

# Galois theory

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ABSTRACT. The notes correspond to the bachelor course **Galois Theory** of the Vrije Universiteit Brussel, Faculty of Sciences, Department of Mathematics and Data Sciences.

## CONTENTS

Introduction	2
1. Lecture – Week 1	3
2. Lecture – Week 2	10
3. Lecture – Week 3	16
4. Lecture – Week 4	19
5. Lecture – Week 5	23
6. Lecture – Week 6	27
7. Lecture – Week 7	31
8. Lecture – Week 8	35
9. Lecture – Week 9	41
10. Lecture – Week 10	46
11. Lecture – Week 11	50
12. Lecture – Week 12	60
Some solutions	65
References	77
Index	78



## Introduction

The notes correspond to the bachelor course **Galois theory** of the Vrije Universiteit Brussel, Faculty of Sciences, Department of Mathematics and Data Sciences. The course is divided into twelve two-hour lectures.

The material is somewhat standard. Basic texts on fields and Galois theory are for example [4] and [5].

As usual, we also mention a set of great expository papers by Keith Conrad, the notes are extremely well-written and useful at every stage of a mathematical career.

The notes include Magma code, which we use to verify examples and offer alternative solutions to certain exercises. Magma [1] is a powerful software tool designed for working with algebraic structures. There is a free online version of Magma available.

Several of the exercise solutions were written by Silvia Properzi.

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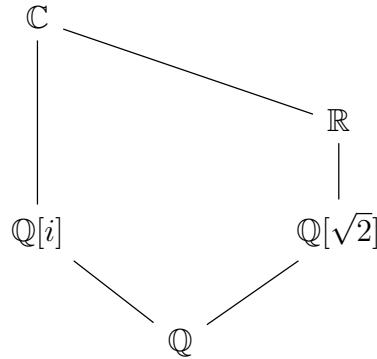
## 1. Lecture – Week 1

**§ 1.1. Fields.** Recall that a **field** is a commutative ring such that  $1 \neq 0$  and every non-zero element is invertible. Examples of (infinite) fields are  $\mathbb{Q}$ ,  $\mathbb{R}$ , and  $\mathbb{C}$ . If  $p$  is a prime number, then  $\mathbb{Z}/p$  is a field.

1.1. EXAMPLE. The abelian group  $\mathbb{Z}/2 \times \mathbb{Z}/2$  is a field with multiplication

$$(a, b)(c, d) = (ac + bd, ad + bc + bd).$$

1.2. EXAMPLE.  $\mathbb{Q}[i] = \{a + bi : a, b \in \mathbb{Q}\}$  and  $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$  are fields.



1.3. EXERCISE. Prove that  $\mathbb{Q}[i]$  and  $\mathbb{Q}[\sqrt{2}]$  are not isomorphic as fields.

If  $R$  is a ring, there exists a unique ring homomorphism  $\mathbb{Z} \rightarrow R$ ,  $m \mapsto m1$ . The image

$$R_{\mathbb{Z}} = \{m1 : m \in \mathbb{Z}\}$$

of this homomorphism is a subring of  $R$  and it is known as the **prime subring** of  $R$ . The kernel is a subgroup of  $\mathbb{Z}$  generated by some  $t \geq 0$ . The integer  $t$  is the **characteristic** of the ring  $R$ .

1.4. EXERCISE. The characteristic of a field is either zero or a prime number.

1.5. EXAMPLE. The characteristic of the field of Example 1.1 is two. Why?

Recall that a commutative ring  $R$  is an **integral domain** if  $xy = 0 \implies x = 0$  or  $y = 0$ . Fields are integral domains.

1.6. EXERCISE. Let  $K$  be a field. Prove that the following statements are equivalent:

- 1)  $K$  is of characteristic zero.
- 2) The additive order of 1 is infinite.
- 3) The additive order of each  $x \neq 0$  is infinite.
- 4) The prime subring of  $K$  is isomorphic to  $\mathbb{Z}$ .

1.7. EXERCISE. Let  $K$  be a field. Prove that the following statements are equivalent:

- 1)  $K$  is of characteristic  $p$ .
- 2) The additive order of 1 is  $p$ .
- 3) The additive order of each  $x \neq 0$  is  $p$ .
- 4) The prime subring of  $K$  is isomorphic to  $\mathbb{Z}/p$ .

Note that finite fields are of prime characteristic.

1.8. PROPOSITION. *Let  $K$  be a field. Then any finite subgroup of  $K^\times = K \setminus \{0\}$  is cyclic.*

PROOF. Let  $G$  be a finite subgroup of  $K^\times$ . Let  $x \in G$  of maximal order  $N$ . We will show that any element of  $G$  is a power of  $x$ . Let  $y \in G$  and  $n = |y|$ . Note that  $X^n - 1 \in K[X]$  has at most  $n$  roots in  $K$ .

We claim that  $n$  divides  $N$ . If not, there exists a prime number  $p$  and two powers  $q = p^\beta$  and  $p^\alpha$  of  $p$  with  $\alpha < \beta$  such that  $q \mid n$  and  $p^\alpha \mid N$ , but  $p^{\alpha+1} \nmid N$ . Let  $z = xy^{n/q}$ . Since  $G$  is abelian,

$$|z| = \text{lcm}\{N, q\} = Np^{\beta-\alpha} > N,$$

a contradiction.

The polynomial  $X^n - 1 \in K[X]$  has  $n$  distinct roots in  $K$ , these are the elements  $x^{kN/n}$  for  $k \in \{0, \dots, n-1\}$ . Since  $y$  has order  $n$ , it has to be one of these roots. Thus  $y = x^{kN/n}$  for some  $k \in \{0, \dots, n-1\}$ .  $\square$

1.9. DEFINITION. A **subfield** of a ring  $R$  is a subring of  $R$  that is also a field.

Note that if  $K$  is a subfield of  $E$ , then the characteristic of  $K$  coincides with the characteristic of  $E$ . Moreover, if  $K \rightarrow L$  is a field homomorphism, then  $K$  and  $L$  have the same characteristic.

1.10. EXERCISE. Let  $K$  be a field of characteristic  $p$ . Prove that  $K \rightarrow K$ ,  $x \mapsto x^{p^n}$ , is a field homomorphism for all  $n \in \mathbb{Z}_{\geq 0}$ .

1.11. DEFINITION. A field  $K$  is **prime** if there are no proper subfields of  $K$ .

Examples of prime fields are  $\mathbb{Q}$  and  $\mathbb{Z}/p$  for a prime number  $p$ .

1.12. PROPOSITION. *Let  $K$  be a field. The following statements hold:*

- 1)  $K$  has a unique subfield which is prime, it is known as the **prime subfield** of  $K$ .
- 2) The prime subfield of  $K$  is either isomorphic to  $\mathbb{Q}$  if the characteristic of  $K$  is zero, or it is isomorphic to  $\mathbb{Z}/p$  for some prime number  $p$  if the characteristic of  $K$  is  $p$ .

PROOF. To prove the first claim let  $L$  be the intersection of all the subfields of  $K$ . Then  $L$  is a subfield of  $K$ . If  $F$  is a subfield of  $L$ , then  $F$  is a subfield of  $K$ . Thus  $L \subseteq F$  and hence  $F = L$ , which proves that  $L$  is prime. If  $L_1$  is a subfield of  $K$  and  $L_1$  is prime, then  $L \subseteq L_1$  and hence  $L = L_1$ .

Let  $K_0$  be the prime field of  $K$ . Suppose that  $K$  is of characteristic  $p > 0$ . Then the ring  $K_{\mathbb{Z}}$  of integers of  $K$  is a field isomorphic to  $\mathbb{Z}/p$  and hence  $K_0 = K_{\mathbb{Z}} \simeq \mathbb{Z}/p$ . Suppose now that the characteristic of  $K$  is zero. Let  $E = \{m1(n1)^{-1} : m, n \in \mathbb{Z}, n \neq 0\}$ . We claim that  $K_0 = E$ . Since  $K_{\mathbb{Z}} \subseteq K_0$ , it follows that  $E \subseteq K_0$ . Hence  $E = K_0$ , as  $E$  is a subfield of  $K$ .  $\square$

1.13. DEFINITION. Let  $E$  be a field and  $K$  be a subfield of  $E$ . Then  $E$  is a **field extension** of  $K$ . We will use the notation  $E/K$ .

Let  $E/K$  be a field extension. Then  $E$  is a  $K$ -vector space with the usual scalar multiplication  $K \times E \rightarrow E$ ,  $(\lambda, x) \mapsto \lambda x$ .

1.14. DEFINITION. The **degree** of an extension  $E$  of  $K$  is the integer  $\dim_K E$ . It will be denoted by  $[E : K]$ .

We say that  $E$  is a **finite extension** of  $K$  if  $[E : K]$  is finite.

1.15. EXAMPLE. Let  $K$  be a field. Then  $[K : K] = 1$ . Conversely, if  $E$  is an extension of  $K$  and  $[E : K] = 1$ , then  $K = E$ . If not, let  $x \in E \setminus K$ . We claim that  $\{1, x\}$  is linearly independent over  $K$ . Indeed, if  $a1 + bx = 0$  for some  $a, b \in K$ , then  $bx = -a$ . If  $b \neq 0$ , then  $x = -a/b \in K$ , a contradiction. If  $b = 0$ , then  $a = 0$ .

We know that  $[\mathbb{C} : \mathbb{R}] = 2$ .

1.16. EXAMPLE. A basis of  $\mathbb{Q}[\sqrt{2}]$  over  $\mathbb{Q}$  is given by  $\{1, \sqrt{2}\}$ . Then  $[\mathbb{Q}[\sqrt{2}] : \mathbb{Q}] = 2$ . The calculations can be easily done by computer:

```
> Q<z> := QuadraticField(2);
> Characteristic(Q);
0
> K := PrimeField(Q);
Rational Field
> Degree(Q);
2
> Basis(Q);
[ 1, z ]
> z^2
2
> One(K) = One(Q);
1 = 1
> G := GaloisGroup(Q);
> G;
Symmetric group G acting on a set of cardinality 2
Order = 2
```

1.17. EXAMPLE. Since  $\mathbb{Q}$  is countable and  $\mathbb{R}$  is not,  $[\mathbb{R} : \mathbb{Q}] > \aleph_0$ . If  $\{x_i : i \in \mathbb{Z}_{>0}\}$  is a countable basis of  $\mathbb{R}$  over  $\mathbb{Q}$ , for each  $n$  consider the  $\mathbb{Q}$ -vector space  $V_n$  generated by  $\{x_1, \dots, x_n\}$ . Then

$$\mathbb{R} = \bigcup_{n \geq 1} V_n,$$

is countable, as each  $V_n$  is countable, a contradiction.

If  $E$  is an extension of  $K$  and  $E$  is finite, then  $[E : K]$  is finite.

1.18. PROPOSITION. Let  $K$  be a finite field. Then  $|K| = p^m$  for some prime number  $p$  and some  $m \geq 1$ .

PROOF. We know the prime subfield  $K_0$  of  $K$  is isomorphic to  $\mathbb{Z}/p$ . In particular,  $|K_0| = p$ . Since  $K$  is finite,  $[K : K_0] = m$  for some  $m$ . If  $\{x_1, \dots, x_m\}$  is a basis of  $K$  over  $K_0$ , then each element of  $K$  can be written uniquely as  $\sum_{i=1}^m a_i x_i$  for some  $a_1, \dots, a_m \in K_0$ . Then there is a bijection between  $K$  and  $K_0^m$  and hence  $|K| = |K_0^m| = p^m$ .  $\square$

We now perform some basic calculations with a finite field of eight elements:

```
> E<x> := FiniteField(8);
> PrimeField(E);
Finite field of size 2
> Degree(E);
3
> Characteristic(E);
2
> #E;
8
> { x : x in E };
{ 1, x, x^2, x^3, x^4, x^5, x^6, 0 }
```

Here is an alternative construction if a field of eight elements as a quotient of a polynomial ring:

```
> R<x> := PolynomialRing(PrimeField(E));
> IsIrreducible(x^3+x+1);
true
> I := ideal< R | x^3+x+1 >;
> IsMaximal(I);
true
> Q<y> := quo< R | I >;
> IsField(Q);
true
> { y : y in Q };
{
    y^2 + 1,
    0,
    y^2,
    y + 1,
    1,
    y,
    y^2 + y + 1,
    y^2 + y
}
> #Q;
8
```

1.19. DEFINITION. Let  $E$  be an extension of  $K$ . A **subextension**  $F/K$  of  $E/K$  is a subfield  $F$  of  $E$  that contains  $K$ , that is  $K \subseteq F \subseteq E$ .

1.20. DEFINITION. Let  $E$  and  $E_1$  be extensions over  $K$ . An **extension homomorphism**

$$E/K \rightarrow E_1/K$$

is a field homomorphism  $\sigma: E \rightarrow E_1$  such that  $\sigma(x) = x$  for all  $x \in K$ .

To describe the homomorphism  $\sigma: E/K \rightarrow E_1/K$  of the extensions over  $K$  one typically writes the commutative diagram

$$\begin{array}{ccc} K & \xlongequal{\quad} & K \\ \downarrow & & \downarrow \\ E & \xrightarrow{\sigma} & E_1 \end{array}$$

We write  $\text{Hom}(E/K, E_1/K)$  to denote the set of homomorphism  $E/K \rightarrow E_1/K$  of extensions of  $K$ . Note that if  $\sigma \in \text{Hom}(E/K, E_1/K)$ , then  $\sigma$  is a  $K$ -linear map, as

$$\sigma(\lambda x) = \sigma(\lambda)\sigma(x) = \lambda\sigma(x)$$

for all  $\lambda \in K$  and  $x \in E$ .

1.21. EXAMPLE. The conjugation map  $\mathbb{C} \rightarrow \mathbb{C}$ ,  $z \mapsto \bar{z}$ , is an endomorphism of  $\mathbb{C}$  as an extension over  $\mathbb{R}$ . Let  $\varphi \in \text{Hom}(\mathbb{C}/\mathbb{R}, \mathbb{C}/\mathbb{R})$ . Then

$$\varphi(x + iy) = \varphi(x) + \varphi(i)\varphi(y) = x + \varphi(i)y$$

for all  $x, y \in \mathbb{R}$ . Since  $\varphi(i)^2 = \varphi(i^2) = \varphi(-1) = -1$ , it follows that  $\varphi(i) \in \{-i, i\}$ . Thus either  $\varphi(x + iy) = x + iy$  or  $\varphi(x + iy) = x - iy$ .

1.22. EXERCISE. Let  $K$  be a field,  $K_0$  be its prime field and  $\sigma: K \rightarrow K$  be a field homomorphism. Prove that  $\sigma \in \text{Hom}(K/K_0, K/K_0)$ .

If  $E/K$  is an extension, then

$$\begin{aligned} \text{Aut}(E/K) &= \{\sigma: E/K \rightarrow E/K \text{ is a bijective extension homomorphism}\} \\ &= \{\sigma: E \rightarrow E : \sigma \text{ is a bijective field homomorphism with } \sigma|_K = \text{id}\} \end{aligned}$$

is a group with composition.

1.23. DEFINITION. Let  $E/K$  be an extension. The **Galois group** of  $E/K$  is the group  $\text{Aut}(E/K)$  and it will be denoted by  $\text{Gal}(E/K)$ .

A typical example:  $\text{Gal}(\mathbb{C}/\mathbb{R}) \simeq \mathbb{Z}/2$ .

As an example, we show with the computer that  $\text{Gal}(\mathbb{Q}[\sqrt{2}]/\mathbb{Q}) \simeq \mathbb{Z}/2$ :

```
> Q<z> := QuadraticField(2);
> G := GaloisGroup(Q);
> G;
Symmetric group G acting on a set of cardinality 2
Order = 2
> GroupName(G);
C2
```

1.24. EXAMPLE. Let  $\theta = \sqrt[3]{2}$  and let  $E = \{a + b\theta + c\theta^2 : a, b, c \in \mathbb{Q}\}$ . Note that

$$a + b\theta + c\theta^2 = 0 \iff a = b = c = 0.$$

Then  $E$  is an extension of  $\mathbb{Q}$  such that  $[E : \mathbb{Q}] = 3$ . We claim that  $\text{Gal}(E/\mathbb{Q})$  is trivial. If  $\sigma \in \text{Gal}(E/\mathbb{Q})$  and  $z = a + b\theta + c\theta^2$ , then  $\sigma(z) = a + b\sigma(\theta) + c\sigma^2(\theta)$ . Since

$$\sigma(\theta)^3 = \sigma(\theta^3) = \sigma(2) = 2,$$

it follows that  $\sigma(\theta) = \theta$  and therefore  $\sigma = \text{id}$ .

1.25. EXERCISE. Prove that the polynomial  $X^3 - 2$  is irreducible in  $\mathbb{Q}[X]$ .

The previous exercise can easily be solved using Magma:

```
> Q<x> := PolynomialRing(Rationals());
> f := x^3-2;
> IsIrreducible(f);
true
```

The following exercise is known as the **Eisenstein's irreducibility criterion**:

1.26. EXERCISE. Let  $f = \sum_{i=0}^n a_i X^i \in \mathbb{Z}[X]$  be a polynomial of degree  $n > 0$ . Assume that there exists a prime number such that  $p \mid a_i$  for all  $i \in \{0, 1, \dots, n-1\}$ ,  $p \nmid a_n$  and  $p^2 \nmid a_0$ . Then  $f$  is irreducible in  $\mathbb{Q}[X]$ .

1.27. BONUS EXERCISE. Let  $A$  be a unique factorization domain and  $K$  be its fraction field. Let  $f = \sum_{i=0}^n a_i X^i \in A[X]$  be a polynomial of degree  $n > 0$ . Assume that there exists a prime element  $p \in A$  such that  $p \mid a_i$  for all  $i \in \{0, 1, \dots, n-1\}$ ,  $p \nmid a_n$  and  $p^2 \nmid a_0$ . Then  $f$  is irreducible in  $K[X]$ .

1.28. EXERCISE. Prove that the polynomials

$$\begin{aligned} f &= X^{10} + 60X^7 + 82X^6 - 36X^3 + 2, \\ g &= 3X^{10} + 15X^2 - 45, \end{aligned}$$

are irreducible in  $\mathbb{Q}[X]$ .

The previous exercise is easy for Magma:

```
> R<x> := PolynomialRing(Rationals());
> g := 3*x^10+15*x^2-45;
> IsIrreducible(g);
true
> f := x^10+60*x^7+82*x^6-36*x^3+2;
> IsIrreducible(f);
true
```

1.29. EXERCISE. Is the polynomial  $f = 3(X^{10} + 5X^2 - 15)$  irreducible in  $\mathbb{Z}[X]$ ?

Let us see what Magma can do here:

```
> R<x> := PolynomialRing(Integers());
> f := 3*x^10+15*x^2-45;
> IsIrreducible(f);
false
> Factorization(f);
[
<3, 1>,
```

```
<x^10 + 5*x^2 - 15, 1>
]
```

If  $E/K$  is an extension and  $S$  is a subset of  $E$ , then there exists a unique smallest subextension  $F/K$  of  $E/K$  such that  $S \subseteq F$ . In fact,

$$F = \bigcap \{T : T \text{ is a subfield of } E \text{ that contains } K \cup S\}$$

If  $L/K$  is a subextension of  $E/K$  such that  $S \subseteq L$ , then  $F \subseteq L$  by definition. The extension  $F$  is known as the **subextension generated by  $S$**  and it will be denoted by  $K(S)$ . If  $S = \{x_1, \dots, x_n\}$  is finite, then  $K(S) = K(x_1, \dots, x_n)$  is said to be of **finite type**.

1.30. EXAMPLE. If  $\{e_1, \dots, e_n\}$  is a basis of  $E$  over  $K$ , then  $E = K(e_1, \dots, e_n)$ .

1.31. EXAMPLE. The field  $\mathbb{Q}(\sqrt{2})$  is precisely the extension of  $\mathbb{R}/\mathbb{Q}$  generated by  $\sqrt{2}$ .

Let  $E/K$  be an extension and  $S$  and  $T$  be subsets of  $E$ . Then

$$K(S \cup T) = K(S)(T) = K(T)(S).$$

If, moreover,  $S \subseteq T$ , then  $K(S) \subseteq K(T)$ .

### § 1.2. Algebraic extensions.

1.32. DEFINITION. Let  $E/K$  be an extension. An element  $x \in E$  is **algebraic** over  $K$  if there exists a non-zero polynomial  $f(X) \in K[X]$  such that  $f(x) = 0$ . If  $x$  is not algebraic over  $K$ , then it is called **transcendental** over  $K$ .

1.33. DEFINITION. An extension  $E/K$  is **algebraic** if every  $x \in E$  is algebraic over  $K$ .

If  $K$  is a field, every  $x \in K$  is algebraic over  $K$ , as  $x$  is a root of  $X - x \in K[X]$ . In particular,  $K/K$  is an algebraic extension.

1.34. EXAMPLE.  $\mathbb{C}/\mathbb{R}$  is an algebraic extension. If  $z \in \mathbb{C} \setminus \mathbb{R}$ , then  $z$  is a root of the polynomial  $X^2 - (z + \bar{z})X + |z|^2 \in \mathbb{R}[X]$ .

If  $F/K$  is an extension  $x \in E$  is algebraic over  $K$  for some field  $E \supseteq F$ , then  $x$  is algebraic over  $F$ .

1.35. EXAMPLE.  $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$  is algebraic, as the number  $a + b\sqrt{2}$  is a root of the polynomial  $X^2 - 2aX + (a^2 - 2b^2) \in \mathbb{Q}[X]$ .

The extension  $\mathbb{C}/\mathbb{Q}$  is not algebraic. For example, Hermite proved that  $e$  is transcendental over  $\mathbb{Q}$ ; see [5, Theorem 24.4]. Lindemann's theorem states that  $\pi$  is not algebraic over  $\mathbb{Q}$ ; see [5, Theorem 24.5].

1.36. EXAMPLE. Let  $a = \sqrt{2}$  and  $b = \sqrt[3]{3}$ . Both  $a$  and  $b$  are algebraic numbers over  $\mathbb{Q}$ . Let us show that  $a + b$  is also algebraic. Let  $f(X) = X^3 - 3 \in \mathbb{Q}[X]$ . Then  $f(b) = 0$ . Note that the polynomial

$$g(X) = f(X - a) = X^3 - 3aX^2 + 3aX - a^3 - 3 \in \mathbb{Q}(a)[X]$$

is such that  $g(a + b) = 0$ . How can we find a polynomial with coefficients in  $\mathbb{Q}$  that vanishes on  $a + b$ ? We do the “conjugation” trick:

$$h(X) = f(X - a)f(X + a) = X^6 - 6X^4 - 6X^3 + 12X^2 - 36X + 1 \in \mathbb{Q}[X].$$

Note that  $h(a + b) = 0$ . How can you prove that  $ab$  is also algebraic over  $\mathbb{Q}$ ?

## 2. Lecture – Week 2

If  $E/K$  is an extension and  $x \in E$  is algebraic over  $K$ , then the evaluation homomorphism  $K[X] \rightarrow E$ ,  $p \mapsto p(x)$ , is not injective. In particular, its kernel is a non-zero ideal. Hence it is generated by a monic polynomial  $f$ .

**2.1. DEFINITION.** Let  $E/K$  be an extension and  $x \in E$  be an algebraic element. The monic polynomial that generates the kernel of  $K[X] \rightarrow E$ ,  $f \mapsto f(x)$ , is known as the **minimal polynomial** of  $x$  over  $K$  and it will be denoted by  $f(x, K)$ . The **degree** of  $x$  over  $K$  is then  $\deg f(x, K)$ .

Some basic properties of the minimal polynomial of an algebraic element:

**2.2. PROPOSITION.** Let  $E/K$  be an extension and  $x \in E$ . Assume that  $x$  is algebraic over  $K$ .

- 1) If  $g \in K[X]$  is non-zero and such that  $g(x) = 0$ , then  $f(x, K)$  divides  $g$  and  $\deg f(x, K) \leq \deg g$ .
- 2)  $f(x, K)$  is irreducible in  $K[X]$ .
- 3) If  $F/K$  is a subextension of  $E/K$ , then  $f(x, F)$  divides  $f(x, K)$ .

**PROOF.** Write  $f = f(x, K)$  to denote the minimal polynomial of  $x$ . To prove 1) note that  $g(x) = 0$  implies that  $g$  belongs to the kernel of the evaluation map, so  $g$  is a multiple of  $f$ . To prove 2) note that if  $f = pq$  for some  $p, q \in K[X]$  such that  $0 < \deg p, \deg q < \deg f$ , then  $f(x) = 0$  implies that either  $p(x) = 0$  or  $q(x) = 0$ , a contradiction. Finally, we prove 3). Since  $f \in K[X] \subseteq F[X]$  and  $f(x) = 0$ , it follows from 1) that  $f(x, F)$  divides  $f$ .  $\square$

Some easy examples:  $f(i, \mathbb{Q}) = X^2 + 1$ ,  $f(i, \mathbb{C}) = X - i$  and  $f(\sqrt[3]{2}, \mathbb{Q}) = X^3 - 2$ . Here is the code:

```
> R<x> := PolynomialRing(Rationals());
> Q<i> := QuadraticField(-1);
> MinimalPolynomial(i);
x^2 + 1
```

A bit harder:  $f(\sqrt[3]{2} + 1, \mathbb{Q}) = X^3 - 3X^2 + 3X - 3$ . Here is the code:

```
> R<x> := PolynomialRing(Rationals());
> E<z> := NumberField(x^3-2);
> MinimalPolynomial(z+1);
x^3 - 3*x^2 + 3*x - 3
```

**2.3. EXAMPLE.** Let us compute  $f(\sqrt{2} + \sqrt{3}, \mathbb{Q})$ . Let  $\alpha = \sqrt{2} + \sqrt{3}$ . Then

$$\begin{aligned} \alpha - \sqrt{2} = \sqrt{3} &\implies (\alpha - \sqrt{2})^2 = 3 \implies \alpha^2 - 2\sqrt{2}\alpha + 2 = 3 \\ &\implies \alpha^2 - 1 = 2\sqrt{2}\alpha \implies (\alpha^2 - 1)^2 = 8\alpha^2 \implies \alpha^4 - 10\alpha^2 + 1 = 0. \end{aligned}$$

Thus  $\alpha$  is a root of  $g = X^4 - 10X^2 + 1$ . To prove that  $g = f(\alpha, \mathbb{Q})$  it is enough to prove that  $g$  is irreducible in  $\mathbb{Q}[X]$ . First note that the roots of  $g$  are  $\sqrt{2} + \sqrt{3}$ ,  $\sqrt{2} - \sqrt{3}$ ,  $-\sqrt{2} + \sqrt{3}$  and  $-\sqrt{2} - \sqrt{3}$ . This means that if  $g$  is not irreducible, then  $g = hh_1$  for some polynomials

$h, h_1 \in \mathbb{Q}[X]$  such that  $\deg h = \deg h_1 = 2$ . This is not possible, as

$$\begin{aligned} (\sqrt{2} + \sqrt{3}) + (\sqrt{2} - \sqrt{3}) &= 2\sqrt{2} \notin \mathbb{Q}, \\ (\sqrt{2} + \sqrt{3}) + (-\sqrt{2} + \sqrt{3}) &= 2\sqrt{3} \notin \mathbb{Q}, \\ (\sqrt{2} + \sqrt{3})(-\sqrt{2} - \sqrt{3}) &= -5 - 2\sqrt{6} \notin \mathbb{Q}. \end{aligned}$$

2.4. PROPOSITION. *Let  $F/K$  be a subextension of  $E/K$ . Then*

$$[E : K] = [E : F][F : K].$$

PROOF. Let  $\{e_i : i \in I\}$  be a basis of  $E$  over  $F$  and  $\{f_j : j \in J\}$  be a basis of  $F$  over  $K$ . If  $x \in E$ , then  $x = \sum_i \lambda_i e_i$  (finite sum) for some  $\lambda_i \in F$ . For each  $i$ ,  $\lambda_i = \sum_j a_{ij} f_j$  (finite sum) for some  $a_{ij} \in K$ . Then  $x = \sum_i \sum_j a_{ij} (f_j e_i)$ . This means that  $\{f_j e_i : i \in I, j \in J\}$  generates  $E$  as a  $K$ -vector space. Let us prove that  $\{f_j e_i : i \in I, j \in J\}$  is linearly independent. If  $\sum_i \sum_j a_{ij} (f_j e_i) = 0$  (finite sum) for some  $a_{ij} \in K$ , then

$$\begin{aligned} 0 = \sum_i \left( \sum_j a_{ij} f_j \right) e_i &\implies \sum_j a_{ij} f_j = 0 \text{ for all } i \in I \\ &\implies a_{ij} = 0 \text{ for all } i \in I \text{ and } j \in J. \end{aligned}$$

□

We state a lemma:

2.5. LEMMA. *If  $A$  is a finite-dimensional commutative algebra over  $K$  and  $A$  is an integral domain, then  $A$  is a field.*

PROOF. Let  $a \in A \setminus \{0\}$ . We need to prove that there exists  $b \in A$  such that  $ab = 1$ . Let  $\theta: A \rightarrow A$ ,  $x \mapsto ax$ . Note that  $\theta$  is  $K$ -linear transformation, as

$$\theta(x+y) = a(x+y) = ax+ay = \theta(x) + \theta(y), \quad \theta(\lambda x) = a(\lambda x) = \lambda(ax) = \lambda\theta(x),$$

for all  $x, y \in A$  and  $\lambda \in K$ . It is injective since  $A$  is an integral domain. Since  $\dim_K A < \infty$ , it follows that  $\theta$  is an isomorphism. In particular,  $\theta(A) = A$ , which implies that there exists  $b \in A$  such that  $1 = ab$ . □

Let  $E/K$  be an extension and  $x \in E$ . Then

$$K[x] = \{f(x) : f \in K[X]\}$$

is a subring of  $E$  that contains  $K$ . Note that  $K[x]$  is a  $K$ -vector space.

More generally, if  $x_1, \dots, x_n \in E$ , then

$$K[x_1, \dots, x_n] = \{f(x_1, \dots, x_n) : f \in K[X_1, \dots, X_n]\}$$

is a subring of  $E$ . Note that  $K[x_1, \dots, x_n]$  is a  $K$ -vector space. Clearly,  $K[x_1, \dots, x_n]$  is a domain and

$$K(x_1, \dots, x_n) = \left\{ \frac{f(x_1, \dots, x_n)}{g(x_1, \dots, x_n)} : f, g \in K[X_1, \dots, X_n] \text{ with } g(x_1, \dots, x_n) \neq 0 \right\}$$

is the extension of  $K$  generated by  $x_1, \dots, x_n$ . Note that

$$K(x_1, \dots, x_n) = (K(x_1, \dots, x_{n-1}))(x_n).$$

The previous construction can be generalized. Let  $I$  be a non-empty set. For each  $i \in I$ , let  $X_i$  be a variable. Consider the polynomial ring  $K[\{X_i : i \in I\}]$  and let  $S = \{x_i : i \in I\}$  be a subset of  $E$ . There exists a unique algebra homomorphism

$$K[\{X_i : i \in I\}] \rightarrow E$$

such that  $X_i \mapsto x_i$  for all  $i \in I$ . The image is denoted by  $K[S]$ . In particular, an element  $z \in K[S]$  is of the form

$$z = h(x_1, \dots, x_n)$$

for a polynomial  $h \in K[X_1, \dots, X_n]$  in finitely many variables  $X_1, \dots, X_n$  and  $x_1, \dots, x_n \in S$ .

**2.6. EXERCISE.** Prove that  $\mathbb{Q}[\sqrt{2}] = \mathbb{Q}(\sqrt{2})$ .

The exercise is not an accident.

**2.7. THEOREM.** *Let  $E/K$  be an extension and  $x \in E \setminus K$ . The following statements are equivalent:*

- 1)  $x$  is algebraic over  $K$ .
- 2)  $\dim_K K[x] < \infty$ .
- 3)  $K[x]$  is a field.
- 4)  $K[x] = K(x)$ .

**PROOF.** We first prove 1)  $\implies$  2). Let  $z \in K[x]$ , say  $z = h(x)$  for some  $h \in K[X]$ . There exists  $g \in K[X]$  such that  $g \neq 0$  and  $g(x) = 0$ . Divide  $h$  by  $g$  to obtain polynomials  $q, r \in K[X]$  such that  $h = gq + r$ , where  $r = 0$  or  $\deg r < \deg g$ . This implies that

$$z = h(x) = g(x)q(x) + r(x) = r(x).$$

If  $\deg g = m$ , then  $r = \sum_{i=0}^{m-1} a_i X^i$  for some  $a_0, \dots, a_{m-1} \in K$ . Thus

$$z = \sum_{i=0}^{m-1} a_i x^i$$

and hence  $K[x] \subseteq \langle 1, x, \dots, x^{m-1} \rangle$ .

The previous lemma proves that 2)  $\implies$  3).

It is trivial that 3)  $\implies$  4).

It remains to prove that 4)  $\implies$  1). Since  $x \neq 0$ ,  $1/x \in K(x) = K[x]$ . There exists  $a_0, \dots, a_n \in K$  such that  $1/x = a_0 + a_1 x + \dots + a_n x^n$ . Thus

$$a_n x^{n+1} + \dots + a_1 x^2 + a_0 x - 1 = 0,$$

and hence  $x$  is a root of  $a_n X^{n+1} + \dots + a_0 X - 1 \in K[X] \setminus \{0\}$ . □

Note that if  $x$  is algebraic over  $K$ , then  $K[x] \simeq K[X]/(f(x, K))$ .

**2.8. EXERCISE.** Let  $E/K$  be an extension and  $x \in E$  be an algebraic element over  $K$ . Prove that the degree of  $x$  over  $K$  is equal to  $[K(x) : K]$ .

**2.9. COROLLARY.** *If  $E/K$  is finite, then  $E/K$  is algebraic.*

PROOF. Let  $n = [E : K]$  and  $x \in E \setminus K$ . The set  $\{1, x, \dots, x^n\}$  has  $n + 1$  elements, so it is linearly dependent. There exist  $a_0, \dots, a_n \in K$ , not all zero, such that

$$a_0 + a_1x + \cdots + a_nx^n = 0.$$

Thus  $x$  is a root of the non-zero polynomial  $a_0 + a_1X + \cdots + a_nX^n \in K[X]$ .  $\square$

In Example 1.36 we proved that  $\sqrt{2} + \sqrt[3]{3}$  and  $\sqrt{2}\sqrt[3]{3}$  are algebraic over  $\mathbb{Q}$ . This can be easily proved now with Corollary 2.9.

2.10. EXERCISE. Let  $E/K$  be an extension and  $a$  and  $b$  be algebraic over  $K$ . Prove that  $a + b$  and  $ab$  are algebraic over  $K$ .

We note that the converse of Corollary 2.9 result does not hold.

2.11. COROLLARY. If  $E/K$  is an extension and  $x_1, \dots, x_n \in E$  are algebraic over  $K$ , then  $K(x_1, \dots, x_n)/K$  is finite and  $K(x_1, \dots, x_n) = K[x_1, \dots, x_n]$ .

PROOF. We proceed by induction on  $n$ . The case  $n = 1$  follows immediately from the theorem. So assume the result holds for some  $n \geq 1$ . Since the extensions

$$K(x_1, \dots, x_n)/K(x_1, \dots, x_{n-1}) \quad \text{and} \quad K(x_1, \dots, x_{n-1})/K$$

are both finite, it follows that  $K(x_1, \dots, x_n)/K$  is finite. Moreover,

$$\begin{aligned} K(x_1, \dots, x_n) &= K(x_1, \dots, x_{n-1})(x_n) \\ &= K(x_1, \dots, x_{n-1})[x_n] = K[x_1, \dots, x_{n-1}][x_n] = K[x_1, \dots, x_n]. \end{aligned} \quad \square$$

2.12. COROLLARY. Let  $E = K(S)$  for some set  $S$ . Then  $E/K$  is algebraic if and only if  $x$  is algebraic over  $K$  for all  $x \in S$ .

PROOF. Let us prove the non-trivial implication. Let  $z \in K(S)$ . In particular, there exists a finite subset  $T \subseteq S$  such that  $z \in K(T)$ . The previous result implies that  $K(T)/K$  is algebraic, and hence  $z$  is algebraic.  $\square$

If  $E/K$  is an extension, let

$$\overline{K}_E = \{x \in E : x \text{ is algebraic over } K\}.$$

2.13. COROLLARY. If  $E/K$  is an extension, then  $\overline{K}_E$  is a subfield of  $E$  that contains  $K$ . Moreover,  $K(\overline{K}_E) = \overline{K}_E$  and  $K(\overline{K}_E)/K$  is algebraic.

PROOF. By definition,  $K(\overline{K}_E)/K$  is algebraic. Thus  $K(\overline{K}_E) \subseteq \overline{K}_E$ . From this, it follows that  $K(\overline{K}_E) = \overline{K}_E$ .  $\square$

The following exercise is now almost trivial:

2.14. EXERCISE. Let  $E/K$  be an extension of finite type; this means that  $E = K(S)$  for some finite set  $S$ . Prove that  $E/K$  is algebraic if and only if  $E/K$  is finite.

Let  $\overline{\mathbb{Q}} = \{\alpha \in \mathbb{C} : \alpha \text{ is algebraic over } \mathbb{Q}\}$ . Then  $\overline{\mathbb{Q}}$  is the field of algebraic numbers. Can you compute  $[\overline{\mathbb{Q}} : \mathbb{Q}]$ ?

2.15. EXERCISE. Prove that  $[\mathbb{Q}[\sqrt[3]{2}] : \mathbb{Q}] = 3$ .

For the previous exercise, you may use Eisenstein's criterion.

2.16. EXERCISE. Let  $E = \mathbb{Q}[i, \sqrt{2}] = \mathbb{Q}[\sqrt{2}][i]$ . Prove that  $[E : \mathbb{Q}] = 4$ .

2.17. EXERCISE. Let  $E = \mathbb{Q}[\sqrt{2}, \sqrt[3]{5}]$ .

- 1) Compute  $[E : \mathbb{Q}]$ .
- 2) Prove that  $E = \mathbb{Q}[\sqrt{2} + \sqrt[3]{5}]$ .
- 3) Find the minimal polynomial of  $\sqrt{2} + \sqrt[3]{5}$  over  $\mathbb{Q}$ .

2.18. EXERCISE. Find the minimal polynomials of  $\sqrt[4]{3}i$  over  $\mathbb{Q}[i]$  and over  $\mathbb{Q}[\sqrt{3}]$ .

2.19. EXERCISE. Find the minimal polynomial of  $\sqrt{2} + \sqrt[3]{5}i$  over  $\mathbb{Q}[i]$ .

Algebraic field extensions form a nice class of extensions. The same happens with finite field extensions.

2.20. PROPOSITION. *Let  $F/K$  be a subextension of  $E/K$ . Then  $E/K$  is algebraic if and only if  $E/F$  and  $F/K$  are algebraic.*

PROOF. If  $E/K$  is algebraic, then  $E/F$  and  $F/K$  are both algebraic, as  $K \subseteq F \subseteq E$ . Let us assume that  $E/F$  and  $F/K$  are both algebraic. Let  $x \in E$  and let  $L$  be the subextension over  $K$  generated by the coefficients of  $f(x, F)$ , the minimal polynomial of  $x$  over  $F$ . Then  $L/K$  is finite, since it is generated by finitely many algebraic elements. Moreover,  $x$  is algebraic over  $L$ . Since

$$[L(x) : K] = [L(x) : L][L : K] < \infty,$$

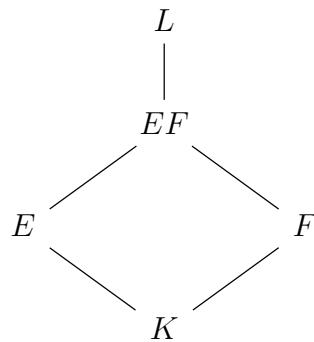
$L(x)/K$  is algebraic. In particular,  $x$  is algebraic over  $K$ . □

2.21. EXERCISE. Let  $F/K$  be a subextension of  $E/K$ . Prove that  $E/K$  is finite if and only if  $E/F$  and  $F/K$  are finite.

Let  $K$  be a field and  $K \subseteq F \subseteq L$  and  $K \subseteq E \subseteq L$  be fields. The **composite** of  $E$  and  $F$  is defined as

$$EF = K(E \cup F) = F(E) = E(F)$$

and it is equal to the smallest field that contains  $E$  and  $F$ . Here is the picture:



2.22. EXERCISE. Let  $E/K$  and  $F/K$  be algebraic field extensions. Prove that

$$EF = \left\{ \sum_{i=1}^m e_i f_i : m \in \mathbb{Z}_{>0}, e_i \in E, f_i \in F \text{ for all } i \in \{1, \dots, m\} \right\}.$$

2.23. EXERCISE. If  $E = \mathbb{Q}(\sqrt{2})$  and  $F = \mathbb{Q}(\sqrt{3})$ , then  $EF = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ . Compute  $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}]$  and  $\mathbb{Q}(\sqrt{2}) \cap \mathbb{Q}(\sqrt{3})$ .

2.24. EXERCISE. Let  $\xi \in \mathbb{C}$  be a primitive cubic root of one. If  $E = \mathbb{Q}(\sqrt[3]{2})$  and  $F = \mathbb{Q}(\xi)$ , then  $EF = \mathbb{Q}(\sqrt[3]{2}, \xi)$ . Compute  $[\mathbb{Q}(\sqrt[3]{2}, \xi) : \mathbb{Q}]$  and  $\mathbb{Q}(\sqrt[3]{2}) \cap \mathbb{Q}(\xi)$ .

2.25. EXERCISE. Let  $E/K$  and  $F/K$  be extensions, where both  $E$  and  $F$  are subfields of a field  $L$ . If  $F/K$  is algebraic, then  $EF/E$  is algebraic.

2.26. EXERCISE. Let  $E/K$  and  $F/K$  be extensions, where both  $E$  and  $F$  are subfields of a field  $L$ . If  $F/K$  is finite, then  $EF/E$  is finite.

The solution to the previous exercise shows, in particular, that  $[EF : E] \leq [F : K]$ .

### 3. Lecture – Week 3

3.1. LEMMA. Let  $\sigma: K \rightarrow L$  be a field homomorphism. Then there exists an extension  $E/K$  and a field isomorphism  $\varphi: E \rightarrow L$  such that  $\varphi|_K = \sigma$ .

PROOF. Note that  $\sigma: K \rightarrow \sigma(K)$  is bijective. Let  $A$  be a set in bijection with  $L \setminus \sigma(K)$  and disjoint with  $K$ . Let  $E = K \cup A$ . If  $\theta: A \rightarrow L \setminus \sigma(K)$  is bijective, then let

$$\varphi: E \rightarrow L, \quad \varphi(x) = \begin{cases} \sigma(x) & \text{if } x \in K, \\ \theta(x) & \text{if } x \in A. \end{cases}$$

Then  $\varphi$  is a bijective map such that  $\varphi|_K = \sigma$ . Transport the operations of  $L$  onto  $E$ , that is to define binary operations on  $E$  as follows:

$$(x, y) \mapsto x \oplus y = \varphi^{-1}(\varphi(x) + \varphi(y)), \quad (x, y) \mapsto x \odot y = \varphi^{-1}(\varphi(x)\varphi(y)).$$

Then, for example,

$$x \oplus y = \varphi^{-1}(\varphi(x) + \varphi(y)) = \varphi^{-1}(\sigma(x) + \sigma(y)) = \varphi^{-1}(\sigma(x+y)) = \varphi^{-1}(\varphi(x+y)) = x+y$$

for all  $x, y \in K$ .  $\square$

If  $\sigma: A \rightarrow B$  is a ring homomorphism, then  $\sigma$  induces a ring homomorphism

$$\bar{\sigma}: A[X] \rightarrow B[X], \quad \sum_i a_i X^i \mapsto \sum_i \sigma(a_i) X^i.$$

3.2. THEOREM. Let  $K$  be a field and  $f \in K[X]$  be such that  $\deg f > 0$ . Then there exists an extension  $E/K$  such that  $f$  admits a root in  $E$ .

PROOF. We may assume that  $f$  is irreducible over  $K$ . Let  $L = K[X]/(f)$  and

$$\pi: K[X] \rightarrow L$$

be the canonical map. Then  $L$  is a field (the reader should explain why). Let  $\sigma: K \rightarrow L$ ,  $a \mapsto \pi(aX^0)$ , and  $g = \bar{\sigma}(f) \in L[X]$ .

We claim that  $\pi(X)$  is a root of  $g$  in  $L$ . Suppose that  $f = \sum_i a_i X^i$ . Then

$$\begin{aligned} g(\pi(X)) &= \bar{\sigma}(f)(\pi(X)) \\ &= \sum_i \sigma(a_i) \pi(X)^i = \sum_i \pi(a_i X^0) \pi(X^i) = \pi\left(\sum_i a_i X^i\right) = \pi(f) = 0. \end{aligned}$$

By Lemma 3.1, there exists an extension  $E/K$  and an isomorphism  $\varphi: E \rightarrow L$  such that  $\varphi|_K = \sigma$ . Note that  $\varphi(x) = 0$  if and only if  $x = 0$ . If  $u = \pi(X)$ , then  $\varphi^{-1}(u)$  is a root of  $f$  in  $E$ , as

$$\begin{aligned} \varphi(f(\varphi^{-1}(u))) &= \varphi\left(\sum_i a_i \varphi^{-1}(u)^i\right) = \varphi\left(\sum_i a_i \varphi^{-1}(u^i)\right) \\ &= \sum_i \varphi(a_i) u^i = \sum_i \sigma(a_i) u^i = g(u) = 0. \end{aligned} \quad \square$$

As a corollary, if  $K$  is a field and  $f_1, \dots, f_n \in K[X]$  are polynomials of positive degree, then there exists an extension  $E/K$  such that each  $f_i$  admits a root in  $E$ . This is proved by induction on  $n$ .

3.3. DEFINITION. A field  $K$  is **algebraically closed** if each  $f \in K[X]$  of positive degree admits a root in  $K$ .

The **fundamental theorem of algebra** states that  $\mathbb{C}$  is algebraically closed. A typical proof uses complex analysis. Later we will give a proof of this result using Galois theory.

3.4. PROPOSITION. *The following statements are equivalent:*

- 1)  $K$  is algebraically closed.
- 2) If  $f \in K[X]$  is irreducible, then  $\deg f = 1$ .
- 3) If  $f \in K[X]$  is non-zero, then  $f$  decomposes linearly in  $K[X]$ , that is

$$f = a \prod_{i=1}^n (X - \alpha_i)^{m_i}$$

for some  $a \in K$  and  $\alpha_1, \dots, \alpha_n \in K$ .

- 4) If  $E/K$  is algebraic, then  $E = K$ .

PROOF. 1)  $\implies$  2)  $\implies$  3) are exercises.

Let us prove that 3)  $\implies$  4). Let  $x \in E$ . Decompose  $f(x, K)$  linearly in  $K[X]$  as

$$f(x, K) = a \prod_{i=1}^n (X - \alpha_i)^{m_i}$$

and evaluate on  $x$  to obtain that  $x = \alpha_j$  for some  $j$ .

To prove that 4)  $\implies$  1) let  $f \in K[X]$  be such that  $\deg f > 0$ . There exists an extension  $E/K$  such that  $f$  has a root  $x$  in  $E$ . The extension  $K(x)/K$  is algebraic and hence  $K(x) = K$ , so  $x \in K$ .  $\square$

### § 3.1. Artin's theorem.

3.5. DEFINITION. An **algebraic closure** of a field  $K$  is an algebraic extension  $C/K$  such that  $C$  is algebraically closed.

For example,  $\mathbb{C}/\mathbb{R}$  is an algebraic closure but  $\mathbb{C}/\mathbb{Q}$  is not.

3.6. PROPOSITION. *Let  $C$  be algebraically closed and  $\sigma: K \rightarrow C$  be a field homomorphism. If  $E/K$  is algebraic, then there exists a field homomorphism  $\varphi: E \rightarrow C$  such that  $\varphi|_K = \sigma$ .*

PROOF. Suppose first that  $E = K(x)$  and let  $f = f(x, K)$ . Let  $\bar{\sigma}(f) \in C[X]$  and let  $y \in C$  be a root of  $\bar{\sigma}(f)$ . If  $z \in E$ , then  $z = g(x)$  for some  $g \in K[X]$ . Let  $\varphi: E \rightarrow C$ ,  $z \mapsto \bar{\sigma}(g)(y)$ .

The map  $\varphi$  is well-defined. If  $z = h(x)$  for some  $h \in K[X]$ , then

$$0 = g(x) - h(x) = (g - h)(x)$$

and thus  $f$  divides  $g - h$ . In particular,  $\bar{\sigma}(f)$  divides  $\bar{\sigma}(g - h) = \bar{\sigma}(g) - \bar{\sigma}(h)$  and hence

$$(\bar{\sigma}(g) - \bar{\sigma}(h))(y) = 0.$$

It is an exercise to show that the map  $\varphi$  is a ring homomorphism.

Let  $a \in K$ . It follows that  $\varphi|_K = \sigma$ , as

$$\varphi(a) = \bar{\sigma}(aX^0)(y) = \sigma(a)$$

Let us now prove the proposition in full generality. Let  $X$  be the set of pairs  $(F, \tau)$ , where  $F$  is a subfield of  $E$  that contains  $K$  and  $\tau: F \rightarrow C$  is a field homomorphism such

that  $\tau|_K = \sigma$ . Note that  $(K, \sigma) \in X$ , so  $X$  is non-empty. Moreover,  $X$  is partially ordered by

$$(F, \tau) \leq (F_1, \tau_1) \iff F \subseteq F_1 \text{ and } \tau_1|_F = \tau.$$

If  $\{(F_i, \tau_i) : i \in I\}$  is a chain in  $X$ , then  $F = \cup_{i \in I} F_i$  is a subfield of  $E$  that contains  $K$ . Moreover, if  $z \in F$ , then  $z \in F_i$  for some  $i \in I$  and then one defines  $\tau(z) = \tau_i(z)$ . It is an exercise to prove that  $\tau$  is well-defined. Since  $(F, \tau) \in X$  is an upper bound, Zorn's lemma implies that there exists a maximal element  $(E_1, \theta) \in X$ . We claim that  $E = E_1$ . If not, let  $z \in E \setminus E_1$ . Since we know the proposition is true for the extension  $E_1(z)/E_1$ , let  $\rho: E_1(z) \rightarrow C$  be a field homomorphism such that  $\rho|_{E_1} = \theta$ . Then, in particular,  $\rho|_K = \sigma$ . This implies that  $(E_1(z), \rho) \in X$  and hence  $(E_1, \theta) < (E_1(z), \rho)$ , a contradiction to the maximality of  $(E_1, \theta)$ .  $\square$

#### 4. Lecture – Week 4

The previous proposition will be used to prove that the algebraic closure always exists.

**4.1. THEOREM (Artin).** *Let  $K$  be a field. Then  $K$  admits an algebraic closure  $C/K$ . If  $C_1/K$  is an algebraic closure, then the extensions  $C/K$  and  $C_1/K$  are isomorphic.*

**PROOF.** Let us first prove the uniqueness. The previous proposition implies the existence of an extension homomorphism  $\varphi: C \rightarrow C_1$ . Let  $y \in C_1$  and  $f = f(y, K)$  be the minimal polynomial of  $y$  in  $K$ . Since  $f$  admits a factorization

$$f = \lambda \prod (X - \alpha_i)^{m_i}$$

in  $C[X]$ , it follows that

$$f = \overline{\varphi}(f) = \varphi(\lambda) \prod (X - \varphi(\alpha_i))^{m_i}$$

Since  $0 = f(y)$ , we conclude that  $y = \varphi(\alpha_j)$  for some  $j$ . In particular,  $\varphi$  is surjective and hence  $\varphi$  is bijective.

We now prove the existence. Let us assume that  $K$  admits an extension  $E/K$  with  $E$  algebraically closed. We will prove later that this extension indeed exists; at the moment, we only want to get an algebraic extension from this setting. Let

$$F = \{x \in E : x \text{ is algebraic over } K\}.$$

Then  $F/K$  is algebraic. Let  $g \in F[X]$  be such that  $\deg g > 0$ . Since  $E$  is algebraically closed,  $g$  admits a root  $\alpha$  in  $E$ . In particular,  $\alpha$  is algebraic over  $F$  and hence  $\alpha$  is algebraic over  $K$ . This implies that  $\alpha \in F$ , thus  $F$  is algebraically closed. This proves that  $F/K$  is an algebraic closure.

Let us prove that there exists an extension  $E_1/K$  such that every polynomial  $f \in K[X]$  with  $\deg f > 0$  has a root in  $E_1$ . Let  $\{f_i : i \in I\}$  be the family of monic irreducible polynomials with coefficients in  $K$ . We may think that  $f_i = f_i(X_i)$ . Let  $R = K[\{X_i : i \in I\}]$  and let  $J$  be the ideal of  $R$  generated by the  $f_i(X_i)$ . We claim that  $J \neq R$ . If not,  $1 \in J$ , so

$$1 = \sum_{j=1}^m g_j f_{i_j}(X_{i_j})$$

for some  $g_1, \dots, g_m \in R$ . There exists an extension  $F/K$  such that  $f_{i_j}$  has a root  $\alpha_j$  in  $F$  for all  $j$ . Let

$$\tau: R \rightarrow F, \quad \tau(X_k) = \begin{cases} \alpha_j & \text{if } k = i_j, \\ 0 & \text{if } k \notin \{i_1, \dots, i_m\}. \end{cases}$$

Then  $\tau$  is a ring homomorphism and

$$1 = \tau(1) = \sum_{j=1}^m \tau(g_j) f_{i_j}(\alpha_j) = 0,$$

a contradiction.

Since  $J$  is a proper ideal, it is contained in a maximal ideal  $M$ . Let  $L = R/M$  and let  $\sigma: K \rightarrow L$  be the composition  $K \hookrightarrow R \rightarrow R/M = L$ , where  $\pi: R \rightarrow R/M$  is the canonical map. As we did before,  $\pi(X_i)$  is a root of  $\overline{\sigma}(f_i)$  for all  $i$ . And there exists an extension  $E_1/K$  such that every  $f_i$  has a root in  $E_1$ . Proceeding in this way, we construct a sequence

$$E_1 \subseteq E_2 \subseteq \dots$$

of fields such that every polynomial of positive degree and coefficients in  $E_k$  admits a root in  $E_{k+1}$ . Let  $E = \cup E_k$ . We claim that  $E$  is algebraically closed. In fact, let  $g \in E[X]$  be such that  $\deg g > 0$ . Then, since  $g \in E_r[X]$  for some  $r$ , it follows that  $g$  has a root in  $E_{r+1} \subseteq E$ .  $\square$

### § 4.1. Splitting fields.

4.2. DEFINITION. Let  $K$  be a field and  $f \in K[X]$  be such that  $\deg f > 0$ . A **splitting field** of  $f$  over  $K$  is a field  $E$  that contains  $K$  and that satisfies the following properties:

- 1)  $f$  factorizes linearly in  $E[X]$ .
- 2) If  $F$  is a field such that  $K \subseteq F \subseteq E$  and  $f$  factorizes linearly in  $F[X]$ , then  $F = E$ .

Easy examples:

4.3. EXAMPLE.  $\mathbb{C}$  is a splitting field of  $X^2 + 1 \in \mathbb{R}[X]$ .

4.4. EXAMPLE.  $\mathbb{Q}[\sqrt{2}]$  is a splitting field of  $X^2 - 2 \in \mathbb{Q}[X]$ .

4.5. EXAMPLE. The splitting field of  $f = X^2 - 2$  over  $\mathbb{Z}/7$  is precisely  $\mathbb{Z}/7$ , as 3 and 4 are the roots of  $f$  in  $\mathbb{Z}/7$ .

4.6. EXAMPLE.  $\mathbb{Q}(\sqrt[3]{2})$  is not a splitting field of  $X^3 - 2 \in \mathbb{Q}[X]$ . However, if  $\omega$  is a primitive cubic root of one, then  $\mathbb{Q}(\sqrt[3]{2}, \omega)$  is a splitting field of the polynomial  $X^3 - 2 \in \mathbb{Q}[X]$ .

4.7. PROPOSITION.  $E$  is a splitting field of  $f \in K[X]$  if and only if  $f$  factorizes linearly in  $E[X]$  and  $E = K(x_1, \dots, x_n)$ , where  $x_1, \dots, x_n$  are the roots of  $f$ .

PROOF. Let  $f = a \prod_{i=1}^r (X - x_i)^{n_i}$  and  $F = K(x_1, \dots, x_r)$  with  $x_1, \dots, x_r \in E$ . Since  $f$  factorizes linearly in  $F[X]$ , it follows that  $F = E$ . Conversely, let  $E = K(x_1, \dots, x_r)$  and assume that  $f$  factorizes linearly in  $F[X]$ . Then, in particular,  $x_1, \dots, x_r \in F$ . Hence  $E \subseteq F$  and  $F = E$ .  $\square$

Note that if  $E = K(x_1, \dots, x_n)$  as in the statement of Proposition 4.7, then it automatically follows that  $f$  factorizes linearly in  $E[X]$ .

One immediately obtains the following consequence: If  $E$  is a splitting field of  $f \in K[X]$ , then  $E/K$  is finite.

4.8. THEOREM. Let  $f \in K[X]$  be such that  $\deg f > 0$ . There exists a (unique up to extension isomorphism) splitting field of  $f$  over  $K$ .

PROOF. Let  $C/K$  be an algebraic closure of  $K$ . Write

$$f = a \prod_{i=1}^r (X - x_i)^{n_i}$$

in  $C[X]$ . Then  $E = K(x_1, \dots, x_r)$  is a splitting field of  $f$  over  $K$ .

Let us prove the uniqueness: if  $E_1/K$  is a splitting field of  $f$  over  $K$ , then  $E_1/K$  is algebraic and thus Proposition 3.6 implies that there exists  $\varphi \in \text{Hom}(E_1/K, C/K)$ , that is  $\varphi: E_1 \rightarrow C$  is a field homomorphism such that  $\varphi|_K$  is the identity. Factorize  $f$  linearly in  $E_1[X]$  and apply  $\overline{\varphi}$ :

$$f = a \prod_{j=1}^s (X - y_j)^{m_j} \implies f = \overline{\varphi}(f) = \varphi(a) \prod_{j=1}^s (X - \varphi(y_j))^{m_j}$$

so  $f$  factorizes linearly in  $\varphi(E_1)[X]$ . Moreover,  $E_1 = K(y_1, \dots, y_s)$  and

$$\varphi(E_1) = K(\varphi(y_1), \dots, \varphi(y_s)).$$

Thus  $\varphi(E_1)$  is a splitting field of  $f$ . Since  $\varphi(E_1) \subseteq E$ , it follows that  $\varphi(E_1) = E$ .  $\square$

**4.9. EXERCISE.** If  $C$  is an algebraic closure of  $K$  and  $\varphi \in \text{Hom}(C/K, C/K)$ , then  $\varphi$  is an isomorphism.

Let  $C$  be an algebraic closure of  $K$  and  $G = \text{Gal}(C/K)$ . The group  $G$  acts on  $C$

$$\sigma \cdot x = \sigma(x), \quad \sigma \in G, x \in C.$$

The orbits are of the form

$$O_G(x) = \{\sigma(x) : \sigma \in G\} = \{y \in C : y = \sigma(x) \text{ for some } \sigma \in G\}$$

The elements  $x, y \in C$  are **conjugate** if  $y = \sigma(x)$  for some  $\sigma \in G$ .

**4.10. PROPOSITION.** Let  $C$  be an algebraic closure of  $K$  and  $x, y \in C$ . Then  $x$  and  $y$  are conjugate if and only if  $f(x, K) = f(y, K)$ . In particular,  $O_G(x)$  is finite.

**PROOF.** Let  $G = \text{Gal}(C/K)$ . If  $x$  and  $y$  are conjugate, say  $y = \sigma(x)$  for some  $\sigma \in G$ , let us write  $g = f(x, K)$  as

$$g = X^n + \sum_{i=0}^{n-1} a_i X^i$$

for some  $n \geq 1$  and  $a_0, \dots, a_{n-1} \in K$ . Then  $0 = g(x) = x^n + \sum_{i=0}^{n-1} a_i x^i$  and hence  $y$  is a root of  $g$ , as

$$\begin{aligned} 0 &= \sigma \left( x^n + \sum_{i=0}^{n-1} a_i x^i \right) = \sigma(x)^n + \sum_{i=0}^{n-1} \sigma(a_i) \sigma(x)^i \\ &= \sigma(x)^n + \sum_{i=0}^{n-1} a_i \sigma(x)^i = y^n + \sum_{i=0}^{n-1} a_i y^i. \end{aligned}$$

Thus  $f(y, K) = g$ .

Conversely, assume that  $f(x, K) = f(y, K)$ . Let  $g = f(x, K) = f(y, K)$  and let

$$\varphi: K[x] \rightarrow K[y], \quad h(x) \mapsto h(y).$$

Let us show that the map  $\varphi$  is well-defined: we need to show that if  $h_1(x) = h_2(x)$ , then

$$h_1(y) = \varphi(h_1(x)) = \varphi(h_2(x)) = h_2(y).$$

If  $h_1(x) = h_2(x)$ , then

$$(h_1 - h_2)(x) = h_1(x) - h_2(x) = 0.$$

This implies that  $g$  divides  $h_1 - h_2$ . In particular,  $h_1(y) = h_2(y)$ .

A straightforward calculation shows that  $\varphi$  is a field homomorphism such that  $\varphi|_K = \text{id}$ , this means that  $\varphi$  is an extension homomorphism such that  $\varphi(x) = y$ . There exists a homomorphism  $\sigma \in \text{Hom}(C/K, C/K)$  such that  $\sigma|_{K[x]} = \varphi$ . Since  $\sigma$  is bijective (why?),  $\sigma(x) = \varphi(x) = y$  and hence  $O_G(x) = O_G(y)$ .  $\square$

4.11. PROPOSITION. Let  $C$  be an algebraic closure of  $K$  and  $x \in C$ . Then

$$f(x, K) = \prod_{y \in O_G(x)} (X - y)^m$$

for some  $m$ .

PROOF. For each  $y \in O_G(x)$  let  $m_y$  be the multiplicity of  $y$  in  $f(x, K)$ . Then, for example,  $f(x, K) = (X - x)^{m_x} g$  for some  $g$ . If  $y \in O_G(x)$ , then  $y = \sigma(x)$  for some  $\sigma \in \text{Gal}(C/K)$ . Since

$$\bar{\sigma}(f(x, K)) = f(x, K) = (X - y)^{m_x} \bar{\sigma}(g),$$

it follows that  $m_y \geq m_x$ . By symmetry, we conclude that  $m_x = m_y$ .  $\square$

The previous proposition shows, in particular, that all the roots of an irreducible polynomial  $f \in K[X]$  in an algebraic closure  $C$  of  $K$  have the same multiplicity. This is not true if  $f$  is not irreducible. Find an example.

4.12. DEFINITION. Let  $K$  be a field and  $\{f_i : i \in I\}$  be a non-empty family of polynomials of positive degree with coefficients in  $K$ . A **splitting field** of  $\{f_i : i \in I\}$  is an extension  $E/K$  such that every  $f_i$  factorizes linearly in  $E[X]$  and if  $F/K$  is a sub extension of  $E/K$  such that every  $f_i$  factorizes linearly in  $F[X]$ , then  $F = E$ .

4.13. EXERCISE. Prove that  $E/K$  is a splitting field of  $\{f_i : i \in I\}$  if and only if every  $f_i$  factorizes linearly in  $E[X]$  and  $E = K(S)$  where  $S = \{\text{roots of } f_i \text{ for all } i\}$ .

4.14. EXERCISE. Prove that if  $E/K$  is a splitting field of  $\{f_i : i \in I\}$ , then  $E/K$  is algebraic. If, moreover,  $I$  is finite, then  $E/K$  is a splitting field of  $\prod_{i \in I} f_i$ .

4.15. EXERCISE. Prove that there exists a splitting field of  $\{f_i : i \in I\}$  and it is unique up to extension isomorphism.

4.16. EXERCISE. Let  $f = X^3 - X - 1 \in (\mathbb{Z}/3)[X]$  and  $E$  be a splitting field of  $f$ . Compute  $[E : \mathbb{Z}/3]$ .

What about the splitting field of  $f = X^3 - X - 1 \in \mathbb{Q}[X]$ ?

4.17. EXERCISE. Let  $f = X^4 - 5X^2 + 5 \in \mathbb{Q}[X]$  and  $E$  be a splitting field of  $f$ . Compute  $[E : \mathbb{Q}]$  and  $\text{Gal}(E/\mathbb{Q})$ .

## 5. Lecture – Week 5

### § 5.1. Normal extensions.

5.1. PROPOSITION. Let  $E/K$  be an algebraic extension and  $\sigma \in \text{Hom}(E/K, E/K)$ . Then  $\sigma$  is bijective.

PROOF. It is enough to prove that  $\sigma$  is surjective. Why?

Let  $x \in E$  and  $C$  be an algebraic closure of  $K$  that contains  $E$ . By Proposition 3.6, there exists a field homomorphism  $\varphi: C \rightarrow C$  such that  $\varphi|_E = \sigma$ . Thus  $\varphi|_K = \sigma|_K = \text{id}_K$ . Let  $G = \text{Gal}(C/K)$ . Then  $\varphi \in G$ . If  $z \in O_G(x)$ , then  $z = \tau(x)$  for some  $\tau \in G$  and hence

$$\varphi(z) = \varphi(\tau(x)) = (\varphi\tau)(x).$$

This implies that  $\varphi(z) \in O_G(x)$  and  $\varphi(O_G(x)) = O_G(x)$ . The restriction  $\sigma|_{E \cap O_G(x)}$  is injective. Then

$$\begin{aligned} \sigma(E \cap O_G(x)) &= \varphi(E \cap O_G(x)) \\ &= \varphi(E) \cap \varphi(O_G(x)) = \sigma(E) \cap O_G(x) \subseteq E \cap O_G(x). \end{aligned}$$

Since  $|E \cap O_G(x)| < \infty$ , it follows that  $E \cap O_G(x) = \sigma(E \cap O_G(x))$  and hence  $x$  belongs to the image of  $\sigma$ .  $\square$

5.2. DEFINITION. Let  $E/K$  be an algebraic extension and  $C$  be an algebraic closure of  $K$  containing  $E$ . Then  $E/K$  is **normal** if  $\sigma(E) \subseteq E$  for all  $\sigma \in \text{Hom}(E/K, C/K)$ .

Note that  $\sigma(E) \subseteq E$  in the previous definition is equivalent to  $\sigma(E) = E$ .

5.3. EXAMPLE. The extension  $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$  is not normal. Why?

Some trivial examples of normal extensions:  $K/K$  is normal and if  $C$  is an algebraic closure of  $K$ , then  $C/K$  is normal.

5.4. EXAMPLE. The extension  $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$  is normal. Every extension generated by algebraic elements of degree two is normal.

5.5. EXERCISE. Let  $\xi$  be a primitive cubic root of one. Then  $\mathbb{Q}(\sqrt[3]{2}, \xi)/\mathbb{Q}$  is normal.

The following result is practical but technical. That is why we leave the proof as an exercise.

5.6. EXERCISE. Prove that the previous definition depends only on  $E$  (and not on the algebraic closure  $C$ ).

Some properties:

5.7. PROPOSITION. Let  $E/K$  be a normal extension and  $f \in K[X]$  be an irreducible polynomial that admits a root  $x$  in  $E$ . Then  $f$  factorizes linearly in  $E$ .

PROOF. We may assume that  $f$  is monic. Let  $C/K$  be an algebraic closure of  $K$  containing  $E$ . Let  $y$  be a root of  $f$  in  $C$ . Since  $f = f(x, K) = f(y, K)$ , it follows that  $y = \sigma(x)$  for some  $\sigma \in \text{Gal}(C/K)$ . Since  $E/K$  is normal,  $\sigma|_E: E \rightarrow C$  is an automorphism of  $E/K$ , that is  $\sigma(E) \subseteq E$ . In particular,  $y \in E$ .  $\square$

Let  $K \subseteq F \subseteq E$  be a tower of fields. If  $E/K$  is normal, then  $E/F$  is normal. However, Note that  $E/K$  normal does not imply  $F/K$  normal, as this would imply that every extension is normal. Moreover,  $E/F$  normal and  $F/K$  normal do not imply  $E/K$  normal.

5.8. EXAMPLE. The extensions  $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}(\sqrt{2})$  and  $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$  are both normal, but the extension  $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$  is not, as the roots of  $X^4 - 2$  are  $\sqrt[4]{2}, -\sqrt[4]{2}, \sqrt[4]{2}i$  and  $-\sqrt[4]{2}i$ .

Recall that if  $C$  is an algebraic closure of  $K$  and  $x \in C$ , then

$$f(x, K) = \prod (X - y)^m,$$

where the product is taken over all  $y \in O_{\text{Gal}(C/K)}(x)$ . If  $E/K$  is normal and  $x \in E$ , then there exists  $m$  such that

$$f(x, K) = \prod (X - y)^m,$$

where the product is taken over all  $y \in O_{\text{Gal}(E/K)}(x)$ .

5.9. PROPOSITION. *Let  $E/K$  and  $F/K$  be extensions. If  $F/K$  is normal, then  $EF/E$  is normal.*

PROOF. Let  $C$  be an algebraic closure of  $E$  containing  $EF$  (this exists because  $EF/E$  is algebraic). Let  $\sigma \in \text{Hom}(EF/E, C/E)$ . We claim that  $\sigma(EF) = EF$ . Let

$$\bar{K} = \{x \in C : x \text{ is algebraic over } K\}.$$

Then  $\bar{K}$  is an algebraic closure over  $K$  and  $F \subseteq \bar{K}$ . Since  $F/K$  is normal and  $\sigma|_F \in \text{Hom}(F/K, \bar{K}/K)$ , it follows that  $\sigma(F) = F$ . If  $z \in EF$ , then  $z = \sum_{i=1}^m e_i f_i$  for some  $e_1, \dots, e_m \in E$  and  $f_1, \dots, f_m \in F$ . Since  $\sigma(e_i) = e_i$  for all  $i$ ,

$$\sigma(z) = \sum_{i=1}^m \sigma(e_i) \sigma(f_i) = \sum_{i=1}^m e_i \sigma(f_i) \in EF. \quad \square$$

What is the relation between normal extensions and splitting fields? The notions look deeply related. The following proposition serves as an explanation:

5.10. PROPOSITION. *Let  $E/K$  be an algebraic extension. Then  $E/K$  is normal if and only if  $E/K$  is the splitting field of a family of polynomials of  $K[X]$  of positive degree.*

PROOF. Assume first that  $E/K$  is a normal extension. Let  $G = \text{Gal}(E/K)$ . If  $x \in E$  and  $f(x, K) = \prod_{y \in O_G(x)} (X - y)^m$ , then  $f(x, K)$  factorizes linearly in  $E[X]$ . Thus  $E/K$  is a splitting field of the family  $\{f(x, K) : x \in E\}$ .

Conversely, assume that  $E/K$  is a splitting field of the family  $\{f_i : i \in I\}$ . Then  $E = K(S)$  where  $S$  is the set of roots of the polynomials  $f_i$ . Let  $C/K$  be an algebraic closure of  $K$  that contains  $E$  and let  $\sigma \in \text{Hom}(E/K, C/K)$ . Let  $x \in S$ . Then  $x$  is a root of some  $f_j = \sum a_k X^k$ . Since  $f_j(x) = 0$ , it follows that  $\sigma(x)$  is a root of  $f_j$ , as

$$f_j(\sigma(x)) = \sum a_k \sigma(x)^k = \sum \sigma(a_k) \sigma(x^k) = \sigma \left( \sum a_k x^k \right) = \sigma(0) = 0.$$

Hence  $\sigma(E) \subseteq E$ .  $\square$

5.11. EXERCISE. Let  $E = \mathbb{Q}[\sqrt[4]{7} + \sqrt{2}]$ .

- 1) Prove that  $E/\mathbb{Q}$  is not normal.
- 2) Compute  $[E : \mathbb{Q}]$ .
- 3) Compute  $\text{Gal}(E/\mathbb{Q})$ .

**§ 5.2. Dedekind's theorem.** Note that every extension homomorphism  $E/K \rightarrow F/K$  is, in particular, a  $K$ -linear map  $E \rightarrow F$ , that is

$$\text{Hom}(E/K, F/K) \subseteq \text{Hom}_K(E, F).$$

If  $F/K$  is an extension and  $V$  is a  $K$ -vector space, the set  $\text{Hom}_K(V, F)$  of  $K$ -linear maps is a vector space over  $F$  with  $(a \cdot f)(v) = af(v)$  for  $a \in F$ ,  $f \in \text{Hom}_K(V, F)$  and  $v \in V$ .

5.12. EXERCISE. Let  $V$  be a  $K$ -vector space. Prove that  $\dim_F \text{Hom}_K(V, F) \geq \dim_K V$ . Moreover, if  $\dim_K V < \infty$ , then  $\dim_F \text{Hom}_K(V, F) = \dim_K V$ .

If  $V$  is a vector space and  $S$  is a (possibly infinite) subset of  $V$ , then  $S$  is linearly independent if every finite subset of  $S$  is linearly independent.

5.13. THEOREM (Dedekind). *Let  $E/K$  and  $F/K$  be extensions and let  $\{\varphi_i : i \in I\}$  be a subset of  $\text{Hom}(E/K, F/K)$ , i.e. a family of extension homomorphisms. Assume that  $\varphi_i \neq \varphi_j$  if  $i \neq j$ . Then the subset  $\{\varphi_i : i \in I\} \subseteq \text{Hom}_K(E, F)$  is linearly independent over  $F$ .*

PROOF. Assume it is not. Let  $\{\varphi_1, \dots, \varphi_n\}$  be linearly dependent over  $F$  with  $n$  minimal. Clearly,  $n > 1$ . Without loss of generality, we may assume that

$$(5.1) \quad \sum_{i=1}^n a_i \varphi_i = 0$$

for some  $a_1, \dots, a_n \in F$  all different from zero. Let  $z \in E \setminus \{0\}$  be such that  $\varphi_1(z) \neq \varphi_2(z)$ . If  $x \in E$ , then

$$0 = \left( \sum_{i=1}^n a_i \varphi_i \right) (xz) = \sum_{i=1}^n a_i \varphi_i(xz) = \sum_{i=1}^n a_i \varphi_i(x) \varphi_i(z) = \left( \sum_{i=1}^n (a_i \varphi_i(z)) \varphi_i \right) (x).$$

Thus

$$\sum_{i=1}^n (a_i \varphi_i(z)) \varphi_i = 0.$$

Since  $\varphi_1(z) \neq 0$ ,

$$(5.2) \quad a_1 \varphi_1 + a_2 \frac{\varphi_2(z)}{\varphi_1(z)} \varphi_2 + \dots + a_n \frac{\varphi_n(z)}{\varphi_1(z)} \varphi_n = 0.$$

Thus, subtracting (5.1) and (5.2),

$$\left( a_2 - a_2 \frac{\varphi_2(z)}{\varphi_1(z)} \right) \varphi_2 + \dots + \left( a_n - a_n \frac{\varphi_n(z)}{\varphi_1(z)} \right) \varphi_n = 0.$$

Since  $a_2 \neq 0$  and  $\varphi_2(z) \neq \varphi_1(z)$ , the scalar  $a_2 - a_2 \frac{\varphi_2(z)}{\varphi_1(z)} \neq 0$  and hence  $\{\varphi_2, \dots, \varphi_n\}$  is linearly dependent, a contradiction.  $\square$

If  $E/K$  and  $F/K$  are extensions, let  $\gamma(E/K, F/K) = |\text{Hom}(E/K, F/K)|$ .

5.14. EXERCISE. Prove the following statements:

- 1)  $\gamma(E/K, F/K) \leq \dim_F \text{Hom}_K(E, F)$ .
- 2) If  $[E : K] < \infty$ , then  $\gamma(E/K, F/K) \leq [E : K]$ .
- 3) If  $x$  is algebraic over  $K$ , then  $\gamma(K(x)/K, F/K) \leq \deg f(x, K)$ .

If  $C$  is an algebraic closure of  $K$ , then we define  $\gamma(E/K) = \gamma(E/K, C/K)$ . This definition does not depend on the algebraic closure.

5.15. EXERCISE. If  $C$  and  $C_1$  are algebraic closures of  $K$ , then

$$|\text{Hom}(E/K, C/K)| = |\text{Hom}(E/K, C_1/K)|.$$

5.16. PROPOSITION. Let  $C$  be an algebraic closure of  $K$  and  $G = \text{Gal}(C/K)$ . If  $x \in C$ , then  $\gamma(K(x)/K) = |O_G(x)|$ .

PROOF. If  $\sigma \in \text{Hom}(K(x)/K, C/K)$ , then there exists  $\phi \in G$  such that  $\phi|_{K(x)} = \sigma$ . Thus

$$\sigma(x) = \phi(x) \in O_G(x).$$

Conversely, if  $y \in O_G(x)$ , then there exists  $\tau \in G$  such that  $y = \tau(x)$ . Hence

$$\tau|_{K(x)} \in \text{Hom}(K(x)/K, C/K)$$

and  $\tau|_{K(x)}(x) = y$ . Since our sets are then in bijective correspondence, the claim follows.  $\square$

5.17. EXERCISE. If  $E/K$  is finite, then  $|\text{Gal}(E/K)| \leq [E : K]$ . Moreover,  $E/K$  is normal if and only if  $|\text{Gal}(E/K)| = \gamma(E/K)$ .

## 6. Lecture – Week 6

If  $t: A \rightarrow B$  is a surjective map, then  $a \sim a_1 \iff t(a) = t(a_1)$  defines an equivalence relation on  $A$ . The set  $\bar{A}$  of equivalence classes is in bijective correspondence with  $B$ ,  $\bar{A} \rightarrow B$ ,  $\bar{a} \mapsto t(a)$ . Moreover, if  $|t^{-1}(\{b\})| = m$  for all  $b \in B$ , then  $|A| = m|\bar{A}| = m|B|$ .

**6.1. PROPOSITION.** *Let  $E/K$  be algebraic and  $F/K$  be a subextension such that  $E/F$  is finite. Then  $\gamma(E/K) = \gamma(E/F)\gamma(F/K)$ .*

**PROOF.** Assume first that  $E = F(x)$ . Let  $C$  be an algebraic closure of  $K$  containing  $E$  and  $G = \text{Gal}(C/F)$ . Let  $f = f(x, F) = \sum b_i X^i$ .

The map

$$\lambda: \text{Hom}(E/K, C/K) \rightarrow \text{Hom}(F/K, C/K), \quad \sigma \mapsto \sigma|_F,$$

is well-defined. It is surjective: if  $\varphi \in \text{Hom}(F/K, C/K)$ , then  $\varphi: F \rightarrow C$  is, in particular, a field homomorphism. Since  $E/F$  is algebraic, by Proposition 3.6 there exists a field homomorphism  $\sigma: E \rightarrow C$  such that  $\sigma|_F = \varphi$ . Since  $\sigma|_K = \varphi|_K = \text{id}$ , in particular  $\sigma \in \text{Hom}(E/K, C/K)$ .

For  $\varphi \in \text{Hom}(F/K, C/K)$ ,

$$\lambda^{-1}(\{\varphi\}) = \{\sigma \in \text{Hom}(E/K, C/K) : \sigma|_F = \varphi\}$$

and let  $R_\varphi$  be the set of roots (in  $C$ ) of the polynomial  $\bar{\varphi}(f) = \sum \varphi(b_i)X^i$ .

**CLAIM.** The map  $\alpha: \lambda^{-1}(\{\varphi\}) \rightarrow R_\varphi$ ,  $\sigma \mapsto \sigma(x)$ , is well-defined.

We need to show that  $\sigma(x)$  is a root of  $\bar{\varphi}(f)$ :

$$\begin{aligned} \bar{\varphi}(f)(\sigma(x)) &= \sum \varphi(b_i)\sigma(x)^i = \sum \sigma(b_i)\sigma(x^i) \\ &= \sum \sigma(b_i x^i) = \sigma\left(\sum b_i x^i\right) = \sigma(f(x)) = \sigma(0) = 0. \end{aligned}$$

**CLAIM.** The map  $\beta: R_\varphi \rightarrow \lambda^{-1}(\{\varphi\})$ ,  $y \mapsto \sigma_y$ , where  $\sigma_y(z) = \bar{\varphi}(h)(y)$  if  $z = h(x)$ , is well-defined.

We need to show that if  $z = h(x)$  and  $z = h_1(x)$  for some  $h, h_1 \in F[X]$ , then

$$\bar{\varphi}(h)(y) = \bar{\varphi}(h_1)(y).$$

The assumptions imply that  $(h - h_1)(x) = 0$  and hence  $f$  divides  $h - h_1$ . Since  $\bar{\varphi}$  is a ring homomorphism,  $\bar{\varphi}(f)$  divides  $\bar{\varphi}(h) - \bar{\varphi}(h_1)$ . This implies  $(\bar{\varphi}(h) - \bar{\varphi}(h_1))(y) = 0$ . We also need to show that  $\sigma_y|_F = \varphi$ : if  $a \in F$ , then write  $a = aX^0 \in F[X]$ . Thus  $\sigma_y(a) = \bar{\varphi}(aX^0)(y) = \varphi(a) \in C$ . It is now an exercise to prove that  $\sigma_y \in \text{Hom}(E/K, C/K)$ .

**CLAIM.**  $|\lambda^{-1}(\{\varphi\})| = |R_\varphi|$ .

For this we need to show that  $\beta$  is the inverse of  $\alpha$ , that is  $\alpha \circ \beta = \text{id}$  and  $\beta \circ \alpha = \text{id}$ . To prove that  $\beta \circ \alpha = \text{id}$  let  $\sigma$  be such that  $\sigma|_F = \varphi$ . Then  $y = \sigma(x) \in R_\varphi$ . Let

$$z = h(x) = \sum a_i x^i \in F[x] = E.$$

Then

$$\bar{\varphi}(h)(y) = \sum \varphi(a_i)y^i = \sum \sigma(a_i)y^i = \sigma\left(\sum a_i x^i\right) = \sigma(z).$$

Conversely, if  $y \in R_\varphi$ , then

$$\alpha(\sigma_y) = \sigma_y(x) = y,$$

as  $\sigma_y(x) = \bar{\varphi}(X)(y) = y$ .

CLAIM. If  $\phi \in \text{Gal}(C/K)$  is such that  $\phi|_F = \varphi$ , then  $|\phi^{-1}(R_\varphi)| = |R_\varphi|$  and

$$O_G(x) = \phi^{-1}(R_\varphi).$$

Let us first prove  $O_G(x) \supseteq \phi^{-1}(R_\varphi)$ . If  $y \in R_\varphi$ , then

$$\begin{aligned} f(\phi^{-1}(y)) &= \sum b_i \phi^{-1}(y^i) = \phi^{-1} \left( \sum \phi(b_i) y^i \right) \\ &= \phi^{-1}(\bar{\varphi}(f)(y)) = \phi^{-1}(0) = 0. \end{aligned}$$

Then  $f(x, F) = f(\phi^{-1}(y), F)$ . By Proposition 4.10,  $\phi^{-1}(y) \in O_G(x)$ .

Now we prove  $O_G(x) \subseteq \phi^{-1}(R_\varphi)$ . Let  $z \in O_G(x)$ . Then  $\bar{\varphi}(f)(\phi(z)) = 0$ , as

$$\begin{aligned} \bar{\varphi}(f)(\phi(z)) &= \sum \varphi(b_i) \phi(z^i) \\ &= \sum \phi(b_i) \phi(z^i) = \phi \left( \sum b_i z^i \right) = \phi(f(z)) = \phi(0) = 0. \end{aligned}$$

Thus  $\phi(z) \in R_\varphi$  and hence  $z \in \phi^{-1}(R_\varphi)$ . It follows that  $|\lambda^{-1}(\{\varphi\})| = |O_G(x)|$  for all  $\varphi$ . By using the argument before the proposition,

$$\begin{aligned} \gamma(E/K) &= |\text{Hom}(E/K, C/K)| \\ &= |O_G(x)| |\text{Hom}(F/K, C/K)| \\ &= |O_G(x)| \gamma(F/K). \end{aligned}$$

Since  $\gamma(E/F) = \gamma(F(x)/F) = |O_G(x)|$  by Proposition 5.16, the claim follows.

For the general case, we assume that  $E = F(x_1, \dots, x_n)$ . We proceed by induction on  $n$ . If  $n = 0$ , then  $E = F$  and the result is trivial. If  $n > 0$ , let  $L = F[x_1, \dots, x_{n-1}]$  and  $E = L(x_n)$ . The case proved implies that  $\gamma(E/F) = \gamma(E/L)\gamma(L/F)$ . By the inductive hypothesis,  $\gamma(L/K) = \gamma(L/F)\gamma(F/K)$ . Thus

$$\gamma(E/F)\gamma(F/K) = \gamma(E/L)\gamma(L/F)\gamma(F/K) = \gamma(E/L)\gamma(L/K) = \gamma(E/K),$$

again using the previous case.  $\square$

### § 6.1. Separable extensions.

6.2. DEFINITION. Let  $E/K$  be an extension and  $x \in E$  an algebraic element over  $K$ . Then  $x$  is **separable** over  $K$  if  $x$  is a simple root of  $f(x, K)$ .

An algebraic extension  $E/K$  is **separable** if every  $x \in E$  is separable over  $K$ . Then  $K/K$  is separable.

6.3. EXERCISE. Prove that an element  $x$  is separable over  $K$  if and only if  $x$  is a simple root of a polynomial with coefficients in  $K$ .

If  $F/K$  is a subextension of  $E/K$  and  $x \in E$  is separable over  $K$ , then  $x$  is separable over  $F$ .

6.4. EXERCISE. If  $C$  is an algebraic closure of  $K$ ,  $x \in C$  and  $G = \text{Gal}(C/K)$ . Prove that the following statements are equivalent:

- 1)  $x$  is separable over  $K$ .
- 2) Every  $y \in O_G(x)$  is separable over  $K$ .
- 3)  $\gamma(K(x)/K) = [K(x) : K] = \deg f(x, K)$ .

Let  $K$  be any field and  $g \in K[X]$ . Let  $z$  be a root of  $g$ . Then  $z$  is a multiple root of  $g$  if and only if  $z$  is a root of  $g'$ .

6.5. EXERCISE. Prove that if  $K$  has characteristic zero or  $K$  is finite, then every algebraic extension of  $K$  is separable.

6.6. EXAMPLE. Let  $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ . Then  $[E : \mathbb{Q}] = 4$  and  $\text{Gal}(E/\mathbb{Q}) \cong C_2 \times C_2$ . The extension  $E/\mathbb{Q}$  is normal, as it is the splitting field of  $(X^2 - 2)(X^2 - 3)$  and it is separable as  $\mathbb{Q}$  has characteristic zero.

6.7. EXAMPLE. Let  $E$  be a splitting field of  $X^4 - 2$  over  $\mathbb{Q}$ . Then  $E/\mathbb{Q}$  is normal and separable. Note that  $E = \mathbb{Q}(\sqrt[4]{2}, i)$ , so

$$[E : \mathbb{Q}] = 8 = |\text{Gal}(E/\mathbb{Q})|.$$

Let us compute  $\text{Gal}(E/\mathbb{Q})$ . If  $\sigma \in \text{Gal}(E/\mathbb{Q})$ , then  $\sigma(\sqrt[4]{2}) \in \{\sqrt[4]{2}, -\sqrt[4]{2}, \sqrt[4]{2}i, -\sqrt[4]{2}i\}$  and  $\sigma(i) \in \{-i, i\}$ . Two examples are

$$\alpha: \begin{cases} \sqrt[4]{2} \mapsto \sqrt[4]{2}i, \\ i \mapsto i, \end{cases} \quad \beta: \begin{cases} \sqrt[4]{2} \mapsto \sqrt[4]{2}, \\ i \mapsto -i. \end{cases}$$

It follows that  $\text{Gal}(E/\mathbb{Q})$  is isomorphic to the group  $\langle \alpha, \beta \rangle$ , which turns out to be isomorphic to the dihedral group of eight elements.

Another consequence: If  $E = K(S)$ , then  $E/K$  is separable if and only if every  $x \in S$  is separable over  $K$ . One first does the case  $E = K(x)$  and then proceeds by induction.

6.8. EXERCISE. Let  $K \subseteq F \subseteq E$  be a tower of fields. Prove that  $E/K$  is separable if and only if  $F/K$  and  $E/F$  are separable.

6.9. EXERCISE. Let  $E/K$  and  $F/K$  be extensions. Prove that if  $F/K$  is separable, then  $EF/E$  is separable.

6.10. EXAMPLE. Let  $p$  be a prime and let  $t$  be transcendental over  $\mathbb{F}_p = \mathbb{Z}/p$ . Let  $K = \mathbb{F}_p(t^p)$  and let  $E = \mathbb{F}_p(t)$ . Then  $E/K$  has degree  $p$  and is not separable.

Indeed,  $E/K$  is algebraic because  $f = f(t, K) = X^p - t^p$  (it is an exercise to check that  $f$  is irreducible in  $K[X]$ ). Moreover,  $f = (X - t)^p \in E[X]$ , and so  $t$  has multiplicity  $p$  in  $f$ . It follows that  $t$  is not separable over  $K$ . Otherwise there is a polynomial  $g \in K[X]$  such that  $t$  is a simple root of  $g$ , and hence, in particular,  $g(t) = 0$ . However, we then must have  $f \mid g$ , and thus  $t$  cannot be a simple root of  $g$ .

Note that, in this case,  $E/K$  is also not normal, as  $\text{Gal}(E/K) = \{\text{id}_E\}$ .

If  $E/K$  is algebraic, then

$$F = \{x \in E : x \text{ is separable over } K\}$$

is a subfield of  $E$  that contains  $K$ . It is known as the **separable closure** of  $K$  with respect to  $E$ . Note that  $F = K(F)$ , as  $K(F)$  is separable because it is generated by separable elements. Moreover,  $F/K$  is separable and  $E/F$  is a **purely inseparable** extension, meaning that for every  $x \in E \setminus F$ ,  $x$  is not separable over  $K$ .

6.11. PROPOSITION. *If  $E/K$  is separable and finite, then  $E = K(x)$  for some  $x \in E$ .*

PROOF. Let us assume that  $K$  is finite. Then  $E$  is finite and hence the multiplicative group  $E^\times = E \setminus \{0\}$  is cyclic, say  $E^\times = \langle x \rangle$ . It follows that  $E = K(x)$ .

Let us now assume that  $K$  is infinite. We first consider the case  $E = K(x, y)$ . The general case  $E = K(x_1, \dots, x_n)$  is left as an exercise, one needs to proceed by induction. Let  $n = [E : K]$  and  $C$  be an algebraic closure of  $K$  containing  $E$ . Write  $\text{Hom}(E/K, C/K) = \{\sigma_1, \dots, \sigma_n\}$ . Let

$$f = \prod_{1 \leq i < j \leq n} ((\sigma_i(y) - \sigma_j(y)) + X(\sigma_i(x) - \sigma_j(x))) \in C[X].$$

Then  $f \neq 0$ , as  $f$  is a product of non-zero polynomials. Since  $K$  is infinite, there exists a non-zero  $c \in K$  such that  $f(c) \neq 0$ . For any  $r, s \in \{1, \dots, n\}$  with  $r \neq s$ ,

$$\sigma_r(y) - \sigma_s(y) + c(\sigma_r(x) - \sigma_s(x)) \neq 0,$$

as  $f(c) \neq 0$ . It follows that  $\sigma_r(y + cx) \neq \sigma_s(y + cx)$ . Thus  $\gamma(K(y + cx)/K) \geq n$ . Now

$$n \geq [K(y + cx) : K] = \gamma(K(y + cx)/K) \geq n,$$

so  $[K(y + cx) : K] = n$  and hence  $K(y + cx) = E$ .  $\square$

For example,  $\mathbb{Q}(\sqrt{2}, i) = \mathbb{Q}(\sqrt{2} + i)$ .

## 7. Lecture – Week 7

7.1. THEOREM (Steinitz). *Let  $E/K$  be a finite extension. Then  $E = K(x)$  for some  $x \in E$  if and only if  $E/K$  admits finitely many subextensions.*

PROOF. We may assume that  $K$  is infinite; otherwise, the result is trivial. We first prove  $\implies$ . Let us assume that  $E = K(x)$  for some  $x$ . We claim that the map

$$\begin{aligned}\Psi: \{F : K \subseteq F \subseteq E\} &\rightarrow \{g \in K[X] : g \text{ is a monic divisor of } f(x, K)\}, \\ F &\mapsto f(x, F),\end{aligned}$$

is injective. Take  $F_0$  such that  $K \subseteq F_0 \subseteq F \subseteq E$  and  $f(x, F) = f(x, F_0)$ . Then

$$[E : F_0] = [F_0(x) : F_0] = \deg f(x, F_0) = m = [F(x) : F] = [E : F]$$

and hence  $F = F_0$ .

In general, let  $F_1$  and  $F_2$  be such that  $K \subseteq F_1, F_2 \subseteq E$  and  $f(x, F_1) = f(x, F_2)$ . Let  $F_0 = F_1 \cap F_2$ . Then  $f = f(x, F_1) = f(x, F_2) \in F_0[X]$  and hence  $f(x, F_0) = f$ . Hence we can apply what we proved before to  $F_0 \subseteq F_1$  and  $F_0 \subseteq F_2$ , to obtain that  $F_1 = F_0 = F_2$ . It follows that  $\Psi$  is injective and hence there are finitely many fields between  $K$  and  $E$ .

Let us prove  $\iff$ . Let us assume that  $E = K(x, y)$ . For each  $a \in K$ , we consider the extension  $K(ay + x)/K$ . By assumption, there exist  $a, b \in K$  such that  $a \neq b$  and

$$K(x + ay) = K(x + by) = L.$$

We claim that  $L = E$ . Note that  $x + ay \in L$  and  $x + by \in L$ , so  $(a - b)y \in L$  and hence, since  $K \subseteq L$ , it follows that  $y \in L$ . Thus  $x \in L$  and therefore  $L = E$ .  $\square$

As a consequence, if  $E/K$  is finite and separable, then  $E/K$  admits finitely many subextensions.

**§ 7.1. Galois extensions.** Let  $E/K$  be an algebraic extension. Assume that  $E = K(S)$  and let  $C$  be an algebraic closure of  $K$  containing  $E$ . Let

$$T = \{y \in C : y \text{ is a root of } f(x, K) \text{ for } x \in S\}$$

and let  $L = K(T)$ . Then  $E \subseteq L$ , as  $S \subseteq T$ . The extension  $L/K$  is normal, as  $L/K$  is a splitting field of the family  $\{f(x, K) : x \in S\}$ . Moreover,  $L$  is the smallest normal extension of  $K$  containing  $E$ . The field  $L$  is the **normal closure** of  $E$  (with respect to  $C$ ).

7.2. EXERCISE. If  $E/K$  is finite, then  $L/K$  is finite

7.3. EXERCISE. If  $E/K$  is separable, then  $L/K$  is separable.

Let  $E/K$  be an extension and  $S \subseteq \text{Gal}(E/K)$  be a subset. The set

$${}^S E = \{x \in E : \sigma(x) = x \text{ for all } \sigma \in S\}$$

is a subfield of  $E$  that contains  $K$ . The subfield  ${}^S E$  is known as the **fixed field** of  $S$ .

7.4. DEFINITION. Let  $E/K$  be an algebraic extension and  $G = \text{Gal}(E/K)$ . Then  $E/K$  is a **Galois extension** if  ${}^G E = K$ .

Clearly,  $K/K$  is a Galois extension. Note that  $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$  is not a Galois extension. Why?

7.5. EXERCISE. Prove that  $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$  is a Galois extension.

7.6. EXERCISE. If the characteristic of  $K$  is different from two, then every quadratic extension of  $K$  is a Galois extension.

7.7. EXERCISE. Let  $E/K$  be an algebraic extension and  $G = \text{Gal}(E/K)$ . Let  $F = {}^G E$ . Prove that  $\text{Gal}(E/F) = G$  and hence  $E/F$  is a Galois extension.

7.8. PROPOSITION. *Let  $E/K$  be an algebraic extension. Then  $E/K$  is a Galois extension if and only if  $E/K$  is normal and separable.*

PROOF. Let  $G = \text{Gal}(E/K)$ . Let us first assume that  $E/K$  is Galois. For  $x \in E$  let

$$f_x = \prod_{y \in O_G(x)} (X - y) = \sum a_i X^i \in E[X].$$

If  $\varphi \in G$ , then

$$\overline{\varphi}(f_x) = \prod_{y \in O_G(x)} (X - \varphi(y)) = f_x,$$

as if  $O_G(x) = \{\sigma_1(x), \dots, \sigma_r(x)\}$ , then  $\varphi(\sigma_i(x)) = (\varphi\sigma_i)(x) = \sigma_j(x)$  for some  $j$ . Since

$$\sum a_i X^i = f_x = \overline{\varphi}(f_x) = \sum \varphi(a_i) X^i,$$

it follows that  $a_i \in {}^G E = K$  for all  $i$ . Thus  $f_x \in K[X]$  and  $E/K$  is a splitting field of the family  $\{f_x : x \in E\}$ . In particular,  $E/K$  is normal. Moreover,  $x$  is a simple root of  $f_x \in K[X]$  and hence  $x$  is separable over  $K$ .

Conversely, let  $x \in {}^G E$ . Since  $E/K$  is normal, then  $f(x, K) = \prod_{y \in O_G(x)} (X - y)^m$  for some  $m$ . Since  $E/K$  is separable,  $m = 1$ . Moreover  $x \in {}^G E$ , so  $O_G(x) = \{x\}$ . Thus  $f(x, K) = \prod_{y \in O_G(x)} (X - y) = X - x$  and  $x \in K$ .  $\square$

7.9. DEFINITION. Let  $K$  be a field and  $f \in K[X]$ . Then  $f$  is **separable** if all roots of  $f$  are simple (in some algebraic closure of  $K$ ).

7.10. PROPOSITION. *Let  $E/K$  be a finite extension. Then  $E/K$  is a Galois extension if and only if  $E$  is a splitting field over  $K$  of a separable polynomial  $f \in K[X]$ .*

PROOF. Let us assume first that  $E/K$  is a Galois extension. Since  $E/K$  is finite and separable,  $E = K(x)$  by Proposition 6.11. Then  $E/K$  is a splitting field of  $f(x, K)$  since  $E/K$  is normal. Since  $E/K$  is separable,  $x$  is separable over  $K$ . Thus  $x$  is a simple root of  $f(x, K)$  and hence  $f(x, K)$  is separable. Conversely, let  $x_1, \dots, x_r$  be the roots of a separable polynomial  $f \in K[X]$ . Then  $E = K(x_1, \dots, x_r)$  is separable and normal.  $\square$

In the previous case,  $\text{Gal}(E/K)$  is known as the **Galois group** of the polynomial  $f$ . The notation is  $\text{Gal}(f, K)$ . If  $n = \deg f$  and  $x_1, \dots, x_n$  are the roots of  $f$ , then any  $\varphi \in \text{Gal}(f, K)$  permutes the roots of  $f$ , that is  $\varphi$  permutes the set  $\{x_1, \dots, x_n\}$ . In particular,  $\text{Gal}(f, K)$  is isomorphic to a subgroup of  $\mathbb{S}_n$  and hence  $|\text{Gal}(f, K)|$  divides  $n!$ .

7.11. PROPOSITION. *Let  $E/K$  be a normal extension and  $F$  be the separable closure of  $K$  with respect to  $E$ . Then  $F/K$  is a Galois extension.*

PROOF. Let  $C/K$  be an algebraic closure such that  $E \subseteq C$ . Let  $\sigma \in \text{Hom}(F/K, C/K)$ , and let  $\varphi \in \text{Hom}(E/K, C/K)$  be such that  $\varphi|_F = \sigma$ . Since  $E/K$  is normal,  $\varphi(E) = E$ . Let  $x \in F$ . Then  $\sigma(x) = \varphi(x) \in E$ . Thus  $f(\sigma(x), K) = f(x, K)$  and  $\sigma(x)$  is separable over  $K$ , which implies that  $\sigma(x) \in F$ . Thus  $F/K$  is normal. Since  $F/K$  is separable, it follows that  $F/K$  is a Galois extension by Proposition 7.8.  $\square$

Some easy facts.

**7.12. EXERCISE.** Let  $E/K$  be a separable extension and  $L/K$  be the normal closure of  $E$  in some algebraic closure  $C$  that contains  $E$ . Prove that  $L/K$  is a Galois extension.

**7.13. EXERCISE.** Let  $E/K$  be a finite extension. Prove that  $E/K$  is Galois if and only if  $[E : K] = |\text{Gal}(E/K)|$ .

For the previous exercise, note that if  $E/K$  is a finite extension, then

$$|\text{Gal}(E/K)| \leq \gamma(E/K) \leq [E : K].$$

The first inequality is equality if and only if  $E/K$  is normal. The second inequality is equality if and only if  $E/K$  is separable.

**7.14. EXERCISE.** Let  $E/K$  be a Galois extension and  $F/K$  be a subextension of  $E/K$ . Prove that  $E/F$  is a Galois extension.

**7.15. THEOREM (Artin).** *Let  $E$  be a field and  $G$  be a finite group of automorphisms of  $E$ . If  $K = {}^G E$ , then  $E/K$  is a Galois extension,  $[E : K] = |G|$  and  $\text{Gal}(E/K) = G$ .*

Before proving the theorem, we need a lemma.

**7.16. LEMMA.** *Let  $E/K$  be a separable extension such that  $\deg f(x, K) \leq m$  for all  $x \in E$ . Then  $E/K$  is finite and  $[E : K] \leq m$ .*

PROOF. Let  $z \in E$  be of maximal degree. If  $x \in E$ , then  $K(x, z)/K$  is separable. There exists  $y$  such that  $K(x, z) = K(y)$ . Then

$$K(z) \subseteq K(x, z) = K(y).$$

Since  $\deg f(z, K) \leq \deg f(y, K)$ ,  $\deg f(z, K) = \deg f(y, K)$ . Hence  $K(y) = K(z)$ . In particular,  $x \in K(z)$  and therefore  $E = K(z)$ .  $\square$

Now we are ready to prove Artin's theorem:

**PROOF OF THEOREM 7.15.** Note that  $G \subseteq \text{Gal}(E/K)$ . Let  $x \in E$  and

$$f_x = \prod_{y \in O_G(x)} (X - y).$$

Since  $f_x \in K[X]$ , the extension  $E/K$  is normal and separable (as it is a decomposition field of a family of separable polynomials), so  $E/K$  is a Galois extension. Moreover,

$$\deg f(x, K) \leq \deg f_x = |O_G(x)| \leq |G|.$$

By the previous lemma,  $E/K$  is finite and  $[E : K] \leq |G|$ . This implies that

$$|\text{Gal}(E/K)| = [E : K] \leq |G|$$

and hence  $|\text{Gal}(E/K)| = |G|$ .  $\square$

7.17. EXAMPLE. Let  $E = K(X, Y)$  and  $\sigma: K[X, Y] \rightarrow E$  be the ring homomorphism given by  $\sigma(X) = Y$  and  $\sigma(Y) = X$ . Note that  $\sigma$  is bijective, as  $\sigma^2 = \text{id}$ . The map  $\sigma$  induces a field homomorphism  $\bar{\sigma}: E \rightarrow E$  such that  $\bar{\sigma}^2 = \text{id}$ . Recall that such a homomorphism is given by  $f/g \mapsto \sigma(f)/\sigma(g)$ . Let  $G = \langle \bar{\sigma} \rangle$ . Then  $|G| = 2$ . We claim that  ${}^G E = K(X+Y, XY)$ . Let  $F = K(X + Y, XY)$ . We only prove that  ${}^G E \subseteq F$ , as the other inclusion is trivial. Artin's theorem implies that  $[E : {}^G E] = 2$  and  $E = F(X)$ , as  $X$  is a root of the polynomial  $Z^2 - (X + Y)Z + XY$ . Then  $[E : F] \leq 2$  and  $[{}^G E : F] = 1$ .

## 8. Lecture – Week 8

**§ 8.1. Galois' correspondence.** A **partially order set** (or **poset**) is a pair  $(X, \leq)$ , where  $X$  is a non-empty set and  $\leq$  is a reflexive, antisymmetric and transitive relation on  $X$ . This means:

- 1)  $x \leq x$  for all  $x \in X$ ,
- 2)  $x \leq y$  and  $y \leq x$  imply  $x = y$ , and
- 3)  $x \leq y$  and  $y \leq z$  imply  $x \leq z$ .

Let  $(X, \leq)$  be a partially ordered set and  $x, y \in X$ . An element  $z$  of a poset  $(X, \leq)$  is an **upper bound** of  $x$  and  $y$  if  $x \leq z$  and  $y \leq z$ . And  $\xi$  is a **least upper bound** of  $x$  and  $y$  if it is an upper bound with  $\xi \leq z$  for every upper bound  $z$  of  $x$  and  $y$ . Similarly, one defines lower bounds and greatest lower bounds.

8.1. **DEFINITION.** A **lattice** is a partially ordered set  $\mathcal{L}$  in which each pair of elements  $x, y \in \mathcal{L}$  has a least upper bound  $x \vee y$  and a greatest lower bound  $x \wedge y$ .

The basic example is the following. Let  $X$  be a set and  $\mathcal{P}(X)$  be the collection of all subsets of  $X$ . The relation  $A \leq B \iff A \subseteq B$  turns  $\mathcal{P}(X)$  into a lattice with  $A \vee B = A \cup B$  and  $A \wedge B = A \cap B$ .

8.2. **EXAMPLE.** Let  $G$  be a group and  $\mathcal{L}(G)$  be the collection of subgroups of  $G$ . The relation

$$H \leq K \iff H \subseteq K$$

turns  $\mathcal{L}(G)$  into a lattice with  $H \vee K = \langle H, K \rangle$  and  $H \wedge K = H \cap K$ .

8.3. **EXAMPLE.** Let  $E/K$  be a field extension and  $\mathcal{L}(E/K)$  be the collection of intermediate fields. The relation

$$F \leq L \iff F \subseteq L$$

turns  $\mathcal{L}(E/K)$  into a lattice with  $F \vee L = FL$  and  $F \wedge L = F \cap L$ .

8.4. **EXAMPLE.** Let  $n \in \mathbb{Z}$  be a positive integer and  $\mathcal{D}(n)$  be the collection of positive integer devisors  $d \in \mathbb{Z}$  of  $n$ . The relation

$$l \leq m \iff l \mid m$$

turns  $\mathcal{D}(n)$  into a lattice with  $l \vee m = \text{lcm}(l, m)$  and  $l \wedge m = \text{gcd}(l, m)$ .

A map  $f: \mathcal{L}_1 \rightarrow \mathcal{L}_2$  between two lattices is said to be **order-reversing** if  $x \leq_1 y$  implies  $f(y) \leq_2 f(x)$ .

We shall need an exercise.

8.5. **EXERCISE.** Let  $\mathcal{L}_1$  and  $\mathcal{L}_2$  be lattices and  $f: \mathcal{L}_1 \rightarrow \mathcal{L}_2$  be a bijection such that  $f$  and its inverse are both order reversing. Then

$$f(x \vee y) = f(x) \wedge f(y), \quad f(x \wedge y) = f(x) \vee f(y)$$

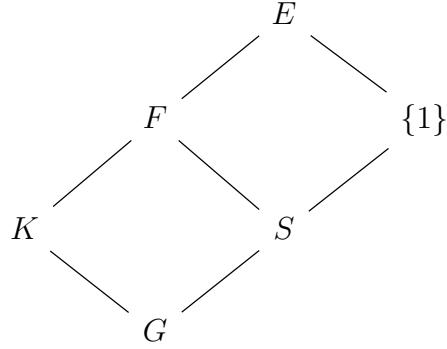
for all  $x, y \in \mathcal{L}_1$ .

8.6. **THEOREM (Galois).** Let  $E/K$  be a finite Galois extension and  $G = \text{Gal}(E/K)$ . There exists a bijective correspondence

$$\mathcal{L}(E/K) = \{F : K \subseteq F \subseteq E \text{ subfields}\} \leftrightarrow \{S : S \text{ is a subgroup of } G\} = \mathcal{L}(G).$$

The correspondence is given by  $\alpha: F \mapsto \text{Gal}(E/F)$  and  $\beta: {}^S E \leftrightarrow S$ . Moreover, the following conditions hold:

- 1)  $\alpha$  and  $\beta$  are order-reversing bijections.
- 2)  $[F : K] = (G : \text{Gal}(E/F))$  and  $(G : S) = [{}^S E : K]$ .
- 3)  $F/K$  is a Galois extension if and only if  $\text{Gal}(E/F)$  is a normal subgroup of  $G$ .



PROOF. Let

$$\begin{aligned} \alpha: \mathcal{L}(E/K) &\rightarrow \mathcal{L}(G), & \alpha(F) &= \text{Gal}(E/F), \\ \beta: \mathcal{L}(G) &\rightarrow \mathcal{L}(E/K), & \beta(S) &= {}^S E. \end{aligned}$$

A routine exercise shows that  $\alpha$  and  $\beta$  are well-defined. We first note that

$$\beta(\alpha(F)) = \beta(\text{Gal}(E/F)) = {}^{\text{Gal}(E/F)} E = F$$

since  $E/F$  is a Galois extension. Moreover,

$$\alpha(\beta(S)) = \alpha({}^S E) = \text{Gal}(E/{}^S E) = S$$

by Artin's theorem, as  $S$  is finite.

It is straightforward to check that  $\alpha$  and  $\beta$  are order-reversing bijections.

Let  $F$  be a subfield of  $E$  containing  $K$  and  $S = \alpha(F)$ . Then

$$[F : K] = \frac{[E : K]}{[E : F]} = \frac{|G|}{|S|} = (G : S).$$

Let  $C$  be an algebraic closure of  $K$  that contains  $E$ . If  $S = \text{Gal}(E/F)$ , then  $F = {}^S E$ .

We need to prove that  $F/K$  is normal if and only if  $S$  is normal in  $G$ . Let us first prove  $\Rightarrow$ . Let  $\tau \in S$  and  $\sigma \in G$ . Since  $F/K$  is normal,  $\sigma|_F \in \text{Aut}(F)$ . Thus  $\sigma^{-1}(F) = F$ . In particular, if  $x \in F$ , then  $\sigma^{-1}(x) \in F$  and

$$\sigma\tau\sigma^{-1}(x) = \sigma\sigma^{-1}(x) = x.$$

Conversely, let  $\varphi \in \text{Hom}(F/K, C/K)$ . There exists  $\Phi: E \rightarrow C$  such that  $\Phi|_F = \varphi$ . Since  $E/K$  is normal,  $\Phi(E) = E$  and hence  $\Phi \in G$ . We claim that  $\varphi(x) \in F$  for all  $x \in F$ . Note that  $F = {}^S E$ , so

$$\tau\varphi(x) = \tau\Phi(x) = \Phi\Phi^{-1}\tau\Phi(x) = \Phi(x) = \varphi(x)$$

for all  $\tau \in S$ , as  $\Phi^{-1}\tau\Phi \in S$ . This means that  $\varphi(x) \in {}^S E = F$ .

Let us compute  $\text{Gal}(F/K)$ . Since  $F/K$  is normal, the map  $\lambda: G \rightarrow \text{Gal}(F/K)$ ,  $\sigma \mapsto \sigma|_F$ , is a surjective group homomorphism such that  $\ker \lambda = S$ . The first isomorphism theorem implies that  $\text{Gal}(F/K) \simeq G/S$ .  $\square$

Some easy consequences.

8.7. EXERCISE. If  $E/K$  is a Galois extension of degree  $n$  and  $p$  is a prime number dividing  $n$ , then  $E/K$  admits a subextension of degree  $n/p$ .

8.8. EXERCISE. If  $E/K$  is a Galois extension of degree  $p^\alpha m$  with  $p$  a prime number coprime with  $m$ , then  $E/K$  admits a subextension of degree  $m$ .

8.9. DEFINITION. An extension  $E/K$  is **abelian** if  $E/K$  is a Galois extension with  $\text{Gal}(E/K)$  abelian.

8.10. EXERCISE. If  $E/K$  is an abelian extension of degree  $n$  and  $d$  divides  $n$ , then  $E/K$  admits a subextension of degree  $d$ .

8.11. DEFINITION. An extension  $E/K$  is **cyclic** if  $E/K$  is a Galois extension with  $\text{Gal}(E/K)$  cyclic.

8.12. EXAMPLE. The extension  $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$  admits exactly three non-trivial subextensions:

$$\mathbb{Q}(\sqrt{2})/\mathbb{Q}, \quad \mathbb{Q}(\sqrt{3})/\mathbb{Q}, \quad \mathbb{Q}(\sqrt{6})/\mathbb{Q},$$

as  $\text{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}) \simeq C_2 \times C_2$ .

8.13. EXAMPLE. Let  $\omega \in \mathbb{C} \setminus \{1\}$  be such that  $\omega^5 = 1$ . Then

$$f(\omega, \mathbb{Q}) = 1 + X + X^2 + X^3 + X^4$$

and  $\mathbb{Q}(\omega)/\mathbb{Q}$  has degree four. Moreover,  $\mathbb{Q}(\omega)/\mathbb{Q}$  is a Galois extension and

$$\text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) \simeq C_4.$$

If  $\sigma \in \text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$ , then  $\sigma(\omega) = \omega^i$  for some  $i \in \{1, \dots, 4\}$ . Moreover, each map  $\omega \mapsto \omega^i$ , for  $i \in \{1, \dots, 4\}$ , induces an automorphism of  $\mathbb{Q}(\omega)/\mathbb{Q}$ . Thus  $|\text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})| = 4$ . Now

$$\sigma_i^k = \text{id} \iff \omega^{i^k} = \sigma_i^k(\omega) = \omega \iff i^k \equiv 1 \pmod{5}.$$

Thus the map  $\sigma_2$  given by  $\omega \mapsto \omega^2$  has order four.

Since  $\text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) = \langle \sigma \rangle$ , where  $\sigma(\omega) = \omega^2$ , is cyclic of order four, the extension  $\mathbb{Q}(\omega)/\mathbb{Q}$  has a unique degree-two subextension  $F/\mathbb{Q}$ . Note that  $|\langle \sigma^2 \rangle| = 2$  and  $\sigma^2(\omega) = \omega^4 = \omega^{-1}$ . Thus  $F = \langle \sigma^2 \rangle \mathbb{Q}(\omega)$ . Let  $\theta = \omega + \omega^{-1}$ . Then

$$\theta^2 = \omega^2 + \omega^3 + 2 = -(1 + \omega + \omega^{-1}) + 2 = 1 - \theta$$

and hence  $\theta$  is a root of  $X^2 + X - 1$ . It follows that

$$\theta \in \{(-1 + \sqrt{5})/2, (-1 - \sqrt{5})/2\}.$$

Therefore  $F = \mathbb{Q}(\sqrt{5})$ .

Let us mention some other consequences (of the fact that the correspondence depends on order-reversing bijections).

8.14. EXERCISE. Let  $E/K$  be a finite Galois extension and  $G = \text{Gal}(E/K)$ . If  $S$  and  $T$  are subgroups of  $G$ , then  ${}^{\langle S, T \rangle} E = {}^S E \cap {}^T E$  and  ${}^{S \cap T} E = {}^S E {}^T E$ .

8.15. EXERCISE. Let  $E/K$  be a finite Galois extension and  $F, L \in L(E/K)$ . Prove that  $\text{Gal}(E/FL) = \text{Gal}(E/F) \cap \text{Gal}(E/L)$  and  $\text{Gal}(E/F \cap L) = \langle \text{Gal}(E/F), \text{Gal}(E/L) \rangle$ .

8.16. EXERCISE. Let  $E/K$  be a finite Galois extension and  $G = \text{Gal}(E/K)$ . Assume that  $G$  is the direct product  $G = S \times T$  of the groups  $S$  and  $T$ . Let  $F = {}^S E$  and  $L = {}^T E$ . Then  $F \cap L = K$  and  $FL = E$ .

8.17. PROPOSITION. Let  $E_1/K, E_2/K$  be Galois extensions. If  $E = E_1 E_2$ , then  $E/K$  is a Galois extension. If, moreover,  $E_1/K$  and  $E_2/K$  are finite, then

$$\theta: \text{Gal}(E/K) \rightarrow \text{Gal}(E_1/K) \times \text{Gal}(E_2/K), \quad \sigma \mapsto (\sigma|_{E_1}, \sigma|_{E_2}),$$

is an injective group homomorphism.

PROOF. Since  $E_1/K$  is algebraic, then  $E_1 E_2/E_2$  is algebraic. Since  $E_2/K$  is algebraic,  $E_1 E_2/K$  is algebraic. Similarly,  $E_1 E_2/K$  is separable.

Let  $C/K$  be an algebraic closure such that  $E_1 E_2 \subseteq C$ . If  $\sigma \in \text{Hom}(E_1 E_2/K, C/K)$ , then  $\sigma(E_1 E_2) \subseteq \sigma(E_1)\sigma(E_2) = E_1 E_2$  (do this calculation as an exercise using the fact that  $E_1/K$  and  $E_2/K$  are normal extensions). Thus  $E_1 E_2/K$  is normal.

If both  $E_1/K$  and  $E_2/K$  are finite, then  $E_1 E_2/K$  is finite.

Then  $\theta$  is a group homomorphism. We claim that the map  $\theta$  is injective. Let  $\sigma \in \ker \theta$ . Then  $\sigma|_{E_i} = \text{id}_{E_i}$  for all  $i \in \{1, 2\}$ . Let  $S = \langle \sigma \rangle \subseteq \text{Gal}(E/K)$  and  $F = {}^S E$ . Then  $E_i \subseteq F$  for all  $i \in \{1, 2\}$  and hence  $E \subseteq F$ . It follows that  $F = E = \{\text{id}\}$  and therefore  $S = \{\text{id}\}$ , so  $\sigma = \text{id}$ .  $\square$

8.18. EXERCISE. Let  $E_1/K, \dots, E_r/K$  be finite Galois extensions such that for each  $j$  one has  $E_j \cap (E_1 \cdots E_{j-1} E_{j+1} \cdots E_r) = K$ . Then

$$\text{Gal}(E/K) \simeq \text{Gal}(E_1/K) \times \cdots \times \text{Gal}(E_r/K).$$

In this case,  $[E : K] = \prod_{i=1}^r [E_i : K]$ .

**§ 8.2. The fundamental theorem of algebra.** We now present an easy proof of the fundamental theorem of algebra based on the ideas of Galois Theory. We need the following well-known facts:

- 1) Every real polynomial of odd degree admits a real root. This means that  $\mathbb{R}$  does not admit extension of odd degree  $> 1$ .
- 2) Every complex number admits a square root in  $\mathbb{C}$ . This means that  $\mathbb{C}$  does not admit degree-two extensions.

8.19. THEOREM. *The field  $\mathbb{C}$  is algebraically closed.*

PROOF. Let  $E/\mathbb{C}$  be an algebraic finite extension. Then  $E/\mathbb{R}$  is finite separable extension of even degree. There exists a Galois extension  $L/\mathbb{R}$  such that  $E \subseteq L$ , so  $[L : \mathbb{R}]$  is even. Let  $G = \text{Gal}(L/\mathbb{R})$ . Then  $|G| = 2^m s$  for some odd number  $s$ . If  $T$  is a 2-Sylow subgroup of  $G$ , then there exists a subextension  $F/\mathbb{R}$  of degree  $s$ . Since  $\mathbb{R}$  does not admit extensions of odd degree  $> 1$ ,  $s = 1$  and hence  $G$  is a 2-group. Since  $L/\mathbb{R}$  is a Galois extension,  $L/\mathbb{C}$  is a Galois extension. In particular,  $|\text{Gal}(L/\mathbb{C})| = 2^{m-1}$ . If  $m > 1$ , let  $U$  be a subgroup of  $\text{Gal}(L/\mathbb{C})$  of order  $2^{m-2}$ . Then  $U$  corresponds to a subextension  $L_1/\mathbb{C}$  of degree two, a contradiction. Hence  $m = 1$  and  $[L : \mathbb{C}] = 1$ , so  $L = \mathbb{C}$  and  $E = \mathbb{C}$ .  $\square$

**§ 8.3. Purely inseparable extensions.** Let  $E/K$  be an algebraic extension. In page 6.1 we defined the **separable closure** of  $K$  with respect to  $E$  as the field

$$F = \{x \in E : x \text{ is separable over } K\}.$$

Note that  $K \subseteq F \subseteq E$  and  $F = K(F)$ . Moreover,  $F/K$  is separable and  $E/F$  is a **purely inseparable** extension, meaning that for every  $x \in E \setminus F$ , the polynomial  $f(x, F)$  is not separable.

The number  $[E : K]_{\text{ins}} = [E : F]$  is known as the **degree of inseparability** of  $E/K$ . Clearly,  $E/K$  is separable if and only if  $[E : K]_{\text{ins}} = 1$  and  $E/K$  is purely inseparable if and only if  $[E : K]_{\text{ins}} = [E : K]$ .

8.20. EXERCISE. Let  $K$  be a field of characteristic  $p > 0$  and  $f \in K[X]$  be irreducible. If  $f$  is not separable, then  $f = g(X^p)$  for some  $g \in K[X]$ .

8.21. PROPOSITION. Let  $K$  be a field of characteristic  $p > 0$  and  $E/K$  be an algebraic extension. The following statements are equivalent:

- 1)  $E/K$  is purely inseparable.
- 2) If  $x \in E$ , then  $x^{p^m} \in K$  for some  $m \geq 0$ .
- 3) If  $x \in E$ , then  $f(x, K) = X^{p^m} - a$  for some  $a \in K$  and  $m \geq 0$ .
- 4)  $\gamma(E/K) = 1$ .

PROOF. We first prove 1)  $\implies$  2). Let  $x \in E$  and  $f = f(x, K)$ . Assume  $x$  is not separable. Then  $f(x) = 0$  and  $f'(x) = 0$ , as  $x$  is not a simple root. By Exercise 8.20,  $f = g(X^p)$  for some  $g \in K[X]$ . We now proceed by induction on the degree of  $x$ . The result is true for elements of degree one. So assume the result holds for the element of degree  $\leq n$  for some  $n \geq 1$ . If  $x \in E$  is such that  $\deg f(x, K) = n + 1$ , then, since  $f(x, K) = g(X^p)$ , the element  $x^p$  has degree  $\leq n$ . By the inductive hypothesis,  $x^{p^{n+1}} = (x^p)^{p^m} \in K$ .

We now prove 2)  $\implies$  3). Let  $x \in E$  and  $m$  be the minimal positive integer such that  $x^{p^m} \in K$ . Then  $x$  is a root of  $X^{p^m} - x^{p^m} \in K[X]$ . Since  $X^{p^m} - x^{p^m} = (X - x)^{p^m}$ , it follows that

$$f(x, K) = (X - x)^r = X^r + \cdots + (-1)^r x^r$$

for some  $r \in \{1, \dots, p^m\}$ . Write  $r = p^s t$  for some integer  $t$  coprime with  $p$  and  $s$  such that  $0 \leq s \leq m$ . Let  $a, b \in \mathbb{Z}$  be such that  $ar + bp^m = p^s$ . Then

$$x^{p^s} = x^{ar+bp^m} = (x^r)^a (x^{p^m})^b \in K.$$

The minimality of  $m$  implies that  $s \geq m$  and hence  $s = m$ . Now  $p^m t = p^s t = r \leq p^m$ , so  $t = 1$ . This means  $f(x, K) = X^{p^m} - x^{p^m}$ .

We now prove 3)  $\implies$  4). Let  $C/K$  be an algebraic closure of  $K$  containing  $E$  and  $x \in E$ . Let  $\sigma \in \text{Hom}(E/K, C/K)$ . We claim that  $\sigma(x) = x$ . Since  $f(x, K) = X^{p^m} - a$ ,

$$(\sigma(x))^{p^m} = \sigma(X^{p^m}) = \sigma(a) = a = x^{p^m}.$$

It follows that  $\sigma(x)$  is a root of  $X^{p^m} - x^{p^m} = (X - x)^{p^m}$ . Thus  $\sigma(x) = x$ .

Finally, we prove that 4)  $\implies$  1). Let  $C$  be an algebraic closure of  $K$  containing  $E$ . Then  $\text{Gal}(E/K) \subseteq \text{Hom}(E/K, C/K) = \{\text{id}\}$ , as  $\gamma(E/K) = 1$ . If  $x \in E$  is separable over  $K$ , then

$$f(x, K) = \prod_{y \in O_{\text{Gal}(E/K)}(x)} (X - y) = X - x \in K[X].$$

Thus  $x \in K$  and hence  $E/K$  is purely inseparable.  $\square$

Some consequences:

8.22. EXERCISE. Let  $K$  be a field of characteristic  $p > 0$  and  $E/K$  be finite and purely inseparable. Then  $[E : K] = p^s$  for some prime number  $p$  and some integer  $s$ . Moreover,  $x^{[E:K]} \in K$ .

For the first part of the previous exercise, write  $E = K(x_1, \dots, x_n)$  and proceed by induction on  $n$ .

8.23. EXERCISE. Let  $K$  be of characteristic  $p > 0$  and  $E/K$  be a finite extension such that  $[E : K]$  is not divisible by  $p$ . Then  $E/K$  is separable.

Let  $K$  be of characteristic  $p > 0$ ,  $E/K$  be finite and  $F$  be the separable closure of  $K$  in  $E$ . Since

$$\gamma(E/K) = \gamma(E/F)\gamma(F/K) = \gamma(F/K),$$

it follows that

$$[E : K] = [E : F]\gamma(E/K) = [E : K]_{\text{ins}}\gamma(E/K).$$

## 9. Lecture – Week 9

### § 9.1. Norm and trace.

9.1. DEFINITION. Let  $E/K$  be a finite extension and  $C/K$  be an algebraic closure that contains  $E$ . Let  $A = \text{Hom}(E/K, C/K)$ . For  $x \in E$  we define the **trace** of  $x$  in  $E/K$  as

$$\text{trace}_{E/K}(x) = [E : K]_{\text{ins}} \sum_{\varphi \in A} \varphi(x)$$

and the **norm** of  $x$  in  $E/K$  as

$$\text{norm}_{E/K}(x) = \left( \prod_{\varphi \in A} \varphi(x) \right)^{[E : K]_{\text{ins}}}.$$

As an optional exercise, one can show that these definitions do not depend on the algebraic closure.

We collect some basic properties as an exercise:

9.2. EXERCISE. Let  $E/K$  be a finite extension. The following statements hold:

- 1) If  $E/K$  is not separable, then  $\text{trace}_{E/K}(x) = 0$  for all  $x \in E$ .
- 2) If  $x \in K$ , then  $\text{trace}_{E/K}(x) = [E : K]x$ .
- 3)  $\text{trace}_{E/K}(x) \in K$  for all  $x \in E$ .
- 4)  $\text{norm}_{E/K}(x) = 0$  if and only if  $x = 0$ .
- 5) If  $x \in K$ , then  $\text{norm}_{E/K}(x) = x^{[E : K]}$ .
- 6)  $\text{norm}_{E/K}(x) \in K$  for all  $x \in E$ .

One proves, moreover, that  $\text{trace}_{E/K}: E \rightarrow K$  satisfies

$$\text{trace}_{E/K}(x + \lambda y) = \text{trace}_{E/K}(x) + \lambda \text{trace}_{E/K}(y)$$

for all  $x, y \in E$  and  $\lambda \in K$ , that is to say that  $\text{trace}_{E/K}: E \rightarrow K$  is a linear form in  $E$ . The norm  $\text{norm}_{E/K}: E^\times \rightarrow K^\times$  is a group homomorphism.

9.3. EXERCISE. Let  $E/K$  be a finite extension and  $x \in E$  be algebraic. If

$$f(x, K) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0,$$

then  $\text{norm}_{E/K}(x) = ((-1)^n a_0)^{[E : K(x)]}$  and  $\text{trace}_{E/K}(x) = -[E : K(x)]a_{n-1}$ .

9.4. EXAMPLE. Let  $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ . Then

$$\begin{aligned} \text{trace}_{E/\mathbb{Q}}(\sqrt{2}) &= 0, & \text{norm}_{E/\mathbb{Q}}(\sqrt{2}) &= 4, \\ \text{trace}_{E/\mathbb{Q}(\sqrt{2})}(\sqrt{2}) &= 2\sqrt{2}, & \text{norm}_{E/\mathbb{Q}(\sqrt{2})}(\sqrt{2}) &= 2. \end{aligned}$$

9.5. EXAMPLE. If  $E/K$  is a finite Galois extension, then

$$\text{trace}_{E/K}(x) = \sum_{\sigma \in \text{Gal}(E/K)} \sigma(x) \quad \text{and} \quad \text{norm}_{E/K}(x) = \prod_{\sigma \in \text{Gal}(E/K)} \sigma(x)$$

for all  $x \in E$ . In particular, since  $E = K(y)$  for some  $y$  by Proposition 6.11,

$$\text{trace}_{E/K}(y) = -a_{n-1} \quad \text{and} \quad \text{norm}_{E/K}(y) = (-1)^n a_0,$$

where  $f(y, K) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$ .

### § 9.2. Finite fields.

In this section,  $p$  will be a prime number.

**9.6. PROPOSITION.** *Let  $m$  be a positive integer. Up to isomorphism, there exists a unique field  $F_{p^m}$  of size  $p^m$ .*

**PROOF.** Let  $C$  be an algebraic closure of the field  $\mathbb{Z}/p$  and let  $F_{p^m} = \{x \in C : x^{p^m} = x\}$  be the set of roots of  $X^{p^m} - X$ . Since the polynomial  $X^{p^m} - X$  has no multiple roots,  $|F_{p^m}| = p^m$ . Moreover,  $F_{p^m}$  is the unique subfield of  $C$  of size  $p^m$ .

To prove the uniqueness, it is enough to note that if  $K$  is a field of  $p^m$  elements, then  $K$  is the splitting field of  $X^{p^m} - X$  over  $\mathbb{Z}/p$ .  $\square$

Let  $K = \mathbb{Z}/p$  and  $C$  be an algebraic closure of  $K$ . We claim that  $C = \bigcup_k F_{p^k}$ . If  $x \in C$ , then  $x$  is algebraic over  $K$ . Since  $K(x)/K$  is finite,  $K(x)$  is a finite field, say  $|K(x)| = p^r$  for some  $r$ . Then  $x^{p^r} = x$  and hence  $x \in F_{p^r}$ .

**9.7. EXERCISE.** Prove the following statements:

- 1) If  $x \in F_{p^r}$ , then  $x^{p^{rk}} = x$  for all  $k \geq 0$ .
- 2)  $F_{p^m} \subseteq F_{p^n}$  if and only if  $m \mid n$ .
- 3)  $F_{p^m} \cap F_{p^n} = F_{p^{\gcd(m,n)}}$ .

**9.8. PROPOSITION.** *Every finite extension of a finite field is cyclic.*

**PROOF.** Let  $K = \mathbb{Z}/p$ . It is enough to show that  $F_{p^n}/F_{p^m}$  is cyclic if  $m$  divides  $n$ .

We first prove that  $F_{p^n}/K$  is cyclic. Let

$$\sigma : F_{p^n} \rightarrow F_{p^n}, \quad x \mapsto x^p.$$

Then  $\sigma \in \text{Gal}(F_{p^n}/K)$  (it is bijective because all field homomorphisms are injective and  $F_{p^n}$  is finite).

Note that  $F_{p^n}/K$  is a Galois extension, as  $F_{p^n}$  is the splitting field over  $K$  of the separable polynomial  $X^{p^n} - X \in K[X]$ . Thus  $|\text{Gal}(F_{p^n}/K)| = [F_{p^n} : K] = n$ .

We claim that  $\sigma$  generated  $\text{Gal}(F_{p^n}/K)$ . Since  $\sigma^i(x) = x^{p^i}$  for all  $i \geq 0$ , in particular,

$$\sigma^n(x) = x^{p^n} = x.$$

Thus  $\sigma^n = \text{id}$  and hence  $|\sigma|$  divides  $n$ . Let  $s = |\sigma|$ . We know that  $F_{p^n}^\times = F_{p^n} \setminus \{0\}$  is cyclic, say  $F_{p^n}^\times = \langle g \rangle$ . Since  $|g| = p^n - 1$ ,

$$g = \sigma^s(g) = g^{p^s}$$

and hence  $p^s \equiv 1 \pmod{p^n - 1}$ . Thus  $p^n - 1$  divides  $p^s - 1$  and hence  $n$  divides  $s$ . Therefore  $n = s$  and  $\text{Gal}(F_{p^n}/K) = \langle \sigma \rangle$ .

For the general case, note that if  $m$  divides  $n$ , then the Galois group  $\text{Gal}(F_{p^n}/F_{p^m})$  is a subgroup of  $\text{Gal}(F_{p^n}/K)$ . Since  $\text{Gal}(F_{p^n}/K)$  is cyclic, the claim follows.  $\square$

If  $K = \mathbb{Z}/p$  and  $m$  divides  $n$ , the subextension  $F_{p^m}$  corresponds to the unique subgroup of index  $m$  of  $\text{Gal}(F_{p^n}/K) = \langle \sigma \rangle$ . This subgroup is  $\langle \sigma^m \rangle$ , where

$$\sigma^m(x) = x^{p^m} = x^{|F_{p^m}|}.$$

Note that  $\text{Gal}(F_{p^n}/F_{p^m}) = \langle \sigma^m \rangle$ . The map  $\sigma^m$  is known as the **Frobenius automorphism**.

**9.9. EXERCISE.** Let  $E/K$  be an extension of finite fields. Then  $E/K$  is cyclic. Moreover,  $\text{Gal}(E/K) = \langle \tau \rangle$ , where  $\tau(x) = x^{|K|}$ .

**§ 9.3. Cyclotomic extensions.** For  $n \geq 1$  let  $G_n(K) = \{x \in K : x^n = 1\}$  be the set of  $n$ -roots of one in  $K$ . Note that  $G_n(K)$  is a cyclic subgroup of  $K^\times$  and that  $|G_n(K)|$  divides  $n$ .

9.10. EXAMPLE.  $G_n(\mathbb{R}) = \{-1, 1\}$  if  $n$  is odd and  $G_n(\mathbb{R}) = \{1\}$  if  $n$  is even.

9.11. EXERCISE. Let  $K$  be a field of characteristic  $p > 0$ . Let  $n = p^s m$  for some  $m$  not divisible by  $p$ . Then  $G_n(K) = G_m(K)$ .

9.12. EXERCISE. Let  $q$  be a prime number. Then  $G_n(\mathbb{Z}/q) \simeq \mathbb{Z}/\gcd(n, q-1)$ .

Similarly, one can prove that if  $K$  is a finite field, then  $G_n(K)$  is a cyclic group of order  $\gcd(n, |K^\times|)$ .

9.13. EXAMPLE. If  $C$  is algebraically closed of characteristic coprime with  $n$ , then  $G_n(C)$  is cyclic of order  $n$ , as  $X^n - 1$  has all its roots in  $C$  and does not contain multiple roots.

Let  $K$  be an algebraically closed field and  $n$  be such that  $n$  is coprime with the characteristic of  $K$ . The set of **primitive  $n$ -roots** is defined as

$$H_n(K) = \{x \in G_n(K) : |x| = n\}.$$

9.14. DEFINITION. Let  $K$  be an algebraically closed field and  $n$  be such that  $n$  is coprime with the characteristic of  $K$ . The  **$n$ -th cyclotomic polynomial** is defined as

$$\Phi_n = \prod_{x \in H_n(K)} (X - x) \in K[X].$$

For  $n \geq 1$  the Euler's function is defined as

$$\varphi(n) = |\{k : 1 \leq k \leq n, \gcd(k, n) = 1\}|.$$

For example,  $\varphi(4) = 2$ ,  $\varphi(8) = \varphi(10) = 4$  and  $\varphi(p) = p - 1$  for every prime  $p$ .

9.15. PROPOSITION. Let  $K$  be an algebraically closed field and  $n$  be such that  $n$  is coprime with the characteristic of  $K$ . Let  $A$  be the prime subring of  $K$ .

- 1)  $\deg \Phi_n = \varphi(n)$ .
- 2)  $\Phi_n \in A[X]$ .

PROOF. The first statement is clear. Let us prove 2) by induction on  $n$ . The case  $n = 1$  is trivial, as  $\Phi_1 = X - 1$ . Assume that  $\Phi_d \in A[X]$  for all  $d$  such that  $d < n$ . In particular,

$$\gamma = \prod_{\substack{d|n \\ d \neq n}} \Phi_d \in A[X].$$

Since  $\gamma$  is monic, it follows that  $\frac{X^n - 1}{\gamma} \in A[X]$ . Now the claim follows from

$$X^n - 1 = \prod_{d|n} \Phi_d = \Phi_n \prod_{\substack{d|n \\ d \neq n}} \Phi_d = \Phi_n \gamma. \quad \square$$

By taking degree in the equality  $X^n - 1 = \prod_{d|n} \Phi_d$  one gets

$$n = \sum_{d|n} \varphi(d).$$

9.16. DEFINITION. Let  $n \geq 2$  and  $K$  be a field of characteristic coprime with  $n$ . A **cyclotomic extension** of  $K$  of index  $n$  is a splitting field of  $X^n - 1$  over  $K$ .

Let  $C$  be an algebraic closure of  $K$  and  $n \geq 2$  be coprime with the characteristic of  $K$ . It follows from Definition 9.16 that a cyclotomic extension of  $K$  of index  $n$  is of the form  $K(\omega)/K$  for some  $\omega \in H_n(C)$ .

9.17. PROPOSITION. *A cyclotomic extension of index  $n$  is abelian and of degree a divisor of  $\varphi(n)$ .*

PROOF. Let  $C$  be an algebraic closure of  $K$  and  $n \geq 2$  be coprime with the characteristic of  $K$ . Let  $\omega \in H_n(C)$  and  $K(\omega)/K$  be a cyclotomic extension. Then  $K(\omega)/K$  is a Galois extension, as it is a splitting field of a separable polynomial. Let  $U = \mathcal{U}(\mathbb{Z}/n)$  be the group of units of  $\mathbb{Z}/n$  and

$$\lambda: \text{Gal}(K(\omega)/K) \rightarrow U, \quad \sigma \mapsto m_\sigma,$$

where  $m_\sigma$  is such that  $\sigma(\omega) = \omega^{m_\sigma}$ . The map  $\lambda$  is well-defined and it is a group homomorphism, as if  $\sigma, \tau \in \text{Gal}(K(\omega)/K)$ , then, since

$$(\tau\sigma)(\omega) = \tau(\sigma(\omega)) = \tau(\omega^{m_\sigma}) = (\omega^{m_\sigma})^{m_\tau} = \omega^{m_\sigma m_\tau},$$

it follows that  $\lambda(\sigma)\lambda(\tau) = \lambda(\sigma\tau)$ . Since  $\lambda$  is injective,  $\text{Gal}(K(\omega)/K)$  is isomorphic to a subgroup of the abelian group  $U$ . Hence  $\text{Gal}(K(\omega)/K)$  is abelian. Moreover,

$$[K(\omega) : K] = |\text{Gal}(K(\omega)/K)|$$

is a divisor of  $|U| = \varphi(n)$ . □

9.18. EXERCISE. Prove that a cyclotomic extension  $K(\omega)/K$  has degree  $\varphi(n)$  if and only if  $\Phi_n$  is irreducible over  $K$ .

Note that  $\Phi_n$  is irreducible over  $\mathbb{Q}$ . Some concrete examples:

$$\Phi_1 = X - 1, \quad \Phi_2 = X + 1, \quad \Phi_3 = X^2 + X + 1, \quad \Phi_6 = X^2 - X + 1.$$

If  $p$  is a prime number, then  $\Phi_p = X^{p-1} + \cdots + X + 1$ .

9.19. EXAMPLE.  $\Phi_5$  is irreducible over  $\mathbb{Z}/2$ . First note that  $\Phi_5 = X^4 + \cdots + X + 1$  does not have roots in  $\mathbb{Z}/2$ . If  $\Phi_5$  is reducible, then, since  $X^2 + X + 1$  is the unique degree-two monic irreducible polynomial over  $\mathbb{Z}/2$ , it follows that

$$\Phi_5 = (X^2 + X + 1)(X^2 + X + 1) = (X^2 + X + 1)^2 = X^4 + X^2 + 1,$$

a contradiction.

9.20. EXERCISE. Prove that  $\Phi_{12} = X^4 - X^2 + 1$  is not irreducible over  $\mathbb{Z}/5$ .

#### § 9.4. Hilbert's theorem 90.

9.21. THEOREM (Hilbert). *Let  $E/K$  be a cyclic extension. Assume that  $\text{Gal}(E/K)$  is generated by  $\tau$ . For  $a \in E$ ,  $\text{norm}_{E/K}(a) = 1$  if and only if  $a = b/\tau(b)$  for some  $b \in E \setminus \{0\}$ .*

PROOF. Let  $n = |\text{Gal}(E/K)|$ . We first prove  $\Leftarrow$ . If  $a = b/\tau(b)$  and  $b \neq 0$ , then

$$\text{norm}_{E/K}(a) = a\tau(a)\tau^2(a)\cdots\tau^{n-1}(a) = \frac{b}{\tau(b)} \frac{\tau(b)}{\tau^2(b)} \cdots \frac{\tau^{n-1}(b)}{\tau^n(b)} = 1.$$

Now we prove  $\implies$ . Let  $a \in E$  be such that  $\text{norm}_{E/K}(a) = 1$ . For  $c \in E$  let

$$\begin{aligned} d_0 &= ac, \\ d_1 &= a\tau(a)\tau(c), \\ d_2 &= a\tau(a)\tau^2(a)\tau^2(c), \\ &\vdots \\ d_{n-1} &= \underbrace{a\tau(a) \cdots \tau^{n-1}(a)}_{=\text{norm}_{E/K}(a)} \tau^{n-1}(c) = \tau^{n-1}(c). \end{aligned}$$

Then

$$a\tau(d_j) = a\tau(a) \cdots \tau^{j+1}(a)\tau^{j+1}(c) = d_{j+1}$$

for all  $j \in \{0, \dots, n-2\}$ . Let  $b = d_0 + \cdots + d_{n-1}$ . We claim that  $b \neq 0$  for some  $c$ . Suppose this is not true, say  $b = 0$  for all  $c$ . Then

$$0 = ac + (a\tau(a))\tau(c) + \cdots + (a\tau(a) \cdots \tau^{n-1}(a))\tau^{n-1}(c)$$

for every  $c \in E$ . This implies that  $a = 0$  by Dedekind's theorem, a contradiction.

So let  $c \in E$  be such that  $b \neq 0$ . Then

$$\begin{aligned} \tau(b) &= \tau(d_0) + \cdots + \tau(d_{n-1}) \\ &= \tau(ac) + \tau(a\tau(a)\tau(c)) + \cdots + \tau(\tau^{n-1}(c)) \\ &= \frac{1}{a}(d_1 + \cdots + d_{n-1}) + \tau^n(c) \\ &= \frac{1}{a}(d_0 + \cdots + d_{n-1}) \\ &= b/a. \end{aligned}$$

□

**9.22. EXERCISE.** Let  $E/K$  be a cyclic extension. Assume that  $\text{Gal}(E/K)$  is generated by  $\tau$ . Prove that for  $a \in E$ ,  $\text{trace}_{E/K}(a) = 0$  if and only if  $a = b - \tau(b)$  for some  $b \in E \setminus \{0\}$ .

## 10. Lecture – Week 10

10.1. COROLLARY. Let  $a, b, c \in \mathbb{Z}$  be such that  $a^2 + b^2 = c^2$ . Then

$$(a, b, c) = \lambda(r^2 - s^2, -2rs, r^2 + s^2)$$

for some  $r, s \in \mathbb{Z}$  and some  $\lambda \in \mathbb{Z}$ .

PROOF. We work with the extension  $\mathbb{Q}(i)/\mathbb{Q}$ . Note that  $\text{Gal}(\mathbb{Q}(i), \mathbb{Q}) = \{\text{id}, \gamma\}$  is cyclic, where  $\gamma: \mathbb{Q}(i) \rightarrow \mathbb{Q}(i)$ ,  $z \mapsto \bar{z}$ , is the complex conjugation. We may assume that  $c \neq 0$ , otherwise  $a = b = 0$  and the result is trivial. Write  $(a/c)^2 + (b/c)^2 = 1$  and let

$$\alpha = (a/c) + (b/c)i \in \mathbb{Q}(i).$$

Then  $\text{norm}_{\mathbb{Q}(i)/\mathbb{Q}}(\alpha) = 1$ . By Hilbert's theorem, there exists  $\beta \in \mathbb{Q}(i) \setminus \{0\}$  such that

$$\alpha = a + bi = \frac{\gamma(\beta)}{\beta}.$$

Note that if  $m \in \mathbb{Z} \setminus \{0\}$ , then  $\frac{\gamma(m\beta)}{m\beta} = \frac{\gamma(\beta)}{\beta}$ . There exists  $m \in \mathbb{Z} \setminus \{0\}$  such that  $m\beta \in \mathbb{Z}[i]$ , say  $m\beta = r + is$  with  $r, s \in \mathbb{Z}$ . Then

$$\alpha = \frac{\gamma(\beta)}{\beta} = \frac{\gamma(m\beta)}{m\beta} = \frac{r - is}{r + is} = \frac{r^2 - s^2 - 2rsi}{r^2 + s^2}.$$

From this the claim follows.  $\square$

10.2. EXERCISE. Let  $A, B \in \mathbb{Z}$  be such that  $A^2 - 4B$  is not a square. Prove that a solution  $(x, y, z) \in \mathbb{Z}^3$  of  $x^2 + Axy + By^2 = z^2$  is proportional to

$$(r^2 - Bs^2, 2rs + As^2, r^2 + Ars + Bs^2).$$

10.3. PROPOSITION. Let  $n \geq 2$  and  $K$  be a field containing a primitive  $n$ -root of one. If  $a \in K^\times$  and  $E/K$  is a splitting field of  $f = X^n - a$ , then  $E/K$  is cyclic of degree  $d$ , where  $d$  divides  $n$ . Moreover,

$$d = \min\{k : a^k \in K^n\},$$

where  $K^n = \{x \in K : x = y^n \text{ for some } y \in K\}$ . Conversely, if  $E/K$  is cyclic of degree  $n$ , then  $E/K$  is a decomposition field of an irreducible polynomial of the form  $X^n - a$  for some  $a \in K^\times$ .

PROOF. A splitting field of  $f$  over  $K$  is of the form  $K(\alpha)$ , where  $\alpha^n = a$ . Thus  $K(\alpha)/K$  is a Galois extension. If  $\sigma \in \text{Gal}(K(\alpha)/K)$ , then  $\sigma(\alpha)$  is a root of  $f$ , so  $\sigma(\alpha) = \omega_\sigma \alpha$ , where  $\omega_\sigma \in G_n(K)$ . This means that there exists an injective map

$$\lambda: \text{Gal}(K(\alpha)/K) \rightarrow G_n(K), \quad \sigma \mapsto \omega_\sigma.$$

Moreover,  $\lambda$  is a group homomorphism, as

$$\sigma\tau(\alpha) = \sigma(\tau(\alpha)) = \sigma(\omega_\tau \alpha) = \omega_\tau \sigma(\alpha) = \omega_\tau \omega_\sigma \alpha.$$

Therefore  $\text{Gal}(K(\alpha)/K)$  is isomorphic to a subgroup of  $G_n(K)$ . In particular,  $\text{Gal}(K(\alpha)/K)$  is cyclic and  $|\text{Gal}(K(\alpha)/K)|$  divides  $n$ .

Let  $d = |\text{Gal}(K(\alpha)/K)|$ . Since  $a = \alpha^n$ ,

$$\text{norm}_{K(\alpha)/K}(\alpha)^n = \text{norm}_{K(\alpha)/K}(a) = a^d.$$

Thus  $a^d \in K^n$ , as  $\text{norm}_{K(\alpha)/K}(\alpha) \in K$ . If  $a^k \in K^n$ , say  $a^k = c^n$  for some  $c \in K$ , then

$$c^n = a^k = (\alpha^n)^k = (\alpha^k)^n \implies \alpha^k = c\omega \in K$$

for some  $\omega \in G_n(K)$ . Thus  $\alpha$  is a root of  $X^k - \alpha^k \in K[X]$  and hence  $k \geq d$ .

Note that  $f(\alpha, K) = X^d - \alpha^d$ .

Let  $E/K$  be cyclic of degree  $n$ . Assume that  $\text{Gal}(E/K) = \langle \sigma \rangle$ . If  $\omega$  is a primitive  $n$ -root of one,

$$\text{norm}_{E/K}(\omega) = \omega^n = 1.$$

By Hilbert's theorem 90, there exists  $b \in E^\times$  such that  $\omega = \sigma(b)/b$ . Thus  $\sigma(b) = \omega b$  and hence  $\sigma^i(b) = \omega^i b$  for all  $i \geq 0$ . Since  $|\{b, \sigma(b), \dots, \sigma^{n-1}(b)\}| = n$ , it follows that  $E = K(b)$ . Moreover,

$$\sigma(b^n) = \sigma(b)^n = (\omega b)^n = b^n$$

and hence  $b^n \in K$ . This means that  $E/K$  is a decomposition field of  $X^n - b^n$ . Note that  $X^n - b^n$  is irreducible, as  $[E : K] = [K(b) : K] = n$ .  $\square$

**10.4. PROPOSITION.** *Let  $K$  be a field of characteristic  $p > 0$ .*

- 1) *Let  $a \in K$  and  $f = X^p - X - a$ . Then  $f$  is irreducible over  $K$  or all the roots of  $f$  belong to  $K$ . In the first case, if  $b$  is a root of  $f$ , then  $K(b)/K$  is a cyclic extension of degree  $p$ .*
- 2) *Every cyclic extension of degree  $p$  is a splitting field of an irreducible polynomial of the form  $X^p - X - a$ .*

**PROOF.** We first prove 1). Let  $K_0$  be the prime field of  $K$ . Note that  $K_0 \simeq \mathbb{Z}/p$ . Let  $b$  be a root of  $f$  and let  $x \in K_0$ . Then

$$f(b+x) = (b+x)^p - (b+x) - a = (b^p - b - a) + (x^p - x) = 0$$

and thus  $\{b+x : x \in K_0\}$  is the set of roots of  $f$ . Note that  $f' = -1$ , so  $f$  has no multiple roots.

We claim that if  $b \notin K$ , then  $f$  is irreducible. If  $f$  is not irreducible, then  $f = gh$  for some  $g, h \in K[X]$  such that  $0 < \deg g < p$ . There exists a subset  $S$  of  $K_0$  such that  $g = \prod_{x \in S} (X - (b+x))$  and hence

$$|S|b + \sum_{x \in S} x = \sum_{x \in S} (b+x) \in K.$$

This implies that  $|S|b \in K$  and hence, since  $|S| \in K^\times$ , it follows that  $b \in K$ .

Since  $K(b)/K$  is a splitting field of a separable polynomial,  $K(b)/K$  is a Galois extension. Moreover,  $|\text{Gal}(K(b)/K)| = [K(b) : K] = p$  and hence  $\text{Gal}(K(b)/K)$  is cyclic.

We now prove 2). Let  $E/K$  be cyclic of degree  $p$ . Assume that  $\text{Gal}(E/K) = \langle \sigma \rangle$ . Since  $\text{trace}_{E/K}(1) = p = 0$ , Hilbert's theorem implies that there exists  $b \in E$  such that  $\sigma(b) = b+1$ . In particular,  $b \notin K$  and thus  $E = K(b)$ . Moreover, since

$$\sigma(b^p - b) = \sigma(b)^p - \sigma(b) = (b+1)^p - (b+1) = b^p - b,$$

it follows that  $b^p - b \in K$ . Thus  $f(b, K) = X^p - X - (b^p - b) \in K[X]$ .  $\square$

**§ 10.1. Symmetric polynomials.** Let  $K$  be a field and  $\{t_1, \dots, t_n\}$  be a commuting set of independent variables. Let  $E = K(t_1, \dots, t_n)$  and  $f = \prod_{i=1}^n (X - t_i) \in E[X]$ . Then

$$f = X^n + \sum_{i=1}^n (-1)^i s_i X^{n-i},$$

where

$$\begin{aligned} s_1 &= t_1 + t_2 + \cdots + t_n, \\ s_2 &= \sum_{1 \leq i < j \leq n} t_i t_j, \\ &\vdots \\ s_n &= t_1 t_2 \cdots t_n. \end{aligned}$$

For example,

$$(X - t_1)(X - t_2)(X - t_3) = X^3 - (t_1 + t_2 + t_3)X^2 + (t_1 t_2 + t_2 t_3 + t_1 t_3)X - t_1 t_2 t_3.$$

The polynomials  $s_1, s_2, \dots, s_n$  are known as the **elementary symmetric polynomials** in the variables  $t_1, \dots, t_n$ . Note that  $\deg s_i = i$ .

Let  $\sigma \in \mathbb{S}_n$  and

$$\alpha_\sigma: K[t_1, \dots, t_n] \rightarrow K[t_1, \dots, t_n], \quad t_i \mapsto t_{\sigma(i)} \quad \text{for all } i.$$

Then  $\alpha_\sigma$  is a bijective homomorphism of  $K$ -algebras. In fact,  $\alpha_\sigma^{-1} = \alpha_{\sigma^{-1}}$ . Note that

$$\alpha_\sigma(h(t_1, \dots, t_n)) = h(t_{\sigma(1)}, \dots, t_{\sigma(n)}).$$

Since  $\alpha_\sigma$  is injective, it induces an element  $\widehat{\sigma} \in \text{Gal}(E/K)$  given by

$$\widehat{\sigma}\left(\frac{h}{g}\right) = \frac{\alpha_\sigma(h)}{\alpha_\sigma(g)}.$$

The map  $\mathbb{S}_n \rightarrow \text{Gal}(E/K)$ ,  $\sigma \mapsto \widehat{\sigma}$ , is an injective group homomorphism. Thus

$$\{\widehat{\sigma} : \sigma \in \mathbb{S}_n\} \simeq \mathbb{S}_n.$$

**10.5. DEFINITION.** Let  $g \in K[t_1, \dots, t_n]$ . Then  $g$  is **symmetric** if  $\widehat{\sigma}(g) = g$  for all  $\sigma \in \mathbb{S}_n$ .

We write  $P$  to denote the set of symmetric polynomials in  $K[t_1, \dots, t_n]$ . Clearly,  $P$  is a subalgebra of  $K[t_1, \dots, t_n]$ . The following statements hold:

- 1)  $K \subseteq P$ .
- 2)  $\sum_{i=1}^n t_i^r \in P$  for all  $r \geq 1$ .
- 3)  $s_i \in P$  for all  $i$ .
- 4)  $K(P) \subseteq {}^G E$ , where  $G = \{\widehat{\sigma} : \sigma \in \mathbb{S}_n\}$ .

Let  $F = K(s_1, s_2, \dots, s_n)$ . Then  $E/F$  is a Galois extension, as it is a splitting field of  $f$ .

**10.6. PROPOSITION.**  $[E : F] \leq n!$ .

**PROOF.** We proceed by induction on  $n$ . The case  $n = 1$  is clear, as  $E = F$ . Assume that  $n > 1$ . Let  $u_1, \dots, u_{n-1}$  be the elementary symmetric polynomials in  $t_1, \dots, t_{n-1}$ . Then

$$s_i = u_i + t_n u_{i-1}$$

for all  $i \in \{1, \dots, n\}$ , where  $u_0 = 1$  and  $u_n = 0$ . Note that  $u_1 = s_1 - t_n$  and  $u_i = s_i - t_n u_{i-1}$  for all  $i$ . Since  $K(s_1, \dots, s_n, t_n) = K(u_1, \dots, u_{n-1}, t_n)$ ,

$$F(t_n) = K(u_1, \dots, u_{n-1}, t_n) = K(t_n)(u_1, \dots, u_{n-1})$$

and

$$[E : F] = [E : F(t_n)][F(t_n) : F] \leq n[E : F(t_n)].$$

Note that  $E = K(t_1, \dots, t_n) = K(t_n)(t_1, \dots, t_{n-1})$ . By the inductive hypothesis,

$$[E : F(t_n)] \leq (n-1)!$$

and hence  $[E : F] \leq n!$ . □

**10.7. THEOREM.**  ${}^G E = F$ .

**PROOF.** By Artin's theorem,

$$[{}^G E : F] = \frac{[E : F]}{[E : {}^G E]} \leq \frac{n!}{[E : {}^G E]} = 1$$

and hence  ${}^G E = F$ . □

**10.8. EXERCISE.** Prove that  $\text{Gal}(E/F) \simeq \mathbb{S}_n$ .

**10.9. EXERCISE.** Prove that  $\{s_1, \dots, s_n\}$  is algebraically independent over  $K$ .

**10.10. EXERCISE.** Prove that every symmetric polynomial in  $t_1, \dots, t_n$  can be written as a rational fraction in  $s_1, \dots, s_n$ .

## 11. Lecture – Week 11

**§ 11.1. Solvable groups.** Let  $G$  be a group. If  $x, y \in G$  we define the **commutator** of  $x$  and  $y$  as

$$[x, y] = xyx^{-1}y^{-1}.$$

Note that  $[x, y] = 1$  if and only if  $xy = yx$ . Moreover,  $[x, y]^{-1} = [y, x]$ . The **commutator (or derived) subgroup**  $[G, G]$  of  $G$  is defined as the subgroup of  $G$  generated by all commutators, i.e.

$$[G, G] = \langle [x, y] : x, y \in G \rangle.$$

This means that every element of  $[G, G]$  is a finite product of commutators, so every element of  $[G, G]$  is of the form  $\prod_{i=1}^m [x_i, y_i]$ . In general, the commutator subgroup is not equal to the set of commutators!

11.1. EXAMPLE. This example is taken from the book [2] of Carmichael. Let  $G$  be the subgroup of  $\mathbb{S}_{16}$  generated by the permutations

$$\begin{aligned} a &= (13)(24), & b &= (57)(68), \\ c &= (911)(10\ 12), & d &= (13\ 15)(14\ 16), \\ e &= (13)(57)(9\ 11), & f &= (12)(34)(13\ 15), \\ g &= (56)(78)(13\ 14)(15\ 16), & h &= (9\ 10)(11\ 12). \end{aligned}$$

Then  $[G, G]$  has order 16. However, the set  $\{[x, y] : x, y \in G\}$  of commutators has 15 elements:

```
> S16 := Sym(16);
> a := S16 ! (1,3)(2,4);
> b := S16 ! (5,7)(6,8);
> c := S16 ! (9,11)(10,12);
> d := S16 ! (13,15)(14,16);
> e := S16 ! (1,3)(5,7)(9,11);
> f := S16 ! (1,2)(3,4)(13,15);
> g := S16 ! (5,6)(7,8)(13,14)(15,16);
> h := S16 ! (9,10)(11,12);
> G := PermutationGroup< 16 | a,b,c,d,e,f,g,h >;
> D := DerivedSubgroup(G);
> #D;
16
> #{ x*y*Inverse(x)*Inverse(y) : x in G, y in G };
15
> c*d in { x : x in D } \
> diff { u*v*Inverse(u)*Inverse(v) : u in G, v in G };
true
```

11.2. EXERCISE. Let  $G$  be a group. Prove the following facts:

- 1)  $G$  is abelian if and only if  $[G, G] = \{1\}$ .
- 2)  $[G, G]$  is a normal subgroup of  $G$ .
- 3)  $G/[G, G]$  is abelian.
- 4) If  $H$  is a subgroup of  $G$  and  $[G, G] \subseteq H$ , then  $H$  is normal in  $G$ .
- 5) If  $H$  is a normal subgroup of  $G$ , then  $G/H$  is abelian if and only if  $[G, G] \subseteq H$ .

11.3. DEFINITION. Let  $G$  be a group. The **derived series** of  $G$  is defined as  $G^{(0)} = G$  and  $G^{(k+1)} = [G^{(k)}, G^{(k)}]$  for  $k \geq 0$ .

11.4. EXERCISE. Prove that  $G^{(k)}$  is normal in  $G$  for all  $k$ .

Why derived series? We cannot explain this here, but let us use the following notation. We write  $G' = [G, G]$ ,  $G'' = [G', G']$ ... Note that

$$G \supseteq G' \supseteq G'' \supseteq \dots$$

11.5. EXERCISE. Let  $n \geq 3$ . Prove that  $[\mathbb{S}_n, \mathbb{S}_n] = \mathbb{A}_n$ .

11.6. EXAMPLE. Let  $K = \{\text{id}, (12)(34), (13)(24), (14)(23)\}$ . Then  $K$  is a normal subgroup of  $\mathbb{A}_4$ . One proves that  $[\mathbb{A}_4, \mathbb{A}_4] = K$ .

A group  $G$  is said to be **simple** if there are no proper non-trivial normal subgroups of  $G$ . If  $p$  is a prime number, then the group  $\mathbb{Z}/p$  of integers modulo  $p$  is a simple group. We will prove later that  $\mathbb{A}_n$  is simple if  $n \geq 5$ .

11.7. EXAMPLE. Let  $n \geq 5$ . Since  $\mathbb{A}_n$  is a non-abelian simple group,  $[\mathbb{A}_n, \mathbb{A}_n] = \mathbb{A}_n$ .

Let us show that  $\mathbb{A}_5$  is a non-abelian simple group. Hence it is not solvable:

```
> G := Alt(5);
> IsAbelian(G);
false
> IsSimple(G);
true
> IsSolvable(G);
false
```

11.8. DEFINITION. A group  $G$  is **solvable** if and only if  $G^{(m)} = \{1\}$  for some  $m$ .

Every abelian group is solvable.

11.9. EXERCISE. Prove that  $\mathbb{S}_n$  is solvable if and only if  $n \leq 4$ .

Let us compute (with the computer software Oscar) the derived series of the symmetric group  $\mathbb{S}_4$ . The calculation shows that  $\mathbb{S}_4$  is solvable:

```
> G := Sym(4);
> DerivedSeries(G);
[
  Symmetric group G acting on a set of cardinality 4
```

```

Order = 24 = 2^3 * 3
(1, 2, 3, 4)
(1, 2),
Permutation group acting on a set of cardinality 4
Order = 12 = 2^2 * 3
(1, 2, 3)
(2, 3, 4),
Permutation group acting on a set of cardinality 4
Order = 4 = 2^2
(1, 4)(2, 3)
(1, 3)(2, 4),
Permutation group acting on a set of cardinality 4
Order = 1
]
> IsSolvable(G);
true

```

11.10. PROPOSITION. Let  $G$  be a group and  $H$  be a subgroup of  $G$ . The following statements hold:

- 1) If  $G$  is solvable, then  $H$  is solvable.
- 2) If  $H$  is normal in  $G$  and  $G$  is solvable, then  $G/H$  is solvable.
- 3) If  $H$  is normal in  $G$  and  $H$  and  $G/H$  are solvable, then  $G$  is solvable.

PROOF. The first statement follows from the fact that  $H^{(i)} \subseteq G^{(i)}$  holds for all  $i$ .

Assume now that  $H$  is normal in  $G$ . Let  $Q = G/H$  and  $\pi: G \rightarrow Q$  be the canonical map. By induction one proves that  $\pi(G^{(i)}) = Q^{(i)}$  for all  $i \geq 0$ . The case where  $i = 0$  is trivial, as  $\pi$  is surjective. If the result holds for some  $i \geq 0$ , then

$$\pi(G^{(i+1)}) = \pi([G^{(i)}, G^{(i)}]) = [\pi(G^{(i)}), \pi(G^{(i)})] = [Q^{(i)}, Q^{(i)}] = Q^{(i+1)}.$$

We now prove 2). Since  $G$  is solvable,  $G^{(n)} = \{1\}$  for some  $n$ . Thus  $Q$  is solvable, as  $Q^n = \pi(G^{(n)}) = \pi(\{1\}) = \{1\}$ .

We finally prove 3). Since  $Q$  is solvable,  $Q^{(n)} = \{1\}$  for some  $n$ . Moreover, since  $\pi(G^{(n)}) = Q^{(n)} = \{1\}$ , it follows that  $G^{(n)} \subseteq H$ . Since  $H$  is solvable,

$$G^{(n+m)} \subseteq (G^{(n)})^{(m)} \subseteq H^{(m)} = \{1\}$$

for some  $m$ . Thus  $G$  is solvable. □

An application:

11.11. PROPOSITION. Let  $G$  be a finite  $p$ -group. Then  $G$  is solvable.

PROOF. Assume the result is not true. Let  $G$  be a finite  $p$ -group of minimal order that is not solvable. Since  $G$  is a  $p$ -group,  $Z(G) \neq \{1\}$ . Since  $|G|$  is minimal,  $G/Z(G)$  is a solvable  $p$ -group. Since  $Z(G)$  is abelian,  $Z(G)$  is solvable. Now  $G$  is solvable by Proposition 11.10. □

Let  $G$  be a group. A subgroup  $N$  of  $G$  is said to be **maximal normal** if  $N$  is a normal subgroup of  $G$  and there is no other normal subgroup of  $G$  containing  $N$ .

11.12. EXERCISE. If a subgroup  $N$  of  $G$  is maximal (for the inclusion) and normal, then it is maximal normal. Show that the converse does not hold.

The following result is a direct consequence of the correspondence theorem:

11.13. EXERCISE. Let  $G$  be a group and  $N$  be a normal subgroup of  $G$ . Prove that  $N$  is maximal normal if and only if  $G/N$  is simple.

Maximal normal subgroups always exist in finite groups (they could be trivial). We can compute maximal normal subgroups as follows:

```
> MaximalNormalSubgroup(Sym(3));
Permutation group acting on a set of cardinality 3
Order = 3
(1, 2, 3)

julia> maximal_normal_subgroups(quaternion_group(8))
3-element Vector{PcGroup}:
Group([ y2, x ])
Group([ y2, y ])
Group([ y2, x*y ])

> MaximalNormalSubgroup(Alt(4));
Permutation group acting on a set of cardinality 4
Order = 4 = 2^2
(1, 2)(3, 4)
(1, 3)(2, 4)
```

11.14. EXERCISE. Let  $G$  be a finite solvable group. Prove that if  $G$  is simple, then  $G$  is cyclic of prime order.

The following result will be important later:

11.15. PROPOSITION. *Every finite solvable group contains a normal subgroup of prime index.*

PROOF. Let  $G$  be a finite solvable group. Let  $M$  be a maximal normal subgroup of  $G$  (there is at least one, as  $G$  is finite). Since  $G/M$  is simple and solvable (see Proposition 11.10),  $G/M$  is cyclic of prime order by Exercise 11.14.  $\square$

We finish this discussion with two important theorems (without proof) about finite solvable groups.

11.16. THEOREM (Burnside). *Let  $p$  and  $q$  be prime numbers. If  $G$  is a group of order  $p^aq^b$ , then  $G$  is solvable.*

The proof appears in courses on the representation theory of finite groups.

11.17. THEOREM (Feit–Thompson). *Every finite group of odd order is solvable.*

The proof of the theorem is extremely hard. It occupies a full volume of **Pacific Journal of Mathematics**, see [3].

**§ 11.2. Simplicity of the alternating simple group.** We will present a family of non-abelian simple groups. We start with some exercises.

11.18. EXERCISE. Let  $G$  be a group. Prove that  $G$  is simple if and only if  $\{(g, g) : g \in G\}$  is a maximal subgroup of  $G \times G$ .

11.19. EXERCISE. Prove that  $\mathbb{A}_n$  is generated by 3-cycles.

11.20. EXERCISE. Compute the commutator subgroup of  $\mathbb{A}_n$  for  $n \geq 2$ .

Note that  $\mathbb{A}_2$  and  $\mathbb{A}_3$  are abelian. For  $\mathbb{A}_4$ , one proves that

$$[\mathbb{A}_4, \mathbb{A}_4] = \{\text{id}, (12)(34), (13)(24), (14)(23)\}.$$

Finally,  $[\mathbb{A}_n, \mathbb{A}_n] = \mathbb{A}_n$  for  $n \geq 5$ .

Let us compute some commutator subgroups (and the inclusion group homomorphism) with the computer:

```
> S3 := Sym(3);
> DerivedSubgroup(S3);
Permutation group acting on a set of cardinality 3
Order = 3
(1, 2, 3)
```

11.21. EXERCISE. Let  $n \geq 3$ . Prove that  $[\mathbb{S}_n, \mathbb{S}_n] = \mathbb{A}_n$ .

Recall that every normal subgroup is a union of conjugacy classes. The group  $\mathbb{A}_5$  has conjugacy classes of sizes 1, 15, 20, 12 and 12. It follows that the only possible normal subgroups of  $\mathbb{A}_5$  are  $\{\text{id}\}$  and  $\mathbb{A}_5$ .

```
> A5 := Alt(5);
> ConjugacyClasses(A5);
Conjugacy Classes of group A5
-----
[1] Order 1      Length 1
    Rep Id(A5)

[2] Order 2      Length 15
    Rep (1, 2)(3, 4)

[3] Order 3      Length 20
    Rep (1, 2, 3)

[4] Order 5      Length 12
    Rep (1, 2, 3, 4, 5)

[5] Order 5      Length 12
    Rep (1, 3, 4, 5, 2)
```

11.22. THEOREM (Jordan). *Let  $n \geq 5$ . Then  $\mathbb{A}_n$  is simple.*

Before proving the theorem, we need some preliminary results.

Every permutation  $\rho \in \mathbb{S}_n$  decomposes as a product of disjoint cycles, say

$$\rho = (a_1 \cdots a_r)(b_1 \cdots b_s) \cdots (c_1 \cdots c_t)$$

where, by convention, we do not write cycles of length one. The cyclic structure of  $\rho$  is, by definition, the ordered sequence of integers  $r, s, \dots, t$ , where, again by convention, we omit fixed points. For example, the cyclic structure of the transposition  $(ab)$  is 2, of  $(abc)(d)$  is 3 and of  $(123)(45)(789a)(bcd)(d)$  is 2,3,3,4.

11.23. LEMMA. *If both permutations  $\rho_1 \in \mathbb{S}_n$  and  $\rho_2 \in \mathbb{S}_n$  have the same cyclic structure, then  $\rho_2 = \sigma\rho_1\sigma^{-1}$  for some  $\sigma \in \mathbb{S}_n$ .*

PROOF. Assume that

$$\begin{aligned} \rho_1 &= (a_1 \cdots a_r)(b_1 \cdots b_s) \cdots (c_1 \cdots c_t), \\ \rho_2 &= (x_1 \cdots x_r)(y_1 \cdots y_s) \cdots (z_1 \cdots z_t). \end{aligned}$$

Let

$$\text{Fix}(\rho_1) = \{x \in \{1, \dots, n\} : \rho_1(x) = x\} = \{k_1, \dots, k_m\}, \quad \text{Fix}(\rho_2) = \{l_1, \dots, l_m\}$$

be the fixed points of the permutations  $\rho_1$  and  $\rho_2$ , respectively. Then

$$\sigma(x) = \begin{cases} x_j & \text{if } x = a_j \text{ for some } j, \\ y_j & \text{if } x = b_j \text{ for some } j, \\ \vdots & \\ z_j & \text{if } x = c_j \text{ for some } j, \\ l_j & \text{if } x = k_j \text{ for some } j, \end{cases}$$

is such that  $\sigma\rho_1\sigma^{-1} = \rho_2$ . □

What happens with the alternating group?

11.24. LEMMA. *If  $\rho_1, \rho_2 \in \mathbb{S}_n$  are conjugate in  $\mathbb{S}_n$  and  $|\text{Fix}(\rho_1)| \geq 2$ , then  $\mu\rho_1\mu^{-1} = \rho_2$  for some  $\mu \in \mathbb{A}_n$ .*

PROOF. Assume that  $\rho_2 = \sigma\rho_1\sigma^{-1}$  for some  $\sigma \in \mathbb{S}_n$ . There are  $a, b \in \{1, \dots, n\}$  such that  $\rho_1(a) = a$ ,  $\rho_1(b) = b$  and  $a \neq b$ . Let

$$\mu = \begin{cases} \sigma & \text{if } \sigma \in \mathbb{A}_n, \\ \sigma(ab) & \text{otherwise.} \end{cases}$$

Then  $\mu \in \mathbb{A}_n$  and  $\mu\rho_1\mu^{-1} = \rho_2$ , as  $(ab)$  commutes with  $\rho_1$ . □

Let us discuss some examples.

11.25. EXAMPLE. If  $\rho_1 = (23)(156)$  and  $\rho_2 = (45)(123)$ , then  $\rho_2 = \sigma\rho_1\sigma^{-1}$  for

$$\sigma = \begin{pmatrix} 123456 \\ 145623 \end{pmatrix}.$$

11.26. EXAMPLE. The permutations  $\rho_1 = (123) \in \mathbb{S}_3$  and  $\rho_2 = (132) \in \mathbb{S}_3$  are conjugate in  $\mathbb{S}_3$ , as  $(123) = \sigma(132)\sigma^{-1}$  if  $\sigma = (23)$ . However,  $\rho_1$  and  $\rho_2$  are not conjugate in  $\mathbb{A}_3$ .

Now we are ready to prove the theorem.

**PROOF OF THEOREM 11.22.** Let  $N \neq \{\text{id}\}$  be a normal subgroup of  $\mathbb{A}_n$ . If  $(abc) \in N$ , then every 3-cycle belongs to  $N$ , because all 3-cycles are conjugate in  $\mathbb{S}_n$ , and the previous lemma states that  $(ijk) = \mu(abc)\mu^{-1} \in N$  for some  $\mu \in \mathbb{A}_n$ . Thus  $N = \mathbb{A}_n$ .

We claim that  $N$  contains a 3-cycle. Since  $N \neq \{\text{id}\}$ , there exists  $\sigma \in N \setminus \{\text{id}\}$ . Let  $m = |\sigma|$  and let  $p$  be a prime number dividing  $m$ . Then  $\tau = \sigma^{m/p}$  has order  $p$  and hence  $\tau = \rho_1 \cdots \rho_s$ , where the  $\rho_j$ 's are disjoint  $p$ -cycles.

If  $p = 2$ , then  $1 = \text{sign}(\tau) = (-1)^s$ . Thus  $s$  is even. Write

$$\tau = (ab)(cd)\rho_3 \cdots \rho_s.$$

Since  $\rho_3 \cdots \rho_s$  commutes with  $(abc)$  and  $(acb)$ ,

$$\underbrace{(abc)\tau(abc)^{-1}\tau^{-1}}_{\in N} = (abc)(ab)(cd)(acb)(ab)(cd) = (ac)(bd).$$

Hence  $(ac)(bd) \in N$ . Let  $e \in \{1, \dots, n\} \setminus \{a, b, c, d\}$ . Then

$$(ae)(bd) = (aec) \underbrace{(ac)(bd)}_{\in N} (aec)^{-1} \in N$$

and therefore

$$(aec) = (ac)(ae) = (ac)(bd)(ae)(bd) \in N.$$

If  $p = 3$ , without loss of generality, we may assume that  $s \geq 2$  (otherwise,  $\tau$  would be a 3-cycle). Then  $\tau = (abc)(def)\rho_3 \cdots \rho_s$ . Since  $(bcd)$  commutes with  $\rho_3 \cdots \rho_s$  and  $N$  is normal in  $\mathbb{A}_n$ ,

$$\underbrace{(bcd)\tau(bcd)^{-1}\tau^{-1}}_{\in N} = (bcd)(abc)(def)(bdc)(acb)(dfe) = (adbce)$$

and therefore

$$(adc) = (adb)(adbce)(adb)^{-1}(adbce)^{-1} \in N.$$

If  $p > 3$ , then  $\tau = (abcd \cdots z)\rho_2 \cdots \rho_s$ . In particular,  $(abc)$  commutes with  $\rho_2 \cdots \rho_s$ . Then

$$(abd) = (abc)\tau(abc)^{-1}\tau^{-1} \in N. \quad \square$$

As an application, we compute the normal subgroups of the symmetric group  $\mathbb{S}_n$ .

11.27. EXERCISE. Compute the list of normal subgroups of  $\mathbb{S}_n$  for  $n \geq 2$ .

### § 11.3. Radical extensions.

11.28. DEFINITION. An extension  $E/K$  is said to be **pure** of type  $m$  if  $E = K(x)$  for some  $x$  such that  $x^m \in K$ .

Note that if  $E = K(x)$  is a pure extension of type  $m$  and  $K$  contains  $m$ -th roots of one, then  $E/K$  is a splitting field of  $X^m - x^m$ .

11.29. DEFINITION. The sequence  $K = R_0 \subseteq R_1 \subseteq \cdots \subseteq R_m$  of fields is said to be a **radical tower** if each  $R_{i+1}/R_i$  is pure. In this case,  $R_m/K$  is a **radical extension**.

Note that radical extensions are finite.

11.30. EXAMPLE. Let  $E$  be a splitting field of  $X^4 - 2$  over  $\mathbb{Q}$ . Then  $E/\mathbb{Q}$  is radical, as  $E = \mathbb{Q}(\sqrt[4]{2}, i)$ .

11.31. EXAMPLE. Let  $\alpha, \beta \in \mathbb{C}$  be such that  $\alpha^2 = 2$  and  $\beta^5 = 1 + \alpha$ . The number  $\sqrt[5]{1 + \sqrt{2}}$  belongs to the radical extension  $\mathbb{Q}(\alpha, \beta)/\mathbb{Q}$ .

11.32. THEOREM. *Let  $K$  be of characteristic zero and  $R/K$  be a radical extension. If  $E/K$  is a subextension of  $R/K$ , then  $\text{Gal}(E/K)$  is solvable.*

PROOF. Without loss of generality, we may assume that  $E/K$  is a Galois extension. To prove this fact, let  $G = \text{Gal}(E/K)$  and  $F = {}^G E$ . Then  $E/F$  is a Galois extension and  $\text{Gal}(E/F) = G$  by Artin's theorem. Thus, replacing  $K$  by  $F$  if needed, we may assume that  $E/K$  is Galois.

Let  $L$  be the normal closure of  $R$  in some algebraic closure  $C$  that contains  $R$ . Note that if  $R = K(x_1, \dots, x_m)$ , then

$$L = K(\{\sigma_i(x_j) : 1 \leq i \leq s, 1 \leq j \leq m\}),$$

where  $\text{Hom}(R/K, C/K) = \{\sigma_1, \dots, \sigma_s\}$ .

CLAIM.  $L/K$  is radical.

Since  $x_j^{a_j} \in K(x_1, \dots, x_{j-1})$  for some integer  $a_j$ ,

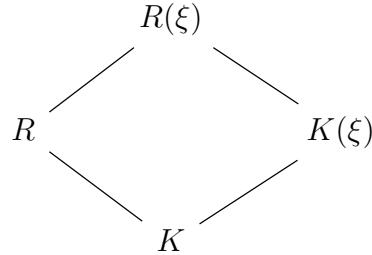
$$\sigma_i(x_j)^{a_j} = \sigma_i(x_j^{a_j}) \in \sigma_i(K(x_1, \dots, x_{j-1})) = K(\sigma_i(x_1), \dots, \sigma_i(x_{j-1}))$$

Thus  $L/K$  is radical and Galois.

We may assume then that  $E/K$  and  $R/K$  are both Galois.

Since  $\text{Gal}(E/K) \simeq \text{Gal}(R/K)/\text{Gal}(R/E)$ , we only need to prove that  $\text{Gal}(R/K)$  is solvable.

For a positive integer  $n$ , let  $\xi$  be a primitive  $n$ -th root of one (in some algebraic closure of  $K$  that contains  $R$ ). Consider the diagram



Then

- 1)  $K(\xi)/K$  and  $R(\xi)/R$  are abelian.
- 2)  $R(\xi)/K$  is Galois.
- 3)  $\text{Gal}(R/K) \simeq \text{Gal}(R(\xi)/K)/\text{Gal}(R(\xi)/R)$ .
- 4)  $\text{Gal}(K(\xi)/K) \simeq \text{Gal}(R(\xi)/K)/\text{Gal}(R(\xi)/K(\xi))$ .

The third item implies that we need to show that  $\text{Gal}(R(\xi)/K)$  is solvable. By the fourth item, it suffices to show that  $\text{Gal}(R(\xi)/K(\xi))$  is solvable (because  $\text{Gal}(K(\xi)/K)$  is abelian and hence solvable).

Since  $R = K(x_1, \dots, x_m)$ ,

$$R(\xi) = K(x_1, \dots, x_m, \xi) = K(\xi)(x_1, \dots, x_m)$$

and hence  $R(\xi)/K(\xi)$  is radical. This means that without loss of generality, we may assume that  $K$  contains primitive  $n$ -roots of one. For example, if  $R = K(x_1, \dots, x_m)$  and  $x_i^{a_i} \in K(x_1, \dots, x_{i-1})$ , then we may assume that  $K$  contains a primitive  $a_i$ -root of one. We proceed by induction on  $m$ . The case  $m = 0$  is trivial. Assume that the claim holds for some  $m \geq 0$ . Let  $L = K(x_1)$ . Then  $L/K$  is a splitting field of  $X^{a_1} - x_1^{a_1}$ , and hence  $L/K$  is a cyclic extension. Thus  $\text{Gal}(L/K)$  is cyclic (and hence, in particular, solvable). Let  $H$  be the subgroup that corresponds to  $L$ , that is  $H = \text{Gal}(R/L)$  (here, we use Galois' correspondence). Then  $H$  is normal in  $\text{Gal}(R/K)$ . Since  $R = K(x_1, \dots, x_m) = L(x_2, \dots, x_m)$ ,  $R/L$  is radical and Galois. By the inductive hypothesis,  $\text{Gal}(R/L)$  is solvable. Since

$$\text{Gal}(L/K) \simeq \text{Gal}(R/K)/\text{Gal}(R/L),$$

it follows that  $\text{Gal}(R/K)$  is solvable.  $\square$

**11.33. DEFINITION.** Let  $f \in K[X]$  and  $E$  be a splitting field of  $f$  over  $K$ . We say that  $f$  is **solvable by radicals** if there is a radical extension  $R/K$  such that  $E \subseteq R$ .

The general polynomial of degree two is solvable by radicals, as its Galois group is solvable (in fact, isomorphic to  $\mathbb{S}_2$ ).

**11.34. EXERCISE.** Prove that  $f = X^2 - s_1X + s_2 \in \mathbb{Q}[X]$  is solvable by radicals.

Theorem 11.32 translates into the following result:

**11.35. EXERCISE.** Let  $K$  be a field of characteristic zero. If  $f \in K[X]$  is solvable by radicals, then  $\text{Gal}(f, K)$  is solvable.

As a consequence, the general polynomial of degree  $n \geq 5$  is not solvable by radicals, as its Galois group is isomorphic to  $\mathbb{S}_5$ .

**11.36. EXAMPLE.** Let  $p$  be a prime number and  $f = X^5 - 2pX + p \in \mathbb{Q}[X]$ . We claim that  $f$  is not solvable by radicals.

By Gauss' theorem, one proves that  $f$  has no rational roots.

Note that  $f' = 5X^4 - 2p$ . Then  $\alpha = \sqrt[4]{2p/5}$  and  $\beta = -\sqrt[4]{2p/5}$  are critical points. Since  $f(\alpha) < 0$  and  $f(\beta) > 0$ , it follows that  $f$  has exactly three real roots. Let  $x_1, x_2 \in \mathbb{C} \setminus \mathbb{R}$  and  $x_3, x_4, x_5 \in \mathbb{R}$  be the roots of  $f$ .

By Eisenstein's theorem,  $f$  is irreducible.

Let  $E/\mathbb{Q}$  be a splitting field of  $f$ . Then  $\text{Gal}(f, \mathbb{Q}) = \text{Gal}(E/\mathbb{Q})$  is isomorphic to a subgroup  $G$  of  $\mathbb{S}_5$ . Since  $f$  is irreducible, 5 divides  $[E : \mathbb{Q}] = |G|$ . In particular, by Cauchy's theorem,  $G$  contains an element  $\sigma$  of order five. This element is a 5-cycle, so without loss of generality, we may assume that  $\sigma = (x_1 x_2 x_3 x_4 x_5)$ . Note that  $(x_1 x_2) \in G$ . Thus  $G \simeq \mathbb{S}_5$  and hence  $G$  is not solvable.

**11.37. EXERCISE.** Let  $f = X^6 + 2X^5 - 5X^4 + 9X^3 - 5X^2 + 2X + 1 \in \mathbb{Q}[X]$ . Prove that  $f$  is solvable by radicals.

It is now time to prove Galois' great theorem on solvability of polynomials.

**11.38. THEOREM (Galois).** *Let  $K$  be a field of characteristic zero and  $f \in K[X]$ . Then  $f$  is solvable by radicals if and only if  $\text{Gal}(f, K)$  is solvable.*

We proved in Theorem 11.32 that solvable polynomials have solvable Galois groups. For the converse, we need two auxiliary results.

11.39. LEMMA. *Let  $E/K$  be a Galois extension of prime degree  $p$ . Assume that  $K$  admits a primitive  $p$ -root of one. Then  $E = K(\beta)$  where  $\beta^p \in K$ .*

PROOF. Assume that  $\text{Gal}(E/K) = \langle \sigma \rangle$ . Let  $\omega \in K$  be a primitive  $p$ -root of one. Then  $\text{norm}_{E/K}(\omega) = \omega^p = 1$ . By Hilbert's theorem,  $\omega = \beta/\sigma(\beta)$  for some  $\beta \in E$ . Note that  $\beta \notin K$ , as  $\omega \neq 1$ . Moreover,

$$\sigma(\beta^p) = (\beta\omega^{-1})^p = \beta^p \in {}^{\text{Gal}(E/K)}E = K.$$

Since  $K \subseteq K(\beta) \subseteq E$  and  $[E : K] = p$ , we conclude that  $E = K(\beta)$  with  $\beta^p \in K$ .  $\square$

11.40. EXERCISE. Let  $E/K$  be a splitting field of  $f \in K[X]$  and  $K^*/K$  be an extension. If  $E^*/K^*$  is a splitting field of  $f$  containing  $E$ , then

$$\text{Gal}(E^*/K^*) \rightarrow \text{Gal}(E/K), \quad \sigma \mapsto \sigma|_E,$$

is an injective group homomorphism.

Now Theorem 11.38 will follow from the following theorem.

11.41. THEOREM. *Let  $K$  be a field of characteristic zero and  $E/K$  be a Galois extension. If  $\text{Gal}(E/K)$  is solvable, then  $E$  can be embedded in a radical extension.*

PROOF. Let  $G = \text{Gal}(E/K)$ . Since  $G$  is solvable, by Proposition 11.15, there exists a normal subgroup  $H$  of  $G$  of prime index  $p$ . Let  $\omega$  be a primitive  $p$ -th root of one. (It exists because  $K$  is a field of characteristic zero.)

We proceed by induction on  $[E : K]$ .

If  $[E : K] = 1$ , there is nothing to prove. So assume that  $[E : K] > 1$ .

We first assume that  $\omega \in K$ . The group  $\text{Gal}(E/{}^H E)$  is solvable, as it is a subgroup of  $G$ . Moreover, since

$$[E : {}^H E] < [E : K],$$

the inductive hypothesis implies that  $E/{}^H E$  can be embedded in a radical extension, so there exists a radical tower

$$(11.1) \quad {}^H E \subseteq R_1 \subseteq R_2 \subseteq \cdots \subseteq R_m,$$

where  $E \subseteq R_m$ . Now  $E/{}^H E$  is a Galois extension, as  $H$  is normal in  $G$ . Moreover,

$$[{}^H E : K] = (G : H) = p.$$

Since  $\omega \in K$ , Lemma 11.39 implies that  ${}^H E = K(\beta)$  for some  $\beta$  such that  $\beta^p \in K$ . The radical tower 11.1 can be extended by adding  $K \subseteq {}^H E$ .

For the general case, let  $K^* = K(\omega)$  and  $E^* = E(\omega)$ . Then  $E^*/K^*$  is a Galois extension with Galois group  $\text{Gal}(E^*/K^*)$ . By Exercise 11.40,  $\text{Gal}(E^*/K^*)$  is solvable. By the previous part,  $E^*$  and  $E$  can be embedded in a radical extension  $R^*/K^*$ , so there exists a radical tower

$$(11.2) \quad K^* \subseteq R_1^* \subseteq R_2^* \subseteq \cdots \subseteq R_n^*.$$

Since  $K^* = K(\omega)$  is a pure extension, the radical tower (11.2) can be extended by adding  $K \subseteq K^*$ .  $\square$

## 12. Lecture – Week 12

**§ 12.1. Group cohomology.** Let  $G$  be a group and  $A$  be a (**left**)  **$G$ -module**. This means that  $A$  is an abelian group together with a map

$$G \times A \rightarrow A, \quad (g, a) \mapsto g \cdot a$$

such that  $1 \cdot a = a$  for all  $a \in A$ ,  $(gh) \cdot a = g \cdot (h \cdot a)$  for all  $g, h \in G$  and  $a \in A$  and  $g \cdot (a + b) = g \cdot a + g \cdot b$  for all  $g \in G$  and  $a, b \in A$ .

12.1. EXAMPLE. The group  $\text{Gal}(\mathbb{C}/\mathbb{R})$  acts on  $\mathbb{C}$  and  $\mathbb{C}^\times$ . Moreover, it acts trivially on  $\mathbb{R}$  and  $\mathbb{R}^\times$ .

More generally, if  $E/K$  is a finite Galois extension, then the Galois group  $\text{Gal}(E/K)$  acts on  $E$  and  $E^\times$ .

12.2. DEFINITION. Let  $G$  be a group and  $M$  and  $N$  be  $G$ -modules. A map  $f: M \rightarrow N$  is a **homomorphism** of  $G$ -modules if  $f(\sigma \cdot m) = \sigma \cdot f(m)$  for all  $m \in M$  and  $\sigma \in G$ .

12.3. DEFINITION. Let  $G$  be a group and  $M$  be a  $G$ -module. The submodule of  **$G$ -invariants** is defined as

$$M^G = \{m \in M : \sigma \cdot m = m \text{ for all } \sigma \in G\}.$$

Note that  $M^G$  is the largest submodule of the  $G$ -module  $M$  where  $G$  acts trivially. For example, if  $G = \text{Gal}(E/K)$ , then  $E^G = K$ .

12.4. PROPOSITION. Let  $G$  be a group. If the sequence of  $G$ -modules and  $G$ -module homomorphism

$$0 \longrightarrow P \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$$

is exact, then

$$0 \longrightarrow P^G \xrightarrow{\alpha^0} M^G \xrightarrow{\beta^0} N^G$$

is exact, where  $\alpha^0$  is the restriction  $\alpha|_{P^G}$  of  $\alpha$  to  $P^G$  and  $\beta^0$  is the restriction  $\beta|_{M^G}$  of  $\beta$  to  $M^G$ .

PROOF. Since  $\alpha$  is injective, the restriction  $\alpha^0$  is injective.

Note that  $\ker \beta^0 = \ker \beta \cap M^G \subseteq \ker \beta$ .

We claim that  $\alpha^0(P^G) = \alpha(P) \cap M^G$ . If  $m \in \alpha(P) \cap M^G$ , then  $\alpha(p) = m$  for some  $p \in P$  and  $\sigma \cdot m = m$ . Since

$$\alpha(p) = m = \sigma \cdot m = \sigma \cdot \alpha(p) = \alpha(\sigma \cdot p),$$

$\sigma \cdot p - p \in \ker \alpha = \{0\}$ . Hence  $\sigma \cdot p = p$  and  $p \in P^G$ . Conversely, if  $m \in \alpha^0(P^G)$ , then  $m = \alpha(p)$  for some  $p \in P^G$ . If  $\sigma \in G$ , then

$$\sigma \cdot m = \sigma \cdot \alpha(p) = \alpha(\sigma \cdot p) = \alpha(p) = m.$$

Hence  $m \in M^G \cap \alpha(P)$ .

Now

$$\alpha^0(P^G) = \alpha(P) \cap M^G = \ker \beta \cap M^G = \ker \beta^0. \quad \square$$

Note that in the previous proposition, we did not prove that the map  $\beta|_{M^G}$  is surjective.

12.5. EXAMPLE. Let  $G = \text{Gal}(\mathbb{C}/\mathbb{R})$ . Consider the following exact sequence of  $G$ -modules:

$$1 \longrightarrow \{-1, 1\} \longrightarrow \mathbb{C}^\times \xrightarrow{\beta} \mathbb{C}^\times \longrightarrow 1$$

where  $\beta(z) = z^2$ . Note that  $\beta$  is surjective. Take invariants to obtain the sequence

$$0 \longrightarrow \{-1, 1\} \longrightarrow \mathbb{R}^\times \xrightarrow{\beta^0} \mathbb{R}^\times$$

where  $\beta^0(x) = x^2$ . Note that  $\beta^0$  is not surjective!

12.6. DEFINITION. Let  $G$  be a group and  $N$  be a  $G$ -module. We define

$$H^0(G, M) = M^G,$$

$$C^1(G, M) = \{\phi: G \rightarrow M : \phi \text{ is a map}\},$$

$$Z^1(G, M) = \{\phi \in C^1(G, M) : \phi(\sigma\tau) = \phi(\sigma) + \sigma \cdot \phi(\tau) \text{ for all } \sigma, \tau \in G\},$$

Note that  $Z^1(G, M)$  is an abelian group with the operation

$$(\phi + \phi_1)(\sigma) = \phi(\sigma) + \phi_1(\sigma).$$

Moreover, if  $\phi \in Z^1(G, M)$ , then  $\phi(1_G) = 0_M$ . To prove this fact, note that

$$\phi(1_G) = \phi(1_G 1_G) = \phi(1_G) + 1_G \cdot \phi(1_G) = \phi(1_G) + \phi(1_G)$$

implies that  $\phi(1_G) = 0_M$ .

12.7. EXAMPLE. Let  $G$  be a group and  $M$  be a  $G$ -module. Fix  $m \in M$ . Then the map  $\phi: G \rightarrow M$ ,  $\phi(\sigma) = \sigma \cdot m - m$ , is an element of  $Z^1(G, M)$ , because

$$\begin{aligned} \phi(\sigma\tau) &= (\sigma\tau) \cdot m - m \\ &= (\sigma\tau) \cdot m - \sigma \cdot m + \sigma \cdot m - m \\ &= \sigma \cdot (\tau \cdot m - m) + \sigma \cdot m - m \\ &= \sigma \cdot \phi(\tau) + \phi(\sigma) \end{aligned}$$

for all  $\sigma, \tau \in G$ .

12.8. DEFINITION. Let  $G$  be a group and  $M$  be a  $G$ -module. The set  $B^1(G, M)$  of **coboundaries** is the set of elements  $\phi \in C^1(G, M)$  such that there is a fixed  $m \in M$  such that  $\phi(\sigma) = \sigma \cdot m - m$  for all  $\sigma \in G$ .

We proved in Example 12.7 that  $B^1(G, M) \subseteq Z^1(G, M)$ . A direct calculation shows that, in fact,  $B^1(G, M)$  is a subgroup of  $Z^1(G, M)$ .

12.9. DEFINITION. Let  $G$  be a group and  $M$  be a  $G$ -module. The **first cohomology group** of  $G$  with coefficients in  $M$  is defined as the quotient

$$H^1(G, M) = Z^1(G, M)/B^1(G, M).$$

12.10. EXAMPLE. If  $G$  acts trivially on  $M$ , then

$$H^0(G, M) = M^G = M, \quad B^1(G, M) = \{0\}, \quad Z^1(G, M) = \text{Hom}(G, M).$$

Hence  $H^1(G, M) \simeq \text{Hom}(G, M)$ .

12.11. EXAMPLE. Let  $G = \text{Gal}(\mathbb{C}/\mathbb{R}) = \{\text{id}, \gamma\}$ , where  $\gamma: \mathbb{C} \rightarrow \mathbb{C}$ ,  $z \mapsto \bar{z}$ , is the complex conjugation. Then

$$H^0(G, \mathbb{R}^\times) = (\mathbb{R}^\times)^G = \mathbb{R}^\times.$$

Since  $G$  acts trivially on  $\mathbb{R}^\times$ ,

$$H^1(G, \mathbb{R}^\times) = \text{Hom}(G, \mathbb{R}^\times) \simeq \text{Hom}(G, \{-1, 1\}) \simeq \mathbb{Z}/2.$$

The following lemma will be useful.

12.12. LEMMA. *Let  $G$  be a group and  $\alpha: M \rightarrow N$  be a homomorphism of  $G$ -modules. Then*

$$\alpha^1: H^1(G, M) \rightarrow H^1(G, N), \quad \phi + B^1(G, M) \mapsto \alpha \circ \phi + B^1(G, N),$$

*is a group homomorphism.*

PROOF. Let us prove that the map  $\alpha^1$  is well-defined. If  $\phi - \phi' \in B^1(G, M)$ , then there exists a fixed  $m \in M$  such that  $(\phi - \phi')(\sigma) = \sigma \cdot m - m$  for all  $\sigma \in G$ . Let  $n = \alpha(m) \in N$ . For  $\sigma \in G$ ,

$$\alpha((\phi - \phi')(\sigma)) = \alpha(\sigma \cdot m - m) = \sigma \cdot \alpha(m) - \alpha(m) = \sigma \cdot n - n.$$

Thus  $\alpha \circ \phi - \alpha \circ \phi' \in B^1(G, N)$ .

We now prove that  $\alpha^1$  is a group homomorphism. If  $\phi, \phi' \in Z^1(G, M)$ , then

$$\begin{aligned} \alpha^1(\phi + B^1(G, M) + \phi' + B^1(G, M)) &= \alpha^1(\phi + \phi' + B^1(G, M)) \\ &= \alpha \circ (\phi + \phi') + B^1(G, N) \\ &= \alpha \circ \phi + \alpha \circ \phi' + B^1(G, N) \\ &= \alpha \circ \phi + B^1(G, N) + \alpha \circ \phi' + B^1(G, N) \\ &= \alpha^1(\phi + B^1(G, M)) + \alpha^1(\phi' + B^1(G, M)). \end{aligned} \quad \square$$

We will provide a detailed proof of the upcoming result. The theorem will be established by applying a **diagram chasing** technique, a widely used tool in homological algebra. The proof is tedious and may seem intricate, but the method essentially involves “chasing” elements around a (commutative) diagram until we attain the desired result.

12.13. THEOREM. *Let  $G$  be a group and*

$$0 \longrightarrow P \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$$

*be an exact sequence of  $G$ -modules and  $G$ -module homomorphism. Then there exists a **connection homomorphism**  $\delta$  such that the sequence*

$$(12.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & H^0(G, P) & \xrightarrow{\alpha^0} & H^0(G, M) & \xrightarrow{\beta^0} & H^0(G, N) \\ & & \overbrace{H^1(G, P) \xrightarrow{\alpha^1} H^1(G, M) \xrightarrow{\beta^1} H^1(G, N)}^{\delta} & & & & \end{array}$$

*of abelian groups and group homomorphisms is exact.*

PROOF. By Proposition 12.4, the long sequence (12.1) is exact at  $H^0(G, P) = P^G$ ,  $H^0(G, M) = M^G$  and  $H^0(G, N) = N^G$ . Note that, in particular,  $\alpha: P \rightarrow \alpha(P)$  is a bijective group homomorphism.

Let us construct the connecting homomorphism  $\delta: H^0(G, N) \rightarrow H^1(G, P)$ . For  $n \in N^G$ , let  $m \in M$  be such that  $\beta(m) = n$ . We define  $\delta(n) = \phi + B^1(G, P)$ , where

$$\phi(\sigma) = \alpha^{-1}(\sigma \cdot m - m).$$

Note that  $\sigma \cdot m - m \in \text{im } \alpha = \ker \beta$ , as

$$\beta(\sigma \cdot m - m) = \sigma \cdot \beta(m) - \beta(m) = \sigma \cdot n - n = 0.$$

Let us prove that the map  $\delta$  is well-defined: if  $m, m' \in M$  are such that

$$\beta(m) = \beta(m') = n,$$

then  $m - m' \in \ker \beta = \alpha(P)$ . For  $\sigma \in G$ , write  $\phi'(\sigma) = \sigma \cdot m' - m'$ . Since  $m - m' = \alpha(p)$  for some  $p \in P$  and  $\alpha^{-1}$  is a homomorphism of  $G$ -modules,

$$\begin{aligned} \phi(\sigma) - \phi'(\sigma) &= \alpha^{-1}(\sigma \cdot m - m) - \alpha^{-1}(\sigma \cdot m' - m') \\ &= \alpha^{-1}(\sigma \cdot (m - m')) - \alpha^{-1}(m - m') \\ &= \alpha^{-1}(\sigma \cdot \alpha(p) - \alpha(p)) \\ &= \sigma \cdot p - p. \end{aligned}$$

Thus  $\phi - \phi' \in B^1(G, P)$ .

Note that  $\phi \in Z^1(G, P)$ , because

$$\begin{aligned} \phi(\sigma\tau) &= \alpha^{-1}((\sigma\tau) \cdot m - m) \\ &= \alpha^{-1}((\sigma\tau) \cdot m - \sigma \cdot m + \sigma \cdot m - m) \\ &= \alpha^{-1}(\sigma \cdot (\tau \cdot m - m)) + \alpha^{-1}(\sigma \cdot m - m) \\ &= \sigma \cdot \phi(\tau) + \phi(\sigma) \end{aligned}$$

holds for all  $\sigma, \tau \in G$ .

We now prove that the sequence (12.1) is exact at  $H^0(G, N) = N^G$ . We need to prove that  $\ker \delta = \text{im } \beta^0$ . To prove  $\supseteq$  note that if  $m \in M^G$  is such that  $\delta(\beta(m)) = \phi + B^1(G, P)$ , then

$$\phi(\sigma) = \alpha^{-1}(\sigma \cdot m - m) = 0.$$

Conversely, if  $n \in \ker \delta$ , then there exists  $m \in M$  such that  $\beta(m) = n$  and

$$\delta(\beta(m)) = \phi + B^1(G, P),$$

where  $\phi \in B^1(G, P)$  and  $\phi(\sigma) = \alpha^{-1}(\sigma \cdot m - m)$  for all  $\sigma \in G$ . Since  $\phi \in B^1(G, P)$ , there exists  $p \in P$  such that  $\phi(\sigma) = \sigma \cdot p - p$  for all  $\sigma \in G$ . Note that

$$\beta(m - \alpha(p)) = \beta(m) - \beta(\alpha(p)) = \beta(m) = n.$$

Moreover,  $m - \alpha(p) \in M^G$ , as  $\sigma \cdot (m - \alpha(p)) = m - \alpha(p)$ . Hence  $n \in \text{im } \beta^0$ .

We now prove that (12.1) is exact at  $H^1(G, P)$ , that is  $\text{im } \delta = \ker \alpha^1$ . To prove  $\subseteq$  note that for  $n \in N^G$ ,  $\delta(n) = \phi + B^1(G, P)$ , where  $\phi(\sigma) = \alpha^{-1}(\sigma \cdot m - m)$  for all  $\sigma \in G$  and some  $m \in M$  such that  $\beta(m) = n$ . In particular,  $\alpha \circ \phi \in B^1(G, M)$ . Then

$$\alpha^1(\phi + B^1(G, P)) = \alpha \circ \phi + B^1(G, M) = B^1(G, M).$$

To prove  $\supseteq$ , let  $\phi + B^1(G, P) \in \ker \alpha^1$ . Then  $\alpha \circ \phi \in B^1(G, M)$ , that is  $\alpha(\phi(\sigma)) = \sigma \cdot m - m$  for all  $\sigma \in G$  and some  $m \in M$ . Note that

$$\delta(\beta(m)) = \psi + B^1(G, P),$$

where  $\psi(\sigma) = \alpha^{-1}(\sigma \cdot m - m)$ . This implies that  $\alpha(\psi(\sigma)) = \alpha(\phi(\sigma))$  for all  $\sigma \in G$ . Since  $\alpha$  is injective,  $\psi = \phi$ . Therefore  $\phi + B^1(G, P)$  belongs to the image of  $\delta$ .

Finally, we prove that the sequence (12.1) is exact at  $H^1(G, M)$ , that is  $\text{im } \alpha^1 = \ker \beta^1$ . To prove  $\subseteq$  note that

$$\beta^1(\alpha^1(\phi + B^1(G, P))) = \beta^1(\alpha \circ \phi + B^1(G, M)) = (\beta \circ \alpha) \circ \phi + B^1(G, N) = B^1(G, N).$$

Conversely, let  $\phi + B^1(G, M) \in \ker \beta_1$ . Then  $\beta \circ \phi \in B^1(G, N)$ . Thus there exists  $n \in N$  such that  $\beta(\phi(\sigma)) = \sigma \cdot n - n$  for all  $\sigma \in G$ . Since  $\beta$  is surjective,  $n = \beta(m)$  for some  $m \in M$ . Hence

$$\beta(\phi(\sigma)) = \sigma \cdot n - n = \sigma \cdot \beta(m) - \beta(m) = \beta(\sigma \cdot m - m).$$

For each  $\sigma \in G$ ,  $\phi(\sigma) - (\sigma \cdot m - m) \in \ker \beta = \text{im } \alpha$ . and therefore  $\phi(\sigma) - (\sigma \cdot m - m) = \alpha(\rho_\sigma)$ . This defines a map  $\rho: G \rightarrow P$ ,  $\sigma \mapsto \rho_\sigma$ . We claim that  $\rho \in Z^1(G, P)$ . If  $\sigma, \tau \in G$ , then

$$\begin{aligned} \alpha(\rho_{\sigma\tau}) &= \phi(\sigma\tau) - ((\sigma\tau) \cdot m - m) \\ &= \phi(\sigma) + \sigma \cdot \phi(\tau) - (\sigma \cdot (\tau \cdot m - m) + \sigma \cdot m - m) \\ &= \alpha(\rho_\sigma) + \sigma \cdot \alpha(\rho_\tau). \end{aligned}$$

Since  $\alpha$  is injective, it follows that  $\rho \in Z^1(G, P)$ . Now

$$\alpha_1(\rho + B^1(G, P)) = \alpha \circ \rho + B^1(G, M) = \phi + B^1(G, M). \quad \square$$

12.14. THEOREM. *Let  $G$  be a finite group and  $M$  be a  $G$ -module. Then*

$$|G|H^1(G, M) = \{0\}.$$

PROOF. Let  $n = |G|$ . It is enough to prove that if  $\phi \in Z^1(G, M)$ , then  $n\phi \in B^1(G, M)$ . Let  $\phi \in Z^1(G, M)$  and  $\sigma \in G$ . Then

$$\phi(\sigma\tau) = \phi(\sigma) + \sigma \cdot \phi(\tau)$$

for all  $\tau \in G$ . Summing over all possible  $\tau \in G$ , we obtain that

$$(12.2) \quad \sum_{\tau \in G} \phi(\tau) = \sum_{\tau \in G} \phi(\sigma\tau) = \sum_{\tau \in G} \sigma \cdot \phi(\tau) + \sum_{\tau \in G} \phi(\sigma) = n\phi(\sigma).$$

Let  $m = -\sum_{\tau \in G} \phi(\tau) \in M$ . Then (12.2) can be rewritten as

$$-m = \sum_{\tau \in G} \phi(\tau) = \sigma \cdot \sum_{\tau \in G} \phi(\tau) + n\phi(\sigma) = -\sigma \cdot m + n\phi(\sigma).$$

Thus  $n\phi(\sigma) = \sigma \cdot m - m$  and hence  $n\phi \in B^1(G, M)$ .  $\square$

12.15. EXERCISE. Let  $G$  be a finite group and  $M$  be a finite  $G$ -module of size coprime to  $|G|$ . Prove that  $H^1(G, M) = \{0\}$ .

12.16. EXERCISE. Let  $G$  be a finite group and  $M$  be a finitely generated  $G$ -module. Prove that  $H^1(G, M)$  is finite.

### Some solutions

**1.3.** Assume that  $\mathbb{Q}[i]$  and  $\mathbb{Q}[\sqrt{2}]$  were isomorphic and let  $\varphi : \mathbb{Q}[i] \rightarrow \mathbb{Q}[\sqrt{2}]$  be a field isomorphism. Then

$$\varphi(i)^2 = \varphi(i^2) = \varphi(-1) = -\varphi(1) = -1.$$

But  $\varphi(i) \in \mathbb{Q}[\sqrt{2}]$  and  $\mathbb{Q}[\sqrt{2}] \subseteq \mathbb{R}$  where every square is positive, a contradiction.

**1.4.** Let  $t > 0$  be the characteristic of a field  $K$  and let  $\varphi : \mathbb{Z} \rightarrow K$ ,  $x \mapsto x1$ . Then, by definition,  $\ker \varphi$  is the ideal generated by  $t$ . On the other hand, the image  $\varphi(\mathbb{Z})$  is a domain, being a subring of a field and is isomorphic to  $\mathbb{Z}/\ker \varphi$ . Therefore,  $\ker \varphi$  is a prime ideal of  $\mathbb{Z}$ , i.e.  $t$  is a prime number.

**1.6.** Let  $\varphi : \mathbb{Z} \rightarrow K$ ,  $x \mapsto x1$ .

We first prove that **1)** implies all the other properties. So suppose that the characteristic of  $K$  is zero, i.e.  $\ker \varphi = \{0\}$ .

Then  $m1 = 0$  if and only if  $m = 0$ , i.e. the order of 1 is infinite.

Let  $0 \neq x \in K$ . If  $mx = 0$ , then  $0 = mx = (m1)x$ . But  $K$  is a field and  $x \neq 0$ , hence  $m1 = 0$ , so  $m \in \ker \varphi = \{0\}$ . Hence the order of  $x$  is infinite.

By definition, the prime subring of  $K$  is the image of  $\varphi$ , which so it is isomorphic to  $\mathbb{Z}/\ker \varphi = \mathbb{Z}$ .

Finally, we prove that **4)** implies **1)**. Take  $m \in \ker \varphi$ . Then  $m1 = 0$ , but 1 has infinite order, hence  $m = 0$ . Therefore  $\ker \varphi = \{0\}$ , i.e.  $K$  has characteristic 0.

**1.7.** Let  $\varphi : \mathbb{Z} \rightarrow K$ ,  $x \mapsto x1$ .

We first prove that **1)** implies all the other properties. So suppose that the characteristic of  $K$  is  $p > 0$  i.e.  $\ker \varphi$  is the ideal generated by  $p$ .

Then  $m1 = 0$  if and only if  $p$  divides  $m$ , i.e. the order of 1 is  $p$ .

Let  $0 \neq x \in K$ . If  $mx = 0$ , then  $0 = mx = (m1)x$ . But  $K$  is a field and  $x \neq 0$ , hence  $m1 = 0$ , so  $p$  divides  $m$ . Hence  $x$  has order  $p$ .

By definition, the prime subring of  $K$  is the image of  $\varphi$ , which so it is isomorphic to  $\mathbb{Z}/\ker \varphi \cong \mathbb{Z}/p$ .

Finally, we prove that **4)** implies **1)**. Take  $m \in \ker \varphi$ . Then  $m1 = 0$ , but 1 has order  $p$ , hence  $p$  divides  $m$ . Therefore  $\ker \varphi$  is generated by  $p$ , i.e.  $K$  has characteristic 0.

**1.10.** Let  $\Phi : K \rightarrow K$  be the map  $x \mapsto x^p$ . Since the map  $x \mapsto x^{p^n}$  is exactly  $\Phi^n$ , it is enough to prove that  $\Phi$  is a field homomorphism.

As  $K$  is commutative under multiplication, for all  $x, y \in K$

$$\Phi(xy) = (xy)^p = x^p y^p = \Phi(x)\Phi(y).$$

Moreover, for all  $x, y \in K$

$$\Phi(x+y) = (x+y)^p \sum_{k=0}^p \binom{p}{k} x^p y^{p-k} = x^p + y^p + \sum_{k=1}^{p-1} \binom{p}{k} x^k y^{p-k},$$

where  $\binom{p}{k} = \frac{p!}{k!(p-k)!}$ , which can also be written as

$$p! = \binom{p}{k} \cdot k! \cdot (p-k)!$$

But  $p$  divides  $p!$ , so  $p$  has to divide at least one factor on the right side. But  $p$  doesn't divide  $i$  for  $1 \leq i \leq p-1$ , therefore if  $k \leq p-1$ ,  $p$  doesn't divide  $k!$  and if  $1 \leq k$ ,  $p$  doesn't divide  $(p-k)!$ . Hence, if  $1 \leq k \leq p-1$ ,  $p$  has to divide  $\binom{p}{k}$  and

$$\sum_{k=1}^{p-1} \binom{p}{k} x^k y^{p-k} = 0.$$

Therefore,  $\Phi$  is a field homomorphism.

**1.22.** By definition  $K_0 = \{m1 : m \in \mathbb{Z}\}$  and  $\sigma : K \rightarrow K$  is a field homomorphism, so  $\sigma(1) = 1$ . Hence, for every  $m \in \mathbb{Z}$

$$\sigma(m1) = m\sigma(1) = m1,$$

i.e.  $\sigma|_{K_0}$  is the identity.

**1.25.** If  $X^3 - 2$  were reducible, since it has degree 3, it would have a linear factor in the decomposition in irreducibles. Therefore it would have a rational root. But the roots of  $X^3 - 2$  are  $\sqrt[3]{2}, \sqrt[3]{2}\xi, \sqrt[3]{2}\xi^2$ , where  $\xi$  is a primitive third root of unity, So all the roots are not in  $\mathbb{Q}$ , a contradiction.

**1.26.** Recall first the following:

1. LEMMA (Gauss' Lemma for integer polynomials). A non-constant polynomial  $f \in \mathbb{Z}[X]$  is irreducible if and only if it is primitive and irreducible in  $\mathbb{Q}[X]$ .

2. LEMMA. Let  $p$  be a prime number and  $\pi_p : \mathbb{Z} \rightarrow \mathbb{Z}/p$  be the canonical homomorphism. The map  $\bar{\pi}_p : \mathbb{Z}[X] \rightarrow \mathbb{Z}/p[X]$ , defined as

$$\bar{\pi}_p : f(X) = \sum_{i=0}^n a_i X^i \mapsto \sum_{i=0}^n \pi_p(a_i) X^i$$

is a ring homomorphism.

Suppose that  $f$  is reducible in  $\mathbb{Q}[X]$ . Then  $\tilde{f} = c^{-1}f$ , where  $c$  is the content of  $f$ , would be reducible and primitive. Hence, by Gauss' Lemma for integer polynomials,  $\tilde{f}$  is also reducible in  $\mathbb{Z}[X]$ . So we can write  $c^{-1}f = \tilde{f} = gh$ , for some polynomials  $g, h \in \mathbb{Z}[X] \setminus \mathcal{U}(\mathbb{Z}[X]) = \mathbb{Z}[X] \setminus \{\pm 1\}$ . Write  $g(X) = \sum_{i=0}^k b_i X^i$  and  $h(X) = \sum_{i=0}^{n-k} c_i X^i$ , where  $k = \deg(g)$  and  $n-k = \deg(h)$ .

First, we prove that  $g$  and  $h$  are non-constant. Suppose for example that  $g$  is constant. Then, denoting by  $c : \mathbb{Z}[X] \rightarrow \mathbb{Z}$  the content map that sends a polynomial to its content, i.e. the gcd of all the coefficients of the polynomial,

$$\pm 1 = c(\tilde{f}) = c(gh) = gc(h).$$

Hence  $g \in \mathcal{U}(\mathbb{Z}) = \{\pm 1\}$ , a contradiction. In the same way, we can prove that  $h$  is also non-constant. Therefore  $0 < k < n$  and  $0 < n-k < n$ .

We now apply  $\pi_p$  to  $f$ . Since  $p$  divides  $a_i$  for all  $i \in \{0, 1, \dots, n-1\}$ , then  $\pi_p(a_i) = 0$  for all  $i \in \{0, 1, \dots, n-1\}$ . Therefore, using also the previous lemma, we have that

$$\pi_p(a_n)X^n = \bar{\pi}_p(f) = \pi_p(c)\bar{\pi}_p(g)\bar{\pi}_p(h) = \bar{\pi}_p(cg)\bar{\pi}_p(h).$$

But  $\mathbb{Z}/p[X]$  is a UFD and  $X$  is irreducible in  $\mathbb{Z}/p[X]$  (see ring and module theory course), so necessarily

$$\bar{\pi}_p(cg) = \pi_p(cb_k)X^k \quad \text{and} \quad \bar{\pi}_p(h) = \pi_p(c_{n-k})X^{n-k}.$$

In particular, since  $g$  and  $h$  are non-constant,  $\overline{\pi_p}(cg)$  and  $\overline{\pi_p}(h)$  both have a constant term equal to 0. Hence  $p$  divides  $cb_0$  and  $c_0$  in  $\mathbb{Z}$ . Therefore,  $p^2$  divides  $cb_0c_0 = a_0$ , a contradiction with the hypothesis.

**1.27.** Recall first the following:

**3. LEMMA (Gauss' Lemma).** Let  $A$  be a unique factorization domain and  $K$  be its fraction field. A non-constant polynomial  $f \in A[X]$  is irreducible if and only if it is primitive and irreducible in  $K[X]$ .

Suppose that  $f$  is reducible in  $K[X]$ . Then  $g = c^{-1}f$ , where  $c$  is the content of  $f$ , would be reducible and primitive. Hence, by Gauss' Lemma,  $g$  is also reducible in  $A[X]$ . So  $c^{-1}f = g = hl$ , for some non-constant polynomials  $h, l \in A[X]$ . Now consider  $\pi : A \rightarrow A/(p)$ ,  $a \mapsto \bar{a}$  the natural surjection. We know that  $\bar{a}_i = 0$  for all  $i \in \{0, 1, \dots, n-1\}$  and  $\bar{a}_n \neq 0$ . Therefore

$$\overline{\pi}(ch)\overline{\pi}(l) = \bar{c}\overline{\pi}(h)\overline{\pi}(l) = \overline{\pi}(f) = \bar{a}_n X^n \in A/(p)[X].$$

But  $A/(p)[X]$  is a UFD so the only possibility is that  $\overline{\pi}(ch) = \bar{d}X^t$  and  $\overline{\pi}(l) = \bar{f}X^s$ , for some  $f, d \in A/(p) \setminus \{\bar{0}\}$  and  $t, s \in \{1, \dots, n-1\}$ . In particular,  $\overline{\pi}(ch)$  and  $\overline{\pi}(l)$  have both constant term equal to 0. Hence  $p$  divides  $ch(0)$  and  $l(0)$  in  $A$ . Therefore  $p^2$  divides  $ch(0)l(0) = f(0)$ , a contradiction.

**1.28.** It is easy to see that  $f$  satisfies the Eisenstein criterion for  $p = 2$  and  $g$  satisfies it for  $p = 5$ .

**1.29.**  $f = 3(X^{10} + 5X^2 - 15)$  is a product of 3 and  $(X^{10} + 5X^2 - 15)$ , which are both non-invertible elements of  $\mathbb{Z}[X]$ . Hence  $f$  is reducible.

**2.6.** Clearly for every field extension  $L/K$  and every  $\alpha \in L$  we have that  $K[\alpha] \subseteq K(\alpha)$ .

Vice versa take  $\frac{a+\sqrt{2}b}{c+\sqrt{2}d} \in \mathbb{Q}(\sqrt{2})$ , then we can write:

$$\frac{a+\sqrt{2}b}{c+\sqrt{2}d} = \frac{(a+\sqrt{2}b)(c-\sqrt{2}d)}{(c+\sqrt{2}d)(c-\sqrt{2}d)} = \frac{ac-2bd+(bc-ad)\sqrt{2}}{c^2-2d^2}.$$

Hence

$$\frac{a+\sqrt{2}b}{c+\sqrt{2}d} = \frac{ac-2bd}{c^2-2d^2} + \frac{bc-ad}{c^2-2d^2}\sqrt{2} \in \mathbb{Q}[\sqrt{2}].$$

**2.8.** Let  $f = f(x, K)$  be the minimal polynomial of  $x$  over  $K$  of degree  $\deg(f) = n$ . We claim that  $\{1, x, \dots, x^{n-1}\}$  is a basis of  $K(x)$  as a  $K$ -vector space.

To prove that  $\{1, x, \dots, x^{n-1}\}$  is a generating set, recall that  $K(x) = K[x]$ , since  $x$  is algebraic over  $K$ . Let  $z \in K(x) = K[x]$ , say  $z = h(x)$  for some  $h \in K[X]$ . Divide  $h$  by  $f$  to obtain polynomials  $q, r \in K[X]$  such that  $h = fq + r$ , where either  $r = 0$  or  $\deg r < \deg f = n$ . Then

$$z = h(x) = f(x)q(x) + r(x) = r(x).$$

Write  $r = \sum_{i=0}^{n-1} c_i X^i$  for some  $c_0, \dots, c_{n-1} \in K$ . Thus  $z = \sum_{i=0}^{n-1} a_i x^i \in \langle 1, x, \dots, x^{n-1} \rangle$ .

We now prove that  $\{1, x, \dots, x^{n-1}\}$  is linearly independent. If not, there exists a linear combination

$$0 = \sum_{i=0}^{n-1} a_i x^i$$

with  $a_0, \dots, a_{n-1} \in K$  not all zero. Then  $h(X) = \sum_{i=0}^{n-1} a_i X^i \in K[X] \setminus \{0\}$  has  $x$  as a root and

$$n = \deg(f) \leq \deg(h) \leq n - 1,$$

a contradiction.

**2.10.** The element  $a$  is algebraic over  $K$ , so, by Theorem 2.7, it has finite degree over  $K$  and  $K[a] = K(a)$ . Similarly,  $b$  is algebraic over  $K$ , so it is also algebraic over  $K(a)$ , hence it has finite degree over  $K(a)$  and  $K(a)[b] = K(a, b)$ . This implies that the extension  $K(a, b)/K$  is a finite extension since it is a tower of finite extensions. Hence, by Corollary 2.9,  $K(a, b)/K$  is an algebraic extension. Therefore, since  $a + b, ab \in K(a, b)$ , this implies that  $a + b$  and  $ab$  are algebraic over  $K$ .

**2.14.** Assume that  $K(S)/K$  is algebraic, then, by Corollary 2.12,  $x$  is algebraic over  $K$  for all  $x \in S$ . By Corollary 2.11, we conclude that  $K(S)/K$  is finite.

On the other hand, if  $K(S)/K$  is finite, then  $K \subseteq K(x) \subseteq K(S)$  for all  $x \in S$ , so  $K(x)/K$  is finite for all  $x \in S$ . Then, by Theorem 2.7,  $x$  is algebraic over  $K$  for all  $x \in S$ . Hence, by Corollary 2.12,  $K(S)/K$  is algebraic.

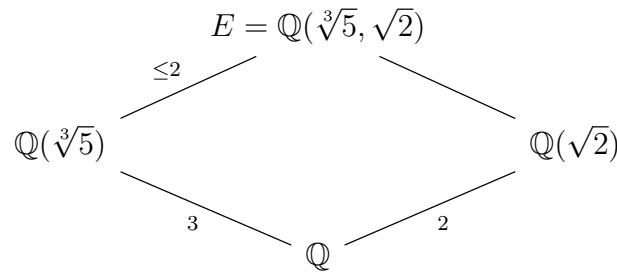
**2.15.**  $\sqrt[3]{2}$  is a root of the monic polynomial  $f = X^3 - 2 \in \mathbb{Q}[X]$ . Therefore  $\sqrt[3]{2}$  is algebraic over  $\mathbb{Q}$  and  $\mathbb{Q}[\sqrt[3]{2}] = \mathbb{Q}(\sqrt[3]{2})$ . In Exercise 1.25, we proved that  $f$  is irreducible in  $\mathbb{Q}[X]$ . Hence  $f$  is the minimal polynomial of  $\sqrt[3]{2}$  over  $\mathbb{Q}$  and, by Theorem 2.7,  $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = \deg f = 3$ .

**2.16.**  $i$  is a root of the monic polynomial  $X^2 + 1 \in \mathbb{Q}[X]$  and  $\sqrt{2}$  is a root of the monic polynomial of  $X^2 - 2 \in \mathbb{Q}[X]$ . So, by Corollary 2.11,  $\mathbb{Q}[i, \sqrt{2}] = \mathbb{Q}(i, \sqrt{2})$  and it is algebraic over  $\mathbb{Q}$ . By Eisenstein's criterion with  $p = 2$ ,  $X^2 - 2$  is irreducible in  $\mathbb{Q}[X]$ , so  $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$ . Since  $i$  is a root of  $X^2 + 1 \in \mathbb{Q}[X]$ , then  $[E : \mathbb{Q}(\sqrt{2})] \leq 2$ . Moreover,  $i \notin \mathbb{R} \supseteq \mathbb{Q}(\sqrt{2})$ . Therefore  $[E : \mathbb{Q}(\sqrt{2})] = 2$  and, by Proposition 2.4,

$$[E : \mathbb{Q}] = [E : \mathbb{Q}(\sqrt{2})][\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 4.$$

**2.17.**

- 1) We know that  $[\mathbb{Q}(\sqrt[3]{5}) : \mathbb{Q}]$  is the same as the degree of the minimal polynomial of  $\sqrt[3]{5}$  over  $\mathbb{Q}$ . Clearly  $\sqrt[3]{5}$  is a root of  $X^3 - 5 \in \mathbb{Q}[X]$ . Moreover, by Eisenstein's criterion with  $p = 5$ , we get that  $X^3 - 5$  is irreducible in  $\mathbb{Q}[X]$ . Hence  $f(\sqrt[3]{5}, \mathbb{Q}) = X^3 - 5$  and  $[\mathbb{Q}(\sqrt[3]{5}) : \mathbb{Q}] = 3$ . We also know that  $X^2 - 2$  is the minimal polynomial of  $\sqrt{2}$  over  $\mathbb{Q}$ . So  $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$ . Therefore we are in the following situation:



so on the one hand

$$[E : \mathbb{Q}] = [E : \mathbb{Q}(\sqrt{2})][\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = [E : \mathbb{Q}(\sqrt{2})]2$$

and on the other hand

$$[E : \mathbb{Q}] = [E : \mathbb{Q}(\sqrt[3]{5})][\mathbb{Q}(\sqrt[3]{5}) : \mathbb{Q}] = [E : \mathbb{Q}(\sqrt[3]{5})]3 \leq 2 \cdot 3 = 6.$$

Therefore 2 and 3 divide  $[E : \mathbb{Q}] \leq 6$ . Hence the only possibility is that  $[E : \mathbb{Q}] = 6$ .

- 2) Clearly  $\mathbb{Q}(\sqrt{2} + \sqrt[3]{5}) \subseteq E$ . On the other hand, let  $\alpha = \sqrt{2} + \sqrt[3]{5}$ . Then

$$5 = (\alpha - \sqrt{2})^3 = \alpha^3 - 3\sqrt{2}\alpha^2 + 6\alpha - 2\sqrt{2},$$

which implies that

$$\sqrt{2} = \frac{6\alpha - 5}{3\alpha^2 + 2} \in \mathbb{Q}(\alpha).$$

Moreover,  $\sqrt[3]{5} = \alpha - \sqrt{2} \in \mathbb{Q}(\alpha)$ , hence  $E = \mathbb{Q}(\alpha)$ .

- 3) From the previous part of this exercise we get that

$$\sqrt{2}(3\alpha^2 + 2) = \alpha^3 + 6\alpha - 5.$$

Hence, squaring both sides of the previous equality, we obtain

$$18\alpha^4 + 24\alpha^2 + 8 = \alpha^6 + 36\alpha^2 + 25 + 12\alpha^4 - 10\alpha^3 - 60\alpha.$$

Therefore  $\alpha$  is a root of the polynomial

$$f(X) = X^6 - 6X^4 - 10X^3 + 12X^2 - 60X + 17.$$

Moreover, from the first part, we know that

$$[E : \mathbb{Q}] = 6 = \deg f.$$

Hence  $f(\alpha, \mathbb{Q}) = f(X)$ .

**2.18.** Let  $\alpha = \sqrt[4]{3}i$ . Observe that  $\alpha^2 = -\sqrt{3}$ , so  $\mathbb{Q}(\sqrt{3}) \subseteq \mathbb{Q}(\alpha)$  and

$$2 \geq \deg f(\alpha, \mathbb{Q}(\sqrt{3})) = [\mathbb{Q}(\alpha) : \mathbb{Q}(\sqrt{3})].$$

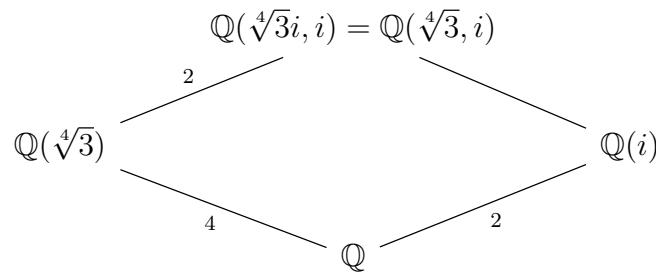
Moreover  $\alpha \notin \mathbb{R}$ , while  $\mathbb{Q}(\sqrt{3}) \subseteq \mathbb{R}$ . So  $[\mathbb{Q}(\alpha) : \mathbb{Q}(\sqrt{3})] > 1$ , hence  $[\mathbb{Q}(\alpha) : \mathbb{Q}(\sqrt{3})] = 2$  and

$$f(\alpha, \mathbb{Q}(\sqrt{3})) = X^2 + \sqrt{3}.$$

Note that the minimal polynomial of  $\alpha$  over  $\mathbb{Q}(i)$  has degree  $[\mathbb{Q}(\sqrt[4]{3}i, i) : \mathbb{Q}(i)]$ . Moreover,  $\mathbb{Q}(\sqrt[4]{3}i, i) = \mathbb{Q}(\sqrt[4]{3}, i)$  and

$$f(\alpha, \mathbb{Q}) = f(\sqrt[4]{3}, \mathbb{Q}) = X^4 + 3,$$

since  $X^4 + 3$  is an irreducible (due to Eisenstein with  $p = 3$ ) monic polynomial that has  $\alpha$  and  $\sqrt[4]{3}$  as roots. Therefore  $[\mathbb{Q}(\sqrt[4]{3}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt[4]{3}i) : \mathbb{Q}] = 4$ . Since  $\mathbb{Q}(\sqrt[4]{3}) \subseteq \mathbb{R}$ , while  $\sqrt[4]{3}i \notin \mathbb{R}$ , we obtain that  $1 < [\mathbb{Q}(\sqrt[4]{3}i, i) : \mathbb{Q}(\sqrt[4]{3})] \leq [\mathbb{Q}(i) : \mathbb{Q}] = 2$ .



Hence  $[\mathbb{Q}(\sqrt[4]{3}i, i) : \mathbb{Q}] = 8$  and  $[\mathbb{Q}(\sqrt[4]{3}i, i) : \mathbb{Q}(i)] = 4$ , which means that  $\deg f(\sqrt[4]{3}i, \mathbb{Q}(i)) = 4$ . But  $f(\sqrt[4]{3}i, \mathbb{Q}(i))$  divides  $f(\alpha, \mathbb{Q}) = X^4 + 3$ , so

$$f(\sqrt[4]{3}i, \mathbb{Q}(i)) = X^4 + 3.$$

**2.19.** Let  $\alpha = \sqrt{2} + i\sqrt[3]{5}$  and  $E = \mathbb{Q}(\alpha, i)$ . The minimal polynomial of  $\alpha$  over  $\mathbb{Q}(i)$  has degree  $[E : \mathbb{Q}(i)]$ . Observe that, since  $\alpha - \sqrt{2} = i\sqrt[3]{5}$ ,

$$\alpha^3 - 3\sqrt{2}\alpha^2 + 6\alpha - 2\sqrt{2} = -i5.$$

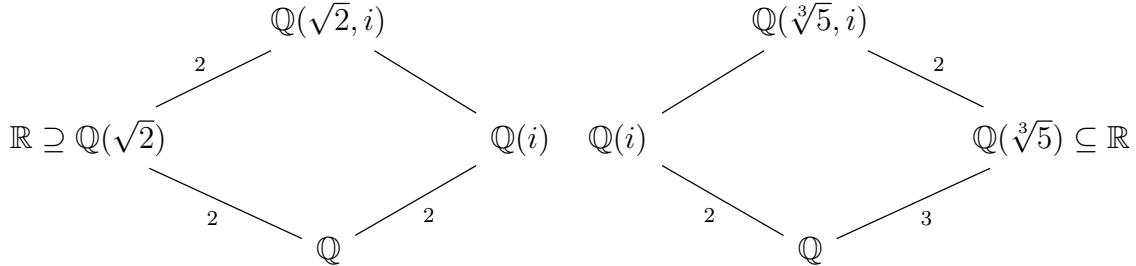
Hence

$$\sqrt{2} = \frac{\alpha^3\alpha^2 + 6\alpha + i5}{3\alpha^2 + 2} \in E$$

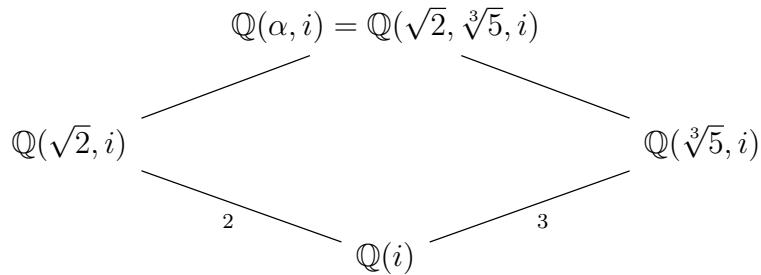
and so also  $\sqrt[3]{5} = \frac{\alpha - \sqrt{2}}{i} \in E$ . Therefore  $E = \mathbb{Q}(\sqrt{2}, \sqrt[3]{5}, i)$ . To compute  $[E : \mathbb{Q}(i)]$  we first compute  $[\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}(i)]$  and  $[\mathbb{Q}(\sqrt[3]{5}, i) : \mathbb{Q}(i)]$ .

We know that  $i$  has degree 2 over  $\mathbb{Q}$ , so  $[\mathbb{Q}(\sqrt[3]{5}, i) : \mathbb{Q}(\sqrt[3]{5})] = 2$  and  $[\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}(\sqrt{2})] = 2$  are both at most 2. Moreover  $\mathbb{Q}(\sqrt[3]{5})$  and  $\mathbb{Q}(\sqrt{2})$  are contained in  $\mathbb{R}$ , while  $i \notin \mathbb{R}$ . Hence

$$[\mathbb{Q}(\sqrt[3]{5}, i) : \mathbb{Q}(\sqrt[3]{5})] = 2 = [\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}(\sqrt{2})].$$



Therefore  $[\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}(i)] = 2$  and  $[\mathbb{Q}(\sqrt[3]{5}, i) : \mathbb{Q}(i)] = 3$ .



So  $[\mathbb{Q}(\alpha, i) : \mathbb{Q}(i)]$  is divisible by 2 and 3 and it is also at most 6. Therefore the degree of the minimal polynomial of  $\alpha$  over  $\mathbb{Q}(i)$  is  $[\mathbb{Q}(\alpha, i) : \mathbb{Q}(i)] = 6$ .

We already got  $\alpha^3 - 3\sqrt{2}\alpha^2 + 6\alpha - 2\sqrt{2} = -i5$ , so  $2(3\alpha^2 + 2)^2 = (\alpha^3\alpha^2 + 6\alpha + i5)^2$ , i.e.

$$\alpha^6 - 6\alpha^4 + 10i\alpha^3 + 12\alpha^2 + 60i\alpha - 33 = 0.$$

This means that  $\alpha$  is a root of the polynomial

$$f(X) = X^6 - 6X^4 + 10iX^3 + 12X^2 + 60iX - 33 \in \mathbb{Q}(i)[X].$$

Since  $f$  is also monic and of degree 6, we can deduce that  $f(\alpha, \mathbb{Q}(i)) = f$ .

**2.21.** By Proposition 2.4, we know that  $[E : K] = [E : F][F : K]$ , so  $[E : K]$  is finite if and only if  $[E : F]$  and  $[F : K]$  are finite.

**2.22.** Let  $P$  be the set  $\{\sum_{i=1}^m e_i f_i : m \in \mathbb{Z}_{>0}, e_i \in E, f_i \in F \text{ for all } i \in \{1, \dots, m\}\}$ . If  $\sum_{i=1}^m e_i f_i \in P$ , it is a  $E$ -linear combination of elements in  $F$ , so in particular it is an element in  $E(F) = EF$ . Hence  $P \subseteq EF$ .

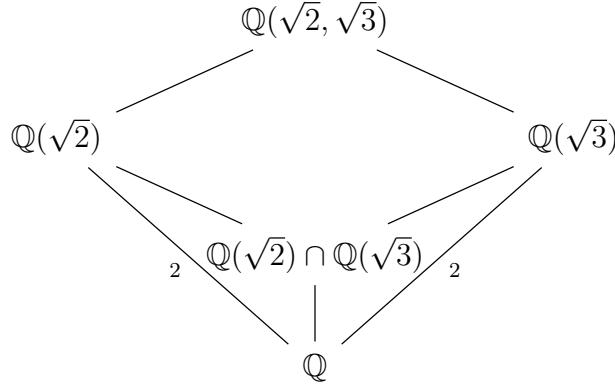
Moreover, since  $E/K$  and  $F/K$  are algebraic extensions, every element in  $E \cup F$  is algebraic over  $K$ . So  $EF = K(E \cup F) = K[E \cup F]$ . Let  $x \in EF$ , then  $x = f(\alpha_1, \dots, \alpha_k)$ , for some polynomial  $f \in K[X_1, \dots, X_k]$  and  $\alpha_1, \dots, \alpha_k \in E \cup F$ . We can then split the polynomial in  $f = p + q$  so that  $x = p(e'_1, \dots, e'_n) + q(f'_1, \dots, f'_m)$ , where  $e'_i \in E$  and  $f'_j \in F$  and  $p, q \in K[X_1, \dots, X_k]$ . Since  $E$  and  $F$  are fields, so closed under multiplication, we can write  $x$  as  $x = \sum_{i=1}^N k_i e_i + \sum_{j=1}^M h_j f_j$ , for some  $k_i, h_j \in K$ ,  $e_i \in E$  and  $f_j \in F$ . Then in particular  $k_i \in K \subseteq F$  and  $h_j \in K \subseteq E$ , hence  $x \in P$  and  $EF \subseteq P$ .

**2.23.** We know that the minimal polynomials over  $\mathbb{Q}$  are  $f(\sqrt{2}, \mathbb{Q}) = X^2 - 2$  and  $f(\sqrt{3}, \mathbb{Q}) = X^2 - 3$ . So  $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{3}) : \mathbb{Q}] = 2$ . Moreover,

$$[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{3})] \leq [\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2.$$

It remains to check whether  $\sqrt{2} \in \mathbb{Q}(\sqrt{3})$  or not. A  $\mathbb{Q}$  basis of  $\mathbb{Q}(\sqrt{3})$  is  $\{1, \sqrt{3}\}$ . If  $\sqrt{2} \in \mathbb{Q}(\sqrt{3})$ , then  $\sqrt{2} = a + b\sqrt{3}$  for some  $a, b \in \mathbb{Q}$ , so  $2 = a^2 + 2ab\sqrt{3} + 3b^2$ . Using the  $\mathbb{Q}$ -linear independence of  $\{1, \sqrt{3}\}$ , we get that  $2ab = 0$  and  $2 = a^2 + 3b^2$ . Therefore either  $a = 0$  and  $2/3 = b^2$ , or  $b = 0$  and  $2 = a^2$ . But both cases are not possible because neither 2 nor  $2/3$  are squares in  $\mathbb{Q}$ .

We conclude that  $\sqrt{2} \notin \mathbb{Q}(\sqrt{3})$ . So  $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{3})] = 2$  and  $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 4$ .



Similarly,

$$[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}(\sqrt{2}) \cap \mathbb{Q}(\sqrt{3})] \leq [\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2,$$

but  $\sqrt{2} \notin \mathbb{Q}(\sqrt{2}) \cap \mathbb{Q}(\sqrt{3})$ . Thus

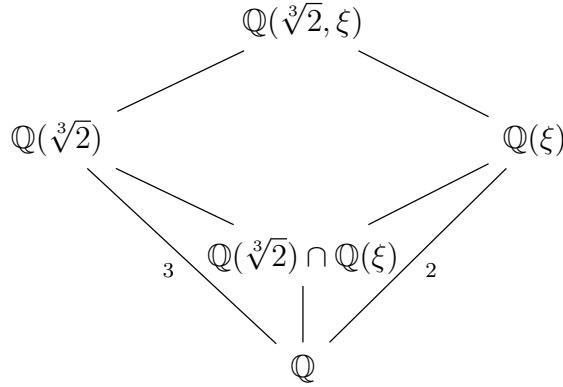
$$[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}(\sqrt{2}) \cap \mathbb{Q}(\sqrt{3})] = 2$$

and hence  $[\mathbb{Q}(\sqrt{2}) \cap \mathbb{Q}(\sqrt{3}) : \mathbb{Q}] = 1$ . Therefore  $\mathbb{Q}(\sqrt{2}) \cap \mathbb{Q}(\sqrt{3}) = \mathbb{Q}$ .

**2.24.** The minimal polynomial of  $\sqrt[3]{2}$  is  $X^3 - 2$  since it is monic, irreducible (by Eisenstein) and has  $\sqrt[3]{2}$  as a root. Hence  $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$ .  $\xi \neq 1$  and it is a root of the polynomial

$$X^3 - 1 = (X - 1)(X^2 + X + 1),$$

so it is a root of  $X^2 + X + 1$ , which is monic and irreducible (it is of degree 2 and the roots are not in  $\mathbb{Q}$ ). Hence  $f(\xi, \mathbb{Q}) = X^2 + X + 1$  and  $[\mathbb{Q}(\xi) : \mathbb{Q}] = 2$ . We also have that  $[\mathbb{Q}(\sqrt[3]{2}, \xi) : \mathbb{Q}] \leq 6$ .



By multiplicity of the degree of extensions, we obtain that 6 has to divide  $[\mathbb{Q}(\sqrt[3]{2}, \xi) : \mathbb{Q}] \leq 6$  and  $[\mathbb{Q}(\sqrt[3]{2}) \cap \mathbb{Q}(\xi) : \mathbb{Q}]$  has to divide 2 and 3. Therefore

$$[\mathbb{Q}(\sqrt[3]{2}, \xi) : \mathbb{Q}] = 6 \text{ and } [\mathbb{Q}(\sqrt[3]{2}) \cap \mathbb{Q}(\xi) : \mathbb{Q}] = 1,$$

which means that  $\mathbb{Q}(\sqrt[3]{2}) \cap \mathbb{Q}(\xi) = \mathbb{Q}$ .

**2.25.** By definition  $EF = E(F)$ . If  $F/K$  is algebraic, then  $EF$  is generated by algebraic elements over  $K$ , so also over  $E$ . Hence  $EF/E$  is an algebraic extension.

**2.26.** If  $F/K$  is finite, then  $F$  is generated by a finite number of algebraic elements over  $K$ . The same elements are algebraic over  $E$  and generate  $EF = E(F)$  over  $E$ . Hence  $EF/E$  is a finite extension and  $[EF : E] \leq [F : K]$ .

**4.9.**  $\varphi$  is a field homomorphism, so it is injective. Take  $\sigma : \varphi(C) \rightarrow C$  a left inverse of  $\varphi$ , i.e such that  $\sigma \circ \varphi = \text{id}$ . Then  $\sigma$  is a surjective field homomorphism (it is actually an isomorphism). Moreover  $C/K$  is algebraic and  $\varphi(C) \subseteq C$ , so, by Proposition 3.6, there exists a field homomorphism  $\psi : C \rightarrow C$  such that  $\psi|_{\varphi(C)} = \sigma$ . But then  $\psi$  is injective and  $\psi(x) = \sigma(\varphi(\psi(x))) = \psi(\varphi(\psi(x)))$ , for every  $x \in C$ . So  $x = \varphi(\psi(x))$  for every  $x \in C$ . Hence  $\varphi(C) = C$ ,  $\psi = \sigma$  and  $\varphi$  is an isomorphism.

**4.16.** We can easily check that  $f$  has no roots in  $\mathbb{Z}/3$ , so, having degree 3, it is irreducible over  $\mathbb{Z}/3$ . Note also that  $f = X^3 - X - 1 = X(X - 1)(X + 1) - 1$ . So if  $\alpha \in E$  is a root of  $f$ , then

$$f(\alpha + 1) = (\alpha + 1)\alpha(\alpha + 2) - 1 = (\alpha + 1)\alpha(\alpha - 1) - 1 = f(\alpha) = 0,$$

$$f(\alpha - 1) = (\alpha - 1)(\alpha - 2)\alpha - 1 = (\alpha - 1)(\alpha + 1)\alpha - 1 = f(\alpha) = 0.$$

This means that  $\alpha + 1$  and  $\alpha - 1$  are also roots of  $f$ . Hence  $E = \mathbb{Z}/3(\alpha)$  and  $f(\alpha, \mathbb{Z}/3) = f$ , so  $[E : \mathbb{Z}/3] = 3$ .

**4.17.** Note that, since  $f = X^4 - 5X^2 + 5$  is an even polynomial if  $\alpha \in \mathbb{C}$  is a root of  $f$ , then also  $-\alpha$  is a root of  $f$ . Hence, given two roots  $\alpha, \beta \in \mathbb{C}$  such that  $\beta \neq -\alpha$ , we have that the decomposition field of  $f$  over  $\mathbb{Q}$  is  $E = \mathbb{Q}(\alpha, -\alpha, \beta, -\beta)$ . But  $-\alpha, -\beta \in \mathbb{Q}(\alpha, \beta) \subseteq E$  and so

$$E = \mathbb{Q}(\alpha, -\alpha, \beta, -\beta) \subseteq \mathbb{Q}(\alpha, \beta) \subseteq \mathbb{Q}(\alpha, -\alpha, \beta, -\beta) = E,$$

which means that  $E = \mathbb{Q}(\alpha, \beta)$ . Moreover we can decompose  $f$  in  $\mathbb{C}[X]$  as

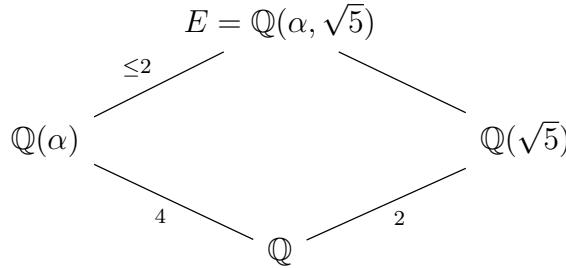
$$(X - \alpha)(X + \alpha)(X - \beta)(X + \beta) = (X^2 - \alpha^2)(X^2 - \beta^2) = X^4 - (\alpha^2 + \beta^2)X^2 + \alpha^2\beta^2.$$

This implies in particular that  $\alpha^2\beta^2 = 5$ , hence  $\beta = \pm\frac{\sqrt{5}}{\alpha} \in \mathbb{Q}(\alpha, \sqrt{5})$ .

Therefore  $E = \mathbb{Q}(\alpha, \beta) \subseteq \mathbb{Q}(\alpha, \sqrt{5})$ . On the other hand  $\sqrt{5} = \pm\alpha\beta \in \mathbb{Q}(\alpha, \beta)$ , hence  $\mathbb{Q}(\alpha, \sqrt{5}) \subseteq \mathbb{Q}(\alpha, \beta) = E$ . So we can conclude that  $E = \mathbb{Q}(\alpha, \sqrt{5})$ . Using the multiplicative of the degree of finite extension we get that

$$[E : \mathbb{Q}] = [E : \mathbb{Q}(\alpha)][\mathbb{Q}(\alpha) : \mathbb{Q}].$$

But  $[\mathbb{Q}(\alpha) : \mathbb{Q}] = \deg(f(\alpha, \mathbb{Q}))$ . Using Eisenstein criterion (Exercise 1.26) with  $p = 5$ , we have that  $f$  is irreducible (and monic), so  $f = f(\alpha, \mathbb{Q})$ . Thus  $[\mathbb{Q}(\alpha) : \mathbb{Q}] = \deg f = 4$ . It remains to compute  $[E : \mathbb{Q}(\alpha)] = [\mathbb{Q}(\alpha, \sqrt{5}) : \mathbb{Q}(\alpha)]$ . We have the following situation:



Observe that  $\mathbb{Q}(\alpha, \sqrt{5})$  is equal to the composite of  $\mathbb{Q}(\alpha)$  and  $\mathbb{Q}(\sqrt{5})$ . We can use the property of composite extension,  $[LF : L] \leq [F : K]$ , to deduce that

$$[\mathbb{Q}(\alpha, \sqrt{5}) : \mathbb{Q}(\alpha)] \leq [\mathbb{Q}(\sqrt{5}) : \mathbb{Q}] = 2.$$

The last equality is because  $f(\sqrt{5}, \mathbb{Q}) = X^2 - 5$ , as it is monic has  $\sqrt{5}$  as a root and it's irreducible (due to Eisenstein's criterion or because it is of degree 2 with 2 non-rational roots). Finally, we want to understand whether  $[\mathbb{Q}(\alpha, \sqrt{5}) : \mathbb{Q}(\alpha)]$  is 1 or 2. Note that  $\alpha^4 - 5\alpha^2 + 5 = 0$ , so we can solve the equation for  $\alpha^2$  as it is a root of  $X^2 - 5X + 5$ , i.e.

$$\alpha^2 = \frac{5 \pm \sqrt{25 - 20}}{2} = \frac{5 \pm \sqrt{5}}{2},$$

hence  $\sqrt{5} = \pm(2\alpha^2 - 5) \in \mathbb{Q}(\alpha)$ . So  $\mathbb{Q}(\alpha, \sqrt{5}) \subseteq \mathbb{Q}(\alpha) \subseteq \mathbb{Q}(\alpha, \sqrt{5})$ , which means that  $E = \mathbb{Q}(\alpha)$  and  $[E : \mathbb{Q}] = [\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$ .

**5.5.** First of all, note that  $\sqrt[3]{2}$  is a root of the polynomial  $f(X) = X^3 - 2$ . To prove that  $\mathbb{Q}(\sqrt[3]{2}, \xi)$  is a normal extension we use Proposition 5.10, so it is enough to prove that  $\mathbb{Q}(\sqrt[3]{2}, \xi)$  is the decomposition field of  $f$ . We know that the decomposition field  $E$  of  $f$  over  $\mathbb{Q}$  is  $\mathbb{Q}$  extended with the roots of  $f$ , i.e.  $E = \mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{2}\xi, \sqrt[3]{2}\xi^2)$ . But it's easy to see that actually

$$\mathbb{Q}(\sqrt[3]{2}, \xi) = \mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{2}\xi, \sqrt[3]{2}\xi^2) = E.$$

The inclusion  $\subseteq$  is because  $\sqrt[3]{2}, \xi = \frac{\sqrt[3]{2}\xi}{\sqrt[3]{2}} \in E$ . Vice versa  $\supseteq$  is due to the fact that the roots of  $f$  are products of  $\sqrt[3]{2}$  and  $\xi$ , elements in  $\mathbb{Q}(\sqrt[3]{2}, \xi)$ .

**5.11.** Let  $\alpha = \sqrt[4]{7} + \sqrt{2}$ . Then  $(\alpha - \sqrt{2})^4 - 7 = 0$ . By expanding the left side, we get

$$0 = \alpha^4 - 4\sqrt{2}\alpha^3 + 12\alpha^2 - 8\sqrt{2}\alpha - 3 = (\alpha^4 + 12\alpha^2 - 3) - (4\alpha^3 + 8\alpha)\sqrt{2}.$$

But  $4\alpha^3 + 8\alpha = 4\alpha(\alpha^2 + 2) \neq 0$ , otherwise  $\alpha \in \{0, \pm i\sqrt{2}\}$ . Therefore  $\sqrt{2} = \frac{\alpha^4 + 12\alpha^2 - 3}{4\alpha^3 + 8\alpha} \in \mathbb{Q}(\alpha)$ .

This allows us to prove that  $\mathbb{Q}(\sqrt{2}, \sqrt[4]{7}) = \mathbb{Q}(\alpha)$ . From the definition of  $\alpha$  it is clear that

$$\mathbb{Q}(\alpha) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt[4]{7}).$$

On the other hand, we just proved that  $\sqrt{2} \in \mathbb{Q}(\alpha)$ . As  $\sqrt[4]{7} = \alpha - \sqrt{2} \in \mathbb{Q}(\alpha)$ , we also see that  $\sqrt[4]{7} \in \mathbb{Q}(\alpha)$ . It follows that  $\mathbb{Q}(\sqrt{2}, \sqrt[4]{7}) \subseteq \mathbb{Q}(\alpha)$ .

Moreover,  $\sqrt{2} \notin \mathbb{Q}(\sqrt[4]{7})$ . Otherwise, as  $[\mathbb{Q}(\sqrt[4]{7}) : \mathbb{Q}] = 4$  and  $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$  we would get that  $[\mathbb{Q}(\sqrt[4]{7}) : \mathbb{Q}(\sqrt{2})] = 2$ . Let  $f(\sqrt[4]{7}, \mathbb{Q}(\sqrt{2})) = X^2 + \beta X + \gamma$ , with  $\beta, \gamma \in \mathbb{Q}(\sqrt{2})$ . So

$$0 = f(\sqrt[4]{7}) = \sqrt{7} + \beta \sqrt[4]{7} + \gamma.$$

Therefore

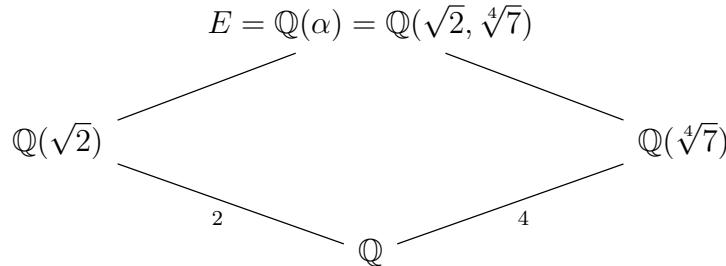
$$\beta^2 \sqrt{7} = (-\sqrt{7} - \gamma)^2 = 7 + 2\gamma\sqrt{7} + \gamma^2.$$

Thus

$$(\beta^2 - 2\gamma)\sqrt{7} = \gamma^2 + 7.$$

But  $\beta^2 - 2\gamma \neq 0$  because  $\gamma^2 + \beta\gamma + \frac{\beta^2}{2} = 0$  holds only for  $\gamma = \frac{\beta}{2}(-1 \pm i) \in \mathbb{C} \setminus \mathbb{R}$ , which is clearly not in  $\mathbb{Q}(\sqrt{2})$ . Thus  $\sqrt{7} = \frac{\gamma^2 + 7}{\beta^2 - 2\gamma} \in \mathbb{Q}(\sqrt{2})$ , a contradiction.

To sum up we have that  $\sqrt{2} \notin \mathbb{Q}(\sqrt[4]{7})$  and



1) We know that  $\sqrt[4]{7} \in \mathbb{Q}(\alpha)$  which has minimal polynomial  $f(\sqrt[4]{7}, \mathbb{Q}) = x^4 - 7$ . One root of this polynomial is  $i\sqrt[4]{7}$ . This root is not in  $\mathbb{Q}(\alpha) \subseteq \mathbb{R}$  as it is in  $\mathbb{C} \setminus \mathbb{R}$ . Therefore  $\mathbb{Q}(\alpha)/\mathbb{Q}$  is not normal by Proposition 5.7.

2) As  $\sqrt{2} \notin \mathbb{Q}(\sqrt[4]{7})$ , we see that  $[\mathbb{Q}(\alpha) : \mathbb{Q}(\sqrt[4]{7})] > 1$ . On the other hand,

$$[\mathbb{Q}(\alpha) : \mathbb{Q}(\sqrt[4]{7})] \leq [\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2,$$

which proves that  $[\mathbb{Q}(\alpha) : \mathbb{Q}(\sqrt[4]{7})] = 2$ . Therefore,

$$[E : \mathbb{Q}] = [\mathbb{Q}(\alpha) : \mathbb{Q}] = [\mathbb{Q}(\alpha) : \mathbb{Q}(\sqrt[4]{7})] \cdot [\mathbb{Q}(\sqrt[4]{7}) : \mathbb{Q}] = 2 \cdot 4 = 8.$$

3) Let  $\sigma \in G = \text{Gal}(E/\mathbb{Q})$ . Since  $E = \mathbb{Q}(\sqrt{2}, \sqrt[4]{7})$  and  $\sqrt{2}$  and  $\sqrt[4]{7}$  are independent because  $\sqrt{2} \notin \mathbb{Q}(\sqrt[4]{7})$ , we know that  $\sigma$  is completely determined by  $\sigma(\sqrt{2})$  and  $\sigma(\sqrt[4]{7})$ . By Proposition 4.10,  $\sigma(\sqrt{2}) \in E$  has to be a root of  $f(\sqrt{2}, \mathbb{Q}) = X^2 - 2$  and  $\sigma(\sqrt[4]{7}) \in E$  has to be a root of  $f(\sqrt[4]{7}, \mathbb{Q}) = X^4 - 7$ . So  $\sigma(\sqrt{2}) = \pm\sqrt{2}$  and, since  $E \subseteq \mathbb{R}$ ,

$$\sigma(\sqrt[4]{7}) \in E \cap \{\sqrt[4]{7}i^j \mid j \in \{0, 1, 2, 3\}\} = \{\pm\sqrt[4]{7}\}.$$

Therefore  $G$  contains 4 elements  $\sigma_{k,l}$  for  $k, l \in \mathbb{Z}/2$  such that  $\sigma_{k,l}(\sqrt{2}) = (-1)^k\sqrt{2}$  and  $\sigma_{k,l}(\sqrt[4]{7}) = (-1)^l\sqrt[4]{7}$ . This gives directly the isomorphism between  $G$  and  $\mathbb{Z}/2 \times \mathbb{Z}/2$ .

**5.12.** Let  $\{v_i : i \in I\}$  be a basis of  $V$  over  $K$ . For each  $i \in I$  let  $f_i : V \rightarrow F$ ,  $f_i(v_j) = \delta_{ij}$ . Then  $\{f_i : i \in I\}$  is linearly independent over  $F$ . In fact, let  $\sum a_i f_i = 0$ , where each  $a_i \in F$ . Then  $a_i = 0$  for almost all  $i$ . If  $j \in I$ , then

$$0 = \left( \sum a_i f_i \right) (v_j) = \sum a_i f_i(v_j) = a_j.$$

Now assume that  $\dim_K V = n$ . Let  $\{v_1, \dots, v_n\}$  be a basis of  $V$  over  $K$ . We claim that  $\{f_1, \dots, f_n\}$  is a basis of  $\text{Hom}_K(V, F)$  over  $F$ . If  $g \in \text{Hom}_K(V, F)$ , then  $g = \sum g(v_i) f_i$ . If  $1 \leq k \leq n$ , then

$$\left( \sum g(v_i) f_i \right) (v_k) = \sum g(v_i) f_i(v_k) = g(v_k).$$

**5.15.** We need to find a bijective map

$$\text{Hom}(E/K, C/K) \rightarrow \text{Hom}(E/K, C_1/K).$$

If  $\sigma \in \text{Hom}(E/K, C/K)$ , then  $\theta^{-1}\sigma \in \text{Hom}(E/K, C_1/K)$ . If  $\varphi \in \text{Hom}(E/K, C_1/K)$ , then  $\theta\varphi \in \text{Hom}(E/K, C/K)$ . The maps  $\sigma \mapsto \theta^{-1}\sigma$  and  $\varphi \mapsto \theta\varphi$  are inverse to each other.

**8.5.** We first prove that for every order reversing function  $\varphi$  and every element  $s, t$  in its domain,

$$\varphi(s \vee t) \leq \varphi(s) \wedge \varphi(t) \text{ and } \varphi(s) \vee \varphi(t) \leq \varphi(s \wedge t).$$

Since that  $s \leq s \vee t$  and  $t \leq s \vee t$  and  $\varphi$  is order reversing, we have that  $\varphi(s \vee t) \leq \varphi(s)$  and  $\varphi(s \vee t) \leq \varphi(t)$ . Hence  $\varphi(s \vee t) \leq \varphi(s) \wedge \varphi(t)$ . Moreover  $s \wedge t \leq s$  and  $s \wedge t \leq t$ . So  $\varphi(s) \leq \varphi(s \wedge t)$  and  $\varphi(t) \leq \varphi(s \wedge t)$ . Hence  $\varphi(s) \vee \varphi(t) \leq \varphi(s \wedge t)$ .

We can now apply this result for  $\varphi = f$ ,  $s = x$  and  $t = y$  obtaining that

$$f(x \vee y) \leq f(x) \wedge f(y), \quad f(x) \vee f(y) \leq f(x \wedge y).$$

On the other hand, for  $\varphi = f^{-1}$ ,  $s = f(x)$  and  $t = f(y)$ , we obtain that

$$f^{-1}(f(x) \vee f(y)) \leq f^{-1}(f(x)) \wedge f^{-1}(f(y)) = x \wedge y,$$

and

$$x \vee y = f^{-1}(f(x)) \vee f^{-1}(f(y)) \leq f^{-1}(f(x) \wedge f(y)).$$

Thus, applying  $f$ , which is order reversing, it implies that

$$f(x \wedge y) \leq f(f^{-1}(f(x) \vee f(y))) = f(x) \vee f(y)$$

and

$$f(x) \wedge f(y) = f(f^{-1}(f(x) \wedge f(y))) \leq f(x \vee y).$$

**8.7.** Since  $E/K$  is a Galois extension, the order of  $\text{Gal}(E/K)$  is precisely  $[E : K] = n$ . So, by Cauchy's Theorem, there exists a subgroup  $S$  of  $\text{Gal}(E/K)$  of order  $p$ . Then, by Galois' Theorem, the subextension  ${}^S E/K$  has degree equal to the index of  $S$ , which is  $n/p$ .

**8.8.** Since  $E/K$  is a Galois extension,  $|\text{Gal}(E/K)| = [E : K] = p^\alpha m$ . So, by Sylow's Theorem, there exists a subgroup  $P$  of  $\text{Gal}(E/K)$  of order  $p^\alpha$ . Then, by Galois' Theorem, the subextension  ${}^P E/K$  has degree  $(\text{Gal}(E/K) : P) = m$ .

8.20. Write  $f = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$ . Then

$$f' = nX^{n-1} + (n-1)a_{n-1}X^{n-2} + \cdots + 2a_2X + a_1.$$

Since  $f$  is not separable,  $f' = 0$ . Thus  $n = ka_k = 0$  in  $K$  for all  $k \in \{0, \dots, n-1\}$ . This implies that  $p$  divides  $k$  whenever  $a_k \neq 0$ . This means that the only terms in  $f$  occur in degree that are multiples of  $p$ . In particular,  $n = pm$  for some  $m$ . Hence

$$f = X^{pm} + a_{p(n-1)}X^{p(m-1)} + \cdots + a_pX^p + a_0 = g(X^p)$$

for some  $g \in K[X]$ .

11.14. If  $G$  is solvable, then  $[G, G]$  is a proper normal subgroup of  $G$ . Since  $G$  is simple,  $[G, G] = \{1\}$  and  $G$  is abelian. Thus  $G$  is cyclic of prime order.

11.18. Assume that  $G$  is simple. Let  $A = G \times \{1\}$ ,  $B = \{1\} \times G$  and  $D = \{(x, x) : x \in G\}$  the diagonal subgroup of  $G \times G$ . Since

$$(g, h) = (g, 1)(1, h) = (gh^{-1}, 1)(h, h)$$

it follows that  $G = AB = AD$ . Let  $M$  be a subgroup of  $G \times G$  that contains  $D$ . Note that

$$M = M \cap (G \times G) = M \cap AD = (M \cap A)D.$$

Similarly,  $M = (M \cap B)D$ . Since  $A$  is normal in  $G \times G$ ,  $M \cap A$  is normal in  $G \times G$  and  $(M \cap A)B$  is normal in  $MB = G \times G$ . Using the second isomorphism theorem, we see that

$$M \cap A \simeq \frac{(M \cap A)B}{B}$$

is a normal subgroup of  $(G \times G)/B \simeq A$ . Since  $A \simeq G$  is simple, either  $M \cap A = \{1\}$  or  $M \cap A = A$ . Thus either  $M = D$  or  $BD = G \times G$ . Therefore  $D$  is maximal.

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## Index

- Algebraic
  - element, 9
  - extension, 9
- Artin's theorem, 19, 33
- Burnside's theorem, 53
- Commutator subgroup, 50
- Cyclotomic polynomial, 43
- Dedekind's theorem, 25
- Degree of an extension, 5
- Derived series, 51
- Eisenstein's criterion, 8
- Element
  - separable, 28
- Euler's  $\phi$  function, 43
- Extension
  - abelian, 37
  - cyclic, 37
  - cyclotomic, 44
  - finite, 5
  - Galois, 31
  - homomorphism, 6
  - of fields, 5
  - of finite type, 13
  - pure, 56
  - radical, 56
  - separable, 28, 29
- Feit–Thompson theorem, 53
- Field
  - fixed, 31
- Field extension, 5
- Fixed field, 31
- Frobenius automorphism, 42
- Galois' theorem, 35, 58
- Group
  - simple, 51
  - solvable, 51
- Hermite's theorem, 9
- Homomorphism of extensions, 6
- Jordan's theorem, 55
- Lattice, 35
  - of subgroups, 35
- Lindemann's theorem, 9
- Minimal polynomial, 10
- Norm, 41
- Order-reversing map, 35
- Partially ordered set, 35
- Poset, 35
- Pure extension, 56
- Radical extension, 56
- Radical tower, 56
- Splitting field, 20, 22
- Steinitz' theorem, 31
- Subextension, 6
- Subfield, 4
- Subgroup
  - maximal normal, 52
- Trace, 41
- Transcendental element, 9