# Radford's theorem

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**Abstract** This is a minicourse on Radford's bosonization theorem.

## 1 Quasitriangular Hopf algebras

**Definition 1.** *A **braided vector space** is a pair $(V,c)$, where $V$ is a vector space and $c \in \mathbf{GL}(V\otimes V)$ is a solution of the braid equation:*

$$(c\otimes \mathrm{id})(\mathrm{id}\otimes c)(c\otimes \mathrm{id}) = (\mathrm{id}\otimes c)(c\otimes \mathrm{id})(\mathrm{id}\otimes c).$$

**Example 2.** Let $V$ be any vector space. Let $\tau\colon V\to V$ be the linear map defined by $\tau(x\otimes y)=y\otimes x$ for all $x,y\in V$. The pair $(V,\tau)$ is a braided vector space.

**Example 3.** Let $G$ be a finite group and $V=\mathbb{K}G$ be the vector space with basis $\{g \mid g\in G\}$. Define $c(g\otimes h)=ghg^{-1}\otimes g$. Then $(V,c)$ is a braided vector space.

**Exercise 4.** Let $(V,c)$ be a braided vector space. Prove that the pairs $(V,\lambda c)$, $(V,c^{-1})$ and $(V,\tau\circ c\circ\tau)$ are also braided vector spaces, where $\lambda$ is any non-zero scalar.

Let $A$ be an algebra (over the field $\mathbb{K}$) and suppose that $R=\sum_{i=1}^n a_i\otimes b_i\in A\otimes A$ is invertible. Define

$$R_{12}=\sum_{i=1}^n a_i\otimes b_i\otimes 1,\quad R_{13}=\sum_{i=1}^n a_i\otimes 1\otimes b_i,\quad R_{23}=\sum_{i=1}^n 1\otimes a_i\otimes b_i.$$

**Definition 5.** *A **quasitriangular** Hopf algebra is a pair $(H,R)$, where $H$ is a Hopf algebra and $R=\sum_i a_i\otimes b_i\in H\otimes H$ is an invertible elemmaent such that the following conditions are satisfied:*

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$$\tau\Delta(h)R = R\Delta(h), \tag{1}$$
$$(\Delta\otimes id)(R) = R_{13}R_{23}, \tag{2}$$
$$(id\otimes\Delta)(R) = R_{13}R_{12} \tag{3}$$

*for all $h \in H$.*

Using Sweedler notation, Equations (1)–(3) can be written as:

$$\sum h_2a_i\otimes h_1b_i = \sum a_ih_1\otimes b_ih_2,$$
$$\sum a_{i,1}\otimes a_{i,2}\otimes b_i = \sum a_i\otimes a_j\otimes b_ib_j,$$
$$\sum a_i\otimes b_{i,1}\otimes b_{i,2} = \sum a_ia_j\otimes b_j\otimes b_i.$$

**Example 6.** Let $H$ be a **cocommutative** Hopf algebra, i.e., $\tau\circ\Delta = \Delta$. The pair $(H,R)$, where $R = 1\otimes 1$, is a quasitriangular Hopf algebra.

**Example 7.** Let $H = \mathbb{C}[\mathbb{Z}_2]$ be the group algebra of $\langle g\rangle \simeq \mathbb{Z}_2$ with the usual Hopf algebra structure. Let

$$R = \frac{1}{2}(1\otimes 1 + 1\otimes g + g\otimes 1 - g\otimes g).$$

Then $(H,R)$ is a quasitriangular Hopf algebra.

**Example 8.** Recall that the Sweedler 4-dimensional algebra $H$ is the algebra (say over $\mathbb{C}$) generated by $x,y$ with relations $x^2 = 1$, $y^2 = 0$ and $xy + yx = 0$. The Hopf algebra structure is given by $\Delta(x) = x\otimes x$, $\Delta(y) = 1\otimes y + y\otimes x$, $\varepsilon(x) = 1$, $\varepsilon(y) = 0$, $S(x) = x$ and $S(y) = xy$. A linear basis for $H$ is $\{1,x,y,xy\}$. Let

$$R_\lambda = \frac{1}{2}(1\otimes 1 + 1\otimes x + x\otimes 1 - x\otimes x) + \frac{\lambda}{2}(y\otimes y + y\otimes xy + xy\otimes xy - xy\otimes y)$$

where $\lambda$ is any scalar. Then $(H,R_\lambda)$ is a quasitriangular Hopf algebra. Observe that $\tau(R_\lambda) = R_\lambda^{-1}$.

**Definition 9.** *A **triangular** Hopf algebra is a quasitriangular Hopf algebra $(H,R)$ such that $\tau(R) = R^{-1}$.*

**Exercise 10.** Let $(H,R)$ be a quasitriangular Hopf algebra with comultiplication $\Delta$ and bijective antipode $S$. Prove that $(H^{\mathrm{cop}}, \tau(R))$ is also a quasitriangular Hopf algebra. (Recall that $H^{\mathrm{cop}}$ is the Hopf algebra structure over $H$ with comultiplication $\Delta^{\mathrm{op}} = \tau\circ\Delta$ and antipode $S^{-1}$.)

**Proposition 11.** *Let $(H,R)$ be a quasitriangular Hopf algebra with bijective antipode. Then*

$$(\varepsilon\otimes\mathrm{id})(R) = (\mathrm{id}\otimes\varepsilon)(R) = 1, \tag{4}$$
$$(S\otimes\mathrm{id})(R) = (\mathrm{id}\otimes S^{-1})(R) = R^{-1}, \tag{5}$$
$$(S\otimes S)(R) = R. \tag{6}$$

*Proof.* We first prove (4). Apply $\varepsilon \otimes \mathrm{id} \otimes \mathrm{id}$ to $(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}$ to obtain

$$R = \sum a_i \otimes b_i = \sum (\varepsilon \otimes \mathrm{id})\Delta(a_i) \otimes b_i = \sum \varepsilon(a_i)a_j \otimes b_i b_j = (\varepsilon \otimes \mathrm{id})(R)R.$$

and the claim follows since $R$ is invertible. The other claim in (4) is similar: one needs to apply $\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon$ to $(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}$.

Now we prove (5). Apply $(m \otimes \mathrm{id})(S \otimes \mathrm{id} \otimes \mathrm{id})$ to $(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}$ to obtain

$$(m \otimes \mathrm{id})(S \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(R) = (\eta\varepsilon \otimes \mathrm{id})(R) = (\varepsilon \otimes \mathrm{id})(R) = 1 \otimes 1.$$

On the other hand

$$1 \otimes 1 = m(S \otimes \mathrm{id})(R_{13}R_{23}) = \sum S(a_i)a_j \otimes b_i b_j = (S \otimes \mathrm{id})(R)R.$$

Hence $(S \otimes \mathrm{id})(R) = R^{-1}$ since $R$ is invertible. To prove $(\mathrm{id} \otimes S^{-1})(R) = R^{-1}$ notice that $(H^{\mathrm{cop}}, \tau(R))$ is a quasitriangular Hopf algebra.

Finally, (6) follows from (5) since

$$(S \otimes S)(R) = (\mathrm{id} \otimes S)(S \otimes \mathrm{id})(R) = (\mathrm{id} \otimes S)(R^{-1}) = R. \qquad \square$$

**Proposition 12.** *Let $(H,R)$ be a quasitriangular Hopf algebra with bijective antipode. Then*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{7}$$

*Proof.* Using (1) and (2) we obtain

$$\begin{aligned} R_{12}R_{13}R_{23} &= R_{12}(\Delta \otimes \mathrm{id})(R) = (\Delta^{\mathrm{op}} \otimes \mathrm{id})(R)R_{12} \\ &= (\tau \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(R)R_{12} = (\tau \otimes \mathrm{id})(R_{13}R_{23})R_{12} = R_{23}R_{13}R_{12}. \end{aligned}$$

This proves the claim. $\square$

**Exercise 13.** Write Equations (4), (5), (6) and (7) using Sweedler notation.

Let $(H,R)$ be a quasitriangular Hopf algebra, and let $V$ and $W$ be two left $H$-modules. Assume that $R = \sum a_i \otimes b_i$ and define the map

$$\begin{aligned} R_{V,W} &: V \otimes W \to W \otimes V \\ &v \otimes w \mapsto \tau_{V,W}\left(R \cdot (v \otimes w)\right) = \sum b_i \cdot w \otimes a_i \cdot v. \end{aligned}$$

The map $R_{V,W}$ is invertible and

$$(R_{V,W})^{-1}(w \otimes v) = R^{-1} \cdot (v \otimes w).$$

**Lemma 14.** *The map $R_{V,W}$ is an isomorphism of $H$-modules.*

*Proof.* First compute

$$\begin{aligned} R_{V,W}\left(h\cdot(v\otimes w)\right) &= \sum \tau_{V,W}\left(R(h_1\cdot v\otimes h_2\cdot w)\right)\\ &= \sum \tau_{V,W}\left((a_ih_1)\cdot v\otimes (b_ih_2)\cdot w\right)\\ &= \sum (b_ih_2)\cdot w\otimes (a_ih_1)\cdot v. \end{aligned}$$

On the other hand,

$$\begin{aligned} h\cdot R_{V,W}(v\otimes w) &= \sum h_1\cdot(b_i\cdot w)\otimes h_2\cdot(a_i\cdot v)\\ &= \sum (h_1b_i)\cdot w\otimes (h_2a_i)\cdot v. \end{aligned}$$

Apply (1) to $h$ and the claim follows. □

**Proposition 15.** *Let $(H,R)$ be a quasitriangular Hopf algebra, and let $V$ and $W$ be two left $H$-modules. Then*

$$\begin{aligned} &(R_{V,W}\otimes \mathrm{id}_U)(\mathrm{id}_V\otimes R_{U,W})(R_{U,V}\otimes \mathrm{id}_W)\\ &\qquad = (\mathrm{id}_W\otimes R_{U,V})(R_{U,W}\otimes \mathrm{id}_V)(\mathrm{id}_U\otimes R_{V,W}). \end{aligned} \tag{8}$$

*Proof.* A direct computation shows that

$$\begin{aligned} &(R_{V,W}\otimes \mathrm{id}_U)(\mathrm{id}_V\otimes R_{U,W})(R_{U,V}\otimes \mathrm{id}_W)(u\otimes v\otimes w)\\ &\qquad = \sum (b_kb_j)\cdot w\otimes (a_kb_i)\cdot v\otimes (a_ja_i)\cdot u. \end{aligned}$$

On the other hand,

$$\begin{aligned} &(\mathrm{id}_W\otimes R_{U,V})(R_{U,W}\otimes \mathrm{id}_V)(\mathrm{id}_U\otimes R_{V,W})(u\otimes v\otimes w)\\ &\qquad = \sum (b_jb_i)\cdot w\otimes (b_ka_i)\cdot v\otimes (a_ka_j)\cdot u \end{aligned}$$

and hence the claim follows from proposition 12. □

**Exercise 16.** Let $(H,R)$ be a quasitriangular Hopf algebra, and let $U$, $V$ and $W$ be three left $H$-modules. Prove that

$$R_{U\otimes V,W} = (R_{U,W}\otimes \mathrm{id}_V)(\mathrm{id}_U\otimes R_{V,W}), \tag{9}$$

$$R_{U,V\otimes W} = (\mathrm{id}_V\otimes R_{U,W})(R_{U,V}\otimes \mathrm{id}_W). \tag{10}$$

Setting $U=V=W$ in proposition 15 we conclude that $R_{V,V}$ is a solution of the braid equation for any left $H$-module $V$.

**Definition 17.** *A Hopf algebra $H$ is called **almost cocommutative** if there exists an invertible elemmaent $R\in H\otimes H$ such that $\tau\Delta(h)R = R\Delta(h)$ for all $h\in H$.*

**Proposition 18.** *Let $(H,R)$ be an almost cocommutative Hopf algebra. Then $S^2$ is an inner automorphism of $H$. More precisely, assume that $R=\sum a_i\otimes b_i$, and let $u=\sum (Sb_i)a_i$. Then $u$ is invertible in $H$ and*

$$S^2h = uhu^{-1} = (Su)^{-1}h(Su)$$

*for all* $h \in H$.

*Proof.* First we prove that $uh = (S^2h)u$ for all $h \in H$. Since $H$ is almost cocommutative, $(R \otimes 1)(h_1 \otimes h_2 \otimes h_3) = (h_2 \otimes h_1 \otimes h_3)(R \otimes 1)$, i.e.,

$$\sum a_ih_1 \otimes b_ih_2 \otimes h_3 = \sum h_2a_i \otimes h_1b_1 \otimes h_3.$$

Then

$$\sum S^2h_3S(b_ih_2)a_ih_1 = \sum (S^2h_3)S(h_1b_i)h_2a_i.$$

Using properties of the antipode and the counit we obtain:

$$\sum S^2h_3S(b_ih_2)a_ih_1 = \sum S(h_2Sh_3)(Sb_i)a_ih_1 = \sum (Sb_i)a_ih = uh.$$

Similarly,

$$\sum (S^2h_3)S(h_1b_i)h_2a_i = \sum S^2h_3(Sb_i)(Sh_1)h_2a_i = \sum (S^2h)(Sb_i)a_i = (S^2h)u$$

and hence $uh = (S^2h)u$ for all $h \in H$.

Now we prove that $u$ is invertible. Write $R^{-1} = \sum c_j \otimes d_j$ and let $v = \sum S^{-1}(d_j)c_j$. Since $uh = (S^2h)u$, we obtain:

$$\begin{aligned} uv = \sum_j u(S^{-1}d_j)c_j &= \sum_j (Sd_j)uc_j \\ &= \sum_{i,j} (Sd_j)(Sb_i)a_ic_j = \sum_{i,j} S(b_id_j)a_ic_j. \end{aligned}$$

Therefore $uv = 1$ since $1 \otimes 1 = RR^{-1} = \sum_{i,j} a_ic_j \otimes b_id_j$. Using $S^2h = uhu^{-1}$ with $h = v$ we obtain $1 = S^2(v)u$ and hence $u$ is invertible.

The formula $S^2h = (Su)^{-1}h(Su)$ follows from applying $S$ to $S^2h = uhu^{-1}$ and replacing $Sh$ by $h$. □

**Corollary 19.** *Let* $(H,R)$ *be an almost cocommutative Hopf algebra. Then the elemmaent* $u(Su)$ *is central in* $H$.

**Exercise 20.** Let $H$ be an almost cocommutative Hopf algebra. Let $V$ and $W$ be two left $H$-modules. Then $V \otimes W \simeq W \otimes V$ as left $H$-modules.

## 2 (Co)actions on (co)algebras

**Definition 21.** *Let H be a Hopf algebra. A left H-**module-algebra** is an algebra A with a left H-module structure such that*

$$\begin{aligned} &h \to (ab) = (h_1 \to a)(h_2 \to b), \\ &h \to 1 = \varepsilon(h)1 \end{aligned}$$

*for all $h \in H$ and $a, b \in A$.*

It is possible to define **right** $H$-module-algebras: it is an algebra with a right $H$-module structure such that $(ab) \leftarrow h = (a \leftarrow h_1)(b \leftarrow h_2)$ and $1 \cdot h = \varepsilon(h)1$ for all $h \in H$ and $a, b \in A$.

**Exercise 22.** Let $H$ be a Hopf algebra. Prove that $H^*$ is an left $H$-module-algebra via $\langle h \rightharpoonup f|x\rangle = \langle f|xh\rangle$ for all $f \in H^*$, $h, x \in H$. Similarly, prove that $H^*$ is a right $H$-module-algebra via $\langle f \leftharpoonup h|x\rangle = \langle f|xh\rangle$.

**Exercise 23.** Let $H$ be a Hopf algebra. Define

$$a \rightarrow x = a_1 x S(a_2) \tag{11}$$

for all $a, x \in H$. Prove that $(H, \rightarrow)$ is a left $H$-module-algebra. The representation 11 is called the **left adjoint representation** of $H$. Similarly, prove that the **right adjoint action**

$$x \leftarrow a = S(a_1) x a_2 \tag{12}$$

gives a right module-algebra over $H$.

Let $G$ be a group and $\mathbb{K}[G]$ be the corresponding Hopf algebra. Then the right adjoint action is given by $a \rightarrow x = axa^{-1}$.

**Example 24.** Let $L$ be a Lie algebra and $U(L)$ be the enveloping algebra with the canonical Hopf algebra structure. The right adjoint action is given by

$$a \rightarrow x = ax - xa.$$

**Exercise 25.** Let $H$ be a bialgebra and let $(A, \rightarrow)$ be an left $H$-module-algebra. There exists an algebra structure on $A \otimes H$ given by

$$(a \otimes h)(b \otimes g) = a(h_1 \rightarrow b) \otimes h_2 g$$

and unit $1 \otimes 1$. This algebra is called the **left smash product** of $A$ and $H$. Observe that the maps $A \rightarrow A \otimes H$, $a \mapsto a \otimes 1$, and $H \rightarrow A \otimes H$, $h \mapsto 1 \otimes h$ are algebra embedings.

**Exercise 26.** Let $H$ be a Hopf algebra and $(A, \leftarrow)$ be an right $H$-module-algebra. Prove that there exists an algebra structure on $H \otimes A$ given by

$$(h \otimes a)(g \otimes b) = hg_1 \otimes (a \leftarrow g_2) b$$

and unit $1 \otimes 1$. This algebra is called the **right smash product** of $H$ and $A$.

**Definition 27.** *Let $H$ be a Hopf algebra. A left $H$-**module-coalgebra** is a coalgebra $C$ with a left $H$-module structure such that*

$$(h \rightarrow c)_1 \otimes (h \rightarrow c)_2 = (h_1 \rightarrow c_1)(h_2 \rightarrow c_2),$$
$$\varepsilon(h \rightarrow c) = \varepsilon(h)\varepsilon(c)$$

*for all $h \in H$ and $c \in C$.*

A **right** $H$-module-coalgebra is a coalgebra $C$ with a right $H$-module structure such that

$$(c \leftarrow h)_1 \otimes (c \leftarrow h)_2 = (c_1 \leftarrow h_1)(c_2 \leftarrow h_2)$$
$$\varepsilon(c \leftarrow h) = \varepsilon(h)\varepsilon(c)$$

for all $h \in H$, $c \in C$.

**Exercise 28.** Let $H$ be a finite-dimensional Hopf algebra. Consider the actions

$$(a \rightharpoonup f)(b) = f(ba), \quad (f \leftharpoonup a)(b) = f(ab)$$

for all $a, b \in H$, $f \in H^*$. The **left coadjoint action** of $H$ on $H^*$ is

$$h \triangleright f = h_1 \rightharpoonup f \leftharpoonup S^{-1}h_2 = f(S^{-1}h_2 ? h_1),$$

where $f(?)$ means the function $x \mapsto f(x)$. Prove that $(H^*)^{\mathrm{cop}}$ is a left $H$-module-coalgebra via the left coadjoint action. Similarly, the **right coadjoint action** of $H$ on $H^*$ is

$$f \triangleleft h = S^{-1}h_1 \rightharpoonup f \leftharpoonup h_2 = f(h_2 ? S^{-1}h_1).$$

Prove that $H$ is a right $(H^*)^{\mathrm{cop}}$-module-coalgebra

**Example 29.** Let $G$ be a finite group and $H = \mathbb{K}G$ be the group Hopf algebra. Then $y \rightharpoonup e_x = e_{xy^{-1}}$ (resp. $e_x \leftharpoonup y = e_{y^{-1}x}$) defines a left (resp. right) $H$-module structure over $H^*$. The left coadjoint action of $H$ over $H^*$ is

$$y \triangleright e_x = y \rightharpoonup e_x \leftharpoonup y^{-1} = e_{xyx^{-1}}.$$

**Exercise 30.** Let $H$ be a Hopf algebra and consider the **left regular action** of $H$ on itself: $h \rightarrow g = gh$ for all $h, g \in H$. Prove that $H$ is a left $H$-module-coalgebra.

Recall that a **left $H$-comodule** is a pair $(V, \delta)$, where $V$ is a vector space and $\delta : V \to H \otimes V$ is a linear map such that

$$(\mathrm{id} \otimes \delta)\delta = (\Delta \otimes \mathrm{id})\delta,$$
$$(\varepsilon \otimes \mathrm{id})\delta = \mathrm{id}.$$

We write $\delta(v) = v_{-1} \otimes v_0$. Similarly, a **right $H$-comodule** is a pair $(V, \delta)$, where $\delta : V \to V \otimes H$ is a linear map such that

$$(\mathrm{id} \otimes \Delta)\delta = (\delta \otimes \mathrm{id})\delta,$$
$$(\mathrm{id} \otimes \varepsilon)\delta = \mathrm{id}.$$

In this case we write $\delta(v) = v_0 \otimes v_1$.

**Definition 31.** *Let $H$ be a Hopf algebra. An algebra $A$ is a said to be a left $H$-**comodule-algebra** if $(A,\delta)$ is a left $H$-comodule and the following properties are satisfied:*

$$\delta(1_A) = 1_H \otimes 1_A,$$
$$\delta(ab) = a_{-1}b_{-1} \otimes a_0 b_0$$

*for all $a,b \in A$. (Here we write $\delta(a) = a_{-1} \otimes a_0 \in H \otimes A$.)*

**Definition 32.** *Let $H$ be a Hopf algebra. A coalgebra $C$ is said to be a left $H$-**comodule-coalgebra** if $(C,\delta)$ is a left $H$-comodule and the following properties are satisfied:*

$$c_{-1}\varepsilon(c_0) = \varepsilon(c)1,$$
$$(c_1)_{-1}(c_2)_{-1} \otimes (c_1)_0 \otimes (c_2)_0 = c_{-1} \otimes (c_0)_1 \otimes (c_0)_2$$

*for all $c \in C$.*

**Exercise 33.** Let $H$ be a Hopf algebra. Consider the **left coadjoint coaction** of $H$ on $H$: $\operatorname{coadj}(h) = h_1 S(h_3) \otimes h_2$ for $h \in H$. Prove that $H$ is a left $H$-comodule-coalgebra via the left coadjoint coaction.

**Exercise 34.** Let $H$ be a Hopf algebra, $C$ be a coalgebra and $f \in \hom(C,H)$ be a coalgebra map with convolution inverse $g$. Prove that $(C,\delta)$ is a left $H$-comodule coalgebra, where $\delta(c) = f(c_1)g(c_3) \otimes c_2$ for all $c \in C$.

**Exercise 35.** Let $H$ be a Hopf algebra, and $(C,\delta)$ be a left $H$-comodule coalgebra. Prove that $C \otimes H$ is a coalgebra with coproduct

$$\Delta(c \otimes h) = (c_1 \otimes c_{2,-1}h_1) \otimes (c_{2,0} \otimes h_2),$$

and counit $\varepsilon(c \times h) = \varepsilon_C(c)\varepsilon_H(h)$ for all $c \in C$, $h \in H$. This coalgebra structure on $C \otimes H$ is called the **left smash coproduct**. Observe that the maps $C \otimes H \to C$, $c \otimes h \mapsto c\varepsilon(h)$, and $C \otimes H \to H$, $c \otimes h \mapsto \varepsilon(c)h$, are coalgebra surjections.

Assume that $C$ is a right $H$-comodule coalgebra. The **right** smash coproduct is then defined by

$$\Delta(h \otimes c) = h_1 \otimes c_{1,0} \otimes h_2 c_{1,1} \otimes c_2$$

for all $h \in H$ and $c \in C$.

## 3 The Drinfeld double

Now we will construct the Drinfeld double of a finite-dimensional Hopf algebra. We first need two very well known actions.

**Exercise 36.** Let $C$ be a coalgebra. There exists a natural left action of $C^*$ on $C$ given by $f \rightharpoonup c = \langle f|c_2\rangle c_1$ for all $f \in C^*$ and $c \in C$. Prove that this action is the transpose of the right multiplication of $C^*$ on itself, i.e.,

$$\langle g|f \rightharpoonup c\rangle = \langle f|c_2\rangle\langle g|c_1\rangle = \langle gf|c\rangle$$

for all $f, g \in C^*$ and $c \in C$. Similarly, there is also a natural right action of $C^*$ on $C$ given by $c \leftharpoonup f = \langle f|c_1\rangle c_2$. As before, this action is the transpose of the left multiplication of $C^*$ on itself:

$$\langle g|c \leftharpoonup f\rangle = \langle fg|c\rangle$$

for all $f, g \in C^*$ and $c \in C$.

**Exercise 37.** Let $A$ be an algebra. Then we define a left action of $A$ on $A^*$ which is the transpose of the right multiplication on $A$: $\langle a \rightharpoonup f|x\rangle = \langle f|xa\rangle$ for all $f \in A^*$ and $a, x \in A$. Similarly, one can define a right action of $A$ on $A^*$ by $\langle f \leftharpoonup a|x\rangle = \langle f|ax\rangle$.

Let $H$ be a Hopf algebra with bijective antipode. The **left coadjoint action** of $H$ on $H^*$ is the action

$$h \triangleright f = h_1 \rightharpoonup f \leftharpoonup S^{-1}h_2 = f(S^{-1}h_2 ? h_1)$$

for all $h \in H$, $f \in H^*$. Notice that $\langle h \triangleright f|x\rangle = \langle f|S^{-1}h_2 x h_1\rangle$. Similarly, one can define the **right coadjoint action** of $H$ on $H^*$ as

$$f \triangleleft h = S^{-1}h_1 \rightharpoonup f \leftharpoonup h_2 = f(h_2 ? S^{-1}h_1)$$

for al $f \in H^*$, $h \in H$. As before, $\langle f \triangleleft h|x\rangle = \langle f|h_2 x S^{-1}h_1\rangle$.

**Exercise 38.** Prove that the left coadjoint action of $H$ on $H^*$ is the transpose of the left adjoint action of $H$ on itself. More precisely, prove that

$$\langle h \triangleright f|x\rangle = \langle f|(\mathrm{ad}_l S^{-1}h)(x)\rangle$$

for all $f \in H^*$ and $h, x \in H$, where $\mathrm{ad}_l(h)(x) = h_1 x(Sh_2)$. Similarly, prove that

$$\langle f \triangleleft h|x\rangle = \langle f|(\mathrm{ad}_r S^{-1}h)(x)\rangle$$

where $\mathrm{ad}_r(h)(x) = (Sh_1)xh_2$

**Exercise 39.** Assume that $H$ is finite-dimensional. We consider the left coadjoint action of $H$ on $H^*$ and the right coadjoint action of $H^*$ on $H$. Prove that

$$\Delta^{\mathrm{cop}}(h \triangleright f) = (h_1 \triangleright f_2) \otimes (h_2 \triangleright f_1) \text{ and } \Delta(h \triangleleft f) = (h_1 \triangleleft f_2) \otimes (h_2 \triangleleft f_1)$$

for all $h \in H$, $f \in H^*$.

**Theorem 40.** *Let $H$ be a finite dimensional Hopf algebra. The* ***Drinfeld double*** *$\mathcal{D}(H)$ of $H$ is a Hopf algebra. It can be realized on the vector space $(H^*)^{cop}\otimes H$ with product*

$$\begin{aligned}(f\otimes h)(f'\otimes h') &= ff'_2\otimes h_2h'\langle f'_3|h_1\rangle\langle f'_1|S^{-1}h_3\rangle\\ &= f(h_1\rightharpoonup f'\leftharpoonup S^{-1}h_3)\otimes h_2h'\\ &= f(h_1\triangleright f'_2)\otimes(h_2\triangleleft f'_1)h',\end{aligned}$$

*unit $1\otimes 1$, coproduct*

$$\Delta(f\otimes h)=f_2\otimes h_1\otimes f_1\otimes h_2,$$

*counit $\varepsilon(f\otimes h)=\varepsilon(f)\varepsilon(h)$ and antipode*

$$\begin{aligned}S(f\otimes h) &= (Sh_2\rightharpoonup Sf_1)\otimes(f_2\rightharpoonup Sh_1)\\ &= (Sf_2\leftharpoonup h_1)\otimes(Sh_2\leftharpoonup Sf_1)\end{aligned}$$

*for $f,f'\in H^*$ and $h,h'\in H$.*

**Exercise 41.** Prove Theorem 40.

**Exercise 42.** Prove that the product of $\mathcal{D}(H)$ is:

$$(f\otimes h)(f'\otimes h')=ff'(S^{-1}(h_3)?h_1)\otimes h_2h'$$

where $f(?)$ means the map $x\mapsto f(x)$.

**Exercise 43.** Let $H$ be a finite-dimensional cocommutative Hopf algebra. Prove that $\mathcal{D}(H)$ is isomorphic (as an algebra) to the smash product on $H^*\otimes H$, see [2, 10.3.10].

**Lemma 44.** *Let $H$ be a finite-dimensional. Assume that $\{h_i\}$ is a basis of $H$ and $\{h^i\}$ is a basis of $H^*$ dual to $\{h_i\}$. Then*

$$R=\sum_i(\varepsilon\otimes h_i)\otimes(h^i\otimes 1) \tag{13}$$

*does not depend on $\{h_i\}$ and $\{h^i\}$.*

*Proof.* Since $H$ is finite-dimensional, the linear map $\Phi: H\otimes H^*\to \mathrm{End}_{\mathbb{K}}(H)$ defined by $\Phi(h\otimes f)(x)=f(x)h$ is an isomorphism. We prove that $\Phi^{-1}(\mathrm{id})=\sum h_i\otimes h^i$ does not depend on the pair of dual basis $\{h_i\}$ and $\{h^i\}$:

$$\Phi(\sum h_i\otimes h^i)(x)=\sum\Phi(h_i\otimes h^i)(x)=\sum h^i(x)h_i=x.$$

Since $R=\varepsilon\otimes\Phi^{-1}(\mathrm{id})\otimes 1$, the claim follows. □

**Theorem 45.** *Let $H$ be a finite-dimensional Hopf algebra. Then $\mathcal{D}(H)$ is a quasitriangular Hopf algebra. More precisely, the quasitriangular structure is given by*

$$R = \sum_i (\varepsilon \otimes h_i) \otimes (h^i \otimes 1), \tag{14}$$

*where $\{h_i\}$ is a basis of $H$ and $\{h^i\}$ is a basis of $H^*$ dual to $\{h_i\}$.*

**Exercise 46.** Prove Theorem 45.

**Corollary 47.** *Let $H$ be a finite-dimensional Hopf algebra. Then $H$ is a subHopf algebra of a quasitriangular Hopf algebra.*

*Proof.* It follows from the fact that $H \simeq \varepsilon_H \otimes H$ is a subalgebra of $\mathcal{D}(H)$. □

**Example 48.** Let $G$ be a finite group, and let $H = \mathbb{K}[G]$ be the group algebra of $G$ with the usual Hopf algebra structure. Let $\{e_g \mid g \in G\}$ be the dual basis of the basis $\{g \mid g \in G\}$ of $H$. The dual algebra $(\mathbb{K}[G]^{\mathrm{op}})^*$ is the algebra $\mathrm{Fun}(G,\mathbb{K})$ with multiplication

$$e_g e_h = \begin{cases} e_g & \text{if } g = h, \\ 0 & \text{otherwise,} \end{cases}$$

for all $g,h \in G$ and unit $\sum_{g \in G} e_g = 1$. The comultiplication is

$$\Delta(e_g) = \sum_{uv=g} e_v \otimes e_u,$$

the counit is

$$\varepsilon(e_g) = \begin{cases} 1 & \text{if } g = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and the antipode is $S(e_g) = e_{g^{-1}}$ for all $g \in G$. Now we describe the Drinfeld double $\mathcal{D}(\mathbb{K}[G])$. A basis of $\mathcal{D}(\mathbb{K}[G])$ is given by

$$\{e_g h \mid (g,h) \in G \times G\}.$$

The product of $\mathcal{D}(\mathbb{K}[G])$ is determined by

$$h e_g = e_{h^{-1}gh} h.$$

The $R$-matrix is

$$R = \sum_{g \in G} g \otimes e_g.$$

## 4 Yetter-Drinfeld modules

**Definition 49.** *Let $H$ be a Hopf algebra. A* ***Yetter-Drifeld module*** *over $H$ is a triple $(V,\rightarrow,\delta)$, where $(V,\rightarrow)$ is a left $H$-module, $(V,\delta)$ is a left $H$-comodule, and such that*

$$\delta(h \rightarrow v) = h_1 v_{-1} S h_3 \otimes h_2 \rightarrow v_0 \tag{15}$$

*for all $h \in H$, $v \in V$. A **morphism** of Yetter-Drinfeld modules is a morphism of left $H$-modules and left $H$-comodules. The category of Yetter-Drinfeld modules will be denoted by ${}^H_H\mathcal{YD}$.*

**Example 50.** Let $H$ be a Hopf algebra with the trivial action and coaction on itself: $h \to x = \varepsilon(h)x$ and $\delta(h) = 1 \otimes h$ for all $h, x \in H$. Then $(H, \to, \delta)$ is a Yetter-Drinfeld module over $H$.

**Example 51.** Let $H$ be a Hopf algebra. Then $(H, \mathrm{adj}, \Delta)$ and $(H, m, \mathrm{coadj})$ are Yetter-Drinfeld modules over $H$.

**Exercise 52.** Prove that the condition (15) is equivalent to

$$h_1 v_{-1} \otimes (h_2 \to v_0) = (h_1 \to v)_{-1} h_2 \otimes (h_1 \to v)_0 \tag{16}$$

for all $h \in H$, $v \in V$.

**Exercise 53.** Let $G$ be a group, and $H$ be the group Hopf algebra of $G$. Assume that $(V, \to)$ is a left $H$-module, and $(V, \delta)$ is a left $H$-comodule. Prove the following statements:

1) $V = \oplus_{g \in G} V_g$, where $V_g = \{v \in V \mid \delta(v) = g \otimes v\}$.
2) The triple $(V, \to, \delta)$ is a Yetter-Drinfeld module if and only if $h \to V_g \subseteq V_{hgh^{-1}}$ for all $g, h \in H$.

**Exercise 54.** Let $V$ and $W$ be two Yetter-Drinfeld modules over $H$. Then $V \otimes W$ is a Yetter-Drinfeld over $H$, where

$$\begin{aligned} h \to (v \otimes w) &= (h_1 \to v) \otimes (h_2 \to w), \\ \delta(v \otimes w) &= v_{-1} w_{-1} \otimes (v_0 \otimes w_0) \end{aligned}$$

for all $h \in H$, $v \in V$, $w \in W$.

Let $H$ be a Hopf algebra with invertible antipode. For any pair $V$ and $W$ of Yetter-Drinfeld modules over $H$, we consider the map

$$\begin{aligned} c_{V,W} : V \otimes W &\to W \otimes V \\ v \otimes w &\mapsto (v_{-1} \to w) \otimes v_0. \end{aligned}$$

**Lemma 55.** *The map $c_{V,W}$ is an isomorphism in ${}^H_H\mathcal{YD}$.*

*Proof.* The map $c$ is invertible and the inverse is

$$\begin{aligned} c_{V,W}^{-1} : W \otimes V &\to V \otimes W \\ w \otimes v &\mapsto v_0 \otimes (S^{-1}(v_{-1}) \to w) \end{aligned}$$

since

$$
\begin{aligned}
c_{V,W}^{-1}c_{V,W}(v\otimes w) &= c_{V,W}^{-1}((v_{-1}\to w)\otimes v_0)\\
&= v_{0,0}\otimes (S^{-1}(v_{0,-1})\to (v_{-1}\to w))\\
&= v_{0,0}\otimes (S^{-1}(v_{0,-1})v_{-1}\to w)\\
&= v_0\otimes (S^{-1}(v_{-1})v_{-2}\to w)\\
&= v_0\otimes (\varepsilon(v_{-1})1\to w)\\
&= v\otimes w,
\end{aligned}
$$

and similarly $c_{V,W}c_{V,W}^{-1}(w\otimes v) = w\otimes v$.

Now we prove that $c_{V,W}$ is a morphism of $H$-modules:

$$
\begin{aligned}
c_{V,W}(h\to (v\otimes w)) &= c_{V,W}(h_1\to v\otimes h_2\to w)\\
&= (h_1\to v)_{-1}\to (h_2\to w)\otimes (h_1\to v)_0\\
&= (h_{11}v_{-1}Sh_{13})\to (h_2\to w)\otimes h_{12}\to v_0\\
&= (h_1v_{-1}(Sh_3)h_4)\to w\otimes h_2\to v_0\\
&= (h_1v_{-1})\to w\otimes h_2\to v_0\\
&= h_1\to (v_{-1}\to w)\otimes h_2\to v_0\\
&= h\to ((v_{-1}\to w)\otimes v_0).
\end{aligned}
$$

To prove that $c_{V,W}$ is a morphism of comodules we need $(\mathrm{id}\otimes c)\delta = \delta c$. We compute:

$$(\mathrm{id}\otimes c)\delta(v\otimes w) = v_{-1}w_{-1}\otimes (v_{0,-1}\to w_0)\otimes v_{0,0}.$$

On the other hand,

$$
\begin{aligned}
\delta(c(v\otimes w) &= \delta(v_{-1}\to w\otimes v_0)\\
&= (v_{-1}\to w)_{-1}v_{0,-1}\otimes (v_{-1}\to w)_0\otimes v_{0,0}\\
&= (v_{-2}\to w)_{-1}v_{-1}\otimes (v_{-2}\to w)_0\otimes v_0\\
&= v_{-2,1}w_{-1}S(v_{-2,3})v_{-1}\otimes (v_{-2,2}\to w_0)\otimes v_0\\
&= v_{-4}w_{-1}S(v_{-2})v_{-1}\otimes (v_{-3}\to w_0)\otimes v_0\\
&= v_{-2}w_{-1}\otimes (v_{-1}\to w_0)\otimes v_0.
\end{aligned}
$$

□

**Exercise 56.** Let $H$ be a Hopf algebra, and let $U$, $V$ and $W$ be three objects of ${}_H^H\mathcal{YD}$. Prove that

$$c_{U\otimes V,W} = (c_{U,W}\otimes \mathrm{id}_V)(\mathrm{id}_U\otimes c_{V,W}), \tag{17}$$
$$c_{U,V\otimes W} = (\mathrm{id}_V\otimes c_{U,W})(c_{U,V}\otimes \mathrm{id}_W). \tag{18}$$

**Exercise 57.** Let $H$ be a Hopf algebra. Prove that

$$c_{V',W'}(f\otimes g) = (g\otimes f)c_{W,V}$$

for all Yetter-Drinfeld modules morphisms $f: V\to V'$ and $g: W\to W'$.

**Theorem 58.** *Let $H$ be a Hopf algebra with invertible antipode, and let $U,V,W$ be Yetter-Drinfeld modules over $H$. Then*

$$\begin{aligned}(c_{V,W}\otimes id_U)(id_V\otimes c_{U,W})(c_{U,V}\otimes id_W)\\ =(id_W\otimes c_{U,V})(c_{U,W}\otimes id_V)(id_U\otimes c_{V,W}).\end{aligned}$$

The proof follows from Exercise 57 with $f=c_{U,V}\otimes \mathrm{id}_W$ and $g=\mathrm{id}_W$ and Exercise 56.

**Exercise 59.** Prove Theorem 58 without using Exercises 57 and 56.

We will also work with the following variation of what a Yetter-Drinfeld module is: An object $V$ in the category ${}_H\mathcal{YD}^H\ {}_H\mathcal{YD}^H$ is a triple $(V,\rightarrow,\delta)$, where $(V,\rightarrow)$ is a left $H$-module, $(V,\delta)$ is a right $H$-comodule, such that

$$h_1\rightarrow v_0\otimes h_2v_1=(h_2\rightarrow v)_0\otimes(h_2\rightarrow v)_1h_1,$$

or equivalently

$$\delta(h\rightarrow v)=h_2\rightarrow v_0\otimes h_3v_1S^{-1}h_1,$$

for all $v\in V$, $h\in H$.

**Exercise 60.** Let $H$ be a finite-dimensional Hopf algebra with bijective antipode. Assume that $(V,\rightarrow,\delta_R)$ is an object of ${}_H\mathcal{YD}^H$ and define

$$\delta_L(v)=S(v_1)\otimes v_0$$

for all $v\in V$. Prove that $(V,\rightarrow,\delta_L)$ is an object of ${}_H^H\mathcal{YD}$. Conversely, if $(V,\rightarrow,\delta_L)$ is an object of ${}_H^H\mathcal{YD}$, define

$$\delta_R(v)=v_0\otimes S^{-1}v_{-1}$$

for all $v\in V$. Prove that $(V,\rightarrow,\delta_R)$ is an object of ${}_H\mathcal{YD}^H$.

There is a deep connection between Yetter-Drinfeld modules and the Drinfeld double. To conclude this section we will prove that there is an equivalence between ${}_H\mathcal{YD}^H$ and ${}_{\mathcal{D}(H)}\mathcal{M}$.

**Exercise 61.** Let $H$ be a finite-dimensional Hopf algebra. Assume that $\{h_i\}$ is a basis of $H$, and let $\{h^i\}$ be its dual basis. Prove that the element

$$\sum h^i\otimes h_i$$

does not depend on the pair of dual basis $\{h_i\}$ and $\{h^i\}$.

**Lemma 62.** *Let $H$ be a finite-dimensional Hopf algebra. Then $V$ is a left $\mathcal{D}(H)$-module if and only if $V$ is a left $H$-module, a left $H^*$-module and*

$$h\cdot(f\cdot v)=f(S^{-1}(h_3)?h_1)\cdot(h_2\cdot v) \tag{19}$$

*for all $h\in H$, $f\in H^*$.*

*Proof.* We compute

$$\begin{aligned}(1\otimes h)\cdot((f\otimes 1)\cdot v) &= ((1\otimes h)(f\otimes 1))\cdot v\\ &= (f(S^{-1}(h_3)?h_1)\otimes h_2)\cdot v\\ &= (f(S^{-1}(h_3)?h_1)\otimes 1)(1\otimes h_2))\cdot v\\ &= f(S^{-1}(h_3)?h_1)\cdot(h_2\cdot v).\end{aligned}$$

and the claim follows. □

**Lemma 63.** *Let $H$ be a finite-dimensional Hopf algebra and assume that $\{h_i\}$ is a basis of $H$, and let $\{h^i\}$ be its dual basis. Let $(V,\cdot)$ be a left $\mathcal{D}(H)$-module. For any $v\in V$ define*

$$\delta(v)=\sum h^i\cdot v\otimes h_i.$$

*Then the triple $(V,\cdot,\delta)$ is an object of ${}_H\mathcal{YD}^H$.*

*Proof.* We prove the compatibility condition

$$\sum h^i\cdot(v\cdot v)\otimes h_i=\sum x_2\cdot(h^i\cdot v)\otimes x_3h_iS^{-1}x_1 \tag{20}$$

for all $x\in H$, $v\in V$. Let $f\in H^*$ and apply $(\mathrm{id}\otimes f)$ to the left hand side of (20) to obtain

$$\sum h^i\cdot(x\cdot v)f(h_i)=f\cdot(x\cdot v).$$

On the other hand, applying $(\mathrm{id}\otimes f)$ to the right hand side of (20) we obtain

$$\begin{aligned}\sum x_2\cdot(h^i\cdot v)f(x_3h_iS^{-1}x_1) &= x_2\cdot\big(f(x_3?S^{-1}x_1)\cdot v\big)\\ &= f(x_3S^{-1}x_{23}?x_{21}S^{-1}x_1)\cdot(x_{22}\cdot v)\\ &= f(x_5S^{-1}x_4?x_2S^{-1}x_1)\cdot(x_3\cdot v)\\ &= f\cdot(x\cdot v)\end{aligned}$$

and the claim follows. □

**Lemma 64.** *Let $H$ be a finite-dimensional Hopf algebra. Let $(V,\cdot,\delta)$ be an object of ${}_H\mathcal{YD}^H$. Then $V$ is a left $\mathcal{D}(H)$-module via*

$$(f\otimes h)\cdot v=\langle f\mid(h\cdot v)_1\rangle(h\cdot v)_0$$

*for all $f\in H^*$, $h\in H$ and $v\in V$.*

*Proof.* By Lemma 62, we need prove that

$$h\cdot(f\cdot v)=\langle f\mid v_1\rangle(h\cdot v_0)$$

for all $f\in H^*$, $h\in H$, $v\in V$. We compute:

$$\begin{aligned} f(S^{-1}h_3?h_1)\cdot(h_2\cdot v) &= \langle f \mid S^{-1}h_3(h_2\cdot v)_1h_1\rangle(h_2\cdot v)_0\\ &= \langle f \mid S^{-1}h_3(h_{23}v_1S^{-1}h_{21})h_1\rangle(h_{22}\cdot v_0)\\ &= \langle f \mid S^{-1}h_5h_4v_1S^{-1}h_2h_1\rangle h_3\cdot v_0\\ &= \langle f \mid v_1\rangle(h\cdot v_0)\end{aligned}$$

and the claim follows. □

**Theorem 65.** *The categories ${}_H\mathcal{YD}^H$ and ${}_{\mathcal{D}(H)}\mathcal{M}$ are equivalent.*

*Proof.* It follows from Lemmas 63 and 64. □

## 5 Monoidal categories

**Definition 66.** *A **monoidal category** is a tuple $(\mathcal{C},\otimes,a,\mathbb{I},l,r)$, where $\mathcal{C}$ is a category, $\otimes:\mathcal{C}\times\mathcal{C}\to\mathcal{C}$ is a funtor, $\mathbb{I}$ is an object of $\mathcal{C}$, $a_{U,V,W}:(U\otimes V)\otimes W\to U\otimes(V\otimes W)$ is a natural isomorphism such that*

$$(\mathrm{id}_U\otimes a_{V,W,X})a_{U,V\otimes W,X}(a_{U,V,W}\otimes\mathrm{id}_X)=a_{U,V,W\otimes X}a_{U\otimes V,W,X} \tag{21}$$

*for all objects $U$,$V$, $W$ of $\mathcal{C}$ and $r_U:U\otimes\mathbb{I}\to U$ and $l_U:\mathbb{I}\otimes U\to U$ are natural isomorphism such that*

$$(\mathrm{id}_V\otimes l_W)a_{V,I,W}=r_V\otimes\mathrm{id}_W \tag{22}$$

*for all objects $U$,$W$ of $\mathcal{C}$.*

**Definition 67.** *A monoidal category $\mathcal{C}$ is called **strict** if the natural isomorphism $a$, $l$ y $r$ are identities.*

**Theorem 68.** *Every monoidal category $\mathcal{C}$ is equivalent to a strict monoidal category.*

*Proof.* See for example [1, Theorem XI.5.3]. □

**Example 69.** Let $H$ be a Hopf algebra. The category of left $H$-modules is a monoidal category. Recall that if $V$ and $W$ are two left $H$-modules, the tensor product of $V$ and $W$ is defined by

$$h\to(v\otimes w)=(h_1\to v)\otimes(h_2\to w)$$

for all $h\in H$, $v\in V$, $w\in W$.

**Example 70.** Let $H$ be a Hopf algebra. The category of left $H$-comodules is a monoidal category. Recall that if $V$ and $W$ are two left $H$-comodules, the tensor product of $V$ and $W$ is defined defined by

$$\delta(v\otimes w)=v_{-1}w_{-1}\otimes(v_0\otimes w_0)$$

for all $v\in V$, $w\in W$.

**Example 71.** Let $H$ be a Hopf algebra with invertible antipode. The category ${}_H\mathcal{YD}^H$ of Yetter-Drinfeld modules is a monoidal category.

**Definition 72.** *A monoidal category $\mathcal{C}$ is **braided** if there exists a natural isomorphism $c : \otimes \to \otimes^{\mathrm{op}}$ such that*

$$c_{U,V\otimes W} = (\mathrm{id}_V \otimes c_{U,W})(c_{U,V} \otimes \mathrm{id}_W), \tag{23}$$
$$c_{U\otimes V,W} = (c_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c_{V,W}) \tag{24}$$

*for all objects $U,V,W$ of $\mathcal{C}$.*

**Definition 73.** *A braided monoidal category is **symmetric** if $c$ satisfies*

$$c_{U,V}c_{V,U} = \mathrm{id}_{U\otimes V}$$

*for all objects $U,V$ of $\mathcal{C}$.*

Remark 74. The naturality of the braiding $c$ means that if $V,W$ are objects of $\mathcal{C}$ then there exists a morphism $c_{V,W} : V\otimes W \to W\otimes V$ such that the diagram

$$\begin{tikzcd} V\otimes W \arrow[r, "c_{V,W}"] \arrow[d, "f\otimes g"'] & W\otimes V \arrow[d, "g\otimes f"] \\ V'\otimes W' \arrow[r, "c_{V',W'}"'] & W'\otimes V' \end{tikzcd}$$

is commutative for all pair of morphisms $f : V \to V'$ y $g : W \to W'$.

**Proposition 75.** *Let $U$, $V$ and $W$ be objects of a braided monoidal category $\mathcal{C}$. Then*

$$\begin{aligned} &(c_{V,W}\otimes id_U)(id_V\otimes c_{U,W})(c_{U,V}\otimes id_W) \\ &\qquad = (id_W\otimes c_{U,V})(c_{U,W}\otimes id_V)(id_U\otimes c_{V,W}). \end{aligned}$$

*Proof.* It follows from Equations (23)–(24) and the diagram

$$\begin{tikzcd} (U\otimes V)\otimes W \arrow[r, "c_{U\otimes V,W}"] \arrow[d, "c_{U,V}\otimes \mathrm{id}_W"] & W\otimes (U\otimes V) \arrow[d, "\mathrm{id}_W\otimes c_{U,V}"] \\ (V\otimes U)\otimes W \arrow[r, "c_{V\otimes U,W}"'] & W\otimes (V\otimes U) \end{tikzcd}$$

obtained from the naturality of the braiding with $f = c_{U,V}\otimes \mathrm{id}_W$ and $g = \mathrm{id}_W$. $\square$

**Example 76.** The category ${}_H^H\mathcal{YD}$ of Yetter-Drinfeld modules is a braided monoidal category.

**Proposition 77.** *Let $H$ be a Hopf algebra. Then $H$ is quasitriangular if and only if ${}_H\mathcal{M}$ is a braided monoidal category.*

*Proof.* We first prove the implication $\implies$. Assume that $H$ is quasitriangular with $R=\sum a_i\otimes b_i$. Let $V$ and $W$ be two left $H$-modules, and define

$$\begin{aligned} c_{V,W}: V\otimes W &\to W\otimes V\\ v\otimes w &\mapsto \sum (b_i\cdot w)\otimes(a_i\cdot v)\end{aligned}$$

Since $R$ is invertible, we assume that $R^{-1}=\sum a_i'\otimes b_i'$. Then $c_{V,W}$ is invertible with inverse

$$\begin{aligned} c^{-1}_{V,W}: W\otimes V &\to V\otimes W\\ w\otimes v &\mapsto \sum (a_i'\cdot v)\otimes(b_i'\cdot w)\end{aligned}$$

For example, we check that $c^{-1}_{V,W}\circ c_{V,W}=\mathrm{id}_{V\otimes W}$:

$$\begin{aligned}(c^{-1}_{V,W}\circ c_{V,W})(v\otimes w) &= \sum c^{-1}_{V,W}((b_i\cdot w)\otimes(a_i\cdot v))\\ &= \sum (a_j'\cdot a_i\cdot v)\otimes(b_j'\cdot b_i\cdot w)\\ &= (1\cdot v)\otimes(1\cdot w)\\ &= v\otimes w.\end{aligned}$$

Similarly we prove that $c_{V,W}\circ c^{-1}_{V,W}=\mathrm{id}_{W\otimes V}$. By Lemma 1, the map $c_{V,W}$ is a morphism of left $H$-modules. We need to to prove that $c_{V,W}$ is a braiding. First we prove that $c$ is natural, i.e.,

$$(g\otimes f)c_{V,W}=c_{V',W'}(f\otimes g)$$

holds for all $f:V\to V'$ and $g:W\to W'$ any two left $H$-module morphisms. We compute:

$$\begin{aligned}(g\otimes f)c_{V,W}(v\otimes w) &= (g\otimes f)\left(\sum b_i\cdot w\otimes a_i\cdot v\right)\\ &= \sum g(b_i\cdot w)\otimes f(a_i\cdot v)\\ &= \sum b_i\cdot g(w)\otimes a_i\cdot f(v)\end{aligned}$$

and on the other hand,

$$\begin{aligned}c_{V',W'}(f\otimes g)(v\otimes w) &= c_{V',W'}(f(v)\otimes g(w))\\ &= \sum b_i\cdot g(w)\otimes a_i\cdot f(v).\end{aligned}$$

To prove Equations (23) and (24) we refer to Exercise (16).

Now we prove the implication $\Longleftarrow$. So assume that ${}_H\mathcal{M}$ is braided and let $c$ be the braiding. Recall that $H$ is a left $H$-module with $h\cdot k=hk$ for all $h,k\in H$. Let

$$R=\tau_{H,H}(c_{H,H}(1\otimes 1))=\sum a_i\otimes b_i.$$

Since $C_{H,H}$ is invertible, $R$ is invertible.

Let $U,V$ be two left $H$-modules and let $v \in V$ and $w \in W$. We consider the maps $f_v : H \to V$, defined by $f_v(h) = h \cdot v$, and $f_w : H \to W$, defined by $f_w(h) = h \cdot w$. By the naturality of $c$ we obtain:

$$c_{V,W}(v \otimes w) = \sum b_i \cdot w \otimes a_i \cdot v. \tag{25}$$

In fact,

$$\begin{aligned} c_{V,W}(v \otimes w) &= c_{V,W}(f_v \otimes f_w)(1 \otimes 1) \\ &= (f_w \otimes f_v)c_{H,H}(1 \otimes 1) \\ &= (f_w \otimes f_v)\tau_{H,H}(R) \\ &= \sum b_i \cdot w \otimes a_i \cdot v. \end{aligned}$$

Since $c_{V,W}$ is a morphism of left $H$-modules,

$$c_{H,H}(h_1 \otimes h_2) = c_{H,H}(h \cdot (1 \otimes 1)) = h \cdot c_{H,H}(1 \otimes 1) = \Delta(h)c_{H,H}(1 \otimes 1).$$

Therefore, using (25) we obtain

$$\begin{aligned} \Delta^{\mathrm{cop}}(h)R &= \tau_{H,H}(\Delta(h)c_{H,H}(1 \otimes 1)) \\ &= \tau_{H,H}(c_{H,H}(h_1 \otimes h_2)) = \sum a_i h_1 \otimes b_i h_2 = R\Delta(h) \end{aligned}$$

for all $h \in H$.

Now using (25) and the equation $c_{U,V\otimes W} = (\mathrm{id}_V \otimes c_{U,W})(c_{U,V} \otimes \mathrm{id}_W)$ we will obtain $(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}$. First we compute:

$$\begin{aligned} c_{H,H\otimes H}(1 \otimes 1 \otimes 1) &= (\mathrm{id}_H \otimes c_{H,H})(c_{H,H} \otimes \mathrm{id}_H)(1 \otimes 1 \otimes 1) \\ &= (\mathrm{id}_H \otimes c_{H,H})(c_{H,H}(1 \otimes 1) \otimes 1) \\ &= (\mathrm{id}_H \otimes c_{H,H})(\tau(R) \otimes 1) \\ &= \sum (\mathrm{id}_H \otimes c_{H,H})(b_i \otimes a_i \otimes 1) \\ &= \sum b_i \otimes c_{H,H}(a_i \otimes 1) \\ &= \sum b_i \otimes b_j \otimes a_j a_i. \end{aligned}$$

Using (25) with $V = H$ and $W = H \otimes H$ one obtains:

$$c_{H,H\otimes H}(1 \otimes 1 \otimes 1) = \sum b_{i,1} \otimes b_{i,2} \otimes a_i$$

and hence $(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}$. Similarly one proves that $(\Delta \otimes \mathrm{id})(R) = R_{12}R_{23}$. □

**Exercise 78.** Prove that a Hopf algebra $H$ is triangular if and only if ${}_H\mathcal{M}$ is symmetric.

Now there is a natural way of defining an algebra in a monoidal category.

**Definition 79.** *Let $\mathcal{C}$ be a monoidal category. An* ***algebra*** *in $\mathcal{C}$ is a triple $(A,m,u)$, where $A$ is an object of $\mathcal{C}$, $m \in \hom(A\otimes A, A)$ and $u \in \hom(\mathbb{I}, A)$ such that*

$$m(\mathrm{id}\otimes m) = m(m\otimes \mathrm{id}),$$
$$m(\mathrm{id}\otimes u) = \mathrm{id} = m(u\otimes \mathrm{id}).$$

*Let $A$ and $B$ be algebras in $\mathcal{C}$ and $f \in \hom(A,B)$. Then $f$ is a* ***morphism*** *(of algebras in $\mathcal{C}$) if $m_B(f\otimes f) = fm_A$ and $fu_A = u_B$. This allows us to define the category $\mathrm{Alg}(\mathcal{C})$ of algebras in $\mathcal{C}$.*

**Example 80.** Let $\mathcal{C} = \mathrm{Vect}(\mathbb{K})$ be the category of $\mathbb{K}$-vector spaces. An algebra $A$ in $\mathcal{C}$ is an algebra in the usual sense.

**Example 81.** Let $\mathcal{C} = {}_H\mathcal{M}$ be the category of left $H$-modules. An algebra $A$ en $\mathcal{C}$ is an object of $\mathcal{C}$ such that $(a_1 \rightharpoonup b)(a_2 \rightharpoonup b') = a \rightharpoonup bb'$ and $a \rightharpoonup 1 = \varepsilon(a)1$ for all $a,b \in A$. Hence an algebra in ${}_H\mathcal{M}$ is a left $H$-module-algebra.

**Example 82.** Let $\mathcal{C} = {}^H\mathcal{M}$ be the category of left $H$-comodules. An algebra $A$ in $\mathcal{C}$ is an object of $\mathcal{C}$ such that $\delta(ab) = a_{-1}b_{-1}\otimes a_0b_0$ for all $a,b\in A$ and $\delta(1) = 1_A \otimes 1_H$. Hence an algebra in ${}^H\mathcal{M}$ is a left $H$-comodule-algebra.

**Example 83.** Let $(\mathcal{C},c)$ be a braided category and let $A$ and $B$ be two algebras in $\mathcal{C}$. Then $A\otimes B$ is an algebra in $\mathcal{C}$ with multiplication

$$m_{A\otimes B} = (m_A\otimes m_B)(\mathrm{id}_A\otimes c_{B,A}\otimes \mathrm{id}_B).$$

Similarly ones defines coalgebras in categories.

**Definition 84.** *Let $\mathcal{C}$ be a monoidal category. A* ***coalgebra*** *$C$ in $\mathcal{C}$ is a triple $(C,\Delta,\varepsilon)$, where $C$ is an object of $\mathcal{C}$, $\Delta \in \hom(C, C\otimes C)$ and $\varepsilon \in \hom(C,\mathbb{I})$ , and the following propositionerties are satisfied:*

$$(\Delta\otimes \mathrm{id})\Delta = (\mathrm{id}\otimes\Delta)\Delta,$$
$$(\mathrm{id}\otimes\varepsilon)\Delta = (\varepsilon\otimes\mathrm{id})\Delta = \mathrm{id}.$$

*Let $C$ and $D$ be two coalgebras in $\mathcal{C}$ and $f\in\hom(C,D)$. Then $f$ is a* ***morphism*** *(of coalgebras in $\mathcal{C}$) if $\Delta_D f = (f\otimes f)\Delta_C$ and $\varepsilon_D f = \varepsilon_C$. This allows us to define the category $\mathrm{Coalg}(\mathcal{C})$ of coalgebras in $\mathcal{C}$.*

**Example 85.** Let $\mathcal{C} = \mathrm{Vect}(\mathbb{K})$ be the category of $\mathbb{K}$-vector spaces. A coalgebra $C$ in $\mathcal{C}$ is a coalgebra in the usual sense.

**Example 86.** A coalgebra $C$ in ${}_H\mathcal{M}$ is an object of $\mathcal{C}$ such that

$$(h\rightharpoonup c)_1\otimes(h\rightharpoonup c)_2 = h_1\rightharpoonup c_1\otimes h_2\rightharpoonup c_2$$

and $\varepsilon(h\rightharpoonup c) = \varepsilon(h)\varepsilon(c)$ for all $h\in H$ and $c\in C$. Hence a coalgebra in ${}_H\mathcal{M}$ is a left $H$-module-coalgebra.

**Example 87.** A coalgebra $C$ in ${}^H\mathcal{M}$ is an object of $\mathcal{C}$ such that

$$c_{1,-1}c_{2,-1} \otimes c_{1,0} \otimes c_{2,0} = c_{-1} \otimes c_{0,1} \otimes c_{0,2}$$

and $c_{-1}\varepsilon_C(c_0) = \varepsilon_C(c)1$ for all $c \in C$. Hence a coalgebra in the category ${}^H\mathcal{M}$ is a left $H$-comodule-coalgebra.

**Example 88.** Let $(\mathcal{C}, c)$ be a braided category and let $C$ and $D$ be two coalgebras in $\mathcal{C}$. Then $C \otimes D$ is an coalgebra in $\mathcal{C}$ with comultiplication

$$\Delta_{C\otimes D} = (\mathrm{id}_C \otimes c_{C,D} \otimes \mathrm{id}_D)(\Delta_C \otimes \Delta_D).$$

Now it is possible to define bialgebras and Hopf algebras in braided monoidal categories.

**Definition 89.** *Let $\mathcal{C}$ be a braided monoidal category with braiding $c$. A bialgebra in $\mathcal{C}$ is a tuple $(B, m, \eta, \Delta, \varepsilon)$, where $(B, m, \eta)$ is an algebra in $\mathcal{C}$, $(B, \Delta, \varepsilon)$ is a coalgebra in $\mathcal{C}$ and such that $\Delta \in \hom(B, B \otimes B)$ and $\varepsilon \in \hom(B, \mathbb{I})$ are morphism of algebras. Here $B \otimes B$ is the algebra in $\mathcal{C}$ given by the product*

$$(m_B \otimes m_B)(\mathrm{id} \otimes c_{B,B} \otimes \mathrm{id}).$$

**Exercise 90.** Let $H$ be a quasitriangular Hopf algebra with $R = \sum a_i \otimes b_i$. Then ${}_H\mathcal{M}$ is a braided monoidal category with braiding

$$c_{V,W}(v \otimes w) = \sum_i b_i \cdot w \otimes a_i \cdot v.$$

Prove that $H$ is a bialgebra in $\mathcal{C}$ if $H$ is an algebra and a coalgebra in ${}_H\mathcal{M}$ and

$$(hh')_1 \otimes (hh')_2 = \sum_i h_1(b_i \cdot h'_1) \otimes (a_i \cdot h_2)h'_2$$

for all $h, h' \in H$.

## 6 Radford biproduct

Our goal is to know when it is possible to make $A \otimes H$ a bialgebra, where the algebra structure is given by the smash product:

$$(a \otimes h)(a' \otimes h') = a(h_1 \rightarrow a') \otimes h_2h'$$

for all $a, a' \in A$, $h, h' \in H$, and the coalgebra structure is the smash coproduct:

$$\Delta(a \otimes h) = (a_1 \otimes a_{2,-1}h_1) \otimes (a_{2,0} \otimes h_2)$$

for all $a \in A$, $h \in H$. This is the **Radford biproduct**.

**Theorem 91 (Radford).** *Let $H$ be Hopf algebra, and let $A$ be an algebra and a coalgebra such that $(A,\rightarrow)$ a left $H$-module-algebra and $(A,\delta)$ a left $H$-comodule-coalgebra. Assume that*

$$A \text{ is a left } H\text{-comodule-algebra}, \tag{26}$$
$$A \text{ is a left } H\text{-module-coalgebra}, \tag{27}$$
$$\varepsilon_A \text{ is a morphism of algebras}, \tag{28}$$
$$\Delta(1_A) = 1_A \otimes 1_A, \tag{29}$$
$$\Delta(aa') = a_1\left(a_{2,-1} \rightarrow a'_1\right) \otimes a_{2,0}a'_2, \tag{30}$$
$$(h_1 \rightarrow a)_{-1} h_2 \otimes (h_1 \rightarrow a)_0 = h_1 a_{-1} \otimes h_2 \rightarrow a_0. \tag{31}$$

*for all $a, a' \in A$, $h \in H$. Then the vector space $A \otimes H$ is a bialgebra with the algebra structure given by the left smash product and the coalgebra is the left smash coproduct. Furthermore, if $A$ has an antipode $S_A$, then $A \otimes H$ is a Hopf algebra with antipode*

$$S(a \otimes h) = (1 \otimes S_H(a_{-1}h))(S_A(a_0) \otimes 1)$$

*for all $a \in A$, $h \in H$.*

*Proof.* We first prove that $\varepsilon$ is a morphism of algebras:

$$\begin{aligned}
\varepsilon((a \otimes h)(a' \otimes h')) &= \varepsilon(a(h_1 \rightarrow a') \otimes h_2h') \\
&= \varepsilon(a(h_1 \rightarrow a')\varepsilon(h_2h') \\
&= \varepsilon(a)\varepsilon(h_1 \rightarrow a')\varepsilon(h_2)\varepsilon(h') \\
&= \varepsilon(a)\varepsilon(h_1)\varepsilon(a')\varepsilon(h_2)\varepsilon(h') \\
&= \varepsilon(a)\varepsilon(h)\varepsilon(a')\varepsilon(h') \\
&= \varepsilon(a \otimes h)\varepsilon(a' \otimes h'),
\end{aligned}$$

and $\varepsilon(1 \otimes 1) = 1$. Now we prove that $\Delta$ is a morphism of algebras. By (29), we need to prove that $\Delta$ is multiplicative. We compute:

$$\begin{aligned}
&\Delta(a \otimes h)\Delta(a' \otimes h') \\
&\quad = (a_1 \otimes a_{2,-1}h_1 \otimes a_{2,0} \otimes h_2)(a'_1 \otimes a'_{2,-1}h'_1 \otimes a'_{2,0} \otimes h'_2) \\
&\quad = (a_1 \otimes a_{2,-1}h_1)(a'_1 \otimes a'_{2,-1}) \otimes (a_{2,0} \otimes h_2)(a'_{2,0} \otimes h'_2) \\
&\quad = a_1((a_{2,-1}h_1)_1 \rightarrow a'_1) \otimes (a_{2,-1}h_1)_2 a'_{2,-1}h'_1 \otimes a_{2,0}(h_{2,1} \rightarrow a'_{2,0}) \otimes h_{2,2}h'_2 \\
&\quad = a_1((a_{2,-1,1}h_{1,1}) \rightarrow a'_1) \otimes a_{2,-1,2}h_{1,2}a'_{2,-1}h'_1 \otimes a_{2,0}(h_{2,1} \rightarrow a'_{2,0}) \otimes h_{2,2}h'_2 \\
&\quad = a_1((a_{2,-1,1}h_1) \rightarrow a'_1) \otimes a_{2,-1,2}h_2a'_{2,-1}h'_1 \otimes a_{2,0}(h_3 \rightarrow a'_{2,0}) \otimes h_4h'_2.
\end{aligned}$$

On the other hand, we compute:

$$
\begin{aligned}
&\Delta((a\otimes h)(a'\otimes h')) \\
&\quad = \Delta(a(h_1 \to a')\otimes h_2h') \\
&\quad = (a(h_1\to a'))_1 \otimes (a(h_1\to a'))_{2,-1}(h_2h')_1 \otimes (a(h_1\to a'))_{2,0}\otimes (h_2h')_2 \\
&\quad = (a(h_1\to a'))_1 \otimes (a(h_1\to a'))_{2,-1}h_2h'_1 \otimes (a(h_1\to a'))_{2,0}\otimes h_3h'_2 \\
&\quad = a_1(a_{2,-1}\to (h_1\to a')_1)\otimes (a_{2,0}(h_1\to a')_2)_{-1}h_2h'_1\otimes (a_{2,0}(h_1\to a')_2)_0\otimes h_2h'_2 \\
&\quad = a_1(a_{2,-1}\to (h_1\to a'_1)\otimes (a_{2,0}(h_2\to a'_2))_{-1}h_3h'_1\otimes (a_{2,0}(h_2\to a'_2))_0\otimes h_4h'_2 \\
&\quad = a_1(a_{2,-1}h_1\to a'_1)\otimes a_{2,0,-1}(h_2\to a'_2)_{-1}h_3h'_1\otimes a_{2,0,0}(h_2\to a'_2)_0\otimes h_4h'_2 \\
&\quad = a_1(a_{2,-1}h_1\to a'_1)\otimes a_{2,0,-1}(h_2a'_{2,-1})h'_1\otimes a_{2,0,0}(h_3\to a'_{2,0})\otimes h_4h'_2 \\
&\quad = a_1(a_{2,-1,1}h_1\to a'_1)\otimes a_{2,-1,2}h_2a'_{2,-1}h'_1\otimes a_{2,0}(h_3\to a'_{2,0})\otimes h_4h'_2.
\end{aligned}
$$

Since $A$ is a left $H$-comodule-coalgebra and $a_{1,-1}a_{2,-1}\otimes a_{1,0}\otimes a_{2,0} = a_{-1}\otimes a_{0,1}\otimes a_{0,2}$, we obtain:

$$
\begin{aligned}
S((a\otimes h)_1)(a\otimes h)_2 &= S(a_1\otimes a_{2,-1}h_1)(a_{2,0}\otimes h_2) \\
&= (1\otimes S_H(a_{1,-1}a_{2,-1}h_1))(S_A(a_{1,0})\otimes 1)(a_{2,0}\otimes h_2) \\
&= S_A(a_{1,0})a_{2,0}\otimes S_H(a_{1,-1}a_{2,-1}h_1)h_2 \\
&= \varepsilon(a_0)1\otimes S_H(a_{-1}h_1)h_2 \\
&= \varepsilon(a)\varepsilon(h)1\otimes 1.
\end{aligned}
$$

Since $a_{-1}\otimes a_{0,-1}\otimes a_{0,0} = a_{-1,1}\otimes a_{-1,2}\otimes a_0$ we obtain:

$$
\begin{aligned}
(a\otimes h)_1S((a\otimes h)_2) &= (a_1\otimes a_{2,-1}h_1)S(a_{2,0}\otimes h_2) \\
&= (a_1\otimes a_{2,-1}h_1)(1\otimes S_H(a_{2,0,-1}h_2))(S_A(a_{2,0,0})\otimes 1) \\
&= a_1S_A(a_{2,0,0})\otimes a_{2,-1}h_1S_H(a_{2,0,-1}h_2) \\
&= a_1S_A(a_{2,0,0})\otimes a_{2,-1}h_1S_H(h_2)S_H(a_{2,0,-1}) \\
&= a_1S_A(a_{2,0})\otimes a_{2,-1,1}S_H(a_{2,-1,2})\varepsilon(h) \\
&= a_1S_A(a_2)\otimes 1\varepsilon(h) \\
&= 1\otimes 1\varepsilon(a)\varepsilon(h). \qquad \square
\end{aligned}
$$

**Exercise 92.** Prove that the Radford biproduct over $A\otimes H$ is commutative if and only if $A$ and $H$ are commutatives and the action $\to$ is trivial. Similarly, the Radford biproduct over $A\otimes H$ is cocommutative if and only if $A$ and $H$ are cocommutative and the coaction $\delta_A$ is trivial.

**Exercise 93.** Prove the converse of Theorem 91: assume that $H$ is a bialgebra, $A$ is a left $H$-module-algebra and a left $H$-comodule coalgebra and the Radford biproduct $A\otimes H$ is a bialgebra. Then (26)–(31) are satisfied.

Similarly, it is possible to put on $H\otimes B$ a bialgebra structure, where the algebra structure is given by the smash product over $H\otimes B$ and the coalgebra is given by the smash coproduct over $H\otimes B$. For that purpose we need $B$ to be a right $H$-module-algebra and a right $H$-comodule-coalgebra. In this case, the necessary and sufficient conditions are:

$$\begin{gathered}
B \text{ is a right } H\text{-comodule-algebra,}\\
B \text{ a right } H\text{-module-coalgebra,}\\
\varepsilon_B \text{ is a morphism of algebras,}\\
\Delta(1_B) = 1_B \otimes 1_B,\\
\Delta(bb') = b_1 b'_{1,0} \otimes (b_2 \leftarrow b'_{1,1}) b'_2,\\
(b_0 \leftarrow h_1) \otimes b_1 h_2 = (b \leftarrow h_2)_0 \otimes h_1 (b \leftarrow h_2)_1.
\end{gathered}$$

A different and important bialgebra structure on $A \otimes H$ is the so-called **Majid product**. Let $A$ be an left $H$-module-algebra and $H$ be a right $A$-comodule-coalgebra. On the vector space $A \otimes H$ we consider the algebra structure given by the smash product on $A \otimes H$ and the coalgebra structure given on $A \otimes H$, i.e.,

$$\begin{aligned}
(a \otimes h)(a' \otimes h') &= a(h_1 \rightarrow a') \otimes h_2 h',\\
\Delta(a \otimes h) &= a_1 \otimes h_{1,0} \otimes a_2 h_{1,1} \otimes h_2.
\end{aligned}$$

Then $A \otimes H$ is a bialgebra if and only if

$$\begin{gathered}
\varepsilon(h \rightarrow a) = \varepsilon_H(h)\varepsilon_A(a),\\
\Delta(h \rightarrow a) = h_{1,0} \rightarrow a_1 \otimes h_{1,1}(h_2 \rightarrow a_2),\\
\delta(1) = 1 \otimes 1,\\
\delta(hh') = h_{1,0} h'_0 \otimes h_{1,1}(h_2 \rightarrow h'_1),\\
h_{2,0} \otimes (h_1 \rightarrow a) h_{2,1} = h_{1,0} \otimes h_{1,1}(h_2 \rightarrow a).
\end{gathered}$$

The following result is known as the Radford's bosonization.

**Theorem 94 (Radford).** *Let $H$ be a Hopf algebra with bijective antipode. There exists a bijective correspondence between*

*1) Hopf algebras $A$ with morphisms $H \xrightarrow{i} A \xrightarrow{p} H$ such that $pi = id_H$.*
*2) Hopf algebras in the category ${}^H_H\mathcal{YD}$.*

*Proof.* Assume (1). We claim that

$$R = A^{\mathrm{co}H} = \{a \in A \mid (\mathrm{id} \otimes p)\Delta(a) = a \otimes 1\}$$

is a Hopf algebra in the category of left Yetter-Drinfeld modules. It is clear that $R$ is a subalgebra of $A$. Now define

$$\begin{aligned}
\Delta_R(r) &= r_1 iSp(r_2) \otimes r_3,\\
S_R(r) &= ip(r_1)S(r_2),\\
h \rightarrow r &= i(h_1) r iS(h_2),\\
\delta(r) &= (p \otimes \mathrm{id})\Delta(r),
\end{aligned}$$

for all $r \in R$ and $h \in H$. We write $\Delta_R(r) = r^1 \otimes r^2$ to distinguish $\Delta_R(r)$ and $\Delta_A(r) = r_1 \otimes r_2$. We claim that $\Delta_R$ is coassociative:

$$\begin{aligned}(\mathrm{id} \otimes \Delta_R)\Delta_R(r) &= (\mathrm{id} \otimes \Delta_R)(r_1 iSp(r_2) \otimes r_3)\\ &= r_1 iSp(r_2) \otimes r_{3,1} iSp(r_{3,2}) \otimes r_{3,3}\\ &= r_1 iSp(r_2) \otimes r_3 iSp(r_4) \otimes r_5.\end{aligned}$$

On the other hand:

$$\begin{aligned}(\Delta_R \otimes \mathrm{id})\Delta_R(r) &= (\Delta_R \otimes \mathrm{id})(r_1 iSp(r_2) \otimes r_3)\\ &= \Delta_R(r_1 iSp(r_2)) \otimes r_3\\ &= [r_1 iSp(r_2)]_1 iSp([r_1 iSp(r_2)]_2) \otimes [r_1 iSp(r_2)]_3 \otimes r_3\\ &= r_{1,1}[iSp(r_2)]_1 iSp(r_{1,2}[iS_H p(r_2)]_2) \otimes r_{1,3}[iSp(r_2)]_3 \otimes r_3\\ &= r_1 iSp(r_6) iSp(r_2 iSp(r_5)) \otimes r_3 iSp(r_4) \otimes r_7\\ &= r_1 iS[p(r_2)Sp(r_5)r_6] \otimes r_3 iSp(r_4) \otimes r_7\\ &= r_1 iSp(r_2) \otimes r_3 iSp(r_4) \otimes r_5.\end{aligned}$$

Hence $R$ is an algebra and a coalgebra.

We claim that $R$ is a left $H$-comodule-algebra, since

$$\delta(1) = (p \otimes \mathrm{id})\Delta(1) = p(1) \otimes 1 = 1 \otimes 1,$$

and

$$\delta(rr') = p(r_1 r'_1) \otimes r_2 r'_2 = p(r_1)p(r'_1) \otimes r_2 r'_2 = r_{-1} r'_{-1} \otimes r_0 r'_0.$$

We claim that $R$ if a left $H$-comodule-coalgebra, since

$$r_{-1}\varepsilon(r_0) = p(r_1)\varepsilon(r_2) = p(r_1\varepsilon(r_2)) = p(r)$$

and since $r \in R$,

$$\varepsilon(r) = (\varepsilon \otimes \mathrm{id})(r \otimes 1) = (\varepsilon \otimes \mathrm{id})(\mathrm{id} \otimes p)\Delta(r) = \varepsilon(r_1)p(r_2) = p(r).$$

Futhermore,

$$\begin{aligned}(r^1)_{-1}(r^2)_{-1} \otimes (r^1)_0 \otimes (r^2)_0 &= p[(r_1 iSpr_2)_1 r_{3,1}] \otimes (r_1 iSp(r_2))_2 \otimes r_{3,2}\\ &= p[r_{1,1} i(Spr_2)_1 r_{3,1}] \otimes r_{1,2} i(Spr_2)_2 \otimes r_{3,2}\\ &= p(r_1 iSpr_4 r_5) \otimes r_2 iSp(r_3) \otimes r_6\\ &= p(r_1) \otimes r_2 iSpr_3 \otimes r_4.\end{aligned}$$

and on the other hand,

$$
\begin{aligned}
r_{-1}\otimes(r_0)^1\otimes(r_0)^2 &= r_{-1}\otimes\Delta_R(r_0)\\
&= p(r_1)\otimes\Delta_R(r_2)\\
&= p(r_1)\otimes r_2 iSp(r_3)\otimes r_4.
\end{aligned}
$$

We claim that $R$ is a left $H$-module-algebra, since

$$
h\to 1 = ih_1 iSh_2 = i(h_1Sh_2) = \varepsilon(h)i(1) = \varepsilon(h)1
$$

and

$$
\begin{aligned}
(h_1\to r)(h_2\to r') &= ih_{1,1}riSh_{1,2}ih_{2,1}r'iSh_{2,2}\\
&= ih_1riSh_2ih_3r'iSh_4\\
&= ih_1r\varepsilon(h_2)r'iSh_3\\
&= ih_1rr'iSh_2\\
&= h\to(rr').
\end{aligned}
$$

We claim that $R$ is a left $H$-module-coalgebra, since

$$
\varepsilon(h\to r) = \varepsilon(ih_1aiSh_2) = \varepsilon(ih_1)\varepsilon(r)\varepsilon(iSh_2) = \varepsilon(h)\varepsilon(r)
$$

and

$$
\begin{aligned}
\Delta_R(h\to r) &= \Delta_R(ih_1riSh_2)\\
&= [ih_1riSh_2]_1 iSp([ih_1riSh_2]_2)\otimes[ih_1riSh_2]_3\\
&= ih_{1,1}r_1iS(h_2)_2iSp(ih_{1,2}r_2iS(h_2)_2\otimes ih_{1,3}r_3iS(h_2)_3\\
&= ih_1r_1iSh_6iSp[ih_2r_2iSh_5]\otimes ih_3r_3iSh_4\\
&= ih_1r_1iSh_6iS[h_2pr_2Sh_5]\otimes ih_3r_3iSh_4\\
&= ih_1r_1iS[h_2pr_2Sh_5h_6]\otimes ih_3r_3iSh_4\\
&= ih_1r_1iS(h_2pr_2)\varepsilon(h_5)\otimes ih_3r_3iSh_4\\
&= ih_1r_1iSpr_2Sh_2\otimes ih_3r_3iSh_4
\end{aligned}
$$

and

$$
\begin{aligned}
h_1\to r^1\otimes h_2\to r^2 &= h_1\to r_1iSpr_2\otimes h_2\to r_3\\
&= ih_{1,1}r_1iSpr_2iSh_{1,2}\otimes ih_{2,1}r_3iSh_{2,2}\\
&= ih_1r_1iSpr_2iSh_2\otimes ih_3r_3iSh_4
\end{aligned}
$$

To prove that $R$ is a bialgebra in ${}^H_H\mathcal{YD}$ it remains to prove that $\Delta_R$ is a morphism in ${}^H_H\mathcal{YD}$. We compute:

$$\begin{aligned}\Delta_R(rr') &= (rr')_1 iSp((rr')_2) \otimes (rr')_3 \\ &= r_1 r'_1 iSp(r_2 r'_2) \otimes r_3 r'_3 \\ &= r_1 r'_1 iSp(r'_2) iSp(r_2) \otimes r_3 r'_3.\end{aligned}$$

On the other hand,

$$\begin{aligned}r^1((r^2)_{-1} \to r'^1) \otimes (r^2)_0 r'^2 &= r_1 iSp(r_2)(r_{3,-1} \to (r'_1 iSp(r'_2)) \otimes r_{3,0} r'_3 \\ &= r_1 iSp(r_2)(p(r_{3,1}) \to r'_1 iSp(r'_2)) \otimes r_{3,2} r'_3 \\ &= r_1 iSp(r_2) i(p(r_3)_1) r'_1 iSp(r'_2) iS(p(r_2)_2) \otimes r_4 r'_3 \\ &= r_1 i[Sp(r_2)p(r_3)] r'_1 iS[p(r'_2) iSp(r_4)] \otimes r_5 r'_3 \\ &= r_1 \varepsilon(r_2) r'_1 iSp(r'_2) iSp(r_3) \otimes r_4 r'_3 \\ &= r_1 r'_1 iSp(r'_2) iSp(r_2) \otimes r_3 r'_3.\end{aligned}$$

Conversely, let $R$ be a Hopf algebra in the category of Yetter-Drinfeld modules. Then the Radford biproduct $R \otimes H$ is a Hopf algebra by Theorem 91. The maps $p : R \otimes H \to H$, defined by $r \otimes h \mapsto \varepsilon(r)h$, and $i : H \to R \otimes H$, defined by $h \mapsto 1 \otimes h$ are Hopf algebra morphisms and $p \circ i = \mathrm{id}$. □

**Exercise 95.** Let $A$ and $H$ be two Hopf algebras such that there exist Hopf algebras morphisms $H \xrightarrow{i} A \xrightarrow{p} H$ such that $pi = \mathrm{id}_H$. Let $R = A^{\mathrm{co}H}$ and consider the map $\omega : A \to R$ defined by $a \mapsto a_1 ip(Sa_2)$.

1) Prove that the maps $\alpha : A \to R \otimes H$, $\alpha(a) = \omega(a_1) \otimes p(a_2)$, and $\beta : R \otimes H \to A$, $r \otimes h \mapsto ri(h)$ are Hopf algebra homomorphisms.
2) Prove that $\alpha \circ \beta = \mathrm{id}_{R \otimes H}$ and $\beta \circ \alpha = \mathrm{id}_A$ and conclude that $A \simeq R \otimes H$ as Hopf algebras.

## 7 Some solutions

**16** We prove (9). A straightforward computation shows that

$$R_{U\otimes V,W}(u\otimes v\otimes w)=\sum b_i\cdot w\otimes a_{i1}\cdot u\otimes a_{i2}\cdot v.$$

On the other hand,

$$\begin{aligned}(R_{U,W}\otimes \mathrm{id}_V)(\mathrm{id}_U\otimes R_{V,W})(u\otimes v\otimes w)\\ &=\sum(R_{U,W}\otimes \mathrm{id}_V)(u\otimes b_i\cdot w\otimes a_i\cdot v)\\ &=\sum(b_jb_i)\cdot w\otimes a_j\cdot w\otimes a_i\cdot w\end{aligned}$$

and the claim follows from Equation (2). The proof for (10) is similar.

**20** Define $\phi: V\otimes W\to W\otimes V$ by $v\otimes w\mapsto R^{-1}\cdot(w\otimes v)$. Then $\phi$ is an isomorphism of left $H$-modules:

$$\begin{aligned}\phi(h\cdot(v\otimes w))&=R^{-1}(h_2\cdot w\otimes h_1\cdot v)\\ &=R^{-1}\tau\Delta(h)(w\otimes v)=\Delta(h)R^{-1}(w\otimes v)=h\cdot\phi(v\otimes w).\end{aligned}$$

**23** First we prove that $(H,\to)$ is a left $H$-module. We compute

$$\begin{aligned}b\to(a\to x)&=b\to(a_1xS(a_2))\\ &=b_1a_1xS(a_2)S(b_2)\\ &=(ba)_1xS((ba)_2)\\ &=(ba)\to x.\end{aligned}$$

Then $(H,\to)$ is a left $H$-module, since it is trivial to prove that $1\to x=x$. To prove that $(H,\to)$ is a left module-algebra over $H$ we compute:

$$a\to=a_1 1S(a_2)=\varepsilon(a)1,$$

and

$$\begin{aligned}(a_1\to x)(a_2\to y)&=(a_{1,1}xS(a_{1,2}))(a_{2,1}yS(a_{2,2}))\\ &=a_1x\varepsilon(a_2)yS(a_3)\\ &=a_1xyS(a_2)\\ &=a\to(xy).\end{aligned}$$

The proof for the right adjoint action is similar.

**25** We first prove that $1\otimes 1$ is the unit:

$$\begin{aligned}&(1\otimes 1)(a\otimes h)=1(1\to a)\otimes 1h=1a\otimes h=a\otimes h,\\ &(a\otimes h)(1\otimes 1)=a(h_1\to 1)\otimes h_2 1=a(\varepsilon(h_1)1)\otimes h_2=a\otimes h.\end{aligned}$$

Now we prove the associativity. A direct computation shows that

$$\begin{aligned}((a\otimes h)(b\otimes g))(c\otimes k) &= (a(h_1\to b)\otimes h_2g)(c\otimes k)\\ &= (a(h_1\to b))((h_2g)_1\to c)\otimes (h_2g)_2k\\ &= a(h_1\to b)(h_2g_1\to c)\otimes (h_3g_2)k.\end{aligned}$$

On the other hand, since $A$ is an $H$-module-algebra,

$$\begin{aligned}(a\otimes h)((b\otimes g)(c\otimes k)) &= (a\otimes h)(b(g_1\to c)\otimes g_2k)\\ &= a(h_1\to (b(g_1\to c)))\otimes h_2(g_2k)\\ &= a(h_1\to b)(h_2\to (g_1\to c))\otimes h_3(g_2k).\end{aligned}$$

**35** We first prove that $\varepsilon$ is the counit:

$$\begin{aligned}(\varepsilon\otimes \mathrm{id})\Delta(c\otimes h) &= (\varepsilon\otimes \mathrm{id})(c_1\otimes c_{2,-1}h_1\otimes c_{2,0}\otimes h_2)\\ &= \varepsilon(c_1\otimes c_{2,-1}h_1)c_{2,0}\times h_2\\ &= \varepsilon_C(c_1)\varepsilon_H(c_{2,-1}h_1)c_{2,0}\otimes h_2\\ &= \varepsilon_C(c_1)\varepsilon_H(h_1)(c_{2,-1})c_{2,0}\otimes \varepsilon_H(h_1)h_2\\ &= c\otimes h,\end{aligned}$$

where the last equality holds since $(\varepsilon_H\otimes \mathrm{id})\delta = \mathrm{id}$ and hence

$$c = (\varepsilon_H\otimes \mathrm{id})\delta(c) = (\varepsilon_H\otimes \mathrm{id})\delta(\varepsilon_C(c_1)c_2) = \varepsilon_C(c_1)\varepsilon_H(c_{2,-1})c_{2,0}.$$

Similarly we obtain that $(\mathrm{id}\otimes\varepsilon)\Delta = \mathrm{id}$. Now we prove the coassociativity:

$$\begin{aligned}(\Delta\otimes \mathrm{id})\Delta(c\otimes h) &= (\Delta\otimes \mathrm{id})((c_1\otimes c_{2,-1}h_1)\otimes (c_{2,0}\otimes h_2))\\ &= \Delta(c_1\otimes c_{2,-1}h_1)\otimes (c_{2,0}\otimes h_2)\\ &= c_{1,1}\otimes c_{1,2,-1}(c_{2,-1}h_1)_1\otimes c_{1,2,0}\otimes (c_{2,-1}h_1)_2\otimes c_{2,0}\otimes h_2\\ &= c_1\otimes c_{2,-1}(c_{3,-1}h_1)_1\otimes c_{2,0}\otimes (c_{3,-1}h_1)_2\otimes c_{3,0}\otimes h_2\\ &= c_1\otimes c_{2,-1}c_{3,-1,1}h_1\otimes c_{2,0}\otimes c_{3,-1,2}h_2\otimes c_{3,0}\otimes h_3\\ &= c_1\otimes c_{2,-1}c_{3,-1}h_1\otimes c_{2,0}\otimes c_{3,0,-1}h_2\otimes c_{3,0,0}\otimes h_3\\ &= c_1\otimes c_{2,-1}c_{3,-2}h_1\otimes c_{2,0}\otimes c_{3,-1}h_2\otimes c_{3,0}\otimes h_3,\end{aligned}$$

where we have used that $C$ is a left $H$-comodule-coalgebra:

$$c_{-1,1}\otimes c_{-1,2}\otimes c_0 = c_{-1}\otimes c_{0,-1}\otimes c_{0,0} = c_{-2}\otimes c_{-1}\otimes c_0 \in H\otimes H\otimes C.$$

On the other hand,

$$
\begin{aligned}
(\mathrm{id}\otimes\Delta)\Delta(c\otimes h) &= (\mathrm{id}\otimes\Delta)((c_1\otimes c_{2,-1}h_1)\otimes(c_{2,0}\otimes h_2))\\
&= c_1\otimes c_{2,-1}h_1\otimes\Delta(c_{2,0}\otimes h_2)\\
&= c_1\otimes c_{2,-1}h_1\otimes c_{2,0,1}\otimes c_{2,0,2,-1}h_{2,1}\otimes c_{2,0,2,0}\otimes h_{2,2}\\
&= c_1\otimes c_{2,-1}h_1\otimes c_{2,0,1}\otimes c_{2,0,2,-1}h_2\otimes c_{2,0,2,0}\otimes h_3\\
&= c_1\otimes c_{2,-1}c_{3,-1}h_1\otimes c_{2,0}\otimes c_{3,0,-1}h_2\otimes c_{3,0,0}\otimes h_3\\
&= c_1\otimes c_{2,-1}c_{3,-2}h_1\otimes c_{2,0}\otimes c_{3,-1}h_2\otimes c_{3,0}\otimes h_3,
\end{aligned}
$$

where we have used that $c_{-1}\otimes c_{0,1}\otimes c_{0,2} = c_{1,-1}c_{2,-1}\otimes c_{1,0}\otimes c_{2,0}$ since $C$ is a left $H$-comodule-coalgebra.

**52** Assume that (15) holds. Then

$$\delta(h_1\to v) = (h_1\to v)_{-1}\otimes(h_1\to v)_0 = h_{1,1}v_{-1}Sh_{1,3}\otimes h_{1,2}\to v_0.$$

Hence

$$(h_1\to v)_{-1}h_2\otimes(h_1\to v)_0 = h_{1,1}v_{-1}Sh_{1,3}h_2\otimes h_{1,2}\to v_0 = h_1v_{-1}\otimes h_2\to v_0.$$

Conversely, assume that (16) holds. Then

$$
\begin{aligned}
&(m\otimes\mathrm{id})(h_{11}v_{-1}\otimes Sh_2\otimes(h_{12}\to v_0))\\
&\quad= (m\otimes\mathrm{id})((h_{11}\to v)_{-1}h_{12}\otimes Sh_2\otimes(h_{11}\to v)_0)\\
&\quad= (h_1\to v)_{-1}h_2Sh_3\otimes(h_1\to v)_0\\
&\quad= (h\to v)_{-1}\otimes(h\to v)_0.
\end{aligned}
$$

**54** To prove the compatibility condition (15) we compute

$$
\begin{aligned}
\delta(h\to(v\otimes w)) &= \delta(h_1\to v\otimes h_2\to w)\\
&= (h_1\to v)_{-1}(h_2\to w)_{-1}\otimes(h_1\to v)_0\otimes(h_2\to w)_0\\
&= (h_1v_{-1}(Sh_3)h_4w_{-1}Sh_6\otimes(h_2\to v_0)\otimes(h_5\to w_0)\\
&= h_1v_{-1}w_{-1}Sh_4\otimes(h_2\to v_0)\otimes(h_3\to w_0)\\
&= h_1v_{-1}w_{-1}Sh_3\otimes h_2\to(v_0\otimes w_0)\\
&= h_1(v\otimes w)_{-1}Sh_3\otimes h_2\to(v\otimes w)_0.
\end{aligned}
$$

**78** Assume first that $H$ es triangular. Then $\tau(R) = R$ and hence $c_{V,W}c_{W,V} = \mathrm{id}_{V\otimes W}$. Conversely, using (25) we obtain

$$1\otimes 1 = c_{H,H}(c_{H,H}(1\otimes 1)) = c_{H,H}(\tau(R)) = \tau(R)R.$$

## References


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