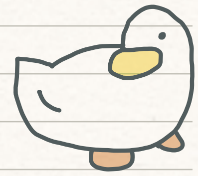
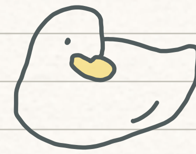


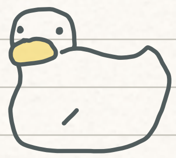
Some Problems in Hopf-Galois Theory

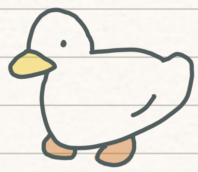
Andrew Darlington



I want to tell you a little bit about my background, what I've been up to and what I'm interested in.

1. Setting & Motivation
2. Applications & problems
3. Further directions





① Setting & Motivation

Galois theory:

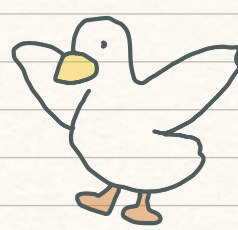
- Galois extension L/K
- $G = \mathrm{Gal}(L/K)$, $|G| = [L:K]$
- $H \leq G \longleftrightarrow K \subseteq F \subseteq L$

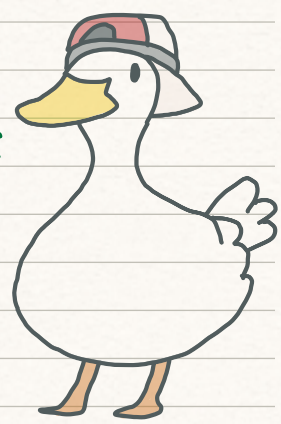
Observations

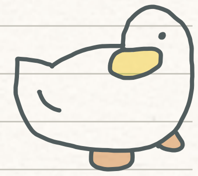
- L is a $K[G]$-module
- $K[G]$ can be endowed with a Hopf algebra structure:

$g, g_1, g_2 \in G$

$m(g_1 \otimes g_2) = g_1 g_2$







Just say:

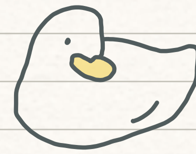
$\iota(1_K) = 1_G$

$\Delta(g) = g \otimes g$ — G is grouplike in $K[G]$

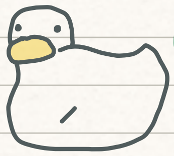
$\varepsilon(g) = 1_K$

$S(g) = g^{-1}$

& extend to $K[G]$



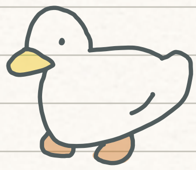
- (The K-linear map)



$$\theta : L \otimes K[G] \cong L[G] \longrightarrow \mathrm{End}_K(L)$$

$$h := \sum a_g g \longmapsto \theta_h(x) = \sum a_g g(x)$$

is bijective

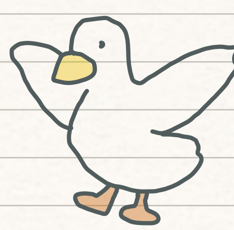


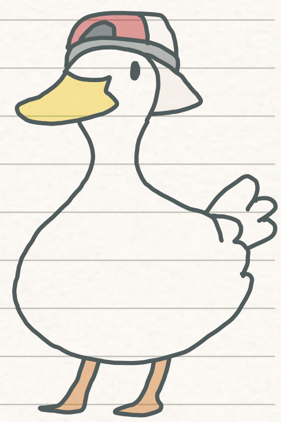
Hopf-Galois theory:

- L/K (not nec. Galois) extension
- H a (cocommutative) K-Hopf alg. acting on L

Def H gives a Hopf-Galois Structure on L/K if

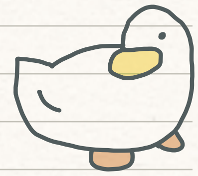
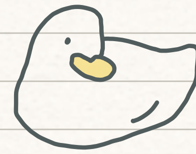
- L is an H-module algebra
  - is sufficiently compatible
- The K-linear map





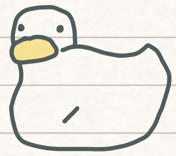
$$\theta : L \otimes H \longrightarrow \mathrm{End}_K(L)$$
$$x \otimes h \longmapsto \theta_{x\otimes h}(y) = x(h \cdot y)$$

is bijective

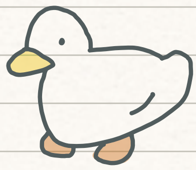




Consequences:

- L/K Galois with group G
  $K[G]$ gives a HGS
  on L/K



- Unique Galois group
  $\rightsquigarrow$ not nec. unique HGS
  ↳ $H, \cdot_1, \cdot_2$ s.t. $(H, \cdot_1)$ & $(H, \cdot_2)$
  give different HGS



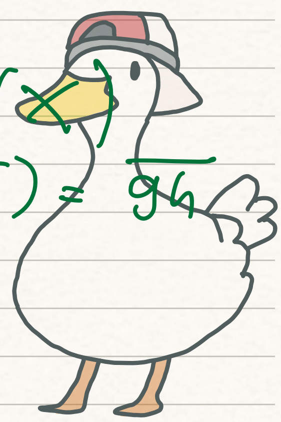
- Applies to several non-Galois
  extensions too
- $I \leq H \rightsquigarrow K \subseteq F \subseteq L$
  injective, not always surjective!

Greither – Pareigis 1987

$$\begin{array}{c} E \\ | \\ L \\ | \\ K \end{array} \quad G' = \mathrm{Gal}(E/L) \text{ (}E\text{ to }L\text{)}, \quad G \text{ (}E\text{ to }K\text{)}$$

$X := G/G'$

$$\lambda : G \longrightarrow \mathrm{Perm}(X)$$
$$g \longmapsto \lambda g(\bar{h}) = \overline{gh}$$

HGS on L/k

$$\text{HGS on } L/k \;\longleftrightarrow\; N \le \mathrm{Perm}(X) \text{ regular}, \quad \lambda_g N \lambda_g^{-1} = N$$

$$H := L[N]^{\lambda(G)} \longmapsfrom N$$

HGS of type N

e.g. L/k Galois

$$K[G] = L[\rho(G)]^{\lambda(G)}$$

right translations

G non-abelian $\Rightarrow K[G] \not\cong L[\lambda(G)]^{\lambda(G)}$

But $\mathrm{Perm}(X)$ can be very large!

Byott's translation

N group $\quad \mathrm{Hol}(N) = N \rtimes \mathrm{Aut}(N)$

$M \le \mathrm{Hol}(N)$ acts on N:

Byott '96 $\quad (\eta, \alpha) \cdot \mu = \eta\alpha(\mu)$

$$\text{HGS of type } N \;\longleftrightarrow\; \text{transitive subgroups of } \mathrm{Hol}(N)$$

L/k with $G \overset{\phi}{\cong} M$, $\phi(G') = M\text{-}\mathrm{stab}(1_N)$, admits HGS of type N $\longmapsfrom M$

+ counting formula!

NB We see a link with skew braces when $|M| = |N|$ (corresponding L/K are Galois)

② Applications & problems

central idea: Look for all HGS on separable extensions of degree n

↳ Gives info about L/K Gal. mod. th.

↳ builds up alg. toolbox

↳ Interesting problem anyway!

Problem 1

n squarefree

results

- Byott '96: n Burnside
- Crespo & Salguero 2020: $n = p^2, 2p$
- D. 2024: $n = pq$

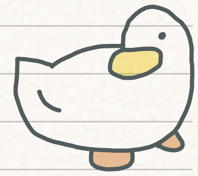
To see a bit of what this looks like [show slide]

- n arbitrary squarefree?

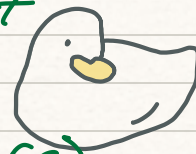
work in progress. $p_i = 2p_{i+1} + 1$

nice results when conditions are imposed.

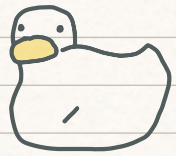
Problem 2: Use a computer?



- Skew braces: - Guarnieri & Vendramin 2017
  - Bardakov, Neshchadim, Yadav 2020 ($n \leq 868$)

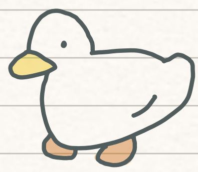


- HGS Galois: - Byott & Vendramin, appendix of Smoktunowicz & Vendramin 2018



- HGS Separable: Crespo & Salguero
  - 2020 ($n \leq 11$)
  - 2021 ($n \leq 31$)

using classification of perm gps.



my work

Skew brace inspired & using MAGMA

HGS on sep. ext. $n \leq 99$
& potential for much further

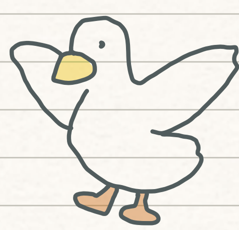
[show on screen]

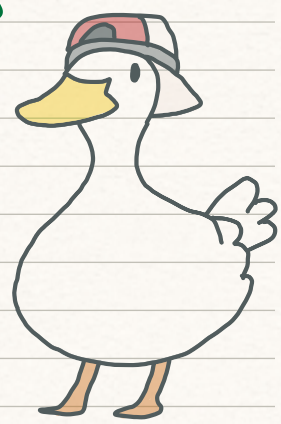
todo

to spot patterns in the data & formulate conjectures

↳ use AI?

↳ How far can we go?





Problem 3

$$\begin{tikzcd}
E \arrow[d, dash] \arrow[d, bend right, dash, "G'"'] \arrow[dr, dash] \arrow[dd, bend right=70, dash, "G"'] & \\
L \arrow[d, dash, "n"] & L' = E^H \arrow[dl, dash, "n"] \\
K &
\end{tikzcd}$$

$H \leq G$, $|G:H| = n$

<u>Question:</u> If L/K admits a HGS, what about L'/K?

time check

$C := \mathrm{Core}_G(H) \rightsquigarrow E^C$ is Galois closure of L'/K

$\Rightarrow G/C \cong \mathrm{Gal}(E^C/K)$

$H/C \cong \mathrm{Gal}(E^C/L')$

<u>revised question:</u>

Does $(G/C, H/C)$ 'appear' as a transitive $M \leq \mathrm{Hol}(N')$ for any $|N'| = n$?

If "no", then we say L/K has the parallel no-HGS property.

results:

- No L/K with $[L:K] = pq$ has parallel no-HGS
- computer examples: degrees 8, 12, 24, 27
- Theorem: If L/K has parallel no-HGS, then we can find an infinite family of such
- Conjecture: $[L:K]$ squarefree $\Rightarrow$ L/K does not admit parallel no-HGS

To do

- Build up intuition of behavior
  - Depends on how close L/K is to being Galois.
- Connections with other properties
  - HG correspondence?

3 Further directions

A <u>skew bracoid</u> is a 5-tuple $(G, \cdot, N, +, \odot)$ s.t.

$(G, \cdot)$, $(N, +)$ groups (not nec. abelian)

$\odot : G \times N \to N$ transitive action with

$$g \odot (\eta + \mu) = (g \odot \eta) - (g \odot e_N) + (g \odot \mu)$$

$$\forall g \in G,\ \forall \eta, \mu \in N$$

Martin-Lyons & Truman 2024:

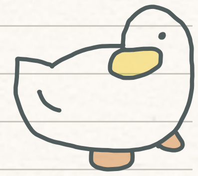
(equiv. classes of) skew bracoids with $(N, +)$

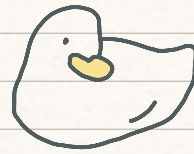
$$\updownarrow$$

$M \leq \mathrm{Hol}(N, +)$ transitive

$$\updownarrow$$

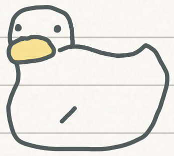
HGS of type N



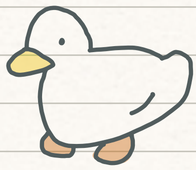


- To what extent can we apply these HGS results to skew braces?

- Are there other properties that make sense in both worlds?



e.g. — with Silvia: different notions of nilpotency in skew braces



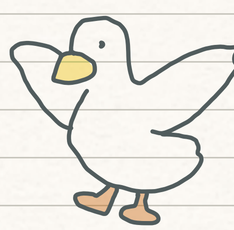
$\{$

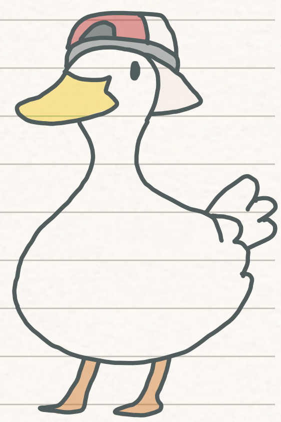
HGS?

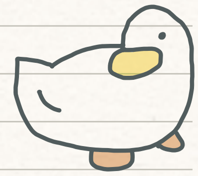
using connection of Senne & Lorenzo

Always looking for cool finite/permutation group facts & tricks!

That's about all I have to say — thanks!

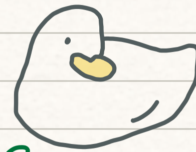






problems

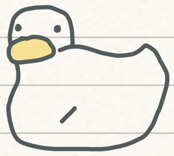
- Complete classification for Squarefree degree?



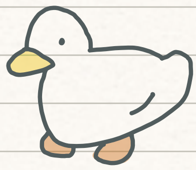
- $M \leq \mathrm{Perm}(N)$ transitive

$\mathrm{Cent}_{\mathrm{Perm}(N)}(M)$ semi-regular

$\mathrm{Stab}(1_N) = 1_M$



Can HGS be classified with Semi-regular subgroups?



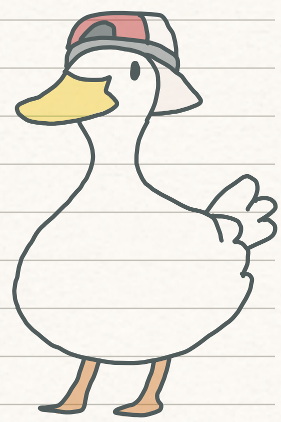
- HGS related to $M \leq \mathrm{Hol}(N)$ is called almost classically Galois if

$\exists\, I \trianglelefteq M$ s.t.

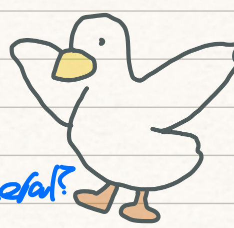
$$M = I \rtimes M' \quad , M' = \mathrm{Stab}_M(1_N)$$

(hence $I \leq \mathrm{Hol}(N)$ is regular)

a.c.g. structures give bijective HG correspondence



looks like the classification of a.c.g. structures is almost the same as finding regular subgroups.



how fesible is this in general?