Radford's theorem

Leandro Vendramin

Abstract This is a minicourse on Radford's bosonization theorem.

1 Quasitriangular Hopf algebras

Definition 1. *A braided vector space is a pair (V, c), where V is a vector space and $c \in \mathbf{GL}(V \otimes V)$ is a solution of the braid equation:*

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c).$$

Example 2. Let V be any vector space. Let $\tau : V \to V$ be the linear map defined by $\tau(x \otimes y) = y \otimes x$ for all $x, y \in V$. The pair (V, τ) is a braided vector space.

Example 3. Let G be a finite group and $V = \mathbb{K}G$ be the vector space with basis $\{g \mid g \in G\}$. Define $c(g \otimes h) = ghg^{-1} \otimes g$. Then (V, c) is a braided vector space.

Exercise 4. Let (V, c) be a braided vector space. Prove that the pairs $(V, \lambda c)$, (V, c^{-1}) and $(V, \tau \circ c \circ \tau)$ are also braided vector spaces, where λ is any non-zero scalar.

Let A be an algebra (over the field $\mathbb{K}$) and suppose that $R = \sum_{i=1}^{n} a_i \otimes b_i \in A \otimes A$ is invertible. Define

$$R_{12} = \sum_{i=1}^{n} a_i \otimes b_i \otimes 1, \quad R_{13} = \sum_{i=1}^{n} a_i \otimes 1 \otimes b_i, \quad R_{23} = \sum_{i=1}^{n} 1 \otimes a_i \otimes b_i.$$

Definition 5. *A quasitriangular Hopf algebra is a pair (H, R), where H is a Hopf algebra and $R = \sum_i a_i \otimes b_i \in H \otimes H$ is an invertible elemmaent such that the following conditions are satisfied:*

Leandro Vendramin
Vrije Universiteit Brussel, e-mail: Leandro.Vendramin@vub.be

$$(\Delta\otimes id)(R)=R_{13}R_{23}, \tag{1}$$
$$(id\otimes\Delta)(R)=R_{13}R_{12}, \tag{2}$$
$$\tau\Delta(h)R=R\Delta(h) \tag{3}$$

for all $h\in H$.

Remark 6. Using Sweedler notation, Equations (3)–(2) can be written as:

$$\sum h_2a_i\otimes h_1b_i=\sum a_ih_1\otimes b_ih_2,$$
$$\sum a_{i,1}\otimes a_{i,2}\otimes b_i=\sum a_i\otimes a_j\otimes b_ib_j,$$
$$\sum a_i\otimes b_{i,1}\otimes b_{i,2}=\sum a_ia_j\otimes b_j\otimes b_i.$$

Example 7. Let H be a cocommutative Hopf algebra, i.e., $\tau\circ\Delta=\Delta$. The pair (H,R), where $R=1\otimes 1$, is a quasitriangular Hopf algebra.

Example 8. Let $H=\mathbb{C}\mathbb{Z}_2$ be the group algebra of $\langle g\rangle\simeq\mathbb{Z}_2$ with the usual Hopf algebra structure. Let

$$R=\frac{1}{2}(1\otimes 1+1\otimes g+g\otimes 1-g\otimes g).$$

Then (H,R) is a quasitriangular Hopf algebra.

Example 9. Recall that the Sweedler 4-dimensional algebra H is the algebra (say over $\mathbb{C}$) generated by x,y with relations $x^2=1$, $y^2=0$ and $xy+yx=0$. The Hopf algebra structure is given by $\Delta(x)=x\otimes x$, $\Delta(y)=1\otimes y+y\otimes x$, $\varepsilon(x)=1$, $\varepsilon(y)=0$, $S(x)=x$ and $S(y)=xy$. A linear basis for H is $\{1,x,y,xy\}$. Let

$$R_\lambda=\frac{1}{2}(1\otimes 1+1\otimes x+x\otimes 1-x\otimes x)+\frac{\lambda}{2}(y\otimes y+y\otimes xy+xy\otimes xy-xy\otimes y)$$

where λ is any scalar. Then (H,R_λ) is a quasitriangular Hopf algebra. Observe that $\tau(R_\lambda)=R_\lambda^{-1}$.

Definition 10. *A triangular Hopf algebra is a quasitriangular Hopf algebra (H,R) such that $\tau(R)=R^{-1}$.*

Exercise 11. Let (H,R) be a quasitriangular Hopf algebra with comultiplication Δ and bijective antipode S. Prove that $(H^{\mathrm{cop}},\tau(R))$ is also a quasitriangular Hopf algebra. (Recall that H^{cop} is the Hopf algebra structure over H with comultiplication $\Delta^{\mathrm{op}}=\tau\circ\Delta$ and antipode S^{-1}.)

Proposition 12. *Let (H,R) be a quasitriangular Hopf algebra with bijective antipode. Then*

$$(\varepsilon\otimes\mathrm{id})(R)=(\mathrm{id}\otimes\varepsilon)(R)=1, \tag{4}$$
$$(S\otimes\mathrm{id})(R)=(\mathrm{id}\otimes S^{-1})(R)=R^{-1}, \tag{5}$$
$$(S\otimes S)(R)=R. \tag{6}$$

Proof. We first prove (4). Apply $\varepsilon\otimes\mathrm{id}\otimes\mathrm{id}$ to $(\Delta\otimes\mathrm{id})(R)=R_{13}R_{23}$ to obtain

$$R=\sum a_i\otimes b_i=\sum(\varepsilon\otimes\mathrm{id})\Delta(a_i)\otimes b_i=\sum\varepsilon(a_i)a_j\otimes b_ib_j=(\varepsilon\otimes\mathrm{id})(R)R.$$

and the claim follows since R is invertible. The other claim in (4) is similar: one needs to apply $\mathrm{id}\otimes\mathrm{id}\otimes\varepsilon$ to $(\mathrm{id}\otimes\Delta)(R)=R_{13}R_{12}$.

Now we prove (5). Apply $(m\otimes\mathrm{id})(S\otimes\mathrm{id}\otimes\mathrm{id})$ to $(\Delta\otimes\mathrm{id})(R)=R_{13}R_{23}$ to obtain

$$(m\otimes\mathrm{id})(S\otimes\mathrm{id}\otimes\mathrm{id})(\Delta\otimes\mathrm{id})(R)=(\eta\varepsilon\otimes\mathrm{id})(R)=(\varepsilon\otimes\mathrm{id})(R)=1\otimes 1.$$

On the other hand

$$1\otimes 1=m(S\otimes\mathrm{id})(R_{13}R_{23})=\sum S(a_i)a_j\otimes b_ib_j=(S\otimes\mathrm{id})(R)R.$$

Hence $(S\otimes\mathrm{id})(R)=R^{-1}$ since R is invertible. To prove $(\mathrm{id}\otimes S^{-1})(R)=R^{-1}$ notice that $(H^{\mathrm{cop}},\tau(R))$ is a quasitriangular Hopf algebra.

Finally, (6) follows from (5) since

$$(S\otimes S)(R)=(\mathrm{id}\otimes S)(S\otimes\mathrm{id})(R)=(\mathrm{id}\otimes S)(R^{-1})=R.$$

This completes the proof.

Proposition 13. *Let (H,R) be a quasitriangular Hopf algebra with bijective antipode. Then*

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}. \tag{7}$$

Proof. Using (3) and (1) we obtain

$$\begin{aligned}R_{12}R_{13}R_{23}&=R_{12}(\Delta\otimes\mathrm{id})(R)=(\Delta^{\mathrm{op}}\otimes\mathrm{id})(R)R_{12}\\&=(\tau\otimes\mathrm{id})(\Delta\otimes\mathrm{id})(R)R_{12}=(\tau\otimes\mathrm{id})(R_{13}R_{23})R_{12}=R_{23}R_{13}R_{12}.\end{aligned}$$

This proves the claim.

Exercise 14. Write Equations (4), (5), (6) and (7) using Sweedler notation.

Let (H,R) be a quasitriangular Hopf algebra, and let V and W be two left H-modules. Assume that $R=\sum a_i\otimes b_i$ and define the map

$$\begin{aligned}R_{V,W}:V\otimes W&\to W\otimes V\\ v\otimes w&\mapsto\tau_{V,W}\left(R\cdot(v\otimes w)\right)=\sum b_i\cdot w\otimes a_i\cdot v.\end{aligned}$$

The map $R_{V,W}$ is invertible and

$$(R_{V,W})^{-1}(w\otimes v)=R^{-1}\cdot(v\otimes w).$$

Lemma 15. *The map $R_{V,W}$ is an isomorphism of H-modules.*

Proof. First compute

$$\begin{aligned}R_{V,W}\left(h\cdot(v\otimes w)\right) &= \sum \tau_{V,W}\left(R(h_1\cdot v\otimes h_2\cdot w)\right)\\ &= \sum \tau_{V,W}\left((a_ih_1)\cdot v\otimes (b_ih_2)\cdot w\right)\\ &= \sum (b_ih_2)\cdot w\otimes (a_ih_1)\cdot v.\end{aligned}$$

On the other hand,

$$\begin{aligned}h\cdot R_{V,W}(v\otimes w) &= \sum h_1\cdot (b_i\cdot w)\otimes h_2\cdot (a_i\cdot v)\\ &= \sum (h_1b_i)\cdot w\otimes (h_2a_i)\cdot v.\end{aligned}$$

Apply (3) to h and the claim follows.

Proposition 16. *Let (H,R) be a quasitriangular Hopf algebra, and let V and W be two left H-modules. Then*

$$\begin{aligned}(R_{V,W}\otimes \mathrm{id}_U)&(\mathrm{id}_V\otimes R_{U,W})(R_{U,V}\otimes \mathrm{id}_W)\\ &= (\mathrm{id}_W\otimes R_{U,V})(R_{U,W}\otimes \mathrm{id}_V)(\mathrm{id}_U\otimes R_{V,W}).\end{aligned}\tag{8}$$

Proof. A direct computation shows that

$$\begin{aligned}(R_{V,W}\otimes \mathrm{id}_U)(\mathrm{id}_V\otimes R_{U,W})&(R_{U,V}\otimes \mathrm{id}_W)(u\otimes v\otimes w)\\ &= \sum (b_kb_j)\cdot w\otimes (a_kb_i)\cdot v\otimes (a_ja_i)\cdot u.\end{aligned}$$

On the other hand,

$$\begin{aligned}(\mathrm{id}_W\otimes R_{U,V})(R_{U,W}\otimes \mathrm{id}_V)&(\mathrm{id}_U\otimes R_{V,W})(u\otimes v\otimes w)\\ &= \sum (b_jb_i)\cdot w\otimes (b_ka_i)\cdot v\otimes (a_ka_j)\cdot u\end{aligned}$$

and hence the claim follows from propositionosition 13.

Exercise 17. Let (H,R) be a quasitriangular Hopf algebra, and let U, V and W be three left H-modules. Prove that

$$R_{U\otimes V,W} = (R_{U,W}\otimes \mathrm{id}_V)(\mathrm{id}_U\otimes R_{V,W}),\tag{9}$$

$$R_{U,V\otimes W} = (\mathrm{id}_V\otimes R_{U,W})(R_{U,V}\otimes \mathrm{id}_W).\tag{10}$$

Setting $U=V=W$ in proposition 16 we conclude that $R_{V,V}$ is a solution of the braid equation for any left H-module V.

Definition 18. *A Hopf algebra H is called* <u>*almost cocommutative*</u> *if there exists an invertible elemmaent $R\in H\otimes H$ such that $\tau\Delta(h)R = R\Delta(h)$ for all $h\in H$.*

Proposition 19. *Let (H,R) be an almost cocommutative Hopf algebra. Then S^2 is an inner automorphism of H. More precisely, assume that $R=\sum a_i\otimes b_i$, and let $u=\sum (Sb_i)a_i$. Then u is invertible in H and*

$$S^2h = uhu^{-1} = (Su)^{-1}h(Su)$$

for all $h \in H$.

Proof. First we prove that $uh = (S^2h)u$ for all $h \in H$. Since H is almost cocommutative, $(R \otimes 1)(h_1 \otimes h_2 \otimes h_3) = (h_2 \otimes h_1 \otimes h_3)(R \otimes 1)$, i.e.,

$$\sum a_ih_1 \otimes b_ih_2 \otimes h_3 = \sum h_2a_i \otimes h_1b_1 \otimes h_3.$$

Then

$$\sum S^2h_3S(b_ih_2)a_ih_1 = \sum (S^2h_3)S(h_1b_i)h_2a_i.$$

Using propositionerties of the antipode and the counit we obtain:

$$\sum S^2h_3S(b_ih_2)a_ih_1 = \sum S(h_2Sh_3)(Sb_i)a_ih_1 = \sum (Sb_i)a_ih = uh.$$

Similarly,

$$\sum (S^2h_3)S(h_1b_i)h_2a_i = \sum S^2h_3(Sb_i)(Sh_1)h_2a_i = \sum (S^2h)(Sb_i)a_i = (S^2h)u$$

and hence $uh = (S^2h)u$ for all $h \in H$.

Now we prove that u is invertible. Assume that $R^{-1} = \sum c_j \otimes d_j$ and let $v = \sum S^{-1}(d_j)c_j$. Since $uh = (S^2h)u$, we obtain:

$$\begin{aligned} uv = \sum_j u(S^{-1}d_j)c_j &= \sum_j (Sd_j)uc_j \\ &= \sum_{i,j} (Sd_j)(Sb_i)a_ic_j = \sum_{i,j} S(b_id_j)a_ic_j. \end{aligned}$$

Therefore $uv = 1$ since $1 \otimes 1 = RR^{-1} = \sum_{i,j} a_ic_j \otimes b_id_j$. Using $S^2h = uhu^{-1}$ with $h = v$ we obtain $1 = S^2(v)u$ and hence u is invertible.

The formula $S^2h = (Su)^{-1}h(Su)$ follows from applying S to $S^2h = uhu^{-1}$ and replacing Sh by h.

Corollary 20. *Let (H,R) be an almost cocommutative Hopf algebra. Then the elemmaent $u(Su)$ is central in H.*

Exercise 21. Let H be an almost cocommutative Hopf algebra. Let V and W be two left H-modules. Then $V \otimes W \simeq W \otimes V$ as left H-modules.

2 Actions on algebras

Definition 22. *Let H be a Hopf algebra. A left $\overline{H\text{-module-algebra}}$ is an algebra A with a left H-module structure such that $h \to \overline{(ab)} = (h_1 \to a)(h_2 \to b)$ and $h \to 1 = \varepsilon(h)1$ for all $h \in H$ and $a,b \in A$.*

Remark 23. It is possible to define $\overline{\text{right}}$ H-module-algebras: it is an algebra with a right H-module structure such that $\overline{(ab)} \leftarrow h = (a \leftarrow h_1)(b \leftarrow h_2)$ and $1 \cdot h = \varepsilon(h)1$ for all $h \in H$ and $a,b \in A$.

Exercise 24. Let H be a Hopf algebra. Prove that H^* is an left H-module-algebra via $\langle h \rightharpoonup f|x\rangle = \langle f|xh\rangle$ for all $f \in H^*$, $h,x \in H$. Similarly, prove that H^* is a right H-module-algebra via $\langle f \leftharpoonup h|x\rangle = \langle f|xh\rangle$.

Exercise 25. Let H be a Hopf algebra. Define

$$a \to x = a_1 x S(a_2) \tag{11}$$

for all $a,x \in H$. Prove that $(H,\to)$ is a left H-module-algebra. The representation 11 is called the left adjoint representation of H. Similarly, prove that the right adjoint action

$$x \leftarrow a = S(a_1) x a_2 \tag{12}$$

gives a right module-algebra over H.

Let G be a group and $\mathbb{K}[G]$ be the corresponding Hopf algebra. Then the right adjoint action is given by $a \to x = axa^{-1}$.

Example 26. Let L be a Lie algebra and $U(L)$ be the enveloping algebra with the canonical Hopf algebra structure. Then the right adjoint action is given by $a \to x = ax - xa$.

Exercise 27. Let H be a bialgebra and let $(A,\to)$ be an left H-module-algebra. There exists an algebra structure on $A \otimes H$ given by

$$(a \otimes h)(b \otimes g) = a(h_1 \to b) \otimes h_2 g$$

and unit $1 \otimes 1$. This algebra is called the left smash product of A and H. Observe that the maps $A \to A \otimes H$, $a \mapsto a \otimes 1$, and $H \to A \otimes H$, $h \mapsto 1 \otimes h$ are algebra embedings.

Exercise 28. Let H be a Hopf algebra and $(A,\leftarrow)$ be an right H-module-algebra. Prove that there exists an algebra structure on $H \otimes A$ given by

$$(h \otimes a)(g \otimes b) = hg_1 \otimes (a \leftarrow g_2) b$$

and unit $1 \otimes 1$. This algebra is called the right smash product of H and A.

3 Actions on coalgebras

Definition 29. *Let H be a Hopf algebra. A left H-module-coalgebra is a coalgebra C with a left H-module structure such that*

$$(h \to c)_1 \otimes (h \to c)_2 = (h_1 \to c_1)(h_2 \to c_2),$$
$$\varepsilon(h \to c) = \varepsilon(h)\varepsilon(c)$$

for all $h \in H$ and $c \in C$.

A right H-module-coalgebra is a coalgebra C with a right H-module structure such that

$$(c \leftarrow h)_1 \otimes (c \leftarrow h)_2 = (c_1 \leftarrow h_1)(c_2 \leftarrow h_2)$$
$$\varepsilon(c \leftarrow h) = \varepsilon(h)\varepsilon(c)$$

for all $h \in H$, $c \in C$.

Exercise 30. Let H be a finite-dimensional Hopf algebra. Consider the actions $(a \rightharpoonup f)(b) = f(ba)$ and $(f \leftharpoonup a)(b) = f(ab)$ for all $a, b \in H$, $f \in H^*$. The left coadjoint action of H on H^* is

$$h \triangleright f = h_1 \rightharpoonup f \leftharpoonup S^{-1}h_2 = f(S^{-1}h_2 ? h_1),$$

where $f(?)$ means the function $x \mapsto f(x)$. Prove that $(H^*)^{\mathrm{cop}}$ is a left H-module-coalgebra via the left coadjoint action. Similarly, the right coadjoint action of H on H^* is

$$f \triangleleft h = S^{-1}h_1 \rightharpoonup f \leftharpoonup h_2 = f(h_2 ? S^{-1}h_1).$$

Prove that H is a right $(H^*)^{\mathrm{cop}}$-module-coalgebra

Example 31. Let G be a finite group and $H = \mathbb{K}G$ be the group Hopf algebra. Then $y \rightharpoonup e_x = e_{xy^{-1}}$ (resp. $e_x \leftharpoonup y = e_{y^{-1}x}$) defines a left (resp. right) H-module structure over H^*. The left coadjoint action of H over H^* is

$$y \triangleright e_x = y \rightharpoonup e_x \leftharpoonup y^{-1} = e_{xyx^{-1}}.$$

Exercise 32. Let H be a Hopf algebra and consider the left regular action of H on itself: $h \rightarrow g = gh$ for all $h, g \in H$. Prove that H is a left H-module-coalgebra.

4 Coactions on algebras

Recall that a left H-comodule is a pair (V, δ), where V is a vector space and $\delta : V \to H \otimes V$ is a linear map such that

$$(\mathrm{id} \otimes \delta)\delta = (\Delta \otimes \mathrm{id})\delta,$$
$$(\varepsilon \otimes \mathrm{id})\delta = \mathrm{id}.$$

We write $\delta(v) = v_{-1} \otimes v_0$. Similarly, a right H-comodule is a pair (V, δ), where $\delta : V \to V \otimes H$ is a linear map such that

$$(\mathrm{id} \otimes \Delta)\delta = (\delta \otimes \mathrm{id})\delta,$$
$$(\mathrm{id} \otimes \varepsilon)\delta = \mathrm{id}.$$

In this case we write $\delta(v) = v_0 \otimes v_1$.

Definition 33. *Let H be a Hopf algebra. An algebra A is a said to be a left <u>H-comodule-algebra</u> if (A,δ) is a left H-comodule and the following properties are satisfied:*

$$\delta(1_A) = 1_H \otimes 1_A,$$
$$\delta(ab) = a_{-1}b_{-1} \otimes a_0 b_0$$

for all $a,b \in A$. (Here we write $\delta(a) = a_{-1} \otimes a_0 \in H \otimes A$.)

5 Coactions on coalgebras

Definition 34. *Let H be a Hopf algebra. A coalgebra C is said to be a left <u>H-comodule-coalgebra</u> if (C,δ) is a left H-comodule and the following properties are satisfied:*

$$c_{-1}\varepsilon(c_0) = \varepsilon(c)1,$$
$$(c_1)_{-1}(c_2)_{-1} \otimes (c_1)_0 \otimes (c_2)_0 = c_{-1} \otimes (c_0)_1 \otimes (c_0)_2$$

for all $c \in C$.

Exercise 35. Let H be a Hopf algebra. Consider the <u>left coadjoint coaction</u> of H on H: $\operatorname{coadj}(h) = h_1 S(h_3) \otimes h_2$ for $h \in H$. Prove that H is a left H-comodule-coalgebra via the left coadjoint coaction.

Exercise 36. Let H be a Hopf algebra, C be a coalgebra and $f \in \hom(C,H)$ be a coalgebra map with convolution inverse g. Prove that (C,δ) is a left H-comodule coalgebra, where $\delta(c) = f(c_1)g(c_3) \otimes c_2$ for all $c \in C$.

Exercise 37. Let H be a Hopf algebra, and (C,δ) be a left H-comodule coalgebra. Prove that $C \otimes H$ is a coalgebra with coproduct

$$\Delta(c \otimes h) = (c_1 \otimes c_{2,-1}h_1) \otimes (c_{2,0} \otimes h_2),$$

and counit $\varepsilon(c \times h) = \varepsilon_C(c)\varepsilon_H(h)$ for all $c \in C$, $h \in H$. This coalgebra structure on $C \otimes H$ is called the <u>left smash coproduct</u>. Observe that the maps $C \otimes H \to C$, $c \otimes h \mapsto c\varepsilon(h)$, and $C \otimes H \to H$, $c \otimes h \mapsto \varepsilon(c)h$, are coalgebra surjections.

Assume that C is a right H-comodule coalgebra. The <u>right</u> smash coproduct is then defined by

$$\Delta(h \otimes c) = h_1 \otimes c_{1,0} \otimes h_2 c_{1,1} \otimes c_2$$

for all $h \in H$ and $c \in C$.

6 The Drinfeld double

Now we will construct the Drinfeld double of a finite-dimensional Hopf algebra. We first need two very well known actions.

Exercise 38. Let C be a coalgebra. There exists a natural left action of C^* on C given by $f \rightharpoonup c = \langle f|c_2\rangle c_1$ for all $f \in C^*$ and $c \in C$. Prove that this action is the transpose of the right multiplication of C^* on itself, i.e.,

$$\langle g|f \rightharpoonup c\rangle = \langle f|c_2\rangle\langle g|c_1\rangle = \langle gf|c\rangle$$

for all $f, g \in C^*$ and $c \in C$. Similarly, there is also a natural right action of C^* on C given by $c \leftharpoonup f = \langle f|c_1\rangle c_2$. As before, this action is the transpose of the left multiplication of C^* on itself:

$$\langle g|c \leftharpoonup f\rangle = \langle fg|c\rangle$$

for all $f, g \in C^*$ and $c \in C$.

Exercise 39. Let A be an algebra. Then we define a left action of A on A^* which is the transpose of the right multiplication on A: $\langle a \rightharpoonup f|x\rangle = \langle f|xa\rangle$ for all $f \in A^*$ and $a, x \in A$. Similarly, one can define a right action of A on A^* by $\langle f \leftharpoonup a|x\rangle = \langle f|ax\rangle$.

Let H be a Hopf algebra with bijective antipode. The <u>left coadjoint action</u> of H on H^* is the action

$$h \triangleright f = h_1 \rightharpoonup f \leftharpoonup S^{-1}h_2 = f(S^{-1}h_2 ? h_1)$$

for all $h \in H$, $f \in H^*$. Notice that $\langle h \triangleright f|x\rangle = \langle f|S^{-1}h_2 x h_1\rangle$. Similarly, one can define the <u>right coadjoint action</u> of H on H^* as

$$f \triangleleft h = S^{-1}h_1 \rightharpoonup f \leftharpoonup h_2 = f(h_2 ? S^{-1}h_1)$$

for al $f \in H^*$, $h \in H$. As before, $\langle f \triangleleft h|x\rangle = \langle f|h_2 x S^{-1}h_1\rangle$.

Exercise 40. Prove that the left coadjoint action of H on H^* is the transpose of the left adjoint action of H on itself. More precisely, prove that

$$\langle h \triangleright f|x\rangle = \langle f|(\mathrm{ad}_l S^{-1}h)(x)\rangle$$

for all $f \in H^*$ and $h, x \in H$, where $\mathrm{ad}_l(h)(x) = h_1 x(Sh_2)$. Similarly, prove that

$$\langle f \triangleleft h|x\rangle = \langle f|(\mathrm{ad}_r S^{-1}h)(x)\rangle$$

where $\mathrm{ad}_r(h)(x) = (Sh_1)xh_2$

Exercise 41. Assume that H is finite-dimensional. We consider the left coadjoint action of H on H^* and the right coadjoint action of H^* on H. Prove that

$$\Delta^{\mathrm{cop}}(h \triangleright f) = (h_1 \triangleright f_2) \otimes (h_2 \triangleright f_1) \text{ and } \Delta(h \triangleleft f) = (h_1 \triangleleft f_2) \otimes (h_2 \triangleleft f_1)$$

for all $h \in H$, $f \in H^*$.

Theorem 42. *Let H be a finite dimensional Hopf algebra. The Drinfeld double $\mathscr{D}(H)$ of H is a Hopf algebra. It can be realized on the vector space $(H^*)^{cop} \otimes H$ with product*

$$\begin{aligned}(f \otimes h)(f' \otimes h') &= f f'_2 \otimes h_2 h' \langle f'_3 | h_1 \rangle \langle f'_1 | S^{-1} h_3 \rangle \\ &= f(h_1 \rightharpoonup f' \leftharpoonup S^{-1} h_3) \otimes h_2 h' \\ &= f(h_1 \triangleright f'_2) \otimes (h_2 \triangleleft f'_1) h',\end{aligned}$$

unit $1 \otimes 1$, coproduct

$$\Delta(f \otimes h) = f_2 \otimes h_1 \otimes f_1 \otimes h_2,$$

counit $\varepsilon(f \otimes h) = \varepsilon(f)\varepsilon(h)$ and antipode

$$\begin{aligned}S(f \otimes h) &= (Sh_2 \rightharpoonup Sf_1) \otimes (f_2 \rightharpoonup Sh_1) \\ &= (Sf_2 \leftharpoonup h_1) \otimes (Sh_2 \leftharpoonup Sf_1)\end{aligned}$$

for $f, f' \in H^*$ and $h, h' \in H$.

Exercise 43. Prove Theorem 42.

Exercise 44. Prove that the product of $\mathscr{D}(H)$ is:

$$(f \otimes h)(f' \otimes h') = f f'(S^{-1}(h_3)?h_1) \otimes h_2 h'$$

where $f(?)$ means the map $x \mapsto f(x)$.

Exercise 45. Let H be a finite-dimensional cocommutative Hopf algebra. Prove that $\mathscr{D}(H)$ is isomorphic (as an algebra) to the smash product on $H^* \otimes H$, see [2, 10.3.10].

Lemma 46. *Let H be a finite-dimensional. Assume that $\{h_i\}$ is a basis of H and $\{h^i\}$ is a basis of H^* dual to $\{h_i\}$. Then*

$$R = \sum_i (\varepsilon \otimes h_i) \otimes (h^i \otimes 1) \tag{13}$$

does not depend on $\{h_i\}$ and $\{h^i\}$.

Proof. Since H is finite-dimensional, the linear map $\Phi : H \otimes H^* \to \mathrm{End}_{\mathbb{K}}(H)$ defined by $\Phi(h \otimes f)(x) = f(x)h$ is an isomorphism. We prove that $\Phi^{-1}(\mathrm{id}) = \sum h_i \otimes h^i$ does not depend on the pair of dual basis $\{h_i\}$ and $\{h^i\}$:

$$\Phi(\sum h_i \otimes h^i)(x) = \sum \Phi(h_i \otimes h^i)(x) = \sum h^i(x) h_i = x.$$

Since $R = \varepsilon \otimes \Phi^{-1}(\mathrm{id}) \otimes 1$, the claim follows.

Theorem 47. *Let H be a finite-dimensional Hopf algebra. Then $\mathscr{D}(H)$ is a quasitriangular Hopf algebra. More precisely, the quasitriangular structure is given by*

$$R = \sum_i (\varepsilon \otimes h_i) \otimes (h^i \otimes 1), \tag{14}$$

where $\{h_i\}$ is a basis of H and $\{h^i\}$ is a basis of H^* dual to $\{h_i\}$.

Exercise 48. Prove Theorem 47.

Corollary 49. *Let H be a finite-dimensional Hopf algebra. Then H is a subHopf algebra of a quasitriangular Hopf algebra.*

Proof. It follows from the fact that $H \simeq \varepsilon_H \otimes H$ is a subalgebra of $\mathscr{D}(H)$.

Example 50. Let G be a finite group, and let $H = \mathbb{K}[G]$ be the group algebra of G with the usual Hopf algebra structure. Let $\{e_g \mid g \in G\}$ be the dual basis of the basis $\{g \mid g \in G\}$ of H. The dual algebra $(\mathbb{K}[G]^{\mathrm{op}})^*$ is the algebra $\mathrm{Fun}(G,\mathbb{K})$ with multiplication

$$e_g e_h = \begin{cases} e_g & \text{if } g = h, \\ 0 & \text{otherwise,} \end{cases}$$

for all $g,h \in G$ and unit $\sum_{g\in G} e_g = 1$. The comultiplication is

$$\Delta(e_g) = \sum_{uv=g} e_v \otimes e_u,$$

the counit is

$$\varepsilon(e_g) = \begin{cases} 1 & \text{if } g = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and the antipode is $S(e_g) = e_{g^{-1}}$ for all $g \in G$. Now we describe the Drinfeld double $\mathscr{D}(\mathbb{K}[G])$. A basis of $\mathscr{D}(\mathbb{K}[G])$ is given by

$$\{e_g h \mid (g,h) \in G \times G\}.$$

The product of $\mathscr{D}(\mathbb{K}[G])$ is determined by

$$h e_g = e_{h^{-1}gh} h.$$

The R-matrix is

$$R = \sum_{g\in G} g \otimes e_g.$$

7 Yetter-Drinfeld modules

Definition 51. *Let H be a Hopf algebra. A* <u>*Yetter-Drifeld module*</u> *over H is a triple $(V,\to,\delta)$, where $(V,\to)$ is a left H-module, (V,δ) is a left H-comodule, and such* *that*

$$\delta(h \rightarrow v) = h_1 v_{-1} S h_3 \otimes h_2 \rightarrow v_0 \tag{15}$$

for all $h \in H$, $v \in V$. A morphism of Yetter-Drinfeld modules is a morphism of left H-modules and left H-comodules. The category of Yetter-Drinfeld modules will be denoted by ${}^H_H\mathscr{YD}$.

Example 52. Let H be a Hopf algebra with the trivial action and coaction on itself: $h \rightarrow x = \varepsilon(h)x$ and $\delta(h) = 1 \otimes h$ for all $h, x \in H$. Then $(H, \rightarrow, \delta)$ is a Yetter-Drinfeld module over H.

Example 53. Let H be a Hopf algebra. Then $(H, \mathrm{adj}, \Delta)$ and (H, m, coadj) are Yetter-Drinfeld modules over H.

Exercise 54. Prove that the condition (15) is equivalent to

$$h_1 v_{-1} \otimes (h_2 \rightarrow v_0) = (h_1 \rightarrow v)_{-1} h_2 \otimes (h_1 \rightarrow v)_0 \tag{16}$$

for all $h \in H$, $v \in V$.

Exercise 55. Let G be a group, and H be the group Hopf algebra of G. Assume that $(V, \rightarrow)$ is a left H-module, and (V, δ) is a left H-comodule.

1. Prove that $V = \oplus_{g \in G} V_g$, where $V_g = \{v \in V \mid \delta(v) = g \otimes v\}$.
2. Prove that the triple $(V, \rightarrow, \delta)$ is a Yetter-Drinfeld module if and only if $h \rightarrow V_g \subseteq V_{hgh^{-1}}$ for all $g, h \in H$.

Exercise 56. Let V and W be two Yetter-Drinfeld modules over H. Then $V \otimes W$ is a Yetter-Drinfeld over H, where

$$\begin{aligned} h \rightarrow (v \otimes w) &= (h_1 \rightarrow v) \otimes (h_2 \rightarrow w), \\ \delta(v \otimes w) &= v_{-1} w_{-1} \otimes (v_0 \otimes w_0) \end{aligned}$$

for all $h \in H$, $v \in V$, $w \in W$.

Let H be a Hopf algebra with invertible antipode. For any pair V and W of Yetter-Drinfeld modules over H, we consider the map

$$\begin{aligned} c_{V,W} : V \otimes W &\rightarrow W \otimes V \\ v \otimes w &\mapsto (v_{-1} \rightarrow w) \otimes v_0. \end{aligned}$$

Lemma 57. *The map $c_{V,W}$ is an isomorphism in ${}^H_H\mathscr{YD}$.*

Proof. The map c is invertible and the inverse is

$$\begin{aligned} c_{V,W}^{-1} : W \otimes V &\rightarrow V \otimes W \\ w \otimes v &\mapsto v_0 \otimes (S^{-1}(v_{-1}) \rightarrow w) \end{aligned}$$

since

$$
\begin{aligned}
c_{V,W}^{-1}c_{V,W}(v\otimes w) &= c_{V,W}^{-1}((v_{-1}\rightarrow w)\otimes v_0)\\
&= v_{0,0}\otimes (S^{-1}(v_{0,-1})\rightarrow (v_{-1}\rightarrow w))\\
&= v_{0,0}\otimes (S^{-1}(v_{0,-1})v_{-1}\rightarrow w)\\
&= v_0\otimes (S^{-1}(v_{-1})v_{-2}\rightarrow w)\\
&= v_0\otimes (\varepsilon(v_{-1})1\rightarrow w)\\
&= v\otimes w,
\end{aligned}
$$

and similarly $c_{V,W}c_{V,W}^{-1}(w\otimes v)=w\otimes v$.

Now we prove that $c_{V,W}$ is a morphism of H-modules:

$$
\begin{aligned}
c_{V,W}(h\rightarrow (v\otimes w)) &= c_{V,W}(h_1\rightarrow v\otimes h_2\rightarrow w)\\
&= (h_1\rightarrow v)_{-1}\rightarrow (h_2\rightarrow w)\otimes (h_1\rightarrow v)_0\\
&= (h_{11}v_{-1}Sh_{13})\rightarrow (h_2\rightarrow w)\otimes h_{12}\rightarrow v_0\\
&= (h_1v_{-1}(Sh_3)h_4)\rightarrow w\otimes h_2\rightarrow v_0\\
&= (h_1v_{-1})\rightarrow w\otimes h_2\rightarrow v_0\\
&= h_1\rightarrow (v_{-1}\rightarrow w)\otimes h_2\rightarrow v_0\\
&= h\rightarrow ((v_{-1}\rightarrow w)\otimes v_0).
\end{aligned}
$$

To prove that $c_{V,W}$ is a morphism of comodules we need $(\mathrm{id}\otimes c)\delta=\delta c$. We compute:

$$(\mathrm{id}\otimes c)\delta(v\otimes w)=v_{-1}w_{-1}\otimes (v_{0,-1}\rightarrow w_0)\otimes v_{0,0}.$$

On the other hand,

$$
\begin{aligned}
\delta(c(v\otimes w) &= \delta(v_{-1}\rightarrow w\otimes v_0)\\
&= (v_{-1}\rightarrow w)_{-1}v_{0,-1}\otimes (v_{-1}\rightarrow w)_0\otimes v_{0,0}\\
&= (v_{-2}\rightarrow w)_{-1}v_{-1}\otimes (v_{-2}\rightarrow w)_0\otimes v_0\\
&= v_{-2,1}w_{-1}S(v_{-2,3})v_{-1}\otimes (v_{-2,2}\rightarrow w_0)\otimes v_0\\
&= v_{-4}w_{-1}S(v_{-2})v_{-1}\otimes (v_{-3}\rightarrow w_0)\otimes v_0\\
&= v_{-2}w_{-1}\otimes (v_{-1}\rightarrow w_0)\otimes v_0.
\end{aligned}
$$

This completes the proof.

Exercise 58. Let H be a Hopf algebra, and let U, V and W be three objects of ${}_H^H\mathscr{YD}$. Prove that

$$
\begin{aligned}
c_{U\otimes V,W} &= (c_{U,W}\otimes \mathrm{id}_V)(\mathrm{id}_U\otimes c_{V,W}), &(17)\\
c_{U,V\otimes W} &= (\mathrm{id}_V\otimes c_{U,W})(c_{U,V}\otimes \mathrm{id}_W). &(18)
\end{aligned}
$$

Exercise 59. Let H be a Hopf algebra. Prove that

$$c_{V',W'}(f\otimes g)=(g\otimes f)c_{W,V}$$

for all Yetter-Drinfeld modules morphisms $f: V \to V'$ and $g: W \to W'$.

Theorem 60. *Let H be a Hopf algebra with invertible antipode, and let U,V,W be Yetter-Drinfeld modules over H. Then*

$$\begin{aligned}(c_{V,W} \otimes id_U)(id_V \otimes c_{U,W})(c_{U,V} \otimes id_W)\\ = (id_W \otimes c_{U,V})(c_{U,W} \otimes id_V)(id_U \otimes c_{V,W}).\end{aligned}$$

Proof. It follows from Exercise (59) with $f = c_{U,V} \otimes \mathrm{id}_W$ and $g = \mathrm{id}_W$, and Exercise (58).

Exercise 61. Prove Theorem 60 without using Exercises 59 and 58.

7.1 The category ${}_H\mathscr{Y}\mathscr{D}^H$

We will also work with the following variation of what a Yetter-Drinfeld module is: An object V in the category ${}_H\mathscr{Y}\mathscr{D}^H$ is a triple $(V, \rightharpoonup, \delta)$, where $(V, \rightharpoonup)$ is a left H-module, (V,δ) is a right H-comodule, such that

$$h_1 \rightharpoonup v_0 \otimes h_2 v_1 = (h_2 \rightharpoonup v)_0 \otimes (h_2 \rightharpoonup v)_1 h_1,$$

or equivalently

$$\delta(h \rightharpoonup v) = h_2 \rightharpoonup v_0 \otimes h_3 v_1 S^{-1} h_1,$$

for all $v \in V$, $h \in H$.

Exercise 62. Let H be a finite-dimensional Hopf algebra with bijective antipode. Assume that $(V, \rightharpoonup, \delta_R)$ is an object of ${}_H\mathscr{Y}\mathscr{D}^H$ and define

$$\delta_L(v) = S(v_1) \otimes v_0$$

for all $v \in V$. Prove that $(V, \rightharpoonup, \delta_L)$ is an object of ${}_H^H\mathscr{Y}\mathscr{D}$. Conversely, if $(V, \rightharpoonup, \delta_L)$ is an object of ${}_H^H\mathscr{Y}\mathscr{D}$, define

$$\delta_R(v) = v_0 \otimes S^{-1} v_{-1}$$

for all $v \in V$. Prove that $(V, \rightharpoonup, \delta_R)$ is an object of ${}_H\mathscr{Y}\mathscr{D}^H$.

7.2 Yetter-Drinfeld modules and the Drinfeld double

Exercise 63. Let H be a finite-dimensional Hopf algebra. Assume that $\{h_i\}$ is a basis of H, and let $\{h^i\}$ be its dual basis. Prove that the element

$$\sum h^i \otimes h_i$$

does not depend on the pair of dual basis $\{h_i\}$ and $\{h^i\}$.

Lemma 64. *Let H be a finite-dimensional Hopf algebra. Then V is a left $\mathscr{D}(H)$-module if and only if V is a left H-module, a left H^*-module and*

$$h\cdot(f\cdot v) = f(S^{-1}(h_3)?h_1)\cdot(h_2\cdot v) \tag{19}$$

for all $h\in H$, $f\in H^*$.

Proof. We compute

$$\begin{aligned}(1\otimes h)\cdot((f\otimes 1)\cdot v) &= ((1\otimes h)(f\otimes 1))\cdot v\\ &= (f(S^{-1}(h_3)?h_1)\otimes h_2)\cdot v\\ &= (f(S^{-1}(h_3)?h_1)\otimes 1)(1\otimes h_2))\cdot v\\ &= f(S^{-1}(h_3)?h_1)\cdot(h_2\cdot v).\end{aligned}$$

and the claim follows.

Lemma 65. *Let H be a finite-dimensional Hopf algebra and assume that $\{h_i\}$ is a basis of H, and let $\{h^i\}$ be its dual basis. Let $(V,\cdot)$ be a left $\mathscr{D}(H)$-module. For any $v\in V$ define*

$$\delta(v) = \sum h^i\cdot v\otimes h_i.$$

Then the triple $(V,\cdot,\delta)$ is an object of ${}_H\mathscr{YD}^H$.

Proof. We prove the compatibility condition

$$\sum h^i\cdot(v\cdot v)\otimes h_i = \sum x_2\cdot(h^i\cdot v)\otimes x_3h_iS^{-1}x_1 \tag{20}$$

for all $x\in H$, $v\in V$. Let $f\in H^*$ and apply $(\mathrm{id}\otimes f)$ to the left hand side of (20) to obtain

$$\sum h^i\cdot(x\cdot v)f(h_i) = f\cdot(x\cdot v).$$

On the other hand, applying $(\mathrm{id}\otimes f)$ to the right hand side of (20) we obtain

$$\begin{aligned}\sum x_2\cdot(h^i\cdot v)f(x_3h_iS^{-1}x_1) &= x_2\cdot\big(f(x_3?S^{-1}x_1)\cdot v\big)\\ &= f(x_3S^{-1}x_{23}?x_{21}S^{-1}x_1)\cdot(x_{22}\cdot v)\\ &= f(x_5S^{-1}x_4?x_2S^{-1}x_1)\cdot(x_3\cdot v)\\ &= f\cdot(x\cdot v)\end{aligned}$$

and the claim follows.

Lemma 66. *Let H be a finite-dimensional Hopf algebra. Let $(V,\cdot,\delta)$ be an object of ${}_H\mathscr{YD}^H$. Then V is a left $\mathscr{D}(H)$-module via*

$$(f\otimes h)\cdot v = \langle f\mid(h\cdot v)_1\rangle(h\cdot v)_0$$

for all $f\in H^*$, $h\in H$ and $v\in V$.

Proof. By Lemma 64, we need prove that

$$h\cdot(f\cdot v)=\langle f\mid v_1\rangle(h\cdot v_0)$$

for all $f\in H^*$, $h\in H$, $v\in V$. We compute:

$$\begin{aligned}
f(S^{-1}h_3?h_1)\cdot(h_2\cdot v) &= \langle f\mid S^{-1}h_3(h_2\cdot v)_1h_1\rangle(h_2\cdot v)_0\\
&= \langle f\mid S^{-1}h_3(h_{23}v_1S^{-1}h_{21})h_1\rangle(h_{22}\cdot v_0)\\
&= \langle f\mid S^{-1}h_5h_4v_1S^{-1}h_2h_1\rangle h_3\cdot v_0\\
&= \langle f\mid v_1\rangle(h\cdot v_0)
\end{aligned}$$

and the claim follows.

Theorem 67. *The categories ${}_H\mathscr{Y}\mathscr{D}^H$ and ${}_{\mathscr{D}(H)}\mathscr{M}$ are equivalent.*

Proof. It follows from Lemmas 65 and 66.

8 Tensor categories

Definition 68. *A monoidal category is a tuple $(\mathscr{C},\otimes,a,\mathbb{I},l,r)$, where $\mathscr{C}$ is a category, $\otimes:\mathscr{C}\times\mathscr{C}\to\mathscr{C}$ is a funtor, $\mathbb{I}$ is an object of $\mathscr{C}$, $a_{U,V,W}:(U\otimes V)\otimes W\to U\otimes(V\otimes W)$ is a natural isomorphism such that*

$$(\mathrm{id}_U\otimes a_{V,W,X})a_{U,V\otimes W,X}(a_{U,V,W}\otimes\mathrm{id}_X)=a_{U,V,W\otimes X}a_{U\otimes V,W,X} \tag{21}$$

for all objects U,V, W of $\mathscr{C}$ and $r_U:U\otimes\mathbb{I}\to U$ and $l_U:\mathbb{I}\otimes U\to U$ are natural isomorphism such that

$$(\mathrm{id}_V\otimes l_W)a_{V,I,W}=r_V\otimes\mathrm{id}_W \tag{22}$$

for all objects U,W of $\mathscr{C}$.

Definition 69. *A monoidal category $\mathscr{C}$ is called strict if the natural isomorphism a, l y r are identities.*

Theorem 70. *Every monoidal category $\mathscr{C}$ is equivalent to a strict monoidal category.*

Proof. See for example [1, Theorem XI.5.3].

Example 71. Let H be a Hopf algebra. The category of left H-modules is a monoidal category. Recall that if V and W are two left H-modules, the tensor product of V and W is defined by

$$h\rightharpoonup(v\otimes w)=(h_1\rightharpoonup v)\otimes(h_2\rightharpoonup w)$$

for all $h\in H$, $v\in V$, $w\in W$.

Example 72. Let H be a Hopf algebra. The category of left H-comodules is a monoidal category. Recall that if V and W are two left H-comodules, the tensor product of V and W is defined defined by

$$\delta(v\otimes w) = v_{-1}w_{-1}\otimes(v_0\otimes w_0)$$

for all $v\in V$, $w\in W$.

Example 73. Let H be a Hopf algebra with invertible antipode. The category ${}_H\mathscr{Y}\mathscr{D}^H$ of Yetter-Drinfeld modules is a monoidal category.

Definition 74. *A monoidal category $\mathscr{C}$ is braided if there exists a natural isomorphism $c:\otimes\to\otimes^{\mathrm{op}}$ such that*

$$c_{U,V\otimes W} = (\mathrm{id}_V\otimes c_{U,W})(c_{U,V}\otimes\mathrm{id}_W), \tag{23}$$
$$c_{U\otimes V,W} = (c_{U,W}\otimes\mathrm{id}_V)(\mathrm{id}_U\otimes c_{V,W}) \tag{24}$$

for all objects U,V,W of $\mathscr{C}$.

Definition 75. *A braided monoidal category is symmetric if c satisfies*

$$c_{U,V}c_{V,U} = \mathrm{id}_{U\otimes V}$$

for all objects U,V of $\mathscr{C}$.

Remark 76. The naturality of the braiding c means that if V,W are objects of $\mathscr{C}$ then there exists a morphism $c_{V,W}: V\otimes W\to W\otimes V$ such that the diagram

$$\begin{tikzcd}
V\otimes W \arrow[r, "c_{V,W}"] \arrow[d, "f\otimes g"'] & W\otimes V \arrow[d, "g\otimes f"] \\
V'\otimes W' \arrow[r, "c_{V',W'}"'] & W'\otimes V'
\end{tikzcd}$$

is commutative for all pair of morphisms $f:V\to V'$ y $g:W\to W'$.

Proposition 77. *Let U, V and W be objects of a braided monoidal category $\mathscr{C}$. Then*

$$\begin{aligned}
&(c_{V,W}\otimes id_U)(id_V\otimes c_{U,W})(c_{U,V}\otimes id_W)\\
&\qquad = (id_W\otimes c_{U,V})(c_{U,W}\otimes id_V)(id_U\otimes c_{V,W}).
\end{aligned}$$

Proof. It follows from Equations (23)–(24) and the diagram

$$\begin{tikzcd}
(U\otimes V)\otimes W \arrow[r, "c_{U\otimes V,W}"] \arrow[d, "c_{U,V}\otimes\mathrm{id}_W"] & W\otimes(U\otimes V) \arrow[d, "\mathrm{id}_W\otimes c_{U,V}"] \\
(V\otimes U)\otimes W \arrow[r, "c_{V\otimes U,W}"'] & W\otimes(V\otimes U)
\end{tikzcd}$$

obtained from the naturality of the braiding with $f = c_{U,V} \otimes \mathrm{id}_W$ and $g = \mathrm{id}_W$.

Example 78. The category ${}^H_H\mathscr{YD}$ of Yetter-Drinfeld modules is a braided monoidal category.

Proposition 79. *Let H be a Hopf algebra. Then H is quasitriangular if and only if ${}_H\mathscr{M}$ is a braided monoidal category.*

Proof. We first prove the implication $\implies$. Assume that H is quasitriangular with $R = \sum a_i \otimes b_i$. Let V and W be two left H-modules, and define

$$\begin{aligned} c_{V,W} : V \otimes W &\to W \otimes V \\ v \otimes w &\mapsto \sum (b_i \cdot w) \otimes (a_i \cdot v) \end{aligned}$$

Since R is invertible, we assume that $R^{-1} = \sum a_i' \otimes b_i'$. Then $c_{V,W}$ is invertible with inverse

$$\begin{aligned} c_{V,W}^{-1} : W \otimes V &\to V \otimes W \\ w \otimes v &\mapsto \sum (a_i' \cdot v) \otimes (b_i' \cdot w) \end{aligned}$$

For example, we check that $c_{V,W}^{-1} \circ c_{V,W} = \mathrm{id}_{V \otimes W}$:

$$\begin{aligned} (c_{V,W}^{-1} \circ c_{V,W})(v \otimes w) &= \sum c_{V,W}^{-1}((b_i \cdot w) \otimes (a_i \cdot v)) \\ &= \sum (a_j' \cdot a_i \cdot v) \otimes (b_j' \cdot b_i \cdot w) \\ &= (1 \cdot v) \otimes (1 \cdot w) \\ &= v \otimes w. \end{aligned}$$

Similarly we prove that $c_{V,W} \circ c_{V,W}^{-1} = \mathrm{id}_{W \otimes V}$. By Lemma 1, the map $c_{V,W}$ is a morphism of left H-modules. We need to to prove that $c_{V,W}$ is a braiding. First we prove that c is natural, i.e.,

$$(g \otimes f) c_{V,W} = c_{V',W'}(f \otimes g)$$

holds for all $f : V \to V'$ and $g : W \to W'$ any two left H-module morphisms. We compute:

$$\begin{aligned} (g \otimes f) c_{V,W}(v \otimes w) &= (g \otimes f)\left(\sum b_i \cdot w \otimes a_i \cdot v\right) \\ &= \sum g(b_i \cdot w) \otimes f(a_i \cdot v) \\ &= \sum b_i \cdot g(w) \otimes a_i \cdot f(v) \end{aligned}$$

and on the other hand,

$$\begin{aligned} c_{V',W'}(f \otimes g)(v \otimes w) &= c_{V',W'}(f(v) \otimes g(w)) \\ &= \sum b_i \cdot g(w) \otimes a_i \cdot f(v). \end{aligned}$$

To prove Equations (23) and (24) we refer to Exercise (17).

Now we prove the implication $\Longleftarrow$. So assume that ${}_H\mathscr{M}$ is braided and let c be the braiding. Recall that H is a left H-module with $h \cdot k = hk$ for all $h,k \in H$. Let

$$R = \tau_{H,H}(c_{H,H}(1 \otimes 1)) = \sum a_i \otimes b_i.$$

Since $C_{H,H}$ is invertible, R is invertible.

Let U,V be two left H-modules and let $v \in V$ and $w \in W$. We consider the maps $f_v : H \to V$, defined by $f_v(h) = h \cdot v$, and $f_w : H \to W$, defined by $f_w(h) = h \cdot w$. By the naturality of c we obtain:

$$c_{V,W}(v \otimes w) = \sum b_i \cdot w \otimes a_i \cdot v. \tag{25}$$

In fact,

$$\begin{aligned} c_{V,W}(v \otimes w) &= c_{V,W}(f_v \otimes f_w)(1 \otimes 1) \\ &= (f_w \otimes f_v)c_{H,H}(1 \otimes 1) \\ &= (f_w \otimes f_v)\tau_{H,H}(R) \\ &= \sum b_i \cdot w \otimes a_i \cdot v. \end{aligned}$$

Since $c_{V,W}$ is a morphism of left H-modules,

$$c_{H,H}(h_1 \otimes h_2) = c_{H,H}(h \cdot (1 \otimes 1)) = h \cdot c_{H,H}(1 \otimes 1) = \Delta(h)c_{H,H}(1 \otimes 1).$$

Therefore, using (25) we obtain

$$\begin{aligned} \Delta^{\mathrm{cop}}(h)R &= \tau_{H,H}(\Delta(h)c_{H,H}(1 \otimes 1)) \\ &= \tau_{H,H}(c_{H,H}(h_1 \otimes h_2)) = \sum a_i h_1 \otimes b_i h_2 = R\Delta(h) \end{aligned}$$

for all $h \in H$.

Now using (25) and the equation $c_{U,V\otimes W} = (\mathrm{id}_V \otimes c_{U,W})(c_{U,V} \otimes \mathrm{id}_W)$ we will obtain $(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}$. First we compute:

$$\begin{aligned} c_{H,H\otimes H}(1 \otimes 1 \otimes 1) &= (\mathrm{id}_H \otimes c_{H,H})(c_{H,H} \otimes \mathrm{id}_H)(1 \otimes 1 \otimes 1) \\ &= (\mathrm{id}_H \otimes c_{H,H})(c_{H,H}(1 \otimes 1) \otimes 1) \\ &= (\mathrm{id}_H \otimes c_{H,H})(\tau(R) \otimes 1) \\ &= \sum (\mathrm{id}_H \otimes c_{H,H})(b_i \otimes a_i \otimes 1) \\ &= \sum b_i \otimes c_{H,H}(a_i \otimes 1) \\ &= \sum b_i \otimes b_j \otimes a_j a_i. \end{aligned}$$

Using (25) with $V = H$ and $W = H \otimes H$ one obtains:

$$c_{H,H\otimes H}(1 \otimes 1 \otimes 1) = \sum b_{i,1} \otimes b_{i,2} \otimes a_i$$

and hence $(\mathrm{id}\otimes\Delta)(R)=R_{13}R_{12}$. Similarly one proves that $(\Delta\otimes\mathrm{id})(R)=R_{12}R_{23}$. This completes the proof.

Exercise 80. Prove that a Hopf algebra H is triangular if and only if ${}_H\mathscr{M}$ is symmetric.

9 Algebras in categories

Definition 81. *Let $\mathscr{C}$ be a monoidal category. An algebra in $\mathscr{C}$ is a triple (A,m,u), where A is an object of $\mathscr{C}$, $m\in\hom(A\otimes A,A)$ and $u\in\hom(\mathbb{I},A)$ such that*

$$m(\mathrm{id}\otimes m)=m(m\otimes\mathrm{id}),$$
$$m(\mathrm{id}\otimes u)=\mathrm{id}=m(u\otimes\mathrm{id}).$$

Let A and B be algebras in $\mathscr{C}$ and $f\in\hom(A,B)$. Then f is a morphism (of algebras in $\mathscr{C}$) if $m_B(f\otimes f)=fm_A$ and $fu_A=u_B$. This allows us to define the category $\mathrm{Alg}(\mathscr{C})$ of algebas in $\mathscr{C}$.

Example 82. Let $\mathscr{C}=\mathrm{Vect}(\mathbb{K})$ be the category of $\mathbb{K}$-vector spaces. An algebra A in $\mathscr{C}$ is an algebra in the usual sense.

Example 83. Let $\mathscr{C}={}_H\mathscr{M}$ be the category of left H-modules. An algebra A en $\mathscr{C}$ is an object of $\mathscr{C}$ such that $(a_1\to b)(a_2\to b')=a\to bb'$ and $a\to 1=\varepsilon(a)1$ for all $a,b\in A$. Hence an algebra in ${}_H\mathscr{M}$ is a left H-module-algebra.

Example 84. Let $\mathscr{C}={}^H\mathscr{M}$ be the category of left H-comodules. An algebra A in $\mathscr{C}$ is an object of $\mathscr{C}$ such that $\delta(ab)=a_{-1}b_{-1}\otimes a_0b_0$ for all $a,b\in A$ and $\delta(1)=1_A\otimes 1_H$. Hence an algebra in ${}^H\mathscr{M}$ is a left H-comodule-algebra.

Example 85. Let $(\mathscr{C},c)$ be a braided category and let A and B be two algebras in $\mathscr{C}$. Then $A\otimes B$ is an algebra in $\mathscr{C}$ with multiplication

$$m_{A\otimes B}=(m_A\otimes m_B)(\mathrm{id}_A\otimes c_{B,A}\otimes\mathrm{id}_B).$$

10 Coalgebras in categories

Definition 86. *Let $\mathscr{C}$ be a monoidal category. A coalgebra C in $\mathscr{C}$ is a triple (C,Δ,ε), where C is an object of $\mathscr{C}$, $\Delta\in\hom(C,C\otimes C)$ and $\varepsilon\in\hom(C,\mathbb{I})$, and the following propositionerties are satisfied:*

$$(\Delta\otimes\mathrm{id})\Delta=(\mathrm{id}\otimes\Delta)\Delta,$$
$$(\mathrm{id}\otimes\varepsilon)\Delta=(\varepsilon\otimes\mathrm{id})\Delta=\mathrm{id}.$$

Let C and D be two coalgebras in $\mathscr{C}$ and $f \in \hom(C,D)$. Then f is a morphism (of coalgebras in $\mathscr{C}$) if $\Delta_D f = (f\otimes f)\Delta_C$ and $\varepsilon_D f = \varepsilon_C$. This allows us to define the category $\mathrm{Coalg}(\mathscr{C})$ *of coalgebras in $\mathscr{C}$.*

Example 87. Let $\mathscr{C} = \mathrm{Vect}(\mathbb{K})$ be the category of $\mathbb{K}$-vector spaces. A coalgebra C in $\mathscr{C}$ is a coalgebra in the usual sense.

Example 88. A coalgebra C in ${}_H\mathscr{M}$ is an object of $\mathscr{C}$ such that

$$(h \to c)_1 \otimes (h \to c)_2 = h_1 \to c_1 \otimes h_2 \to c_2$$

and $\varepsilon(h \to c) = \varepsilon(h)\varepsilon(c)$ for all $h \in H$ and $c \in C$. Hence a coalgebra in ${}_H\mathscr{M}$ is a left H-module-coalgebra.

Example 89. A coalgebra C in ${}^H\mathscr{M}$ is an object of $\mathscr{C}$ such that

$$c_{1,-1}c_{2,-1} \otimes c_{1,0} \otimes c_{2,0} = c_{-1} \otimes c_{0,1} \otimes c_{0,2}$$

and $c_{-1}\varepsilon_C(c_0) = \varepsilon_C(c)1$ for all $c \in C$. Hence a coalgebra in the category ${}^H\mathscr{M}$ is a left H-comodule-coalgebra.

Example 90. Let $(\mathscr{C}, c)$ be a braided category and let C and D be two coalgebras in $\mathscr{C}$. Then $C\otimes D$ is an coalgebra in $\mathscr{C}$ with comultiplication

$$\Delta_{C\otimes D} = (\mathrm{id}_C \otimes c_{C,D} \otimes \mathrm{id}_D)(\Delta_C \otimes \Delta_D).$$

11 Bialgebras and Hopf algebras in categories

Definition 91. *Let $\mathscr{C}$ be a braided monoidal category with braiding c. A bialgebra in $\mathscr{C}$ is a tuple $(B,m,\eta,\Delta,\varepsilon)$, where (B,m,η) is an algebra in $\mathscr{C}$, (B,Δ,ε) is a coalgebra in $\mathscr{C}$ and such that $\Delta \in \hom(B, B\otimes B)$ and $\varepsilon \in \hom(B,\mathbb{I})$ are morphism of algebras. Here $B\otimes B$ is the algebra in $\mathscr{C}$ given by the product*

$$(m_B \otimes m_B)(\mathrm{id} \otimes c_{B,B} \otimes \mathrm{id}).$$

Exercise 92. Let H be a quasitriangular Hopf algebra with $R = \sum a_i \otimes b_i$. Then ${}_H\mathscr{M}$ is a braided monoidal category with braiding

$$c_{V,W}(v\otimes w) = \sum_i b_i \cdot w \otimes a_i \cdot v.$$

Prove that H is a bialgebra in $\mathscr{C}$ if H is an algebra and a coalgebra in ${}_H\mathscr{M}$ and

$$(hh')_1 \otimes (hh')_2 = \sum_i h_1(b_i \cdot h'_1) \otimes (a_i \cdot h_2)h'_2$$

for all $h, h' \in H$.

12 Radford biproduct and bosonization

12.1 Radford biproduct

Our goal is to know when it is possible to make $A\otimes H$ a bialgebra, where the algebra structure is given by the smash product:

$$(a\otimes h)(a'\otimes h')=a(h_1\rightarrow a')\otimes h_2h'$$

for all $a,a'\in A$, $h,h'\in H$, and the coalgebra structure is the smash coproduct:

$$\Delta(a\otimes h)=(a_1\otimes a_{2,-1}h_1)\otimes(a_{2,0}\otimes h_2)$$

for all $a\in A$, $h\in H$. This is the so-called Radford biproduct.

Theorem 93. *Let H be Hopf algebra, and let A be an algebra and a coalgebra such that $(A,\rightarrow)$ a left H-module-algebra and (A,δ) a left H-comodule-coalgebra. Assume that*

$$\begin{aligned}
&A \text{ is a left } H\text{-comodule-algebra}, && (26)\\
&A \text{ is a left } H\text{-module-coalgebra}, && (27)\\
&\varepsilon_A \text{ is a morphism of algebras}, && (28)\\
&\Delta(1_A)=1_A\otimes 1_A, && (29)\\
&\Delta(aa')=a_1\left(a_{2,-1}\rightarrow a'_1\right)\otimes a_{2,0}a'_2, && (30)\\
&(h_1\rightarrow a)_{-1}h_2\otimes(h_1\rightarrow a)_0=h_1a_{-1}\otimes h_2\rightarrow a_0. && (31)
\end{aligned}$$

for all $a,a'\in A$, $h\in H$. Then the vector space $A\otimes H$ is a bialgebra with the algebra structure given by the left smash product and the coalgebra is the left smash coproduct.

Furthermore, if A has an antipode S_A, then $A\otimes H$ is a Hopf algebra with antipode

$$S(a\otimes h)=(1\otimes S_H(a_{-1}h))(S_A(a_0)\otimes 1)$$

for all $a\in A$, $h\in H$.

Proof. We first prove that ε is a morphism of algebras:

$$\begin{aligned}
\varepsilon((a\otimes h)(a'\otimes h'))&=\varepsilon(a(h_1\rightarrow a')\otimes h_2h')\\
&=\varepsilon(a(h_1\rightarrow a')\varepsilon(h_2h')\\
&=\varepsilon(a)\varepsilon(h_1\rightarrow a')\varepsilon(h_2)\varepsilon(h')\\
&=\varepsilon(a)\varepsilon(h_1)\varepsilon(a')\varepsilon(h_2)\varepsilon(h')\\
&=\varepsilon(a)\varepsilon(h)\varepsilon(a')\varepsilon(h')\\
&=\varepsilon(a\otimes h)\varepsilon(a'\otimes h'),
\end{aligned}$$

and $\varepsilon(1\otimes 1)=1$. Now we prove that Δ is a morphism of algebras. By (29), we need to prove that Δ is multiplicative. We compute:

$$\begin{aligned}
&\Delta(a\otimes h)\Delta(a'\otimes h')\\
&\quad=(a_1\otimes a_{2,-1}h_1\otimes a_{2,0}\otimes h_2)(a'_1\otimes a'_{2,-1}h'_1\otimes a'_{2,0}\otimes h'_2)\\
&\quad=(a_1\otimes a_{2,-1}h_1)(a'_1\otimes a'_{2,-1})\otimes(a_{2,0}\otimes h_2)(a'_{2,0}\otimes h'_2)\\
&\quad=a_1((a_{2,-1}h_1)_1\to a'_1)\otimes(a_{2,-1}h_1)_2a'_{2,-1}h'_1\otimes a_{2,0}(h_{2,1}\to a'_{2,0})\otimes h_{2,2}h'_2\\
&\quad=a_1((a_{2,-1,1}h_{1,1})\to a'_1)\otimes a_{2,-1,2}h_{1,2}a'_{2,-1}h'_1\otimes a_{2,0}(h_{2,1}\to a'_{2,0})\otimes h_{2,2}h'_2\\
&\quad=a_1((a_{2,-1,1}h_1)\to a'_1)\otimes a_{2,-1,2}h_2a'_{2,-1}h'_1\otimes a_{2,0}(h_3\to a'_{2,0})\otimes h_4h'_2.
\end{aligned}$$

On the other hand, we compute:

$$\begin{aligned}
&\Delta((a\otimes h)(a'\otimes h'))\\
&\quad=\Delta(a(h_1\to a')\otimes h_2h')\\
&\quad=(a(h_1\to a'))_1\otimes(a(h_1\to a'))_{2,-1}(h_2h')_1\otimes(a(h_1\to a'))_{2,0}\otimes(h_2h')_2\\
&\quad=(a(h_1\to a'))_1\otimes(a(h_1\to a'))_{2,-1}h_2h'_1\otimes(a(h_1\to a'))_{2,0}\otimes h_3h'_2\\
&\quad=a_1(a_{2,-1}\to(h_1\to a')_1)\otimes(a_{2,0}(h_1\to a')_2)_{-1}h_2h'_1\otimes(a_{2,0}(h_1\to a')_2)_0\otimes h_2h'_2\\
&\quad=a_1(a_{2,-1}\to(h_1\to a'_1)\otimes(a_{2,0}(h_2\to a'_2))_{-1}h_3h'_1\otimes(a_{2,0}(h_2\to a'_2))_0\otimes h_4h'_2\\
&\quad=a_1(a_{2,-1}h_1\to a'_1)\otimes a_{2,0,-1}(h_2\to a'_2)_{-1}h_3h'_1\otimes a_{2,0,0}(h_2\to a'_2)_0\otimes h_4h'_2\\
&\quad=a_1(a_{2,-1}h_1\to a'_1)\otimes a_{2,0,-1}(h_2a'_{2,-1})h'_1\otimes a_{2,0,0}(h_3\to a'_{2,0})\otimes h_4h'_2\\
&\quad=a_1(a_{2,-1,1}h_1\to a'_1)\otimes a_{2,-1,2}h_2a'_{2,-1}h'_1\otimes a_{2,0}(h_3\to a'_{2,0})\otimes h_4h'_2.
\end{aligned}$$

Since A is a left H-comodule-coalgebra and $a_{1,-1}a_{2,-1}\otimes a_{1,0}\otimes a_{2,0}=a_{-1}\otimes a_{0,1}\otimes a_{0,2}$, we obtain:

$$\begin{aligned}
S((a\otimes h)_1)(a\otimes h)_2&=S(a_1\otimes a_{2,-1}h_1)(a_{2,0}\otimes h_2)\\
&=(1\otimes S_H(a_{1,-1}a_{2,-1}h_1))(S_A(a_{1,0})\otimes 1)(a_{2,0}\otimes h_2)\\
&=S_A(a_{1,0})a_{2,0}\otimes S_H(a_{1,-1}a_{2,-1}h_1)h_2\\
&=\varepsilon(a_0)1\otimes S_H(a_{-1}h_1)h_2\\
&=\varepsilon(a)\varepsilon(h)1\otimes 1.
\end{aligned}$$

Since $a_{-1}\otimes a_{0,-1}\otimes a_{0,0}=a_{-1,1}\otimes a_{-1,2}\otimes a_0$ we obtain:

$$\begin{aligned}
(a\otimes h)_1S((a\otimes h)_2)&=(a_1\otimes a_{2,-1}h_1)S(a_{2,0}\otimes h_2)\\
&=(a_1\otimes a_{2,-1}h_1)(1\otimes S_H(a_{2,0,-1}h_2))(S_A(a_{2,0,0})\otimes 1)\\
&=a_1S_A(a_{2,0,0})\otimes a_{2,-1}h_1S_H(a_{2,0,-1}h_2)\\
&=a_1S_A(a_{2,0,0})\otimes a_{2,-1}h_1S_H(h_2)S_H(a_{2,0,-1})\\
&=a_1S_A(a_{2,0})\otimes a_{2,-1,1}S_H(a_{2,-1,2})\varepsilon(h)\\
&=a_1S_A(a_2)\otimes 1\varepsilon(h)\\
&=1\otimes 1\varepsilon(a)\varepsilon(h).
\end{aligned}$$

This completes the proof.

Exercise 94. Prove that the Radford biproduct over $A\otimes H$ is commutative if and only if A and H are commutatives and the action $\rightarrow$ is trivial. Similarly, the Radford biproduct over $A\otimes H$ is cocommutative if and only if A and H are cocommutative and the coaction δ_A is trivial.

Exercise 95. Prove the converse of Theorem 93: assume that H is a bialgebra, A is a left H-module-algebra and a left H-comodule coalgebra and the Radford biproduct $A\otimes H$ is a bialgebra. Then (26)–(31) are satisfied.

Remark 96. Similarly, it is possible to put on $H\otimes B$ a bialgebra structure, where the algebra structure is given by the smash product over $H\otimes B$ and the coalgebra is given by the smash coproduct over $H\otimes B$. For that purpose we need B to be a right H-module-algebra and a right H-comodule-coalgebra. In this case, the necessary and sufficient conditions are:

$$
\begin{gathered}
B \text{ is a right } H\text{-comodule-algebra},\\
B \text{ a right } H\text{-module-coalgebra},\\
\varepsilon_B \text{ is a morphism of algebras},\\
\Delta(1_B)=1_B\otimes 1_B,\\
\Delta(bb')=b_1b'_{1,0}\otimes(b_2\leftarrow b'_{1,1})b'_2,\\
(b_0\leftarrow h_1)\otimes b_1h_2=(b\leftarrow h_2)_0\otimes h_1(b\leftarrow h_2)_1.
\end{gathered}
$$

Remark 97. A different and important bialgebra structure on $A\otimes H$ is the so-called Majid product. Let A be an left H-module-algebra and H be a right A-comodule-coalgebra. On the vector space $A\otimes H$ we consider the algebra structure given by the smash product on $A\otimes H$ and the coalgebra structure given on $A\otimes H$, i.e.,

$$
\begin{aligned}
(a\otimes h)(a'\otimes h')&=a(h_1\rightarrow a')\otimes h_2h',\\
\Delta(a\otimes h)&=a_1\otimes h_{1,0}\otimes a_2h_{1,1}\otimes h_2.
\end{aligned}
$$

Then $A\otimes H$ is a bialgebra if and only if

$$
\begin{gathered}
\varepsilon(h\rightarrow a)=\varepsilon_H(h)\varepsilon_A(a),\\
\Delta(h\rightarrow a)=h_{1,0}\rightarrow a_1\otimes h_{1,1}(h_2\rightarrow a_2),\\
\delta(1)=1\otimes 1,\\
\delta(hh')=h_{1,0}h'_0\otimes h_{1,1}(h_2\rightarrow h'_1),\\
h_{2,0}\otimes(h_1\rightarrow a)h_{2,1}=h_{1,0}\otimes h_{1,1}(h_2\rightarrow a).
\end{gathered}
$$

12.2 Bosonization

Theorem 98. *Let H be a Hopf algebra with bijective antipode. There exists a bijective correspondence between*

1. *Hopf algebras A with morphisms $H \xrightarrow{i} A \xrightarrow{p} H$ such that $pi = id_H$.*
2. *Hopf algebras in the category ${}^H_H\mathscr{Y}\mathscr{D}$.*

Proof. Assume (1). We claim that

$$R = A^{\mathrm{co}H} = \{a \in A \mid (\mathrm{id} \otimes p)\Delta(a) = a \otimes 1\}$$

is a Hopf algebra in the category of left Yetter-Drinfeld modules. It is clear that R is a subalgebra of A. Now define

$$\begin{aligned} \Delta_R(r) &= r_1 iSp(r_2) \otimes r_3, \\ S_R(r) &= ip(r_1)S(r_2), \\ h \to r &= i(h_1)riS(h_2), \\ \delta(r) &= (p \otimes \mathrm{id})\Delta(r), \end{aligned}$$

for all $r \in R$ and $h \in H$. We write $\Delta_R(r) = r^1 \otimes r^2$ to distinguish $\Delta_R(r)$ and $\Delta_A(r) = r_1 \otimes r_2$. We claim that Δ_R is coassociative:

$$\begin{aligned} (\mathrm{id} \otimes \Delta_R)\Delta_R(r) &= (\mathrm{id} \otimes \Delta_R)(r_1 iSp(r_2) \otimes r_3) \\ &= r_1 iSp(r_2) \otimes r_{3,1} iSp(r_{3,2}) \otimes r_{3,3} \\ &= r_1 iSp(r_2) \otimes r_3 iSp(r_4) \otimes r_5. \end{aligned}$$

On the other hand:

$$\begin{aligned} (\Delta_R \otimes \mathrm{id})\Delta_R(r) &= (\Delta_R \otimes \mathrm{id})(r_1 iSp(r_2) \otimes r_3) \\ &= \Delta_R(r_1 iSp(r_2)) \otimes r_3 \\ &= [r_1 iSp(r_2)]_1 iSp([r_1 iSp(r_2)]_2) \otimes [r_1 iSp(r_2)]_3 \otimes r_3 \\ &= r_{1,1}[iSp(r_2)]_1 iSp(r_{1,2}[iS_H p(r_2)]_2) \otimes r_{1,3}[iSp(r_2)]_3 \otimes r_3 \\ &= r_1 iSp(r_6) iSp(r_2 iSp(r_5)) \otimes r_3 iSp(r_4) \otimes r_7 \\ &= r_1 iS[p(r_2)Sp(r_5)r_6] \otimes r_3 iSp(r_4) \otimes r_7 \\ &= r_1 iSp(r_2) \otimes r_3 iSp(r_4) \otimes r_5. \end{aligned}$$

Hence R is an algebra and a coalgebra.

We claim that R is a left H-comodule-algebra, since

$$\delta(1) = (p \otimes \mathrm{id})\Delta(1) = p(1) \otimes 1 = 1 \otimes 1,$$

and

$$\delta(rr') = p(r_1 r'_1) \otimes r_2 r'_2 = p(r_1)p(r'_1) \otimes r_2 r'_2 = r_{-1} r'_{-1} \otimes r_0 r'_0.$$

We claim that R if a left H-comodule-coalgebra, since

$$r_{-1}\varepsilon(r_0)=p(r_1)\varepsilon(r_2)=p(r_1\varepsilon(r_2))=p(r)$$

and since $r\in R$,

$$\varepsilon(r)=(\varepsilon\otimes\mathrm{id})(r\otimes 1)=(\varepsilon\otimes\mathrm{id})(\mathrm{id}\otimes p)\Delta(r)=\varepsilon(r_1)p(r_2)=p(r).$$

Futhermore,

$$\begin{aligned}(r^1)_{-1}(r^2)_{-1}\otimes(r^1)_0\otimes(r^2)_0&=p[(r_1iSpr_2)_1r_{3,1}]\otimes(r_1iSp(r_2))_2\otimes r_{3,2}\\&=p[r_{1,1}i(Spr_2)_1r_{3,1}]\otimes r_{1,2}i(Spr_2)_2\otimes r_{3,2}\\&=p(r_1iSpr_4r_5)\otimes r_2iSp(r_3)\otimes r_6\\&=p(r_1)\otimes r_2iSpr_3\otimes r_4.\end{aligned}$$

and on the other hand,

$$\begin{aligned}r_{-1}\otimes(r_0)^1\otimes(r_0)^2&=r_{-1}\otimes\Delta_R(r_0)\\&=p(r_1)\otimes\Delta_R(r_2)\\&=p(r_1)\otimes r_2iSp(r_3)\otimes r_4.\end{aligned}$$

We claim that R is a left H-module-algebra, since

$$h\rightarrow 1=ih_1iSh_2=i(h_1Sh_2)=\varepsilon(h)i(1)=\varepsilon(h)1$$

and

$$\begin{aligned}(h_1\rightarrow r)(h_2\rightarrow r')&=ih_{1,1}riSh_{1,2}ih_{2,1}r'iSh_{2,2}\\&=ih_1riSh_2ih_3r'iSh_4\\&=ih_1r\varepsilon(h_2)r'iSh_3\\&=ih_1rr'iSh_2\\&=h\rightarrow(rr').\end{aligned}$$

We claim that R is a left H-module-coalgebra, since

$$\varepsilon(h\rightarrow r)=\varepsilon(ih_1aiSh_2)=\varepsilon(ih_1)\varepsilon(r)\varepsilon(iSh_2)=\varepsilon(h)\varepsilon(r)$$

and

$$\begin{aligned}
\Delta_R(h \to r) &= \Delta_R(ih_1 riSh_2) \\
&= [ih_1 riSh_2]_1 iSp([ih_1 riSh_2]_2) \otimes [ih_1 riSh_2]_3 \\
&= ih_{1,1} r_1 iS(h_2)_2 iSp(ih_{1,2} r_2 iS(h_2)_2 \otimes ih_{1,3} r_3 iS(h_2)_3 \\
&= ih_1 r_1 iSh_6 iSp[ih_2 r_2 iSh_5] \otimes ih_3 r_3 iSh_4 \\
&= ih_1 r_1 iSh_6 iS[h_2 pr_2 Sh_5] \otimes ih_3 r_3 iSh_4 \\
&= ih_1 r_1 iS[h_2 pr_2 Sh_5 h_6] \otimes ih_3 r_3 iSh_4 \\
&= ih_1 r_1 iS(h_2 pr_2)\varepsilon(h_5) \otimes ih_3 r_3 iSh_4 \\
&= ih_1 r_1 iSpr_2 Sh_2 \otimes ih_3 r_3 iSh_4
\end{aligned}$$

and

$$\begin{aligned}
h_1 \to r^1 \otimes h_2 \to r^2 &= h_1 \to r_1 iSpr_2 \otimes h_2 \to r_3 \\
&= ih_{1,1} r_1 iSpr_2 iSh_{1,2} \otimes ih_{2,1} r_3 iSh_{2,2} \\
&= ih_1 r_1 iSpr_2 iSh_2 \otimes ih_3 r_3 iSh_4
\end{aligned}$$

To prove that R is a bialgebra in ${}^H_H\mathscr{Y}\mathscr{D}$ it remains to prove that Δ_R is a morphism in ${}^H_H\mathscr{Y}\mathscr{D}$. We compute:

$$\begin{aligned}
\Delta_R(rr') &= (rr')_1 iSp((rr')_2) \otimes (rr')_3 \\
&= r_1 r'_1 iSp(r_2 r'_2) \otimes r_3 r'_3 \\
&= r_1 r'_1 iSp(r'_2) iSp(r_2) \otimes r_3 r'_3.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
r^1((r^2)_{-1} \to r'^1) \otimes (r^2)_0 r'^2 &= r_1 iSp(r_2)(r_{3,-1} \to (r'_1 iSp(r'_2)) \otimes r_{3,0} r'_3 \\
&= r_1 iSp(r_2)(p(r_{3,1}) \to r'_1 iSp(r'_2)) \otimes r_{3,2} r'_3 \\
&= r_1 iSp(r_2) i(p(r_3)_1) r'_1 iSp(r'_2) iS(p(r_2)_2) \otimes r_4 r'_3 \\
&= r_1 i[Sp(r_2)p(r_3)] r'_1 iS[p(r'_2) iSp(r_4)] \otimes r_5 r'_3 \\
&= r_1 \varepsilon(r_2) r'_1 iSp(r'_2) iSp(r_3) \otimes r_4 r'_3 \\
&= r_1 r'_1 iSp(r'_2) iSp(r_2) \otimes r_3 r'_3.
\end{aligned}$$

Conversely, let R be a Hopf algebra in the category of Yetter-Drinfeld modules. Then the Radford biproduct $R \otimes H$ is a Hopf algebra by Theorem 93. The maps $p : R \otimes H \to H$, defined by $r \otimes h \mapsto \varepsilon(r)h$, and $i : H \to R \otimes H$, defined by $h \mapsto 1 \otimes h$ are Hopf algebra morphisms and $p \circ i = \mathrm{id}$.

Exercise 99. Let A and H be two Hopf algebras such that there exist Hopf algebras morphisms $H \xrightarrow{i} A \xrightarrow{p} H$ such that $pi = \mathrm{id}_H$. Let $R = A^{\mathrm{co}H}$ and consider the map $\omega : A \to R$ defined by $a \mapsto a_1 ip(Sa_2)$. Prove that the maps $\alpha : A \to R \otimes H$, $\alpha(a) = \omega(a_1) \otimes p(a_2)$ and $\beta : R \otimes H \to A$, $r \otimes h \mapsto ri(h)$, are Hopf algebra morphisms. Furthermore, prove that $\alpha \circ \beta = \mathrm{id}_{R \otimes H}$ and $\beta \circ \alpha = \mathrm{id}_A$. Conclude that $A \simeq R \otimes H$ as Hopf algebras.

13 Some solutions

17 We prove (9). A straightforward computation shows that

$$R_{U\otimes V,W}(u\otimes v\otimes w)=\sum b_i\cdot w\otimes a_{i1}\cdot u\otimes a_{i2}\cdot v.$$

On the other hand,

$$\begin{aligned}(R_{U,W}\otimes \mathrm{id}_V)&(\mathrm{id}_U\otimes R_{V,W})(u\otimes v\otimes w)\\ &=\sum(R_{U,W}\otimes \mathrm{id}_V)(u\otimes b_i\cdot w\otimes a_i\cdot v)\\ &=\sum(b_jb_i)\cdot w\otimes a_j\cdot w\otimes a_i\cdot w\end{aligned}$$

and the claim follows from Equation (1). The proof for (10) is similar.

21 Define $\phi: V\otimes W\to W\otimes V$ by $v\otimes w\mapsto R^{-1}\cdot(w\otimes v)$. Then ϕ is an isomorphism of left H-modules:

$$\begin{aligned}\phi(h\cdot(v\otimes w))&=R^{-1}(h_2\cdot w\otimes h_1\cdot v)\\ &=R^{-1}\tau\Delta(h)(w\otimes v)=\Delta(h)R^{-1}(w\otimes v)=h\cdot\phi(v\otimes w).\end{aligned}$$

25 First we prove that $(H,\to)$ is a left H-module. We compute

$$\begin{aligned}b\to(a\to x)&=b\to(a_1xS(a_2))\\ &=b_1a_1xS(a_2)S(b_2)\\ &=(ba)_1xS((ba)_2)\\ &=(ba)\to x.\end{aligned}$$

Then $(H,\to)$ is a left H-module, since it is trivial to prove that $1\to x=x$. To prove that $(H,\to)$ is a left module-algebra over H we compute:

$$a\to=a_1 1S(a_2)=\varepsilon(a)1,$$

and

$$\begin{aligned}(a_1\to x)(a_2\to y)&=(a_{1,1}xS(a_{1,2}))(a_{2,1}yS(a_{2,2}))\\ &=a_1x\varepsilon(a_2)yS(a_3)\\ &=a_1xyS(a_2)\\ &=a\to(xy).\end{aligned}$$

The proof for the right adjoint action is similar.

27 We first prove that $1\otimes 1$ is the unit:

$$\begin{aligned}&(1\otimes 1)(a\otimes h)=1(1\to a)\otimes 1h=1a\otimes h=a\otimes h,\\ &(a\otimes h)(1\otimes 1)=a(h_1\to 1)\otimes h_2 1=a(\varepsilon(h_1)1)\otimes h_2=a\otimes h.\end{aligned}$$

Now we prove the associativity. A direct computation shows that

$$\begin{aligned}((a\otimes h)(b\otimes g))(c\otimes k) &= (a(h_1\to b)\otimes h_2 g)(c\otimes k)\\ &= (a(h_1\to b))((h_2 g)_1\to c)\otimes (h_2 g)_2 k\\ &= a(h_1\to b)(h_2 g_1\to c)\otimes (h_3 g_2)k.\end{aligned}$$

On the other hand, since A is an H-module-algebra,

$$\begin{aligned}(a\otimes h)((b\otimes g)(c\otimes k)) &= (a\otimes h)(b(g_1\to c)\otimes g_2 k)\\ &= a(h_1\to (b(g_1\to c)))\otimes h_2(g_2 k)\\ &= a(h_1\to b)(h_2\to (g_1\to c))\otimes h_3(g_2 k).\end{aligned}$$

37 We first prove that ε is the counit:

$$\begin{aligned}(\varepsilon\otimes \mathrm{id})\Delta(c\otimes h) &= (\varepsilon\otimes \mathrm{id})(c_1\otimes c_{2,-1}h_1\otimes c_{2,0}\otimes h_2)\\ &= \varepsilon(c_1\otimes c_{2,-1}h_1)c_{2,0}\times h_2\\ &= \varepsilon_C(c_1)\varepsilon_H(c_{2,-1}h_1)c_{2,0}\otimes h_2\\ &= \varepsilon_C(c_1)\varepsilon_H(h_1)(c_{2,-1})c_{2,0}\otimes \varepsilon_H(h_1)h_2\\ &= c\otimes h,\end{aligned}$$

where the last equality holds since $(\varepsilon_H\otimes \mathrm{id})\delta = \mathrm{id}$ and hence

$$c = (\varepsilon_H\otimes \mathrm{id})\delta(c) = (\varepsilon_H\otimes \mathrm{id})\delta(\varepsilon_C(c_1)c_2) = \varepsilon_C(c_1)\varepsilon_H(c_{2,-1})c_{2,0}.$$

Similarly we obtain that $(\mathrm{id}\otimes\varepsilon)\Delta = \mathrm{id}$. Now we prove the coassociativity:

$$\begin{aligned}(\Delta\otimes \mathrm{id})\Delta(c\otimes h) &= (\Delta\otimes \mathrm{id})((c_1\otimes c_{2,-1}h_1)\otimes (c_{2,0}\otimes h_2))\\ &= \Delta(c_1\otimes c_{2,-1}h_1)\otimes (c_{2,0}\otimes h_2)\\ &= c_{1,1}\otimes c_{1,2,-1}(c_{2,-1}h_1)_1\otimes c_{1,2,0}\otimes (c_{2,-1}h_1)_2\otimes c_{2,0}\otimes h_2\\ &= c_1\otimes c_{2,-1}(c_{3,-1}h_1)_1\otimes c_{2,0}\otimes (c_{3,-1}h_1)_2\otimes c_{3,0}\otimes h_2\\ &= c_1\otimes c_{2,-1}c_{3,-1,1}h_1\otimes c_{2,0}\otimes c_{3,-1,2}h_2\otimes c_{3,0}\otimes h_3\\ &= c_1\otimes c_{2,-1}c_{3,-1}h_1\otimes c_{2,0}\otimes c_{3,0,-1}h_2\otimes c_{3,0,0}\otimes h_3\\ &= c_1\otimes c_{2,-1}c_{3,-2}h_1\otimes c_{2,0}\otimes c_{3,-1}h_2\otimes c_{3,0}\otimes h_3,\end{aligned}$$

where we have used that C is a left H-comodule-coalgebra:

$$c_{-1,1}\otimes c_{-1,2}\otimes c_0 = c_{-1}\otimes c_{0,-1}\otimes c_{0,0} = c_{-2}\otimes c_{-1}\otimes c_0\in H\otimes H\otimes C.$$

On the other hand,

$$\begin{aligned}(\mathrm{id}\otimes\Delta)\Delta(c\otimes h) &= (\mathrm{id}\otimes\Delta)((c_1\otimes c_{2,-1}h_1)\otimes(c_{2,0}\otimes h_2))\\ &= c_1\otimes c_{2,-1}h_1\otimes\Delta(c_{2,0}\otimes h_2)\\ &= c_1\otimes c_{2,-1}h_1\otimes c_{2,0,1}\otimes c_{2,0,2,-1}h_{2,1}\otimes c_{2,0,2,0}\otimes h_{2,2}\\ &= c_1\otimes c_{2,-1}h_1\otimes c_{2,0,1}\otimes c_{2,0,2,-1}h_2\otimes c_{2,0,2,0}\otimes h_3\\ &= c_1\otimes c_{2,-1}c_{3,-1}h_1\otimes c_{2,0}\otimes c_{3,0,-1}h_2\otimes c_{3,0,0}\otimes h_3\\ &= c_1\otimes c_{2,-1}c_{3,-2}h_1\otimes c_{2,0}\otimes c_{3,-1}h_2\otimes c_{3,0}\otimes h_3,\end{aligned}$$

where we have used that $c_{-1}\otimes c_{0,1}\otimes c_{0,2}=c_{1,-1}c_{2,-1}\otimes c_{1,0}\otimes c_{2,0}$ since C is a left H-comodule-coalgebra.

54 Assume that (15) holds. Then

$$\delta(h_1\to v)=(h_1\to v)_{-1}\otimes(h_1\to v)_0=h_{1,1}v_{-1}Sh_{1,3}\otimes h_{1,2}\to v_0.$$

Hence

$$(h_1\to v)_{-1}h_2\otimes(h_1\to v)_0=h_{1,1}v_{-1}Sh_{1,3}h_2\otimes h_{1,2}\to v_0=h_1v_{-1}\otimes h_2\to v_0.$$

Conversely, assume that (16) holds. Then

$$\begin{aligned}&(m\otimes\mathrm{id})(h_{11}v_{-1}\otimes Sh_2\otimes(h_{12}\to v_0))\\ &\quad=(m\otimes\mathrm{id})((h_{11}\to v)_{-1}h_{12}\otimes Sh_2\otimes(h_{11}\to v)_0)\\ &\quad=(h_1\to v)_{-1}h_2Sh_3\otimes(h_1\to v)_0\\ &\quad=(h\to v)_{-1}\otimes(h\to v)_0.\end{aligned}$$

56 To prove the compatibility condition (15) we compute

$$\begin{aligned}\delta(h\to(v\otimes w)) &= \delta(h_1\to v\otimes h_2\to w)\\ &= (h_1\to v)_{-1}(h_2\to w)_{-1}\otimes(h_1\to v)_0\otimes(h_2\to w)_0\\ &= (h_1v_{-1}(Sh_3)h_4w_{-1}Sh_6\otimes(h_2\to v_0)\otimes(h_5\to w_0)\\ &= h_1v_{-1}w_{-1}Sh_4\otimes(h_2\to v_0)\otimes(h_3\to w_0)\\ &= h_1v_{-1}w_{-1}Sh_3\otimes h_2\to(v_0\otimes w_0)\\ &= h_1(v\otimes w)_{-1}Sh_3\otimes h_2\to(v\otimes w)_0.\end{aligned}$$

80 Assume first that H es triangular. Then $\tau(R)=R$ and hence $c_{V,W}c_{W,V}=\mathrm{id}_{V\otimes W}$. Conversely, using (25) we obtain

$$1\otimes1=c_{H,H}(c_{H,H}(1\otimes1))=c_{H,H}(\tau(R))=\tau(R)R.$$

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