

# Trusses: between braces and rings

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**Abstract** An algebraic structure is a collection of sets with operations. Typical and most widespread across mathematics are systems such as a semigroup, monoid, group, ring, field, associative algebra, vector space or module. In this course we will study some simple algebraic systems which have recently gained prominent position in algebra and topology such as braces, racks or quandles (sets with two operations interacting with each other in prescribed ways). In particular we will explore a little known fact (first described nearly 100 years ago by Pruefer and Baer) that one can give a definition of a group without requesting existence of the neutral element and inverses by using a ternary rather than a binary operation (i.e. an operation with three rather than the usual two inputs). A set with such a suitable ternary operation is known as a heap. By picking an element in a heap, the ternary operation is reduced to the binary group operation, for which the chosen element is the neutral element (the resulting group is known as a retract). We will study properties and examples of heaps and relate them to the properties of corresponding groups (retracts). Next we will look at heaps with an additional binary operation that distributes over the ternary heap operation, known as trusses, relate them to both rings and braces, and study their properties and applications.

## 1 Introduction

The notes correspond to a 8-hours mini-course taught by Prof. Tomasz Brzeziński at the Department of Mathematics at ULB, Brussels, in May 2022. The course is addressed to Master and PhD students, researchers and any one else with an interest in (new) algebraic structures. Notes by Leandro Vendramin. Please send comments and corrections to: `Leandro.Vendramin@vub.be`

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## 2 Algebraic structures

We start with some basic definitions from universal algebra. We refer to [1] for more details.

**Definition 2.1.** An **algebraic structure** is a set  $A$  with a collection of maps (called operations)  $\alpha_i: A^{|\alpha_i|} \rightarrow A$  for  $i \in I$ . By convention,  $A^0 = \{*\}$ . This algebraic structure on  $A$  will be denoted by  $(A, (\alpha_i)_{i \in I})$ . The number  $|\alpha_i| \in \mathbb{Z}_{\geq 0}$  is the **arity** of the operation  $\alpha_i$ .

For example, let  $A$  be a set and  $\alpha: A^{|\alpha|} \rightarrow A$  be a map. If  $|\alpha| = 0$ , then  $\alpha$  is a nullary operation. If  $|\alpha| = 1$ , then  $\alpha$  is a unary operation. If  $|\alpha| = 2$ , then  $\alpha$  is a binary operation.

**Example 2.2.** A semigroup is a set  $A$  with an operation  $A \times A \rightarrow A$ ,  $(a, b) \mapsto ab$ . Thus a semigroup is an algebraic structure.

Other examples of algebraic structures are monoids, groups, vector spaces.

**Definition 2.3.** We say that the algebraic structures  $(A, (\alpha_i)_{i \in I})$  and  $(B, (\beta_i)_{i \in I})$  have the **same type** if  $|\alpha_i| = |\beta_i|$  for all  $i \in I$ .

**Definition 2.4.** Let  $(A, (\alpha_i)_{i \in I})$  and  $(B, (\beta_i)_{i \in I})$  be algebraic structures of the same type. A map  $f: A \rightarrow B$  is a **homomorphism** of algebraic structures if for every  $i \in I$  the diagram

$$\begin{array}{ccc} A^{|\alpha_i|} & \xrightarrow{\alpha_i} & A \\ f^{|\alpha_i|} \downarrow & & \downarrow f \\ B^{|\beta_i|} & \xrightarrow{\beta_i} & B \end{array}$$

is commutative.

If  $f: A \rightarrow B$  is a map and  $X \subseteq A$  is a subset, we write  $f|_X$  to denote the restriction of  $f$  on  $X$ , that is the map  $f: X \rightarrow B$ ,  $x \mapsto f(x)$ .

**Exercise 2.5.** If  $f: (A, (\alpha_i)_{i \in I}) \rightarrow (B, (\beta_i)_{i \in I})$  is a homomorphism, then the **image**  $(\text{img}(f), (\beta_i|_{\text{img}(f)})_{i \in I})$  of  $f$  is an algebraic structure of the same type.

**Definition 2.6.** A **congruence** on  $(A, (\alpha_i)_{i \in I})$  is an equivalence relation  $R$  on  $A$  such that for every  $i \in I$  one has

$$a_k R b_k \quad \forall k \in \{1, \dots, |\alpha_i|\} \implies \alpha_i(a_1, \dots, a_{|\alpha_i|}) R \alpha_i(b_1, \dots, b_{|\alpha_i|}).$$

**Exercise 2.7.** Prove the following statements:

1) If  $R$  is a congruence on  $(A, (\alpha_i)_{i \in I})$ , then  $(A/R, (\overline{\alpha_i})_{i \in I})$ , where

$$\overline{\alpha_i}(\overline{a_1}, \dots, \overline{a_{|\alpha_i|}}) = \overline{\alpha_i(a_1, \dots, a_{|\alpha_i|})},$$

is an algebraic structure of the same type.

2) If  $f: (A, (\alpha_i)_{i \in I}) \rightarrow (B, (\beta_i)_{i \in I})$  is a homomorphism, then

$$a(\ker f)b \iff f(a) = f(b)$$

is a congruence. This is known as the **kernel relation**  $\ker f$ .

3) Every congruence is a kernel relation.

### 3 Heaps

**Definition 3.1.** A **heap** is a set  $H$  with a ternary operation  $H \times H \times H \rightarrow H$ ,  $(x, y, z) \mapsto [x, y, z]$ , such that for all  $a, b, c, d, e \in H$ ,

$$[[a, b, c], d, e] = [a, b, [c, d, e]], \quad (1)$$

$$[a, a, b] = [b, a, a] = b. \quad (2)$$

Equality (2) is known as Malcev's identity.

**Definition 3.2.** A heap  $H$  is **abelian** if  $[a, b, c] = [c, b, a]$  for all  $a, b, c \in H$ .

Homomorphism of heaps are defined in the usual way. Heaps and heap homomorphism for a category. It will be denoted by **Hp**.

The empty set  $\emptyset$  is a heap. However, we will only work with non-empty heaps.

**Example 3.3.** If  $G$  is a group, then  $[a, b, c] = ab^{-1}c$  turns  $G$  into a heap  $H(G)$ . Note that the heap  $G$  is abelian if and only if  $H(G)$  is abelian. Moreover, if  $f: G \rightarrow G_1$  is a homomorphism of groups, then  $H(f): H(G) \rightarrow H(G_1)$ ,  $x \mapsto f(x)$ , is a homomorphism of heaps.

**Example 3.4.** Let  $G$  be a group and  $H$  be a subgroup of  $G$ . For every  $a \in G$ , the coset  $aH = \{ah : h \in H\}$  is a heap with  $[ax, ay, az] = axy^{-1}z$ .

Recall that a **groupoid** is a small category  $\mathcal{C}$  in which every morphism is an isomorphism. We write  $\text{Obj}(\mathcal{C})$  to denote the set of objects of  $\mathcal{C}$ . If  $A, B \in \text{Obj}(\mathcal{C})$ , then  $\mathcal{C}(A, B)$  will be the set of morphisms  $A \rightarrow B$ .

**Example 3.5.** Let  $\mathcal{C}$  be a groupoid and  $A, B \in \text{Obj}(\mathcal{C})$ . Then  $\mathcal{C}(A, B)$  is a heap with  $[f, g, h] = f \circ g^{-1} \circ h$ . Note that this is well-defined, as

$$A \xrightarrow{h} B \xleftarrow[g]{g^{-1}} A \xrightarrow{f} B$$

An **affine space** (over a field  $F$ ) is a set  $A$  with a free and transitive action of an  $F$ -vector space  $\vec{A}$ . This means that there is a map  $A \times \vec{A} \rightarrow A$ ,  $(a, v) \mapsto a + v$ , such that the following condition hold:

1)  $a + (v + w) = (a + v) + w$  for all  $a \in A$  and  $v, w \in \vec{A}$ .

- 2)  $a + 0 = a$  for all  $a \in A$ .  
 3) For every  $a, b \in A$  there exists a unique  $\vec{ab} \in \vec{A}$  such that  $b = a + \vec{ab}$ .

**Example 3.6.** Let  $A$  be an affine space. Then  $A$  is an abelian heap with the operation  $[a, b, c] = a + \vec{bc}$ .

We first prove Malcev's identities: Clearly,  $[a, a, b] = a + \vec{ab} = b$ , as  $\vec{ab}$  is the unique vector that sends  $a$  to  $b$ . Similarly,  $[b, a, a] = b$ , as  $\vec{aa} = 0$ .

We claim that  $\overrightarrow{a(b+v)} = \vec{ab} + v$  for all  $a, b \in A$  and  $v \in \vec{A}$ . In fact, using Malcev's identities,

$$a + \overrightarrow{a(b+v)} = b + v = (a + \vec{ab}) + v = a + (\vec{ab} + v).$$

Now we compute

$$\begin{aligned} [a, b, [c, x, y]] &= [a, b, c + \vec{xy}] = a + \overrightarrow{b(c + \vec{xy})} \\ &= a + (\vec{bc} + \vec{xy}) = (a + \vec{bc}) + \vec{xy} = [[a, b, c], x, y]. \end{aligned}$$

To prove that the heap is abelian we first note that since

$$a + (\vec{ab} + \vec{ba}) = b + \vec{ba} = a,$$

it follows that  $\vec{ab} + \vec{ba} = 0$ . Since

$$c = b + \vec{bc} = (a + \vec{ab}) + \vec{bc},$$

it follows that

$$[c, b, a] = c + \vec{ba} = a + \vec{ab} + \vec{bc} + \vec{ba} = a + \vec{bc} = [a, b, c].$$

We now summarize the relationship between groups and heaps. Let **Grp** denote the category of groups and group homomorphism.

**Theorem 3.7.**

- 1) The assignment  $G \rightarrow H(G)$  and  $f \mapsto H(f)$  is a functor from **Grp** to **Hp**.
- 2) For any heap  $H$  and any  $e \in H$ , the operation  $ab = [a, e, b]$  turns  $H$  into a group. This group is known as the **retract**  $G(H, e)$  of  $H$  at  $e$ .
- 3) If  $f: H \rightarrow H'$  is a heap homomorphism, then for all  $e \in H$  and  $e' \in H'$  the maps

$$\begin{aligned} f_e^{e'}: G(H, e) &\rightarrow G(H', e'), & a &\mapsto [f(a), f(e), e'], \\ f_e^e: G(H, e) &\rightarrow G(H', e'), & a &\mapsto [e', f(e), f(a)], \end{aligned}$$

are group homomorphisms.

- 4) If  $H$  is a heap and  $e \in H$ , then  $H(G(H, e)) = H$ .
- 5) If  $G$  is a group and  $x \in G$ , then  $G(H(G), x) \simeq G$ .

*Sketch of the proof.* Routine calculations prove 1), see Exercise 3.3.

Let us prove 2). By using (2) we obtain that  $e$  is the identity of  $G(H, e)$ . For example,  $ae = [a, e, e] = a$ . If  $a \in H$ , then  $a^{-1} = [e, a, e]$ . In fact,

$$aa^{-1} = [a, e, a^{-1}] = [a, e, [e, a, e]] = [[a, e, e], a, e] = [a, a, e] = e.$$

The associativity is left as an exercise.

3) Let us prove that  $f_e^{e_1}$  is a group homomorphism. Let  $a, b \in H(G)$ . On the one hand,

$$\begin{aligned} f_e^{e_1}(ab) &= f_e^{e_1}([a, e, b]) \\ &= [f([a, e, b]), f(e), e_1] \\ &= [[f(a), f(e), f(b)], f(e), e_1] \\ &= [f(a), f(e), [f(b), f(e), e_1]]. \end{aligned}$$

On the other hand,

$$\begin{aligned} f_e^{e_1}(a)f_e^{e_1}(b) &= [f(a), f(e), e_1][f(b), f(e), e_1] \\ &= [[f(a), f(e), e_1], e_1, [f(b), f(e), e_1]] \\ &= [f(a), f(e), [e_1, e_1, [f(b), f(e), e_1]]] \\ &= [f(a), f(e), [f(b), f(e), e_1]]. \end{aligned}$$

The other equality is similar.

We prove 4). We start with a heap  $H$ . Fix  $e \in H$  and construct the group  $G(H, e)$  with multiplication  $(x, y) \mapsto xy = [x, e, y]$ . Now we construct the heap  $H(G(H, e))$  with operation  $(a, b, c) \mapsto ab^{-1}c$ . Recall that  $b^{-1} = [e, b, e]$ . Thus

$$\begin{aligned} ab^{-1}c &= [ab^{-1}, e, c] = [[a, e, b^{-1}], e, c] \\ &= [[a, e, [e, b, e]], e, c] = [[[a, e, e], b, e], e, c] \\ &= [[a, b, e], e, c] = [a, b, [e, e, c]] = [a, b, c]. \end{aligned}$$

To prove 5) recall that  $G$  has multiplication  $(a, b) \mapsto ab$ . Then  $H(G)$  is a heap with  $[a, b, c] = ab^{-1}c$  and for  $x \in G$ ,  $G(H(G), x)$  is a group with multiplication  $(a, b) \mapsto a \cdot b = [a, x, b] = ax^{-1}b$ . The map  $f: G \rightarrow G(H(G), x)$ ,  $a \mapsto ax$  is a group homomorphism, as

$$f(ab) = (ab)x = (ax)x^{-1}(bx) = f(a)x^{-1}f(b) = f(a) \cdot f(b).$$

Moreover,  $f$  is bijective with inverse  $G(H(G), x) \rightarrow G$ ,  $a \mapsto ax^{-1}$ .  $\square$

As an application of Theorem 3.7 we can quickly prove the several properties of heaps. However, we will present heap-theoretic proofs.

For  $n \in \mathbb{Z}_{\geq 2}$  let  $\text{Sym}_n$  be the symmetric group in  $n$  letters.

**Theorem 3.8.** *Let  $H$  be a heap and  $a, b, c, d, e \in H$ . Then the following statements hold:*

- 1) If  $[a, b, c] = d$ , then  $a = [d, c, b]$ .
- 2)  $[a, b, [c, d, e]] = [a, [d, c, b], e]$ .
- 3) In  $[a, b, c] = d$ , any three elements determine the fourth one.
- 4)  $a = b$  if and only if  $[a, b, c] = c$  for all  $c \in H$ .

*Proof.* Let us start with 1). Using (1) and (2),

$$a = [a, b, b] = [a, b, [c, c, b]] = [[a, b, c], c, b] = [d, c, b].$$

We now prove 2). Let  $a, b, c, d, e \in H$ . Then (1) and (2) imply that

$$a = [[a, b, c], c, b] = [[[a, b, c], d, d], c, b] = [[a, b, c], d, [d, c, b]].$$

Using 1) and Malcev's identities we obtain that

$$[a, b, c] = [a, [d, c, b], d] = [a, [d, c, b], [e, e, d]] = [[a, [d, c, b], e], e, d].$$

Again by using 1) we conclude that

$$[a, [d, c, b], e] = [[a, b, c], d, e].$$

Let us prove 3). Let  $a, b, c \in H$  and  $d = [a, b, c]$ . In 1) we obtained that  $a = [d, c, b]$ . Similarly,

$$c = [b, b, c] = [[b, a, a], b, c] = [b, a, [a, b, c]] = [b, a, d].$$

Now using 2) we obtain

$$b = [b, a, a] = [c, c, [b, a, a]] = [c, [a, b, c], a] = [c, d, a].$$

To prove 4) note that, if  $a = b$ , then  $[a, b, c] = [a, a, c] = c$  for all  $c \in H$  by Malcev's identity. Conversely, if  $[a, b, c] = c$  for all  $c \in H$ , in particular,  $a = [a, b, b] = b$  by Malcev's identity.  $\square$

**Exercise 3.9.** Let  $H$  be a heap and  $a, b \in H$ . Prove that  $a = b$  if and only if there exists  $c \in H$  such that  $[a, b, c] = c$ .

**Exercise 3.10.** In an abelian heap,

$$[x_1, y_1, x_2, y_2, \dots, x_n, y_n, x_{n+1}] = [x_{\sigma(1)}, y_{\tau(1)}, \dots, x_{\sigma(n)}, y_{\tau(n)}, x_{\sigma(n+1)}]$$

for all  $\sigma \in \text{Sym}_{n+1}$  and  $\tau \in \text{Sym}_n$ .

The previous exercise and Malcev's identity (2) give a useful trick that avoids painful calculations in the context of abelian heaps. Let us do a concrete example:

$$[a, b, c, d, b] = [a, b, [c, d, b]] = [a, b, [b, d, c]] = [[a, b, b], d, c] = [a, d, c].$$

**Definition 3.11.** Let  $H$  be a heap. A non-empty subset  $S$  of  $H$  is a **subheap** if  $[s, s_1, s_2] \in S$  for all  $s, s_1, s_2 \in S$ .

If  $H$  is a heap and  $S$  is a subheap of  $H$ , we define a **subheap relation** as follows:

$$a \sim_S b \iff [a, b, s] \in S \text{ for some } s \in S.$$

Note that  $a \sim_S b$  if and only if  $[a, b, s] \in S$  for all  $s \in S$ .

**Proposition 3.12.** *Let  $H$  be a heap and  $S$  be a subheap. Then  $\sim_S$  is an equivalence relation.*

*Proof.* Let  $a, b, c \in S$ . Then  $a \sim_S a$ , as  $[a, a, a] = a$  by Malcev's identity. If  $a \sim_S b$ , then  $[a, b, s] \in S$  for some  $s \in S$ . Thus, since

$$[b, a, [a, b, s]] = [[b, a, a], b, s] = [b, b, s] = s \in S,$$

we obtain that  $b \sim_S a$ . Finally, assume that  $a \sim_S b$  and  $b \sim_S c$ . We know that  $c \sim_S a$ , so  $[c, a, s] \in S$  for some  $s \in S$ . Thus

$$[a, c, [c, a, s]] = [[a, c, c], a, s] = [a, a, s] = s \in S$$

and hence  $a \sim_S c$ . □

Let  $H$  be a heap and  $S \subseteq H$  be a subheap. For  $a \in H$  the orbit of  $a$  is the set

$$\bar{a} = \{c \in H : c \sim_S a\}$$

We write  $H/S$  to denote the set of equivalence classes.

**Theorem 3.13.** *Let  $H$  be a heap and  $S$  be a subheap of  $H$ . The following statements hold:*

- 1) *For every  $s \in S$ ,  $\bar{s} = S$ .*
- 2) *For every  $a \in H$ ,  $\bar{a}$  is a subheap of  $H$ .*
- 3) *For every  $a, b \in H$ , the map*

$$\tau_b^a : H \rightarrow H, \quad c \mapsto [c, b, a],$$

*is an automorphism of heaps.*

- 4) *For every  $a, b \in H$ ,  $\bar{a} \simeq \bar{b}$  as heaps.*
- 5) *For every  $a \in H$ ,  $\sim_S = \sim_{\bar{a}}$ .*

*Proof.* Let us first prove 1). Let  $s \in S$ . To prove that  $\bar{s} \subseteq S$  let  $a \in H$  be such that  $a \sim_S s$ . Then  $[a, s, t] = u \in S$  for some  $t \in S$ . By Theorem 3.8,  $a = [u, t, s] \in S$ , as  $S$  is a subheap. Conversely, if  $a \in S$ , then, in particular,  $[a, s, s] \in S$ . Thus  $a \sim_S s$ .

We now prove 2). Let  $x, y, z \in S$  be such that  $x, y, z \in \bar{a}$ . Since  $x \sim_S a$  and  $y \sim_S a$ , it follows that  $x \sim_S y$ , so there exists  $s \in S$  such that  $[x, y, s] \in S$ . Since

$$[[x, y, z], z, s] = [x, y, [z, z, s]] = [x, y, s] \in S,$$

one has  $[x, y, z] \sim_S z$ . Now the claim follows since  $z \sim_S a$ .

Let us prove 3). On the one hand,

$$\tau_b^a([x, y, z]) = [[x, y, z], b, a] = [x, y, [z, b, a]].$$

On the other hand, using Theorem 3.7 and Malcev's identities:

$$\begin{aligned} [\tau_b^a(x), \tau_b^a(y), \tau_b^a(z)] &= [[x, b, a], [y, b, a], [z, b, a]] \\ &= [x, b, [a, [y, b, a], [z, b, a]]] \\ &= [x, b, [a, [a, b, y], [z, b, a]]] \\ &= [x, b, [[a, a, b], [y, [z, b, a]]] \\ &= [[x, b, b], [y, [z, b, a]]] \\ &= [x, y, [z, b, a]]. \end{aligned}$$

The map  $\tau_b^a$  is bijective with inverse  $\tau_a^b$ , as, for example,

$$\tau_b^a \tau_a^b(c) = \tau_b^a([c, a, b]) = [[c, a, b], b, a] = [c, a, [b, b, a]] = [c, a, a] = c.$$

4) Let  $a, b \in H$ . The map  $\tau_a^b$  is an automorphism of heaps such that  $\tau_a^b(a) = b$ . We claim that  $\tau_a^b(\bar{a}) = \bar{b}$ . Let  $x \in H$  be such that  $x \sim_S a$ . In particular,  $a \sim_S x$ , so  $[a, x, s] \in S$  for some  $s \in S$ . Now  $\tau_a^b(x) \sim_S b$ , as

$$\begin{aligned} [\tau_a^b(x), b, [a, x, s]] &= [[x, a, b], b, [a, x, s]] \\ &= [x, a, [b, b, [a, x, s]]] = [x, a, [a, x, s]] = s \in S. \end{aligned}$$

5) Assume first that  $x \sim_{\bar{a}} y$ . Then there exists  $b \in \bar{a}$  such that  $[x, y, b] \sim_S a$ . Since  $[x, y, b] \sim_S a$  and  $b \sim_S a$ , it follows that  $[x, y, b] \sim_S b$ . Thus there exists  $s \in S$  such that

$$[x, y, s] = [x, y, [b, b, s]] = [[x, y, b], b, s] \in S.$$

That is  $x \sim_S y$ . Conversely, assume now that  $x \sim_S y$ . Then there exists  $s \in S$  such that

$$[[x, y, b], b, s] = [x, y, [b, b, s]] = [x, y, s] \in S.$$

Let  $b \in \bar{a}$ . Since  $[x, y, b] \sim_S b$ , it follows that  $[x, y, b] \sim_S a$  and hence  $x \sim_{\bar{a}} y$ .  $\square$

**Definition 3.14.** A subheap  $S$  of  $H$  is **normal** if for every  $a \in H$  and  $s, e \in S$  there exists  $t \in S$  such that  $[a, e, s] = [t, e, a]$ .

Note a subheap  $S$  of  $H$  is normal if and only if for every  $a \in H$  and  $s, e \in H$  one has  $[[a, e, s], a, e] \in S$ .

**Exercise 3.15.** Prove that a subheap  $S$  of  $H$  is normal if and only if  $S$  is a normal subgroup of  $G(H, e)$  for every  $e \in S$ .

**Theorem 3.16.** Let  $f: H \rightarrow K$  be a homomorphism of heaps. For  $c \in H$  let

$$\bar{c} = \{b \in H : f(b) = f(c)\}$$



be the equivalence class of  $c$  with respect to  $\ker(f)$ . Then  $\bar{c}$  is a subheap of  $H$ .

*Proof.* To prove that  $\bar{c}$  is a subheap first note that  $c \in \bar{c}$ , so  $\bar{c}$  is non-empty. Now let  $x, y, z \in \bar{c}$ . Then  $f(x) = f(y) = f(z) = [x, y, z] \in \bar{c}$ , as

$$f([x, y, z]) = [f(x), f(y), f(z)] = [f(c), f(c), f(c)] = f(c).$$

To prove that  $\bar{c}$  is normal let  $x, y \in \bar{c}$  and  $a \in H$ . Since  $f(x) = f(y) = f(c)$ ,

$$\begin{aligned} f([x, y, s]) &= [f(x), f(y), f(s)] = [f(c), f(c), f(s)] \\ &= [f(a), f(c), f(c)], f(a), f(c) \\ &= [f(a), f(a), f(c)] = f(c). \end{aligned} \quad \square$$

**Definition 3.17.** Let  $f: H \rightarrow K$  be a homomorphism of heaps and  $e \in f(H)$ . An  $e$ -kernel of  $f$  is defined as the set

$$\ker_e f = \{a \in H : f(a) = e\}$$

**Theorem 3.18.** Let  $f: H \rightarrow K$  be a homomorphism of heaps and  $e \in f(H)$ .

- 1)  $\ker_e f$  is a normal subheap of  $H$ .
- 2)  $a \sim_{\ker_e f} b$  if and only if  $f(a) = f(b)$ .
- 3) For every  $e, e_1 \in f(H)$ ,  $\ker_e f \simeq \ker_{e_1} f$ .

*Proof.* Let us prove 1). By definition,  $\ker_e f$  is non-empty. If  $x, y, z \in \ker_e f$ , then  $f(x) = f(y) = f(z) = e$ . Since  $f$  is a heap homomorphism,

$$f([x, y, z]) = [f(x), f(y), f(z)] = [e, e, e] = e$$

and hence  $[x, y, z] \in \ker_e f$ . To prove that  $\ker_e f$  is normal note that

$$f([a, x, y]) = [f(a), f(x), f(y)] = [f(a), e, e] = e.$$

Now we prove 2). Let  $S = \ker_e f$ . Then

$$\begin{aligned} a \sim_S b &\iff [a, b, s] \in S \text{ for some } s \in S \\ &\iff f([a, b, s]) = e \text{ for some } s \in H \text{ such that } f(s) = e \\ &\iff [f(a), f(b), e] = e \\ &\iff f(a) = f(b), \end{aligned}$$

by Exercise 3.9.

Finally we prove 3). Let  $x, y \in H$  be such that  $f(x) = e$  and  $f(y) = e_1$ . Note that

$$\ker_e f = \{a \in H : f(a) = e\} = \{a \in H : f(a) = f(x)\} = \bar{x}$$

Similarly,  $\ker_{e_1} f = \bar{y}$ . We know that both  $\bar{x}$  and  $\bar{y}$  are subheaps of  $H$ . They are isomorphic, as  $\tau_x^y(\bar{x}) = \bar{y}$ . In fact, if  $z \in \bar{x}$ , then  $\tau_x^y(z) \in \bar{y}$ , as

$$f(\tau_x^y(z)) = f([z, x, y]) = [f(z), f(x), f(y)] = [f(x), f(x), f(y)] = f(y). \quad \square$$

We leave the proof of the following result as an exercise.

**Corollary 3.19.** *Every congruence of heaps is a subheap relation with respect to a normal subheap.*

We now discuss universal differences between heaps and groups.

**Definition 3.20.** *Let  $X$  be a set. A **free heap**...*

## 4 Trusses

**Definition 4.1.** *A left (resp. right) **skew truss** is a heap  $(T, [-, -, -])$  with an associative binary operation such that*

$$a[b, c, d] = [ab, ac, ad] \quad (\text{resp. } [a, b, c]d = [ad, bd, cd]).$$

*If  $T$  is an abelian heap, then we drop skew from the terminology. A **truss** is a left and right truss.*

**Example 4.2.** The set  $2\mathbb{Z} + 1$  is a truss with

**Example 4.3.**  $\frac{2\mathbb{Z}+1}{2\mathbb{Z}+1}$ .

**Example 4.4.** Let  $H$  be an abelian heap. The set  $\text{End}(H)$  of heap endomorphisms  $H \rightarrow H$  is a truss with

$$[f, g, h](a) = [f(a), g(a), h(a)]$$

Let  $F$  be a field. An associative  $F$ -algebra is an  $F$ -vector space  $A$  with a bilinear associative multiplication. In particular,  $A$  is a ring. What if  $A$  is replaced by an affine space?

**Example 4.5.** Let  $F$  be a field...  $F$  acts on  $A$  by  $(\lambda, a, b) \mapsto a + \lambda \overrightarrow{ab}$ . An affine transformation is a pair  $(f, \overrightarrow{f})$ , where  $f: A \rightarrow B$  is a map and  $\overrightarrow{f}: \overrightarrow{A} \rightarrow \overrightarrow{B}$  is a linear transformation such that

$$\overrightarrow{f}(\overrightarrow{ab}) = \overrightarrow{f(a)f(b)}$$

An affine space with an  $F$ -affine multiplication is a truss...

**Example 4.6.** A (left) skew ring is a (left) truss with

$$a[b, c, d] = a(b - c + d) = ab - ac + ad = [ab, ac, ad].$$

A (left) skew brace is a triple  $(A, +, \circ)$ , where  $(A, +)$  and  $(A, \circ)$  are groups and  $a \circ (b + c) = a \circ b - a + a \circ c$  holds for all  $a, b, c \in A$ .

**Proposition 4.7.** *Let  $B$  be a skew brace. Then  $(B, +)$  is a heap and  $(H(B), \circ)$  is a skew truss.*

*Proof.*

□

**Example 4.8.** A skew left truss is a skew...  $a + b = [a, 1, b]$ .  $G(T, 1)$  is a skew brace.

## References

1. P. M. Cohn. *Universal algebra*, volume 6 of *Mathematics and its Applications*. D. Reidel Publishing Co., Dordrecht-Boston, Mass., second edition, 1981.