2. Since $|N|$ and $(G : N)$ are coprime, there exist $r,s \in \mathbb{Z}$ such that $r|N| + s(G : N) = 1$. Thus

$$n = n^{r|N|+s(G:N)} = (n^s)^{(G:N)} = \theta(n^s).$$

Let $H = \ker\theta$. We claim that H is a complement for N. We know that H is a subgroup of G. Since, by Lemma 7.2.1,

$$|N| = |\theta(G)| = (G : H) = \frac{|G|}{|H|},$$

it follows that $N \cap H = \{1\}$ because $|N|$ and $(G : N) = |H|$ are coprime. Since $|NH| = |N||H| = |G|$, we conclude that $G = NH$ and hence H is a complement of N in G.

We now prove that two complements of N in G are conjugate. Let K be a complement of N in G. Since $NK = G$ and $N \cap K = \{1\}$, it follows that K is a transversal of N in G. Let $m = d(T,K) \in N$. Since $\theta|_N$ is surjective, there exists $n \in N$ such that $\theta(n) = m$. By Lemma 7.2.2, for each $k \in K$,

$$kmk^{-1} = kd(T,K)k^{-1} = d(kT,kK) = d(kT,K) = d(kT,T)d(T,K) = \theta(k)m$$

holds. Since N is abelian, $\theta(n^{-1}) = m^{-1}$ and hence

$$\begin{aligned}\theta(nkn^{-1}) &= \theta(n)n\theta(kn^{-1})n^{-1} = m\theta(kn^{-1})\\ &= m\theta(k)k\theta(n^{-1})k^{-1} = m\theta(k)km^{-1}k^{-1} = 1.\end{aligned}$$

Therefore $nKn^{-1} \subseteq H = \ker\theta$. Since $|K| = (G : N) = |H|$, $nKn^{-1} = H$. □

The general version of Schur–Zassenhaus' theorem does not need N to be abelian.

thm:SchurZassenhaus

Theorem 7.2.4 (Schur–Zassenhaus). *Let G be a finite group and let N be a normal subgroup of G. If $|N|$ and $(G : N)$ are coprime, then N admits a complement in G.*

Proof. We proceed by induction on $|G|$. Suppose that $|G| > 1$ and the result holds for all finite groups of order $< |G|$. If there exists a proper subgroup K of G such that $NK = G$, then, since $(K : K \cap N) = (G : N)$ is coprime with $|N|$, it is coprime with $|K \cap N|$. Moreover, $K \cap N$ is normal in K. By the inductive hypothesis, $K \cap N$ admits a complement in K. Hence there exists a subgroup H of K such that

$$|H| = (K : K \cap N) = (G : N).$$

Therefore H is a complement of N in G.

Hence we may assume that G has no proper subgroups K such that $NK = G$. We may also assume that $N \neq \{1\}$, otherwise G is a complement of $\{1\}$ in G. Note that $N \subseteq \Phi(G)$, because if M is a maximal subgroup of G such that $N \not\subseteq M$, then $NM = G$, in contradiction with our assumption. By Frattini's theorem 4.3.8, $\Phi(G)$ is nilpotent and thus so is N. Since N is a non-trivial nilpotent group, we have that $Z(N) \neq \{1\}$. Since N is normal in G and $Z(N)$ is characteristic in N, $Z(N)$ is normal in G. Let $\pi\colon G \to G/Z(N)$ the canonical map. We have that

$$(\pi(G) : \pi(N)) = \frac{|\pi(G)|}{|\pi(N)|} = \frac{|G/Z(N)|}{|N/Z(N)|} = (G : N)$$

is coprime with $|N|$, and thus it is coprime with $|\pi(N)|$. By the inductive hypothesis, $\pi(N)$ admits a complement in $G/Z(N)$. Let K be a subgroup of G such that $Z(N) \subseteq K$ and $\pi(K)$ is a complement of $\pi(N)$ in $\pi(G)$. Since $\pi(G) = \pi(N)\pi(K) = \pi(NK)$, we have that $NK = G$. By assumption, we get that $K = G$ and

$$\pi(Z(N)) = \pi(N) \cap \pi(K) = \pi(N) \cap \pi(G) = \pi(N).$$

Hence $N = Z(N)$ is abelian. By Theorem 7.2.3, we get a contradiction. Hence N admits a complement in G, and the result follows by induction. □

urZassenhaus:conjugacion

Theorem 7.2.5. *Let G be a finite group and N be a normal subgroup of G such that $|N|$ and $(G : N)$ are coprime. If either N or G/N is solvable, then all complements of N in G are conjugate.*

Proof. Let G be a minimal counterexample, so there are complements K_1 and K_2 of N in G such that K_1 and K_2 are not conjugate and $|G|$ is minimal with this property.

Claim. Each subgroup U of G satisfies the hypotheses of the theorem with respect to the normal subgroup $U \cap N$.

Since N is normal in G, the subgroup $U \cap N$ is normal in U. Moreover, $|U \cap N|$ and $(U : U \cap N)$ are coprime, as $|U \cap N|$ divides $|N|$ and $(U : U \cap N) = (UN : N)$ divides $(G : N)$. If G/N is solvable, then $U/U \cap N$ is solvable since $U/U \cap N$ is isomorphic to a subgroup of G/N. If N is solvable, the subgroup $U \cap N$ is solvable. This proves the claim.

Claim. Let L be a normal subgroup of G and let $\pi\colon G \to G/L$ be the canonical map. Then $\pi(G)$ satisfies the hypotheses of the theorem with respect to $\pi(N) = \pi(NL)$. In this case, if H is a complement of N in G, then $\pi(H)$ is a complement of $\pi(N)$ in $\pi(G)$.

If N is solvable, then $\pi(N)$ is solvable. If G/N is solvable, then the group $\pi(G)/\pi(N) \cong G/NL$ is solvable. Moreover, $(\pi(G) : \pi(N)) = \frac{|G/L|}{|NL/L|} = (G : NL)$ divides the index $(G : N)$ of N in G.

If H is a complement of N in G, $|\pi(H)|$ and $|\pi(N)|$ are coprime. Thus $\pi(H)$ is a complement of $\pi(N)$, as $\pi(G) = \pi(NH) = \pi(N)\pi(H)$ and $\pi(N) \cap \pi(H) = \{1\}$. Thus the claim follows.

Claim. N is minimal normal in G.

Let $M \neq \{1\}$ be a normal subgroup of G such that $M \subseteq N$. Let $\pi\colon G \to G/M$ be the canonical map. The group $\pi(G)$ satisfies the hypotheses of the theorem with respect to the normal subgroup $\pi(N)$. By the minimality of $|G|$, there exists $x \in G$ such that $\pi(xK_1x^{-1}) = \pi(K_2)$. The subgroup $U = MK_2$ satisfies the hypotheses of the theorem with respect to the normal subgroup $U \cap N$. Since $xK_1x^{-1} \cup K_2 \subseteq U$, we conclude that xK_1x^{-1} and K_2 are complements of $U \cap N$ in U. Thus $U = MK_2 = G$, as xK_1x^{-1} and K_2 are not conjugate and $|G|$ is minimal. Therefore $M = N$, as

$$|K_2| = \frac{|K_2|}{|M \cap K_2|} = (MK_2 : M) = (G : M) = \frac{|NK_2|}{|M|} = (N : M)|K_2|.$$

Hence N is a minimal normal subgroup of G.

Claim. N is not solvable and G/N is solvable.

Otherwise, by Lemma 5.0.1, N is minimal normal and hence abelian. This yields a contradiction with Theorem 7.2.3.

Let $\pi\colon G \to G/N$ be the canonical map and let S be a subgroup of G such that $N \subseteq S$ and $\pi(S)$ is minimal normal in $\pi(G) = G/N$. By Lemma 5.0.1, $\pi(S)$ is a p-group for some prime number p. Since $G = NK_1 = NK_2$ and $N \subseteq S$, Lemma 4.3.3 implies that

$$S = N(S \cap K_1) = N(S \cap K_2).$$

Thus $S \cap K_1$ and $S \cap K_2$ are complements of N in S. Since

$$\pi(S) = \pi(S \cap K_1) = \pi(S \cap K_2)$$

is a p-group, p divides $|S|$. The group S satisafies the hypotheses of the theorem with respect to the normal subgroup N, so $|N|$ and $(S : N)$ are coprime. If $p \mid |N|$, then $p \nmid (S : N) = |p(S)|$, a contradiction. Therefore $p \nmid |N|$. This implies that $S \cap K_1$ and $S \cap K_2$ are Sylow p-subgroups of S, as

$$|S \cap K_1| = (S : N) = |S \cap K_2|.$$

By Sylow's theorem, there exists $s \in S$ such that

$$S \cap sK_1s^{-1} = S \cap K_2.$$

In particular, $S \neq G$. Let

$$L = S \cap K_2 = S \cap sK_1s^{-1} \neq \{1\}.$$

Since S is normal in G, it follows that $sK_1s^{-1} \cup K_2 \subseteq N_G(L)$ (because L is normal both in sK_1s^{-1} and in K_2). The subgroups $sK_1s^{-1} \subseteq N_G(L)$ and $K_2 \subseteq N_G(L)$ are complements of $N \cap N_G(L)$ in $N_G(L)$. Thus $N_G(L) = G$ by the minimality of $|G|$. Therefore L is normal in G. Let $\pi_L : G \to G/L$ be the canonical map. Since both $\pi_L(K_1)$ and $\pi_L(K_2)$ are complements of $\pi_L(N)$ in $\pi_L(G)$, the minimality of $|G|$ implies that there exists $g \in G$ such that $\pi_L(gK_1g^{-1}) = \pi_L(K_2)$, so there exists $g \in G$ such that $(gK_1g^{-1})L = K_2L$. Thus $gK_1g^{-1} \cup K_2 \subseteq \langle K_2, L\rangle = K_2$, as $L \subseteq K_2$. In conclusion, $gK_1g^{-1} = K_2$, a contradiction. therefore the result follows. □

By Feit–Thompson's theorem (Theorem 5.0.25) , we do not need to assume that N or G/N is solvable. Indeed, since every group of odd order is solvable and $|N|$ and $(G : N)$ are coprime, it follows that either $|N|$ or $(G : N)$ is odd.

Corollary 7.2.6. *Let G be a finite group and let N be a normal subgroup of G of order n. Suppose that either N or G/N is solvable. If $|G : N| = m$ is coprime with n and m_1 is a divisor of m, then every subgroup of G of order m_1 is contained in some subgroup of order m of G.*

Proof. By Schur–Zassenhaus' theorem, there exists a complement H of N in G. Hence $|H| = m$. Let H_1 be a subgroup of G such that $|H_1| = m_1$. Since n and m are coprime,

$$\frac{|H||N||H_1|}{|H \cap NH_1|} = \frac{|H||NH_1|}{|H \cap NH_1|} = |H(NH_1)| = |G| = |NH| = |N||H|.$$

Hence $m_1 = |H_1| = |H \cap NH_1|$, and thus H_1 and $H \cap NH_1$ are complements of N in NH_1. By Theorem 7.2.5, there exists $g \in NH_1$ such that

$$H_1 = g(H \cap NH_1)g^{-1} \subseteq gHg^{-1},$$

and clearly $|gHg^{-1}| = |H| = m$. □

7.3 Bijective 1-cocycles

Let A be an additive group and G be a group and let $G \times A \to A$, $(g,a) \mapsto g \cdot a$, be a left action of G on A by automorphisms. This means that the action of G on A satisfies $g \cdot (a+b) = g \cdot a + g \cdot b$ for all $g \in G$ and $a, b \in A$. A *bijective* 1*-cocyle* is a bijective map $\pi \colon G \to A$ such that

$$\pi(gh) = \pi(g) + g \cdot \pi(h) \tag{7.1}$$

eq:1cocycle

for all $g, h \in G$. We now prove the equivalence between skew braces and bijective 1-cocycles.

thm:1cocycle

Theorem 7.3.1. *Over any additive group A the following data are equivalent:*

1) *A group G and a bijective 1-cocycle $\pi \colon G \to A$.*
2) *A brace structure over A.*

Proof. Consider on A a second group structure given by

$$a \circ b = \pi(\pi^{-1}(a)\pi^{-1}(b)) = a + \pi^{-1}(a) \cdot b$$

for all $a, b \in A$. Since G acts on A by automorphisms,

$$\begin{aligned} a \circ (b+c) &= \pi(\pi^{-1}(a)\pi^{-1}(b+c)) = a + \pi^{-1}(a) \cdot (b+c) \\ &= a + \pi^{-1}(a) \cdot b + \pi^{-1}(a) \cdot c = a \circ b - a + a \circ c \end{aligned}$$

holds for all $a, b, c \in A$.

Conversely, assume that the additive group A has a brace structure. Let G be the multiplicative group of A and $\pi = \mathrm{id}$. By Proposition 6.1.12, $a \mapsto \lambda_a$ is a group homomorphism from G to $\mathrm{Aut}(A,+)$ and hence G acts on A by automorphisms. Then (7.1) holds and therefore $\pi \colon G \to A$ is a bijective 1-cocycle. □

exa:d8q8

Example 7.3.2. Let

$$D_4 = \langle r, s : r^4 = s^2 = 1, srs = r^{-1} \rangle$$

be the dihedral group of eight elements and let

$$Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$$

be the quaternion group of eight elements. Let $\pi : Q_8 \to D_4$ be given by

$$\begin{aligned} 1 &\mapsto 1, & -1 &\mapsto r^2, & -k &\mapsto r^3 s, & k &\mapsto rs, \\ i &\mapsto s, & -i &\mapsto r^2 s, & j &\mapsto r^3, & -j &\mapsto r. \end{aligned}$$

Since π is bijective, a straightforward calculation shows that D_4 with

$$x + y = xy, \quad x \circ y = \pi(\pi^{-1}(x)\pi^{-1}(y))$$

is a skew two-sided brace with additive group isomorphic to D_4 and multiplicative group isomorphic to Q_8.

Exercises

xca:lifting **7.3.1.** Let $E: \ 1 \longrightarrow K \longrightarrow G \xrightarrow{p} Q \longrightarrow 1$ be an extension.

1) If $\ell\colon Q \to G$ is a lifting, then $\ell(Q)$ is a transversal of $\ker p$ in G.
2) Each transversal of $\ker p$ in G induces a lifting $\ell\colon Q \to G$.
3) If $\ell\colon Q \to G$ is a lifting, then $\ell(xy)\ker p = \ell(x)\ell(y)\ker p$.

xca:1cocycle **7.3.2.** Let $\varphi\colon Q \to K$ be a 1-cocycle.

1) $\varphi(1) = 1$.
2) $\varphi(y^{-1}) = (y^{-1}\cdot\phi(y))^{-1} = y^{-1}\cdot\phi(y)^{-1}$.
3) The set $\ker\varphi = \{x \in Q : \varphi(x) = 1\}$ is a subgroup of Q.

xca:ker1cocycle **7.3.3.** Let $\delta \in \mathrm{Der}(Q,K)$.

1) Prove that δ is injective if and only if $\ker\delta = \{1\}$.
2) Prove that if δ is bijective, then K admits a complement Δ in $K \rtimes Q$ such that $K \rtimes Q = K \rtimes \Delta = \Delta Q$ and $Q \cap \Delta = \{1\}$.

7.4 Notes

In the case of skew braces of abelian type, Theorem 7.3.1 is implicit in the work of Rump, see [65, 66] or [27]. Similar results appear in the work of Etingof, Schedler and Soloviev [39], Lu, Yan and Zhu [57] and Soloviev [73]. In [38] Etingof and Gelaki give a method of constructing finite-dimensional complex semisimple triangular Hopf algebras. They show how any non-abelian group which admits a bijective 1-cocycle gives rise to a semisimple minimal triangular Hopf algebra which is not a group algebra.

Chapter 8
The structure skew brace of a solution

structure_brace

To prove that the structure group $G(X,r)$ of a solution (X,r) is a skew brace, we follow the proof of Lu, Yan and Zhu. They use the language of braided groups.

8.1 Braided groups and skew braces

braidedgroup

Definition 8.1.1. A *braided group* is a pair (G,r), where G is a group with operation $m\colon G\times G\to G$, $m(x,y)=xy$, and $r\colon G\times G\to G\times G$ is a bijective map such that

1) $r(xy,z)=(\mathrm{id}\times m)r_1r_2(x,y,z)$ for all $x,y,z\in G$,
2) $r(x,yz)=(m\times\mathrm{id})r_2r_1(x,y,z)$ for all $x,y,z\in G$,
3) $r(1,x)=(x,1)$ and $r(x,1)=(1,x)$ for all $x\in G$, and
4) $m\circ r=m$.

The map r is called a *braiding operator* on G.

lem:braidedsolYBE

Lemma 8.1.2. *Let (G,r) be a braided group. We write $r(x,y)=(x\triangleright_r y, x\triangleleft_r y)$, for all $x,y\in G$. Let $\xi\colon G\to\mathbb{S}_G$ be the map defined by $\xi(x)(y)=x\triangleright_r y$ for all $x,y\in G$. Let $\eta\colon G\to\mathbb{S}_G$ be the map defined by $\eta(x)(y)=y\triangleleft_r x$ for all $x,y\in G$. Then ξ is a left action of G on itself, η is a right action of G on itself and (G,r) is a solution to the YBE.*

Proof. By the definition of a braiding operator, we have that

$$\begin{aligned} r(xy,z) =& (\mathrm{id}\times m)r_1r_2(x,y,z)\\ =& (\mathrm{id}\times m)r_1(x, y\triangleright_r z, y\triangleleft_r z)\\ =& (\mathrm{id}\times m)(x\triangleright_r (y\triangleright_r z), x\triangleleft_r (y\triangleright_r z), y\triangleleft_r z)\\ =& (x\triangleright_r (y\triangleright_r z), (x\triangleleft_r (y\triangleright_r z))(y\triangleleft_r z)), \end{aligned}$$

and thus $(xy)\triangleright_r z=(x\triangleright_r (y\triangleright_r z)$. Since $r(1,x)=(x,1)$, we have that the map ξ is a left action of G on itself. By the definition of a braiding operator, we have that

$$\begin{aligned} r(x,yz) =& (m\times \mathrm{id})r_2r_1(x,y,z)\\ =& (m\times \mathrm{id})r_2(x\triangleright_r y, x\triangleleft_r y, z)\\ =& (m\times \mathrm{id})(x\triangleright_r y,(x\triangleleft_r y)\triangleright_r z,(x\triangleleft_r y)\triangleleft_r z)\\ =& ((x\triangleright_r y)((x\triangleleft_r y)\triangleright_r z),(x\triangleleft_r y)\triangleleft_r z), \end{aligned} \tag{8.1}$$

LYZequa1

and thus $x\triangleleft_r (yz) = (x\triangleleft_r y)\triangleleft_r z$. Since $r(x,1) = (1,x)$, we have that the map η is a right action of G on itself. Furthermore, $xy = (x\triangleright_r y)(x\triangleleft_r y)$, for all $x,y\in G$. Hence, by Theorem 3.1.15, (G,r) is a solution to the YBE. □

thm:braidedbraces

Theorem 8.1.3. *Let G be a group. Then there is a bijection between the set of braiding operators on G and the set of skew brace structures on the set G with multiplicative group G.*

Proof. Let $\mathcal{BO}(G)$ be the set of all the braiding operators on G. Let $\mathcal{B}(G)$ be the set of all the skew brace structures $(G,+,\circ)$ on the group G, where $x\circ y = xy$, for all $x,y\in G$. For $r\in\mathcal{BO}(G)$, we write $r(x,y) = (x\triangleright_r y, x\triangleleft_r y)$, for all $x,y\in G$. We define

$$f\colon \mathcal{BO}(G)\to\mathcal{B}(G)$$

by $f(r) = (G,+_r,\circ)$, where $x+_r y = x(x^{-1}\triangleright_r y)$, for all $x,y\in G$ and $r\in\mathcal{BO}(G)$. Now, by (8.1) and Lemma 8.1.2, we have

$$\begin{aligned} x\triangleright_r (y+_r z) =& x\triangleright_r (y(y^{-1}\triangleright_r z))\\ =& (x\triangleright_r y)((x\triangleleft_r y)\triangleright_r (y^{-1}\triangleright_r z))\\ =& (x\triangleright_r y)(((x\triangleleft_r y)y^{-1})\triangleright_r z))\\ =& (x\triangleright_r y)(((x\triangleright_r y)^{-1}x)\triangleright_r z))\\ =& (x\triangleright_r y)+(x\triangleright_r z). \end{aligned} \tag{8.2}$$

LYZequa

Note that $1+_r x = 1(1^{-1}\triangleright_r x) = x$ and $x+_r 1 = x(x^{-1}\triangleright_r 1) = x1 = x$ for all $x\in G$. By Lemma 8.1.2, for every $x\in G$, we have

$$x+_r (x\triangleright_r x^{-1}) = x(x^{-1}\triangleright_r (x\triangleright_r x^{-1})) = xx^{-1} = 1$$

and by (8.1),

$$\begin{aligned} (x\triangleright_r x^{-1})+_r x =& (x\triangleright_r x^{-1})((x\triangleright_r x^{-1})^{-1}\triangleright_r x)\\ =& (x\triangleright_r x^{-1})((x\triangleleft_r x^{-1})\triangleright_r x)\\ =& x\triangleright_r (x^{-1}x) = 1. \end{aligned}$$

By Lemma 8.1.2 and (8.2), we have that

$$\begin{aligned} (x+_r y)+_r z =& x(x^{-1}\triangleright_r y)(((x^{-1}\triangleright_r y)^{-1}x^{-1})\triangleright_r z)\\ =& x(x^{-1}\triangleright_r y)((x^{-1}\triangleright_r y)^{-1}\triangleright_r (x^{-1}\triangleright_r z))\\ =& x((x^{-1}\triangleright_r y)+_r (x^{-1}\triangleright_r z)) \end{aligned}$$

$$= x(x^{-1} \triangleright_r (y +_r z)) = x +_r (y +_r z)$$

for all $x, y, z \in G$. Hence $(G, +_r)$ is a group. By Lemma 8.1.2,

$$x + (x \triangleright_r y) = x(x^{-1} \triangleright_r (x \triangleright_r y)) = xy$$

for all $x, y \in G$. By (8.2), we have

$$\begin{aligned} x(y +_r z) &= x +_r (x \triangleright_r (y +_r z)) \\ &= x +_r (x \triangleright_r y) +_r (x \triangleright_r z) \\ &= xy -_r x +_r xz, \end{aligned}$$

for all $x, y, z \in G$. Hence $(G, +_r, \circ)$ is a skew brace and f is a well-defined map. Note that the lambda map of this skew brace is defined by $\lambda_x(y) = -_r x +_r xy = x \triangleright_r y$, for all $x, y \in G$.

Now we define $g : \mathcal{B}(G) \to \mathcal{BO}(G)$, by $g(G, +, \circ) = r_+$ and

$$r_+(x, y) = (-x + xy, (-x + xy)^{-1}xy) = (\lambda_x(y), \mu_y(x)),$$

for all $(G, +, \circ) \in \mathcal{B}(G)$ and $x, y \in G$. We shall prove that r_+ is a braiding operator on G. Note that, by Proposition 6.1.12,

$$\begin{aligned} r_+(xy, z) &= (\lambda_{xy}(z), \mu_z(xy)) \\ &= (\lambda_x\lambda_y(z), \lambda_x(\lambda_y(z))^{-1}xyz) \\ &= (\lambda_x\lambda_y(z), \lambda_x(\lambda_y(z))^{-1}x\lambda_y(z)\lambda_y(z)^{-1}yz) \\ &= (\lambda_x\lambda_y(z), \mu_{\lambda_y(z)}(x)\mu_z(y)) \\ &= (\mathrm{id} \times m)(r_+)_1(r_+)_2(x, y, z). \end{aligned}$$

By Propositions 6.1.12 and 6.1.13,

$$\begin{aligned} r_+(x, yz) &= (\lambda_x(yz), \mu_{yz}(x)) \\ &= (\lambda_x(y + \lambda_y(z)), \mu_z\mu_y(x)) \\ &= (\lambda_x(y) + \lambda_x\lambda_y(z), \mu_z\mu_y(x)) \\ &= (\lambda_x(y)\lambda_{\lambda_x(y)^{-1}xy}(z), \mu_z\mu_y(x)) \\ &= (\lambda_x(y)\lambda_{\mu_y(x)}(z), \mu_z\mu_y(x)) \\ &= (m \times \mathrm{id})(r_+)_2(r_+)_1(x, y, z). \end{aligned}$$

We also have that

$$r_+(1, x) = (\lambda_1(x), \mu_x(1)) = (x, 1),$$
$$r_+(x, 1) = (\lambda_x(1), \mu_1(x)) = (1, x),$$

and

$$\lambda_x(y)\mu_y(x) = \lambda_x(y)\lambda_x(y)^{-1}xy = xy,$$

for all $x, y \in G$. Hence r_+ is a braiding operator on G, thus g is a well-defined map. Now it is easy to check that $f \circ g = \mathrm{id}_{\mathcal{B}(G)}$ and $g \circ f = \mathrm{id}_{\mathcal{BO}(G)}$. □

8.2 The structure group of a solution

Let (X,r) be a solution to the YBE. Write as usual $r(x,y) = (\sigma_x(y), \tau_y(x))$. The *structure group* of (X,r) is the group

$$G(X,r) = \mathrm{gr}(X : xy = \sigma_x(y)\tau_y(x), \text{ for all } x, y \in X).$$

In this section we shall see that there is a natural braiding operator on $G(X,r)$ induced by r that satisfies a nice universal property.

To do so we prove first that there are interesting extensions of the solution (X,r).

prop:extendsol

Proposition 8.2.1. *Let (X,r) be a solution to the YBE. Let $X' = \{x' \mid x \in X\}$ be a copy of X and let Y be the disjoint union of X and X'. We write $r(x,y) = (\sigma_x(y), \tau_y(x))$ and $r^{-1}(x,y) = (\widehat{\sigma}_x(y), \widehat{\tau}_y(x))$. We define $r' \colon Y \times Y \to Y \times Y$ by $r'(u,v) = (\sigma'_u(v), \tau'_v(u))$, for all $u, v \in Y$, where*

$$\sigma'_x(y) = \sigma_x(y),\ \sigma'_{x'}(y) = \sigma_x^{-1}(y),\ \sigma'_x(y') = \widehat{\tau}_x^{-1}(y)',\ \sigma'_{x'}(y') = \widehat{\tau}_x(y)'$$

and

$$\tau'_x(y) = \tau_x(y),\ \tau'_{x'}(y) = \tau_x^{-1}(y),\ \tau'_x(y') = \widehat{\sigma}_x^{-1}(y)',\ \tau'_{x'}(y') = \widehat{\sigma}_x(y)',$$

for all $x, y \in X$. Then (Y,r') is a solution to the YBE. Furthermore, (Y,r') is involutive if and only if (X,r) is involutive.

Proof. Let $x, y \in X$. Since

$$r(\tau_y^{-1}(x), y) = (\sigma_{\tau_y^{-1}(x)}(y), x) \text{ and } r(y, \sigma_y^{-1}(x)) = (x, \tau_{\sigma_y^{-1}(x)}(y)),$$

we have that

$$\widehat{\tau}_x^{-1}(y) = \sigma_{\tau_y^{-1}(x)}(y) \text{ and } \widehat{\sigma}_x^{-1}(y) = \tau_{\sigma_y^{-1}(x)}(y).$$

Similarly one can see that

$$\tau_x^{-1}(y) = \widehat{\sigma}_{\widehat{\tau}_y^{-1}(x)}(y) \text{ and } \sigma_x^{-1}(y) = \widehat{\tau}_{\widehat{\sigma}_y^{-1}(x)}(y).$$

Now it is easy to check that r' is bijective and

$$(r')^{-1}(x,y) = (\widehat{\sigma}_x(y), \widehat{\tau}_y(x)),\ (r')^{-1}(x',y) = (\widehat{\sigma}_x^{-1}(y), \sigma_y^{-1}(x)'),$$

$$(r')^{-1}(x,y') = (\tau_x^{-1}(y)', \widehat{\tau}_y^{-1}(x)),\ (r')^{-1}(x',y') = (\sigma_x(y)', \tau_y(x)')$$

for all $x, y \in X$.

Hence, to show that (Y, r') is a solution to the YBE, by Lemma 3.1.5 it is enough to prove that

(a) $\sigma'_u\sigma'_v = \sigma'_{\sigma'_u(v)}\sigma'_{\tau'_v(u)}$,
(b) $\sigma'_{\tau'_{\sigma'_u(v)}(w)}\tau'_v(u) = \tau'_{\sigma'_{\tau'_u(w)}(v)}\sigma'_w(u)$,
(c) $\tau'_u\tau'_v = \tau'_{\tau'_u(v)}\tau'_{\sigma'_v(u)}$,

for all $u, v, w \in Y$.

Let $x, y, z \in X$. Since (X, r^{-1}) is a solution to the YBE, by Lemma 3.1.5, we have that

$$\begin{aligned}
\sigma'_x\sigma'_y(z) =& \sigma_x\sigma_y(z) = \sigma_{\sigma_x(y)}\sigma_{\tau_y(x)}(z)\\
=& \sigma'_{\sigma'_x(y)}\sigma'_{\tau'_y(x)}(z),\\
\sigma'_x\sigma'_y(z') =& \sigma'_x(\widehat{\tau}_y^{-1}(z)') = \widehat{\tau}_x^{-1}\widehat{\tau}_y^{-1}(z)'\\
=& \widehat{\tau}^{-1}_{\sigma_x(y)}\widehat{\tau}^{-1}_{\tau_y(x)}(z)'\\
=& \sigma'_{\sigma'_x(y)}\sigma'_{\tau'_y(x)}(z').
\end{aligned}$$

Hence $\sigma'_x\sigma'_y = \sigma'_{\sigma'_x(y)}\sigma'_{\tau'_y(x)}$. We also have that

$$\begin{aligned}
\sigma'_x\sigma'_{y'}(z) =& \sigma_x\sigma_y^{-1}(z) = \sigma^{-1}_{\sigma_{\tau_y^{-1}(x)}(y)}\sigma_{\tau_y^{-1}(x)}(z)\\
=& \sigma^{-1}_{\widehat{\tau}_x^{-1}(y)}\sigma_{\tau_y^{-1}(x)}(z) = \sigma'_{\sigma'_x(y')}\sigma'_{\tau'_{y'}(x)}(z),\\
\sigma'_x\sigma'_{y'}(z') =& \sigma'_x(\widehat{\tau}_y(z)') = \widehat{\tau}_x^{-1}\widehat{\tau}_y(z)'\\
=& \widehat{\tau}_{\widehat{\tau}_x^{-1}(y)}\widehat{\tau}^{-1}_{\sigma_{\tau_x^{-1}(y)}(x)}(z)' = \widehat{\tau}_{\widehat{\tau}_x^{-1}(y)}\widehat{\tau}^{-1}_{\widehat{\tau}_y^{-1}(x)}(z)'\\
=& \sigma'_{\sigma'_x(y')}\sigma'_{\tau'_{y'}(x)}(z').
\end{aligned}$$

Hence $\sigma'_x\sigma'_{y'} = \sigma'_{\sigma'_x(y')}\sigma'_{\tau'_{y'}(x)}$. Similarly one can check that $\sigma'_{x'}\sigma'_y = \sigma'_{\sigma'_{x'}(y)}\sigma'_{\tau'_y(x')}$ and $\sigma'_{x'}\sigma'_{y'} = \sigma'_{\sigma'_{x'}(y')}\sigma'_{\tau'_{y'}(x')}$. Hence (a) follows. By a symmetric argument (c) follows.

Now we prove (b). By Lemma 3.1.5, we have

$$\begin{aligned}
\sigma'_{\tau'_{\sigma'_{x'}(y)}(z)}\tau'_y(x') =& \sigma'_{\tau'_{\sigma_x^{-1}(y)}(z)}\left(\widehat{\sigma}_y^{-1}(x)'\right)\\
=& \sigma'_{\tau_{\sigma_x^{-1}(y)}(z)}\left(\tau_{\sigma_x^{-1}(y)}(x)'\right)\\
=& \left(\widehat{\tau}^{-1}_{\tau_{\sigma_x^{-1}(y)}(z)}\tau_{\sigma_x^{-1}(y)}(x)\right)'\\
=& \left(\sigma_{\tau^{-1}_{\tau_{\sigma_x^{-1}(y)}(x)}\tau_{\sigma_x^{-1}(y)}(z)}\tau_{\sigma_x^{-1}(y)}(x)\right)'\\
=& \left(\sigma_{\tau_y\tau_x^{-1}(z)}\tau_{\sigma_x^{-1}(y)}(x)\right)'
\end{aligned}$$

$$
\begin{aligned}
&= \left(\sigma_{\tau_{\sigma_x \sigma_x^{-1}(y)} \tau_x^{-1}(z)} \tau_{\sigma_x^{-1}(y)}(x) \right)' \\
&= \left(\tau_{\sigma_{\tau_x \tau_x^{-1}(z)} \sigma_x^{-1}(y)} \sigma_{\tau_x^{-1}(z)}(x) \right)' \\
&= \tau'_{\sigma'_{\tau'_{x'}(z)}(y)} \sigma'_z(x'),
\end{aligned}
$$

where the last equality follows by a symmetric argument. By similar calculations one get that

$$\tau'_{\sigma'_{\tau'_x(y')}(z')} \sigma'_{y'}(x) = \sigma'_{\tau'_{\sigma'_x(z')}(y')} \tau'_{z'}(x).$$

Note that,

$$
\begin{aligned}
\sigma'_{\tau'_{\sigma'_x(y')}(z)} \tau'_{y'}(x) &= \sigma'_{\tau'_{\widehat{\tau}_x^{-1}(y)'}(z)} \tau_y^{-1}(x) \\
&= \sigma_{\tau_{\sigma_{\tau_y^{-1}(x)}(y)}^{-1}(z)} \tau_y^{-1}(x)
\end{aligned}
$$

and

$$
\begin{aligned}
\tau'_{\sigma'_{\tau'_x(z)}(y')} \sigma'_z(x) &= \tau'_{\widehat{\tau}_{\tau_x(z)}^{-1}(y)'} \sigma_z(x) \\
&= \tau^{-1}_{\sigma_{\tau_y^{-1} \tau_x(z)}(y)} \sigma_z(x) \\
&= \tau^{-1}_{\sigma_{\tau_{\tau_y^{-1}(x)} \tau^{-1}_{\sigma_{\tau_y^{-1}(x)}(y)}(z)}(y)} \sigma_z(x),
\end{aligned}
$$

by Lemma 3.1.5.

Since, by Lemma 3.1.5,

$$
\begin{aligned}
&\tau_{\sigma_{\tau_{\tau_y^{-1}(x)} \tau^{-1}_{\sigma_{\tau_y^{-1}(x)}(y)}(z)}(y)} \sigma_{\tau^{-1}_{\sigma_{\tau_y^{-1}(x)}(y)}(z)} \tau_y^{-1}(x) \\
&\qquad = \sigma_{\tau_{\sigma_{\tau_y^{-1}(x)}(y)} \tau^{-1}_{\sigma_{\tau_y^{-1}(x)}(y)}(z)} \tau_y \tau_y^{-1}(x) \\
&\qquad = \sigma_z(x),
\end{aligned}
$$

we have that

$$\sigma'_{\tau'_{\sigma'_x(y')}(z)} \tau'_{y'}(x) = \tau'_{\sigma'_{\tau'_x(z)}(y')} \sigma'_z(x).$$

By a symmetric argument one can prove that

$$\tau'_{\sigma'_{\tau'_x(y')}(z)} \sigma'_{y'}(x) = \sigma'_{\tau'_{\sigma'_x(z)}(y')} \tau'_z(x).$$

By a similar calculations one get that

$$\tau'_{\sigma'_{\tau'_{x'}(y)}(z')} \sigma'_y(x') = \sigma'_{\tau'_{\sigma'_{x'}(z')}(y)} \tau'_{z'}(x')$$

and

$$\sigma'_{\tau'_{\sigma'_{x'}(y)}(z')}\tau'_y(x') = \tau'_{\sigma'_{\tau'_{x'}(z')}(y)}\sigma'_{z'}(x').$$

Hence, by Lemma 3.1.5, (c) follows. Therefore (Y, r') is a solution to the YBE.

By the definition of r', it is clear that if $(r')^2 = \mathrm{id}_{Y^2}$, then $r^2 = \mathrm{id}_{X^2}$. Suppose that $r^2 = \mathrm{id}_{X^2}$. We have that

$$(r')^2(x, y) = r^2(x, y) = (x, y)$$

for all $x, y \in X$. Since

$$r'(x', y) = (\sigma'_{x'}(y), \tau'_y(x')) = (\sigma_x^{-1}(y), \sigma_y^{-1}(x)')$$

and

$$r'(x, y') = (\sigma'_x(y'), \tau'_{y'}(x)) = (\tau_x^{-1}(y)', \tau_y^{-1}(x))$$

for all $x, y \in X$, we have that

$$(r')^2(x', y) = (\tau^{-1}_{\sigma_x^{-1}(y)}\sigma_y^{-1}(x)', \tau^{-1}_{\sigma_y^{-1}(x)}\sigma_x^{-1}(y))$$

and

$$(r')^2(x, y') = (\sigma^{-1}_{\tau_x^{-1}(y)}\tau_y^{-1}(x), \sigma^{-1}_{\tau_y^{-1}(x)}\tau_x^{-1}(y))'$$

for all $x, y \in X$. Note that

$$\sigma_x\tau_{\sigma_y^{-1}(x)}(y) = \sigma_x\sigma^{-1}_{\sigma_y(\sigma_y^{-1}(x))}(y) = y$$

and

$$\tau_x\sigma_{\tau_y^{-1}(x)}(y) = \tau_x\tau^{-1}_{\tau_y(\tau_y^{-1}(x))}(y) = y$$

for all $x, y \in X$. Hence

$$(r')^2(x', y) = (x', y) \text{ and } (r')^2(x, y') = (x, y')$$

for all $x, y \in X$. Finally,

$$\begin{aligned}(r')^2(x', y') =& r'(\sigma'_{x'}(y'), \tau'_{y'}(x')) = r'(\tau_x(y)', \sigma_y(x)')\\ =& (\tau_{\tau_x(y)}\sigma_y(x)', \sigma_{\sigma_y(x)}\tau_x(y)')\\ =& (\tau_{\tau_x(y)}\tau^{-1}_{\tau_x(y)}(x)', \sigma_{\sigma_y(x)}\sigma^{-1}_{\sigma_y(x)}(y)') = (x', y')\end{aligned}$$

for all $x, y \in X$. Therefore $(r')^2 = \mathrm{id}_{Y^2}$, and the result follows. □

We use the following notation. For every solution (X, r) of the YBE,

$$r_k = \mathrm{id}^{\times(k-1)} \times r \times \mathrm{id}^{\times(n-k-1)} \colon X^n \to X^n$$

for all $1 \leq k < n$ and $n \geq 3$.

thm:solfreemonoid

Theorem 8.2.2. *Let (X,r) be a solution of the YBE. Let $\mathrm{FM}(X)$ be the free monoid on X. Let $m_k\colon X^k \to \mathrm{FM}(X)$ be the map defined by $m_k(x_1,\dots,x_k) = x_1\cdots x_k$ for every positive integer k. Let $r_{FM}\colon \mathrm{FM}(X)\times\mathrm{FM}(X) \to \mathrm{FM}(X)\times\mathrm{FM}(X)$ be the map defined by $r_{FM}(x_1\cdots x_n, y_1\cdots y_k)$*

$$= (m_k \times m_n)(r_k\cdots r_2r_1)(r_{k+1}\cdots r_3r_2)\cdots(r_{n+k-1}\cdots r_{n+1}r_n)(x_1,\dots,x_n,y_1,\dots,y_k)$$

for all $x_1,\dots,x_n,y_1,\dots,y_k \in X$ and all positive integers n,k, $r_{FM}(1,a) = (a,1)$ and $r_{FM}(a,1) = (1,a)$ for all $a \in \mathrm{FM}(X)$. Then $(\mathrm{FM}(X), r_{FM})$ is a solution to the YBE. Furthermore, $(\mathrm{FM}(X), r_{FM})$ is involutive if and only if (X,r) is involutive.

Proof. Note that $r_{FM}(x,y) = r_1(x,y) = r(x,y)$ for all $x,y \in X$. Hence if $(r_{FM})^2 = \mathrm{id}_{\mathrm{FM}(X)^2}$, the $r^2 = \mathrm{id}_{X^2}$.

Note that

$$r_{l+m}r_m = r_m r_{l+m} \tag{8.3}$$

eq:SFcom

for all $1 \le m$ and $2 \le l$. Hence

$$\begin{aligned} r_m(r_{m+l}\cdots r_{m+1}r_m) =& r_{m+l}\cdots r_{m+2}r_m r_{m+1}r_m \\ =& (r_{m+l}\cdots r_{m+2}r_{m+1}r_m)r_{m+1}. \end{aligned} \tag{8.4}$$

eq:solfree

Suppose that $r^2 = \mathrm{id}_{X^2}$. By using (8.3) and (8.4), we have that

$$\begin{aligned} &(r_k\cdots r_2r_1)(r_{k+1}\cdots r_3r_2)\cdots(r_{n+k-1}\cdots r_{n+1}r_n) \\ &\cdot(r_n\cdots r_2r_1)(r_{n+1}\cdots r_3r_2)\cdots(r_{k+n-1}\cdots r_{k+1}r_k) \\ &= (r_k\cdots r_2r_1)(r_{k+1}\cdots r_3r_2)\cdots(r_{n+k-1}\cdots r_{n+1}) \\ &\cdot(r_{n-1}\cdots r_2r_1)(r_{n+1}\cdots r_3r_2)\cdots(r_{k+n-1}\cdots r_{k+1}r_k) \\ &= (r_k\cdots r_2r_1)(r_{k+1}\cdots r_3r_2)\cdots(r_{n+k-2}\cdots r_{n-1}) \\ &\cdot(r_{n-1}\cdots r_2r_1)(r_{n+k-1}\cdots r_{n+1})(r_{n+1}\cdots r_3r_2)\cdots(r_{k+n-1}\cdots r_{k+1}r_k) \\ &= \cdots = (r_k\cdots r_2)(r_{k+1}\cdots r_3)\cdots(r_{n+k-1}\cdots r_{n+1}) \\ &\cdot(r_{n+1}\cdots r_3r_2)\cdots(r_{k+n-1}\cdots r_{k+1}r_k) \\ &= \cdots = (r_k)(r_{k+1})\cdots(r_{n+k-1}) \\ &\cdot(r_{k+n-1}\cdots r_{k+1}r_k) = \mathrm{id}_{X^{k+n}} \end{aligned}$$

for positive integers n,k. Hence $(r_{FM})^2 = \mathrm{id}_{\mathrm{FM}(X)^2}$.

We write $r_{FM}(a,b) = (\sigma_a(b), \tau_b(a))$ for all $a,b \in \mathrm{FM}(X)$. We have that

$$\begin{aligned} &(\mathrm{id}\times r_{FM})(r_{FM}\times\mathrm{id})(\mathrm{id}\times r_{FM})(1,a,b) \\ &\qquad = (\mathrm{id}\times r_{FM})(r_{FM}\times\mathrm{id})(1,\sigma_a(b),\tau_b(a)) \\ &\qquad = (\mathrm{id}\times r_{FM})(\sigma_a(b),1,\tau_b(a)) = (\sigma_a(b),\tau_b(a),1) \end{aligned}$$

and

$$\begin{aligned}(r_{FM}\times\mathrm{id})(\mathrm{id}\times r_{FM})(r_{FM}\times\mathrm{id})(1,a,b)&\\ =(r_{FM}\times\mathrm{id})(\mathrm{id}\times r_{FM})(a,1,b)&\\ =(r_{FM}\times\mathrm{id})(a,b,1)=(\sigma_a(b),\tau_b(a),1)&\end{aligned}$$

Similarly one can check that

$$(\mathrm{id}\times r_{FM})(r_{FM}\times\mathrm{id})(\mathrm{id}\times r_{FM})(a,1,b)=(r_{FM}\times\mathrm{id})(\mathrm{id}\times r_{FM})(r_{FM}\times\mathrm{id})(a,1,b)$$

and

$$(\mathrm{id}\times r_{FM})(r_{FM}\times\mathrm{id})(\mathrm{id}\times r_{FM})(a,b,1)=(r_{FM}\times\mathrm{id})(\mathrm{id}\times r_{FM})(r_{FM}\times\mathrm{id})(a,b,1).$$

Hence to prove that

$$(\mathrm{id}\times r_{FM})(r_{FM}\times\mathrm{id})(\mathrm{id}\times r_{FM})=(r_{FM}\times\mathrm{id})(\mathrm{id}\times r_{FM})(r_{FM}\times\mathrm{id})$$

we need to show that

$$\begin{aligned}&(r_{k+t}\cdots r_{1+t})\cdots(r_{k+t+n-1}\cdots r_{t+n})(r_t\cdots r_2r_1)\cdots(r_{n+t-1}\cdots r_n)\\ &\quad\cdot(r_{n+t}\cdots r_{n+1})\cdots(r_{n+k+t-1}\cdots r_{n+k})(x_1,\ldots,x_n,y_1,\ldots,y_k,z_1,\ldots,z_t)\\ &=(r_t\cdots r_2r_1)\cdots(r_{k+t-1}\cdots r_k)(r_{k+t}\cdots r_{k+1})\cdots(r_{n+k+t-1}\cdots r_{n+k})\\ &\quad\cdot(r_k\cdots r_2r_1)\cdots(r_{n+k-1}\cdots r_n)(x_1,\ldots,x_n,y_1,\ldots,y_k,z_1,\ldots,z_t)\end{aligned}$$

for all positive integers n,k,t and $x_1,\ldots,x_n,y_1,\ldots,y_k,z_1,\ldots,z_t\in X$.

By using (8.3) and (8.4), we have that

$$\begin{aligned}&(r_{t+k}\cdots r_{t+1})\cdots(r_{t+k+n-1}\cdots r_{t+n})(r_t\cdots r_2r_1)\cdots(r_{n+t-1}\cdots r_n)\\ &\quad\cdot(r_{n+t}\cdots r_{n+1})\cdots(r_{n+k+t-1}\cdots r_{n+k})\\ &=(r_{t+k}\cdots r_{t+1}r_t\cdots r_2r_1)\cdots(r_{t+k+n-1}\cdots r_{t+n}r_{n+t-1}\cdots r_n)\\ &\quad\cdot(r_{n+t}\cdots r_{n+1})\cdots(r_{n+k+t-1}\cdots r_{n+k})\\ &=(r_t\cdots r_2r_1)(r_{t+k}\cdots r_{t+1}r_t\cdots r_2r_1)\\ &\quad\cdots(r_{t+k+n-1}\cdots r_{t+n}r_{n+t-1}\cdots r_n)\\ &\quad\cdot(r_{n+t+1}\cdots r_{n+2})\cdots(r_{n+k+t-1}\cdots r_{n+k})\\ &=(r_t\cdots r_2r_1)\cdots(r_{k+t-1}\cdots r_k)\\ &\quad\cdot(r_{t+k}\cdots r_{t+1}r_t\cdots r_2r_1)\cdots(r_{t+k+n-1}\cdots r_{t+n}r_{n+t-1}\cdots r_n)\\ &=(r_t\cdots r_2r_1)\cdots(r_{k+t-1}\cdots r_k)\\ &\quad\cdot(r_{k+t}\cdots r_{k+1}r_k\cdots r_2r_1)\cdots(r_{n+k+t-1}\cdots r_{n+k}r_{n+k-1}\cdots r_n)\\ &=(r_t\cdots r_2r_1)\cdots(r_{k+t-1}\cdots r_k)\\ &\quad\cdot(r_{k+t}\cdots r_{k+1})\cdots(r_{n+k+t-1}\cdots r_{n+k})(r_k\cdots r_2r_1)\cdots(r_{n+k-1}\cdots r_n).\end{aligned}$$

Hence

$$(\mathrm{id}\times r_{FM})(r_{FM}\times\mathrm{id})(\mathrm{id}\times r_{FM})=(r_{FM}\times\mathrm{id})(\mathrm{id}\times r_{FM})(r_{FM}\times\mathrm{id}).$$

By the definition of the free monoid on X, every m_k is bijective. Hence by the definition of r_{FM}, since r is bijective, we have that r_{FM} is bijective.

Note that $\sigma_1 = \mathrm{id}$ and $\tau_1 = \mathrm{id}$. We know that the restrictions of σ_x and τ_x on X for all $x \in X$ are bijective. We shall prove, by induction on k, that the restrictions of σ_x and τ_x on the set X_k of words of length k for all $x \in X$ are bijective. For $k = 0$, $\sigma_x(1) = 1$ and $\tau_x(1) = 1$, by the definition of r_{FM}. Suppose that $k \geq 1$ and that the restrictions of σ_x and τ_x on the set X_l of words of length $l \leq k$ for all $x \in X$ are bijective. Let $x, y_1, \ldots, y_{k+1} \in X$. By the definition of r_{FM} we have that

$$\sigma_x(y_1 \cdots y_{k+1}) = \sigma_x(y_1 \cdots y_k)\sigma_{\tau_{y_k} \cdots \tau_{y_1}(x)}(y_{k+1}).$$

Let $z_1, \ldots, z_{k+1} \in X$ such that $\sigma_x(y_1 \cdots y_{k+1}) = \sigma_x(z_1 \cdots z_{k+1})$. Hence

$$\sigma_x(y_1 \cdots y_k) = \sigma_x(z_1 \cdots z_k) \text{ and } \sigma_{\tau_{y_k} \cdots \tau_{y_1}(x)}(y_{k+1}) = \sigma_{\tau_{z_k} \cdots \tau_{z_1}(x)}(z_{k+1}).$$

By the inductive hypothesis the restriction of σ_x on X_k is bijective. Hence $y_1 \cdots y_k = z_1 \cdots z_k$, and thus $y_i = z_i$ for all $1 \leq i \leq k$. Now

$$\sigma_{\tau_{y_k} \cdots \tau_{y_1}(x)}(y_{k+1}) = \sigma_{\tau_{z_k} \cdots \tau_{z_1}(x)}(z_{k+1}) = \sigma_{\tau_{y_k} \cdots \tau_{y_1}(x)}(z_{k+1})$$

and thus $y_{k+1} = z_{k+1}$. Therefore the restriction of σ_x on X_{k+1} is injective. Let $c_1, \ldots, c_{k+1} \in X$. By the inductive hypothesis there exist $x_1, \ldots, x_k \in X$ such that

$$\sigma_x(x_1 \cdots x_k) = c_1 \cdots c_k.$$

There exists $x_{k+1} \in X$ such that

$$\sigma_{\tau_{x_k} \cdots \tau_{x_1}(x)}(x_{k+1}) = z_{k+1}.$$

Hence

$$\sigma_x(x_1 \cdots x_{k+1}) = c_1 \cdots c_{k+1},$$

and thus the restriction of σ_x on X_{k+1} is bijective. Therefore, by induction, σ_x is bijective for all $x \in X$.

By the definition of r_{FM} we have that

$$\tau_x(y_1 \cdots y_{k+1}) = \tau_{\sigma_{y_2} \cdots \sigma_{y_{k+1}}(x)}(y_1)\tau_x(y_2 \cdots y_{k+1}).$$

Let $z_1, \ldots, z_{k+1} \in X$ such that $\tau_x(y_1 \cdots y_{k+1}) = \tau_x(z_1 \cdots z_{k+1})$. Hence

$$\tau_x(y_2 \cdots y_{k+1}) = \tau_x(z_2 \cdots z_{k+1}) \text{ and } \tau_{\sigma_{y_2} \cdots \sigma_{y_{k+1}}(x)}(y_1) = \tau_{\sigma_{z_2} \cdots \sigma_{z_{k+1}}(x)}(z_1).$$

By the inductive hypothesis the restriction of τ_x on X_k is bijective. Hence $y_2 \cdots y_{k+1} = z_2 \cdots z_{k+1}$, and thus $y_i = z_i$ for all $2 \leq i \leq k+1$. Now

$$\tau_{\sigma_{y_2} \cdots \sigma_{y_{k+1}}(x)}(y_1) = \tau_{\sigma_{z_2} \cdots \sigma_{z_{k+1}}(x)}(z_1) = \tau_{\sigma_{y_2} \cdots \sigma_{y_{k+1}}(x)}(z_1)$$

and thus $y_1 = z_1$. Therefore the restriction of τ_x on X_{k+1} is injective. Let $c_1, \ldots, c_{k+1} \in X$. By the inductive hypothesis there exist $x_2, \ldots, x_{k+1} \in X$ such that

$$\tau_x(x_2 \cdots x_{k+1}) = c_2 \cdots c_{k+1}.$$

There exists $x_1 \in X$ such that

$$\tau_{\sigma_{x_2} \cdots \sigma_{x_{k+1}}(x)}(x_1) = z_1.$$

Hence

$$\tau_x(x_1 \cdots x_{k+1}) = c_1 \cdots c_{k+1},$$

and thus the restriction of τ_x on X_{k+1} is bijective. Therefore, by induction, τ_x is bijective for all $x \in X$.

Let $x_1, \ldots, x_{n+1}, y_1, \ldots, y_{k+1} \in X$. We shall prove that

$$\sigma_{x_1 \cdots x_n}(y_1 \cdots y_k) = \sigma_{x_1} \cdots \sigma_{x_n}(y_1 \cdots y_k)$$

and

$$\tau_{x_1 \cdots x_n}(y_1 \cdots y_k) = \tau_{x_n} \cdots \tau_{x_1}(y_1 \cdots y_k)$$

by induction on $n+k$. Note that by the definition of r_{FM},

$$\sigma_{x_1 \cdots x_n}(y) = \sigma_{x_1} \cdots \sigma_{x_n}(y) \text{ and } \tau_{x_1 \cdots x_n}(y) = \tau_{x_n} \cdots \tau_{x_1}(y)$$

for all positive integer n and all $y \in \{1\} \cup X_1$. Suppose that $k, n \geq 1$ and

$$\sigma_{x_1 \cdots x_l}(y_1 \cdots y_t) = \sigma_{x_1} \cdots \sigma_{x_l}(y_1 \cdots y_t)$$

and

$$\tau_{x_1 \cdots x_l}(y_1 \cdots y_t) = \tau_{x_l} \cdots \tau_{x_1}(y_1 \cdots y_t)$$

for all positive integers l, t such that $l + t \leq n + k$. Clearly we may assume that $n > 1$. By the definition of r_{FM} and the inductive hypothesis, we have that

$$\begin{aligned}
\sigma_{x_1 \cdots x_n}(y_1 \cdots y_{k+1}) =& \sigma_{x_1 \cdots x_n}(y_1 \cdots y_k)\sigma_{\tau_{y_k} \cdots \tau_{y_1}(x_1 \cdots x_n)}(y_{k+1}) \\
=& \sigma_{x_1 \cdots x_n}(y_1 \cdots y_k)\sigma_{\tau_{y_1 \cdots y_k}(x_1 \cdots x_n)}(y_{k+1}) \\
=& \sigma_{x_1 \cdots x_n}(y_1 \cdots y_k)\sigma_{\tau_{\sigma_{x_2} \cdots \sigma_{x_n}(y_1 \cdots y_k)}(x_1)\tau_{y_1 \cdots y_k}(x_2 \cdots x_n)}(y_{k+1}) \\
=& \sigma_{x_1}\sigma_{x_2 \cdots x_n}(y_1 \cdots y_k)\sigma_{\tau_{\sigma_{x_2} \cdots \sigma_{x_n}(y_1 \cdots y_k)}(x_1)\sigma_{\tau_{y_1 \cdots y_k}(x_2 \cdots x_n)}}(y_{k+1}) \\
=& \sigma_{x_1}\left(\sigma_{x_2 \cdots x_n}(y_1 \cdots y_k)\sigma_{\tau_{y_1 \cdots y_k}(x_2 \cdots x_n)}(y_{k+1})\right) \\
=& \sigma_{x_1}\left(\sigma_{x_2 \cdots x_n}(y_1 \cdots y_k)\sigma_{\tau_{y_k} \cdots \tau_{y_1}(x_2 \cdots x_n)}(y_{k+1})\right) \\
=& \sigma_{x_1}\sigma_{x_2 \cdots x_n}(y_1 \cdots y_{k+1}) \\
=& \sigma_{x_1} \cdots \sigma_{x_n}(y_1 \cdots y_{k+1})
\end{aligned}$$

and

$$\begin{aligned}
\tau_{x_1\cdots x_n}(y_1\cdots y_{k+1}) =& \tau_{\sigma_{y_2}\cdots\sigma_{y_{k+1}}(x_1\cdots x_n)}(y_1)\tau_{x_1\cdots x_n}(y_2\cdots y_{k+1})\\
=& \tau_{\sigma_{y_2\cdots y_{k+1}}(x_1\cdots x_n)}(y_1)\tau_{x_1\cdots x_n}(y_2\cdots y_{k+1})\\
=& \tau_{\sigma_{y_2\cdots y_{k+1}}(x_1\cdots x_{n-1})\sigma_{\tau_{x_{n-1}}\cdots\tau_{x_1}(y_2\cdots y_{k+1})}(x_n)}(y_1)\tau_{x_1\cdots x_n}(y_2\cdots y_{k+1})\\
=& \tau_{\sigma_{\tau_{x_1}\cdots x_{n-1}}(y_2\cdots y_{k+1})}(x_n)\tau_{\sigma_{y_2\cdots y_{k+1}}(x_1\cdots x_{n-1})}(y_1)\\
& \cdot\tau_{x_n}\tau_{x_1\cdots x_{n-1}}(y_2\cdots y_{k+1})\\
=& \tau_{x_n}\left(\tau_{\sigma_{y_2\cdots y_{k+1}}(x_1\cdots x_{n-1})}(y_1)\tau_{x_1\cdots x_{n-1}}(y_2\cdots y_{k+1})\right)\\
=& \tau_{x_n}\left(\tau_{\sigma_{y_2}\cdots\sigma_{y_{k+1}}(x_1\cdots x_{n-1})}(y_1)\tau_{x_1\cdots x_{n-1}}(y_2\cdots y_{k+1})\right)\\
=& \tau_{x_n}\tau_{x_1\cdots x_{n-1}}(y_1\cdots y_{k+1})\\
=& \tau_{x_n}\cdots\tau_{x_1}(y_1\cdots y_{k+1}).
\end{aligned}$$

Therefore, by induction,

$$\sigma_{x_1\cdots x_n}=\sigma_{x_1}\cdots\sigma_{x_n}\text{ and }\tau_{x_1\cdots x_n}=\tau_{x_n}\cdots\tau_{x_1}$$

for all positive integer n. Hence σ_a and τ_a are bijective for all $a\in FM(X)$, and thus $(FM(X),r_{FM})$ is a solution to the YBE. □

thm:LYZ9 **Theorem 8.2.3.** *Let (X,r) be a solution to the YBE. Let $i\colon X\to G(X,r)$ be the natural map. Then there exists a unique braiding operator r_G on $G(X,r)$ such that $r_G(i\times i)=(i\times i)r$. Furthermore, if (H,s) is a braided group and $j\colon X\to H$ is a map such that $s(j\times j)=(j\times j)r$, then there exists a unique group homomorphism $f\colon G(X,r)\to H$ such that $s(f\times f)=(f\times f)r_G$ and $j=fi$. Furthermore, if $r^2=\mathrm{id}_{X^2}$, then $r_G^2=\mathrm{id}_{G(X,r)^2}$.*

Proof. We use the notation of Proposition 8.2.1. Thus, by Proposition 8.2.1, (Y,r') is a solution to the YBE. By Theorem 8.2.2, we extend (Y,r') to a solution $(FM(Y),r'_{FM})$ to the YBE. We write $r'_{FM}(a,b)=(\sigma'_a(b),\tau'_b(a))$ for all $a,b\in FM(Y)$. Note that, by the definition of r'_{FM},

$$\sigma'_{ab}=\sigma'_a\sigma'_b,\quad \tau'_{ab}=\tau'_b\tau'_a,$$

$$\sigma'_c(ab)=\sigma'_c(a)\sigma'_{\tau'_a(c)}(b)\quad\text{and}\quad\tau'_c(ab)=\tau'_{\sigma'_b(c)}(a)\tau'_c(b)$$

for all $a,b,c\in FM(Y)$. Hence it is easy to check that r'_{FM} satisfies conditions (1), (2) and (3) of the Definition 8.1.1.

Note that $G(X,r)=FM(Y)/\sim$, where $\sim$ is the equivalence relation on $FM(Y)$ generated by $axx'b\sim ab$, $ax'xb\sim ab$ and $axyb\sim a\sigma_x(y)\tau_y(x)b$ for all $a,b\in FM(Y)$ and $x,y\in X$.

We define $r_G(\overline{a},\overline{b})=(\overline{\sigma'_a(b)},\overline{\tau'_b(a)})$ for all $a,b\in FM(Y)$, where $\overline{a}\in G(X.r)$ is the the $\sim$-class of $a\in FM(Y)$. By Lemma 3.1.5,

$$\sigma'_{xy}=\sigma'_x\sigma'_y=\sigma'_{\sigma_x(y)}\sigma'_{\tau_y(x)}=\sigma'_{\sigma_x(y)\tau_y(x)}$$

and

$$\tau'_{xy} = \tau'_y \tau'_x = \tau'_{\tau_y(x)} \tau'_{\sigma_x(y)} = \tau'_{\sigma_x(y)\tau_y(x)}$$

for all $x, y \in X$. Hence, to prove that r_G is well-defined it is enough to check that

$$\mathrm{id} = \sigma'_{xx'} = \sigma_{x'x} = \tau'_{xx'} = \tau'_{x'x},$$

$$\sigma'_a(bxx'c) \sim \sigma'_a(bc) \sim \sigma'_a(bx'xc),$$

$$\tau'_a(bxx'c) \sim \tau'_a(bc) \sim \tau'_a(bx'xc),$$

$$\sigma'_a(bxyc) \sim \sigma'_a(b\sigma_x(y)\tau_y(x)c)$$

and

$$\tau'_a(bxyc) \sim \tau'_a(b\sigma_x(y)\tau_y(x)c)$$

for all $a, b, c \in FM(Y)$ and $x, y \in X$. We shall prove that $\sigma'_{xx'}(a) = a$ for all $x \in X$ and $a \in FM(Y)$ by induction on the length of a. Let $x, y \in X$. We have that

$$\sigma'_{xx'}(y) = \sigma'_x \sigma'_{x'}(y) = \sigma_x \sigma_x^{-1}(y) = y,$$

$$\sigma'_{xx'}(y') = \sigma'_x \sigma'_{x'}(y') = \sigma'_x(\widehat{\tau}_x(y)') = \widehat{\tau}_x^{-1}\widehat{\tau}_x(y)' = y'.$$

Let n be a positive integer and let $y_1, \ldots, y_{n+1} \in Y$. Suppose that $\sigma_{xx'}(a) = a$ for all $x \in X$ and all $a \in FM(Y)$ of length $\leq n$. By the inductive hypothesis, we have that

$$\begin{aligned}\sigma'_{xx'}(y_1 \cdots y_{n+1}) =& \sigma'_{xx'}(y_1 \cdots y_n)\sigma'_{\tau'_{y_n} \cdots \tau'_{y_1}(xx')}(y_{n+1}) \\ =& y_1 \cdots y_n \sigma'_{\tau'_{y_n} \cdots \tau'_{y_1}(xx')}(y_{n+1}).\end{aligned}$$

Note that $\tau'_y(xx') = \tau'_{\sigma'_{x'}(y)}(x)\tau'_y(x') = \tau_{\sigma_x^{-1}(y)}(x)\widehat{\sigma}_y^{-1}(x)' = \widehat{\sigma}_y^{-1}(x)\widehat{\sigma}_y^{-1}(x)'$ and

$$\begin{aligned}\tau'_{y'}(xx') =& \tau'_{\sigma'_{x'}(y')}(x)\tau'_{y'}(x') = \tau'_{\widehat{\tau}_x(y)'}(x)\widehat{\sigma}_y(x)' \\ =& \tau^{-1}_{\widehat{\tau}_x(y)}(x)\widehat{\sigma}_y(x)' = \widehat{\sigma}_{\widehat{\tau}_x^{-1}(\widehat{\tau}_x(y))}(x)\widehat{\sigma}_y(x)' \\ =& \widehat{\sigma}_y(x)\widehat{\sigma}_y(x)'\end{aligned}$$

for all $x, y \in X$. Hence $\sigma'_{xx'}(y_1 \cdots y_{n+1}) = y_1 \cdots y_{n+1}$, and thus, by induction, $\sigma'_{xx'}(a) = a$ for all $x \in X$ and all $a \in FM(Y)$. Similarly one can check that $\sigma'_{x'x} = \mathrm{id} = \tau'_{xx'} = \tau'_{x'x}$ for all $x \in X$.

Let $x \in X$ and $a, b, c \in FM(Y)$. We have

$$\begin{aligned}\sigma'_a(bxx'c) =& \sigma'_a(bxx')\sigma'_{\tau'_{bxx'}(a)}(c) \\ =& \sigma'_a(b)\sigma'_{\tau'_b(a)}(xx')\sigma'_{\tau'_{xx'}\tau'_b(a)}(c) \\ =& \sigma'_a(b)\sigma'_{\tau'_b(a)}(xx')\sigma'_{\tau'_b(a)}(c)\end{aligned}$$

and

$$\sigma'_a(bc) = \sigma'_a(b)\sigma'_{\tau'_b(a)}(c).$$

Note that $\sigma'_{y'}(xx') = \sigma'_{y'}(x)\sigma'_{\tau'_x(y')}(x') = \sigma_y^{-1}(x)\sigma'_{\widehat{\sigma}_x^{-1}(y)'}(x') = \sigma_y^{-1}(x)\widehat{\tau}_{\widehat{\sigma}_x^{-1}(y)}(x)' = \sigma_y^{-1}(x)\sigma_y^{-1}(x)'$ and

$$\begin{aligned}\sigma'_y(xx') =& \sigma'_y(x)\sigma'_{\tau'_x(y)}(x') = \sigma_y(x)\sigma'_{\tau_x(y)}(x')\\ =& \sigma_y(x)\widehat{\tau}^{-1}_{\tau_x(y)}(x)' = \sigma_y(x)\sigma_{\tau_x^{-1}(\tau_x(y))}(x)'\\ =& \sigma_y(x)\sigma_y(x)'\end{aligned}$$

for all $x, y \in X$. Hence

$$\sigma'_a(bxx'c) \sim \sigma'_a(bc)$$

for all $x \in X$ and $a, b, c \in FM(Y)$. Similarly one can check that

$$\sigma'_a(bx'xc) \sim \sigma'_a(bc)$$

and

$$\tau'_a(bxx'c) \sim \tau'_a(bc) \sim \tau'_a(bx'xc)$$

for all $x \in X$ and $a, b, c \in FM(Y)$.

Let $x, y \in X$ and $a, b, c \in FM(Y)$. We have

$$\begin{aligned}\sigma'_a(bxyc) =& \sigma'_a(bxy)\sigma'_{\tau'_{bxy}(a)}(c)\\ =& \sigma'_a(b)\sigma'_{\tau'_b(a)}(xy)\sigma'_{\tau'_{bxy}(a)}(c)\end{aligned}$$

and

$$\begin{aligned}\sigma'_a(b\sigma_x(y)\tau_y(x)c) =& \sigma'_a(b\sigma_x(y)\tau_y(x))\sigma'_{\tau'_{b\sigma_x(y)\tau_y(x)}(a)}(c)\\ =& \sigma'_a(b)\sigma'_{\tau'_b(a)}(\sigma_x(y)\tau_y(x))\sigma'_{\tau'_{\sigma_x(y)\tau_y(x)}\tau'_b(a)}(c)\\ =& \sigma'_a(b)\sigma'_{\tau'_b(a)}(\sigma_x(y)\tau_y(x))\sigma'_{\tau'_{xy}\tau'_b(a)}(c)\\ =& \sigma'_a(b)\sigma'_{\tau'_b(a)}(\sigma_x(y)\tau_y(x))\sigma'_{\tau'_{bxy}(a)}(c).\end{aligned}$$

Note that

$$\begin{aligned}\sigma'_z(xy) =& \sigma'_z(x)\sigma'_{\tau'_x(z)}(y)\\ =& \sigma_z(x)\sigma_{\tau_x(z)}(y),\end{aligned} \tag{8.5}$$

eq:LYZ1

$$\begin{aligned}\sigma'_{z'}(xy) =& \sigma'_{z'}(x)\sigma'_{\tau'_x(z')}(y)\\ =& \sigma_z^{-1}(x)\sigma_{\widehat{\sigma}_x^{-1}(z)'}(y)\\ =& \sigma_z^{-1}(x)\sigma^{-1}_{\widehat{\sigma}_x^{-1}(z)}(y)\end{aligned} \tag{8.6}$$

eq:LYZ2

and by Lemma 3.1.5,

$$\begin{aligned}\sigma'_z(\sigma_x(y)\tau_y(x)) =&\sigma'_z(\sigma_x(y))\sigma'_{\tau'_{\sigma_x(y)}(z)}(\tau_y(x))\\ =&\sigma_z\sigma_x(y)\sigma_{\tau_{\sigma_x(y)}(z)}(\tau_y(x))\\ =&\sigma_{\sigma_z(x)}\sigma_{\tau_x(z)}(y)\tau_{\sigma_{\tau_x(z)}(y)}(\sigma_z(x))\end{aligned} \tag{8.7}$$ eq:LYZ3

and

$$\begin{aligned}\sigma'_{z'}(\sigma_x(y)\tau_y(x)) =&\sigma'_{z'}(\sigma_x(y))\sigma'_{\tau'_{\sigma_x(y)}(z')}(\tau_y(x))\\ =&\sigma_z^{-1}\sigma_x(y)\sigma'_{\tau'_{\sigma'_x(y)}(z')}(\tau'_y(x))\\ =&\sigma_{\sigma_z^{-1}(x)}\sigma^{-1}_{\tau_{\sigma_z^{-1}(x)}(z)}(y)\tau'_{\sigma'_{\tau'_x(z')}(y)}(\sigma'_{z'}(x))\\ =&\sigma_{\sigma_z^{-1}(x)}\sigma^{-1}_{\widehat{\sigma}_x^{-1}(z)}(y)\tau_{\sigma^{-1}_{\widehat{\sigma}_x^{-1}(z)}(y)}(\sigma_z^{-1}(x))\end{aligned} \tag{8.8}$$ eq:LYZ4

for all $x,y,z\in X$. Hence, by (8.5), (8.6), (8.7) and (8.8), we have that

$$\sigma'_a(bxyc)\sim\sigma'_a(b\sigma_x(y)\tau_y(x)c)$$

for all $x,y\in X$ and $a,b,c\in FM(Y)$. Similarly one can check that

$$\tau'_a(bxyc)\sim\tau'_a(b\sigma_x(y)\tau_y(x)c)$$

for all $x,y\in X$ and $a,b,c\in FM(Y)$. Therefore r_G is well-defined. Now, by the definition of r_G, Proposition 8.2.1 and Theorem 8.2.2, if $r^2=\mathrm{id}_{X^2}$, then $r_G^2=\mathrm{id}_{G(X,r)^2}$.

Since r'_{FM} satisfies conditions (1), (2) and (3) of the Definition 8.1.1, r_G also satisfies these conditions. To prove that r_G is a braiding operator on $G(X,r)$, it is enough to show that

$$\overline{ab}=\overline{\sigma'_a(b)\tau'_b(a)} \tag{8.9}$$ eq:LYZ5

for all $a,b\in FM(Y)$. We shall prove (8.9) by induction on the length of ab. For $b=1$, clearly $a=\sigma'_a(1)\tau'_1(a)$, and thus (8.9) holds in this case. For $a=1$, we have that $b=\sigma'_1(b)\tau'_b(1)$, and thus (8.9) holds in this case. Hence we may assume that $a\neq 1$ and $b\neq 1$. Note that

$$\begin{aligned}\sigma'_x(y)\tau'_y(x) =&\sigma_x(y)\tau_y(x)\sim xy,\\ \sigma'_x(y')\tau'_{y'}(x) =&\widehat{\tau}_x^{-1}(y)'\tau_y^{-1}(x)\sim\widehat{\tau}_x^{-1}(y)'\tau_y^{-1}(x)yy'\\ \sim&\widehat{\tau}_x^{-1}(y)'\sigma_{\tau_y^{-1}(x)}(y)\tau_y(\tau_y^{-1}(x))y'\\ =&\widehat{\tau}_x^{-1}(y)'\widehat{\tau}_x^{-1}(y)xy'\sim xy',\\ \sigma'_{x'}(y)\tau'_y(x') =&\sigma_x^{-1}(y)\widehat{\sigma}_y^{-1}(x)'\sim x'x\sigma_x^{-1}(y)\widehat{\sigma}_y^{-1}(x)'\\ \sim&x'\sigma_x(\sigma_x^{-1}(y))\tau_{\sigma_x^{-1}(y)}(x)\widehat{\sigma}_y^{-1}(x)'\\ =&x'y\widehat{\sigma}_y^{-1}(x)\widehat{\sigma}_y^{-1}(x)'\sim x'y,\\ \sigma'_{x'}(y')\tau'_{y'}(x') =&\widehat{\tau}_x(y)'\widehat{\sigma}_y(x)'\sim x'x\widehat{\tau}_x(y)'\widehat{\sigma}_y(x)'\end{aligned}$$

$$\begin{aligned}&\sim x'\sigma'_x(\widehat{\tau}_x(y)')\tau'_{\widehat{\tau}_x(y)'}(x)\widehat{\sigma}_y(x)'\\&=x'y'\tau^{-1}_{\widehat{\tau}_x(y)}(x)\widehat{\sigma}_y(x)' = x'y'\widehat{\sigma}_{\widehat{\tau}_x^{-1}(\widehat{\tau}_x(y))}(x)\widehat{\sigma}_y(x)'\\&=x'y'\widehat{\sigma}_y(x)\widehat{\sigma}_y(x)' \sim x'y'\end{aligned}$$

for all $x, y \in X$. Hence (8.9) holds for all $a, b \in FM(Y)$ such that the length of ab is ≤ 2. Let $a, b \in FM(Y)$ be elements such that the length of ab is $n > 2$ and suppose that

$$cd \sim \sigma'_c(d)\tau'_d(c)$$

for all $c, d \in FM(Y)$ such that the length of cd is $< n$. We may assume that $a \neq 1$ and $b \neq 1$. Hence there exist $y, z \in Y$ and $c, d \in FM(Y)$ such that $a = cy$ and $b = zd$. Suppose that $c = 1$. In this case, by the inductive hypothesis, we have

$$\begin{aligned}ab =& yzd \sim \sigma'_y(z)\tau'_z(y)d \sim \sigma'_y(z)\sigma'_{\tau'_z(y)}(d)\tau'_d(\tau'_z(y))\\=&\sigma'_y(zd)\tau'_{zd}(y) = \sigma'_a(b)\tau'_b(a).\end{aligned}$$

Suppose that $c \neq 1$. In this case, by the inductive hypothesis, we have

$$\begin{aligned}ab =& cyb \sim c\sigma'_y(b)\tau'_b(y) \sim \sigma'_c(\sigma'_y(b))\tau'_{\sigma'_y(b)}(c)\tau'_b(y)\\=&\sigma'_{cy}(b)\tau'_b(cy) = \sigma'_a(b)\tau'_b(a).\end{aligned}$$

Hence, by induction, (8.9) follows. Therefore r_G is a braiding operator on $G(X, r)$.

Note that $r_G(i \times i)(x, y) = r_G(\bar{x}, \bar{y}) = (\overline{\sigma_x(y)}, \overline{\tau_y(x)}) = (i \times i)r(x, y)$ for all $x, y \in X$. Let (H, s) be a braided group and let $j : X \to H$ be a map such that $s(j \times j) = (j \times j)r$. We write $s(\bar{a}, \bar{b}) = (\bar{a} \triangleright_s \bar{b}, \bar{a} \triangleleft_s \bar{b})$ for all $a, b \in FM(Y)$. Since

$$j(x)j(y) = (j(x) \triangleright_s j(y))(j(x) \triangleleft_s j(y)) = j(\sigma_x(y))j(\tau_y(x))$$

for all $x, y \in Y$, there exists a unique group homomorphism $f : G(X, r) \to H$ such that $fi = j$. We shall prove that

$$s(f \times f)(\bar{a}, \bar{b}) = (f \times f)r_G(\bar{a}, \bar{b})$$

for all $a, b \in FM(Y)$ by induction on the length of ab. We know that

$$s(f \times f)(\bar{a}, 1) = s(f(\bar{a}), 1) = (1, f(\bar{a})) = (f \times f)(1, \bar{a}) = (f \times f)r_G(\bar{a}, 1)$$

and

$$s(f \times f)(1, \bar{a}) = s(1, f(\bar{a})) = (f(\bar{a}), 1) = (f \times f)(\bar{a}, 1) = (f \times f)r_G(1, \bar{a})$$

for all $a \in FM(Y)$. Note that

$$s(f \times f)(\bar{x}, \bar{y}) = s(j(x), j(y)) = (j(\sigma_x(y), \tau_y(x)) = (f \times f)r_G(\bar{x}, \bar{y})$$

for all $x, y \in X$. Since

$$\begin{aligned}\left(f(\bar{y}), f\left(\overline{\widehat{\sigma}_y^{-1}(x)}\right)\right) =& (f\times f)(i\times i)r(x,\sigma_x^{-1}(y))\\ =& (j\times j)r(x,\sigma_x^{-1}(y)) = s(j\times j)(x,\sigma_x^{-1}(y))\\ =& (j(x)\rhd_s j(\sigma_x^{-1}(y)), j(x)\lhd_s j(\sigma_x^{-1}(y))),\end{aligned}$$

we have that

$$f(\bar{x})\rhd_s f\left(\overline{\sigma_x^{-1}(y)}\right) = j(x)\rhd_s j(\sigma_x^{-1}(y)) = f(\bar{y}),$$

and thus

$$f\left(\overline{x'}\right)\rhd_s f(\bar{y}) = f(\bar{x})^{-1}f(\bar{y}) = f\left(\overline{\sigma_x^{-1}(y)}\right) = f\left(\overline{\sigma'_{x'}(y)}\right)$$

for all $x, y\in X$. Furthermore

$$\begin{aligned}f\left(\overline{\sigma'_{x'}(y)}\right)\left(f\left(\overline{x'}\right)\lhd_s f(\bar{y})\right) =& \left(f\left(\overline{x'}\right)\rhd_s f(\bar{y})\right)\left(f\left(\overline{x'}\right)\lhd_s f(\bar{y})\right)\\ =& f\left(\overline{x'}\right)f(\bar{y}) = f\left(\overline{x'y}\right)\\ =& f\left(\overline{\sigma'_{x'}(y)\tau'_y(x')}\right)\\ =& f\left(\overline{\sigma'_{x'}(y)}\right)f\left(\overline{\tau'_y(x')}\right).\end{aligned}$$

Hence

$$f\left(\overline{x'}\right)\lhd_s f(\bar{y}) = f\left(\overline{\tau'_y(x')}\right),$$

and thus

$$s(f\times f)\left(\overline{x'},\bar{y}\right) = (f\times f)r_G\left(\overline{x'},\bar{y}\right)$$

for all $x, y\in X$. Similarly one can check that

$$s(f\times f)\left(\bar{x},\overline{y'}\right) = (f\times f)r_G\left(\bar{x},\overline{y'}\right)$$

for all $x, y\in X$.

Since

$$\begin{aligned}\left(f\left(\overline{y'}\right), f\left(\overline{\tau'_{(\sigma')_x^{-1}(y')}(x)}\right)\right) =& (f\times f)r_G\left(\bar{x},\overline{(\sigma')_x^{-1}(y')}\right)\\ =& s(f\times f)\left(\bar{x},\overline{(\sigma')_x^{-1}(y')}\right)\\ =& \left(f(\bar{x})\rhd_s f\left(\overline{(\sigma')_x^{-1}(y')}\right), f(\bar{x})\lhd_s f\left(\overline{(\sigma')_x^{-1}(y')}\right)\right),\end{aligned}$$

we have that

$$f(\bar{x})\rhd_s f\left(\overline{(\sigma')_x^{-1}(y')}\right) = f\left(\overline{y'}\right),$$

and thus

$$f\left(\overline{x'}\right)\rhd_s f\left(\overline{y'}\right) = f(\bar{x})^{-1}f\left(\overline{y'}\right) = f\left(\overline{(\sigma')_x^{-1}(y')}\right) = f\left(\overline{\sigma'_{x'}(y')}\right)$$

for all $x, y \in X$. Furthermore

$$\begin{aligned} f\left(\overline{\sigma'_{x'}(y')}\right)\left(f\left(\overline{x'}\right) \lhd_s f\left(\overline{y'}\right)\right) &= \left(f\left(\overline{x'}\right) \rhd_s f\left(\overline{y'}\right)\right)\left(f\left(\overline{x'}\right) \lhd_s f\left(\overline{y'}\right)\right) \\ &= f\left(\overline{x'}\right) f\left(\overline{y'}\right) = f\left(\overline{x'y'}\right) \\ &= f\left(\overline{\sigma'_{x'}(y')\tau'_{y'}(x')}\right) \\ &= f\left(\overline{\sigma'_{x'}(y')}\right) f\left(\overline{\tau'_{y'}(x')}\right). \end{aligned}$$

Hence

$$f\left(\overline{x'}\right) \lhd_s f\left(\overline{y'}\right) = f\left(\overline{\tau'_{y'}(x')}\right),$$

and thus

$$s(f \times f)\left(\overline{x'}, \overline{y'}\right) = (f \times f) r_G\left(\overline{x'}, \overline{y'}\right)$$

for all $x, y \in X$. Hence

$$s(f \times f)\left(\bar{a}, \bar{b}\right) = (f \times f) r_G\left(\bar{a}, \bar{b}\right)$$

for all $a, b \in FM(Y)$ such that the length of ab is ≤ 2. Let $a, b \in FM(Y)$ be elements such that the length of ab is $n > 2$ and suppose that

$$s(f \times f)\left(\bar{c}, \bar{d}\right) = (f \times f) r_G\left(\bar{c}, \bar{d}\right)$$

for all $c, d \in FM(Y)$ such that the length of cd is $< n$. We may assume that $a \neq 1$ and $b \neq 1$. Hence there exist $y, z \in Y$ and $c, d \in FM(Y)$ such that $a = cy$ and $b = zd$. Suppose that $c \neq 1$. In this case, by the inductive hypothesis,

$$\begin{aligned} s(f \times f)(\bar{a}, \bar{b}) &= s(f(\bar{c}) f(\bar{y}), f(\bar{b})) \\ &= (\mathrm{id} \times m) s_1 s_2 (f(\bar{c}), f(\bar{y}), f(\bar{b})) \\ &= (\mathrm{id} \times m)(f \times f \times f)(r_G)_1 (r_G)_2 (\bar{c}, \bar{y}, \bar{b}) \\ &= (f \times f)(\mathrm{id} \times m)(r_G)_1 (r_G)_2 (\bar{c}, \bar{y}, \bar{b}) \\ &= (f \times f) r_G(\bar{c}\bar{y}, \bar{b}) = (f \times f) r_G(\bar{a}, \bar{b}). \end{aligned}$$

Suppose that $c = 1$. In this case, by the inductive hypothesis,

$$\begin{aligned} s(f \times f)(\bar{a}, \bar{b}) &= s(f(\bar{a}), f(\bar{z}) f(\bar{d})) \\ &= (m \times \mathrm{id}) s_2 s_1 (f(\bar{a}), f(\bar{z}), f(\bar{d})) \\ &= (m \times \mathrm{id})(f \times f \times f)(r_G)_2 (r_G)_1 (\bar{a}, \bar{z}, \bar{d}) \\ &= (f \times f)(m \times \mathrm{id})(r_G)_2 (r_G)_1 (\bar{a}, \bar{z}, \bar{d}) \\ &= (f \times f) r_G(\bar{a}, \bar{z}\bar{d}) = (f \times f) r_G(\bar{a}, \bar{b}). \end{aligned}$$

Hence, by induction, we have that

$$s(f\times f) = (f\times f)r_G.$$

The uniqueness of r_G follows by the above argument taking $H = G(X,r)$ and $j = i$, because in this case $f = \mathrm{id}$, and thus $s = r_G$. □

thm:GVbraces

Theorem 8.2.4. *Let (X,r) be a solution to the YBE. Let $i\colon X \to G(X,r)$ be the natural map. Then there exists a unique structure of skew brace on $G(X,r)$ with multiplicative group the structure group $G(X,r)$ such that $\lambda_{i(x)}(i(y)) = i(\sigma_x(y))$ for all $x,y \in X$. Furthermore, if $(B,+,\circ)$ is a skew brace and $j\colon X \to B$ is a map such that $\lambda_{j(x)}(j(y)) = j(\sigma_x(y))$ and $j(x)j(y) = j(\sigma_x(y))j(\tau_y(x))$ for all $x,y \in X$, then there exists a unique homomorphism of skew braces $f\colon G(X,r) \to B$ such that $fi = j$.*

Proof. By Theorem 8.2.3, there exists a unique braiding operator r_G on $G(X,r)$ such that $r_G(i\times i) = (i\times i)r$. Consider the structure of skew brace on $G(X,r)$ associated to the braided group $(G(X,r),r_G)$ by the bijective correspondence described in the proof of Theorem 8.1.3. Thus the addition on $G(X,r)$ is defined by $\bar{a}+\bar{b} = \bar{a}\overline{(\sigma')_a^{-1}(b)}$ for all $a,b \in FM(Y)$. Hence

$$\begin{aligned}\lambda_{i(x)}(i(y)) =& -i(x)+i(x)i(y) = i(x)(i(x)^{-1}+i(y))\\ =& i(x)i(x)^{-1}\overline{(\sigma')_{x'}^{-1}(y)}\\ =& \overline{\sigma_x(y)} = i(\sigma_x(y))\end{aligned}$$

for all $x,y \in X$.

Let $(B,+,\circ)$ be a skew brace and $j\colon X \to B$ is a map such that $\lambda_{j(x)}(j(y)) = j(\sigma_x(y))$ and $j(x)j(y) = j(\sigma_x(y))j(\tau_y(x))$ for all $x,y \in X$. By the definition of $G(X,r)$ there exists a unique group homomorphism $f\colon G(X,r) \to B$ such that $fi = j$. Let $r_B\colon B\times B \to B\times B$ be the map defined by $r_B(u,v) = (\lambda_u(v), \lambda_u(v)^{-1}uv)$. By Theorem 6.1.23, (B,r_B) is a solution to the YBE. Furthermore, it is easy to check that r_B is a braiding operator on the multiplicative group of B. Note that

$$\begin{aligned}r_B(j\times j)(x,y) =& (\lambda_{j(x)}j(y), \lambda_{j(x)}j(y)^{-1}j(x)j(y))\\ =& (j(\sigma_x(y)), j(\tau_y(x))) = (j\times j)r(x,y)\end{aligned}$$

for all $x,y \in X$. By Theorem 8.2.3, $r_B(f\times f) = (f\times f)r_G$. In particular, $f\left(\lambda_{\bar{a}}(\bar{b})\right) = \lambda_{f(\bar{a})}(f(\bar{b}))$ for all $a,b \in FM(Y)$. Hence

$$\begin{aligned}f(\bar{a}+\bar{b}) =& f\left(\bar{a}\overline{(\sigma')_a^{-1}(b)}\right)\\ =& f(\bar{a})f\left(\overline{(\sigma')_a^{-1}(b)}\right)\\ =& f(\bar{a})f\left(\lambda_{(\bar{a})^{-1}}(\bar{b})\right)\\ =& f(\bar{a})\lambda_{f(\bar{a})^{-1}}(f(\bar{b})) = f(\bar{a})+f(\bar{b})\end{aligned}$$

for all $a,b \in FM(Y)$. Hence f is a homomorphism of skew braces. □

Definition 8.2.5. Given a solution (X,r) to the YBE, its structure skew brace is the skew brace structure on the group $G(X,r)$ given in Theorem 8.2.4.

thm:involstruct

Theorem 8.2.6. *Let (X,r) be an involutive solution of the YBE. Then the additive group of the structure skew brace $G(X,r)$ is the additive free abelian group on X.*

Proof. The structure group $G(X,r)$ is

$$G(X,r) = \mathrm{gr}(X : xy = \sigma_x(y)\tau_y(x) \text{ for all } x,y \in X).$$

Since $xy = x+\lambda_x(y) = x+\sigma_x(y)$ and (X,r) is involutive, we have that $\sigma_x(y)\tau_y(x) = \sigma_x(y)+\lambda_{\sigma_x(y)}(\tau_y(x) = \sigma_x(y)+\sigma_{\sigma_x(y)}(\tau_y(x)) = \sigma_x(y)+x$, for all $x,y \in X$. Hence the additive group of $G(X,r)$ is generated by X and it is abelian.

Let $\mathbb{Z}^{(X)}$ be the additive free abelian group with basis X. Thus the elements of $\mathbb{Z}^{(X)}$ are finite sums of the form

$$z_1x_1 + \cdots + z_nx_n,$$

with $z_1,\ldots,z_n \in \mathbb{Z}$ and $x_1,\ldots,x_n \in X$. Furthermore, if

$$z_1x_1 + \cdots + z_nx_n = 0$$

and $x_1,\ldots,x_n \in X$ are n distinct elements, then $z_1 = \cdots = z_n = 0$. Consider the semidirect product $\mathbb{Z}^{(X)} \rtimes \mathbb{S}_X$, with respect to the natural action of $\mathbb{S}_X$ on $\mathbb{Z}^{(X)}$ given by the rule

$$\sigma(z_1x_1 + \cdots + z_nx_n) = z_1\sigma(x_1) + \cdots + z_n\sigma(x_n)$$

for all $\sigma \in \mathbb{S}_X$, $z_1,\ldots,z_n \in \mathbb{Z}$ and $x_1,\ldots,x_n \in X$. Let $i\colon X \to G(X,r)$ be the natural map. We define a map $j\colon X \to \mathbb{Z}^{(X)} \rtimes \mathbb{S}_X$ by $j(x) = (x,\sigma_x)$ for all $x \in X$. Note that, since (X,r) is involutive,

$$\begin{aligned} j(x)j(y) =& (x,\sigma_x)(y,\sigma_y) = (x+\sigma_x(y),\sigma_x\sigma_y)\\ =& (\sigma_x(y)+x,\sigma_{\sigma_x(y)}\sigma_{\tau_y(x)})\\ =& (\sigma_x(y)+\sigma_{\sigma_x(y)}(\tau_y(x)),\sigma_{\sigma_x(y)}\sigma_{\tau_y(x)})\\ =& (\sigma_x(y),\sigma_{\sigma_x(y)})(\tau_y(x),\sigma_{\tau_y(x)})\\ =& j(\sigma_x(y))j(\tau_y(x)) \end{aligned}$$

for all $x,y \in X$. Hence there exists a unique homomorphism of groups $f\colon G(X,r) \to \mathbb{Z}^{(X)} \rtimes \mathbb{S}_X$, from the structure group of (X,r) (the multiplicative group of $G(X,r)$) to the semidirect product $\mathbb{Z}^{(X)} \rtimes \mathbb{S}_X$, such that $f(i(x)) = j(x) = (x,\sigma_x)$ for all $x \in X$. In particular, the map i is injective and thus we identify x and $i(x)$. Note that with this identification, for every $a \in G(X,r)$, the restriction of λ_a to X is in the subgroup $\langle \sigma_x : x \in X\rangle$ of $\mathbb{S}_X$. We denote by σ_a the restriction of λ_a to X for all $a \in G(X,r)$. Since $f(x^{-1}) = (x,\sigma_x)^{-1} = (-\sigma_x^{-1}(x),\sigma_x^{-1})$ and $\sigma_x^{-1} = \sigma_{x^{-1}} = \sigma_{-\sigma_{x^{-1}}(x)}$, we have that $f(x^{-1}) = (-\sigma_x^{-1}(x),\sigma_{-\sigma_x^{-1}(x)})$. We shall prove that

$$f(\epsilon_1x_1 + \cdots + \epsilon_nx_n) = (\epsilon_1x_1 + \cdots + \epsilon_nx_n, \sigma_{\epsilon_1x_1+\cdots+\epsilon_nx_n}) \tag{8.10}$$

eq:keyinv

for all $x_1, \dots, x_n \in X$ and $\epsilon_1, \dots, \epsilon_n \in \{-1, 1\}$, by induction on n. Note that in $G(X,r)$ we have that

$$(-x)^{-1} = -\lambda_{(-x)^{-1}}(-x) = \lambda^{-1}_{(-x)}(x) \in X$$

for all $x \in X$. Hence

$$\begin{aligned} f(-x) =& f(\lambda^{-1}_{(-x)}(x))^{-1} = (-\sigma^{-1}_{\lambda^{-1}_{(-x)}(x)}(\lambda^{-1}_{(-x)}(x)), \sigma_{-\sigma^{-1}_{\lambda^{-1}_{(-x)}(x)}(\lambda^{-1}_{(-x)}(x))}) \\ =& (-\sigma^{-1}_{(-x)^{-1}}(\lambda^{-1}_{(-x)}(x)), \sigma_{-\sigma^{-1}_{(-x)^{-1}}(\lambda^{-1}_{(-x)}(x))}) \\ =& (-\lambda_{-x}(\lambda^{-1}_{(-x)}(x)), \sigma_{-\lambda_{-x}(\lambda^{-1}_{(-x)}(x))}) \\ =& (-x, \sigma_{-x}) \end{aligned}$$

for all $x \in X$. Hence (8.10) holds for $n = 1$. Suppose that (8.10) holds for some $n \geq 1$. By the inductive hypothesis, we have that

$$\begin{aligned} & f(\epsilon_1 x_1 + \cdots + \epsilon_{n+1} x_{n+1}) = f((\epsilon_1 x_1 + \cdots + \epsilon_n x_n)\lambda^{-1}_{\epsilon_1 x_1 + \cdots + \epsilon_n x_n}(\epsilon_{n+1} x_{n+1})) \\ &= f(\epsilon_1 x_1 + \cdots + \epsilon_n x_n) f(\lambda^{-1}_{\epsilon_1 x_1 + \cdots + \epsilon_n x_n}(\epsilon_{n+1} x_{n+1})) \\ &= (\epsilon_1 x_1 + \cdots + \epsilon_n x_n, \sigma_{\epsilon_1 x_1 + \cdots + \epsilon_n x_n})(\lambda^{-1}_{\epsilon_1 x_1 + \cdots + \epsilon_n x_n}(\epsilon_{n+1} x_{n+1}), \sigma_{\lambda^{-1}_{\epsilon_1 x_1 + \cdots + \epsilon_n x_n}(\epsilon_{n+1} x_{n+1})}) \\ &= (\epsilon_1 x_1 + \cdots + \epsilon_n x_n, \sigma_{\epsilon_1 x_1 + \cdots + \epsilon_n x_n})(\epsilon_{n+1} \sigma^{-1}_{\epsilon_1 x_1 + \cdots + \epsilon_n x_n}(x_{n+1}), \sigma_{\lambda^{-1}_{\epsilon_1 x_1 + \cdots + \epsilon_n x_n}(\epsilon_{n+1} x_{n+1})}) \\ &= (\epsilon_1 x_1 + \cdots + \epsilon_n x_n + \epsilon_{n+1} x_{n+1}, \sigma_{\epsilon_1 x_1 + \cdots + \epsilon_n x_n} \sigma_{\lambda^{-1}_{\epsilon_1 x_1 + \cdots + \epsilon_n x_n}(\epsilon_{n+1} x_{n+1})}) \\ &= (\epsilon_1 x_1 + \cdots + \epsilon_n x_n + \epsilon_{n+1} x_{n+1}, \sigma_{(\epsilon_1 x_1 + \cdots + \epsilon_n x_n)\lambda^{-1}_{\epsilon_1 x_1 + \cdots + \epsilon_n x_n}(\epsilon_{n+1} x_{n+1})}) \\ &= (\epsilon_1 x_1 + \cdots + \epsilon_n x_n + \epsilon_{n+1} x_{n+1}, \sigma_{\epsilon_1 x_1 + \cdots + \epsilon_n x_n + \epsilon_{n+1} x_{n+1}}). \end{aligned}$$

Hence (8.10) holds for all n by induction. Therefore one can easily see that $\pi_1 f \colon G(X.r) \to \mathbb{Z}^{(X)}$, where $\pi_1 \colon \mathbb{Z}^{(X)} \rtimes \mathbb{S}_X \to \mathbb{Z}^{(X)}$ is the natural map, is an isomorphism from the additive group of $G(X,r)$ to $\mathbb{Z}^{(X)}$, and the result follows. □

Exercises

8.2.1. Check the missing steps of the proofs of all the results of this chapter.

Chapter 9
Bieberbach groups

Bieberbach

9.1 Left ordered groups

A group G is *left ordered* if there is a total ordering $\leq$ on G such that $x \leq y$ implies $zx \leq zy$ for all $x, y, z \in G$. In this case we say that G is a left ordered group with respect to the total order $\leq$. Similarly one defines right ordered groups.

A group G is *ordered* if there is a total ordering $\leq$ on G such that $x \leq y$ implies $zx \leq zy$ and $xz \leq yz$ for all $x, y, z \in G$.

Example 9.1.1. The group $\mathbb{Z}$ is an ordered group with respect to the natural order.

Example 9.1.2. If G is a left ordered group and H is a subgroup of G, then H is left ordered.

Remark 9.1.3. Let G be a left ordered group with respect to the total order $\leq$. We define another total order $\leq'$ on G by

$$x \leq' y \text{ if and only if } x^{-1} \leq y^{-1}$$

for all $x, y \in G$. Note that if $x \leq' y$ then

$$(xz)^{-1} = z^{-1}x^{-1} \leq z^{-1}y^{-1} = (yz)^{-1},$$

and thus $xz \leq' yz$ for all $z \in G$. Hence G is a right ordered group with respect to the total order $\leq'$.

prop:LOgroup1

Proposition 9.1.4. *Let G be a group and let $N \trianglelefteq G$. If N and G/N are left ordered groups, then G is left ordered.*

Proof. Suppose that N is a left ordered group with respect to the total order $\leq_N$ and G/N is a left ordered group with respect to the total order $\leq_{\bar{G}}$. We define

$$x \leq y \Longleftrightarrow \begin{cases} 1 \leq_N x^{-1}y & \text{if } xN = yN, \\ xN \leq_{\bar{G}} yN & \text{otherwise,} \end{cases}$$

for all $x, y \in G$. A straightforward computation shows that then G is a left ordered group with respect to the total order $\leq$. □

Example 9.1.5. Let us show that $G = \mathrm{gr}(x, y : xyx^{-1} = y^{-1})$ is left ordered. Let $f : G \to \mathbb{Z}$ be given by $x \mapsto 1$ and $y \mapsto 0$. Then $\ker f = \langle y \rangle$. The map

$$\{x, y\} \to \mathbf{GL}_2(\mathbb{C}), \quad x \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

induces a group homomorphism $G \to \mathbf{GL}_2(\mathbb{C})$. In particular, y has infinite order and hence $\langle y \rangle \cong \mathbb{Z}$. Hence $\ker f$ and $G/\ker f$ are left ordered groups and, by Proposition 9.1.4, G is a left ordered group. We shall see that G is not an ordered group. Suppose that G is an ordered group with respect to the total order $\leq$. Suppose that $1 \leq y$. Then we have

$$1 = xx^{-1} \leq xyx^{-1} = y^{-1},$$

and thus $y \leq yy^{-1} = 1$, a contradiction because $y \neq 1$. Hence $y \leq 1$. But then we have that

$$y^{-1} = xyx^{-1} \leq xx^{-1} = 1,$$

and thus $1 = yy^{-1} \leq y$, a contradiction. Therefore G is not an ordered group.

The previous example is the Baumslag–Solitar group $B(1, -1)$. Recall that for $n, m \in \mathbb{Z}$ the Baumslag–Solitar's group is defined as the group

$$B(m, n) = \mathrm{gr}(a, b : ba^m b^{-1} = a^n).$$

The map

$$\{a, b\} \to \mathbf{GL}_2(\mathbb{C}), \quad a \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad b \mapsto \begin{pmatrix} \frac{1}{m} & 0 \\ 0 & \frac{1}{n} \end{pmatrix}$$

induces a group homomorphism $B(m, n) \to \mathbf{GL}_2(\mathbb{C})$.

A group G is said to be *poly-*$\mathbb{Z}$ if it has a finite subnormal series

$$\{1\} = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_n = G$$

such that $N_{i+1}/N_i \cong \mathbb{Z}$ for all $1 \leq i < n$. By Proposition 9.1.4 and induction on n, it is easy to see that every poly-$\mathbb{Z}$ group is left ordered.

A group G is said to be *indicable* if there exists a non-trivial group homomorphism $G \to \mathbb{Z}$, and G is said to be *locally indicable* if every non-trivial finitely generated subgroup of G is indicable.

Theorem 9.1.6 (Burns–Hale). *Let G be a group. Then G is left ordered if and only if for each finitely generated non-trivial subgroup H of G there exists a left ordered group L and a non-trivial group homomorphism $H \to L$.*

Proof. If G is a left ordered group, take $L = G$.

Conversely, suppose that for each finitely generated non-trivial subgroup H of G there exists a left ordered group L and a non-trivial group homomorphism $H \to L$. We claim that for all $\{x_1, \ldots, x_n\} \subseteq G \setminus \{1\}$ there exist $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$ such that

$$1 \notin S(x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n}),$$

where $S(x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n})$ denotes the semigroup generated by the set $\{x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n}\}$. We proceed by induction on n. If $n = 1$, then $x_1 \in G \setminus \{1\}$. Let $\epsilon_1 = 1$. If $1 \in S(x_1)$, then x_1 is an element of finite order and hence $\langle x_1 \rangle \to L$ is the trivial homomorphism for every left ordered group L. Hence $1 \notin S(x_1)$. Now assume that the claim holds for some $n \geq 1$. Let $\{x_1, \ldots, x_{n+1}\} \subseteq G \setminus \{1\}$. By assumption, there exists a non-trivial group homomorphism $h\colon \langle x_1, \ldots, x_{n+1} \rangle \to L$ for some left ordered group L. In particular, $h(x_i) \neq 1$ for some $i \in \{1, \ldots, n\}$. Without loss of generality we may assume that there exists an integer $1 \leq k \leq n+1$ such that $h(x_j) \neq 1$ for all $j \in \{1, \ldots, k\}$ and $h(x_j) = 1$ for all $j > k$. Suppose that L is left ordered with respect to a total order $\leq$. Since $h(x_j) \neq 1$ for all $j \leq k$, there are elements $\epsilon_j \in \{-1, 1\}$ such that $1 \leq h(x_j^{\epsilon_j})$ for all $j \leq k$. By the inductive hypothesis, there are elements $\epsilon_{k+1}, \ldots, \epsilon_{n+1} \in \{-1, 1\}$ such that $1 \notin S(x_{k+1}^{\epsilon_{k+1}}, \ldots, x_{n+1}^{\epsilon_{n+1}})$. Note that for every $x \in S(x_1^{\epsilon_1}, \ldots, x_{n+1}^{\epsilon_{n+1}}) \setminus S(x_{k+1}^{\epsilon_{k+1}}, \ldots, x_{n+1}^{\epsilon_{n+1}})$, $1 \leq h(x) \neq 1$. Hence $1 \notin S(x_1^{\epsilon_1}, \ldots, x_{n+1}^{\epsilon_{n+1}})$, and the claim follows by induction.

Consider the set

$\mathcal{F} = \{(F, f) : F$ is a finite subset of $G \setminus \{1\}$ and $f\colon F \to \{-1, 1\}$ such that for every finite subset B of $G \setminus \{1\}$ containing F, there exists a map $g\colon B \to \{-1, 1\}$ such that $1 \notin S(a^{g(a)} : a \in B)$ and $g(x) = f(x)$ for all $x \in F\}$.

Let $C = \{(A, f) : A \subseteq G \setminus \{1\}$ and $f\colon A \to \{-1, 1\}$ such that $(F, f|_F) \in \mathcal{F}$ for all finite subset F of $A\}$. We define and order on C by $(A, f) \leq (B, g)$ if and only if $A \subseteq B$ and $g(a) = f(a)$ for all $a \in A$, i. e. $f = g|_A$. Note that there is a unique map $f_\emptyset\colon \emptyset \to \{-1, 1\}$. We have shown that $(\emptyset, f_\emptyset) \in C$. Hence $C \neq \emptyset$. Furthermore, it is easy to see that every chain of elements in C has an upper bound in C. Thus, by Zorn's lemma, there exists a maximal element $(A, f) \in C$. Suppose that $A \neq G \setminus \{1\}$. Let $x \in G \setminus (A \cup \{1\})$. Let $g_1\colon A \cup \{x\} \to \{-1, 1\}$ and $g_{-1}\colon A \cup \{x\} \to \{-1, 1\}$ be the maps defined $g_i(a) = f(a)$ for all $a \in A$ and $g_i(x) = i$ for $i \in \{-1, 1\}$. By the maximality of (A, f), we have that $(A \cup \{x\}, g_i) \notin C$. Hence there exist finite subsets F_1 and F_{-1} of $A \cup \{x\}$ and finite subsets B_1 and B_{-1} of $G \setminus \{1\}$ such that $F_1 \subseteq B_1$, $F_{-1} \subseteq B_{-1}$, $1 \in S(a^{h_1(a)} : a \in B_1)$ and $1 \in S(a^{h_{-1}(a)} : a \in B_2)$ for all $h_1\colon B_1 \to \{-1, 1\}$ and all $h_{-1}\colon B_{-1} \to \{-1, 1\}$ such that $g_i(a) = h_i(a)$ for all $a \in F_i$. Let $C = \bigcup_{i \in \{-1,1\}} (A \cap F_i)$. Note that $C \cup \{x\} = F_1 \cup F_{-1} \subseteq B_1 \cup B_{-1}$. Since $(C, f|_C) \in \mathcal{F}$, there exists $h\colon B_1 \cup B_{-1} \to \{-1, 1\}$ such that $1 \notin S(a^{h(a)} : a \in B_1 \cup B_{-1})$ and $h(a) = f(a)$ for all $a \in C$. Let $i = h(x) \in \{-1, 1\}$. We have that $h(x) = g_i(x)$, and thus $h(a) = g_i(a)$ for all $a \in F_i$, a contradiction because $S(a^{h(a)} : a \in B_i) \subseteq S(a^{h(a)} : a \in B_1 \cup B_{-1})$. Therefore $A = G \setminus \{1\}$.

Let $P = \{a \in G \setminus \{1\} : f(a) = 1\}$. Note that if $b \in G \setminus \{1\}$ then

$$1 \notin S(b^{f(b)}, (b^{-1})^{f(b^{-1})}),$$

and thus $f(b)f(b^{-1}) = -1$. Hence G is the disjoint union of P, $P^{-1} = \{a^{-1} : a \in P\}$ and $\{1\}$. Note that for all $a, b \in P$, $1 \notin S(a, b, (ab)^{f(ab)})$. Hence $f(ab) = 1$ and thus $ab \in P$. This proves that P is a subsemigroup of G. We define a binary relation $\leq$ on G by, for all $a, b \in G$,

$$a \leq b \text{ if and only if } a^{-1}b \in P \cup \{1\}.$$

It is straightforward to check that $\leq$ is a total order on G and that G is a left ordered group with respect to $\leq$. □

An immediate corollary:

cor:LIimpliesLO **Corollary 9.1.7.** *Locally indicable groups are left ordered groups.*

9.2 The unique product property and diffuse groups

A group G satisfies the *unique product property* if for all finite non-empty subsets A and B there exists $g \in G$ such that $g = ab$ for unique elements $a \in A$ and $b \in B$.

Proposition 9.2.1. *A group with the unique product property is torsion-free.*

Proof. Assume that G has torsion and let $g \in G$ be an element of order $n \geq 2$. Let $A = B = \langle g \rangle$. Then $g^i g^j = g^{i+1} g^{j-1}$ for all i, j, and $g^i \neq g^{i+1}$, so G cannot have the unique product property. □

Proposition 9.2.2. *A left ordered group satisfies the unique product property.*

Proof. Let G be a left ordered group with respect to a total order $\leq$ and $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_m\}$ be non-empty subsets of G. We may assume that $a_1 < a_2 < \cdots < a_n$ and $b_1 < b_2 < \cdots < b_m$. Let a_i be the unique element in A such that $a_i b_m$ is the largest element in AB. Note that $a_j b_k \leq a_j b_m \leq a_i b_m$. Furthermore, if $a_j b_k = a_i b_m$, then $a_j b_k = a_j b_m$ and thus $b_k = b_m$ and $a_j = a_i$. Hence G satisfies the unique product property. □

A group G satisfies the *double unique product property* if for any two given finite non-empty subsets A and B of G such that $|A| + |B| > 2$ there exist at least two unique products in AB, i. e. there exist two distinct elements $g_1, g_2 \in G$ such that $g_1 = a_1 b_1$ for unique elements $a_1 \in A$ and $b_1 \in B$, and $g_2 = a_2 b_2$ for unique elements $a_2 \in A$ and $b_2 \in B$.

thm:Strojnowski **Theorem 9.2.3 (Strojnowski).** *Let G be a group. the following conditions are equivalent:*

1) *G satisfies the double unique product property.*
2) *For all finite non-empty subset $A \subseteq G$ there exists at least a unique product in $AA = \{a_1 a_2 : a_1, a_2 \in A\}$.*
3) *G satisfies the unique product property.*

Proof. The implication (1) $\implies$ (2) is trivial.

We shall prove (2) $\implies$ (3). Suppose that G satisfies (2). Let A, B be finite non-empty subsets of G. Let $C = BA$. By (2), there exist a unique $g \in G$ such that $g = (b_1 a_1)(b_2 a_2)$ for unique $b_1 a_1, b_2 a_2 \in C$, where $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Note

that this implies that $a_1b_2 = ab$ for $a \in A$ and $b \in B$ if and only if $a = a_1$ and $b = b_2$. Hence G satisfies the unique product property.

We shall prove (3) $\implies$ (1). Suppose that G satisfies the unique product property, but it does not satisfy the double unique product property. Thus there exist finite non-empty subsets $A, B \subseteq G$ with $|A|+|B| > 2$ and there is a unique $g \in G$ such that $g = ab$ for unique elements $a \in A$ and $b \in B$. Let $C = a^{-1}A$ and $D = Bb^{-1}$. Then $1 \in C \cap D$. Note that if $c \in C$, $d \in D$ and $cd \neq 1$, then there exist $a_1 \in A$ and $b_1 \in B$ such that $c = a^{-1}a_1$, $d = b_1b^{-1}$ and $ab \neq a_1b_1$. Hence there exist $a_2 \in A \setminus \{a_1\}$ and $b_2 \in B \setminus \{b_1\}$ such that $a_1b_1 = a_2b_2$. Let $c_1 = a^{-1}a_2$ and $d_1 = b_2b^{-1}$. We have that $c \neq c_1$, $d \neq d_1$ and

$$cd = a^{-1}a_1b_1b^{-1} = a^{-1}a_2b_2b^{-1} = c_1d_1.$$

Let $E = D^{-1}C$ and $F = DC^{-1}$. Every element of EF is of the form $(d_1^{-1}c_1)(d_2c_2^{-1})$, where $c_1, c_2 \in C$ and $d_1, d_2 \in D$. Suppose that $c_1d_2 \neq 1$. We have seen that then there exist $c_3 \in C \setminus \{c_1\}$ and $d_3 \in D \setminus \{d_2\}$ such that $c_1d_2 = c_3d_3$. Hence $d_1^{-1}c_3 \in E \setminus \{d_1^{-1}c_1\}$, $d_3c_2^{-1} \in F \setminus \{d_2c_2^{-1}\}$ and

$$(d_1^{-1}c_1)(d_2c_2^{-1}) = (d_1^{-1}c_3)(d_3c_2^{-1}).$$

Suppose that $c_2d_1 \neq 1$. Then there exist $c_4 \in C \setminus \{c_2\}$ and $d_4 \in D \setminus \{d_1\}$ such that $c_2d_1 = c_4d_4$. Hence $d_4^{-1} \cdot 1 \in E \setminus \{d_1^{-1} \cdot 1\}$, $1 \cdot c_4^{-1} \in F \setminus \{1 \cdot c_2^{-1}\}$ and

$$(d_1^{-1} \cdot 1)(1 \cdot c_2^{-1}) = (d_4^{-1} \cdot 1)(1 \cdot c_4^{-1}).$$

Since $|C|+|D| = |A|+|B| > 2$, either there exists $c \in C \setminus \{1\}$ or there exists $d \in D \setminus \{1\}$. In the first case, we have

$$(1 \cdot 1)(1 \cdot 1) = (1 \cdot c)(1 \cdot c^{-1}),$$

and in the second case, we have

$$(1 \cdot 1)(1 \cdot 1) = (d^{-1} \cdot 1)(d \cdot 1).$$

Thus, we have found two finite non-empty subsets $E, F \subseteq G$ such that for every $e \in E$ and $f \in F$, there exist $e_1 \in E \setminus \{e\}$ and $f_1 \in F \setminus \{f\}$ such that $ef = e_1f_1$, a contradiction, because G satisfies the unique product property. Therefore G also satisfies the double unique product property. □

In general, it is difficult to check whether a group satisfies the unique product property.

Definition 9.2.4. A group G is said to be *diffuse* if, given any finite non-empty subset $A \subseteq G$, there exists $a \in A$ such that for every $g \in G \setminus \{1\}$ either $ga \notin A$ or $g^{-1}a \notin A$. Such an element $a \in A$ satisfying this property is called an *extreme point* of A.

Proposition 9.2.5 (Linnell,Witte Morris). *Let G be a diffuse group. Then, given any finite subset $A \subseteq G$ such that $|A| \geq 2$, there exist two distinct extreme points of A.*

Proof. Suppose that there exists a finite subset $A \subseteq G$ with a unique extreme point $a \in A$ and $|A| \geq 2$. Let $B = a^{-1}A$. We shall see that 1 is the unique extreme point of B. Let $b \in B$ be an extreme point of B. Thus, for every $g \in G \setminus \{1\}$, either $gb \notin B$ or $g^{-1}b \notin B$. We have that $b = a^{-1}a_1$ for some $a_1 \in A$. Hence either $ga^{-1}a_1 \notin a^{-1}A$ or $g^{-1}a^{-1}a_1 \notin a^{-1}A$, and thus, either $aga^{-1}a_1 \notin A$ or $ag^{-1}a^{-1}a_1 \notin A$. Hence a_1 is an extreme point of A, and thus $a_1 = a$. This proves that $b = a^{-1}a_1 = 1$ is the only extreme point of B. Let $C = B \cup B^{-1}$, where $B^{-1} = \{b^{-1} : b \in B\}$. Let $b \in B \setminus \{1\}$. Since b is not an extreme point of C, there exists $g \in G \setminus \{1\}$ such that $gb, g^{-1}b \in B \subseteq C$. Hence b is not an extreme point of C. Note that

$$(b^{-1}g^{-1}b)b^{-1} = (gb)^{-1} \in B^{-1} \subseteq C \text{ and } (b^{-1}gb)b^{-1} = (g^{-1}b)^{-1} \in B^{-1} \subseteq C.$$

Hence b^{-1} is not an extreme point of C. Since $|B| = |A| \geq 2$, there exists $b \in B \setminus \{1\}$. Since $b \cdot 1, b^{-1} \cdot 1 \in C$, we have that 1 is not an extreme point of C. Hence C has no extreme points, a contradiction. Therefore the result follows. □

lem:LOimpliesdiffuse

Lemma 9.2.6. *Let G be a left ordered group. Then G is diffuse.*

Proof. Suppose that G is left ordered with respect to a total order $\leq$. Let $A \subseteq G$ be a finite non-empty subset. Let $a_1 \in A$ such that $a_1 \leq a$ for all $a \in A$. We shall prove that a_1 is an extreme point of A. Suppose that there exists $g \in G \setminus \{1\}$ such that $ga_1, g^{-1}a_1 \in A$. Hence $a_1 \leq ga_1$ and $a_1 \leq g^{-1}a_1$. Hence

$$g^{-1}a_1 \leq g^{-1}ga_1 = a_1 \leq g^{-1}a_1,$$

and thus $g^{-1}a_1 = a_1$, a contradiction because $g \neq 1$. Thus a_1 is an extreme point of A. Therefore G is diffuse. □

lemma:difuso=>2up

Lemma 9.2.7. *Let G be a diffuse group. Then G satisfies the unique product property.*

Proof. Suppose that G does not satisfy the unique product property. Thus, there exist finite non-empty subsets $A, B \subseteq G$ such that for all $(a, b) \in A \times B$, there exists $(a_1, b_1) \in A \times B \setminus \{(a, b)\}$ such that $ab = a_1b_1$. Let $C = AB = \{ab : a \in A,\ b \in B\}$. Let $c \in C$. There exist two distinct elements $(a, b), (a_1, b_1) \in A \times B$ such that $c = ab = a_1, b_1$. Note that $aa_1^{-1} \neq 1$ and

$$aa_1^{-1}c = aa_1^{-1}a_1b_1 = ab_1 \in C \text{ and } (aa_1^{-1})^{-1}c = a_1a^{-1}ab = a_1b \in C.$$

Hence c is not an extreme point of C. Therefore the result follows. □

Note that the above result shows that every diffuse group G is torsion-free.

9.3 The transfer map

Let G be a group and H be a finite index subgroup. We will define a group homomorphism $G \to H/[H,H]$, known as the *transfer map* of G on H.

lem:sigma **Lemma 9.3.1.** *Let G be a group and H be a subgroup of finite index $n = (G : H)$. Let $S = \{s_1, \dots, s_n\}$ and $T = \{t_1, \dots, t_n\}$ be left transversals of H in G. If $g \in G$, there exist unique $h_{1,g}, \dots, h_{n,g} \in H$ and a permutation $\sigma_g \in \mathbb{S}_n$ such that*

$$gt_i = s_{\sigma_g(i)} h_{i,g}, \quad i \in \{1, \dots, n\}.$$

Furthermore, if $s_i = t_i$ for all i, then $\sigma_{g_1 g_2} = \sigma_{g_1}\sigma_{g_2}$ and $h_{i,g_1g_2} = h_{\sigma_{g_2}(i),g_1} h_{i,g_2}$ for all $g_1, g_2 \in G$ and all i.

Proof. If $i \in \{1, \dots, n\}$, then there exists a unique $j \in \{1, \dots, n\}$ such that $gt_i \in s_j H$. Thus there exists a unique $h_{i,g} \in H$ such that $gt_i = s_j h_{i,g}$. Take $\sigma_g(i) = j$ and thus there is a well-defined map $\sigma_g \colon \{1, \dots, n\} \to \{1, \dots, n\}$. To prove that $\sigma_g \in \mathbb{S}_n$ it is enough to check that σ_g is injective. If $\sigma_g(i) = \sigma_g(k) = j$, since $gt_i = s_j h_{i,g}$ and $gt_k = s_j h_{k,g}$, it follows that $t_i^{-1} t_k = h_{i,g}^{-1} h_{k,g} \in H$. Hence $i = k$, as $t_i H = t_k H$.

Suppose that $s_i = t_i$ for all i. Let $g_1, g_2 \in G$. We have that

$$t_{\sigma_{g_1g_2}(i)} h_{i,g_1g_2} = g_1 g_2 t_i = g_1 t_{\sigma_{g_2}(i)} h_{i,g_2} = t_{\sigma_{g_1}\sigma_{g_2}(i)} h_{\sigma_{g_2}(i),g_1} h_{i,g_2}.$$

Hence $\sigma_{g_1g_2}(i) = \sigma_{g_1}\sigma_{g_2}(i)$ and $h_{i,g_1g_2} = h_{\sigma_{g_2}(i),g_1} h_{i,g_2}$, thus the result follows. □

Let G be a group and H be a subgroup of G of finite index n. If $T = \{t_1, \dots, t_n\}$ is a left transversal of H in G, we define the map

$$\nu_T : G \to H/[H,H], \quad \nu_T(g) = \prod_{i=1}^{n} \bar{h}_i$$

where $gt_i = t_j h_i$ and $\bar{h} = h[H,H] \in H/[H,H]$. Note that the product is well-defined since $H/[H,H]$ is an abelian group. We now prove that the map does not depend on the transversal.

lem:nu_T **Lemma 9.3.2.** *Let G be a group and H be a subgroup of G of finite index. if T and S are left transversals of H in G, then $\nu_T = \nu_S$.*

Proof. Assume that $n = (G : H), T = \{t_1, \dots, t_n\}$ and $S = \{s_1, \dots, s_n\}$, where $s_i = t_i k_i$, for some $k_i \in H$. Let $g \in G$. There exist $\sigma_g \in \mathbb{S}_n$ and $h_{i,g} \in H$ such that $gs_i = s_{\sigma_g(i)} h_{i,g}$ for all i. Let $l_{i,g} = k_{\sigma_g(i)} h_{i,g} k_i^{-1} \in H$. Then

$$gt_i = gs_i k_i^{-1} = s_{\sigma_g(i)} h_{i,g} k_i^{-1} = t_{\sigma_g(i)} k_{\sigma_g(i)} h_{i,g} k_i^{-1} = t_{\sigma_g(i)} l_{i,g}$$

for all $i \in \{1, \dots, n\}$. Moreover,

$$s_{\sigma_g(i)}^{-1} g s_i = k_{\sigma_g(i)}^{-1} t_{\sigma_g(i)}^{-1} g t_i k_i.$$

Since $H/[H,H]$ is abelian,

$$\begin{aligned}\nu_S(g) &= \prod_{i=1}^{n} \overline{s_{\sigma_g(i)}^{-1} g s_i} = \prod_{i=1}^{n} \bar{k}_{\sigma_g(i)}^{-1} \overline{t_{\sigma_g(i)}^{-1} g t_i} \bar{k}_i \\ &= \prod_{i=1}^{n} \bar{k}_{\sigma(i)}^{-1} \prod_{i=1}^{n} \bar{k}_i \prod_{i=1}^{n} \overline{t_{\sigma(i)}^{-1} g t_i} = \prod_{i=1}^{n} \overline{t_{\sigma(i)}^{-1} g t_i} = \nu_T(g). \qquad \square\end{aligned}$$

By Lemma 9.3.2, if H is a finite-index subgroup of G, the map

$$\nu\colon G \to H/[H,H], \quad \nu(g) = \nu_T(g),$$

where T is some left transversal of H in G, is well-defined.

theorem:transfer

Theorem 9.3.3. *Let G be a group and H be a finite-index subgroup of G. Then $\nu(xy) = \nu(v)\nu(y)$ for all $x, y \in G$.*

Proof. Let $T = \{t_1, \dots, t_n\}$ be a left transversal of H in G, where $n = (G : H)$. By Lemma 9.3.1, for every $g \in G$ there exist unique elements $h_{1,g}, \dots, h_{n,g} \in H$ and a permutation $\sigma_g \in \mathbb{S}_n$ such that $g t_i = t_{\sigma_g(i)} h_{i,g}$. Furthermore $\sigma_{xy} = \sigma_x \sigma_y$ and $h_{i,xy} = h_{\sigma_y(i),x} h_{i,y}$ for all $x, y \in G$ and all i. Since $H/[H,H]$ is abelian,

$$\nu(xy) = \prod_{i=1}^{n} h_{i,xy} = \prod_{i=1}^{n} h_{\sigma_y(i),x} h_{i,y} = \prod_{i=1}^{n} h_{\sigma_y(i),x} \prod_{i=1}^{n} h_{i,y} = \nu(x)\nu(y). \qquad \square$$

If G is a group and H is a finite-index subgroup of G, the *transfer homomorphism* is the group homomorphism $\nu\colon G \to H/[H,H]$, $\nu(g) = \nu_T(g)$, for some left transversal T of H in G.

lem:evaluation

Lemma 9.3.4. *Let G be a group and H be a subgroup of G with $(G : H) = n$. Let $T = \{t_1, \dots, t_n\}$ be a left transversal of H in G. For each $g \in G$ there exist a positive integer m, m distinct elements $s_1, \dots, s_m \in T$ and positive integers $n_1, \dots, n_m$ such that*

$$s_i^{-1} g^{n_i} s_i \in H, \quad n_1 + \cdots + n_m = n \quad \text{and} \quad \nu(g) = \prod_{i=1}^{m} s_i^{-1} g^{n_i} s_i.$$

Proof. Let $g \in G$. For each i there exist $h_1, \dots, h_n \in H$ and $\sigma \in \mathbb{S}_n$ such that $g t_i = t_{\sigma(i)} h_i$. Write σ as a product

$$\sigma = \alpha_{k+1} \cdots \alpha_m$$

of disjoint cycles and $|\{j : \sigma(j) = j\}| = k$. Note that if $\sigma(j) = j$, then $g t_j = t_j h_j$, and thus $t_j^{-1} g t_j \in H$. Thus we take $\{s_1, \dots, s_k\} = \{t_j : \sigma(j) = j\}$ and $n_1 = \cdots = n_k = 1$.

Fix $i \in \{k+1, \dots, m\}$ and write $\alpha_i = (j_1 \cdots j_{n_i})$. Since

$$g t_{j_k} = t_{\sigma(j_k)} h_{j_k} = \begin{cases} t_{j_1} h_{n_k} & \text{if } k = n_i, \\ t_{j_{k+1}} h_k & \text{otherwise,} \end{cases}$$

it follows that

$$\begin{aligned} t_{j_1}^{-1} g^{n_i} t_{j_1} &= t_{j_1}^{-1} g^{n_i-1} g t_{j_1} \\ &= t_{j_1}^{-1} g^{n_i-1} t_{j_2} h_{j_1} \\ &= t_{j_1}^{-1} g^{n_i-2} g t_{j_2} h_{j_1} \\ &= t_{j_1}^{-1} g^{n_i-2} t_{j_3} h_{j_2} h_{j_1} \\ &\vdots \\ &= t_{j_1}^{-1} g t_{j_{n_i}} h_{j_{n_i-1}} \cdots h_{j_2} h_{j_1} \\ &= t_{j_1}^{-1} t_{j_1} h_{j_{n_i}} \cdots h_{j_2} h_{j_1} \in H. \end{aligned}$$

So we let $s_i = t_{j_1} \in T$. Now the claim follows, since $\nu(g) = \bar{h}_1 \cdots \bar{h}_n$. □

prop:v(g)=g^n

Proposition 9.3.5. *Let G be a group and H be a central subgroup of index n. Then $\nu(g) = g^n$ for all $g \in G$.*

Proof. Let $g \in G$. Let T be a left transversal of H in G. By Lemma 9.3.4, there exist $s_1, \ldots, s_m \in T$ and positive integers $n_1, \ldots, n_m$ such that $\sum_{i=1}^m n_i = n$, $s_i^{-1} g^{n_i} s_i \in H$ and $\nu(g) = \prod_{i=1}^m s_i^{-1} g^{n_i} s_i$. Since H is central in G, then it is normal in G. Thus

$$g^{n_i} = s_i (s_i^{-1} g^{n_i} s_i) s_i^{-1} \in H \subseteq Z(G)$$

and hence

$$\nu(g) = \prod_{i=1}^m s_i^{-1} g^{n_i} s_i = \prod_{i=1}^m g^{n_i} = g^{\sum_{i=1}^m n_i} = g^n.$$ □

The next result is an application to finite groups.

prop:semidirecto

Proposition 9.3.6. *Let G be a finite group and H a central subgroup of index n, where n is coprime with $|H|$. Then $G \cong N \times H$.*

Proof. Let $\nu \colon G \to H$ be the transfer homomorphism. By Proposition 9.3.5, $\nu(g) = g^n$ for all $g \in G$. In particular, since the order of $h \in H$ and n are coprime, there exists a positive integer k such that $h^{kn} = h$. Hence ν is surjective. Let $N = \ker \nu$. Then $|G| = |N||H| = n|H|$. Hence $|N| = n$. Since n and $|H|$ are coprime, we have that $N \cap H = \{1\}$ and $G = NH \cong N \times H$, because N and H are normal subgroups of G. □

An application to infinite groups.

thm:Zfiniteindex

Theorem 9.3.7. *Let G be a torsion-free group that contains a finite-index subgroup isomorphic to $\mathbb{Z}$. Then $G \simeq \mathbb{Z}$.*

Proof. We may assume that G contains a finite-index normal subgroup isomorphic to $\mathbb{Z}$. Indeed, if H is a finite-index subgroup of G such that $H \simeq \mathbb{Z}$, and T is a left transversal of H in G, then $K = \bigcap_{x \in T} xHx^{-1}$ is a finite-index normal subgroup of G.

Since $K \subseteq H \subseteq G$ and $(G : K) = (G : H)(H : K)$ is finite, we have that $K \cong \mathbb{Z}$. The action of G on K by conjugation induces a group homomorphism $\epsilon\colon G \to \mathrm{Aut}(K)$. Since $\mathrm{Aut}(K) \cong \mathrm{Aut}(\mathbb{Z}) = \{-1, 1\}$, there are two cases to consider.

Assume first that $\epsilon(g) = \mathrm{id}$ for all $g \in G$. In this case, $K \subseteq Z(G)$, let $\nu\colon G \to K$ be the transfer homomorphism. By Proposition 9.3.5, $\nu(g) = g^n$, where $n = (G : K)$. Since G has no torsion, ν is injective. Thus $G \cong \mathbb{Z}$ because it is isomorphic to a non-trivial subgroup of K.

Assume now that there exists $g \in G$ such that $\epsilon(g) \neq \mathrm{id}$. Let $N = \ker\epsilon \neq G$. Since $K \cong \mathbb{Z}$ is abelian, $K \subseteq N$. The result proved in the previous paragraph applied to $\epsilon|_N$ implies that $N \cong \mathbb{Z}$, as N contains a finite-index subgroup isomorphic to $\mathbb{Z}$. Let $g \in G \setminus N$. Since N is normal in G, G acts by conjugation on N and hence there exists a group homomorphism $c_g \in \mathrm{Aut}(N) \simeq \{-1, 1\}$ defined by $c_g(n) = gng^{-1}$ for all $n \in N$. Since $K \subseteq N$ and g acts non-trivially on K,

$$c_g(n) = gng^{-1} = n^{-1}$$

for all $n \in N$. Since $g^2 \in N$,

$$g^2 = gg^2g^{-1} = g^{-2}.$$

Therefore $g^4 = 1$, a contradiction since $g \neq 1$ and G has no torsion. Therefore $\epsilon(g) = \mathrm{id}$ for all $g \in G$ and the result follows. □

9.4 Bieberbach groups

We begin this section with an elementary property of the subgroups of finite index.

lem:fgfisubgroup

Lemma 9.4.1. *Let G be a group. Let H be a subgroup of finite index of G. Then H is finitely generated if and only if G is finitely generated.*

Proof. Suppose that H is finitely generated. Let A be a finite subset of H such that $H = \langle A\rangle$. Let T be a left transversal of H in G. It is clear that $G = \langle T \cup A\rangle$, and thus G is finitely generated.

Conversely, suppose that G is finitely generated. Assume that $1 \in T$. Let X be a finite subset of G such that $G = \langle X\rangle$. We assume that $x \in X$ if and only if $x^{-1} \in X$. Let $F = T \cup X = \{f_1, f_2, \ldots, f_n\}$, with $f_1 = 1$. For $f_i, f_j \in F$, there exist $t_{i,j} \in T$ and $h_{i,j} \in H$ such that $f_i f_j = t_{i,j} h_{i,j}$. Let $W = \langle h_{i,j} : 1 \leq i, j \leq n\rangle$. Let $g \in G$. There exist $f_{i_1}, \ldots, f_{i_m} \in X \subseteq F$ such that $g = f_{i_1} \cdots f_{i_m}$. We shall prove that $g \in \bigcup_{t\in T} tW$, by induction on m. If $m = 1$, then $g = f_{i_1} = t_{i_1,1} h_{i_1,1} \in \bigcup_{t\in T} tW$. Suppose that $m > 1$ and that every product of $m - 1$ elements of F belongs to $\bigcup_{t\in T} tW$. We have that

$$\begin{aligned} g =& f_{i_1} \cdots f_{i_m} \\ =& f_{i_1} \cdots f_{i_{m-2}} t_{i_{m-1},i_m} h_{i_{m-1},i_m} \in \bigcup_{t\in T} tW, \end{aligned}$$

by the inductive hypothesis. By induction, $G = \bigcup_{t\in T} tW$. Since $tW \subseteq tH$ and $G = \bigcup_{t\in T} tH$ is a disjoint union, we have that $tW = tH$ for all $t \in T$. In particular $H = W$ is finitely generated. □

The following result is due to Schur.

lem:Schurcenter

Lemma 9.4.2. *Let G be a group such that $(G : Z(G)) = n < \infty$. Then $[G,G]$ is finite and $|[G,G]| \leq n^{n(n-1)}$.*

Proof. Let $\nu\colon G \to Z(G)$ be the transfer homomorphism. By Proposition 9.3.5, $\nu(g) = g^n$ for all $g \in G$. Since $[G,G] \in \ker\nu$, we have that $h^n = 1$ for all $h \in [G,G]$. Let $T = \{t_1,\dots,t_n\}$ be a left transversal of $Z(G)$ in G. Let

$$A = \{[g,h] : g,h \in G\}.$$

Note that

$$A = \{[t_i,t_j] : 1 \leq i,j \leq n\}.$$

Since $[t_i,t_j]^{-1} = [t_j,t_i]$, every element of $[G,G]$ is a product of elements of A. Note that $|A \setminus \{1\}| = k \leq n(n-1)$. Let $A \setminus \{1\} = \{a_1,\dots,a_k\}$. We shall prove that

$$[G,G] = \{a_1^{n_1}\cdots a_k^{n_k} : 0 \leq n_i < n \text{ for all } i = 1,\dots,k\}.$$

Let $B = \{a_1^{n_1}\cdots a_k^{n_k} : 0 \leq n_i < n \text{ for all } i = 1,\dots,k\}$. Let $h \in [G,G]$. There exist $a_{i_1},\dots,a_{i_m} \in A \setminus \{1\}$ such that $h = a_{i_1}\cdots a_{i_m}$. We shall prove that $h = a_1^{n_1}\cdots a_k^{n_k} \in B$, for some non-negative integers $n_1,\dots,n_k$ such that $n_1+\cdots+n_k \leq m$, by induction on m. For $m = 1$, it is clear that $h = a_{i_1} \in B$. Suppose that $m > 1$ and that $a_{j_1}\cdots a_{j_l} = a_1^{n_1}\cdots a_k^{n_k} \in B$, for some non-negative integers $n_1,\dots,n_k < n$ such that $n_1+\cdots+n_k \leq l$, for all $l < m$. Since

$$[x,y][u,v] = [u,v][u,v]^{-1}[x,y][u,v] = [u,v][[u,v]^{-1}x[u,v],[u,v]^{-1}y[u,v]] \tag{9.1}$$

eq:Bieberbach1

for all $x,y,u,v \in G$, we may assume that i_1 is the smallest positive integer in all the representations of h as a product of m elements in $A \setminus \{1\}$. By the inductive hypothesis

$$a_{i_2}\cdots a_{i_m} = a_1^{n_1}\cdots a_k^{n_k} \in B,$$

for some non-negative integers $n_1,\dots,n_k < n$ such that $n_1+\cdots+n_k \leq m-1$. Hence, by the choice of i_1 and (9.1), we have that $h = a_{i_1}a_1^{n_1}\cdots a_k^{n_k} = a_{i_1}^{n_{i_1}+1}a_{i_1+1}^{n_{i_1+1}}\cdots a_k^{n_k}$. Note that $n_{i_1}+1 \leq n$ and $n_{i_1}+1+n_{i_1+1}+\cdots+n_k \leq m$. If $n_{i_1}+1 < n$, then clearly $h \in B$. If $n_{i_1}+1 = n$, then

$$h = a_{i_1+1}^{n_{i_1+1}}\cdots a_k^{n_k} \in B.$$

Hence, by induction, $[G,G] = B$. Clearly $|[G,G]| \leq n^k \leq n^{n(n-1)}$, and the result follows. □

The *FC-center* of a group G is the set

$$\Delta(G) = \{g \in G : (G : C_G(g)) < \infty\}.$$

Note that $\Delta(G)$ is the set of all elements in G that have finitely many conjugates, i. e.

$$\Delta(G) = \{g \in G : |\{xgx^{-1} : x \in G\}| < \infty\}.$$

lem:Delta(G)

Lemma 9.4.3. *Let G be a group. Then $\Delta(G)$ and $T(\Delta(G)) = \{g \in \Delta(G) : o(g) < \infty\}$ are a characteristic subgroups of G and $\Delta(G)/T(\Delta(G))$ is an abelian torsion-free group.*

Proof. Note that $1 \in T(\Delta(G))$. Let $g, h \in \Delta(G)$. Then $xgh^{-1}x^{-1} = (xgx^{-1})(xhx^{-1})^{-1}$ for all $x \in G$. Since g and h have finitely many conjugates, we get that gh^{-1} also have finitely many conjugates. Hence $\Delta(G)$ is a subgroup of G. Let $\alpha \in \mathrm{Aut}(G)$. Then $x\alpha(g)x^{-1} = \alpha(\alpha^{-1}(x)g\alpha^{-1}(x)^{-1})$, for all $x \in G$. Hence $\alpha(g) \in \Delta(G)$ and thus $\Delta(G)$ is a characteristic subgroup of G. Let $H = \langle g, h\rangle$. Note that $\Delta(H) = H$ and $Z(H) = C_H(g) \cap C_H(h)$. Thus $(H : Z(H)) < \infty$. By Lemma 9.4.2, $[H, H]$ is finite. Hence $T(H)$ is a finite subgroup. In particular, if $g, h \in T(\Delta(G))$, then $\langle g, h\rangle \subseteq T(\Delta(G))$, and this shows that $T(\Delta(G))$ is a subgroup of G. Clearly $T(\Delta(G))$ is characteristic in $\Delta(G)$. Since $\Delta(G)$ is characteristic in G, it follows that $T(\Delta(G))$ is characteristic in G. Since $[g, h] \in T(\Delta(G))$ for all $g, h \in \Delta(G)$, it is clear that $\Delta(G)/T(\Delta(G))$ is torsion-free abelian. □

A group G is said to be a *Bieberbach* group if it is a finitely generated torsion-free group with an abelian group of finite index.

prop:Bieberbach2

Proposition 9.4.4. *Let G be a Bieberbach group. Then $\Delta(G)$ is a torsion-free finitely generated abelian normal subgroup of G of finite index. Futhermore $A \subseteq \Delta(G)$ for every abelian subgroup A of G of finite index.*

Proof. Let A be an abelian subgroup of G of finite index. Note that for every $a \in A$, $A \subseteq C_G(a)$. Hence $A \subseteq \Delta(G)$. By Lemmas 9.4.3 and 9.4.1, the result follows. □

The *dimension* of a Bieberbach group G is the rank of $\Delta(G)$.

lem:SubBieberbeach

Lemma 9.4.5. *Let G be a Bieberbach group. Then every subgroup H of G is a Bieberbach group and* $\dim(H) \leq \dim(G)$.

Proof. Let H be a subgroup of G. Then $(H : H \cap \Delta(G)) < \infty$. Let $n = \dim(G)$. Thus $\Delta(G) \cong \mathbb{Z}^n$. Hence $H \cap \Delta(G)$ is a finitely generated subgroup of $\Delta(G)$, and thus $H \cap \Delta(G) \cong \mathbb{Z}^m$ for some $m \leq n$. Since $(H : H \cap \Delta(G)) < \infty$ and $H \cap \Delta(G)$ is finitely generated, we have that H also is finitely generated by Lemma 9.4.1. Hence H is a Bieberbach group. Note that $H \cap \Delta(G) \subseteq \Delta(H)$. Since $(\Delta(H) : H \cap \Delta(G)) < \infty$, the rank of $\Delta(H)$ is m. Thus $\dim(H) = m \leq n = \dim(G)$. □

lem:semidirectBieberbach

Lemma 9.4.6. *Let G be a group with a torsion-free abelian normal subgroup A of finite index. Then G is isomorphic to a subgroup H of the semidirect product $A \rtimes_\alpha G/A$, where the action α is naturally induced by conjugation. Furthermore*

$$H \cap (A \times \{A\}) = \{a^n : a \in A\} \times \{A\},$$

where $n = (G : A)$.

Proof. Let $n = (G : A)$ and let T be a transversal of A in G such that $1 \in T$. For each $g \in G$ there exist a unique $a_g \in A$ and a unique $t_g \in T$ such that $g = a_g t_g$. We denote $a_{t_g t_h}$ by $c(t_g, t_h)$ for all $g, h \in G$. Hence

$$t_g t_h = c(t_g, t_h) t_{gh}$$

for all $g, h \in G$. Note that $c(t_1, t) = c(1, t) = 1 = c(t, 1) = c(t, t_1)$ for all $t \in T$. The associativity of G implies that

$$c(t_{g_1}, t_{g_2}) c(t_{g_1 g_2}, t_{g_3}) = t_1 c(t_{g_2}, t_{g_3}) t_1^{-1} c(t_{g_1}, t_{g_2 g_3})$$

for all $g_1, g_2, g_3 \in G$. Since A is abelian and $n = (G : A)$, multiplying this equality for all $g_3 \in T$, we have that

$$c(t_{g_1}, t_{g_2})^n \prod_{t \in T} c(t_{g_1 g_2}, t) = t_{g_1} \left(\prod_{t \in T} c(t_{g_2}, t) \right) t_{g_1}^{-1} \prod_{t \in T} c(t_{g_1}, t). \tag{9.2}$$

eq:SemiB1

for all $g_1, g_2 \in G$. Let $f \colon G \to A \rtimes_\alpha G/A$ be the map defined by

$$f(g) = f(a_g t_g) = \left(a_g^n \prod_{t \in T} c(t_g, t), gA \right)$$

for all $g \in G$. It is clear that $f(g) \in A \times \{A\}$ if and only if $t_g = 1$, and in this case $f(g) = (g^n, A)$. Note that for $g_1, g_2 \in G$ we have

$$\begin{aligned} f(g_1 g_2) &= f(a_{g_1} t_{g_1} a_{g_2} t_{g_2}) \\ &= f(a_{g_1} t_{g_1} a_{g_2} t_{g_1}^{-1} c(t_{g_1}, t_{g_2}) t_{g_1 g_2}) \\ &= \left(a_{g_1}^n t_{g_1} a_{g_2}^n t_{g_1}^{-1} c(t_{g_1}, t_{g_2})^n \prod_{t \in T} c(t_{g_1 g_2}, t), g_1 g_2 A \right) \end{aligned}$$

and

$$\begin{aligned} f(g_1) f(g_2) &= f(a_{g_1} t_{g_1}) f(a_{g_2} t_{g_2}) \\ &= \left(a_{g_1}^n \prod_{t \in T} c(t_{g_1}, t), g_1 A \right) \left(a_{g_2}^n \prod_{t \in T} c(t_{g_2}, t), g_2 A \right) \\ &= \left(a_{g_1}^n \prod_{t \in T} c(t_{g_1}, t) t_{g_1} a_{g_2}^n t_{g_1}^{-1} t_{g_1} \left(\prod_{t \in T} c(t_{g_2}, t) \right) t_{g_1}^{-1}, g_1 g_2 A \right). \end{aligned}$$

By (9.2), we get that $f(g_1 g_2) = f(g_1) f(g_2)$. Hence f is a homomorphism of groups. Let $g \in \ker f$. Then $gA = A$, and thus $t_g = 1$. Hence

$$(1, A) = f(g) = \left(a_g^n \prod_{t \in T} c(1, t), A \right) = (a_g^n, A),$$

and, since A is torsion-free, we get that $a_g = 1$. Hence f is injective and the result follows. □

We denote by $\mathrm{Isom}(\mathbb{R}^n)$ the group of isometries of the euclidean space $\mathbb{R}^n$. Let $T(\mathbb{R}^n)$ be the subgroup of translations of $\mathrm{Isom}(\mathbb{R}^n)$ and let $O(\mathbb{R}^n)$ be the subgroup of orthogonal automorphism of $\mathbb{R}^n$. It is known that $T(\mathbb{R}^n)$ is normal in $\mathrm{Isom}(\mathbb{R}^n)$, $\mathrm{Isom}(\mathbb{R}^n) = T(\mathbb{R}^n)O(\mathbb{R}^n)$ and $T(\mathbb{R}^n) \cap O(\mathbb{R}^n) = \{\mathrm{id}\}$. Hence every $g \in \mathrm{Isom}(\mathbb{R}^n)$ can be written uniquely as $g = tf$ with $t \in T(\mathbb{R}^n)$ and $f \in O(\mathbb{R}^n)$.

thm:GeomBieberbach

Theorem 9.4.7. *Let G be an n-dimensional Bieberbach group. Then G is isomorphic to a subgroup H of the group* $\mathrm{Isom}(\mathbb{R}^n)$ *such that $\Delta(G)$ corresponds to the translations of H, that is $\Delta(H) = H \cap T(\mathbb{R}^n)$.*

Proof. By Lemma 9.4.6, it is enough to show that $\Delta(G) \rtimes G/\Delta(G)$ is isomorphic to a subgroup H of $\mathrm{Isom}(\mathbb{R}^n)$ and $\Delta(H) = H \cap T(\mathbb{R}^n)$. Since G is an n-dimensional Bieberbach group, there is an isomorphism $f\colon \Delta(G) \to \mathbb{Z}^n$. Let $\sigma\colon G \to \mathrm{Aut}(\mathbb{Z}^n)$ be the map defined by $\sigma(g) = \sigma_g$ and $\sigma_g(a_1, \dots a_n) = f(gf^{-1}(a_1, \dots, a_n)g^{-1})$ for all $g \in G$ and $(a_1, \dots, a_n) \in \mathbb{Z}^n$. It is straightforward to check that $\sigma_g \in \mathrm{Aut}(\mathbb{Z}^n)$ and that σ is a homomorphism of groups. Note that $\ker\sigma = C_G(\Delta(G))$. By Proposition 9.4.4, $\Delta(G) = C_G(\Delta(G))$. Hence $\sigma(G) \cong G/\Delta(G)$. Note that every automorphism of $\mathbb{Z}^n$ can be naturally extended to an automorphism of $\mathbb{R}^n$. Hence we can think $\sigma(G)$ as a finite subgroup of $\mathrm{Aut}(\mathbb{R}^n)$.

Let $\langle u \mid v\rangle$ be the standard inner product of $u, v \in \mathbb{R}^n$. We define a new inner product on $\mathbb{R}^n$ by

$$\langle u \mid v\rangle' = \sum_{g \in \sigma(G)} \langle g(u) \mid g(v)\rangle$$

for all $u, v \in \mathbb{R}^n$. Let $e_1', \dots, e_n'$ be an orthonormal basis for $\langle . \mid .\rangle'$. Let $\alpha \in \mathrm{Aut}(\mathbb{R}^n)$ be the map defined by $\alpha(a_1, \dots, a_n) = \sum_{i=1}^n a_i e_i'$. Note that

$$\langle u \mid v\rangle = \langle \alpha(u) \mid \alpha(v)\rangle'$$

for all $u, v \in \mathbb{R}^n$. If $g \in \sigma(G)$, then

$$\begin{aligned}\langle \alpha^{-1}g\alpha(u), \alpha^{-1}g\alpha(v)\rangle =& \langle g\alpha(u), g\alpha(v)\rangle' \\ =& \langle \alpha(u), \alpha(v)\rangle' = \langle u, v\rangle\end{aligned}$$

for all $u, v \in \mathbb{R}^n$. Hence $\alpha^{-1}g\alpha$ is in the orthogonal group of $\mathbb{R}^n$ for all $g \in \sigma(G)$.

Let $\Psi\colon \Delta(G) \rtimes G/\Delta(G) \to \mathrm{Isom}(\mathbb{R}^n)$ be the map defined by

$$\Psi(h, g\Delta(G))(a_1, \dots, a_n) = \alpha^{-1}(f(h)) + \alpha^{-1}\sigma(g)\alpha(a_1, \dots, a_n)$$

for all $h \in \Delta(g)$, $g \in G$ and $(a_1, \dots, a_n) \in \mathbb{R}^n$. For $h_1, h_2 \in \Delta(G)$, $g_1, g_2 \in G$ and $(a_1, \dots, a_n) \in \mathbb{R}^n$, we have that

$$\begin{aligned}
&\Psi(h_1, g_1\Delta(G))\Psi(h_2, g_2\Delta(G))(a_1, \dots, a_n) \\
&\qquad = \Psi(h_1, g_1\Delta(G))(\alpha^{-1}(f(h_2)) + \alpha^{-1}\sigma(g_2)\alpha(a_1, \dots, a_n)) \\
&\qquad = \alpha^{-1}(f(h_1)) + \alpha^{-1}\sigma(g_1)\alpha(\alpha^{-1}(f(h_2)) + \alpha^{-1}\sigma(g_2)\alpha(a_1, \dots, a_n)) \\
&\qquad = \alpha^{-1}(f(h_1)) + \alpha^{-1}\sigma(g_1)(f(h_2)) + \alpha^{-1}\sigma(g_1)\sigma(g_2)\alpha(a_1, \dots, a_n)) \\
&\qquad = \alpha^{-1}(f(h_1)) + \alpha^{-1}(f(g_1 h_2 g_1^{-1})) + \alpha^{-1}\sigma(g_1 g_2)\alpha(a_1, \dots, a_n)) \\
&\qquad = \alpha^{-1}(f(h_1 g_1 h_2 g_1^{-1})) + \alpha^{-1}\sigma(g_1 g_2)\alpha(a_1, \dots, a_n)) \\
&\qquad = \Psi(h_1 g_1 h_2 g_1^{-1}, g_1 g_2\Delta(G))(a_1, \dots, a_n) \\
&\qquad = \Psi((h_1, g_1\Delta(G))(h_2, g_2\Delta(G))(a_1, \dots, a_n).
\end{aligned}$$

Hence Ψ is a homomorphism of groups. Let $h \in \Delta(G)$ and $g \in G$ such that $\Psi(h, g\Delta(G)) = \mathrm{id}$. In particular,

$$(0, \dots, 0) = \Psi(h, g\Delta(G))(0, \dots 0) = \alpha^{-1}(f(h)),$$

and thus $h = 1$. Now we have that

$$\alpha^{-1}(f(x)) = \Psi(1, g\Delta(G))(\alpha^{-1}(f(x))) = \alpha^{-1}\sigma_g(f(x)) = \alpha^{-1}(f(gxg^{-1}))$$

for all $x \in \Delta(G)$. Hence $g \in C_G(\Delta(G)) = \Delta(G)$. Therefore $g\Delta(G) = \Delta(G)$. This shows that Ψ is injective, and the result follows. □

hm:leftorderedBieberbach

Theorem 9.4.8. *Let G be a Bieberbach group. Then the following conditions are equivalent.*

1) *Every non-trivial subgroup of G has non-trivial center.*
2) *G is locally indicable.*
3) *G is a poly-$\mathbb{Z}$ group.*
4) *G is left ordered.*
5) *G is diffuse.*

Proof. (1) $\implies$ (2). Suppose that every non-trivial subgroup of G has a non-trivial center. Let H be a non-trivial finitely generated subgroup of G. Let $z \in Z(H) \setminus \{1\}$. By Lemma 9.4.5, H is a Bieberbach group. Let $\nu\colon H \to \Delta(H)$ be the transfer homomorphism. Let $n = (H : \Delta(H))$. Then $\nu(z) = z^n \neq 1$. Hence $\nu(H)$ is an infinite finitely generated abelian group. Therefore there exists a surjective homomorphism of groups $f\colon \nu(H) \to \mathbb{Z}$ and thus $f\nu\colon H \to \mathbb{Z}$ is a surjective homomorphism of groups. This shows that G is locally indicable.

(2) $\implies$ (3). Suppose that G is locally indicable. We shall show that G is a poly-$\mathbb{Z}$ group by induction on the dimension of G. If G has dimension 1, then by Theorem 9.3.7, $G \cong \mathbb{Z}$. Suppose that $\dim(G) = n > 1$ and that every locally indicable Bieberbach group of dimension $n-1$ is a poly-$\mathbb{Z}$ group. Since G is locally indicable and finitely generated, there exists a surjective homomorphism $f\colon G \to \mathbb{Z}$. Let $H = \ker f$. Let $g \in G$ be an element such that $f(g) = 1$. Note that $G = H\langle g\rangle$ and $H \cap \langle g\rangle = \{1\}$. By Lemma 9.4.5, H is a Bieberbach group. Let $A_1 = H \cap \Delta(G)$ and $A_2 = \langle g\rangle \cap \Delta(G)$. Note that A_1A_2 is an abelian subgroup of finite index in G.

Since A_1 is a subgroup of finite index of H, we have that the rank of A_1 is the dimension of H. Since A_2 is a non-trivial subgroup of $\langle g \rangle$, we have that $A_2 \cong \mathbb{Z}$. Hence $\dim(G) = \dim(H) + 1$. Therefore H has dimension $n-1$ an it is loccally indicable. By the inductive hypothesis, H is a poly-$\mathbb{Z}$ group. Therefore, G also is a poly-$\mathbb{Z}$ group. Thus the result follows by induction.

(3) $\implies$ (4). Since $\mathbb{Z}$ is an ordered group, the result is an easy consequence of Proposition 9.1.4.

(4) $\implies$ (5). This follows by Lemma 9.2.6

(5) $\implies$ (1). Suppose that there exists a non-trivial subgroup H of G such that $Z(H) = \{1\}$. We shall prove that H is not diffuse, and therefore G is not diffuse and the result follows.

By Lemma 9.4.5, H is a Bieberbach group. Let $n = \dim(H)$. By Theorem 9.4.7, H is isomorphic to a subgroup H_1 of $\mathrm{Isom}(\mathbb{R}^n)$ such that $\Delta(H)$ corresponds to the translations of H_1.

Note that if $u \in \mathbb{R}^n$ and $r \in \mathbb{R}$ is positive, then

$$B(u,r) = \{t \in \Delta(H_1) : \|t(0) - u\| < r\}$$

is finite. Since $(H_1 : \Delta(H_1)) = n$,

$$B'(0,r) = \{h \in H : \|h(0)\| < r\}$$

is finite. We shall prove that $B'(0,r)$ has no extreme points for all sufficiently large $r > 0$.

Let $r_0 > 0$ such that $B(u, r_0) \neq \emptyset$ for all $u \in \mathbb{R}^n$. Note that every element $h \in H_1$ can be uniquely written as $h = t_h f_h$ with $t_h \in T(\mathbb{R}^n)$ and $f_h \in O(\mathbb{R}^n)$. Furthermore $H_1/\Delta(H_1) \cong (T(\mathbb{R}^n)H_1)/T(\mathbb{R}^n) \cong \langle f_h : h \in H_1 \rangle = \{f_h : h \in H_1\}$.

Let $F = \{v \in \mathbb{R}^n : f_g(v) = v \text{ for all } g \in H_1\}$. Let $v \in F$. Let

$$w = \sum_{t \in B(v,r_0)} t(0) = \left(\prod_{t \in B(v,r_0)} t \right)(0).$$

Suppose that $v = (v_1, \ldots, v_n) \neq (0, \ldots, 0)$. Since $sv \in F$ for all $s \in \mathbb{R}$, we may assume that $v_i > r_0$. If $t \in B(v, r_0)$ and $t(0) = (a_1, \ldots, a_n)$, then $0 < v_i - r_0 < a_i < v_i + r_0$. Hence $w \neq 0$. Note that if $t \in B(v, r_0)$, then

$$\|f_g t f_g^{-1}(0) - v\| = \|f_g t(0) - f_g(v)\| = \|t(0) - v\| < r_0$$

for all $g \in H_1$. Now we have,

$$\begin{aligned} f_g(w) &= \sum_{t \in B(v,r_0)} f_g(t(0)) \\ &= \sum_{t \in B(v,r_0)} f_g t f_g^{-1}(0) \end{aligned}$$

$$= \sum_{t \in B(v,r_0)} t(0) = w \neq 0.$$

Furthermore,

$$\begin{aligned} g\left(\prod_{t \in B(v,r_0)} t\right) g^{-1}(0) &= f_g\left(\prod_{t \in B(v,r_0)} t\right) f_g^{-1}(0) \\ &= \sum_{t \in B(v,r_0)} f_g t f_g^{-1}(0) \\ &= \sum_{t \in B(v,r_0)} t(0) \\ &= \left(\prod_{t \in B(v,r_0)} t\right)(0) \neq 0, \end{aligned}$$

for all $g \in H_1$. Hence $\prod_{t \in B(v,r_0)} t \in Z(H_1) \setminus \{\mathrm{id}\}$, a contradiction because $Z(H_1) = \{\mathrm{id}\}$. Hence $F = \{(0, \ldots, 0)\}$.

Hence, for every nonzero vector $u \in \mathbb{R}^n$, there exists $h_u \in H_1$ such that $f_{h_u}(u) \neq u$, and thus

$$\|f_{h_u}(u) + u\| < 2\|u\|.$$

Hence, there exists $0 < \delta_u < 1$ such that

$$\|f_{h_u}(u) + u\| \leq 2\delta_u \|u\|.$$

Let $S = \{u \in \mathbb{R}^n : \|u\| = 1\}$. Let $V_u = \{v \in \mathbb{R}^n : \|u - v\| < \frac{1-\delta_u}{2}\}$. Note that

$$\begin{aligned} \|f_{h_u}(v) + v\| &\leq \|f_{h_u}(v) - f_{h_u}(u)\| + \|f_{h_u}(u) + u\| + \|v - u\| \\ &< 1 - \delta_u + 2\delta_u = 1 + \delta_u. \end{aligned}$$

Since S is compact and $S \subseteq \bigcup_{u \in S} V_u$, there exists a finite subset $K \subseteq S$ such that

$$S \subseteq \bigcup_{u \in K} V_u.$$

Let $\delta = \max\{\frac{1+\delta_u}{2} : u \in K\}$. Note that $0 < \delta < 1$. Then it is easy to check that for all $u \in \mathbb{R}^n$ there exists $h_u \in H_1 \setminus \Delta(H_1)$ such that

$$\|f_{h_u}(u) + u\| \leq 2\delta \|u\|.$$

Let

$$r > \frac{r_0}{1 - \delta}.$$

Let $h \in B'(0, r)$. Hence $\|h(0)\| < r$. Let $u = h(0)$. We have seen that there exists $h \in H_1 \setminus \Delta(H_1)$ such that

$$\|f_{h_u}(u)+u\| \leq 2\delta\|u\|.$$

Let $v = \frac{1}{2}(-f_{h_u}(u)+u)$. Let $t \in B(v-h_u(0), r_0)$. Since $\|u\| = \|f_{h_u}(u)\| < r$, we have that

$$\|u-t(0)-h_u(0)\| \leq \|u-v\| + \|v-t(0)-h_u(0))\| \leq \delta r + r_0 < r$$

and

$$\|f_{h_u}(u)+t(0)+h_u(0)\| \leq \|f_{h_u}(u)+v\| + \|-v+t(0)+h_u(0)\| \leq \delta r + r_0 < r.$$

Let $g = th_u$. Note that $g \in H_1$. Since $f_{h_u} \neq \mathrm{id}$, we have that $g \neq 1$. Note that

$$\|gh(0)\| = \|g(u)\| = \|t(0)+h_u(0)+f_{h_u}(u)\| < r$$

and

$$\begin{aligned}\|g^{-1}h(0)\| =& \|h_u^{-1}(0) - f_{h_u}^{-1}t(0) + f_{h_u}^{-1}(u)\| \\ =& \|-f_{h_u}h_u(0) - f_{h_u}^{-1}t(0) + f_{h_u}^{-1}(u)\| = \|-h_u(0)-t(0)+u\| < r.\end{aligned}$$

Hence $gh, g^{-1}h \in B'(0,r)$, and the result follows. □

thm:finiteinvol

Theorem 9.4.9. *Let (X,r) be a finite involutive solution of the YBE. Then the structure group $G(X,r)$ is a solvable Bieberbach group.*

Proof. Since $\sigma_x\sigma_y = \sigma_{\sigma_x(y)}\sigma_{\tau_y(x)}$ for all $x, y \in X$, there exists a unique homomorphism of groups $\phi\colon G(X,r) \to \langle \sigma_x : x \in X\rangle \subseteq \mathbb{S}_X$ such that $\phi(x) = \sigma_x$ for all $x \in X$. By Theorem 8.2.6, the additive group of $G(X,r)$ is free with basis X (the natural map $i\colon X \to G(X,r)$ is injective and we identify x with $i(x)$ for all $x \in X$. Hence $\ker(\phi) = \mathrm{Soc}(G(X,r))$ is an abelian subgroup of $G(X,r)$ and $G(X,r)/\mathrm{Soc}(G(X,r))$ is a finite skew brace of abelian type. By Theorem 6.2.17, it follows that $G(X,r)$ is solvable.

Since $G(X,r)$ has an abelian subgroup of finite index, to prove that $G(X,r)$ is a Bieberbach group, it is enough to show that $G(X,r)$ is torsion free. Suppose that $G(X,r)$ is not torsion free. Hence there exists $g \in G(X,r)$ of prime order $p \leq |X| = n$. Since $g^1 = g + \lambda_g(g) + \cdots + \lambda_{g^{n-1}}(g)$ and the additive group of the skew brace $G(X,r)$ is torsion free, we have that $\lambda_g(g) \neq g$. Hence λ_g has order p. Let $X = \{x_1, \ldots, x_n\}$. The restriction of λ_g to X is a product of cycles of length p in $\mathbb{S}_X$. Hence we may assume that there exists a non-negative integer m such that

$$\lambda_g(x_{kp+i} = \begin{cases} x_{kp+i+1} & \text{if } 1 \leq i < p \\ x_{kp+1} & \text{if } i = p \end{cases}$$

for all $0 \leq k \leq m$ and $\lambda_g(x_t) = x_t$ for all $mp + p < t \leq n$. Since $g^p = 1$, in the additive group of the skew brace $G(X,r)$ we have that $g = z_1x_1 + \cdots + z_nx_n$ for some $z_1, \ldots, z_n \in \mathbb{Z}$ and

$$g + \lambda_g(g) + \cdots + \lambda_{g^{n-1}}(g) = 0.$$

Hence

$$\sum_{i=1}^{p} z_{kp+1} = 0 \quad \text{and} \quad z_t = 0$$

for all $0 \leq k \leq m$ and $mp + p < t \leq n$. Thus $g = \sum_{i=1}^{mp+p} z_i x_i$. Let

$$h_k = \sum_{j=1}^{p-1} \left(\sum_{i=1}^{j} z_{kp+i} \right) x_{kp+j}$$

for all $0 \leq k \leq m$. Let $h = \sum_{k=0}^{m} h_k$. We have that

$$\begin{aligned}
gh =& g + \lambda_g(h) = \sum_{k=0}^{m} \sum_{j=1}^{p} z_{kp+j} x_{kp+j} + \sum_{k=0}^{m} \sum_{j=1}^{p-1} \left(\sum_{i=1}^{j} z_{kp+i} \right) \lambda_g(x_{kp+j}) \\
=& \sum_{k=0}^{m} \sum_{j=1}^{p} z_{kp+j} x_{kp+j} + \sum_{k=0}^{m} \sum_{j=1}^{p-1} \left(\sum_{i=1}^{j} z_{kp+i} \right) x_{kp+j+1} \\
=& \sum_{k=0}^{m} z_{kp+1} x_{kp+1} + \sum_{k=0}^{m} \sum_{j=1}^{p-1} \left(\sum_{i=1}^{j+1} z_{kp+i} \right) x_{kp+j+1} \\
=& \sum_{k=0}^{m} \sum_{j=0}^{p-1} \left(\sum_{i=1}^{j+1} z_{kp+i} \right) x_{kp+j+1} \\
=& h + \sum_{k=0}^{m} \left(\sum_{i=1}^{p} z_{kp+i} \right) x_{kp+p} \\
=& h.
\end{aligned}$$

Hence $g = 1$, a contradiction. Therefore the structure group $G(X, r)$ is torsion free and the result follows. □

9.5 Exercises

9.5.1. Let G be a group and $\{N_\alpha : \alpha\}$ be a collection of normal subgroups of G such that $\cap_\alpha N_\alpha = \{1\}$. Prove that if G/N_α is a left ordered group for all α, then G is a left ordered group.

9.5.2. Let K be a field. Prove that if G is a group satisfying the unique product property, then every invertible element of $K[G]$ is of the form ag for $a \in K \setminus \{0\}$ and $g \in G$ (i. e. $K[G]$ has only trivial units).

xca:[x,y]^n=1

9.5.3. Let G be a group such that $(G : Z(G)) = n$. Prove that $[x, y]^n = 1$ for all $x, y \in G$.

9.5.4. Let H be a central subgroup of a finite group G. Prove that if $|H|$ and $|G/H|$ are coprime, then $G \cong H \times G/H$.

9.5.5. Let G be a group. Prove that $\Delta(G)$ is a characteristic subgroup of the group G.

9.6 Open problems

Open problem 9.6.1. Let G be a group satisfying the unique product property. Is G diffuse?

Notes

Chapter 10
Garside groups

Garside

A

In this chapter prove that the structure group of a finite involutive solution is a Garside group.

A *monoid* is a non-empty set M provided with an associative binary operation $M \times M \to M$, $(x, y) \mapsto xy$, and an identity element. A monoid M is said to be *cancellative* if

$$xy = xz \implies y = z \quad \text{and} \quad xy = zy \implies x = z$$

for all $x, y, z \in M$.

Definition 10.0.1. A *Garside monoid* is a pair (M, Δ), where M is a cancellative monoid such that

1) There exists a map $d\colon M \to \mathbb{N}$ such that $d(xy) \geq d(x) + d(y)$ and $d(x) \neq 0$ if $x \neq 1$.
2)
3) Δ is a Garside element of M...
4) The family of all divisors of Δ is finite.

Definition 10.0.2. A group G is said to be a *Garside group* if...

Structure groups of involutive solutions are Garside groups.

thm:Chouraqui

Theorem 10.0.3. *Let (X, r) be an involutive solution. Then $G(X, r)$ is a Garside group.*

Proof. □

At this point it is easy to prove the following important result of Gateva–Ivanova and Van den Bergh.

thm:torsion_free **Theorem 10.0.4.** *Let (X,r) be an involutive solution. Then $G(X,r)$ has no torsion. In particular, $G(X,r)$ is a Bieberbach group.*

Proof. □

As a consequence we obtain the following result on linear representations of the structure group of an involutive solution.

thm:ESS **Theorem 10.0.5.**

Proof. □

thm:D **Theorem 10.0.6.**

Proof. □

Exercises

Open problems

Notes

Theorem 10.0.3 was proved by Chouraqui in [31]. Our proof is based on the work of Dehornoy [35] and the presentation of Cedó's survey [24].

Theorem 10.0.4 was proved by Gateva–Ivanova and Van den Bergh in [44] using somewhat different methods.

Chapter 11
Left nilpotent skew braces

invariant

11.1 Invariant subgroups

We say that a group G acts on a group K by automorphism if the (left) action

$$G \times K \to K, \quad (g,x) \mapsto g \cdot x,$$

satisfies $g \cdot (xy) = (g \cdot x)(g \cdot y)$ for all $g \in G$ and $x, y \in K$. In this case, the map $\alpha : G \to \mathrm{Aut}(K)$, defined by $\alpha(g)(k) = g \cdot k$ for all $g \in G$ and $k \in K$, is a homomorphism of groups. Consider the semidirect product $\Gamma = K \rtimes_\alpha G$. We will identify K with $K \times \{1\}$ and G with $\{1\} \times G$ in Γ. With this identification we have that $g \cdot k = gkg^{-1}$ for all $g \in G$ and $k \in K$. The group

$$C_K(G) = \{x \in K : g \cdot x = x \text{ for all } g \in G\}$$

acts on the set of G-orbits by left multiplication. Indeed, if $x \in K$ and $c \in C_K(G)$, then $g \cdot c = c$ for all $g \in G$. Thus

$$\begin{aligned} c(G \cdot x) &= \{c(g \cdot x) : g \in G\} \\ &= \{(g \cdot c)(g \cdot x) : g \in G\} = \{g \cdot (cx) : g \in G\} = G \cdot (cx). \end{aligned}$$

The following theorem goes back to Deaconescu and Walls [34]. Our proof is that of Isaacs, see [48].

thm:DeaconescuWalls

Theorem 11.1.1 (Deaconescu–Walls). *Let G be a group acting by automorphism on a finite group K. Let $C = C_K(G)$ and $N = C \cap [G,K]$, where $[G,K]$ is the subgroup of K generated by $[g,x] = (g \cdot x)x^{-1}$ for all $g \in G$ and $x \in K$. Then the index $(C : N)$ divides the number of G-orbits of K.*

Proof. The group C acts by left multiplication on the set Ω of G-orbits on K. Let $X = G \cdot x \in \Omega$ be an orbit and C_X be the stabilizer of X in C. If $c \in C_X$, then $cX = X$. In particular, if $c \in C_X$ and $x \in X$, then $cx = g \cdot x$ for some $g \in G$. Thus

$$c = (g \cdot x)x^{-1} = [g,x] \in [G,K]$$

and hence $C_X \subseteq N$.

To prove that $(C : N)$ divides the size of Ω, decompose Ω as a disjoint union of C-orbits. Then it is enough to show that $(C : N)$ divides the size of each C-orbit. If $X \in \Omega$, then $C \cdot X$ has size

$$(C : C_X) = (C : N)(N : C_X).$$

Thus $(C : N)$ divides the size of $C \cdot X$. □

cor:Z(K)subset[K,K]

Corollary 11.1.2. *Let K be a non-trivial finite group with k conjugacy classes. If $|Z(K)|$ and k are coprime, then $Z(K) \subseteq [K, K]$.*

Proof. Let the group K acts on K by conjugation, which is an action by automorphism. Deaconescu–Walls' theorem implies that $(Z(K) : Z(K) \cap [K, K])$ divides k. Since k and $|Z(K)|$ are coprime, it follows that $Z(K) = Z(K) \cap [K, K] \subseteq [K, K]$. □

Let K be a group and $f \in \mathrm{Aut}(K)$. Then f is *central* if $f(x)x^{-1} \in Z(K)$ for all $x \in K$. Note that $f \in \mathrm{Aut}(K)$ is central if and only if $f \in C_{\mathrm{Aut}(K)}(\mathrm{Inn}(K))$.

Corollary 11.1.3. *Let K be a finite group with k conjugacy classes and c central automorphisms. If $\gcd(|K|, kc) = 1$, then $[K, K] = Z(K)$.*

Proof. By Corollary 11.1.2, $Z(K) \subseteq [K, K]$.

Let us prove that $Z(K) \supseteq [K, K]$. Let $G = C_{\mathrm{Aut}(K)}(\mathrm{Inn}(K))$. Since $\gcd(|K|, kc) = 1$ and, by Deaconescu–Walls' theorem, $(C_K(G) : C_K(G) \cap [G, K])$ divides c, it follows that $C_K(G) = C_K(G) \cap [G, K]$. Since $[K, K] \subseteq C_K(G)$, as

$$a \cdot [x, y] = [(a \cdot x)x^{-1}x, (a \cdot y)y^{-1}y] = [x, y]$$

for all $a \in G$, $x, y \in K$ and $[G, K] \subseteq Z(K)$, we conclude that

$$[K, K] \subseteq C_K(G) = C_K(G) \cap [G, K] \subseteq [G, K] \subseteq Z(K).$$ □

Corollary 11.1.4. *Let p be a prime number. If K is a group with p conjugacy classes, then $Z(K) \subseteq [K, K]$ or $|K| = p$.*

Proof. Let K acts on K by conjugation. Since every element of $C = Z(K)$ form a conjugacy class, $|C| \leq p$. If $|C| = p$, then $K = C = Z(K)$ has p elements. Otherwise, $\gcd(|C|, p) = 1$ and, by Corollary 11.1.2, $Z(K) = C \subseteq [K, K]$. □

Let G and K be groups such that G acts on K by automorphisms. A subgroup H of K is *G-invariant* if $g \cdot H \subseteq H$ for all $g \in G$.

Now we will develop Sylow theory for invariant subgroups.

Lemma 11.1.5 (Glauberman). *Let G and K be finite groups of coprime order, where at least one of G or K is solvable. Assume that G acts on K by automorphisms and that G acts on a set Ω, K acts transitively on Ω and*

$$(g \cdot x) \cdot (g \cdot \omega) = g \cdot (x \cdot \omega)$$

for all $g \in G$, $x \in K$ and $\omega \in \Omega$. The following statements hold:

1) *There exists a G-invariant element of Ω.*
2) *If $\omega, \omega_1 \in \Omega$ are G-invariant elements, then $c \cdot \omega = \omega_1$ for some $c \in C_K(G)$.*

Proof. We demonstrate the first claim. Let $\Gamma = K \rtimes G$. We identify K and G with the subgroups $K \times \{1\}$ and $\{1\} \times G$ of Γ respectively. Each $\gamma \in \Gamma$ can be written uniquely as $\gamma = xg$ for $x \in K$ and $g \in G$. Thus Γ acts on Ω by

$$(xg) \cdot \omega = x \cdot (g \cdot \omega), \quad x \in K, g \in G, \omega \in \Omega.$$

To prove that this is an action we use the compatibility condition to compute

$$\begin{aligned}(xg) \cdot ((x_1 g_1) \cdot \omega) &= (xg) \cdot (x_1 \cdot (g_1 \cdot \omega)) \\ &= x \cdot (g \cdot (x_1 \cdot (g_1 \cdot \omega))) = x \cdot ((g \cdot x_1) \cdot (g \cdot (g_1 \cdot \omega))) \\ &= (x(g \cdot x_1)) \cdot ((gg_1) \cdot \omega) = (x(g \cdot x_1)(gg_1)) \cdot \omega \\ &= ((xg)(x_1 g_1)) \cdot \omega.\end{aligned}$$

Let $\omega \in \Omega$ and $U = \Gamma_\omega$ be the stabilizer of ω in Γ. Let $\gamma \in \Gamma$. Since K acts transitively on Ω, there exists $x \in K$ such that $\gamma \cdot \omega = x \cdot \omega$. Thus $x^{-1}\gamma \in U$ and hence $\gamma \in xU \subseteq KU$, and therefore $KU = \Gamma$.

Since K is normal in Γ, it follows that $U \cap K$ is normal in U. Moreover,

$$(U : U \cap K) = (KU : K) = (\Gamma : K) = |G|$$

is coprime with $|U \cap K|$. By Schur–Zassenhaus' theorem (Theorem 7.2.4), there exists a complement H of $U \cap K$ in U. Since

$$|H| = (U : U \cap K) = |G|,$$

it follows that H is also a complement of K in Γ. Since G is a complement of K in Γ, by Theorem 7.2.5, H and G are conjugate in Γ, this means $G = \gamma H \gamma^{-1}$ for some $\gamma \in \Gamma$. Since $H \subseteq U$, it follows that H stabilizes ω and hence $\gamma \cdot \omega$ is a G-invariant element.

Let us prove the second claim. Let $\omega, \omega_1 \in \Omega$ be G-invariant elements. By assumption, K acts transitively on Ω and thus the set

$$X = \{x \in K : x \cdot \omega = \omega_1\}$$

is non-empty.

Claim. The group G acts on X.

If $x \in X$, then $x \cdot \omega = \omega_1$. By applying $g \in G$ to this equality we obtain that $g \cdot (x \cdot \omega) = g \cdot \omega_1$. By the compatibility condition and using that ω and ω_1 are both G-invariant elements,

$$(g \cdot x) \cdot \omega = (g \cdot x) \cdot (g \cdot \omega) = g \cdot (x \cdot \omega) = g \cdot \omega_1 = \omega_1.$$

This proves that $G \cdot X \subseteq X$ and hence G acts on X.

Claim. There exists a G-invariant element of X. (This completes the proof, as if $x \in X$ is G-invariant, then $x \in K$ is such that $x \cdot \omega = \omega_1$ and $g \cdot x = x$ for all $g \in G$, i.e. $x \in C_K(G)$.)

Let $H = K_{\omega_1}$ be the stabilizer of ω_1 in K. Note that H is a subgroup of K. The group H acts transitively on X by left multiplication: if $h \in H$ and $x \in X$, then

$$(hx) \cdot \omega = h \cdot (x \cdot w) = h \cdot \omega_1 = \omega_1.$$

Note also that $(g \cdot h) \cdot \omega_1 = (g \cdot h) \cdot (g \cdot \omega_1) = g \cdot (h \cdot \omega_1) = g \cdot \omega_1 = \omega_1$. Thus the action of G on K restricts to H and hence G acts on H by automorphisms. The orders of G and H are coprime and either G or H is solvable. The group G acts on X and the compatibility condition holds:

$$g \cdot (h \cdot x) = g \cdot (hx) = (g \cdot h)(g \cdot x) = (g \cdot h) \cdot (g \cdot x),$$

as G acts on K by automorphisms. Hence the first part of the lemma implies the existence of a G-invariant element of X. □

Note that by Feit–Thompson's theorem, the solvability of G or K is not really needed since at least one of $|G|$ or $|K|$ is of odd order.

We now prove Sylow's theorems for invariant subgroups.

thm:InvariantSylow

Theorem 11.1.6. *Let G and K be finite groups of coprime order. Assume that G acts by automorphisms on K, and that at least one of G or K is solvable. Let p be a prime number.*

1) *There exists a Sylow p-group of K which is G-invariant.*
2) *If S and T are G-invariant Sylow p-subgroups, then there exists $c \in C_K(G)$ such that $cSc^{-1} = T$.*

Proof. Let us prove the first claim. By Sylow's theorems, the group K acts transitively by conjugation on the non-empty set $\Omega = \mathrm{Syl}_p(K)$. Since G acts on K by automorphisms, G acts on Ω. Let us check the compatibility condition. Since $g \cdot P \in \Omega$ for all $g \in G$ and $P \in \mathrm{Syl}_p(K)$, K acts by conjugation on Ω and G acts by automorphisms on K,

$$(g \cdot x) \cdot (g \cdot P) = (g \cdot x)(g \cdot P)(g \cdot x)^{-1} = g \cdot (xPx^{-1}) = g \cdot (x \cdot P),$$

for all $g \in G$, $x \in K$ and $P \in \mathrm{Syl}_p(K)$. The first part of Glauberman's lemma implies that there exists a Sylow p-subgroup of K that is G-invariant.

Let us now prove the second claim. Let S and T be G-invariant Sylow p-subgroups of K, so S and T are G-invariant elements of Ω. By the second part of Glauberman's lemma, there exits $c \in C_K(G)$ such that $c \cdot S = cSc^{-1} = T$. □

Corollary 11.1.7. *Let G and K be finite groups of coprime orders. Assume that G acts by automorphisms on K and at least one of K or G is solvable. Let P be a G-invariant p-subgroup of K for some prime number p. Then P is contained in some G-invariant Sylow p-subgroup of K.*

Proof. We may assume that P is a maximal G-invariant p-subgroup of K. Let $N = N_K(P)$. Note that N is G-invariant: if $n \in N$ and $g \in G$, then

$$(g \cdot n)P(g \cdot n)^{-1} = g \cdot (nPn^{-1}) = g \cdot P = P$$

since P is G-invariant.

We claim that $P \in \mathrm{Syl}_p(K)$. By the first part of Theorem 11.1.6, there exists $S \in \mathrm{Syl}_p(N)$ that is G-invariant. By Sylow's theorem, some conjugate of P is contained in S, say $xPx^{-1} \subseteq S$ for some $x \in N$, and hence

$$P = xPx^{-1} \subseteq S$$

since P is normal in N. Since P is a maximal G-invariant p-subgroup of K, it follows that $P = S$ and thus $P \in \mathrm{Syl}_p(N)$. Let $Q \in \mathrm{Syl}_p(K)$ be such that $P \subseteq Q$. If $P \neq Q$, then $Q \cap N$ is a p-subgroup of N such that $P \subsetneq Q \cap N$, by the normalizer condition (Lemma 4.1.9), a contradiction because $P \in \mathrm{Syl}_p(N)$. Hence $P = Q$ and therefore $P \in \mathrm{Syl}_p(K)$. □

thm:K=C_K(G)[G,K]

Theorem 11.1.8. *Let G and K be finite groups of coprime order. Assume that G acts by automorphisms on K and that at least one of K or G is solvable. Then $K = C_K(G)[G,K]$.*

Proof. Note that $[G,K]$ is a normal subgroup of K and it is G-invariant. In fact, for $k_1, k_2 \in K$ and $g_1, g_2 \in G$, we have

$$\begin{aligned} k_1[g_1,k_2]k_1^{-1} =& k_1(g_1 \cdot k_2)k_2^{-1}k_1^{-1} \\ =& (g_1 \cdot (g_1^{-1} \cdot k_1))(g_1 \cdot k_2)k_2^{-1}(g_1^{-1} \cdot k_1^{-1})(g_1^{-1} \cdot k_1)k_1^{-1} \\ =& (g_1 \cdot ((g_1^{-1} \cdot k_1)k_2))k_2^{-1}(g_1^{-1} \cdot k_1)^{-1}(g_1^{-1} \cdot k_1)k_1^{-1} \\ =& [g_1, (g_1^{-1} \cdot k_1)k_2][g_1^{-1}, k_1] \in [G,K], \end{aligned}$$

and

$$(g_1 \cdot [g_2,k_1])[g_2,k_1]^{-1} = [g_1,[g_2,k_1]] \in [G,K].$$

Hence G acts on $K/[G,K]$ and K acts transitively on $K/[G,K]$ by left multiplication. Furthermore,

$$\begin{aligned} g_1 \cdot (k_1 \cdot k_2[G,K]) =& g_1 \cdot (k_1k_2[G,K]) = (g_1 \cdot (k_1k_2))[G,K] \\ =& (g_1 \cdot k_1)(g_1 \cdot k_2)[G,K] = (g_1 \cdot k_1) \cdot (g_1 \cdot (k_2[G,K])). \end{aligned}$$

Note that for every $g \in G$ and $k \in K$, we have that

$$g \cdot (k[G,K]) = (g \cdot k)[G,K] = kk^{-1}(g \cdot k)k^{-1}k[G,K] = k[G,K].$$

By Glauberman's lemma, for every $k \in K$, there exists $c \in C_K(G)$ such that $k[G,K] = c[G,K] \subseteq C_K(G)[G,K]$, and thus $K = C_K(G)[G,K]$. □

cor:coprimeaction

Corollary 11.1.9. *Let G and K be finite groups of coprime order. Assume that G acts by automorphisms on K. Then $[G,[G,K]] = [G,K]$.*

Proof. It is clear that $[G,[G,K]] \subseteq [G,K]$. Let $g \in G$ and $k \in K$. We shall prove that $[g,k] \in [G,[G,K]]$. Consider the group $A = \langle g \rangle$. Since A is abelian, by Theorem 11.1.8, we have that $K = C_K(A)[A,K]$. Hence there exists $c \in C_K(A)$ and $x \in [A,K]$ such that $k = xc$. Now we have that

$$\begin{aligned}[g,k] =& (g \cdot k)k^{-1} = (g \cdot x)(g \cdot c)c^{-1}x^{-1} \\ =& (g \cdot x)x^{-1} = [g,x] \in [G,[G,K]].\end{aligned}$$

Hence $[G,[G,K]] = [G,K]$. □

An action of a group G on a group K by automorphisms is said to be *faithful* if, for $g \in G$, $g \cdot x = x$ for all $x \in K$ implies that $g = 1$. Note that if the action is faithful, then $C_G(K) = \{1\}$.

The *kernel* of an action is the (normal) subgroup N of elements that act trivially. In this case, the quotient group G/N acts by $\pi(g) \cdot x = g \cdot x$ and the action is faithful. Note that the kernel of the action is the subgroup $C_G(K)$ of $K \rtimes G$. Note that $C_G(K)$ is isomorphic to the largest subgroup N of G such that $[N,K] = \{1\}$. If $N = C_G(K)$, then N is normal in G. In this case, the quotient group G/N acts on K by $g \cdot x = \pi(g) \cdot x$.

It will be convenient to introduce the following notaiton.

Let $[G,\dots,G,K]_1 = [G,K]$ and $[G,\dots,G,K]_{m+1} = [G,\dots,G,[G,K]]_m$ for $m \geq 1$. Note that

$$[G,\dots,G,K]_m = [\underbrace{G,\dots,G}_{m\text{-copies}},K]$$

for $m \geq 1$.

Theorem 11.1.10. *Let G and K be groups such that G acts by automorphisms on K. If $[G,\dots,G,K]_m = \{1\}$, then $G^{(m-1)} \subseteq C_G(K)$. In particular, if the action is faithful, then G is solvable and its derived series has length $\leq m-1$.*

Proof. Let us prove that $G^{(m-1)} \subseteq C_G(K)$ by induction on m. If $m = 1$, then $[G,K] = \{1\}$ by assumption and thus $G^{(0)} = G = C_G(K)$. Assume now that $m > 1$ and the result holds for $m-1$. Let $L = [G,K]$. Then

$$\{1\} = [G,\dots,G,K]_m = [G,\dots,G,L]_{m-1}.$$

Since L is G-invariant, G acts on L. By the inductive hypothesis, $G^{(m-2)} \subseteq C_G(L)$, which implies that $[G^{(m-2)},L] = \{1\}$. Thus

$$[G^{(m-2)},[G^{(m-2)},K]] \subseteq [G^{(m-2)},[G,K]] = [G^{(m-2)},L] = \{1\}.$$

Moreover, since $[G^{(m-2)},K] = [K,G^{(m-2)}]$, it follows that

$$[G^{(m-2)},[K,G^{(m-2)}]] = \{1\}.$$

The three-subgroups Lemma with $X = Y = G^{(m-2)}$ and $Z = K$ implies that

$$\{1\} = [K, [G^{(m-2)}, G^{(m-2)}]] = [K, G^{(m-1)}].$$

Hence $G^{(m-1)} \subseteq C_G(K)$ and the claim follows by induction. In particular, if the action is faithful, then $C_G(K) = \{1\}$ and the result follows. □

thm:Hallnilpotent

Theorem 11.1.11 (Hall). *Let G be a finite group that acts on a finite group K by automorphisms. If $[G, \dots, G, K]_m = \{1\}$ for some m, then there exists a positive integer n such that $[[G, \dots, G, G]_n, K] = \{1\}$. Furthermore, if the action of G on K is faithful, then G is nilpotent.*

Proof. We proceed by induction on $|K|$. We may assume that K is non-trivial. Let

$$G = \gamma_1(G) \supseteq \gamma_2(G) \supseteq \cdots \supseteq \gamma_n(G) \supseteq \cdots$$

be the lower central series of G. Since G is finite, this sequence stabilizes. Let $n \in \mathbb{N}$ be the smallest positive integer such that $\gamma_n(G) = \gamma_{n+k}(G)$ for all k.

Claim. $[G, K] \neq K$.

In fact, if $[G, K] = K$, then

$$\{1\} = [G, \dots, G, K]_m = [G, \dots, G, K]_{m-1} = \cdots = [G, K] = K,$$

a contradiction.

Claim. $\gamma_n(G)$ acts trivially on K.

By assumption,

$$\{1\} = [G, \dots, G, K]_m = [G, \dots, G, [G, K]]_{m-1}.$$

Since $[G, K]$ is a proper G-invariant subgroup of K, the inductive hypothesis implies that $\gamma_n(G)$ acts trivially on $[G, K]$ and hence $[\gamma_n(G), [G, K]] = [\gamma_n(G), [K, G]] = \{1\}$.

In order to use the three-subgroups lemma, we need to show that

$$[G, [\gamma_n(G), K]] = \{1\}.$$

Suppose that $[\gamma_n(G), K] \neq \{1\}$. Since $[\gamma_n(G), K] \subseteq K$ and $[G, \dots, G, K]_m = \{1\}$, we have that there exists a positive integer $l \leq m$ such that $[G, \dots, G, [\gamma_n(G), K]]_l = \{1\}$ and $[G, \dots, G, [\gamma_n(G), K]]_{l-1} \neq \{1\}$. Let $C = C_{[\gamma_n(G), K]}(G)$. Then

$$[G, \dots, G, [\gamma_n(G), K]]_{l-1} \subseteq C$$

and thus C is a non-trivial subgroup of K and G acts trivially on C. Since the group $\gamma_n(G)$ acts trivially on $[G, K]$, it follows that $[\gamma_n(G), [G, K]] = \{1\}$. Thus

$$[K, [\gamma_n(G), [G, K]] = [K, \{1\}] = \{1\}$$

and hence, since $[G,K]$ is normal in K, it follows that

$$[\gamma_n(G),[[G,K],K]] \subseteq [\gamma_n(G),[G,K]] = \{1\}.$$

By the three-subgroups lemma with $X = K$, $Y = \gamma_n(G)$ and $Z = [G,K]$,

$$[[G,K],[\gamma_n(G),K]] = [[G,K],[K,\gamma_n(G)]] = \{1\}.$$

By definition, $C \subseteq [\gamma_n(G),K]$ and $[C,G] = \{1\}$. Thus

$$[C,[G,K]] \subseteq [[\gamma_n(G),K],[G,K]] = \{1\}$$

and

$$[K,[C,G]] = [K,\{1\}] = \{1\}.$$

By using the three-subgroups lemma with $X = K$, $Y = C$ and $Z = G$, it follows that $[G,[C,K]] = \{1\}$, so G centralizes $[C,K]$. Since $C \subseteq [\gamma_n(G),K]$ and $[\gamma_n(G),K]$ is a normal subgroup of K, it follows that $[C,K] \subseteq [\gamma_n(G),K]$. But

$$[C,K] \subseteq C_{[\gamma_n(G),K]}(G) = C$$

and therefore C is normal in K.

Let $\pi\colon K \to K/C$ be the canonical map. Since $|K/C| < |K|$ and

$$[G,\dots,G,\pi(K)]_m = \pi([G,\dots,G,K]_m) = \{1\},$$

the inductive hypothesis implies that

$$\{1\} = [\gamma_n(G),\pi(K)] = \pi([\gamma_n(G),K]).$$

Thus $[\gamma_n(G),K] \subseteq C$. In particular,

$$[G,[\gamma_n(G),K]] = \{1\},$$

as G acts trivially on C. Then by the three-subgroups lemma,

$$\{1\} = [K,[G,\gamma_n(G)]] = [K,\gamma_{n+1}(G)] = [K,\gamma_n(G)],$$

a contradiction. Therefore $[\gamma_n(G),K] = \{1\}$ and the result follows by induction. □

For the next result we need a lemma.

lem:[G,K]abelian

Lemma 11.1.12. *Let G and K be groups such that G acts on K by automorphisms. If $[G,[G,K]] = \{1\}$, then $[G,K]$ is abelian.*

Proof. Recall that we consider K and G as subgroups of $\Gamma = K \rtimes G$. Since $[G,[G,K]] = \{1\}$, it follows that $[K,[[G,[G,K]]] = \{1\}$. Moreover, $[G,K]$ is normal in K and thus

$$[G,[[G,K],K]] \subseteq [G,[G,K]] = \{1\}.$$

The three-subgroups lemma with $X = K$, $Y = G$ and $Z = [G,K]$ implies that

$$\{1\} = [[G,K],[K,G]] = [[G,K],[G,K]],$$

so the commutator subgroup of the group $[G,K]$ is trivial. This means that $[G,K]$ is abelian. □

thm:[G,K]_pgroup

Theorem 11.1.13. *Let p be a prime number and G be a finite p-group that acts by automorphisms on a finite group K. If $[G,\dots,G,K]_m = \{1\}$ for some m, then $[G,K]$ is a p-group.*

Proof. We proceed by induction on $|K|$. We may assume that $K \neq \{1\}$. Then, since $[G,\dots,G,K]_m = \{1\}$, it follows that $[G,K] \neq K$. Let $L = [G,K]$. Then L is a G-invariant proper normal subgroup of K. Since

$$[G,\dots,G,L]_{m-1} = [G,\dots,G,K]_m = \{1\},$$

the inductive hypothesis on L implies that $[G,L]$ is p-group. Note that $[G,L]$ is a normal subgroup of L. Then $[G,L] \subseteq O_p(L)$ and since $O_p(L)$ is characteristic in L and L is normal in K, it follows that $O_p(L)$ is normal in K. Since L is G-invariant and $O_p(L)$ is characteristic in L, we have that $O_p(L)$ is G-invariant. Let $\pi\colon K \to K/O_p(L)$ be the canonical map. Then G acts by automorphisms on $\pi(K)$. Since $[G,L] \subseteq O_p(L)$, we have that $[G,[G,\pi(K)]] = \{1\}$. By Lemma 11.1.12, $[G,\pi(K)]$ is abelian. Let $g \in G \setminus \{1\}$ and $k \in K$. We have that g has order p^n for some positive integer n. Since $[G,[G,\pi(K)]] = \{1\}$, we have that

$$\begin{aligned} 1 =& [\pi(k), g^{p^n}] = [\pi(k), g^{p^n-1}]g^{p^n-1}[\pi(k),g](g^{p^n-1})^{-1} \\ =& [\pi(k), g^{p^n-1}][\pi(k),g] = \cdots = [\pi(k),g]^{p^n}. \end{aligned}$$

Hence $[G,\pi(K)] \cong ([G,K]O_p(L))/O_p(L) = L/O_p(L)$ is an abelian p-group. Therefore $[G,K]$ is a p-group, and the result follows by induction. □

hm:general[G,K]nilpotent

Theorem 11.1.14. *Let G and K be finite groups such that G acts on K by automorphisms. If $[G,\dots,G,K]_m = \{1\}$ for some m, then $[G,K]$ is nilpotent.*

Proof. We proceed by induction on $|G|$. If $G = \{1\}$, then $[G,K] = \{1\}$ and there is nothing to demonstrate. So we may assume that $G \neq \{1\}$. Let G_1 be a proper subgroup of G. Since

$$[G_1,\dots,G_1,K]_m \subseteq [G,\dots,G,K]_m = \{1\},$$

the inductive hypothesis implies that $[G_1,K]$ is nilpotent. Since $[G_1,K]$ is a normal subgroup of K, by Fitting's theorem, $[G_1,K] \subseteq F(K)$, the Fitting subgroup of K. Let $g \in G$. If the order of g is a power of a prime p, then by Theorem 11.1.13, $[\langle g\rangle, K]$ is a p-group and it is normal in K. By Fitting's theorem $[\langle g\rangle, K] \subseteq F(K)$. If the order of g is not a power of any prime, then the order of g is mn for some coprime integers $m,n > 1$. There exist integers a,b such that $am + bn = 1$. Hence $g = g^{am}g^{bn}$ and $\langle g^{am}\rangle$ and $\langle g^{bn}\rangle$ are proper subgroups of G. Hence

$$[\langle g\rangle, K] = [\langle g^{am}\rangle, K][\langle g^{bn}\rangle, K] \subseteq F(K).$$

Therefore $[G, K] \subseteq F(K)$ and the result follows by induction and Fitting's theorem. □

11.2 Left nilpotent skew braces

Definition 11.2.1. Let A be a skew brace. One defines $A^1 = A$ and for $n \geq 1$

$$A^{n+1} = A * A^n = \langle a * x : a \in A, x \in A^n\rangle_+.$$

The sequence $A^1 \supseteq A^2 \supseteq A^3 \supseteq \cdots \supseteq A^n \supseteq \cdots$ is the *left series* of A.

pro:left_series **Proposition 11.2.2.** *Let A be a skew brace. Each A^n is a left ideal of A.*

Proof. We proceed by induction on n. The case $n = 1$ is trivial, so we may assume that the result is true for some $n \geq 1$. Let $a, b \in A$ and $x \in A^n$. By the inductive hypothesis, $\lambda_a(x) \in A^n$ and hence

$$\begin{aligned}\lambda_a(x * b) &= \lambda_a(\lambda_x(b) - b) = \lambda_a\lambda_x(b) - \lambda_a(b)\\ &= \lambda_{a\circ x\circ a'}(\lambda_a(b)) - \lambda_a(b) = (a \circ x \circ a') * \lambda_a(b) \in A^{n+1}.\end{aligned} \tag{11.1}$$

eq:another_trick

This implies that $\lambda_a(A^{n+1}) \subseteq A^{n+1}$. Thus the result follows. □

Definition 11.2.3. A skew brace A is said to be *left nilpotent* if $A^m = \{0\}$ for some $m \geq 1$.

Some basic properties of left nilpotent skew braces appear in Exercises 11.2.3–11.2.5.

pro:IcapFix **Proposition 11.2.4.** *Let A be a left nilpotent skew brace and I be a non-zero left ideal of A. Then $I \cap \mathrm{Fix}(A) \neq \{0\}$. In particular,* $\mathrm{Fix}(A) \neq \{0\}$.

Proof. Let $m = \max\{k : I \cap A^k \neq \{0\}\}$. Since $A * (I \cap A^m) \subseteq I \cap A^{m+1} = \{0\}$, it follows that there exists a non-zero $x \in I \cap A^m$ such that $a * x = 0$ for all $a \in A$. Thus $0 \neq x \in \mathrm{Fix}(A) \cap I$. For the second claim, apply the first case with $I = A$. □

Let A be a skew brace. Let $A^{[1]} = A$ and for $n \geq 1$ let $A^{[n+1]}$ be the additive subgroup of A generated by elements from $\{A^{[i]} * A^{[n+1-i]} : 1 \leq i \leq n\}$. One easily proves by induction that $A^{[k]} \supseteq A^{[k+1]}$ for all $k \geq 1$.

pro:Smoktunowicz **Proposition 11.2.5.** *Let A be a skew brace. Each $A^{[n]}$ is a left ideal of A.*

Proof. Each $A^{[n]}$ is a subgroup of $(A, +)$. Since $A * A^{[n]} \subseteq A^{[n+1]} \subseteq A^{[n]}$, the claim follows from Proposition 6.2.8. □

There exists a skew brace A such that $A^{[n]} = A^{[n+1]} \neq \{0\}$ for some positive integer n and $A^{[n+2]} = \{0\}$.

exa:funny **Example 11.2.6.** Let

$$G = \langle r, s : r^8 = s^2 = 1, srs = r^7 \rangle \simeq \mathbb{D}_{16},$$
$$K = \langle a, b : 8a = 2b = 0, a + b = b + a \rangle \simeq \mathbb{Z}/(8) \times \mathbb{Z}/(2).$$

The group G acts by automorphisms on K via

$$r \cdot a = a + b, \quad r \cdot b = 4a + b, \quad s \cdot a = 3a, \quad s \cdot b = 4a + b.$$

A direct calculation shows that the map $\pi \colon G \to K$ given by

$$\begin{array}{llll} 1 \mapsto 0, & r \mapsto a, & r^2 \mapsto 2a + b, & r^3 \mapsto 7a + b, \\ r^4 \mapsto 4a, & r^5 \mapsto 5a, & r^6 \mapsto 6a + b, & r^7 \mapsto 3a + b, \\ rs \mapsto 6a, & r^2 s \mapsto 7a, & r^3 s \mapsto b, & r^4 s \mapsto 5a + b, \\ r^5 s \mapsto 2a, & r^6 s \mapsto 3a, & r^7 s \mapsto 4a + b, & s \mapsto a + b, \end{array}$$

is a bijective 1-cocycle. Therefore there exists a skew brace A with additive group isomorphic to K and multiplicative group isomorphic to G. The addition of A is that of K and the multiplication is given by

$$x \circ y = \pi(\pi^{-1}(x)\pi^{-1}(y)), \quad x, y \in K.$$

Since

$$a * a = -a + a \circ a - a = -a + (2a + b) - a = b,$$
$$(5a + b) * a = -(5a + b) + (5a + b) \circ a - a = -(5a + b) + b - a = 2a,$$

it follows that $A^{[2]}$ contains $\langle 2a, b \rangle_+ = \{0, 2a, 4a, 6a, b, 2a + b, 4a + b, 6a + b\}$, the additive subgroup of $(A, +)$ generated by $2a$ and b. Therefore $A^{[2]} = \langle 2a, b \rangle_+$ since $A^{[2]} \neq A$. Routine calculations prove that

$$A^{[3]} = \{0, 2a + b, 4a, 6a + b\}, \qquad A^{[4]} = A^{[5]} = \{0, 4a\}, \qquad A^{[6]} = \{0\}.$$

Definition 11.2.7. For a skew brace A let $\ell_1(a) = a$ and $\ell_{k+1}(a) = a * \ell_k(a)$ for $n \geq 1$. The skew brace A is said to be *left nil* if for every $a \in A$ there exists a positive integer $n = n(a)$ such that $\ell_n(a) = 0$.

Definition 11.2.8. A skew brace A is said to be *strongly nilpotent* if there is a positive integer n such that $A^{[n]} = 0$.

Definition 11.2.9. A skew brace A is said to be *strongly nil* if for every $a \in A$ there is a positive integer $n = n(a)$ such that any $*$-product of n copies of a is zero.

We first prove that if both groups of a finite skew brace A are nilpotent, then A can be decomposed as a direct product of skew braces of prime-power size.

sum **Lemma 11.2.10.** *Let A be a skew brace such that the additive group is a direct sum of ideals I_1, I_2, that is $A = I_1 + I_2$ and $I_1 \cap I_2 = \{0\}$. Then the map $f : A \to I_1 \times I_2$ defined by $f(a_1 + a_2) = (a_1, a_2)$, for all $a_1 \in I_1$ and $a_2 \in I_2$, is an isomorphism of skew braces.*

Proof. The operations of the skew brace $I_1 \times I_2$ are defined component-wise. Clearly f is an isomorphism of the additive groups of A and $I_1 \times I_2$. Let $a_1 \in I_1$ and $a_2 \in I_2$. Since I_1 and I_2 are ideals we have that

$$a_1 + a_2 - a_1 - a_2, a_1 * a_2, a_2 * a_1 \in I_1 \cap I_2 = \{0\},$$

thus $a_1 + a_2 = a_2 + a_1$ and $a_1 \circ a_2 = a_1 + a_2 = a_2 \circ a_1$. Hence

$$\begin{aligned} f((a_1 + a_2) \circ (b_1 + b_2)) &= f(a_1 \circ a_2 \circ b_1 \circ b_2) = f(a_1 \circ b_1 \circ a_2 \circ b_2) \\ &= f(a_1 \circ b_1 + a_2 \circ b_2) = (a_1 \circ b_1, a_2 \circ b_2) \\ &= (a_1, a_2) \circ (b_1, b_2) = f(a_1 + a_2) \circ f(b_1 + b_2), \end{aligned}$$

for all $a_1, b_1 \in I_1$ and $a_2, b_2 \in I_2$. □

thm:direct **Theorem 11.2.11.** *Let n be a positive integer. Let A be a skew brace such that the additive group is a direct sum of ideals $I_1, \ldots, I_n$, that is every element $a \in A$ is uniquely written as $a = a_1 + \cdots + a_n$, with $a_j \in I_j$ for all j. Then the map*

$$f : A \to I_1 \times \cdots \times I_n, \quad f(a_1 + \cdots + a_n) = (a_1, \ldots, a_n),$$

for all $a_j \in I_j$, is an isomorphism of skew braces.

Proof. We shall prove the result by induction on n. For $n = 1$, it is clear. Suppose that $n > 1$ and that the result is true for $n - 1$. Let $A_1 = I_1 + \cdots + I_{n-1}$. Then A_1 is an ideal of A and A is the direct sum of the ideals A_1 and I_n. By Lemma 11.2.10, the map $f_1 : A \to A_1 \times I_n$ defined by $f(a + a_n) = (a, a_n)$, for all $a \in A_1$ and $a_n \in I_n$, is an isomorphism of skew braces. By the inductive hypothesis, the map

$$f_2 : A_1 \to I_1 \times \cdots \times I_{n-1}, \quad f_2(a_1 + \cdots + a_{n-1}) = (a_1, \ldots, a_{n-1}),$$

is an isomorphism of skew braces. Therefore $f = (f_2 \times \mathrm{id}) f_1 : A \to I_1 \times \cdots \times I_n$ is an isomorphism of skew braces and $f(a_1 + \cdots + a_n) = (a_1, \ldots, a_n)$, for all $a_j \in I_j$. The result then follows. □

cor:product **Corollary 11.2.12.** *Let A be a finite skew brace such that $(A, +)$ and $(A, \circ)$ are nilpotent. Let $I_1, \ldots, I_n$ be the distinct Sylow subgroups of the additive group of A. Then $I_1, \ldots, I_n$ are ideals of A and the map*

$$f : A \to I_1 \times \cdots \times I_n, \quad f(a_1 + \cdots + a_n) = (a_1, \ldots, a_n),$$

for all $a_j \in I_j$, is an isomorphism of skew braces.

Proof. Since $(A, +)$ is nilpotent, for every prime divisor p of the order of A, there is a unique Sylow p-subgroup I of $(A, +)$. Hence I is a normal subgroup of $(A, +)$, and

$\lambda_a(b) \in I$ for all $a \in A$ and $b \in I$. Thus I is a left ideal of A and thus it is a Sylow p-subgroup of $(A,\circ)$. Since $(A,\circ)$ is nilpotent, I is the unique Sylow p-subgroup of $(A,\circ)$ and, thus, it is normal in $(A,\circ)$. Therefore I is an ideal of A. Hence $I_1,\dots,I_n$ are ideals of A and clearly the additive group of A is the direct sum of $I_1,\dots,I_n$. The result follows by Theorem 11.2.11. □

Let A be a skew brace. Let G be the multiplicative group of A and K be the additive group of A. The group G acts on K by automorphisms. Let Γ be the semidirect product $\Gamma = K \rtimes G$. The operation of Γ is

$$(x,g)(y,h) = (x+\lambda_g(y), g\circ h).$$

Identifying each $g \in G$ with $(0,g) \in \Gamma$ and each $x \in K$ with $(x,0) \in \Gamma$,

$$\begin{aligned}[g,x] = gxg^{-1}x^{-1} &= (0,g)(x,0)(0,g')(-x,0)\\ &= (\lambda_g(x),g)(-\lambda_g^{-1}(x),g') = (\lambda_g(x)-x,0) = \lambda_g(x)-x = g*x.\end{aligned}$$

Note that $A^2 = [G,K]$ and $A^{n+1} = [G,\dots,G,K]_n$ for $n > 0$.

pro:pgroups **Proposition 11.2.13.** *Let p be a prime and A be skew brace of size p^m. Then A is left nilpotent.*

Proof. Let G be the multiplicative group of A and K be the additive group of A. Since the semidirect product $\Gamma = K \rtimes G$ is a p-group, it is nilpotent. Thus there exists a positive integer k such that $[\Gamma,\Gamma,\dots,\Gamma]_k = \{(0,1)\}$. Since $A^{k+1} = [G,\dots,G,K]_k \subseteq [\Gamma,\dots,\Gamma]_k$, it follows that A is left nilpotent. □

thm:A2 **Theorem 11.2.14.** *Let A be a finite left nilpotent skew brace. Then the following statements hold:*

1) *The additive group of A^2 is nilpotent.*
2) *The multiplicative group of $A/\ker\lambda$ is nilpotent.*

Proof. Let G be the multiplicative group of A and K be the additive group of A. As above we consider G and K as subgroups of $\Gamma = K \rtimes G$. Since A is left nilpotent, there exists a positive integer n such that $A^n = \{0\}$. Hence $[G,\dots,G,K]_{n-1} = A^n = \{0\}$. By Theorem 11.1.14, $[G,K]$ is nilpotent, i. e. the additive group of A^2 is nilpotent.

Let C be the centralizer of K in G. Note that C is a normal subgroup of G. Since $[G,\dots,G,K]_m = \{1\}$, by Theorem 11.1.11, there exists a positive integer n such that $[[G,\dots,G]_n,K] = \{1\}$. Hence $[G,\dots,G]_n \subseteq C$. Now G/C is nilpotent. Note that

$$\begin{aligned}C &= \{g \in G \mid gxg^{-1} = x, \text{ for all } x \in K\}\\ &= \{g \in A \mid \lambda_g(x) = x, \text{ for all } x \in A\} = \ker\lambda.\end{aligned}$$

Thus G/C is the multiplicative group of $A/\ker\lambda$, and the result follows. □

lem:sylow_leftideals **Lemma 11.2.15.** *Let A be a finite skew brace with nilpotent additive group. Let p and q distinct prime numbers and let P and Q be Sylow subgroups of $(A,+)$ of sizes p^n and q^m, respectively. Then P, Q and $P+Q$ are left ideals of A.*

Proof. Let us first prove that P is a left ideal. Since $(A,+)$ is nilpotent, P is a normal subgroup of $(A,+)$. Let $a \in A$ and $x \in P$. Then $\lambda_a(x) \in P$ since λ_a is a group homomorphism. Similarly one proves that Q is a left ideal. From this it follows that $P+Q$ is a left ideal. □

The following is based on [70, Theorem 5(1)]. However, the proof is completely different.

thm:P*Q=0

Theorem 11.2.16. *Let A be a finite skew left brace with nilpotent additive group. Let p and q distinct prime numbers and let A_p and A_q be Sylow subgroups of $(A,+)$ of sizes p^n and q^m, respectively. If p does not divide q^t-1 for all $t \in \{1,\dots,m\}$, then $A_p * A_q = 0$. In particular, $\lambda_x(y) = y$ for all $x \in A_p$ and $y \in A_q$.*

Proof. By Lemma 11.2.15 A_p, A_q and $A_p + A_q$ are left ideals of A. In particular, $A_p + A_q$ is a skew subbrace of A and A_p and A_q are Sylow subgroups of $(A_p + A_q, \circ)$. By Sylow's theorem, the number n_p of Sylow p-subgroups of the multiplicative group of $A_p + A_q$ is

$$n_p = [A_p + A_q : N] \equiv 1 \bmod p,$$

where $N = \{g \in A_p + A_q : g \circ A_p \circ g' = A_p\}$ is the normalizer of A_p in the multiplicative group of $A_p + A_q$. Since $[A_p + A_q : N] = q^s$ for some $s \in \{0,\dots,m\}$ and p does not divide $q^t - 1$ for all $t \in \{1,\dots,n\}$, it follows that $s = 0$ and hence A_p is a normal subgroup of the multiplicative group of $A_p + A_q$. Thus A_p is an ideal of the skew brace $A_p + A_q$. Since A_p is an ideal of $A_p + A_q$ and A_q is a left ideal, we have that $A_p * A_q \subseteq A_p \cap A_q = 0$, and the result follows. □

Corollary 11.2.17. *Let A be a skew brace of size $p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, where $p_1 < p_2 < \cdots < p_k$ are prime numbers and $\alpha_1,\dots,\alpha_k$ are positive integers. Assume that the additive group of A is nilpotent. Let A_j be the Sylow p_j- subgroups of the additive group of A. Assume that, for some $j \le k$, p_j does not divide $p_i^{t_i} - 1$ for all $t_i \in \{1,\dots \alpha_i\}$ for all $i \neq j$. Then* $\mathrm{Soc}(A_j) \subseteq \mathrm{Soc}(A)$.

Proof. Write $A = A_1 + \cdots + A_k$. Let $a \in \mathrm{Soc}(A_j)$ and $b \in A$. Hence there exist elements $b_k \in A_k$ such that $b = b_1 + \cdots + b_k$. By Theorem 11.2.16, $\lambda_a(b_i) = b_i$, for all $i \neq j$. Then $\lambda_a(b) = \lambda_a(b_1) + \cdots + \lambda_a(b_k) = b_1 + \cdots + b_k = b$ and hence $a \in Soc(A)$. Thus the result follows. □

A finite group G is said to be *p-nilpotent* for a prime p if G has a normal p-complement. It is an easy exercise to show that a finite group is nilpotent if and only if it is p-nilpotent for every prime p.

Let A be a skew brace. For subsets X and Y of A we define inductively $L_0(X,Y) = Y$ and $L_{n+1}(X,Y) = X * L_n(X,Y)$ for $n \ge 0$.

Definition 11.2.18. Let p be a prime number. A finite skew brace A of nilpotent type is said to be *left p-nilpotent* if there exists $n \ge 1$ such that $L_n(A, A_p) = 0$, where A_p is the Sylow p-subgroup of $(A,+)$.

lem:factorization

Lemma 11.2.19. *Let A be a skew brace such that its additive group is the direct product of the left ideals B and C. Then $A*(B+C) = A*B + A*C$. Moreover, if $A = \oplus_{i=1}^{n} B_i$ where the B_i are left ideals, then*

$$A * \sum_{i=1}^{n} B_i = \sum_{i=1}^{n} A * B_i.$$

Proof. Let $a \in A$, $b \in B$ and $c \in C$. Then

$$a*(b+c) = a*b + b + a*c - b = a*b + a*c$$

holds for all $a \in A$, $b \in B$ and $c \in C$. The second part follows by induction. □

pro:left_p

Proposition 11.2.20. *Let A be a finite skew brace of nilpotent type. Then A is left nilpotent if and only if A is left p-nilpotent for all $p \in \pi(A)$.*

Proof. For each $p \in \pi(A)$ there exists $n(p) \in \mathbb{N}$ such that $L_{n(p)}(A, A_p) = \{0\}$. Let $n = \max\{n(p) : p \in \pi(A)\}$. Then $L_n(A, A_p) = \{0\}$ for all $p \in \pi(A)$. Since A is of nilpotent type, the additive group $(A,+)$ is isomorphic to the direct sum of the A_p for $p \in \pi(A)$. Then Lemma 11.2.19 implies that

$$L_n(A,A) = \sum_{p \in \pi(A)} L_n(A, A_p) = \{0\}.$$

The other implication is trivial. □

We now recall some notation about commutators. Given a brace A, the group $(A,\circ)$ acts on $(A,+)$ by automorphisms. If in the semidirect product $(A,+) \rtimes (A,\circ)$ we identify a with $(0,a)$ and b with $(b,0)$, then

$$\begin{aligned}[a,b] &= (0,a)(b,0)(0,a)^{-1}(b,0)^{-1} = (0,a)(b,0)(0,a')(-b,0)\\ &= (\lambda_a(b),a)(-\lambda_{a'}(b),a') = (\lambda_a(b)-b,0)\\ &= (a*b,0)\end{aligned}$$

Under this identification, we write $[X,Y] = X*Y$ for any pair of subsets $X, Y \subseteq A$. Then the iterated commutator satisfies

$$[X,\dots,X,Y] = [X,[X,\dots,[X,Y]\dots]] = L_n(X,Y),$$

where the subset X appears n times.

In a finite brace A of nilpotent type we denote by $A_{p'}$ the p-complement of A for every prime p.

thm:left_p

Theorem 11.2.21. *Let A be a finite brace of nilpotent type. The following statements are equivalent:*

1) *A is left p-nilpotent.*
2) *$A_{p'} * A_p = \{0\}$.*

3) *The group $(A,\circ)$ is p-nilpotent.*

Proof. We first prove that (1) implies (2). Since A is left p-nilpotent, there exists $n \in \mathbb{N}$ such that $L_n(A_{p'}, A_p) \subseteq L_n(A, A_p) = \{0\}$. Since $(A_{p'}, \circ)$ acts by automorphisms on $(A_p, +)$ and this is a coprime action, it follows from Corollary 11.1.9 that

$$L_1(A_{p'}, A_p) = A_{p'} * A_p = A_{p'} * (A_{p'} * A_p) = L_2(A_{p'}, A_p).$$

By induction one then proves that $A_{p'} * A_p = L_n(A_{p'}, A_p) = \{0\}$.

We now prove that (2) implies (3). It is enough to prove that $(A_{p'}, \circ)$ is a normal subgroup of $(A, \circ)$. By using Lemma 11.2.19,

$$A_{p'} * A = A_{p'} * (A_p + A_{p'}) = (A_{p'} * A_p) + (A_{p'} * A_{p'}) \subseteq A_{p'}.$$

since A'_p is a left ideal of A and $A_{p'} * A_p = \{0\}$. Then $A_{p'}$ is an ideal of A by Proposition 6.2.9. In particular, $(A_{p'}, \circ)$ is a normal subgroup of $(A, \circ)$.

Finally we prove that (3) implies (1). We need to prove that $L_n(A_p, A_p) = 0$ for some n. Since $(A, \circ)$ is p-nilpotent, there exists a normal p-complement that is a characteristic subgroup of $(A, \circ)$. This group is $A_{p'}$ and hence $A_{p'}$ is an ideal of A. Then $A_{p'} * A_p \subseteq A_{p'} \cap A_p = 0$. We now prove that $L_n(A, A_p) = L_n(A_p, A_p)$ for all $n \geq 0$. The case where $n = 0$ is trivial, so assume that the result holds for some $n \geq 0$. By the inductive hypothesis,

$$L_{n+1}(A, A_p) = A * L_n(A, A_p) = A * L_n(A_p, A_p).$$

Thus it is enough to prove that $A * L_n(A_p, A_p) \subseteq A_p * L_n(A_p, A_p)$. Let $a \in A$ and $b \in L_n(A_p, A_p)$. Write $a = x \circ y$ for $x \in A_p$ and $y \in A_{p'}$. Then

$$a * b = (x \circ y) * b = x * (y * b) + y * b + x * b = x * b \in A_p * L_n(A_p, A_p)$$

since $A_{p'} * A_p = 0$. The skew left brace A_p is left nilpotent by Proposition 11.2.13, so there exists $n \in \mathbb{N}$ such that $L_n(A_p, A_p) = 0$, and the result follows. □

thm:Smoktuwonicz

Theorem 11.2.22. *Let A be a finite skew brace of nilpotent type. Then, A is left nilpotent if and only if the multiplicative group of A is nilpotent.*

Proof. By Proposition 11.2.20, A is left nilpotent if and only if A is left p-nilpotent for every prime p. By Theorem 11.2.21, this is equivalent to the p-nilpotency of the multiplicative group of A for every prime p. Therefore the result follows. □

Exercises

11.2.1. Let G and K be groups such that G acts on K by automorphisms. Then $[G, K]$ is the unique smallest G-invariant subgroup of K such that the induced action of G on $K/[G, K]$ is trivial.

11.2.2. Let G and K be groups such that G acts faithfully on K by automorphisms. If $[G,[G,K]] = \{1\}$, then G is abelian.

prob:LN_direct

11.2.3. Let $A_1, \ldots, A_k$ be left nilpotent skew braces. Prove that $A_1 \times \cdots \times A_k$ is left nilpotent.

prob:LN_surj

11.2.4. Let $f\colon A \to B$ be a surjective homomorphism of skew braces. Prove that if A is left nilpotent, then B is left nilpotent.

prob:LN_sub

11.2.5. Let A be a left nilpotent skew brace and $B \subseteq A$ be a sub brace. Prove that B is left nilpotent.

11.2.6. Let A be finite skew brace such that $A^3 = 0$. Prove that A^2 is a trivial skew brace of abelian type.

Open problems

prob:nil=>leftnilpotent

Open problem 11.2.1. Let A be a finite left nil skew brace. Is A left nilpotent? And if furthermore A is of nilpotent type, is A left nilpotent?

tronglynil=>stronglynilp

Open problem 11.2.2. Let A be a finite strongly nil skew brace. Is A strongly nilpotent?

Notes

The left series of a skew brace was defined by Rump [65] in the context of skew braces of abelian type. Precisely in that paper he proved Proposition 11.2.13 by a different method in the case of skew braces of abelian type.

Strongly nilpotent skew braces of abelian type were defined by Smoktunowicz in [71]. These definitions extend to skew left braces, see [30].

Theorem 11.2.11 was proved by Byott in the context of Hopf–Galois extensions [19].

Theorem 12.1.8 was proved by Smoktunowicz in [71] for skew braces of abelian type and it was extended to nilpotent type in [30].

Theorem 11.2.22 was proved by Smoktunowicz in [71, Theorem 1.1] for skew braces of abelian type. The generalization to skew braces of nilpotent type appeared in [30, Theorem 4.8]. The proof presented in this chapter appeared in [1] and it is heavily based on the ideas of Ballester–Bolinches, Meng and Romero [58].

Theorem 11.2.21 was proved by Ballester–Bolinches, Meng and Romero for skew braces of abelian type.

Open problem 11.2.1 was proved in the case of skew braces of abelian type by Smoktunowicz [70].

Chapter 12
Right nilpotent braces

In this chapter we study right nilpotency of skew braces. We will see later that this algebraic notion translates into an important combinatorial property of solutions.

12.1 Right series

Definition 12.1.1. Let A be a skew brace. One defines $A^{(1)} = A$ and for $n \geq 1$

$$A^{(n+1)} = A^{(n)} * A = \langle x * a : x \in A^{(n)}, a \in A \rangle_+,$$

where $\langle X \rangle_+$ denotes the subgroup of the additive group of A generated by the subset X. The sequence

$$A^{(1)} \supseteq A^{(2)} \supseteq A^{(3)} \supseteq \cdots \supseteq A^{(n)} \supseteq \cdots$$

is the *right series* of A.

pro:right_series

Proposition 12.1.2. *Let A be a skew brace. Each $A^{(n)}$ is an ideal of A.*

Proof. We want to prove that for each $n \in \mathbb{N}$, $A^{(n)}$ is a normal subgroup of $(A,+)$, that $\lambda_a(A^{(n)}) \subseteq A^{(n)}$ for all $a \in A$ and that $A^{(n)}$ is a normal subgroup of $(A,\circ)$. We proceed by induction on n. The case $n = 1$ is trivial. We assume that the claim is true for some $n \geq 1$. We first prove that $A^{(n+1)}$ is a normal subgroup of $(A,+)$. Let $a, b \in A$ and $x \in A^{(n)}$. Then $a + x * b - a \in A^{(n+1)}$ since

$$\begin{aligned} a + x * b - a &= a + \lambda_x(b) - b - a \\ &= a + \lambda_x(b) - (a + b) = a + \lambda_x(-a + a + b) - (a + b) \\ &= a + \lambda_x(-a) + \lambda_x(a + b) - (a + b) = -x * a + x * (a + b). \end{aligned}$$

Now we prove that $A^{(n+1)}$ is an ideal. Let $a, b \in A$ and $x \in A^{(n)}$. Then, by (11.1),

$$\lambda_a(x * b) = (a \circ x \circ a') * \lambda_a(b) \in A^{(n+1)}$$

since $a \circ x \circ a' \in A^{(n)}$ by the inductive hypothesis. From this it immediately follows that $\lambda_a(A^{(n+1)}) \subseteq A^{(n+1)}$. Now let $y \in A^{(n+1)}$. By using the formula (6.12) of Proposition 6.2.9 one obtains that $a \circ y \circ a' = a + \lambda_a(y + y * a') - a \in A^{(n+1)}$. Thus the result follows by induction. □

The second term of the right series is particularly important.

Proposition 12.1.3. *Let A be a skew brace. Then $A^{(2)}$ is the smallest ideal of A such that $A/A^{(2)}$ is a trivial brace.*

Proof. Proposition 12.1.2 shows that $A^{(2)}$ is an ideal of A. Let I be an ideal of A and $\pi\colon A \to A/I$ be the canonical map. Then A/I is trivial as a brace if and only if $\lambda_a(b) - b \in I$ for all $a, b \in A$. Since this condition is equivalent to $A^{(2)} \subseteq I$, the claim follows. □

Corollary 12.1.4. *Let A be a finite skew brace of size p^n for some prime number p and some positive integer n. Then either A is the trivial brace of order p or it is not simple.*

Proof. By Proposition 11.2.13, A is left nilpotent. In particular, if $A \neq 0$, then $A^{(2)} = A^2 \neq A$. Since A^2 is an ideal either A is not simple or $A^2 = 0$. Assume that $A^2 = 0$. In this case, $a \circ b = a + b$ for all $a, b \in A$. Therefore the derived subgroup $[A, A]_+$ of the additive group of A is a proper an ideal of A. Hence, either A is not simple or $[A, A]_+ = 0$. Assume that $A^2 = [A, A]_+ = 0$. In this case A is a trivial skew brace of abelian type and the result follows. □

Definition 12.1.5. A brace A is said to be *right nilpotent* if $A^{(m)} = \{0\}$ for some $m \geq 1$.

Let us review some basic properties of right nilpotent braces.

right_nilpotent:quotient

Lemma 12.1.6. *Let A and B be skew braces.*

1) *If $f\colon A \to B$ is a surjective homomorphism of skew braces and A is right nilpotent, then B is right nilpotent.*
2) *B is a sub brace of A and A is right nilpotent, then B is right nilpotent.*

Proof. To prove the first claim we proceed by induction on k. The case $k = 1$ is trivial. Let us assume that the result is valid for some $k \geq 1$. Since $f(A^{(k)}) = B^{(k)}$,

$$f(A^{(k+1)}) = f(A^{(k)} * A) = f(A^{(k)}) * f(A) = B^{(k)} * B = B^{(k+1)}.$$

The second claim also follows by induction, as $B^{(k)} \subseteq A^{(k)}$ for all k. □

thm:IcapSoc

Theorem 12.1.7. *Let A be a right nilpotent skew brace of nilpotent type and I be a non-zero ideal of A. Then $I \cap \mathrm{Soc}(A) \neq \{0\}$. In particular,* $\mathrm{Soc}(A) \neq \{0\}$.

Proof. Since $(A,+)$ is nilpotent and each $I \cap A^{(k)}$ is a normal subgroup of $(A,+)$, it follows from Theorem 4.1.14 that $I \cap A^{(k)} \cap Z(A,+) \neq \{0\}$ whenever $I \cap A^{(k)} \neq \{0\}$. Let $m = \max\{k \in \mathbb{N} : I \cap A^{(k)} \cap Z(A,+) \neq \{0\}\}$. Since

$$(I \cap A^{(m)} \cap Z(A,+)) * A \subseteq I \cap (A^{(m)} * A) = I \cap A^{(m+1)} = \{0\},$$

it follows that $I \cap A^{(m)} \cap Z(A,+) \subseteq I \cap \mathrm{Soc}(A)$. For the second claim just take $I = A$ in the first claim. □

The relation between the sequence of the $A^{[n]}$ and the left and right series is given in the following theorem.

thm:equivalence

Theorem 12.1.8. *Let A be a skew brace. The following statements are equivalent:*

1) $A^{[\alpha]} = \{0\}$ *for some* $\alpha \in \mathbb{N}$.
2) $A^{(\beta)} = \{0\}$ *and* $A^{\gamma} = \{0\}$ *for some* $\beta, \gamma \in \mathbb{N}$.

Proof. To prove that (1) $\implies$ (2) one proves that $A^n \subseteq A^{[n]}$ and $A^{(n)} \subseteq A^{[n]}$ for all positive integer n. Let us prove that (2) $\implies$ (1). We proceed by induction on β. If $\beta \in \{1,2\}$, then $0 = A^{(2)} = A^2 = A^{[2]}$ and the result is true. Fix $\beta \in \mathbb{N}$ and suppose that the result holds for this β, so for every γ there exists $\alpha = \alpha(\gamma)$ depending on γ such that $A^{[\alpha]} = 0$. We need to show that $A^{(\beta+1)} = 0$ and $A^{\gamma} = 0$ imply that $A^{[n]}$ for some n. Let $n > \alpha(\gamma)$. Every element of $A^{[n]}$ is a sum of elements from $A^{[i]} * A^{[j]}$, where $i + j = n$ and $1 \leq i \leq n-1$. Note that if $\alpha(\gamma) \leq i \leq n-1$, $a_i \in A^{[i]}$ and $a_{n-i} \in A^{[n-i]}$, then by the inductive hypothesis applied to the quotient $A/A^{(\beta)}$, $a_i \in A^{(\beta)}$ and thus $a_i * a_{n-i} \in A^{(\beta+1)} = 0$. Hence we may assume that the elements of $A^{[n]}$ are sums of elements from $A^{[i]} * A^{[j]}$ for $1 \leq i < \alpha(\gamma)$ and $j \geq n - \alpha(\gamma)$ such that $i + j = n$. Then

$$A^{[n]} \subseteq A * A^{[n-\alpha(\gamma)]} \subseteq A^2.$$

Applying the same argument for $n' = n - \alpha(\gamma)$ we obtain that $A^{[n']} \subseteq A * A^{[n'-\alpha(\gamma)]}$ provided that $n' > \alpha(\gamma)$. Therefore

$$A^{[n]} \subseteq A * (A * A^{[n-2\alpha(\gamma)]}) \subseteq A^3$$

provided that $n > 2\alpha(\gamma)$. Continuing in this way we obtain that $A^{[n]} \subseteq A^k$ provided that $n > (k-1)\alpha(\gamma)$. Then it follows that $A^{[(\gamma-1)\alpha(\gamma)+1]} \subseteq A^{\gamma} = \{0\}$. □

12.2 Right nilpotent skew braces

We now present some characterizations of right nilpotent skew braces. We first need some definitions.

defn:s-series

Definition 12.2.1. Let A be a skew brace. A *s-series* of A is a sequence

$$A = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n = \{0\}$$

of ideals of A such that $I_{j-1}/I_j \subseteq \mathrm{Soc}(A/I_j)$ for each $j \in \{1,\dots,n\}$.

defn:socle_series

Definition 12.2.2. Let A be a skew brace. We define $\mathrm{Soc}_0(A) = \{0\}$ and, for $n \geq 1$, $\mathrm{Soc}_n(A)$ is the ideal of A containing $\mathrm{Soc}_{n-1}(A)$ such that

$$Soc_n(A)/\mathrm{Soc}_{n-1}(A) = \mathrm{Soc}(A/\mathrm{Soc}_{n-1}(A)).$$

Hence

$$\{0\} = \mathrm{Soc}_0(A) \subseteq \mathrm{Soc}_1(A) \subseteq \cdots \subseteq \mathrm{Soc}_n(A) \subseteq \dots$$

is a series of ideals of the skew brace A. It is called the *socle series* of A.

lem:socle_series

Lemma 12.2.3. *Let A be a skew brace and let $A = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n = \{0\}$ be a s-series for A. Then $A^{(i+1)} \subseteq I_i$ for all i.*

Proof. We proceed by induction on i. The case $i = 0$ is trivial, so let us assume that the result holds for some $i \geq 0$. Let $\pi\colon A \to A/I_{i+1}$ be the canonical map. Since $\pi(I_i) \subseteq \mathrm{Soc}(A/I_{i+1})$, $\pi(I_i * A) = \pi(I_i) * \pi(A) = \{0\}$ and hence $I_i * A \subseteq I_{i+1}$. The inductive hypothesis then implies that $A^{(i+2)} = A^{(i+1)} * A \subseteq I_i * A \subseteq I_{i+1}$. Thus the result follows. □

lem:socn

Lemma 12.2.4. *Let A be a skew brace. Then A admits a s-series if and only if there exists a positive integer n such that $A = \mathrm{Soc}_n(A)$.*

Proof. Suppose that there exists a positive integer n such that $A = \mathrm{Soc}_n(A)$. Then

$$A = \mathrm{Soc}_n(A) \supseteq \mathrm{Soc}_{n-1}(A) \supseteq \cdots \supseteq \mathrm{Soc}_0(A) = \{0\},$$

is a s-series. Conversely, suppose that A admits a s-series. Let

$$A = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n = \{0\}$$

be a s-series of A. We shall prove that $I_{n-j} \subseteq \mathrm{Soc}_j(A)$ by induction on j. For $j = 0$, $I_n = \{0\} = \mathrm{Soc}_0(A)$. Suppose that $j > 0$ and $I_{n-j+1} \subseteq \mathrm{Soc}_{j-1}(A)$. Since $I_{n-j}/I_{n-j+1} \subseteq \mathrm{Soc}(A/I_{n-j+1})$, $I_{n-j} * A \subseteq I_{n-j+1} \subseteq \mathrm{Soc}_{j-1}(A)$, by the induction hypothesis. Furthermore, for all $x \in A$ and all $y \in I_{n-j}$,

$$x + y - x - y \in I_{n-j+1} \subseteq \mathrm{Soc}_{j-1}(A).$$

Therefore $I_{n-j} \subseteq \mathrm{Soc}_j(A)$. Hence $A = I_0 = \mathrm{Soc}_n(A)$ and the result follows. □

pro:right_nilpotent

Proposition 12.2.5. *A skew brace of nilpotent type is right nilpotent if and only if it admits a s-series.*

Proof. Let A be a skew brace of nilpotent type. If A admits a s-series, then A is right nilpotent by Lemma 12.2.3. Conversely, suppose that A is right nilpotent. There exists a positive integer such that $A^{(m)} = \{0\}$. We shall prove that A admits a s-series by induction on m. For $m = 1$, $A = A^{(1)} = \{0\}$ is a s-series. Suppose that $m > 1$ and that the result is true for $m - 1$. Consider $\bar{A} = A/A^{(m-1)}$. Since $\bar{A}^{(m-1)} = \{0\}$, by the induction hypothesis $\bar{A}$ admits a s-series. Thus there is a sequence

$$A = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n = A^{(m-1)}$$

of ideals of A such that $I_{j-1}/I_j \subseteq \mathrm{Soc}(A/I_j)$ for each $j \in \{1,\dots,n\}$. Since $A^{(m)} = \{0\}$, we have that $A^{(m-1)} \subseteq \ker(\lambda)$. Since A is of nilpotent type, there exists a positive integer s such that $\gamma_s^+(A) = \{0\}$, where $\gamma_i^+(A)$ denotes the lower central series of the additive group of A, that is $\gamma_1^+(A) = A$ and $\gamma_{i+1}^+(A) = [A, \gamma_i^+(A)]_+$, for all positive integers i. Let $I_{n+j-1} = A^{(m-1)} \cap \gamma_j^+(A)$ for $j = 1,\dots,s$. Note that I_{n+j-1} is a normal subgroup of the additive group of A invariant by λ_x, for all $x \in A$, and $I_{n+j-1} * A = \{0\}$, for all $j = 1,\dots,s$, because $A^{(m-1)} \subseteq \ker(\lambda)$. By Proposition 6.2.9, I_{n+j-1} is an ideal of A, for all $j = 1,\dots,s$. Note that $I_{n+j-1}/I_{n+j} \subseteq Z(A/I_{n+j}, +)$, for all $j = 1,\dots,s-1$. Therefore, since $I_{n+j-1} \subseteq \ker(\lambda)$, we have that $I_{n+j-1}/I_{n+j} \subseteq \mathrm{Soc}(A/I_{n+j})$, for all $j = 1,\dots,s-1$. Hence

$$A = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n = A^{(m-1)} \supseteq I_{n+1} \supseteq \cdots \supseteq I_{n+s-1} = \{0\}$$

is a s-series of A, and the result follows by induction. □

pro:A/Soc

Proposition 12.2.6. *Let A be a skew brace such that $A/\mathrm{Soc}(A)$ is right nilpotent. Then A is right nilpotent.*

Proof. Note that $(A/\mathrm{Soc}(A))^{(k)} = \{0\}$ if and only if $A^{(k)} \subseteq \mathrm{Soc}(A)$ by the definition of the quotient brace. Then $A^{(k+1)} = A^{(k)} * A \subseteq \mathrm{Soc}(A) * A = \{0\}$ as required. □

Definition 12.2.7. A skew brace A has *finite multipermutation level* if the sequence S_n defined as $S_1 = A$ and $S_{n+1} = S_n/\mathrm{Soc}(S_n)$ for $n \geq 1$, reaches zero.

newmulti

Proposition 12.2.8. *Let A be a skew brace. Then A has finite multipermutation level if and only if A admits a s-series.*

Proof. Let $S_1 = A$ and $S_{n+1} = S_n/\mathrm{Soc}(S_n)$ for $n \geq 1$. By induction on n we prove that $S_{n+1} \simeq A/\mathrm{Soc}_n(A)$. For $n = 0, 1$, it is clear since $\mathrm{Soc}_0(A) = \{0\}$ and $\mathrm{Soc}_1(A) = \mathrm{Soc}(A)$. Suppose that $n > 1$ and the result is true for $n-1$. Hence, by the inductive hypothesis,

$$\begin{aligned} S_n &= S_{n-1}/\mathrm{Soc}(S_{n-1}) \\ &\simeq (A/\mathrm{Soc}_{n-2}(A))/\mathrm{Soc}(A/\mathrm{Soc}_{n-2}(A)) \\ &= (A/\mathrm{Soc}_{n-2}(A))/(\mathrm{Soc}_{n-1}(A)/\mathrm{Soc}_{n-2}(A)) \simeq A/\mathrm{Soc}_{n-1}(A). \end{aligned}$$

Therefore $S_n = 0$ if and only if $A = \mathrm{Soc}_{n-1}(A)$. Now Lemma 12.2.4 applies. □

thm:mpl&right_nilpotent

Theorem 12.2.9. *Let A be a skew brace. Then A has finite multipermutation level, if and only if A is right nilpotent and $(A,+)$ is nilpotent.*

Proof. Suppose that A has finite multipermutation level. We proceed by induction on the multipermutation level n. The case $n = 1$ is trivial. Let A be a skew left brace of finite multipermutation level $n+1$. Since $A/\mathrm{Soc}(A)$ has multipermutation level n, the inductive hypothesis implies that $(A/\mathrm{Soc}(A))^{(m)} = \{0\}$ for some

m and $(A/\mathrm{Soc}(A),+)$ is nilpotent. This implies that $A^{(m)} \subseteq \mathrm{Soc}(A)$ and hence $A^{(m+1)} = \{0\}$, furthermore, since $\mathrm{Soc}(A)$ is central in $(A,+)$, we have that $(A,+)$ is nilpotent. Conversely, suppose that A is right nilpotent and $(A,+)$ is nilpotent. By Proposition 12.2.5, A admits a s-series. Thus the result follows from Proposition 12.2.8. □

The following example shows that the assumption on the nilpotency of the additive group of the skew brace is needed in Theorem 12.2.9.

ex:trivial **Example 12.2.10.** Let A be a non-zero skew brace such that $a \circ b = a + b$ for all $a, b \in A$. Then $A^{(2)} = \{0\}$, thus A is right nilpotent. If $Z(A,+) = \{0\}$, then $\mathrm{Soc}(A) = \{0\}$ and A does not have finite multipermutation level. For example, we can take $(A,+) = (A,\circ)$ any non-abelian simple group.

Let A be a skew brace. For subsets X and Y of A we define inductively $R_0(X,Y) = X$ and $R_{n+1}(X,Y)$ as the additive subgroup generated by $R_n(X,Y) * Y$ and $[R_n(X,Y),Y]_+$ for $n \geq 0$.

lem:R:inclusion **Lemma 12.2.11.** *Let I be an ideal of a skew brace A. Then $R_{n+1}(I,A) \subseteq R_n(I,A)$ for all $n \geq 0$.*

Proof. We proceed by induction on n. The case $n = 0$ is trivial as I is an ideal of A. Let us assume that the claim holds for some $n \geq 0$. Since by the inductive hypothesis $R_n(I,A) * A \subseteq R_{n-1}(I,A) * A \subseteq R_n(I,A)$ and

$$[R_n(I,A),A]_+ \subseteq [R_{n-1}(I,A),A]_+ \subseteq R_n(I,A),$$

it follows that $R_{n+1}(I,A) \subseteq R_n(I,A)$. □

lem:R:ideal **Proposition 12.2.12.** *Let I be an ideal of a skew brace A. Then each $R_n(I,A)$ is an ideal of A.*

Proof. We proceed by induction on n. The case where $n = 0$ follows from the fact that I is an ideal of A. So assume that the result holds for some $n \geq 0$. We first prove that $R_{n+1}(I,A)$ is a normal subgroup of $(A,+)$. Let $a, b \in A$ and $x \in R_n(I,A)$. Then

$$a + x * b - a = -x * a + x * (a + b) \in R_{n+1}(I,A),$$

by definition. Since, moreover,

$$\begin{aligned} &a + (x + b - x - b) - a \\ &\qquad = (a + x - a) + (a + b - a) - (a + x - a) - (a + b - a) \in R_{n+1}(I,A) \end{aligned}$$

by the inductive hypothesis, it follows that $R_{n+1}(I,A)$ is a normal subgroup of $(A,+)$.

We now prove that

$$\lambda_a(R_{n+1}(I,A)) \subseteq R_{n+1}(I,A) \tag{12.1}$$ eq:lambda

for all $a \in A$. Using the inductive hypothesis and that each $\lambda_a \in \mathrm{Aut}(A,+)$, by (11.1), we have that

$$\lambda_a(x * b) = (a \circ x \circ a') * \lambda_a(b) \in R_{n+1}(I, B)$$

and

$$\begin{aligned}\lambda_a([R_n(I,A), A]_+) &\subseteq [\lambda_a(R_n(I,A)), \lambda_a(A)]_+ \\ &\subseteq [R_n(I,A), A]_+ \subseteq R_{n+1}(I,A),\end{aligned}$$

and thus (12.1) follows.

Since $R_{n+1}(I,A) \subseteq R_n(I,A)$ by Lemma 12.2.11,

$$R_{n+1}(I,A) * A \subseteq R_n(I,A) * A \subseteq R_{n+1}(I,A).$$

Hence the claim follows from Proposition 6.2.9. □

lem:mbbr **Lemma 12.2.13.** *Let A be a skew brace, X be a subset of A and $n, m \in \mathbb{N}$. Then $R_m(X,A) \subseteq \mathrm{Soc}_n(A)$ if and only if $X \subseteq \mathrm{Soc}_{m+n}(A)$.*

Proof. We proceed by induction on m. The case where $m = 0$ is trivial, so assume that the result is valid for some $m \geq 0$. Note that $R_{m+1}(X,A) \subseteq \mathrm{Soc}_n(A)$ is equivalent to $R_m(X,A) * A \subseteq \mathrm{Soc}_n(A)$ and $[R_m(X,A), A]_+ \subseteq \mathrm{Soc}_n(A)$. By the definition of the socle series, this is equivalent to $R_m(X,A) \subseteq \mathrm{Soc}_{n+1}(A)$, which is equivalent to $X \subseteq \mathrm{Soc}_{m+n+1}(A)$ by the inductive hypothesis. □

Lemma 12.2.14. *A skew brace A of nilpotent type is right nilpotent if and only if $R_n(A,A) = \{0\}$ for some $n \in \mathbb{N}$.*

Proof. By Lemma 12.2.13, $R_n(A,A) = 0$ if and only if $A = \mathrm{Soc}_n(A)$. By Lemma 12.2.4 and Proposition 12.2.5, the latter is equivalent to A being right nilpotent. □

Recall that a finite group G is said to be *p-nilpotent* if G has a normal p-complement. One proves that this subgroup is characteristic in G.

Definition 12.2.15. Let p be a prime number. A finite skew brace A of nilpotent type is said to be *right p-nilpotent* if there exists $n \geq 1$ such that $R_n(A_p, A) = \{0\}$, where A_p is the Sylow p-subgroup of $(A,+)$.

pro:soc_n **Proposition 12.2.16.** *Let A be a finite skew brace of nilpotent type and $p \in \pi(A)$. Then $A_p \subseteq \mathrm{Soc}_n(A)$ for some $n \geq 1$ if and only if A is right p-nilpotent.*

Proof. By Lemma 12.2.13, $R_n(A_p, A) = \{0\}$ if and only if $A_p \subseteq \mathrm{Soc}_n(A)$. □

Proposition 12.2.17. *A finite skew brace A of nilpotent type is right nilpotent if and only if A is right p-nilpotent for all $p \in \pi(A)$.*

Proof. Assume first that A is right nilpotent. By Lemma 12.2.4 and Proposition 12.2.5, there exists $n \in \mathbb{N}$ such that $A_p \subseteq A = \mathrm{Soc}_n(A)$ for all $p \in \pi(A)$. Hence the claim follows from Lemma 12.2.4. Assume now that A is right p-nilpotent for all $p \in \pi(A)$. This means that for each $p \in \pi(A)$ there exists $n(p) \in \mathbb{N}$ such that $A_p \subseteq \mathrm{Soc}_{n(p)}(A)$. Let $n = \max\{n(p) : p \in \pi(A)\}$. Then $A_p \subseteq \mathrm{Soc}_n(A)$ for all $p \in \pi(A)$. Since $\mathrm{Soc}_n(A)$ is an ideal of A and A is of nilpotent type, $A = \oplus_{p \in \pi(A)} A_p \subseteq \mathrm{Soc}_n(A)$. Hence A is right nilpotent by Lemma 12.2.4 and Proposition 12.2.5. □

lem:A_p:ideal

Lemma 12.2.18. *Let A be a finite skew brace of nilpotent type. If $(A,\circ)$ has a normal Sylow p-subgroup for some $p \in \pi(A)$, then A_p is an ideal of A.*

Proof. Since the group $(A,+)$ is nilpotent, there exists a unique normal Sylow p-subgroup A_p of $(A,+)$. Thus A_p is a left ideal of A. Then A_p is a Sylow p-subgroup of $(A,\circ)$, normal by hypothesis and hence A_p is an ideal of A. □

lem:Soc(A_p)

Lemma 12.2.19. *Let A be a finite skew brace of nilpotent type. If $(A,\circ)$ has a normal Sylow p-subgroup for some $p \in \pi(A)$, then $\operatorname{Soc}(A_p) = \operatorname{Soc}(A) \cap A_p$. In particular, $\operatorname{Soc}(A_p)$ is an ideal of A.*

Proof. By Lemma 12.2.18, A_p is an ideal of A. Clearly $\operatorname{Soc}(A_p) \supseteq \operatorname{Soc}(A) \cap A_p$, so we only need to prove that $\operatorname{Soc}(A_p) \subseteq \operatorname{Soc}(A) \cap A_p$. If $a \in \operatorname{Soc}(A_p)$, then $a \in Z(A_p,+) = Z(A,+) \cap A_p$ and $a * b = 0$ for all $b \in A_p$. Let $c \in A$ and write $c = x + y$, where $x \in A_p$ and $y \in A_{p'}$. Since

$$a * c = a * (x+y) = a * x + x + a * y - x = x + a * y - x \in A_p \cap A_{p'} = 0$$

and $a \in Z(A,+)$, the lemma is proved. □

thm:right_p

Theorem 12.2.20. *Let A be a finite skew left brace of nilpotent type. If $(A,\circ)$ has an abelian normal Sylow p-subgroup for some $p \in \pi(A)$, then A is right p-nilpotent.*

Proof. Let us assume that the result does not hold and let A be a counterexample of minimal size. We may assume that A is non-trivial, i.e. $\operatorname{Soc}(A) \neq A$. By Lemma 12.2.18, A_p is an ideal of A.

Since $\lambda_a \in \operatorname{Aut}(A_p,+)$, $\lambda_a(Z(A_p,+)) \subseteq Z(A_p,+)$ and hence $Z(A_p,+)$ is a left ideal of A_p.

By Lemma 12.2.19, $\operatorname{Soc}(A_p)$ is an ideal of A. Furthermore, since $(A_p,\circ)$ is abelian,

$$\begin{aligned}\operatorname{Soc}(A_p) &= \{a \in A_p : a * b = 0 \text{ for all } b \in A_p\} \cap Z(A_p,+)\\ &= \{a \in A_p : a \circ b = a + b \text{ for all } b \in A_p\} \cap Z(A_p,+)\\ &= \{a \in A_p : b \circ a = b + a \text{ for all } b \in A_p\} \cap Z(A_p,+)\\ &= \operatorname{Fix}(A_p) \cap Z(A_p,+).\end{aligned}$$

Since $|A_p| = p^m$ for some $m \geq 1$, the skew left brace A_p is left nilpotent by Proposition 11.2.13, and, moreover, $Z(A_p,+)$ is a non-zero subgroup of $(A_p,+)$. Then $\operatorname{Soc}(A_p) = \operatorname{Fix}(A_p) \cap Z(A_p,+) \neq \{0\}$ by Proposition 11.2.4. In particular,

$$\{0\} \neq \operatorname{Soc}(A_p) \subseteq \operatorname{Soc}(A).$$

Lemma 12.2.19 implies that $I = \operatorname{Soc}(A_p)$ is a non-trivial ideal of A. Then A/I is a brace of nilpotent type such that $0 < |A/I| < |A|$. The minimality of $|A|$ implies that A/I is right p-nilpotent and therefore $R_n(A_p/I, A/I) = \{0\}$ for some n. That is $R_n(A_p, A) \subseteq I \subseteq \operatorname{Soc}(A)$. By Lemma 12.2.13, $R_{n+1}(A_p, A) = \{0\}$. Then A is right p-nilpotent, a contradiction. □

Recall that a group G has the *Sylow tower property* if there exists a normal series $1 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G$ such that each quotient G_i/G_{i-1} is isomorphic to a Sylow subgroup of G, in particular G_1 is a normal Sylow subgroup of G. We also recall that *A–groups* are finite groups whose Sylow subgroups are abelian.

cor:STP+abelian

Corollary 12.2.21. *Let A be a finite skew brace of nilpotent type. Assume that $(A,\circ)$ has the Sylow tower property and that all Sylow subgroups of $(A,\circ)$ are abelian. Then A is right nilpotent.*

Proof. Assume that the result is not true and let A be a counterexample of minimal size. Since $(A,\circ)$ has the Sylow tower property, there exists a normal Sylow p-subgroup A_p of $(A,\circ)$. Then A_p is a non-zero ideal of A and one proves that

$$\{0\} \neq \operatorname{Soc}(A_p) = \operatorname{Soc}(A) \cap A_p \subseteq \operatorname{Soc}(A).$$

The group $(A/\operatorname{Soc}(A),\circ)$ has abelian Sylow subgroups and has the Sylow tower property. Since A is a non-trivial skew left brace, $0 < |A/\operatorname{Soc}(A)| < |A|$, and therefore $A/\operatorname{Soc}(A)$ is right nilpotent by the minimality of $|A|$. By Proposition 12.2.6, A is right nilpotent, a contradiction. □

There are examples of right nilpotent skew braces of abelian type where the multiplicative group contains a non-abelian Sylow subgroup or does not have the Sylow tower property:

Example 12.2.22. The operation $a \circ b = a + 3^a b$ turns $\mathbb{Z}/8$ into a right nilpotent skew brace of abelian type with multiplicative group isomorphic to the quaternion group. This example appears in [7].

Example 12.2.23. Let $G = \mathbb{A}_4 \times \mathbb{S}_3$. Each Sylow subgroups of G is abelian, so it follows from [**?**, Theorem 2.1] that there exists a skew brace of abelian type with multiplicative group isomorphic to G. The group G does not have the Sylow tower property. The database of left braces of [45] shows that there are only four left braces with multiplicative group isomorphic to G, all with additive group isomorphic to $C_6 \times C_6 \times C_2$. However, only one of these four braces is not right nilpotent.

Corollary 12.2.24. *Let A be a finite skew brace of nilpotent type. If all Sylow subgroups of the multiplicative group of A are cyclic, then A is right nilpotent.*

Proof. Since all Sylow subgroups of $(A,\circ)$ are cyclic, the group $(A,\circ)$ is supersolvable and hence it has the Sylow tower property. Then the claim follows from Corollary 12.2.21. □

Exercises

12.2.1. Let $A_1, \ldots, A_k$ be right nilpotent skew braces. Prove that $A_1 \times \cdots \times A_k$ is right nilpotent.

12.2.2. Let A be a right nilpotent skew brace of nilpotent type and I be a minimal ideal of A. Prove that $I \subseteq \mathrm{Soc}(A)$.

12.2.3. Let I be an ideal of a skew brace A such that $I \cap A^2 = \{0\}$. Prove that I is a trivial skew brace.

12.2.4.

Open problems

tion:rightnil=>rightnilp

Open problem 12.2.1. For a skew brace A let $\rho_1(a) = a$ and $\rho_{k+1}(a) = \rho_k(a) * a$ for $n \geq 1$. The skew brace A is said to be *right nil* if for every $a \in A$ there exists a positive integer $n = n(a)$ such that $\rho_n(a) = 0$.

Let A be a finite right nil skew brace. Is A right nilpotent?

Notes

The right series first appeared in the work of Rump [65], where it was proved Proposition 12.1.2 for braces of abelian type. The proof of the general case appeared in [30].

The socle series was introduced by Rump [65] for skew braces of abelian type.

Theorem 12.2.20 was proved by Meng, Ballester–Bolinches and Romero in the case of skew braces of abelian type [58]. For general skew braces the proof appeared in [1].

Chapter 13
Multipermutation solutions

MP

13.1 The permutation group of a solution

Let (X,r) be a solution to the YBE. We write

$$r(x,y) = (\sigma_x(y), \tau_y(x)) \quad \text{and} \quad r^{-1}(x,y) = (\widehat{\sigma}_x(y), \widehat{\tau}_y(x))$$

for all $x,y \in X$. Consider the structure group $G(X,r)$ of the solution (X,r). Let $i\colon X \to G(X,r)$ be the natural map.

The *permutation group* of (X,r) is the subgroup

$$\mathcal{G}(X,r) = \langle (\sigma_x, \tau_x^{-1}) : x \in X \rangle \subseteq \mathbb{S}_X \times \mathbb{S}_X.$$

Since

$$\sigma_x\sigma_y = \sigma_{\sigma_x(y)}\sigma_{\tau_y(x)} \quad \text{and} \quad \tau_x^{-1}\tau_y^{-1} = \tau_{\sigma_x(y)}^{-1}\tau_{\tau_y(x)}^{-1}$$

for all $x,y \in X$, there exists a unique group homomorphism $h\colon G(X,r) \to \mathcal{G}(x,r)$ such that $hi(x) = (\sigma_x, \tau_x^{-1})$ for all $x \in X$. By Theorem 8.2.4, $G(X,r)$ has a unique structure of skew brace with multiplicative group the structure group $G(X,r)$ and $\lambda_{i(x)}(i(y)) = i(\sigma_x(y))$ for all $x \in X$. We shall see that $\ker h$ is an ideal of the skew brace $G(X,r)$. Note that

$$\mu_{i(y)}(i(x)) = \lambda_{i(x)}(i(y))i(x)i(y) = i(\sigma_x(y))i(x)i(y) = i(\tau_y(x))$$

for all $x,y \in X$, by the defining relations of $G(X,r)$. Let $a \in \ker h$. There exists $x_1,\dots,x_n \in X$ and $\epsilon_1,\dots,\epsilon_n \in \{-1,1\}$ such that $a = i(x_1)^{\epsilon_1}\cdots i(x_n)^{\epsilon_n}$. Hence

$$(\mathrm{id}_X, \mathrm{id}_X) = h(a) = (\sigma_{x_1}^{\epsilon_1}\cdots\sigma_{x_n}^{\epsilon_n}, \tau_{x_1}^{-\epsilon_1}\cdots\tau_{x_n}^{-\epsilon_n}).$$

Therefore

$$\lambda_a(i(z)) = i(\sigma_{x_1}^{\epsilon_1}\cdots\sigma_{x_n}^{\epsilon_n}(z)) = i(z)$$

and

$$\mu_a^{-1}(i(z)) = i(\tau_{x_1}^{-\epsilon_1}\cdots\tau_{x_n}^{-\epsilon_n}(z)) = i(z)$$

for all $z \in X$ and thus

$$\ker h \subseteq \ker \lambda \cap \ker \mu = \mathrm{Soc}(G(X,r)),$$

by Propositon 6.2.14.

If X is a solution, we consider over X the relation

$$x \sim y \iff \sigma_x = \sigma_y \text{ and } \tau_x = \tau_y.$$

Then $\sim$ is an equivalence relation. Let $\overline{X}$ be the set of equivalence classes and $[x]$ denote the equivalence class of x.

Proposition 13.1.1. *Let (X,r) be a solution. Then $(\overline{X},\overline{r})$, where*

$$\overline{r}([x],[y]) = ([\sigma_x(y)],[\tau_y(x)]),$$

is a solution.

Proof. We first prove that $\overline{r}$ is well-defined. Let $x, y \in X$ be such that $x \sim y$ and let $z \in X$. Since (X,r) is a solution, Lemma 3.1.5 implies that

$$\sigma_{\sigma_x(z)} \circ \sigma_{\sigma_z(x)} = \sigma_x \circ \sigma_z = \sigma_y \circ \sigma_z = \sigma_{\sigma_y(z)} \circ \sigma_{\sigma_z(x)},$$

it follows that $\sigma_{\sigma_z(x)} = \sigma_{\sigma_z(y)}$ and hence $\sigma_z(x) \sim \sigma_z(y)$. Similarly $\tau_{\tau_z(x)} = \tau_{\tau_z(y)}$ and therefore $\overline{r}$ is well-defined.

We now prove that $\overline{r}$ is invertible.

... □

In the case of involutive solutions, it follows from Proposition 3.3.6 that $\sigma_x = \sigma_y$ if and only if $\tau_x = \tau_y$.

Definition 13.1.2. Let (X,r) be a solution. The solution $\mathrm{Ret}(X,r) = (\overline{X},\overline{r})$ induced by the equivalence relation $\sim$ is the *retraction* of (X,r).

We define inductively $\mathrm{Ret}^0(X,r) = (X,r)$, $\mathrm{Ret}^1(X,r) = \mathrm{Ret}(X,r)$ and

$$\mathrm{Ret}^{n+1}(X,r) = \mathrm{Ret}(\mathrm{Ret}^n(X,r)) \quad n \geq 1.$$

Definition 13.1.3. A solution (X,r) is said to be of *multipermutation level n* if n is the smallest non-negative integer such that $|\mathrm{Ret}^n(X,r)| = 1$. The solution (X,r) is said to be *irretractable* if $\mathrm{Ret}(X,r) = (X,r)$.

The trivial solution over the set with one element is a multipermutation of level zero. Permutation solutions are multipermutation solutions of level one.

Example 13.1.4.

Example 13.1.5.

Table 13.1: Involutive solutions of size ≤ 10.

tab:INV_mp

n	2	3	4	5	6	7	8	9	10
solutions	2	5	23	88	595	3456	34530	321931	4895272
multipermutation	2	5	21	84	554	3295	32155	305916	4606440
irretractable	0	0	2	4	9	13	191	685	3590

For size ≤ 7 the numbers of Table 13.1 coincide with those in [39] but there are some differences for solutions of size eight.

Table 13.2: Non-involutive solutions of size ≤ 8.

tab:mp

n	2	3	4	5	6	7	8
solutions	2	21	230	3519	100071	4602720	422449480
multipermutation		15	206	3165	95517	4461805	416725250
irretractable		6	24	98	514	2659	17370

thm:CJKAV **Theorem 13.1.6.** *Let (X,r) be a finite multipermutation solution. If $|X| > 1$, then r has even order.*

Proof. Since $(X,r) \to \mathrm{Ret}(X,r), x \mapsto [x]$ is a homomorphism of solutions, it follows that the order of the solution $\overline{r}$ divides de orden of r. Assume that (X,r) has multipermutation level n. There exists a homomorphism of solutions $(X,r) \to \mathrm{Ret}^{n-1}(X,r)$, thus it is enough to prove the theorem in the case where $r(x,y) = (\sigma(y), \tau(x))$ for commuting permutations σ and τ, i.e. multipermutation solutions of level one. If r has order $2k+1$, then

$$(x,y) = r^{2k+1}(x,y) = (\sigma^{k+1}\tau^k(y), \sigma^k\tau^{k+1}(x)).$$

This implies that $\sigma^{k+1}\tau^k(y) = x$ for all $x, y \in X$. This equality in particular implies that $x = y$ because $\sigma^{k+1}\tau^k$ is a permutation, a contradiction. □

The connection between the socle of a brace and the retract of a solution was discovered by Rump in the case of involutive solutions and braces of abelian type, see [65].

pro:add_cyclic **Proposition 13.1.7.** *Let A be a brace and (A,r) be its associated solution. Then the retraction* $\mathrm{Ret}(A,r)$ *is the canonical solution associated with the quotient brace* $A/\mathrm{Soc}(A)$.

Proof. The equivalence relation $\sim$ on A is defined as $a \sim b$ if and only if $\lambda_a = \lambda_b$ and $\mu_a = \mu_b$. Let $\overline{A}$ be the set of equivalence classes. The equivalence class of an element a is then

$$\begin{aligned}[a] &= \{b \in A : a \sim b\} = \{b \in A : \lambda_a = \lambda_b, \mu_a = \mu_b\} \\ &= \{b \in A : a' \circ b \in \ker\lambda \cap \ker\mu\} = \{b \in A : a' \circ b \in \mathrm{Soc}(A)\},\end{aligned}$$

by Proposition 6.2.14. This means that $[a] = [b]$ if and only if $\pi(a) = \pi(b)$, where $\pi\colon A \to A/\mathrm{Soc}(A)$, $x \mapsto x \circ \mathrm{Soc}(A)$, is the canonical brace homomorphism. Moreover, $A/\mathrm{Soc}(A) = \overline{A}$ as sets. Now we compute the retraction of (A,r):

$$\begin{aligned}\overline{r}([a],[b]) &= ([\lambda_a(b)],[\mu_b(a)]) = (\pi(\lambda_a(b)),\pi(\mu_b(a)))\\ &= \left(\lambda_{\pi(a)}(\pi(b)),\mu_{\pi(b)}(\pi(a))\right) = \left(\lambda_{[a]}([b]),\mu_{[b]}([a])\right).\end{aligned}$$

Therefore $\mathrm{Ret}(A,r) = (A/\mathrm{Soc}(A),\overline{r})$. □

Now...

pro:mpl **Proposition 13.1.8.** *Let (X,r) and (Y,s) be solutions. Each surjective homomorphism of solutions $f\colon (X,r) \to (Y,s)$ induces a surjective homomorphism of solutions* $\mathrm{Ret}(X,r) \to \mathrm{Ret}(Y,s)$.

Proof. Write $r(x,y) = (\sigma_x(y),\tau_y(x))$ and $s(x,y) = (\lambda_x(y),\mu_y(x))$. Let $x,x_1 \in X$ be such that $x \sim x_1$. If $z \in X$, then

$$\lambda_{f(x)}f(z) = f(\sigma_x(z)) = f(\sigma_{x_1}(z)) = \lambda_{f(x_1)}f(z).$$

Since f is surjective, it follows that $\lambda_{f(x)} = \lambda_{f(x_1)}$. A similar calculation proves that $\mu_{f(x)} = \mu_{f(x_1)}$. If $\pi\colon (Y,s) \to \mathrm{Ret}(Y,r)$, $y \mapsto [y]$, is the canonical map, the composition $\pi \circ f\colon (X,r) \to \mathrm{Ret}(Y,s)$ is a surjective homomorphism of solutions. Therefore the map $\mathrm{Ret}(X,r) \to \mathrm{Ret}(Y,s)$, $[x] \mapsto \pi(f(x))$, is then a well-defined surjective homomorphism of solutions. □

pro:mpl_subsol **Proposition 13.1.9.** *Let (X,r) be a solution of finite multipermutation level m and $Y \subseteq X$ be such that $r(Y \times Y) \subseteq Y \times Y$. Then the* subsolution $(Y, r|_{Y\times Y})$ *is of finite multipermutation level $\leq m$.*

Proof. □

Theorem 13.1.10. *Let (X,r) be a solution. The following statements are equivalent:*

1) *(X,r) has finite multipermutation level.*
2) *$\left(\mathcal{G}(X,r), r_{\mathcal{G}(X,r)}\right)$ has finite multipermutation level.*
3) *$\left(G(X,r), r_{G(X,r)}\right)$ has finite multipermutation level.*

Proof. Let us first prove that (2) implies (1). The map $X \to \mathcal{G}(X,r), x \mapsto (\lambda_x,\mu_x^{-1})$, is a homomorphism of solutions that induces an injective homomorphism of solutions $\mathrm{Ret}(X,r) \to \left(\mathcal{G}(X,r), r_{\mathcal{G}(X,r)}\right)$. Since $\left(\mathcal{G}(X,r), r_{\mathcal{G}(X,r)}\right)$ has finite multipermutation level, (X,r) has finite multipermutation level by Proposition 13.1.9.

Let us now prove that (3) implies (2). The canonical map $G(X,r) \to \mathcal{G}(X,r)$ yields a surjective homomorphism of solutions. Then Proposition 13.1.8 applies.

...

□

The following result appeared in [39].

Proposition 13.1.11. *Let (X,r) be a finite involutive solution. If the additive group of the brace $\mathcal{G}(X,r)$ is cyclic, then (X,r) is multipermutation.*

Proof. Let (X,r) be a counterexample of minimal cardinality. If K is the additive group of $\mathcal{G}(X,r)$, then K is finite and cyclic. Write G for the multiplicative group of $\mathcal{G}(X,r)$. Since $|\operatorname{Aut}(K)| = \varphi(|K|) < |K|$, where φ is the Euler function, the group homomorphism $\lambda\colon G \to \operatorname{Aut}(K)$ has a non-trivial kernel, so $\operatorname{Soc}(\mathcal{G}(X,r))$ is non-zero. This implies that (X,r) is retractable. Since $\mathcal{G}(X,r)/\operatorname{Soc}(\mathcal{G}(X,r))$ is a brace with cyclic additive group and $\operatorname{Ret}(X,r)$ is an involutive solution, the minimality of $|X|$ implies that $\operatorname{Ret}(X,r)$ is a multipermutation solution, and hence so is (X,r), a contradiction. □

The converse of the previous proposition does not hold.

Example 13.1.12. Let $X = \{1,2,3,4\}$ and $r(x,y) = (\varphi_x(y), \varphi_y(x))$, where

$$\varphi_1 = \varphi_2 = \mathrm{id}, \quad \varphi_3 = (34), \quad \varphi_4 = (12)(34).$$

Then (X,r) is an involutive multipermutation solution. One easily checks that $\mathcal{G}(X,r) \simeq C_2 \times C_2$.

A similar idea proves the following result:

thm:mul_cyclic

Theorem 13.1.13. *Let (X,r) be a finite involutive solution. If the multiplicative group of the brace $\mathcal{G}(X,r)$ is cyclic, then (X,r) is multipermutation.*

Proof. Let (X,r) be a counterexample of minimal cardinality. Write K for the additive group of $\mathcal{G}(X,r)$ and $G = \langle g \rangle$ for the multiplicative group of $\mathcal{G}(X,r)$. Since the image of the group homomorphism $\lambda\colon G \to \operatorname{Aut}(K)$ is cyclic generated by λ_g and $|\lambda_g| < |G|$ by Horosevskii's theorem, see [47, Corollary 3.3], it follows that λ has a non-trivial kernel, so $\operatorname{Soc}(\mathcal{G}(X,r))$ is non-zero. This implies that (X,r) is retractable. Since $\mathcal{G}(X,r)/\operatorname{Soc}(\mathcal{G}(X,r))$ is a brace with cyclic additive group and $\operatorname{Ret}(X,r)$ is an involutive solution, the minimality of $|X|$ implies that $\operatorname{Ret}(X,r)$ is a multipermutation solution, and hence so is (X,r), a contradiction. □

The previous result does not hold in the case of arbitrary solutions.

Example 13.1.14. Let $X = \{1,2,3,4,5,6\}$ and $r(x,y) = (\sigma_x(y), \tau_y(x))$, where

$$\begin{aligned}
&\sigma_1 = \mathrm{id}, && \sigma_2 = \mathrm{id}, && \sigma_3 = \mathrm{id},\\
&\sigma_4 = (23)(56), && \sigma_5 = (23)(56), && \sigma_6 = (23)(56),\\
&\tau_1 = \mathrm{id}, && \tau_2 = (456), && \tau_3 = (465),\\
&\tau_4 = \mathrm{id}, && \tau_5 = (465), && \tau_6 = (456).
\end{aligned}$$

The brace $\mathcal{G}(X,r)$ has multiplicative group isomorphic to $\mathbb{S}_3$ and additive group isomorphic to the cyclic group of order six.

We will see later that Theorem 13.1.13 is true for braces of nilpotent type. The following example appears in the work of Rump [65].

pro:radical

Proposition 13.1.15. *Let A be a finite non-trivial radical ring. Then $\operatorname{Soc}(A) \neq \{0\}$ and (A, r_A) is an involutive multipermutation solution.*

Proof. Let A be a counterexample of minimal size. This means that $\mathrm{Soc}(A) = \{0\}$ and all two-sided braces of abelian type of size $< |A|$ have non-trivial socle. Since A is finite, there exists a non-zero minimal left ideal I of A. Recall that A is a radical ring with product $a * b = \lambda_a(b) - b$. Since A is a radical ring, A is a nil ring, which implies by Nakayama's lemma that $I * A = \{0\}$. This means that if $x \in I$, then $x \in \mathrm{Soc}(A)$, as $0 = x * a = \lambda_x(a) - a$ for all $a \in A$. In particular, $\mathrm{Soc}(A) \neq \{0\}$, a contradiction. □

Proposition 13.1.15 has a nice application. The results appeared first in [26]. The proof presented here is from [27].

thm:CJO_abelian

Theorem 13.1.16. *Let (X,r) be a finite involutive solution. If the multiplicative group of the brace $\mathcal{G}(X,r)$ is abelian, then (X,r) is multipermutation.*

Proof. □

In [42], Gateva–Ivanova conjectured that finite involutive square-free solutions are retractable.

In [43] Gateva–Ivanova asked when...

Right nilpotency...

The following theorem characterizes multipermutation involutive solutions in terms of left orderability of groups. A group G is said to be *left ordered* if it admits a total ordering $<$ such that

$$x < y \implies zx < zy$$

for all $x, y, z \in G$. Torsion-free abelian groups, free groups and braid groups are left ordered groups. See [32] for more information on ordered groups.

thm:BCV

Theorem 13.1.17. *Let (X,r) be a finite involutive solution. The following statements are equivalent:*

1) *(X,r) is a multipermutation solution.*
2) *$G(X,r)$ is poly-$\mathbb{Z}$.*
3) *$G(X,r)$ is left orderable.*
4) *$G(X,r)$ is diffuse.*

Proof. □

Recall that a group G has the *unique product property* if for all finite non-empty subsets A and B of G there exists $x \in G$ that can be written uniquely as $x = ab$ with $a \in A$ and $b \in B$.

It is natural to ask when $G(X,r)$ has the unique product property. By Theorem 13.1.17, if (X,r) is an involutive multipermutation solution, then $G(X,r)$ has the unique product property since $G(X,r)$ is left orderable.

pro:4-19

Example 13.1.18. Let $X = \{1,2,3,4\}$ and $r(x,y) = (\sigma_x(y), \tau_y(x))$ be the irretractable involutive solution given by

$$\sigma_1 = (12), \qquad \sigma_2 = (1324), \qquad \sigma_3 = (34), \qquad \sigma_4 = (1423),$$
$$\tau_1 = (14), \qquad \tau_2 = (1243), \qquad \tau_3 = (23), \qquad \tau_4 = (1342).$$

We claim that the group $G(X,r)$ with generators x_1, x_2, x_3, x_4 and relations

$$x_1^2 = x_2x_4, \qquad x_1x_3 = x_3x_1, \qquad x_1x_4 = x_4x_3,$$
$$x_2x_1 = x_3x_2, \qquad x_2^2 = x_4^2, \qquad x_3^2 = x_4x_2.$$

does not have the unique product property. Let $x = x_1x_2^{-1}$ and $y = x_1x_3^{-1}$ and

$$S = \{x^2y, y^2x, xyx^{-1}, (y^2x)^{-1}, (xy)^{-2}, y, (xy)^2x, (xy)^2, (xyx)^{-1}, yxy, y^{-1}, x, xyx, x^{-1}\}. \quad (13.1)$$

eq:Promislow

To prove that $G(X,r)$ does not have the unique product property it is enough to prove that each $s \in S^2 = \{s_1s_2 : s_1, s_2 \in S\}$ admits at least two different decompositions of the form $s = ab = uv$ for $a, b, u, v \in S$. To perform these calculations we use the injective group homomorphism $G \to \mathbf{GL}(5,\mathbb{Z})$ given by

$$x_1 \mapsto \begin{pmatrix} 0&1&0&0&1\\ 1&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \qquad x_2 \mapsto \begin{pmatrix} 0&0&0&1&0\\ 0&0&1&0&1\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&1 \end{pmatrix},$$
$$x_3 \mapsto \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&1&1\\ 0&0&1&0&0\\ 0&0&0&0&1 \end{pmatrix}, \qquad x_4 \mapsto \begin{pmatrix} 0&0&1&0&0\\ 0&0&0&1&0\\ 0&1&0&0&0\\ 1&0&0&0&1\\ 0&0&0&0&1 \end{pmatrix}.$$

This faithful representation of $G(X,r)$ allows us to compute all possible products of the form s_1s_2 for all $s_1, s_2 \in S$. By inspection, each element of S^2 admits at least two different representations.

thm:CJO_mp

Theorem 13.1.19. *Let A be a finite brace of abelian type with multiplicative group G. Then there exists a finite solution (X,r) such that $\mathrm{Ret}(X,r)$ is isomorphic to (A,r_A) and $\mathcal{G}(X,r) \simeq G$.*

Proof. Let $X = A \times \mathbb{Z}/(2)$. For $a, b \in A$, let

$$\varphi_{(a,0)}(b,0) = (b,0), \qquad \varphi_{(a,0)}(b,1) = (b,1),$$
$$\varphi_{(a,1)}(b,0) = (a' \circ b, 0), \qquad \varphi_{(a,1)}(b,1) = (\lambda_a^{-1}(b), 1).$$

The maps $\varphi_{(a,\epsilon)}$ are invertible for all $a \in A$ and $\epsilon \in \mathbb{Z}/(2)$. In fact,

$$\varphi_{(a,0)}^{-1} = \mathrm{id}, \quad \varphi_{(a,1)}^{-1}(b,\epsilon) = \begin{cases} a \circ b & \text{if } \epsilon = 0, \\ \lambda_a(b) & \text{if } \epsilon = 1. \end{cases}$$

So we need to check that

$$((a,\epsilon_1)\cdot(b,\epsilon_2))\cdot((a,\epsilon_1)\cdot(c,\epsilon_3)) = ((b,\epsilon_2)\cdot(a,\epsilon_1))\cdot((b,\epsilon_2)\cdot(c,\epsilon_3)) \quad (13.2)$$

eq:CJO_tocheck

holds for all $a,b,c \in A$ and $\epsilon_1,\epsilon_2,\epsilon_3 \in \mathbb{Z}/(2)$. There are several cases to consider. Let us assume first that $(\epsilon_1,\epsilon_2,\epsilon_3) = (1,1,0)$. Then (13.2) turns out to be

$$(\lambda_a^{-1}(b)' \circ a' \circ c, 0) = (\lambda_b^{-1}(a)' \circ b' \circ c, 0),$$

which holds for all $a,b \in A$, as

$$\lambda_a^{-1}(b)' \circ a' \circ c = (a \circ \lambda_a^{-1}(b))' \circ c = (a+b)' \circ c = (b+a)' \circ c = \lambda_b^{-1}(a)' \circ b' \circ c$$

because A is of abelian type. Let us now deal with the case $(\epsilon_1,\epsilon_2,\epsilon_3) = (1,1,1)$. In this case Equality (13.2) turns out to be equivalent to

$$(\lambda_a^{-1}(b),1) \cdot (\lambda_a^{-1}(c),1) = (\lambda_b^{-1}(a),1) \cdot (\lambda_b^{-1}(c),1).$$

Since A is of abelian type,

$$\begin{aligned}(\lambda_a^{-1}(b),1) \cdot (\lambda_a^{-1}(c),1) &= \lambda^{-1}_{\lambda_a^{-1}(b)} \lambda_a^{-1}(c) = \lambda^{-1}_{a \circ \lambda_a^{-1}(b)}(c) = \lambda^{-1}_{a+b}(c) \\ &= \lambda^{-1}_{b+a}(c) = \lambda^{-1}_{\lambda_b^{-1}(a)} \lambda_b^{-1}(c) = (\lambda_b^{-1}(a),1) \cdot (\lambda_b^{-1}(c),1).\end{aligned}$$

The other cases are easier and require straightforward calculations.

Let $\psi\colon G \to \mathbb{S}_X$, $\psi(g) = \varphi_{(g',1)}$. Since

$$\psi(g)(0,0) = \varphi_{(g',1)}(0,0) = (g \circ 0, 0) = (g,0),$$

it follows that ψ is injective. Moreover, ψ is a group homomorphism, as

$$\begin{aligned}\psi(a)\psi(b)(c,0) &= \psi(a)\varphi_{(b,1)}(c,0) = \psi(a)(b \circ c, 0) \\ &= \varphi_{(a,1)}(b \circ c, 0) = (a \circ b \circ c, 0) = \varphi_{(a \circ b,1)}(c,0) = \psi_{(a \circ b)}(c,0)\end{aligned}$$

and

$$\begin{aligned}\psi(a)\psi(b)(c,1) &= \varphi_{(a',1)}\varphi_{(b',1)}(c,1) = \varphi_{(a',1)}(\lambda_{b'}^{-1}(c),1) \\ &= (\lambda_{a'}^{-1}\lambda_{b'}^{-1}(c),1) = (\lambda_{a \circ b}(c),1) = \varphi_{(a \circ b,1)}(c,1) = \psi(a \circ b)(c,1).\end{aligned}$$

Since ψ is an injective group homomorphism,

$$G \simeq \psi(G) \simeq \langle \psi(a) : a \in A \rangle = \langle \varphi_{(a,1)} : a \in A \rangle \simeq \mathcal{G}(X,r).$$

Consider the equivalence relation on X given by $x \sim y$ if and only if $\varphi_x = \varphi_y$. As usual $[x]$ denotes the equivalence class of the element $x \in X$ and $\overline{X}$ is the set of equivalence classes. A straightforward computation shows that $(a,0) \sim (0,1)$ for all $a \in A$. This implies that

$$\overline{p}\colon \overline{X} \to A, \quad \overline{p}([(a,\epsilon)]) = \begin{cases} 0 & \text{if } \epsilon = 0, \\ a & \text{if } \epsilon = 1, \end{cases}$$

is a well-defined surjective map. We claim that $\overline{p}$ is injective. Let $(a,\epsilon_1) \in \overline{X}$ and $(b,\epsilon_2) \in \overline{X}$ be such that $\overline{p}([(a,\epsilon_1)]) = \overline{p}([(b,\epsilon_2)])$. Since $[(a,0)] = [(0,1)]$ for all $a \in A$, we only need to consider the case where $\epsilon_1 = \epsilon_2 = 1$. In this case,

$$a = \overline{p}([(a,\epsilon_1)]) = \overline{p}([(b,\epsilon_2)]) = b.$$

Thus $\overline{p}$ is bijective. Now

$$\begin{aligned} r_A(\overline{p}[(a,1)],\overline{p}[(b,1)]) &= r_A(a,b) = (\lambda_a(b),\mu_b(a)) = (\overline{p}[\lambda_a(b),1],\overline{p}[\mu_b(a),1]) \\ &= (\overline{p}\varphi_{(a,1)}(b,1),\overline{p}...). \end{aligned}$$

□

Theorem 13.1.20. *Let A be...*

Proof. □

Exercises

prob:bounded_mpl

13.1.1. Let (X,r) be a solution of finite multipermutation level m. Prove that any homomorphic image of (X,r) is a solution of finite multipermutation level $\leq m$.

prob:4-13

13.1.2. Let $X = \{1,2,3,4\}$ and $r(x,y) = (\sigma_x(y),\tau_y(x))$ be the irretractable involutive solution given by

$$\begin{aligned} &\sigma_1 = (34), && \sigma_2 = (1324), && \sigma_3 = (1423), && \sigma_4 = (12), \\ &\tau_1 = (24), && \tau_2 = (1432), && \tau_3 = (1234), && \tau_4 = (13). \end{aligned}$$

Prove that $G(X,r)$ does not have the unique product property.

Open problems

Open problem 13.1.1. Does the group $G(X,r)$... A linear representation of this group is...

Notes

Multipermutation involutive solutions were introduced in [39]. The notion was extended to the non-involutive case in [56].

Theorem 13.1.6 was proved in...

Proposition 13.1.8 was proved in [27] for involutive solutions. The general case goes back to...

Theorem 13.1.17 combines several results. The implication...

Theorem 13.1.19 appears in [27].

Non-involutive multipermutation solutions...

The set (13.1) appears in the work of Promislow [62]. Exercise 13.1.2 appears in the book of Jespers and Okniński, see [51, Example 8.2.14].

Chapter 14
Factorizations

factorizations

In Chapter 6 we found that groups with an exact factorization produce skew braces. In this chapter we will study a different relationship between factorizations of groups and and skew braces.

Factorizations of groups

A group G is said to be *factorized through subgroups* A and B if $G = AB$. We remark that we do not assume that $A \cap B = \{1\}$.

A group G is *metabelian* if $[G,G]$ is abelian. Thus G is metabelian if and only if there is a normal subgroup K of G such that K and G/K are abelian. The groups $\mathbb{S}_3$ and $\mathbb{A}_4$ are metabelian.

A straightforward calculation shows that the following formulas hold:

$$[a,bc] = [a,b]b[a,c]b^{-1},$$
$$[ab,c] = a[b,c]a^{-1}[a,c].$$

The following theorem is considered the most satisfying result about group factorization. The proof is based on a surprisingly short and smart calculation with commutators.

theorem:Ito

Theorem 14.0.1 (Itô). *Let $G = AB$ be a factorization of G through abelian subgroups A and B. Then G is metabelian.*

Proof. Since $G = AB$, it follows that $AB = BA$. Let us prove that $[A,B]$ is a normal subgroup of G. Let $a, a_1, \alpha, \alpha_1 \in A$ and $b, b_1, \beta, \beta_1 \in B$ be such that $\alpha b \alpha^{-1} = b_1 a_1$, $\beta a \beta^{-1} = a_2 b_2$. Since

$$\alpha[a,b]\alpha^{-1} = a(\alpha b\alpha^{-1})a^{-1}(\alpha b^{-1}\alpha^{-1}) = ab_1a_1a^{-1}a_1^{-1}b_1^{-1} = [a,b_1] \in [A,B]$$
$$\beta[a,b]\beta^{-1} = (\beta a\beta^{-1})\beta b\beta^{-1}(\beta a^{-1}\beta^{-1})b^{-1} = a_2b_2bb_2^{-1}a_2^{-1}b^{-1} = [a_2,b] \in [A,B],$$

it follows that $[A,B]$ is a normal subgroup of G.

Now we prove that $[A,B]$ is abelian. Since

$$\beta\alpha[a,b]\alpha^{-1}\beta^{-1} = \beta[a,b_1]\beta^{-1} = (\beta a\beta^{-1})b_1(\beta a^{-1}\beta^{-1})b_1^{-1} = [a_2,b_1],$$
$$\alpha\beta[a,b]\beta^{-1}\alpha^{-1} = \alpha[a_2,b]\alpha^{-1} = a_2(\alpha b\alpha^{-1})a_2^{-1}(\alpha b\alpha^{-1}) = [a_2,b_1],$$

a direct calculation shows that

$$[\alpha^{-1},\beta^{-1}][a,b] = [a,b][\alpha^{-1},\beta^{-1}].$$

Since two arbitrary generators of $[A,B]$ commute, the group $[A,B]$ is abelian.

Finally we note that $[G,G] = [A,B]$. Since $[A,B]$ is normal in G,

$$[a_1b_1,a_2b_2] = a_1[a_2,b_1]^{-1}a_1^{-1}a_2[a_1,b_2]a_2^{-1} \subseteq [A,B]. \qquad \square$$

Sysak found the following generalization of Itô's theorem:

Theorem 14.0.2 (Sysak). *If A and B are abelian subgroups of G and H is a subgroup of G contained in the set AB, then H is metabelian.*

The proof appears in [74].

There are several other interesting results in the theory of factorizable groups. Another important result that is worth mentioning is the following theorem.

Theorem 14.0.3 (Kegel–Wielandt). *Let G be a finite group. If there are nilpotent subgroups A and B of G such that $G = AB$, then G is solvable.*

The proof appears for example in [3, Theorem 2.4.3].

The theorem of Kegel–Wielandt turns out to be the main tool in the proof of the following result on the structure of finite braces. This proves a conjecture of Byott, see [20].

thm:mul_nilpotent

Theorem 14.0.4. *Let A be a finite brace with nilpotent multiplicative group. Then the additive group of A is solvable.*

Proof. Let K be the additive group of A and let G be the multiplicative group of A. The group $\Gamma = K \rtimes G$ has multiplication

$$(k_1,g_1)(k_2,g_2) = (k_1 + \lambda_{g_1}(k_2), g_1 \circ g_2)),$$

for all $k_1,k_1 \in K$ and $g_1,g_2 \in G$. Let $f\colon G \to \Gamma$, $g \mapsto (g,g)$. Note that

$$f(g_1 \circ g_2) = (g_1 \circ g_2, g_1 \circ g_2) = (g_1 + \lambda_{g_1}(g_2), g_1 \circ g_2) = (g_1,g_1)(g_2,g_2) = f(g_1)f(g_2).$$

Hence f is a group homomorphism and $f(G)$ is nilpotent. Since

$$(k,g) = (k,k)(0,k' \circ g) \in f(G)G,$$

for all $k \in K$ and $g \in G$, we have that $\Gamma = f(G)G$ is a product of two nilpotent subgroups. By the theorem of Kegel–Wielandt, Γ is solvable. Since K is a subgroup of Γ, K also is solvable. $\square$

Factorizations of skew braces

It turns out to be interesting to study factorization of braces.

Definition 14.0.5. Let A be a skew brace and let B and C be left ideals of A. We say that A admits a *factorization* through B and C if $A = B + C$.

Note that if a skew brace A admits a factorization through B and C, then it follows that

$$A = B + C = C + B = B \circ C = C \circ B.$$

Now we prove an analog of Itô's theorem in the context of skew braces. It turns out that one needs to consider factorizations through strong left ideals. We also need the following definition:

Definition 14.0.6. A skew brace A is said to be *meta-trivial* if $A^{(2)}$ is a trivial brace.

Clearly a skew brace A is meta-trivial if and only there is an ideal I of A such that I and A/I are trivial skew braces.

lem:calculations

Lemma 14.0.7. *Let A be a skew brace such that $A = B + C$, where B and C are left ideals. If B and C are trivial skew braces then, for any $b, \beta \in B$ and $c, \gamma \in C$, the following statements hold:*

1) $\lambda_{\beta\circ\gamma} = \lambda_{\gamma\circ\beta}$,
2) $(c+b)\circ\beta - \beta = c + b + c * \beta$,
3) $b \circ c \circ b' \circ c' = b \circ c - c \circ b = b + \lambda_b(c) - \lambda_c(b) - c \in \ker\lambda$.

Proof. To prove (1) put $c_1 = \lambda_\beta(c) \in C$ and $b_1 = \lambda_\gamma(b) \in B$. As B and C are trivial braces, $\lambda_\beta(b+c) = \lambda_\beta(b) + \lambda_\beta(c) = b + c_1$ and similarly $\lambda_\gamma(b+c) = b_1 + c$. Then $\lambda_{\beta\circ\gamma}(b+c) = b_1 + c_1 = \lambda_{\gamma\circ\beta}(b+c)$.

Let us prove (2). As B is a trivial brace, it follows from (6.3) that

$$\begin{aligned}(c+b)\circ\beta - \beta &= (c \circ \lambda_{c'}(b))\circ\beta - \beta \\ &= c \circ (\lambda_{c'}(b)\circ\beta) - \beta \\ &= c \circ (\lambda_{c'}(b) + \beta) - \beta \\ &= c \circ \lambda_{c'}(b) - c + c \circ \beta - \beta \\ &= c + b + c * \beta.\end{aligned}$$

Part (3) follows from the following computation

$$\begin{aligned}b \circ c \circ b' \circ c' &= (b \circ c) + \lambda_{b\circ c}(b' + \lambda_{b'}(c')) \\ &= b + \lambda_b(c) + \lambda_{b\circ c}(b') + \lambda_{b\circ c\circ b'}(c') \\ &= b + \lambda_b(c) + \lambda_c(b') + \lambda_{b\circ b'}(c') \\ &= b + \lambda_b(c) + \lambda_c(-b) - c \\ &= b \circ c - c \circ b.\end{aligned}$$

Moreover, by (1) it follows that $b \circ c \circ b' \circ c' \in \ker\lambda$. □

lem:hardworkfactoriz

Lemma 14.0.8. *Let A be a skew brace such that $A = B + C$ is a factorization through left ideals B and C. If B and C are trivial skew braces, then:*

1) *$B * C$ and $C * B$ are strong left ideals of A,*
2) *$B * C$ and $C * B$ are trivial skew braces, and*
3) *$A^{(2)} = C * B + B * C = B * C + C * B$.*

Proof. Since C is a left ideal, it follows that $B * C \subseteq C$. Let $b, \beta \in B$ and $c, \gamma \in C$. As C is trivial, it follows that

$$\begin{aligned}\lambda_{b\circ c}(\beta * \gamma) &= \lambda_b(\beta * \gamma)\\ &= \lambda_b\lambda_\beta(\gamma) - \lambda_b(\gamma)\\ &= \lambda_{b\circ\beta\circ b'}\lambda_b(\gamma) - \lambda_b(\gamma)\\ &= (b \circ \beta \circ b') * \lambda_b(\gamma) \in B * C.\end{aligned}$$

Hence $B * C$ is a left ideal and a trivial skew brace.

Let $a \in A$, $b \in B$ and $c \in C$. Write $a = b_1 + c_1$, with $b_1 \in B$ and $c_1 \in C$. Then

$$\begin{aligned}a + (b * c) - a &= a + \lambda_b(c) - c - a\\ &= -(b * a) + b * (a + c)\\ &= -(b * (b_1 + c_1)) + b * (b_1 + c_1 + c).\end{aligned} \tag{14.1}$$

eq:7

As $B + C = C + B$, it follows that for any $\beta \in B$ and $\gamma \in C$, there exist $\beta_1 \in B$ and $\gamma_1 \in C$ such that $\beta + \gamma = \gamma_1 + \beta_1$. Hence, for any $b \in B$ it holds that

$$b * (\beta + \gamma) = b * (\gamma_1 + \beta_1) = b * \gamma_1 + \gamma_1 + b * \beta_1 - \gamma_1 = b * \gamma_1,$$

as B is trivial. Applying this on (14.1) it follows that $B * C$ is a normal subgroup of $(A, +)$. This proves (1) and (2) for $B * C$. The proof for $C * B$ is similar.

Now we show that $A^{(2)} \subseteq C * B + B * C$. Let $b, b_1 \in B$ and $c, c_1 \in C$. Then

$$\begin{aligned}(b \circ c) * (b_1 + c_1) &= (b \circ c) * b_1 + b_1 + (b \circ c) * c_1 - b_1\\ &= \lambda_{b\circ c}(b_1) - b_1 + b_1 + b * (c * c_1) + c * c_1 + b * c_1 - b_1\\ &= \lambda_c(b_1) - b_1 + b_1 + b * c_1 - b_1\\ &= c * b_1 + b_1 + b * c_1 - b_1 \in C * B + B * C.\end{aligned}$$

Clearly $C * B + B * C \subseteq A^{(2)}$ and thus $A^{(2)} = C * B + B * C = B * C + C * B$. □

thm:Ito_braces

Theorem 14.0.9. *Let A be a skew brace. If $A = B + C$ is a factorization through strong left ideals B and C that are trivial skew braces, then A is right nilpotent of class at most three. In particular, A is meta-trivial.*

Proof. By Lemma 14.0.8, $B * C$ and $C * B$ are strong left ideals of A, and both are trivial skew braces. Furthermore,

$$A^{(2)} = B * C + C * B = (B * C) \circ (C * B).$$

It remains to show that $A^{(2)}$ acts trivially on A, i. e. $\lambda_a(b) = b$, for all $a \in A^{(2)}$ and $b \in A$. Note that this is equivalent to $A^{(3)} = \{0\}$. We first show that $B * C$ acts trivially on A. Since $B * C$ is trivial as skew brace, it is enough to show that $\lambda_{b*c}(a) = a$, for all $b \in B$, $c \in C$ and $a \in A$. For that purpose, let $b \in B$, $c \in C$ and $a \in A$. Write $a = \beta + \gamma$, where $\beta \in B$ and $\gamma \in C$. Then

$$(b * c) * (\beta + \gamma) = (b * c) * \beta + \beta + (b * c) * \gamma - \beta = (b * c) * \beta,$$

as C is a trivial brace and $b * c \in C$. By Lemma 14.0.7(3),

$$(b \circ c - c \circ b) + \beta = (b \circ c - c \circ b) \circ \beta = (b + \lambda_b(c) - \lambda_c(b) - c) \circ \beta.$$

Since $(B, +)$ is a normal subgroup of $(A, +)$,

$$b \circ c - c \circ b = b + \lambda_b(c) - \lambda_c(b) - c = \lambda_b(c) - c + b_1$$

for some $b_1 \in B$. By Lemma 14.0.7(2),

$$\begin{aligned}(b \circ c - c \circ b) + \beta &= (\lambda_b(c) - c + b_1) \circ \beta \\ &= \lambda_b(c) - c + b_1 + (b * c) * \beta + \beta\end{aligned}$$

and therefore $(b * c) * \beta = 0$. Thus $B * C$ acts trivially on A. As $(C, +)$ also is a normal subgroup of $(A, +)$, it follows by symmetry that $C * B$ acts trivially on A. Hence $A^{(2)} = (B * C) \circ (C * B)$ acts trivially on A. □

Corollary 14.0.10. *Let A be a skew brace. Assume that $A = B + C$, where B and C are (not necessarily strong) left ideals, which are trivial as braces. Then A has a meta-trivial ideal I such that A/I is a trivial skew brace.*

Proof. By Lemma 14.0.8, the ideal $A^{(2)}$ has a factorization through the strong left ideals $B * C$ and $C * B$, which are trivial skew braces. By Theorem 14.0.9, $A^{(2)}$ is meta-trivial and hence the claim follows. □

Theorem 14.0.9 has application to involutive solutions.

thm:MP **Theorem 14.0.11.** *Let (X, r) be an involutive non-degenerate (not necessarily finite) solution of the Yang–Baxter equation with $|X| \geq 2$. If the brace of abelian type $\mathcal{G}(X, r)$ admits a factorization through left ideals, which are trivial as left braces, then (X, r) is a multipermutation solution of level at most three.*

Proof. Let $A = \mathcal{G}(X, r)$ and $G = G(X, r)$. Then Theorem 14.0.9 yields $A^{(m)} = 0$ for some $m \leq 3$. Because $G/\mathrm{Soc}(G) \cong A$ as left braces, we get $G^{(m)} \subseteq \mathrm{Soc}(G)$, and thus $G^{(m+1)} = 0$. Hence G is a right nilpotent left brace of class at most four and, by [**?**, Proposition 6], (G, r_G) is a multipermutation solution of level at most three. Therefore, by [43, Theorem 5.15], (X, r) is a multipermutation solution of level at most three. □

This shows that properties of the involutive non-degenerate set-theoretic solution (X,r) are not completely determined by the group theory of the additive and multiplicative groups of the left brace $\mathcal{G}(X,r)$.

exa:B(8,27)

Example 14.0.12. Let $X = \{1,2,3,4\}$ and $r(x,y) = (\sigma_x(y), \tau_y(x))$ be the irretractable involutive non-degenerate solution given by

$$\sigma_1 = (34), \quad \sigma_2 = (1324), \quad \sigma_3 = (1423), \quad \sigma_4 = (12),$$
$$\tau_1 = (24), \quad \tau_2 = (1432), \quad \tau_3 = (1234), \quad \tau_4 = (13).$$

The associated left brace $\mathcal{G}(X,r)$ has additive group C_2^3 and multiplicative group D_8. Furthermore, $\mathcal{G}(X,r)$ is not right nilpotent. Hence it is impossible to decompose the left brace $\mathcal{G}(X,r)$ as in Theorem 14.0.11.

Example 14.0.13. The left brace $B(8,26)$ has the same additive and multiplicative groups as the brace $\mathcal{G}(X,r)$ of Example 14.0.12 but it has a factorization as in Theorem 14.0.11. This shows that $B(8,26)$ is right nilpotent.

Exercises

14.0.1. Let G be a metabelian group. Prove that the following statements hold.

1) If H is a subgroup of G, then H is metabelian.
2) If $f\colon G \to H$ is a group homomorphism, then $f(H)$ is metabelian.

14.0.2. Prove that $\mathbf{SL}_2(3)$ is metabelian.

prob:decomposable

14.0.3. Let A be a brace. If there exists a proper strong left ideal I, then (A, r_A) is decomposable as $A = I \cup A \setminus I$.

prob:Ito_relaxed

14.0.4. Prove that the assumptions of Theorem 14.0.9 cannot be relaxed.

prob:Ito_version2

14.0.5. Let A be a non-zero brace that has a factorization $A = B + C$ through left ideals B and C, where both are trivial as braces. If B is a strong left ideal of A, then B or C contains a non-zero ideal I of A that acts trivially on A.

prob:mul_abelian

14.0.6. Let A be a brace with abelian multiplicative group. Prove that the the additive group of A is meta-abelian.

prob:mul_cyclic

14.0.7. Let A be a finite brace with cyclic multiplicative group. Prove that the the additive group of A is supersolvable.

Open problems

problem:Byott

Open problem 14.0.1. Let A be a brace with solvable additive group. Is the multiplicative group of A solvable?

Notes

Theorem 14.0.9 was proved by Jespers, Kubat, Antwerpen and Vendramin in [50]. Exercises 14.0.4 and 14.0.5 also appear in there. One cannot expect a naive result similar to that of Kegel–Wielandt in the context of braces.

Theorem 14.0.4 was proved by Tsang and Qin in [75]. Exercises 14.0.6 and 14.0.7 also appear in [75].

Problem 14.0.1 was formulated by Byott in [20].

Chapter 15
Transitive groups

transitive

A

B

The classification of transitive groups of small degree can be used to produce quandles...

thm:quandles

Theorem 15.0.1.

Proof. □

C

Definition 15.0.2. A finite solution (X,r) is said to be **decomposable** if there is a decomposition $X = X_1 \cup X_2$ of X into a disjoint union of non-empty subsets X_1 and X_2 such that $r(X_1 \times X_1) \subseteq X_1 \times X_2$ and $r(X_2 \times X_2) \subseteq X_2 \times X_2$. A solution (X,r) is then **indecomposable** if it is not decomposable.

If (X,r) is a finite decomposable solution and $X = X_1 \cup X_2$ is a decomposition, then the restrictions $r|_{X_1 \times X_1}$ and $r|_{X_2 \times X_2}$ are solutions. Moreover, it follows that $r(X_1 \times X_2) \subseteq X_2 \times X_1$ and $r(X_2 \times X_1) \subseteq X_1 \times X_2$, see Exercise 15.0.1.

Proposition 15.0.3. *A finite solution (X,r) is indecomposable if and only if the group*

$$\langle \sigma_x, \tau_x : x, y \in X \rangle$$

acts transitively on X.

Proof. Let us assume that $X = X_1 \cup X_2$ is a decomposition of X into non-empty orbits... □

Note that this group is in general not isomorphic to the permutation group of the solution.

Definition 15.0.4. A finite solution (X,r) is said to be **simple** if $|X|>1$ and for every surjective homomorphism $f\colon (X,r)\to(Y,s)$ of solutions either f is an isomorphism or $|Y|=1$.

Example 15.0.5.

Example 15.0.6.

Example 15.0.7.

p:simple=>indecomposable

Proposition 15.0.8. *Let (X,r) be a finite simple solution. If $|X|>2$, then (X,r) is indecomposable.* *involutive?*

Proof. Let us assume that (X,r) is decomposable. Decompose $X=X_1\cup X_2$ for non-empty disjoint subsets X_1 and X_2 of X such that $r(X_i\times X_i)\subseteq X_i\times X_i$ for $i\in\{1,2\}$. Let $Y=\{1,2\}$ and $s\colon Y\times Y\to Y\times Y$, $s(x,y)=(y,x)$. Since $X=X_1\cup X_2$ is a decomposition, it follows that $r(X_i\times X_j)\subseteq X_j\times X_i$ for all $i,j\in\{1,2\}$. Why? Let $f\colon X\to Y$, $f(x)=i$ if $x\in X_i$. Since f is then a surjective homomorphism of solutions and f is not an isomorphism (because $|X|>2$), the simplicity of (X,r) implies that $|Y|=1$, a contradiction. □

Proposition 15.0.9. *Let (X,r) be a finite simple* *involutive* *solution. If $|X|$ is not a prime number, then (X,r) is irretractable.*

Proof. Let us assume that (X,r) is retractable. Let $(X,r)\to \mathrm{Ret}(X,r)$, $x\mapsto[x]$, be the canonical map. Since it is a surjective homomorphism of solutions and (X,r) is retractable, the simplicity of (X,r) implies that $|\mathrm{Ret}(X,r)|=1$. Therefore (X,r) is a permutation solution, say $r(x,y)=(\sigma(y),\tau(x))$ for some commuting permutations $\sigma\colon X\to X$ and $\tau\colon X\to X$. Since $|X|>2$, the solution (X,r) is indecomposable by Proposition 15.0.8. This implies that σ is a cycle of length $|X|$ and $\tau=\sigma^k$ for some $k\in\mathbb{Z}$. Let us assume that $\sigma=(x_1\cdots x_n)$, where $n=|X|$. Since n is not a prime number, $n=dm$ for some $1<d<n$. Let $Y=\mathbb{Z}/(d)$ and $s\colon Y\times Y\to Y\times Y$, $s(i,j)=(j+1,i+k)$. Then (Y,s) is a solution. The map $f\colon X\to Y$, $f(x_i)=i \bmod d$ satisfies $f(\tau_{x_j}(x_i))=i+k$ and

$$f(\sigma_{x_i}(x_j))=\begin{cases} f(x_{j+1}) & \text{if } j<n,\\ 1 & \text{if } j=n.\end{cases}$$

Then a straightforward computation shows that f is a surjective homomorphism of solutions, a contradiction. □

The previous proposition cannot be extended to the non-involutive case.

Example 15.0.10. Let $X=\{1,\dots,6\}$. The permutation solution with permutations $\sigma=(153)(264)$ and $\tau=(12)(34)(56)$ is indecomposable.

D

For an additive group A, the **holomorph** of A is the semidirect product

$$\mathrm{Hol}(A) = A \rtimes \mathrm{Aut}(A).$$

This means that the operation is

$$(a,f)(b,g) = (a + f(b), f \circ g), \quad a,b \in A, \quad f,g \in \mathrm{Aut}(A).$$

Every subgroup G of $\mathrm{Hol}(A)$ acts on A by

$$(x,f) \cdot a = \pi_1((x,f)(a,\mathrm{id})) = \pi_1(x + f(a), f) = x + f(a), \quad a,x \in A, \quad f \in \mathrm{Aut}(A),$$

where $\pi_1 \colon \mathrm{Hol}(A) \to A$, $(a,f) \mapsto a$.

Exercise 15.0.11. The group $\mathrm{Hol}(A)$ acts transitively on A and the stabilizer $a \in A$ is isomorphic to $\mathrm{Aut}(A)$.

A subgroup G of $\mathrm{Hol}(A)$ is said to be *regular* if its acts regularly on A, this means that given $a,b \in A$ there exists a unique $(x,f) \in G$ such that

$$b = (x,f) \cdot a = x + f(a).$$

lem:bijective

Lemma 15.0.12. *If G is a regular subgroup of* $\mathrm{Hol}(A)$*, then $\pi_1 \colon G \to A$ is bijective.*

Proof. We first prove that restriction restriction $\pi_1|_G$ of π_1 onto G is injective. Let $(a,f) \in G$ and $(b,g) \in G$ be such that $\pi_1(a,f) = \pi_1(b,g)$. Then $a = b$. Since G is a subgroup,

$$(-f^{-1}(a), f^{-1}) = (a,f)^{-1} \in G, \quad (-g^{-1}(a), g^{-1}) = (a,g)^{-1} \in G,$$

and hence $f = g$ since

$$(-f^{-1}(a), f^{-1}) \cdot a = 0 = (-g^{-1}(a), g^{-1}) \cdot a$$

and G is a regular subgroup. Now we prove that $\pi_1|_G$ is surjective. Let $a \in A$. Since G is regular, there exists $(x,f) \in G$ such that $x + f(a) = (x,f) \cdot a = 0$, so $(-f(a), f) \in G$ for some $f \in \mathrm{Aut}(A)$. Then $(a, f^{-1}) = (-f(a), f)^{-1} \in G$ and $\pi_2|_G(a, f^{-1}) = a$. □

Now we establish an important connection between braces and regular subgroups.

thm:regular

Theorem 15.0.13. *If A is a brace, then $\Delta = \{(a, \lambda_a) : a \in A\}$ is a regular subgroup of* $\mathrm{Hol}(A,+)$*. Conversely, if A is an additive group and G is a regular subgroup of* $\mathrm{Hol}(A)$*, then A is a brace with*

$$a \circ b = a + f(b),$$

where $(\pi_1|_G)^{-1}(a) = (a,f) \in G$.

Proof. Assume first that A is a brace. Using (6.3) and that λ is a group homomorphism, it follows that $\Delta = \{(a, \lambda_a) : a \in A\}$ is a subgroup of $\mathrm{Hol}(A, +)$, as

$$(a,\lambda_a)^{-1} = (\lambda_a^{-1}(-a), \lambda_a^{-1}) = (a', \lambda_{a'}) \in \Delta,$$
$$(a,\lambda_a)(b,\lambda_b) = (a + \lambda_a(b), \lambda_a \circ \lambda_b) = (a \circ b, \lambda_{a\circ b}) \in \Delta.$$

To see that Δ is a regular subgroup, note that $(c, \lambda_c) \cdot a = b$ implies that $c = b \circ a'$, as $(A, \circ)$ is a group.

Assume now that A is an additive group and that G is a regular subgroup of $\mathrm{Hol}(A)$. By Lemma 15.0.12, the restriction $\pi_1|_G$ is bijective. Use the bijection $\pi_1|_G$ to transport the operation of G into A:

$$a \circ b = \pi_1|_G \left((\pi_1|_G)^{-1}(a)(\pi_1|_G)^{-1}(b) \right) = a + f(b),$$

where $a, b \in A$ and $(\pi_1|_G)^{-1}(a) = (a, f) \in G$. Then $(A, \circ)$ is a group isomorphic to G and moreover A is a brace, as

$$\begin{aligned} a \circ (b + c) &= a + f(b + c) = a + f(b) + f(c) \\ &= a + f(b) - a + a + f(c) = a \circ b - a + a \circ c \end{aligned}$$

holds for all $a, b, c \in A$. □

The following lemma is from [13].

lem:BNY **Lemma 15.0.14.** *Let A be a group. If H and K are conjugate regular subgroups of* $\mathrm{Hol}(A)$*, then H and K are conjugate by an automorphism of A.*

Proof. Assume that H and K are conjugate in $\mathrm{Hol}(A)$. Let $(b, g) \in \mathrm{Hol}(A)$ be such that $(b, g)^{-1} H (b, g) = K$. Since $b \in A$, the regularity of H implies that there exists $(a, f) \in H$ such that $a + f(b) = 0$. Since $(a, f) \in H$,

$$\begin{aligned} K = (b,g)^{-1} H (b,g) &= (b,g)^{-1}(a,f)^{-1} H (a,f)(b,g) \\ &= (0, f \circ g)^{-1} H (0, f \circ g) = (f \circ g)^{-1} H (f \circ g). \end{aligned}$$ □

pro:regular **Proposition 15.0.15.** *Let A be an additive group. There exists a bijective correspondence between isomorphism classes of brace structures with additive group A and conjugacy classes of regular subgroups of* $\mathrm{Hol}(A)$.

Proof. Assume that the additive group A has two isomorphic brace structures given by $(a, b) \mapsto a \circ b$ and $(a, b) \mapsto a \times b$. Let $f \colon A \to A$ be a bijective map such that $f(a + b) = f(a) + f(b)$ and $f(a \circ b) = f(a) \times f(b)$ for all $a, b \in A$. We claim that the regular subgroups $\{(a, \lambda_a) : a \in A\}$ and $\{(a, \mu_a) : a \in A\}$, where $\lambda_a(b) = -a + a \circ b$ and $\mu_a(b) = -a + a \times b$, are conjugate. Since f is an isomorphism of braces,

$$f \circ \lambda_a \circ f^{-1} = \mu_{f(a)}$$

for all $a \in A$. This implies that $(0,f)(a,\lambda_a)(0,f)^{-1} = (f(a),\mu_{f(a)})$ for all $a \in A$ and hence and the first claim follows.

Conversely, let H and K be conjugate regular subgroups of $\mathrm{Hol}(A)$. Since H and K are conjugate in $\mathrm{Hol}(A)$, by Lemma 15.0.14 there exists $\varphi \in \mathrm{Aut}(A)$ such that $\varphi H \varphi^{-1} = K$. The brace structure on A corresponding to the subgroup H is given by $a \circ b = a + f(b)$, where $(a,f) = (\pi_1|_H)^{-1}(a) \in H$, see Lemma 15.0.12. Since

$$\varphi(f,a)\varphi^{-1} = (\varphi(a), \varphi \circ f \circ \varphi^{-1}) \in K,$$

it follows that $(\pi_1|_K)^{-1}(\varphi(a)) = (\varphi(a), \varphi \circ f \circ \varphi^{-1})$. Since $\varphi \in \mathrm{Aut}(A)$,

$$\begin{aligned}\varphi(a) \times \varphi(b) &= \varphi(a) + (\varphi \circ f \circ \varphi^{-1})(\varphi(b)) \\ &= \varphi(a) + \varphi(f(b)) = \varphi(a + f(b)) = \varphi(a \circ b)\end{aligned}$$

and hence the braces corresponding to H and K are isomorphic. □

Now we present algorithm used to enumerate braces. It is based on Theorem 15.0.13. The use of Lemma 15.0.14 in Proposition 15.0.15 significantly improves the performance.

alg:regular

Algorithm 15.0.16. Let A be a finite group. To construct all braces with additive group A we proceed as follows:

1) Compute the holomorph $\mathrm{Hol}(A)$ of A.
2) Compute the list of regular subgroups of $\mathrm{Hol}(A)$ of order $|A|$ up to conjugation.
3) For each representative G of regular subgroups of $\mathrm{Hol}(A)$ construct the map $p\colon A \to G$ given by $a \mapsto (a,f) \in G$. Then the set A is a brace with additive group A and multiplication given by $a \circ b = p^{-1}(p(a)p(b))$ for all $a,b \in A$.

To enumerate all isomorphism classes of braces structures with a fixed additive group the third step of Algorithm 15.0.16 is not needed. Algorithm 15.0.16 can be used to compute the number $s(n)$ of non-isomorphic braces of size n. With small modifications it could be used to compute the number $a(n)$ of non-isomorphic braces of abelian type of size n, or the number of non-isomorphic radical rings, or the number of non-isomorphic braces of nilpotent type. Some values for $s(n)$ and $a(n)$ appear in Table 15.1.

Open problems

Open problem 15.0.1. Estimate $s(n)$ and $a(n)$ for $n \to \infty$.

Open problem 15.0.2. Construct and enumerate braces of size 64, 96, 128 and 160.

Table 15.1: The number of non-isomorphic braces.

n	1	2	3	4	5	6	7	8	9	10	11	12
$a(n)$	1	1	1	4	1	2	1	27	4	2	1	10
$s(n)$	1	1	1	4	1	6	1	47	4	6	1	38
n	13	14	15	16	17	18	19	20	21	22	23	24
$a(n)$	1	2	1	357	1	8	1	11	2	2	1	96
$s(n)$	1	6	1	1605	1	49	1	43	8	6	1	855
n	25	26	27	28	29	30	31	32	33	34	35	36
$a(n)$	4	2	37	9	1	4	1	25281	1	2	1	46
$s(n)$	4	6	101	29	1	36	1	1223061	1	6	1	400
n	37	38	39	40	41	42	43	44	45	46	47	48
$a(n)$	1	2	2	106	1	6	1	9	4	2	1	1708
$s(n)$	1	6	8	944	1	78	1	29	4	6	1	66209
n	49	50	51	52	53	54	55	56	57	58	59	60
$a(n)$	4	8	1	11	1	80	2	91	2	2	1	28
$s(n)$	4	51	1	43	1	1028	12	815	8	6	1	418
n	61	62	63	64	65	66	67	68	69	70	71	72
$a(n)$	1	2	11	?	1	4	1	11	1	4	1	489
$s(n)$	1	6	11	?	1	36	1	43	1	36	1	17790
n	73	74	75	76	77	78	79	80	81	82	83	84
$a(n)$	1	2	5	9	1	6	1	1985	804	2	1	34
$s(n)$	1	6	14	29	1	78	1	74120	8436	6	1	606
n	85	86	87	88	89	90	91	92	93	94	95	96
$a(n)$	1	2	1	90	1	16	1	9	2	2	1	195971
$s(n)$	1	6	1	800	1	294	1	29	8	6	1	?
n	97	98	99	100	101	102	103	104	105	106	107	108
$a(n)$	1	8	4	51	1	4	1	106	2	2	1	494
$s(n)$	1	53	4	711	1	36	1	944	8	6	1	11223
n	109	110	111	112	113	114	115	116	117	118	119	120
$a(n)$	1	6	2	1671	1	6	1	11	11	2	1	395
$s(n)$	1	94	8	65485	1	78	1	43	47	6	1	22711
n	121	122	123	124	125	126	127	128	129	130	131	132
$a(n)$	4	2	1	9	49	36	1	?	2	4	1	24
$s(n)$	4	6	1	29	213	990	1	?	8	36	1	324
n	133	134	135	136	137	138	139	140	141	142	143	144
$a(n)$	1	2	37	108	1	4	1	27	1	2	1	10215
$s(n)$	1	6	101	986	1	36	1	395	1	6	1	3013486
n	145	146	147	148	149	150	151	152	153	154	155	156
$a(n)$	1	2	9	11	1	19	1	90	4	4	2	40
$s(n)$	1	6	123	43	1	401	1	800	4	36	12	782
n	157	158	159	160	161	162	163	164	165	166	167	168
$a(n)$	1	2	1	209513	1	1374	1	11	2	2	1	443
$s(n)$	1	6	1	?	1	45472	1	43	12	6	1	28505

tab:braces

Notes

Theorem 15.0.13 was first observed by Catino and Rizzo [23] and Bachiller [8].

Algorithm 15.0.16 and most of the numbers of Table 15.1 appeared in [45]. It should be noted that the number of braces of size 57 of [45] is incorrect; the correct

value is $s(57) = 8$, as Table 15.1 shows. Lemma 15.0.14 appears in [13] and it is needed to compute the number $a(n)$ of isomorphism classes of braces of abelian type of size $n \in \{32, 81, 96, 144, 160, 162\}$ and the number $s(n)$ of isomorphism classes of braces of size $n \in \{32, 54, 80, 81, 108, 112, 120, 136, 144, 147, 150, 152, 162, 168\}$.

Exercises

prob:decomposition

15.0.1. Let (X, r) be a finite decomposable solution and $X = X_1 \cup X_2$ be a decomposition. Prove that $r(X_1 \times X_2) \subseteq X_2 \times X_1$ and $r(X_2 \times X_1) \subseteq X_1 \times X_2$. Solution?

15.0.2. Let (X, r) be a finite involutive permutation solution. Prove that (X, r) is indecomposable if and only if σ is a cycle of length $|X|$.

Open problems

Open problem 15.0.3. Construct indecomposable (involutive) solutions of small size.

Notes

With some variations Theorem 15.0.1 appears in several places, see for example... Algorithms based on this theorem were used in ... and ... to construct and enumerate indecomposable quandles of small size.

Indecomposable quandles are...

Chapter 16
Involutive solutions

Isolutions

A

Let (X,r) be an involutive solution to the YBE. We know that its permutation group $\mathcal{G}(X,r)$ has a natural structure of brace of abelian type. If (Y,r') is a solution isomorphic to (X,r), then $\mathcal{G}(Y,r') \cong \mathcal{G}(X,r)$ as braces. Thus the classification of the involutive solutions can be done in two steps:

1) Classify all the braces of abelian type.
2) For each brace B of abelian type, classify all the involutive solutions (X,r) such that $\mathcal{G}(X,r) \cong B$.

In this chapter we focus on the second step. To this end we will use the next result about extensions of braces.

extensions

Proposition 16.0.1. *([7, Theorem 2.1]) Let H be an abelian group and B be a brace of abelian type. Suppose that $\sigma\colon (B,\circ) \longrightarrow \mathrm{Aut}(H,+)$ is an injective morphism and $\eta\colon (H,+) \longrightarrow (B,+)$ is a surjective morphism such that they satisfy $\eta(\sigma(g)(h)) = \lambda_g(\eta(h))$ for all $g \in B$ and $h \in H$. Then, the multiplication in H defined by*

$$h_1 \cdot h_2 := h_1 + \sigma(\eta(h_1))(h_2),$$

for $h_1, h_2 \in H$, defines a structure of brace on H such that η is a morphism of braces with $\mathrm{Soc}(H) = \ker(\eta)$ and $H/\mathrm{Soc}(H) \cong B$ as braces.

Two of these structures, determined by σ, η and σ', η' respectively, are isomorphic if and only if there exists an $F \in \mathrm{Aut}(H,+)$ such that

$$\sigma'(\eta'(h)) = F^{-1} \circ \sigma(\eta(F(h))) \circ F,$$

for all $h \in H$.

Conversely, suppose that G is a brace. Then, the map $\sigma\colon (G/\mathrm{Soc}(G),\circ) \longrightarrow \mathrm{Aut}(G,+)$, induced by the map $\lambda\colon (G,\circ) \longrightarrow \mathrm{Aut}(G,+)$, and the natural map $\eta\colon G \longrightarrow G/\mathrm{Soc}(G)$ satisfy the above properties.

Proof. We shall see that $(H,\cdot)$ is a group. Let $a,b,c \in H$. We have

$$\begin{aligned} a\cdot(b\cdot c) &= a\cdot(b+\sigma(\eta(b))(c)) = a+\sigma(\eta(a))(b+\sigma(\eta(b))(c)) \\ &= a+\sigma(\eta(a))(b)+\sigma(\eta(a))(\sigma(\eta(b))(c)) \\ &= a+\sigma(\eta(a))(b)+\sigma(\eta(a)\circ\eta(b))(c) \\ &= a+\sigma(\eta(a))(b)+\sigma(\eta(a)+\lambda_{\eta(a)}(\eta(b)))(c) \\ &= a+\sigma(\eta(a))(b)+\sigma(\eta(a)+\eta(\sigma(\eta(a))(b)))(c) \\ &= (a+\sigma(\eta(a))(b))+\sigma(\eta(a+\sigma(\eta(a))(b)))(c) \\ &= (a\cdot b)\cdot c. \end{aligned}$$

Note that, $0\cdot a = 0+\sigma(\eta(0))(a) = \sigma(0)(a) = \mathrm{id}(a) = a$ and $a\cdot 0 = a+\sigma(\eta(a))(0) = a+0 = a$. Furthermore, $a\cdot(-\sigma(\eta(a)')(a)) = a+\sigma(\eta(a))(-\sigma(\eta(a)')(a)) = a-\sigma(0)(a) = a-a = 0$ and

$$\begin{aligned} (-\sigma(\eta(a)')(a))\cdot a &= -\sigma(\eta(a)')(a)+\sigma(\eta(-\sigma(\eta(a)')(a)))(a) \\ &= -\sigma(\eta(a)')(a)+\sigma(-\eta(\sigma(\eta(a)')(a)))(a) \\ &= -\sigma(\eta(a)')(a)+\sigma(-\lambda_{\eta(a)'}(\eta(a)))(a) \\ &= -\sigma(\eta(a)')(a)+\sigma(\eta(a)')(a) = 0. \end{aligned}$$

Hence $(H,\cdot)$ is a group. Now we have

$$\begin{aligned} a\cdot(b+c) &= a+\sigma(\eta(a))(b+c) = a+\sigma(\eta(a))(b)+\sigma(\eta(a))(c) \\ &= a+\sigma(\eta(a))(b)-a+a+\sigma(\eta(a))(b+c) = a\cdot b-a+a\cdot c. \end{aligned}$$

Thus $(H,+,\cdot)$ is a left brace. Note that $\eta(a\cdot b) = \eta(a+\sigma(\eta(a))(b)) = \eta(a)+\eta(\sigma(\eta(a))(b)) = \eta(a)+\lambda_{\eta(a)}(\eta(b)) = \eta(a)\circ\eta(b)$. Hence η is a morphism of braces. Since σ is injective,

$$\mathrm{Soc}(H) = \{a\in H \mid a\cdot b = a+b\forall b\in H\} = \{a\in H \mid \sigma(\eta(a))(b) = b\forall b\in H\} = \{a\in H \mid \eta(a) = 0\} = \ker(eta).$$

Therefore, since η is surjective, $H/\mathrm{Soc}(\eta)\cong B$ as braces.

Let $\sigma'\colon (B,\circ)\longrightarrow \mathrm{Aut}(H,+)$ be an injective morphism and let $\eta'\colon (H,+)\longrightarrow (B,+)$ be a surjective morphism such that they satisfy $\eta'(\sigma'(g)(h)) = \lambda_g(\eta'(h))$ for all $g\in B$ and $h\in H$. Define the multiplication $\circ$ in H by

$$h_1\circ h_2 := h_1+\sigma'(\eta'(h_1))(h_2),$$

for $h_1,h_2\in H$. Note that $F\in\mathrm{Aut}(H,+)$ is an isomorphism from $(H,+,\circ)$ to $(H,+,\cdot)$ if and only if

$$F(h_1\circ h_2) = F(h_1)\cdot F(h_2),$$

which is equivalent to

$$F(h_1+\sigma'(\eta'(h_1))(h_2)) = F(h_1)+\sigma(\eta(F(h_1)))(F(h_2)).$$

Hence F is an isomorphism if and only if

$$F^{-1} \circ \sigma(\eta(h)) \circ F = \sigma'(\eta'(h)),$$

for all $h \in H$. This proves the first part of the result. The second part follows easily. □

The next result gives a first way to construct involutive solutions to the YBE.

BenDavid

Proposition 16.0.2. *Let B be a brace of abelian type, and let X be a set. Suppose that $\eta\colon \mathbb{Z}^{(X)} \longrightarrow (B,+)$ is a surjective morphism, and that $\sigma\colon (B,\cdot) \longrightarrow \mathrm{Aut}(\mathbb{Z}^{(X)})$ is an injective morphism such that $\sigma(a)\,|_X$ is a bijection of X for all $a \in B$, and $\eta(\sigma(a)(m)) = \lambda_a(\eta(m))$ for all $a \in B$ and $m \in \mathbb{Z}^{(X)}$. Let r be the map*

$$\begin{array}{rccc} r\colon & X\times X & \longrightarrow & X\times X \\ & (x,\ y) & \mapsto & (f_x(y),\ f^{-1}_{f_x(y)}(x)), \end{array}$$

where $f_x(y) := \sigma(\eta(x))(y)$. Then (X,r) is an involutive solution to the YBE and $\mathcal{G}(X,r) \cong B$ as braces. Moreover, any involutive solution (Z,t) of the YBE with $\mathcal{G}(Z,t) \cong B$ is of this form.

Proof. By Proposition 16.0.1, the abelian group $\mathbb{Z}^{(X)}$ with the multiplication defined by

$$x \cdot y := x + \sigma(\eta(x))(y),$$

for $x, y \in \mathbb{Z}^X$, is a brace of abelian type and η becomes a homomorphism of braces. Note that for $x, y \in X \subseteq \mathbb{Z}^{(X)}$, we have that

$$\lambda_x(y) = xy - x = x + \sigma(\eta(x))(y) - x = \sigma(\eta(x))(y) = f_x(y).$$

Therefore (X,r) is an involutive solution to the YBE because it is the restriction of the solution to the YBE associated with the brace $\mathbb{Z}^{(X)}$ (cf. [**?**, Lemma 2]). In fact the brace $\mathbb{Z}^{(X)}$ is equal to the brace $G(X,r)$. Recall that the addition of the brace

$$\mathcal{G}(X,r) = \{\sigma(\eta(m))|_X \mid m \in \mathbb{Z}^{(X)}\}$$

is defined by $\sigma(\eta(m_1))|_X + \sigma(\eta(m_2))|_X = \sigma(\eta(m_1 + m_1))|_X$, for all $m_1, m_2 \in \mathbb{Z}^{(X)}$. Therefore, the map $B \longrightarrow \mathcal{G}(X,r)$ defined by $a \mapsto \sigma(a)|_X$ is an isomorphism of braces.

On the other hand, observe that, if (Z,t) is a solution to the YBE and $\xi\colon B \longrightarrow \mathcal{G}(Z,t)$ is an isomorphism of braces, then the unique homomorphism $\eta\colon (\mathbb{Z}^{(Z)},+) \longrightarrow (B,+)$ such that $\eta(z) = \xi^{-1}(\phi(z))$, for all $z \in Z$, where $\phi\colon G(Z,t) \longrightarrow \mathcal{G}(Z,t)$ is the natural projection, is surjective, and the map $\sigma : (B,\cdot) \longrightarrow \mathrm{Aut}(\mathbb{Z}^{(Z)})$, where $\sigma(a)$ is the unique automorphism of $\mathbb{Z}^{(Z)}$ such that $\sigma(a)(z) = \xi(a)(z)$, for all $z \in Z$, is an injective homomorphism. Furthermore, for $a \in B$, there exists $m \in \mathbb{Z}^{(Z)}$ such that $a = \eta(m)$ and, since ϕ is a homomorphism of braces,

$$\begin{aligned}\eta(\sigma(a)(z)) &= \eta(\xi(a)(z)) = \xi^{-1}(\phi(\xi(\eta(m))(z)))\\ &= \xi^{-1}(\phi(\phi(m)(z))) = \xi^{-1}(\phi(mz-m)) = \xi^{-1}(\phi(m)\phi(z)-\phi(m))\\ &= \eta(m)\eta(z)-\eta(m) = a\eta(z)-a\\ &= \lambda_a(\eta(z)),\end{aligned}$$

for all $z \in Z$. Therefore $\eta(\sigma(a)(n)) = \lambda_a(\eta(n))$, for all $a \in B$ and all $n \in \mathbb{Z}^{(Z)}$. Now we have that

$$\sigma(\eta(x))(y) = \xi(\eta(x))(y) = \phi(x)(y),$$

for all $x, y \in Z$. Hence (Z,t) is exactly the same solution to the YBE as the solution obtained by the given construction using the maps η and σ. □

This proposition reduces the problem of finding all the involutive solutions to the YBE to the problem of finding maps η and σ with the required properties. Note that the map η is determined by its restriction to X and, since η is surjective, $\eta(X)$ should generate $(B,+)$. Note also that the monomorphism σ factors trough the symmetric group $\mathbb{S}_X$, that is $\sigma = \sigma_1\sigma_2$ where $\sigma_2 \colon (B,\cdot) \longrightarrow \mathbb{S}_X$ is a monomorphism and $\sigma_1 \colon \mathbb{S}_X \longrightarrow \mathrm{Aut}(\mathbb{Z}^{(X)})$ is the natural map. Therefore the problem of finding all the solutions to the YBE is reduced to the problem of finding all the faithful $(B,\cdot)$-sets X and all the homomorphisms of $(B,\cdot)$-sets $X \longrightarrow (B,+)$ such that the image of X generates $(B,+)$. In the next sections we give a method that describes how to solve this problem using only the structure of the brace B.

16.1 Construction of solutions

sec3

As we have seen in the previous section, given a brace B of abelian type, to construct an involutive solution (X,r) to the YBE such that $\mathcal{G}(X,r) \cong B$, it is enough to find two maps $\eta \colon X \longrightarrow B$ and $\sigma \colon B \longrightarrow \mathbb{S}_X$ satisfying some properties. In this section, we show how to construct the sets X and the maps η and σ using only information about the structure of the brace B.

Recall that $\lambda : (B,\cdot) \to \mathrm{Aut}(B,+)$ is an action. The stabiliser of $a \in B$ by the action λ is denoted $\mathrm{St}(a)$. For $a \in B$, let $B_a = \{\lambda_b(a) \mid b \in B\}$ be the orbit of a, and let $\mathcal{O} = \{B_a \mid a \in B\}$ be the set of orbits of the action λ. For each $i \in \mathcal{O}$, choose an element $a_i \in i$. Let I be a subset of $\mathcal{O}$, such that $Y = \bigcup_{i\in I} i$ (and thus $Y = \{\lambda_b(a_i) \mid i \in I,\ b \in B\}$) satisfies $B = \langle Y\rangle_+$, the additive subgroup generated by Y. For each $i \in I$, let J_i be a non-empty set and let $\{K_{i,j}\}_{j\in J_i}$ be a family of subgroups of $\mathrm{St}(a_i)$ such that

$$\bigcap_{i\in I}\bigcap_{j\in J_i}\bigcap_{b\in B} bK_{i,j}b^{-1} = \{1\}.$$

Note that if one of the subgroups $K_{i,j}$ is trivial, then this last condition is satisfied.

The next result is the main result of this section.

main **Theorem 16.1.1.** *With the above notation, let $X := \bigsqcup_{i\in I}\bigsqcup_{j\in J_i} B/K_{i,j}$ be the disjoint union of the sets of left cosets $B/K_{i,j}$. Let $\eta\colon X \longrightarrow B$ be the map defined by $\eta(bK_{i,j}) = \lambda_b(a_i)$, and define $\sigma\colon B \longrightarrow \mathbb{S}_X$ to be the natural action of $(B,\cdot)$ on X given by left multiplications on the cosets in $B/K_{i,j}$; i.e. $\sigma(c)(bK_{i,j}) := cbK_{i,j}$. Then σ is injective and $\eta(\sigma(a)(x)) = \lambda_a(\eta(x))$, for all $x \in X$. Moreover (X,r), where r is the map*

$$\begin{aligned} r\colon\ X\times X &\longrightarrow & X\times X\\ (x,y) &\mapsto & (f_x(y),\ f^{-1}_{f_x(y)}(x)), \end{aligned}$$

with $f_x(y) = \sigma(\eta(x))(y)$, is an involutive solution to the YBE such that $\mathcal{G}(X,r) \cong B$ as braces.

Furthermore, any involutive solution (Z,t), with $\mathcal{G}(Z,t) \cong B$ as braces, is isomorphic to such a solution.

Proof. First we shall prove σ is injective. Let $c \in B$ be an element such that $\sigma(c) = \mathrm{id}$. Hence $cbK_{i,j} = bK_{i,j}$, for all $b \in B$, $i \in I$ and $j \in J_i$. Thus $c \in \bigcap_{i\in I}\bigcap_{j\in J_i}\bigcap_{b\in B} bK_{i,j}b^{-1} = \{1\}$. Therefore σ is injective. Let $a,b \in B$. We have that

$$\begin{aligned} \eta(\sigma(a)(bK_{i,j})) &= \eta(abK_{i,j}) = \lambda_{ab}(a_i)\\ &= \lambda_a(\lambda_b(a_i)) = \lambda_a(\eta(bK_{i,j})). \end{aligned}$$

Hence $\eta(\sigma(a)(x)) = \lambda_a(\eta(x))$, for all $x \in X$. Hence, by Proposition 16.0.2, (X,r) is an involutive solution to the YBE and $\mathcal{G}(X,r) \cong B$ as braces.

Let (Z,t) be an involutive solution to the YBE such that $\mathcal{G}(Z,t) \cong B$ as braces. Let $\xi\colon \mathcal{G}(Z,t) \longrightarrow B$ be an isomorphism of braces. Let $\eta = \xi\circ\phi$, where $\phi\colon G(Z,t) \longrightarrow \mathcal{G}(Z,t)$ is the natural projection. Then $\eta(Z)$ is a subset of B, invariant by λ which generates B additively. Let $Y = \eta(Z)$.

We also have an injective morphism $\sigma : (B,\cdot) \longrightarrow \mathrm{Aut}(\mathbb{Z}^{(Z)})$, such that $\sigma(b)(z) = \xi^{-1}(b)(z)$, for all $b \in B$ and $z \in Z$. Therefore Z is a left B-set with the action induced by σ. Let $a \in B$ and $z \in Z$. Let $g \in G(Z,t)$ such that $\phi(g) = \xi^{-1}(a)$. We have

$$\begin{aligned} \eta(\sigma(a)(z)) &= \xi(\phi(\xi^{-1}(a)(z))) = \xi(\phi(\phi(g)(z)))\\ &= \xi(\phi(gz-g)) = \xi(\phi(g))\xi(\phi(z)) - \xi(\phi(g))\\ &= a\eta(z) - a = \lambda_a(\eta(z)). \end{aligned}$$

Therefore the restriction $\eta|_Z\colon Z \longrightarrow Y$ of η is a B-map.

Let $Y = \bigcup_{i\in I} i$ be the decomposition of Y as disjoint union of orbits under the action λ. Since $\eta|_Z$ is a surjective B-map, for all $i \in I$, the action σ splits $(\eta|_Z)^{-1}(i)$ into orbits: $(\eta|_Z)^{-1}(i) = \bigcup_{j\in J_i} Z_{i,j}$ and $\eta(Z_{i,j}) = i$. So we have $Z = \bigcup_{i\in I}\bigcup_{j\in J_i} Z_{i,j}$, where $\eta(Z_{i,j}) = i$ for all i,j.

For each $i \in I$, we choose an element $a_i \in i$, and for each $j \in J_i$, we choose $z_{i,j} \in Z_{i,j}$ such that $\eta(z_{i,j}) = a_i$. Note that $\mathrm{St}(z_{i,j}) \le \mathrm{St}(a_i) \le B$, since any $b \in \mathrm{St}(z_{i,j})$ satisfies $\sigma(b)(z_{i,j}) = z_{i,j}$ and, applying η, we obtain $a_i = \eta(z_{i,j}) = \eta(\sigma(b)(z_{i,j})) = \lambda_b(\eta(z_{i,j})) = \lambda_b(a_i)$, so $b \in \mathrm{St}(a_i)$.

Recall that the maps $f_i\colon i \to B/\mathrm{St}(a_i)$ and $f_{i,j}\colon Z_{i,j} \to B/\mathrm{St}(z_{i,j})$ defined by $f_i(\lambda_b(a_i)) = b\,\mathrm{St}(a_i)$ and $f_{i,j}(\sigma(b)(z_{i,j})) = b\,\mathrm{St}(z_{i,j})$ are isomorphisms of B-sets. Then, $\eta\,|_{Z_{i,j}}$ is determined by the canonical projection

$$\begin{array}{rccc} \pi\colon & B/\mathrm{St}(z_{i,j}) & \longrightarrow & B/\mathrm{St}(a_i) \\ & t\,\mathrm{St}(z_{i,j}) & \mapsto & t\,\mathrm{St}(a_i), \end{array}$$

that is,

$$\begin{aligned} \eta(f_{i,j}^{-1}(b\,\mathrm{St}(z_{i,j}))) &= \eta(\sigma(b)(z_{i,j})) = \lambda_b(\eta(z_{i,j})) \\ &= \lambda_b(a_i) = f_i^{-1}(b\,\mathrm{St}(a_i)) \\ &= f_i^{-1}(\pi(b\,\mathrm{St}(z_{i,j}))). \end{aligned}$$

Note that $\sigma(c)(z) = z$, for all $z \in Z$, if and only if $cb\,\mathrm{St}(z_{i,j}) = b\,\mathrm{St}(z_{i,j})$, for all $b \in B$ and all i, j, equivalently, $c \in \bigcap_{i,j}\bigcap_{b\in B} b\,\mathrm{St}(z_{i,j})b^{-1}$. Since σ is injective, we have $\bigcap_{i,j}\bigcap_{b\in B} b\,\mathrm{St}(z_{i,j})b^{-1} = \{1\}$. Put $H_i := \mathrm{St}(a_i)$ and $K_{i,j} := \mathrm{St}(z_{i,j})$. Let (X,r) be the solution defined as in the statement of the theorem. We shall prove that $(X,r) \cong (Z,t)$. Let $f\colon Z \longrightarrow X$ be the map defined by $f(z) = f_{i,j}(z)$, for all $z \in Z_{i,j}$. Clearly f is bijective. Let $x, z \in Z$. We may assume that $z \in Z_{i,j}$ and $x \in Z_{p,q}$. Therefore there exist $b_1, b_2 \in B$ such that $z = \sigma(b_1)(z_{i,j})$ and $x = \sigma(b_2)(z_{p,q})$. We have

$$\begin{aligned} f(\phi(x)(z)) &= f(\sigma(\eta(x))(z)) = f_{i,j}(\sigma(\eta(x))(\sigma(b_1)(z_{i,j}))) \\ &= f_{i,j}(\sigma(\eta(x)b_1)(z_{i,j})) = \eta(x)b_1\,\mathrm{St}(z_{i,j}) \\ &= \eta(x)b_1K_{i,j} = \eta(\sigma(b_2)(z_{p,q}))b_1K_{i,j} \\ &= \lambda_{b_2}(\eta(z_{p,q}))b_1K_{i,j} = \lambda_{b_2}(a_p)b_1K_{i,j} \\ &= f_{b_2K_{p,q}}(b_1K_{i,j}) \\ &= f_{f(x)}(f(z)). \end{aligned}$$

Therefore f is an isomorphism of solutions to the YBE. □

As a consequence of Theorem 16.1.1, given a brace B of abelian type, to construct all the involutive solutions (X,r) of the YBE such that $\mathcal{G}(X,r) \cong B$ as braces one can proceed as follows:

1) Find the decomposition of B as disjoint union of orbits, $B = \bigcup_{i\in K} B_i$, by the action $\lambda\colon (B,\cdot) \longrightarrow \mathrm{Aut}(B,+)$. Then choose one element a_i in each orbit B_i for all $i \in K$.

2) Find all the subsets I of K such that the subset $Y = \bigcup_{i\in I} B_i$ generates the additive group of B.

3) Given such an Y, find for each $i \in I$ a non-empty family $\{K_{i,j}\}_{j\in J_i}$ of subgroups of $\mathrm{St}(a_i)$ such that $\bigcap_{i,j}\bigcap_{b\in B} bK_{i,j}b^{-1} = \{1\}$. Note that the $K_{i,j}$ could be equal for different (i,j).

4) Construct a solution as in the statement of Theorem 16.1.1 using the families $\{K_{i,j}\}_{j\in J_i}$, for $i \in I$.

Note that by Theorem 16.1.1, any involutive solution (X,r) to the YBE such that $\mathcal{G}(X,r) \cong B$ (as braces) is isomorphic to one constructed in this way. It could happen that different solutions of the YBE constructed in this way from a brace B of abelian type are in fact isomorphic. In the next section we characterize when two of these solutions are isomorphic.

16.2 Isomorphism of solutions

Let B be a brace of abelian type and let O, I, a_i, J_i, $K_{i,j}$ be as in the beginning of Section 16.1. Let (X,r) be the solution to the YBE of the statement of Theorem 16.1.1.

Let $I' \subseteq O$ such that $Y' = \bigcup_{i' \in I'} i'$ satisfy $B = \langle Y' \rangle_+$. For each $i' \in I'$, let $\{L_{i',j'}\}_{j' \in J'_{i'}}$ be a non-empty family of subgroups of $\mathrm{St}(a_{i'})$ such that

$$\bigcap_{i' \in I'} \bigcap_{j' \in J'_{i'}} \bigcap_{b \in B} bL_{i',j'}b^{-1} = \{1\}.$$

Let (X',r') be the corresponding solution to the YBE defined as in the statement of Theorem 16.1.1, that is

$$r'(b_1L_{i'_1,j'_1}, b_2L_{i'_2,j'_2}) = (\lambda_{b_1}(a_{i'_1})b_2L_{i'_2,j'_2}, \lambda_{\lambda_{b_1}(a_{i'_1})b_2}(a_{i'_2})^{-1}b_1L_{i'_1,j'_1}).$$

We shall characterize when (X,r) and (X',r') are isomorphic in the following result.

isomorphism

Theorem 16.2.1. *The solutions (X,r) and (X',r') are isomorphic if and only if there exist an automorphism ψ of the left brace B, a bijective map $\alpha\colon I \to I'$, a bijective map $\beta_i\colon J_i \to J'_{\alpha(i)}$ and $z_{i,j} \in B$, for each $i \in I$ and $j \in J_i$, such that*

$$\psi(a_i) = \lambda_{z_{i,j}}(a_{\alpha(i)}) \quad \text{and} \quad \psi(K_{i,j}) = z_{i,j}L_{\alpha(i),\beta_i(j)}z_{i,j}^{-1},$$

for all $i \in I_1$ and $j \in J_i$.

Proof. Suppose that there exist an automorphism ψ of the brace B, a bijective map $\alpha\colon I \to I'$, a bijective map $\beta_i\colon J_i \to J'_{\alpha(i)}$ and $z_{i,j} \in B$, for $i \in I$, and for $j \in J_i$, such that

$$\psi(a_i) = \lambda_{z_{i,j}}(a_{\alpha(i)}) \quad \text{and} \quad \psi(K_{i,j}) = z_{i,j}L_{\alpha(i),\beta_i(j)}z_{i,j}^{-1}.$$

Observe that we also have $\psi(\lambda_b(a_i)) = \lambda_{\psi(b)z_{i,j}}(a_{\alpha(i)})$ for every $b \in B$, because ψ is a morphism of braces. We define $F\colon X \to X'$ by $F(bK_{i,j}) = \psi(b)z_{i,j}L_{\alpha(i),\beta_i(j)}$, for all $i \in I$, $j \in J_i$ and $b \in B$. Since $\psi(K_{i,j}) = z_{i,j}L_{\alpha(i),\beta_i(j)}z_{i,j}^{-1}$, we get that F is well defined. It is easy to check that F is an isomorphism of the solutions (X,r) and (X',r').

Conversely, suppose that there exists an isomorphism $F\colon X \to X'$ of the solutions (X,r) and (X',r'). We can write $F(bK_{i,j}) = \varphi(bK_{i,j})L_{\alpha(b,i,j),\beta(b,i,j)}$, for some maps $\varphi\colon X \to B$, $\alpha\colon X \to I'$ and $\beta\colon X \to \bigcup_{i' \in I'} J'_{i'}$. We shall prove that $\alpha(b,i,j) =$

$\alpha(1,i,k)$ and $\beta(b,i,j)=\beta(1,i,j)$, for all $b\in B$, $i\in I$ and $j,k\in J_i$. Since F is a morphism of solutions to the YBE, we have

$$F(\lambda_{b_1}(a_{i_1})b_2K_{i_2,j_2})=\lambda_{\varphi(b_1K_{i_1,j_1})}(a_{\alpha(b_1,i_1,j_1)})F(b_2K_{i_2,j_2}), \tag{16.1}$$

for all $b_1,b_2\in B$, $i_1,i_2\in I$, $j_1\in J_{i_1}$ and $j_2\in J_{i_2}$. Hence

$$\begin{aligned}&\varphi(\lambda_{b_1}(a_{i_1})b_2K_{i_2,j_2})L_{\alpha(\lambda_{b_1}(a_{i_1})b_2,i_2,j_2),\beta(\lambda_{b_1}(a_{i_1})b_2,i_2,j_2)}\\&=\lambda_{\varphi(b_1K_{i_1,j_1})}(a_{\alpha(b_1,i_1,j_1)})\varphi(b_2K_{i_2,j_2})L_{\alpha(b_2,i_2,j_2),\beta(b_2,i_2,j_2)},\end{aligned}$$

for all $b_1,b_2\in B$, $i_1,i_2\in I$, $j_1\in J_{i_1}$ and $j_2\in J_{i_2}$. Thus $\alpha(\lambda_{b_1}(a_{i_1})b_2,i_2,j_2)=\alpha(b_2,i_2,j_2)$ and $\beta(\lambda_{b_1}(a_{i_1})b_2,i_2,j_2)=\beta(b_2,i_2,j_2)$. Since $B=\langle Y\rangle_+$ and Y is B-invariant (by the action λ), we know that Y also generates the multiplicative group of B. Therefore $\alpha(b_2,i_2,j_2)=\alpha(1,i_2,j_2)$ and $\beta(b_2,i_2,j_2)=\beta(1,i_2,j_2)$. Note also that

$$\lambda_{\varphi(b_1K_{i_1,j_1})}(a_{\alpha(b_1,i_1,j_1)})F(b_2K_{i_2,j_2})=\lambda_{\varphi(b_1K_{i_1,j})}(a_{\alpha(b_1,i_1,j)})F(b_2K_{i_2,j_2}),$$

for all $b_1,b_2\in B$, $i_1,i_2\in I$, $j_1,j\in J_{i_1}$ and $j_2\in J_{i_2}$. Since $\bigcap_{i',j'}\bigcap_{b\in B}bL_{i',j'}b^{-1}=\{1\}$, we have that

$$\lambda_{\varphi(b_1K_{i_1,j_1})}(a_{\alpha(b_1,i_1,j_1)})=\lambda_{\varphi(b_1K_{i_1,j})}(a_{\alpha(b_1,i_1,j)}),$$

for all $b_1\in B$, $i_1\in I$ and $j_1,j\in J_{i_1}$. Therefore $a_{\alpha(b_1,i_1,j_1)},a_{\alpha(b_1,i_1,j)}\in\alpha(1,i_1,k)$, for all $b_1\in B$, $i_1\in I$ and $j_1,j\in J_{i_1}$ and thus $\alpha(b,i,j)=\alpha(1,i,k)$, for all $b\in B$, $i\in I_1$ and $j,k\in J_i$. For each $i\in I$ we choose an element $j_i\in J_i$. Since F is bijective, the map $I\to I'$ defined by $i\mapsto\alpha(1,i,j_i)$ is bijective and for each $i\in I$ the map $J_i\to J'_{\alpha(1,i,j_i)}$ defined by $j\mapsto\beta(1,i,j)$ is bijective. We shall see that there exists an automorphism ψ of the left brace B such that

$$\psi(\lambda_b(a_i))=\lambda_{\psi(b)\varphi(K_{i,j_i})}(a_{\alpha(1,i,j_i)})$$

and

$$\psi(K_{i,j})=\varphi(K_{i,j_i})L_{\alpha(1,i,j_i),\beta(1,i,j)}\varphi(K_{i,j_i})^{-1},$$

for all $b\in B, i\in I$ and $j\in J_i$. Let $1=\lambda_{b_1}(a_{i_1})^{\varepsilon_1}\cdots\lambda_{b_m}(a_{i_m})^{\varepsilon_m}$, for some $b_1,\dots,b_m\in B$, $i_1,\dots,i_m\in I$ and $\varepsilon_1,\dots,\varepsilon_m\in\{1,-1\}$. By (16.1), we have

$$\begin{aligned}F(bK_{i,j})&=F(\lambda_{b_1}(a_{i_1})^{\varepsilon_1}\cdots\lambda_{b_m}(a_{i_m})^{\varepsilon_m}bK_{i,j})\\&=\lambda_{\varphi(b_1K_{i_1,j_{i_1}})}(a_{\alpha(1,i_1,j_{i_1})})^{\varepsilon_1}F(\lambda_{b_2}(a_{i_2})^{\varepsilon_2}\cdots\lambda_{b_m}(a_{i_m})^{\varepsilon_m}bK_{i,j})\\&=\lambda_{\varphi(b_1K_{i_1,j_{i_1}})}(a_{\alpha(1,i_1,j_{i_1})})^{\varepsilon_1}\cdots\lambda_{\varphi(b_mK_{i_m,j_{i_m}})}(a_{\alpha(1,i_m,j_{i_m})})^{\varepsilon_m}F(bK_{i,j}),\end{aligned}$$

for all $b\in B$, $i\in I$ and $j\in J_i$. Since $\bigcap_{i',j'}\bigcap_{b\in B}bL_{j,l}b^{-1}=\{1\}$, we have that $\lambda_{\varphi(b_1K_{i_1,j_{i_1}})}(a_{\alpha(1,i_1,j_{i_1})})^{\varepsilon_1}\cdots\lambda_{\varphi(b_mK_{i_m,j_{i_m}})}(a_{\alpha(1,i_m,j_{i_m})})^{\varepsilon_m}=1$. Therefore there exists a unique morphism $\psi\colon B\to B$ of multiplicative groups such that $\psi(\lambda_b(a_i))=\lambda_{\varphi(bK_{i,j_i})}(a_{\alpha(1,i,j_i)})$. Since Y generates the multiplicative group of B, by (16.1) one can see that

$$\varphi(bK_{i,j})L_{\alpha(1,i,j_i),\beta(1,i,j)} = F(bK_{i,j}) = \psi(b)\varphi(K_{i,j})L_{\alpha(1,i,j_i),\beta(1,i,j)}.$$

Therefore, since $L_{\alpha(1,i,j_i),\beta(1,i,j)} \subseteq \mathrm{St}(a_{\alpha(1,i,j_i)})$, we have

$$\lambda_{\varphi(bK_{i,j})}(a_{\alpha(1,i,j_i)}) = \lambda_{\psi(b)\varphi(K_{i,j})}(a_{\alpha(1,i,j_i)}).$$

Hence $\psi(\lambda_b(a_i)) = \lambda_{\psi(b)\varphi(K_{i,j_i})}(a_{\alpha(1,i,j_i)})$. Now we have that

$$\begin{aligned}\psi(b+a_i) &= \psi(b\lambda_{b^{-1}}(a_i)) = \psi(b)\psi(\lambda_{b^{-1}}(a_i))\\ &= \psi(b)\lambda_{\psi(b)^{-1}\varphi(K_{i,j_i})}(a_{\alpha(1,i,j_i)})\\ &= \psi(b)\lambda_{\psi(b)^{-1}}(\lambda_{\varphi(K_{i,j_i})}(a_{\alpha(1,i,j_i)}))\\ &= \psi(b)+\lambda_{\varphi(K_{i,j_i})}(a_{\alpha(1,i,j_i)})\\ &= \psi(b)+\psi(\lambda_1(a_i)) = \psi(b)+\psi(a_i).\end{aligned}$$

Now it is easy to see that ψ is a morphism of braces. Since F is bijective and $F(bb'K_{i,j}) = \psi(b)F(b'K_{i,j})$, it follows that ψ is bijective. Furthermore $b \in K_{i,j}$ if and only if

$$\begin{aligned}\varphi(K_{i,j})L_{\alpha(1,i,j_i),\beta(1,i,j)} &= F(K_{i,j}) = F(bK_{i,j}) = \psi(b)F(K_{i,j})\\ &= \psi(b)\varphi(K_{i,j})L_{\alpha(1,i,j_i),\beta(1,i,j)}.\end{aligned}$$

Therefore the result follows. □

Summarizing, the last theorem says that two solutions constructed as in Theorem 16.1.1 are isomorphic if we can find an automorphism of the brace B that brings each $K_{i,j}$ to one $L_{i',j'}$, taking into account that maybe the $L_{i',j'}$'s are permuted (that is the reason for the α and β_i maps), and that maybe we have chosen another element of the orbit in the process (that is the reason why the image a_i is $\lambda_{z_{i,j}}(a_{\alpha(i)})$ and not just $a_{\alpha(i)}$, and it is also the reason why the $L_{\alpha(i),\beta_i(j)}$ is conjugated by $z_{i,j}$).

The following is an example of how to use Theorem 16.1.1 and Theorem 16.2.1 to compute all the finite solutions associated to a given finite left brace up to isomorphism. We use the easiest examples of braces: trivial braces of order p, where p is a prime. Recall that a brace B is trivial if $ab = a+b$ for all $a,b \in B$.

bracesp

Example 16.2.2. Consider the trivial brace over $G = \mathbb{Z}/(p)$. Then, the orbits are $\{\alpha\}$ for every $\alpha \in \mathbb{Z}/(p)$. Since any orbit has only one element, we have $\mathrm{St}(\alpha) = G$, and the possible $K_{i,j}$'s in this case are 0 and G.

Let Y be a subset of $\mathbb{Z}/(p)$ with at least a nonzero element. Let $K_{\alpha,j} = G$ for $\alpha \in Y$ and $j \in \{1,\dots,k_\alpha\}$, and let $K'_{\alpha,k} = 0$ for $\alpha \in Y$ and $k \in \{1,\dots,m_\alpha\}$, where k_α and m_α are non-negative integers such that $k_\alpha + m_\alpha > 0$. Write $G/K_{\alpha,j} = \{y_{\alpha,j}\}$, and $G/K'_{\alpha,k} = \{y^1_{\alpha,k},\dots,y^p_{\alpha,k}\}$, where $y^l_{\alpha,k} = l + K'_{\alpha,k}$. Assume that at least one m_α is positive. Then the corresponding solution of the YBE is (X,r), where

$$X = \bigcup_{\alpha\in Y}\left(\left(\bigcup_{1\le j\le k_\alpha}\{y_{\alpha,j}\}\right)\cup\left(\bigcup_{1\le k\le m_\alpha}\{y^1_{\alpha,k},\dots,y^p_{\alpha,k}\}\right)\right)$$

and $r(x,y) = (\sigma_x(y), \sigma^{-1}_{\sigma_x(y)}(x))$, with the sigma maps given by

$$\sigma_{y_{\alpha,j}} = \sigma_{y^l_{\alpha,k}} = \tau^\alpha, \text{ for all } \alpha \in Y, \text{ for all } j,k \text{ and for all } l \in \{1,\dots,p\},$$

where $\tau \in \mathbb{S}_X$ is the product of all the cycles $(y^1_{\alpha,k}, y^2_{\alpha,k}, \dots, y^p_{\alpha,k})$ for any $\alpha \in Y$ and $k \in \{1,\dots,m_\alpha\}$.

Finally observe that, in this case, $\mathrm{Aut}(G,+,\cdot) = \mathrm{Aut}(G,+) \cong (\mathbb{Z}/(p))^*$, and the effect of an automorphism of G over a solution is to change $\sigma_{y_{\alpha,j}} = \sigma_{y^l_{\alpha,k}} = \tau^\alpha$ to the isomorphic solution $\sigma_{y_{\alpha,j}} = \sigma_{y^l_{\alpha,k}} = \tau^{A\alpha}$, where $A \in (\mathbb{Z}/(p))^*$.

Exercises

16.2.1.

16.2.2.

16.2.3.

16.2.4.

Notes

The first example of a simple non-trivial brace of abelian type is due to Bachiller [9]. In the same papar Bachiller introduced the matched product of two left braces of abelian type. One can see that the definition of matched product of two left braces is the same that in the case of braces of abelian type. In [11], Bachiller, Cedó, Jespers and Okniński introduced the matched product of more than two left braces of abelian type (this can be generaliced to arbitrari left braces without changes). In the same paper the authors construct several families of simple non-trivial left braces of abelian type using matched products.

The asymmetric product of left braces of abelian type was intruduced by Catino, Colazzo and Stefanelli in [21], and Theorem 17.0.7 is due to them ([21, Theorem 3]). In [12] Bachiller, Cedó, Jespers and Okniński used the asymmetric product to construct to costruct new families of simple non-trivial left braces of abelian type. In fact, every known simple non-trivial left brace is an asymmetric product (see [12, 28, 29]). Theorem 17.0.8 is based on [11, Theorem 3.6] and [12, Theorem 6.2]. The two concrete constructions of simple non-trivial simple braces of abelian type appear in [11, 28].

Chapter 17
Simple braces

Sbraces

A

As we have seen in Chapter **??**, the classification of finite involutive solutions is reduced to the classification of the finite braces of abelian type. A first step in the classification of the finite braces of abelian type is the classification of the finite simple braces of abelian type.

Definition 17.0.1. A non-zero brace B is *simple* if $\{0\}$ and B are the only ideals of B.

Example 17.0.2. Let G be a simple additive group. Then the trivial brace $(G,+,\circ)$, is a simple brace. These are the simple trivial braces.

Thus the classification of the finite simple trivial braces is equivalent to the classification of the finite simple groups. In particular, the finite simple trivial braces of abelian type are the trivial braces of prime cardinality.

By Theorem 6.2.17, the multiplicative group of a finite brace of nilpotent type is solvable. Let B be a non-zero finite brace of nilpotent type. Let $p_1,\dots,p_k$ be the distinct prime divisors of $|B|$. Let B_{p_i} be the Sylow p_i-subgroup of the additive group of B, for $i=1,\dots,k$. Note that B_{p_i} is a left ideal of B, and the multiplicative groups of these left ideals B_{p_i} form a Sylow system of the multiplicative group of B. Thus

$$B=\prod_{i=1}^{k} B_{p_i}.$$

Note that for $a\in B_{p_i}$, $b\in B_{p_j}$, where $i\neq j$, there exist unique $c\in B_{p_j}$ and $d\in B_{p_i}$ such that $a\circ b=c\circ d$. On the other hand,

$$a\circ b=\lambda_a(b)\circ\lambda_a(b)'\circ a\circ b=\lambda_a(b)\circ(b'+\lambda_{b'}(a')-b')'.$$

Hence $c=\lambda_a(b)$ and $d=(b'+\lambda_{b'}(a')-b')'$.

lem:nilpotentsimple

Lemma 17.0.3. *Let B be a finite simple brace of nilpotent type with nilpotent multiplicative group. Then B is a trivial brace of prime order.*

Proof. Let p be a prime divisor of $|B|$. Let B_p be the Sylow p-subgroup of the additive group of B. We know that B_p is a left ideal of B. In particular, B_p also is a Sylow p-subgroup of the multiplicative group of B. Since the multiplicative group of B is nilpotent, we have that B_p is an ideal of B. Since B is simple, $B = B_p$. Let G be the multiplicative group of B and let A be the additive group of B. Then the semidirect product $H = A \rtimes G$ via the lambda map is a p-group, and thus it is nilpotent. Let $a \in A$ and $g \in G$. We have that

$$(a,0)(0,g)(-a,0)(0,g') = (a,g)(-a,g') = (a - \lambda_g(a),0) = (-(g*a),0).$$

Therefore $[\{0\} \times G, A \times \{0\}] = B * B \times \{0\}$. Since H is nilpotent, it is clear that $B * B \neq B$. Since $B * B$ is an ideal of B and B is simple, it follows that $B * B = \{0\}$. Hence B is a trivial brace. Since B is simple, $|B| = p$. □

The above result shows that if B is a finite non-trivial simple brace of abelian type, then there exist two distinct prime divisors of $|B|$. To understand the structure of B we need to study the structure of the left ideals B_p, where B_p is the Sypow p-subgroup of $(B,+)$, and the restriction of the lambda map to B_p acting on B_q, where p,q are distinct prime divisors of $|B|$.

Let B be a finite simple brace of abelian type of order $p^n q^m$ for distinct primes p,q and positive integers n,m. Let B_p and B_q be the Sylow p-subgroup and the Sylow q-subgroup of the additive group of B respectively. Let $\alpha\colon B_q \to \mathrm{Aut}(B_p,+)$ and $\beta\colon B_p \to \mathrm{Aut}(B_q,+)$ be restictions of the lambda map of B. Let $a \in B_p$ and $b \in B_q$ and suppose that $(a,b) \neq (0,0)$. Since B is simple, we have that the ideal generated by $a+b$ is B. In particular, if $a \neq 0$, then the ideal generated by a is B. Let I_0 be a minimal nonzero ideal of B_p. Clearly I_0 is not an ideal of B. This means that there exist $d \in B_q$ and $c \in I_0$ such that $c * d = (\lambda_c - \mathrm{id})(d) = (\beta_c - id)(d) \neq 0$. Let J_0 be the ideal of B_q generated by $\{(\beta_c - id)(d) \mid c \in I_0,\ d \in B_q\}$. Let I_1 be the ideal of B_p generated by

$$I_0 \cup \{(\alpha_d - id)(c) \mid d \in J_0,\ c \in B_p\}.$$

Let J_1 be the ideal of B_q generated by

$$J_0 \cup \{(\beta_c - id)(d) \mid c \in I_1,\ d \in B_q\}.$$

For $k > 1$ let I_k be the ideal of B_p generated by

$$I_{k-1} \cup \{(\alpha_d - id)(c) \mid d \in J_{k-1},\ c \in B_p\},$$

and let J_K be the ideal of B_q generated by

$$J_{k-1} \cup \{(\beta_c - id)(d) \mid c \in I_k,\ d \in B_q\}.$$

Then there exists a positive integer k such that $I_k = B_p$ and $J_k = B_q$.

We shall study the main constructions of the finite simple braces of abelian type of order $p^n q^m$ for distinct primes p, q and positive integers n, m.

Matched product of braces

Let B be a left brace. Suppose that $(B,+)$ is the direct product of two left ideals B_1 and B_2 of B. Note that for $a,x \in B_1$ and $b,y \in B_2$, we have

$$\begin{aligned}\lambda_{a+b}(x+y) &= \lambda_{a+b}(x) + \lambda_{a+b}(y)\\ &= \lambda_a \lambda_{\lambda_a^{-1}(b)}(x) + \lambda_a \lambda_{\lambda_a^{-1}(b)}(y)\\ &= \lambda_b \lambda_{\lambda_b^{-1}(a)}(x) + \lambda_b \lambda_{\lambda_b^{-1}(a)}(y).\end{aligned}$$

Let $\alpha\colon (B_2,\circ) \to \mathrm{Aut}(B_1,+)$ and $\beta\colon (B_1,\circ) \to \mathrm{Aut}(B_2,+)$ be the restrictions of the lambda map, i.e.

$$\alpha_b(x) = \lambda_b(x) \text{ and } \beta_a(y) = \lambda_a(y),$$

where $\alpha_b = \alpha(b)$ and $\beta_a = \beta(a)$, for all $a,x \in B_1$ and $b,y \in B_2$. Then

$$\alpha_b \lambda^{(1)}_{\alpha_b^{-1}(a)} = \lambda^{(1)}_a \alpha_{\beta_a^{-1}(b)}$$

and

$$\beta_a \lambda^{(2)}_{\beta_a^{-1}(b)} = \lambda^{(2)}_b \beta_{\alpha_b^{-1}(a)},$$

for all $a \in B_1$ and $b \in B_2$, where $\lambda^{(1)}$ is the lambda map of B_1 and $\lambda^{(2)}$ is the lambda map of B_2.

def:matchedpair

Definition 17.0.4. Let B_1 and B_2 be two left braces. Let $\alpha\colon (B_2,\circ) \to \mathrm{Aut}(B_1,+)$ and $\beta\colon (B_1,\circ) \to \mathrm{Aut}(B_2,+)$ be group homomorphisms. We say that $(B_1, B_2, \alpha, \beta)$ is a *matched pair* of left braces if

$$\alpha_b \lambda^{(1)}_{\alpha_b^{-1}(a)} = \lambda^{(1)}_a \alpha_{\beta_a^{-1}(b)}$$

and

$$\beta_a \lambda^{(2)}_{\beta_a^{-1}(b)} = \lambda^{(2)}_b \beta_{\alpha_b^{-1}(a)},$$

where $\alpha_b = \alpha(b)$ and $\beta_a = \beta(a)$, for all $a \in B_1$ and $b \in B_2$, and where $\lambda^{(1)}$ is the lambda map of B_1 and $\lambda^{(2)}$ is the lambda map of B_2.

lemma:lambda

Lemma 17.0.5. *Let $(B,+)$ be a group (not necessarily abelian). Let*

$$\lambda\colon B \to \mathrm{Aut}(B.+), \quad a \mapsto \lambda_a,$$

be a map such that $\lambda_a \circ \lambda_b = \lambda_{a+\lambda_a(b)}$, for all $a,b \in B$. Then A with the sum $+$ and the product defined by $a \circ b := a + \lambda_a(b)$, for all $a,b \in B$, is a left brace.

Proof. Note that, if 0 is the neutral element of $(B,+)$, then $\lambda_0 \circ \lambda_0 = \lambda_{\lambda_0(0)} = \lambda_0$. Hence $\lambda_0 = \mathrm{id}$. Since $a \circ 0 = a + \lambda_a(0) = a + 0 = a$ and $0 \circ a = 0 + \lambda_0(a) = a$, we have that 0 is the neutral element of $(B,\circ)$. Since $\lambda_a \circ \lambda_{\lambda_a^{-1}(-a)} = \lambda_{a-a} = \lambda_0 = \mathrm{id}$, we get that $\lambda_a^{-1} = \lambda_{\lambda_a^{-1}(-a)}$. Now $a \circ \lambda_a^{-1}(-a) = a - a = 0$ and $\lambda_a^{-1}(-a) \circ a = \lambda_a^{-1}(-a) + \lambda_{\lambda_a^{-1}(-a)}(a) = \lambda_a^{-1}(-a) + \lambda_a^{-1}(a) = \lambda_a^{-1}(-a+a) = \lambda_a^{-1}(0) = 0$. Hence $\lambda_a^{-1}(-a)$ is the simmetric element of a in $(B,\circ)$. We shall check the associativity property:

$$\begin{aligned} a \circ (b \circ c) &= a + \lambda_a(b + \lambda_b(c)) = a + \lambda_a(b) + \lambda_a(\lambda_b(c)) \\ &= a + \lambda_a(b) + \lambda_{a+\lambda_a(b)}(c) = (a \circ b) + \lambda_{a\circ b}(c) \\ &= (a \circ b) \circ c. \end{aligned}$$

Hence $(B,\circ)$ is a group. Note that

$$\begin{aligned} a \circ (b + c) &= a + \lambda_a(b + c) = a + \lambda_a(b) + \lambda_a(c) \\ &= a + \lambda_a(b) - a + a + \lambda_a(c) = a \circ b - a + a \circ c. \end{aligned}$$

Therefore $(B,+,\circ)$ is a left brace, and the result follows. □

thm:matchedproduct

Theorem 17.0.6. *Let $(B_1, B_2, \alpha, \beta)$ be a matched pair of left braces. Then $B_1 \times B_2$ is a left brace with sum given by*

$$(a,b) + (a',b') = (a + a'.b + b')$$

and lambda map given by

$$\lambda_{(a,b)}(a',b') := \left(\lambda_a^{(1)} \alpha_{\beta_a^{-1}(b)}(a'), \lambda_b^{(2)} \beta_{\alpha_b^{-1}(a)}(b')\right).$$

Moreover $B_1 \times \{0\}$ and $\{0\} \times B_2$ are left ideals isomorphic to B_1 and B_2 as left braces respectively. This left brace is the matched product *of B_1 and B_2, and it is denoted by $B_1 \bowtie B_2$.*

Conversely, if B is a left brace such that $(B,+)$ is the direct product of two left ideals L_1 and L_2 of B, then $(L_1, L_2, \alpha, \beta)$ is a matched pair of left braces, where α and β are the restrictions of the lambda map of B, and B is isomorphic to the corresponding matched product $L_1 \bowtie L_2$.

Proof. To prove the first part, by Lemma 17.0.5, it is enough to check that

$$\lambda_{(a,b)} \lambda_{(a',b')} = \lambda_{(a,b)+\lambda_{(a,b)}(a',b')},$$

for all $a, a' \in B_1$ and $b, b' \in B_2$. Let $x \in B_1$ and $y \in B_2$. We have that

$$\begin{aligned} \lambda_{(a,b)} \lambda_{(a',b')}(x,y) &= \lambda_{(a,b)} \left(\lambda_{a'}^{(1)} \alpha_{\beta_{a'}^{-1}(b')}(x), \lambda_{b'}^{(2)} \beta_{\alpha_{b'}^{-1}(a')}(y)\right) \\ &= \left(\lambda_a^{(1)} \alpha_{\beta_a^{-1}(b)} \lambda_{a'}^{(1)} \alpha_{\beta_{a'}^{-1}(b')}(x), \lambda_b^{(2)} \beta_{\alpha_b^{-1}(a)} \lambda_{b'}^{(2)} \beta_{\alpha_{b'}^{-1}(a')}(y)\right) \end{aligned}$$

and

$$\lambda_{(a,b)+\lambda_{(a,b)}(a',b')}(x,y) = \lambda_{(a+\lambda_a^{(1)}\alpha_{\beta_a^{-1}(b)}(a'),b+\lambda_b^{(2)}\beta_{\alpha_b^{-1}(a)}(b'))}(x,y)$$
$$= \left(\lambda^{(1)}_{a+\lambda_a^{(1)}\alpha_{\beta_a^{-1}(b)}(a')}{}^{\alpha_{\beta^{-1}_{a+\lambda_a^{(1)}\alpha_{\beta_a^{-1}(b)}(a')}(b+\lambda_b^{(2)}\beta_{\alpha_b^{-1}(a)}(b'))}(x),\right.$$
$$\left.\lambda^{(2)}_{b+\lambda_b^{(2)}\beta_{\alpha_b^{-1}(a)}(b')}{}^{\beta_{\alpha^{-1}_{b+\lambda_b^{(2)}\beta_{\alpha_b^{-1}(a)}(b')}(a+\lambda_a^{(1)}\alpha_{\beta_a^{-1}(b)}(a'))}(y)\right).$$

Thus, it is enough to check the following equalities:

$$\lambda_a^{(1)}\alpha_{\beta_a^{-1}(b)}\lambda_{a'}^{(1)}\alpha_{\beta_{a'}^{-1}(b')} = \lambda^{(1)}_{a+\lambda_a^{(1)}\alpha_{\beta_a^{-1}(b)}(a')}{}^{\alpha_{\beta^{-1}_{a+\lambda_a^{(1)}\alpha_{\beta_a^{-1}(b)}(a')}(b+\lambda_b^{(2)}\beta_{\alpha_b^{-1}(a)}(b'))} \tag{17.1}$$

eq:matched1

and

$$\lambda_b^{(2)}\alpha_{\alpha_b^{-1}(a)}\lambda_{b'}^{(2)}\beta_{\alpha_{b'}^{-1}(a')} = \lambda^{(2)}_{b+\lambda_b^{(2)}\beta_{\alpha_b^{-1}(a)}(b')}{}^{\beta_{\alpha^{-1}_{b+\lambda_b^{(2)}\beta_{\alpha_b^{-1}(a)}(b')}(a+\lambda_a^{(1)}\alpha_{\beta_a^{-1}(b)}(a'))}. \tag{17.2}$$

eq:matched2

We shall prove (17.1). The equality (17.2) is proved similarly. We have

$$\begin{aligned}
&\lambda^{(1)}_{a+\lambda_a^{(1)}\alpha_{\beta_a^{-1}(b)}(a')}{}^{\alpha_{\beta^{-1}_{a+\lambda_a^{(1)}\alpha_{\beta_a^{-1}(b)}(a')}(b+\lambda_b^{(2)}\beta_{\alpha_b^{-1}(a)}(b'))}\\
&= \lambda^{(1)}_{a\circ\alpha_{\beta_a^{-1}(b)}(a')}{}^{\alpha_{\beta^{-1}_{a\circ\alpha_{\beta_a^{-1}(b)}(a')}(b+\lambda_b^{(2)}\beta_{\alpha_b^{-1}(a)}(b'))}\\
&= \lambda^{(1)}_{a\circ\alpha_{\beta_a^{-1}(b)}(a')}{}^{\alpha_{\beta^{-1}_{a\circ\alpha_{\beta_a^{-1}(b)}(a')}(b)+\beta^{-1}_{a\circ\alpha_{\beta_a^{-1}(b)}(a')}(\lambda_b^{(2)}\beta_{\alpha_b^{-1}(a)}(b'))}\\
&= \lambda^{(1)}_{a\circ\alpha_{\beta_a^{-1}(b)}(a')}{}^{\alpha_{\beta^{-1}_{a\circ\alpha_{\beta_a^{-1}(b)}(a')}(b)+\lambda^{(2)}_{\beta^{-1}_{a\circ\alpha_{\beta_a^{-1}(b)}(a')}}{}^{\beta^{-1}_{\alpha_b^{-1}(a\circ\alpha_{\beta_a^{-1}(b)}(a'))}\beta_{\alpha_b^{-1}(a)}(b')}\\
&= \lambda^{(1)}_{a\circ\alpha_{\beta_a^{-1}(b)}(a')}{}^{\alpha_{\beta^{-1}_{a\circ\alpha_{\beta_a^{-1}(b)}(a')}(b)\circ\beta^{-1}_{\alpha_b^{-1}(a\circ\alpha_{\beta_a^{-1}(b)}(a'))}\beta_{\alpha_b^{-1}(a)}(b')}\\
&= \lambda^{(1)}_{a\circ\alpha_{\beta_a^{-1}(b)}(a')}{}^{\alpha_{\beta^{-1}_{a\circ\alpha_{\beta_a^{-1}(b)}(a')}(b)\circ\beta^{-1}_{\alpha_b^{-1}(a+\lambda_a^{(1)}\alpha_{\beta_a^{-1}(b)}(a'))}\beta_{\alpha_b^{-1}(a)}(b')}\\
&= \lambda^{(1)}_{a\circ\alpha_{\beta_a^{-1}(b)}(a')}{}^{\alpha_{\beta^{-1}_{a\circ\alpha_{\beta_a^{-1}(b)}(a')}(b)\circ\beta^{-1}_{\alpha_b^{-1}(a+\alpha_b\lambda^{(1)}_{\alpha_b^{-1}(a)}(a'))}\beta_{\alpha_b^{-1}(a)}(b')}\\
&= \lambda^{(1)}_{a\circ\alpha_{\beta_a^{-1}(b)}(a')}{}^{\alpha_{\beta^{-1}_{a\circ\alpha_{\beta_a^{-1}(b)}(a')}(b)\circ\beta^{-1}_{\alpha_b^{-1}(a)+\lambda^{(1)}_{\alpha_b^{-1}(a)}(a')}\beta_{\alpha_b^{-1}(a)}(b')}\\
&= \lambda^{(1)}_{a\circ\alpha_{\beta_a^{-1}(b)}(a')}{}^{\alpha_{\beta^{-1}_{a\circ\alpha_{\beta_a^{-1}(b)}(a')}(b)\circ\beta^{-1}_{\alpha_b^{-1}(a)\circ a'}\beta_{\alpha_b^{-1}(a)}(b')}\\
&= \lambda^{(1)}_{a\circ\alpha_{\beta_a^{-1}(b)}(a')}{}^{\alpha_{\beta^{-1}_{a\circ\alpha_{\beta_a^{-1}(b)}(a')}(b)\circ\beta^{-1}_{a'}(b')}
\end{aligned}$$

$$= \lambda_a^{(1)} \lambda^{(1)}_{\alpha_{\beta_a^{-1}(b)}(a')} \alpha_{\beta^{-1}_{\alpha_{\beta_a^{-1}(b)}(a')}\beta_a^{-1}(b)} \alpha_{\beta_{a'}^{-1}(b')}$$
$$= \lambda_a^{(1)} \alpha_{\beta_a^{-1}(b)} \lambda_{a'}^{(1)} \alpha_{\beta_{a'}^{-1}(b')}.$$

This finishes the proof of the first part.

The second part follows easily by the argument above Definition 17.0.4. □

The matched product of two braces can be generalized to the mastched product of more than two braces following the case of braces of abelian type (see [11] for the matched product of several braces of abelian type).

Let B be a finite left brace of nilpoent type. Let $B_1, \ldots, B_n$ be the Sylow subgroups of $(B,+)$. Then $(B.+)$ is the direct product of these Sylow subgroups and, moreover, $B_1, \ldots, B_n$ are left ideals of B. Thus B is the matched product of $B_1, \ldots, B_n$. In particular, every simple non-trivial left braces of abelian type is the matched product of two or more left braces of abelian type of order a power of a prime. But, in fact all the known simple non-trivial left braces of abelian type are constructed using another kind of product, the so called asymmetric product.

Asymmetric product of braces of abelian type

Let S and T be two (additive) abelian groups. Recall that a (normalized) *symmetric 2-cocyle* on T with values in S is a map $b\colon T \times T \longrightarrow S$ such that

cond0 (i) $b(0,0) = 0$;
cond1 (ii) $b(t_1,t_2) = b(t_2,t_1)$;
cond2 (iii) $b(t_1+t_2,t_3) + b(t_1,t_2) = b(t_1,t_2+t_3) + b(t_2,t_3)$, for all $t_1,t_2,t_3 \in T$.

As a consequence, we get that $b(t,0) = b(0,t) = 0$, for all $t \in T$.

ccs **Theorem 17.0.7.** *Let T and S be two left braces of abelian type. Let $b\colon T \times T \longrightarrow S$ be a symmetric 2-cocycle on $(T,+)$ with values in $(S,+)$, and let $\alpha\colon (S,\circ) \longrightarrow \mathrm{Aut}(T,+,\circ)$ be a homomorphism of groups such that*

$$s \cdot b(t_2,t_3) + b(t_1 \circ \alpha_s(t_2+t_3), t_1) = b(t_1 \circ \alpha_s(t_2), t_1 \circ \alpha_s(t_3)) + s, \quad (17.3)$$

where $\alpha_s = \alpha(s)$, for all $s \in S$ and $t_1,t_2,t_3 \in T$. Then the addition and multiplication over $T \times S$ given by

$$(t_1,s_1) + (t_2,s_2) = (t_1+t_2,\ s_1+s_2+b(t_1,t_2)),$$
$$(t_1,s_1) \circ (t_2,s_2) = (t_1 \circ \alpha_{s_1}(t_2),\ s_1 \circ s_2),$$

define a structure of left brace on $T \times S$. We call this left brace the asymmetric product *of T by S (via b and α) and denote it by $T \rtimes_\circ S$.*

Proof. It is easy to check that $(T \times S,+)$ is an abelian group and that $(T \times S,\circ)$ is a group. In fact, $(T \times S,+)$ is the extension of the abelian groups S and T via the

2-cocycle b, and $(T\times S,\circ)$ is the semidirect product $T\rtimes_\alpha S$ of T by S via α. Thus it is enough to check that

$$(t_1,s_1)\circ((t_2,s_2)+(t_3,s_3))=(t_1,s_1)\circ(t_2,s_2)-(t_1,s_1)+(t_1,s_1)\circ(t_3,s_3).$$

We have that

$$\begin{aligned}
&(t_1,s_1)\circ(t_2,s_2)-(t_1,s_1)+(t_1,s_1)\circ(t_3,s_3)\\
&=(t_1\circ\alpha_{s_1}(t_2),s_1\circ s_2)-(t_1,s_1)+(t_1\circ\alpha_{s_1}(t_3),s_1\circ s_3)\\
&=(t_1\circ\alpha_{s_1}(t_2)-t_1,s_1\circ s_2-s_1-b(t_1\circ\alpha_{s_1}(t_2)-t_1,t_1))+(t_1\circ\alpha_{s_1}(t_3),s_1\circ s_3)\\
&=(t_1\circ\alpha_{s_1}(t_2)-t_1+t_1\circ\alpha_{s_1}(t_3),s_1\circ s_2-s_1-b(t_1\circ\alpha_{s_1}(t_2)-t_1,t_1)+s_1\circ s_3\\
&\qquad+b(t_1\circ\alpha_{s_1}(t_2)-t_1,t_1\circ\alpha_{s_1}(t_3)))\\
&=(t_1\circ(\alpha_{s_1}(t_2)+\alpha_{s_1}(t_3)),s_1\circ(s_2+s_3)-b(t_1\circ\alpha_{s_1}(t_2)-t_1,t_1)\\
&\qquad+b(t_1\circ\alpha_{s_1}(t_2)-t_1,t_1\circ\alpha_{s_1}(t_3)))\\
&=(t_1\circ(\alpha_{s_1}(t_2+t_3)),s_1\circ(s_2+s_3)+b(t_1\circ\alpha_{s_1}(t_2),t_1\circ\alpha_{s_1}(t_3))\\
&\qquad-b(t_1\circ\alpha_{s_1}(t_2)-t_1+t_1\circ\alpha_{s_1}(t_3),t_1))\\
&=(t_1\circ(\alpha_{s_1}(t_2+t_3)),s_1\circ(s_2+s_3)+b(t_1\circ\alpha_{s_1}(t_2),t_1\circ\alpha_{s_1}(t_3))\\
&\qquad-b(t_1\circ(\alpha_{s_1}(t_2)+\alpha_{s_1}(t_3)),t_1))\\
&=(t_1\circ(\alpha_{s_1}(t_2+t_3)),s_1\circ(s_2+s_3)+b(t_1\circ\alpha_{s_1}(t_2),t_1\circ\alpha_{s_1}(t_3))\\
&\qquad-b(t_1\circ\alpha_{s_1}(t_2+t_3)),t_1))\\
&=(t_1\circ(\alpha_{s_1}(t_2+t_3)),s_1\circ(s_2+s_3)-s_1+s_1\circ b(t_2,t_3))\\
&=(t_1\circ(\alpha_{s_1}(t_2+t_3)),s_1\circ(s_2+s_3+b(t_2,t_3))\\
&=(t_1,s_1)\circ(t_2+t_3,s_2+s_3+b(t_2,t_3))\\
&=(t_1,s_1)\circ((t_2,s_2)+(t_3,s_3)).
\end{aligned}$$

Therefore the result follows. □

Note that the lambda map of $T\rtimes_\circ S$ is defined by

$$\lambda_{(t_1,s_1)}(t_2,s_2)=\left(\lambda_{t_1}\alpha_{s_1}(t_2),\ \lambda_{s_1}(s_2)-b(\lambda_{t_1}\alpha_{s_1}(t_2),t_1)\right),\tag{17.4}$$

and its socle is

$$\mathrm{Soc}(T\rtimes_\circ S)=\{(t,s)\mid \lambda_s=\mathrm{id}_S,\ \lambda_t\circ\alpha_s=\mathrm{id}_T,\ b(t,t')=0\text{ for all }t'\in T\}.$$

Moreover, the subset $T\times\{0\}$ is a normal subgroup of $(T\rtimes_\circ S,\circ)$, and $\{0\}\times S$ is a left ideal of $T\rtimes_\circ S$.

A particular case of this theorem is when we assume that b is a symmetric bilinear form. In this case, conditions (17)-(17) are automatic, and condition (17.3) becomes

$$\lambda_s(b(t_2,\ t_3))=b(\lambda_{t_1}\alpha_s(t_2),\ \lambda_{t_1}\alpha_s(t_3)),$$

which is equivalent to the two conditions:

$$\lambda_s(b(t_2,\ t_3)) = b(\alpha_s(t_2),\ \alpha_s(t_3)), \tag{17.5}$$ condAlpha

$$b(t_2,\ t_3) = b(\lambda_{t_1}(t_2),\ \lambda_{t_1}(t_3)), \tag{17.6}$$ condLambda

for all $s \in S$ and $t_1, t_2, t_3 \in T$. Furthermore, if T and S are trivial left braces of abelian type, then these two conditions (17.5) and (17.6) reduce to

$$b(t_2,\ t_3) = b(\alpha_s(t_2),\ \alpha_s(t_3)),$$

for all $s \in S$ and $t_1, t_2, t_3 \in T$, i.e. α_s is in the orthogonal group of the bilinear form b, for all $s \in S$.

Construction of simple braces of abelian type

Let p, q be two distinct primes. Consider the trivial left braces over the abelian groups $T = (\mathbb{Z}/(p))^{n_1} \times (\mathbb{Z}/(q))^{n_2}$ and $S = \mathbb{Z}/(p) \times \mathbb{Z}/(q)$. Let $b_1 \colon (\mathbb{Z}/(p))^{n_1} \times (\mathbb{Z}/(p))^{n_1} \to \mathbb{Z}/(p)$ and $b_2 \colon (\mathbb{Z}/(q))^{n_2} \times (\mathbb{Z}/(q))^{n_2} \to \mathbb{Z}/(q)$ be two non-degenerate symmetric bilinear forms. Let c_1 be an element of order q in the orthogonal group of b_1 and let c_2 be an element of order p in the orthogonal group of b_2. Let $\alpha \colon S \to \mathrm{Aut}(T)$ be the map defined by $\alpha(i,j) = \alpha_{(i,j)}$ and $\alpha_{(i,j)}(u,v) = (c_1^j(u), c_2^i(v))$, for all $(i,j) \in S$ and $(u,v) \in T$. Let $b \colon T \times T \to S$ be the map defined by $b((u_1,v_1),(u_2,v_2)) = (b_1(u_1,u_2), b_2(v_1,v_2))$, for all $(u_1,v_1),(u_2,v_2) \in T$. Then

$$\begin{aligned} b(\alpha_{(i,j)}(u_1,v_1), \alpha_{(i,j)}(u_2,v_2)) &= (b_1(c_1^j(u_1), c_1^j(u_2)), b_2(c_2^i(v_1), c_2^i(v_2))) \\ &= (b_1(u_1,u_2), b_2(v_1,v_2)). \end{aligned}$$

Hence we can consider the asymmetric product $T \rtimes_\circ S$ of T by S (via b and α).

thm:simplebrace

Theorem 17.0.8. *With the above notation, the left brace $T \rtimes_\circ S$ is simple if and only if $c_1 - \mathrm{id}$ and $c_2 - \mathrm{id}$ are invertible.*

Proof. Let $J = \{((u,v),(i,j)) \mid u \in \mathrm{im}(c_1 - \mathrm{id}),\ v \in \mathrm{im}(c_2 - \mathrm{id}),\ (i,j) \in S\}$. We shall prove that J is an ideal of $T \rtimes_\circ S$. Clearly J is a subgroup of $(T \rtimes_\circ S, +)$. Let $((u,v),(i,j)) \in J$ and $((u',v'),(k,l)) \in T \rtimes_\circ S$. We have that

$$\begin{aligned} &\lambda_{((u',v'),(k,l))}((u,v),(i,j)) \\ &= -((u',v'),(k,l)) + ((u',v'),(k,l)) \circ ((u,v),(i,j)) \\ &= ((c_1^l(u), c_2^k(v)), (i - b_1(c_1^l(u),u'), j - b_2(c_2^k(v),v'))) \in J. \end{aligned}$$

Hence J is a left ideal of $T \rtimes_\circ S$. Furthermore

$$\begin{aligned} &((u,v),(i,j)) * ((u',v'),(k,l)) \\ &= -((u,v),(i,j)) + ((u,v),(i,j)) \circ ((u',v'),(k,l)) - ((u',v'),(k,l)) \\ &= ((c_1^j(u'), c_2^i(v')), (k - b_1(c_1^j(u'),u), l - b_2(c_2^{ik}(v'),v))) - ((u',v'),(k,l)) \\ &= ((c_1^j(u') - u', c_2^i(v') - v'), (-b_1(c_1^j(u'),u) - b_1(c_1^j(u') - u', u'), -b_2(c_2^{ik}(v'),v) \end{aligned}$$

$$-b_2(c_2^i(v')-v',v'))) \in J.$$

Thus J is an ideal of $T \rtimes_\circ S$. Therefore if $T \rtimes_\circ S$ is simple, then $c_1 - \mathrm{id}$ and $c_2 - \mathrm{id}$ are invertible.

Conversely, suppose that $c_1 - \mathrm{id}$ and $c_2 - \mathrm{id}$ are invertible. Let I be a nonzero ideal of $T \rtimes_\circ S$. Let $((u,v),(i,j)) \in I$ be a nonzero element. Suppose first that $(u,v) = (0,0)$. In this case, $(i,j) \neq (0,0)$. We may assume that $i \neq 0$. Note that $q((0,0),(i,j)) = ((0,0),(qi,0)) \in I$. Hence we may assume that $j = 0$. Let $i' \in \mathbb{Z}/(p)$ the inverse of i. Then $i'((0,0),(i,0)) = ((0,0),(1,0)) \in I$. Now,

$$\begin{aligned}&((0,0),(1,0)) * ((u',v'),(k,l))\\ &= ((0,(c_2-\mathrm{id})(v')),(b_1(u',u'),-b_2((c_2-\mathrm{id})(v'),v'))) \in I,\end{aligned}$$

for all $((u',v'),(k,l)) \in T \rtimes_\circ S$. Thus for every $w \in (\mathbb{Z}/(q))^{n_2}$ there exists $l_w \in \mathbb{Z}/(q)$ such that $((0,w),(0,l_w)) \in I$. Let $l \in \mathbb{Z}/(q)$. Note that if $w \neq 0$, since b_2 is non-degenerate, there exists $w' \in (\mathbb{Z}/(q))^{n_2}$ such that $b_2(w,w') = l_w - l$. Hence $\lambda_{((0,w'),(0,0))}((0,w),(0,l)) = ((0,w),(0,l-b_2(w,w'))) = ((0,w),(0,l)) \in I$, for all $w \in (\mathbb{Z}/(q))^{n_2} \setminus \{0\}$ and $l \in \mathbb{Z}/(q)$. Thus, if $w \neq 0$, then $((0,w),(0,l)) - ((0,w),(0,0)) = ((0,0),(0,l)) \in I$, for all $l \in \mathbb{Z}/(q)$. In particular, $((0,0),(0,1)) \in I$. Now by a similar argument one can see that $((u',0),(k,0)) \in I$, for all $u' \in (\mathbb{Z}/(p))^{n_1}$ and $k \in \mathbb{Z}/(p)$. Now $((u',0),(k,0)) + ((0,w),(0,l)) = ((u',w),(k,l)) \in I$, for all $u' \in (\mathbb{Z}/(p))^{n_1}$, $w \in (\mathbb{Z}/(q))^{n_2}$, $k \in \mathbb{Z}/(p)$ and $l \in \mathbb{Z}/(q)$. Thus $I = T \rtimes_\circ S$ in this case.

Suppose now that $(u,v) \neq (0,0)$. In this case we may assume that $u \neq 0$. Then $q((u,v),(i,j)) = ((qu,0),(i',j')) \in I$, for some $(i',j') \in S$. Furthermore

$$q((qu,0),(i',j')) = ((q^2u,0),(i'',0)) \in I,$$

for some $i'' \in \mathbb{Z}/(p)$. Since $q^2u \neq 0$ and b_1 is non-degenerate, there exists $u' \in (\mathbb{Z}/(p))^{n_1}$ such that $b_1(q^2u,u') = i'' - k$. Hence $\lambda_{((u',0),(0,0))}((q^2u,0),(i'',0)) = ((q^2u,0),(i''-b_1(q^2u,u'))) = ((q^2u,0),(k,0)) \in I$, for all $k \in \mathbb{Z}/(p)$, and then $((q^2u,0),(1,0)) - ((q^1u,0),(0,0)) = ((0,0),(1,0)) \in I$. Now, as above, one can prove that $I = T \rtimes_\circ S$, and the result follows. □

We shall give now some concrete constructions of simple left braces of abelian type using Theorem 17.0.8. Let p,q be two distinct primes. Let n,m be two positive integers. Let

$$D_1 = \begin{pmatrix} 0 & 0 & \dots & 0 & -1 \\ 1 & 0 & \dots & 0 & -1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & -1 \\ 0 & \dots & 0 & 1 & -1 \end{pmatrix},\ E_1 = \begin{pmatrix} 1-q & 1 & \dots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & 1-q \end{pmatrix} \in M_{q-1}(\mathbb{Z}/(p))$$

and

$$D_2 = \begin{pmatrix} 0 & 0 & \dots & 0 & -1 \\ 1 & 0 & \dots & 0 & -1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & -1 \\ 0 & \dots & 0 & 1 & -1 \end{pmatrix},\ E_2 = \begin{pmatrix} 1-p & 1 & \dots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & 1-p \end{pmatrix} \in M_{p-1}(\mathbb{Z}/(q)).$$

Note that D_1 is the companion matrix of the polynomial $1+x+\cdots+x^{q-1}$ and thus it has order q. Similarly D_2 has order p. It is easy to check that $D_1^t E_1 D_1 = E_1$ and $D_2^t E_2 D_2 = E_2$. Consider in $M_{n_1(q-1)}(\mathbb{Z}/(p))$ the block diagonal matrices with n_1 blocks of degree $q-1$:

$$C_1 = \begin{pmatrix} D_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & D_1 \end{pmatrix} \text{ and } B_1 = \begin{pmatrix} E_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & E_1 \end{pmatrix}.$$

Similarly consider in $M_{n_2(p-1)}(\mathbb{Z}/(q))$ the block diagonal matrices with n_2 blocks of degree $p-1$:

$$C_2 = \begin{pmatrix} D_2 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & D_2 \end{pmatrix} \text{ and } B_2 = \begin{pmatrix} E_2 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & E_2 \end{pmatrix}.$$

Let $T = (\mathbb{Z}/(p))^{n_1(q-1)} \times (\mathbb{Z}/(q))^{n_2(p-1)}$ and $S = \mathbb{Z}/(p) \times \mathbb{Z}/(q)$ be the trivial left braces over these abelian groups. Let $b\colon T \times T \to S$ be the symmetric linear map defined by

$$b((u_1, v_1), (u_2, v_2)) = (u_1 B_1 u_2^t, v_1 B_2 v_2^t),$$

for all $u_1, u_2 \in (\mathbb{Z}/(p))^{n_1(q-1)}$ and $v_1, v_2 \in (\mathbb{Z}/(q))^{n_2(p-1)}$. Note that $\det(B_1) = (-q)^{q-2} \neq 0$ and $\det(B_2) = (-p)^{p-2} \neq 0$, thus the symmetric biliear forms associated to these matrices asre non-degenerate. Let $\alpha\colon S \to \mathrm{Aut}(T)$ be the map defined by $\alpha(i,j) = \alpha_{(i,j)}$ and $\alpha_{(i,j)}(u,v) = (u(C_1^t)^j, v(C_2^t)^i)$, for all $(i,j) \in S$ and $(u,v) \in T$. Since $C_k^t B_k C_k = B_k$, for $k = 1, 2$, we can consider the asymmetric product $T \rtimes_\circ S$ of T by S (via b and α). Since $C_k - I$, where I is the identity matrix, is invertible, by Theorem 17.0.8, $T \rtimes_\circ S$ is a simple left brace of abelian type.

Another construction is based in the following fact. Let D an invertible matrix in $M_n(K)$, where K is a field. Consider the block matrices

$$C = \begin{pmatrix} D^t & 0 \\ 0 & D^{-1} \end{pmatrix} \in M_{2n}(K) \quad \text{and} \quad B = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \in M_{2n}(K).$$

Then one can check that $C^t B C = B$. Thus we can take any invertible matrix $D_1 \in M_{m_1}(\mathbb{Z}/(p))$ of order q and any matrix $D_2 \in M_{m_2}(\mathbb{Z}/(q))$ of order p, such that

$D_k - I$ also is invertible for $k = 1, 2$. Consider in $M_{2n_1m_1}(\mathbb{Z}/(p))$ the block square matrices with $2n_1$ blocks of degree m_1 in the diagonal:

$$C_1 = \begin{pmatrix} D_1^t & 0 & 0 & \dots & 0 & 0 \\ 0 & D_1^{-1} & 0 & \dots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \dots & 0 & D_1^t & 0 \\ 0 & 0 & \dots & 0 & 0 & D_1^{-1} \end{pmatrix} \text{ and } B_1 = \begin{pmatrix} 0 & I_{m_1} & 0 & \dots & 0 & 0 \\ I_{m_1} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & I_{m_1} \\ 0 & 0 & \dots & 0 & I_{m_1} & 0 \end{pmatrix}.$$

Similarly consider in $M_{2n_2m_2}(\mathbb{Z}/(q))$ the block square matrices with $2n_2$ blocks of degree m_2 in the diagonal:

$$C_2 = \begin{pmatrix} D_2^t & 0 & 0 & \dots & 0 & 0 \\ 0 & D_2^{-1} & 0 & \dots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \dots & 0 & D_2^t & 0 \\ 0 & 0 & \dots & 0 & 0 & D_2^{-1} \end{pmatrix} \text{ and } B_1 = \begin{pmatrix} 0 & I_{m_2} & 0 & \dots & 0 & 0 \\ I_{m_2} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & I_{m_2} \\ 0 & 0 & \dots & 0 & I_{m_2} & 0 \end{pmatrix}.$$

Let $T = (\mathbb{Z}/(p))^{2n_1m_1} \times (\mathbb{Z}/(q))^{2n_2m_2}$ and $S = \mathbb{Z}/(p) \times \mathbb{Z}/(q)$ be the trivial left braces over these abelian groups. Let $b\colon T \times T \to S$ be the symmetric linear map defined by

$$b((u_1, v_1), (u_2, v_2)) = (u_1 B_1 u_2^t, v_1 B_2 v_2^t),$$

for all $u_1, u_2 \in (\mathbb{Z}/(p))^{2n_1m_1}$ and $v_1, v_2 \in (\mathbb{Z}/(q))^{2n_2m_2}$. Let $\alpha\colon S \to \mathrm{Aut}(T)$ be the map defined by $\alpha(i, j) = \alpha_{(i,j)}$ and $\alpha_{(i,j)}(u, v) = (u(C_1^t)^j, v(C_2^t)^i)$, for all $(i, j) \in S$ and $(u, v) \in T$. Since $C_k^t B_k C_k = B_k$, for $k = 1, 2$, we can consider the asymmetric product $T \rtimes_\circ S$ of T by S (via b and α). Since $C_k - I$, where I is the identity matrix, is invertible, by Theorem 17.0.8, $T \rtimes_\circ S$ is a simple left brace of abelian type.

For example, for $p = 2$ and $q = 31$, with the first construction we obtain simple left braces of abelian type of order $2^{30n_1}31^{n_2}$ for all positive integers n_1, n_2. With the second construction, taking

$$D_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \in M_5(\mathbb{Z}/(2)),$$

we obtain simple left braces of abelian type of order $2^{10n_1}31^{2n_2}$ for all positive integers n_1, n_2. Using the two methods one can easily construct simple left braces of abelian type of order $2^{10n_1}31^{n_2}$ for all positive integers n_1, n_2.

Exercises

17.0.1.

17.0.2.

17.0.3.

17.0.4.

Notes

The first example of a simple non-trivial brace of abelian type is due to Bachiller [9]. In the same papar Bachiller introduced the matched product of two left braces of abelian type. One can see that the definition of matched product of two left braces is the same that in the case of braces of abelian type. In [11], Bachiller, Cedó, Jespers and Okniński introduced the matched product of more than two left braces of abelian type (this can be generaliced to arbitrari left braces without changes). In the same paper the authors construct several families of simple non-trivial left braces of abelian type using matched products.

The asymmetric product of left braces of abelian type was intruduced by Catino, Colazzo and Stefanelli in [21], and Theorem 17.0.7 is due to them ([21, Theorem 3]). In [12] Bachiller, Cedó, Jespers and Okniński used the asymmetric product to construct to costruct new families of simple non-trivial left braces of abelian type. In fact, every known simple non-trivial left brace is an asymmetric product (see [12, 28, 29]). Theorem 17.0.8 is based on [11, Theorem 3.6] and [12, Theorem 6.2]. The two concrete constructions of simple non-trivial simple braces of abelian type appear in [11, 28].

References

1. E. Acri, R. Lutowski, and L. Vendramin. Retractability of solutions to the Yang-Baxter equation and p-nilpotency of skew braces. *Internat. J. Algebra Comput.*, 30(1):91–115, 2020.
2. O. Akgün, M. Mereb, and L. Vendramin. Enumeration of set-theoretic solutions to the Yang-Baxter equation. *Math. Comp.*, 91(335):1469–1481, 2022.
3. B. Amberg, S. Franciosi, and F. de Giovanni. *Products of groups*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1992. Oxford Science Publications.
4. S. A. Amitsur. Nil radicals. Historical notes and some new results. In *Rings, modules and radicals (Proc. Internat. Colloq., Keszthely, 1971)*, pages 47–65. Colloq. Math. Soc. János Bolyai, Vol. 6, 1973.
5. N. Andruskiewitsch and M. Graña. From racks to pointed Hopf algebras. *Adv. Math.*, 178(2):177–243, 2003.
6. M. Ashford and O. Riordan. Counting racks of order n. *Electron. J. Combin.*, 24(2):Paper No. 2.32, 20, 2017.
7. D. Bachiller. Classification of braces of order p^3. *J. Pure Appl. Algebra*, 219(8):3568–3603, 2015.
8. D. Bachiller. Counterexample to a conjecture about braces. *J. Algebra*, 453:160–176, 2016.
9. D. Bachiller. Extensions, matched products, and simple braces. *J. Pure Appl. Algebra*, 222(7):1670–1691, 2018.
10. D. Bachiller. Solutions of the Yang-Baxter equation associated to skew left braces, with applications to racks. *J. Knot Theory Ramifications*, 27(8):1850055, 36, 2018.
11. D. Bachiller, F. Cedó, E. Jespers, and J. Okniński. Iterated matched products of finite braces and simplicity; new solutions of the Yang-Baxter equation. *Trans. Amer. Math. Soc.*, 370(7):4881–4907, 2018.
12. D. Bachiller, F. Cedó, E. Jespers, and J. Okniński. Asymmetric product of left braces and simplicity; new solutions of the Yang-Baxter equation. *Commun. Contemp. Math.*, 21(8):1850042, 30 pp., 2019.
13. V. G. Bardakov, M. V. Neshchadim, and M. K. Yadav. Computing skew left braces of small orders. *Internat. J. Algebra Comput.*, 30(4):839–851, 2020.
14. G. M. Bergman. Errata: "A ring primitive on the right but not on the left". *Proc. Amer. Math. Soc.*, 15:1000, 1964.
15. G. M. Bergman. A ring primitive on the right but not on the left. *Proc. Amer. Math. Soc.*, 15:473–475, 1964.
16. S. R. Blackburn. Enumerating finite racks, quandles and kei. *Electron. J. Combin.*, 20(3):Paper 43, 9, 2013.
17. M. Brešar. *Introduction to noncommutative algebra*. Universitext. Springer, Cham, 2014.
18. E. Brieskorn. Automorphic sets and braids and singularities. In *Braids (Santa Cruz, CA, 1986)*, volume 78 of *Contemp. Math.*, pages 45–115. Amer. Math. Soc., Providence, RI, 1988.

19. N. P. Byott. Nilpotent and abelian Hopf-Galois structures on field extensions. *J. Algebra*, 381:131–139, 2013.
20. N. P. Byott. Solubility criteria for Hopf-Galois structures. *New York J. Math.*, 21:883–903, 2015.
21. F. Catino, I. Colazzo, and P. Stefanelli. Regular subgroups of the affine group and asymmetric product of radical braces. *J. Algebra*, 455:164–182, 2016.
22. F. Catino, I. Colazzo, and P. Stefanelli. Skew left braces with non-trivial annihilator. *J. Algebra Appl.*, 18(2):1950033, 23, 2019.
23. F. Catino and R. Rizzo. Regular subgroups of the affine group and radical circle algebras. *Bull. Aust. Math. Soc.*, 79(1):103–107, 2009.
24. F. Cedó. Left braces: solutions of the Yang-Baxter equation. *Adv. Group Theory Appl.*, 5:33–90, 2018.
25. F. Cedó, T. Gateva-Ivanova, and A. Smoktunowicz. Braces and symmetric groups with special conditions. *J. Pure Appl. Algebra*, 222(12):3877–3890, 2018.
26. F. Cedó, E. Jespers, and J. Okniński. Retractability of set theoretic solutions of the Yang-Baxter equation. *Adv. Math.*, 224(6):2472–2484, 2010.
27. F. Cedó, E. Jespers, and J. Okniński. Braces and the Yang-Baxter equation. *Comm. Math. Phys.*, 327(1):101–116, 2014.
28. F. Cedó, E. Jespers, and J. Okniński. An abundance of simple left braces with abelian multiplicative Sylow subgroups. *Rev. Mat. Iberoam.*, 36(5):1309–1332, 2020.
29. F. Cedó, E. Jespers, and J. Okniński. Every finite abelian group is a subgroup of the additive group of a finite simple left brace. *J. Pure Appl. Algebra*, 225(1):106476, 10, 2021.
30. F. Cedó, A. Smoktunowicz, and L. Vendramin. Skew left braces of nilpotent type. *Proc. Lond. Math. Soc. (3)*, 118(6):1367–1392, 2019.
31. F. Chouraqui. Garside groups and Yang-Baxter equation. *Comm. Algebra*, 38(12):4441–4460, 2010.
32. A. Clay and D. Rolfsen. *Ordered groups and topology*, volume 176 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2016.
33. P. M. Cohn. *Algebra, 2nd edition*, volume 1. John Wiley & Sons, New York, 1982.
34. M. Deakonesku and G. L. Uolls. On the orbits of automorphism groups. *Sibirsk. Mat. Zh.*, 46(3):533–537, 2005.
35. P. Dehornoy. Set-theoretic solutions of the Yang-Baxter equation, RC-calculus, and Garside germs. *Adv. Math.*, 282:93–127, 2015.
36. V. G. Drinfel′d. On some unsolved problems in quantum group theory. In *Quantum groups (Leningrad, 1990)*, volume 1510 of *Lecture Notes in Math.*, pages 1–8. Springer, Berlin, 1992.
37. M. Elhamdadi and S. Nelson. *Quandles—an introduction to the algebra of knots*, volume 74 of *Student Mathematical Library*. American Mathematical Society, Providence, RI, 2015.
38. P. Etingof and S. Gelaki. A method of construction of finite-dimensional triangular semisimple Hopf algebras. *Math. Res. Lett.*, 5(4):551–561, 1998.
39. P. Etingof, T. Schedler, and A. Soloviev. Set-theoretical solutions to the quantum Yang-Baxter equation. *Duke Math. J.*, 100(2):169–209, 1999.
40. W. Feit and J. G. Thompson. Solvability of groups of odd order. *Pacific J. Math.*, 13:775–1029, 1963.
41. The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.11.1*, 2021.
42. T. Gateva-Ivanova. A combinatorial approach to the set-theoretic solutions of the Yang-Baxter equation. *J. Math. Phys.*, 45(10):3828–3858, 2004.
43. T. Gateva-Ivanova. Set-theoretic solutions of the Yang-Baxter equation, braces and symmetric groups. *Adv. Math.*, 338:649–701, 2018.
44. T. Gateva-Ivanova and M. Van den Bergh. Semigroups of I-type. *J. Algebra*, 206(1):97–112, 1998.
45. L. Guarnieri and L. Vendramin. Skew braces and the Yang-Baxter equation. *Math. Comp.*, 86(307):2519–2534, 2017.
46. J. Hoste and P. D. Shanahan. An enumeration process for racks. *Math. Comp.*, 88(317):1427–1448, 2019.

47. I. M. Isaacs. *Finite group theory*, volume 92 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.
48. I. M. Isaacs. Group actions and orbits. *Arch. Math. (Basel)*, 98(5):399–401, 2012.
49. N. Jacobson. The radical and semi-simplicity for arbitrary rings. *Amer. J. Math.*, 67:300–320, 1945.
50. E. Jespers, L. . Kubat, A. Van Antwerpen, and L. Vendramin. Factorizations of skew braces. *Math. Ann.*, 375(3-4):1649–1663, 2019.
51. E. Jespers and J. Okniński. *Noetherian semigroup algebras*, volume 7 of *Algebra and Applications*. Springer, Dordrecht, 2007.
52. A. Konovalov, A. Smoktunowicz, and L. Vendramin. On skew braces and their ideals. *Exp. Math.*, 30(1):95–104, 2021.
53. G. Köthe. Die Struktur der Ringe, deren Restklassenring nach dem Radikal vollständig reduzibel ist. *Math. Z.*, 32(1):161–186, 1930.
54. J. Krempa. Logical connections between some open problems concerning nil rings. *Fund. Math.*, 76(2):121–130, 1972.
55. I. Lau. An associative left brace is a ring. *J. Algebra Appl.*, 19(9):2050179, 6, 2020.
56. V. Lebed and L. Vendramin. On structure groups of set-theoretic solutions to the Yang-Baxter equation. *Proc. Edinb. Math. Soc. (2)*, 62(3):683–717, 2019.
57. J.-H. Lu, M. Yan, and Y.-C. Zhu. On the set-theoretical Yang-Baxter equation. *Duke Math. J.*, 104(1):1–18, 2000.
58. H. Meng, A. Ballester-Bolinches, and R. Esteban-Romero. Left braces and the quantum Yang-Baxter equation. *Proc. Edinb. Math. Soc. (2)*, 62(2):595–608, 2019.
59. P. P. Nielsen. Simplifying Smoktunowicz's extraordinary example. *Comm. Algebra*, 41(11):4339–4350, 2013.
60. J. Pakianathan and K. Shankar. Nilpotent numbers. *Amer. Math. Monthly*, 107(7):631–634, 2000.
61. R. Plemmons. Construction and analysis of non-equivalent finite semigroups. In *Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967)*, pages 223–228. Pergamon, Oxford, 1970.
62. S. D. Promislow. A simple example of a torsion-free, nonunique product group. *Bull. London Math. Soc.*, 20(4):302–304, 1988.
63. J. H. Przytycki. Distributivity versus associativity in the homology theory of algebraic structures. *Demonstratio Math.*, 44(4):823–869, 2011.
64. W. Rump. A decomposition theorem for square-free unitary solutions of the quantum Yang-Baxter equation. *Adv. Math.*, 193(1):40–55, 2005.
65. W. Rump. Braces, radical rings, and the quantum Yang-Baxter equation. *J. Algebra*, 307(1):153–170, 2007.
66. W. Rump. The brace of a classical group. *Note Mat.*, 34(1):115–144, 2014.
67. A. Smoktunowicz. Polynomial rings over nil rings need not be nil. *J. Algebra*, 233(2):427–436, 2000.
68. A. Smoktunowicz. On some results related to Köthe's conjecture. *Serdica Math. J.*, 27(2):159–170, 2001.
69. A. Smoktunowicz. Some results in noncommutative ring theory. In *International Congress of Mathematicians. Vol. II*, pages 259–269. Eur. Math. Soc., Zürich, 2006.
70. A. Smoktunowicz. A note on set-theoretic solutions of the Yang-Baxter equation. *J. Algebra*, 500:3–18, 2018.
71. A. Smoktunowicz. On Engel groups, nilpotent groups, rings, braces and the Yang-Baxter equation. *Trans. Amer. Math. Soc.*, 370(9):6535–6564, 2018.
72. A. Smoktunowicz and L. Vendramin. On skew braces (with an appendix by N. Byott and L. Vendramin). *J. Comb. Algebra*, 2(1):47–86, 2018.
73. A. Soloviev. Non-unitary set-theoretical solutions to the quantum Yang-Baxter equation. *Math. Res. Lett.*, 7(5-6):577–596, 2000.
74. Y. P. Sysak. Products of almost abelian groups. In *Investigations of groups with restrictions for subgroups (Russian)*, pages 81–85, iii. Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev, 1988.

75. C. Tsang and C. Qin. On the solvability of regular subgroups in the holomorph of a finite solvable group. *Internat. J. Algebra Comput.*, 30(2):253–265, 2020.
76. P. Vojtěchovský and S. Y. Yang. Enumeration of racks and quandles up to isomorphism. *Math. Comp.*, 88(319):2523–2540, 2019.
77. M. Wada. Group invariants of links. *Topology*, 31(2):399–406, 1992.
78. A. Weinstein and P. Xu. Classical solutions of the quantum Yang-Baxter equation. *Comm. Math. Phys.*, 148(2):309–343, 1992.
79. E. Zelmanov. *Nil rings and periodic groups*. KMS Lecture Notes in Mathematics. Korean Mathematical Society, Seoul, 1992. With a preface by Jongsik Kim.

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