Nichols algebras

Leandro Vendramin

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Contents

Introduction

The goal of this work is to review some fundamental problems in the theory of Nichols algebras of group type, i.e. Nichols algebras over Yetter-Drinfeld modules over groups. Most of these problems appeared in [**?**]. We refer to [**?**] for the basic theory of Nichols algebras.

Problems.

(1) Classify finite-dimensional Nichols algebras.
(2) Give an optimal set of defining relations for finite-dimensional Nichols algebras.

Sections 1 and 2 are devoted to recall some basic preliminaries. In Sections 3 and 4 we will study the problem over *indecomposable* braided vector spaces. In this context Fomin–Kirillov algebras appear naturally and form one of the main examples to study. In Sections 5–5 we will review the classification of finite-dimensional Nichols algebras over *decomposable* braided vector spaces of group type and sketch the proof in the case of two simple summands.

1. Braided vector spaces and Nichols algebras

We start with the definition of Nichols algebras. For that purpose we need first to define braided vector spaces. Fix a field $\mathbb{K}$.

1.1. A **braided vector space** is a pair (V, c), where V is a vector space and $c \in \mathbf{GL}(V \otimes V)$ is a solution of the **braid equation**:

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c).$$

1.2. Example (braided vector spaces of diagonal type). Let V be a complex vector space with basis $x_1, x_2, \dots, x_n$. Let $q_{ij} \in \mathbb{C}^{\times}$, where $1 \leqslant i, j \leqslant n$. Define $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$. Then (V, c) is a braided vector space.

Using the theory of Weyl groupoids and generalized root systems [**?**], Heckenberger classified finite-dimensional Nichols algebras over complex braided vector spaces of diagonal type

[**?**, **?**, **?**, **?**]. Heckenberger's results have applications in the classification problem of finite-dimensional pointed Hopf algebras (over the complex numbers) with abelian coradical of order coprime to 210 [**?**]. Angiono found a minimal set of defining relations [**?**, **?**].

For the classification of finite-dimmensional Nichols algebras of braided vector spaces of diagonal type in arbitrary characteristic we refer to [**?**] and [**?**].

1.3. Example. Let G be a finite group and $V = \mathbb{C}G$ be the complex vector space with basis $\{g : g \in G\}$. Define $c(g \otimes h) = ghg^{-1} \otimes g$ for $g, h \in G$. Then (V, c) is a braided vector space.

1.4. A braided vector space yields a special type of (braided) Hopf algebra called the **Nichols algebra** of (V, c). To define Nichols algebras we need the Artin braid group $\mathbb{B}_n$. This group can be presented with generators $\sigma_1, \dots, \sigma_{n-1}$ and relations

$$\begin{aligned} &\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} && \text{for } 1 \leqslant i \leqslant n-2,\\ &\sigma_i\sigma_j = \sigma_j\sigma_i && \text{for } |i-j| > 1. \end{aligned}$$

Recall that the symmetric group $\mathbb{S}_n$ can be presented with generators $\tau_1, \dots, \tau_{n-1}$ and relations

$$\begin{aligned} &\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1} && \text{for } 1 \leqslant i \leqslant n-2,\\ &\tau_i\tau_j = \tau_j\tau_i && \text{for } |i-j| > 1,\\ &\tau_i^2 = 1 && \text{for } 1 \leqslant i \leqslant n-1. \end{aligned}$$

Hence there exists a surjection $\mathbb{B}_n \to \mathbb{S}_n$ defined by $\sigma_i \mapsto \tau_i$.

Lemma (Matsumoto). *There exists a set-theoretical section* $\mu : \mathbb{S}_n \to \mathbb{B}_n$, $\tau_i \mapsto \sigma_i$, *such that* $\mu(xy) = \mu(x)\mu(y)$ *if* $\operatorname{length}(xy) = \operatorname{length}(x) + \operatorname{length}(y)$.

Remark. Let (V, c) be a braided vector space, and $n \in \mathbb{N}$. Define

$$c_i = \mathrm{id}_{V^{\otimes(i-1)}} \otimes c \otimes \mathrm{id}_{V^{\otimes(n-i-1)}} \in \operatorname{Aut}(V^{\otimes n})$$

for $1 \leqslant i \leqslant n-1$, i.e.,

$$c_i \cdot (v_1 \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes v_{i-1} \otimes c(v_i \otimes v_{i+1}) \otimes v_{i+2} \otimes \cdots \otimes v_n.$$

The operators c_i $(1 \leqslant i \leqslant n-1)$ satisfy the defining-relations of the braid group and hence $\rho_n : \mathbb{B}_n \to \operatorname{Aut}(V^{\otimes n})$, defined by $\rho_n(\sigma_i) = c_i$, is a representation of $\mathbb{B}_n$ into $V^{\otimes n}$.

1.5. Let (V, c) be a braided vector space. The **Nichols algebra** of (V, c) is

$$\mathfrak{B}(V, c) = \bigoplus_n T^n(V)/\ker \mathfrak{S}_n = \mathbb{K} \oplus V \oplus \bigoplus_{n \geqslant 2} V^{\otimes n}/\ker \mathfrak{S}_n,$$

where $\mathfrak{S}_n$ is the **quantum symmetrizer**:

$$\mathfrak{S}_n = \sum_{\sigma \in \mathbb{S}_n} \rho_n \mu(\sigma).$$

Let us compute some quantum symmetrizers:

$$\begin{aligned} \mathfrak{S}_2 &= \mathrm{id} + c,\\ \mathfrak{S}_3 &= \mathrm{id} + c_1 + c_2 + c_1c_2 + c_2c_1 + c_1c_2c_1, \end{aligned}$$

where $c_1 = c \otimes \mathrm{id}$ and $c_2 = \mathrm{id} \otimes c$.

1.6. Examples. Let V be a complex vector space and let $\mathrm{flip} : V \otimes V \to V \otimes V$ be the linear map defined by $x \otimes y \mapsto y \otimes x$. The Nichols algebra of the braided vector space (V, flip) is the Symmetric Algebra $S(V)$. The Nichols algebra of the braided vector space $(V, -\mathrm{flip})$ is the Exterior Algebra $\Lambda(V)$.

1.7. Our braided vector spaces will be a particular family of G-modules. Recall that a **Yetter-Drinfeld module** (over $\mathbb{K}G$) is a $\mathbb{K}G$-module $V = \oplus_{g \in G} V_g$ such that $hV_g \subseteq V_{hgh^{-1}}$ for all $g, h \in G$. The braiding is $c : V \otimes V \to V \otimes V$ is the map defined by $c(u \otimes v) = xv \otimes u$ if $\deg u = x$.

1.8. Over the complex numbers, the category of Yetter-Drinfeld modules over G is semisimple. Furthermore, simple Yetter-Drinfeld modules over G are parametrized by pairs (g^G, ρ), where g^G denotes the conjugacy class of $g \in G$ and (ρ, M) is an irreducible representation of the centralizer $C_G(g)$.

Let us describe the simple Yetter-Drinfeld modules over G. Let $\{x_1, \dots, x_n\}$ be a set of representatives of left cosets of $C_G(g)$ in G. Then the simple Yetter-Drinfeld modules over G are

$$M(g^G, \rho) = \mathrm{Ind}_{C_G(g)}^G \rho = \bigoplus_{i=1}^n x_i \otimes_{\mathbb{C}C_G(g)} M$$

with the induced action $y(x \otimes m) = yx \otimes m$ for $x, y \in G$ and $m \in M$, and the coaction $\delta(x_i \otimes m) = x_i g x_i^{-1} \otimes (x_i \otimes m)$ for $m \in W$. The braiding is

$$c\left((x_i \otimes m) \otimes (x_j \otimes m')\right) = (x_i g x_i^{-1} x_j \otimes m') \otimes (x_i \otimes m).$$

Thus over the complex numbers every simple Yetter-Drinfeld module over G can be written as $V = \oplus_{x \in g^G} V_x$, where $V_x = \{v \in V : \delta(v) = x \otimes v\}$ and $V_g = 1 \otimes M$. For all $x \in g^G$, V_x is a simple $C_G(x)$-module and $yV_x \subseteq V_{yxy^{-1}}$ for all $y \in G$.

2. A crash course on quandles

2.1. A **quandle** is a pair $(X, \triangleright)$, where X is a set and $\triangleright \colon X \times X \to X$ is a binary operation such that the maps $\varphi_x \colon X \times X \to X$, given by $y \mapsto x \triangleright y$, are bijective for each $x \in X$, $x \triangleright x = x$ for all $x \in X$, and $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$ for all $x, y, z \in X$.

2.2. Example. Let G be a group and X be a union of conjugacy classes. Then X with $x \triangleright y = xyx^{-1}$ for all $x, y \in X$ is a quandle.

2.3. The **inner group** of X is the group $\mathrm{Inn}(X) = \langle \varphi_x : x \in X \rangle$. A quandle is called **indecomposable** if Inn(X) acts transitively in X. Using the classification of transitive groups, indecomposable quandles of small size were classified in [?]. Let $q(n)$ be the number of indecomposable quandles (up to isomorphism) of size n. Table 1 shows some values of $q(n)$.

Table 1. The number of isomorphism classes of indecomposable quandles.

n	1	2	3	4	5	6	7	8	9	10	11	12
$q(n)$	1	0	1	1	3	2	5	3	8	1	9	10
n	13	14	15	16	17	18	19	20	21	22	23	24
$q(n)$	11	0	7	9	15	12	17	10	9	0	21	42
n	25	26	27	28	29	30	31	32	33	34	35	36
$q(n)$	34	0	65	13	27	24	29	17	11	0	15	73
n	37	38	39	40	41	42	43	44	45	46	47	
$q(n)$	35	0	13	33	39	26	41	9	45	0	45	

2.4. To describe a finite quandle X we may assume that $X = \{1, \dots, n\}$ for some $n \in \mathbb{N}$ and then write $X\colon \varphi_1 \ \dots \ \varphi_n$ to denote the quandle structure on X given by the permutations $\varphi_1, \dots, \varphi_n$. The following quandles are very important:

$$
\begin{aligned}
Z_T^{4,1} &: \ (243)\ (134)\ (142)\ (123)\ \mathrm{id} \\
Z_2^{2,2} &: \ (24)\ (13)\ (24)\ (13) \\
Z_3^{3,1} &: \ (23)\ (13)\ (12)\ \mathrm{id} \\
Z_3^{3,2} &: \ (23)(45)\ (13)(45)\ (12)(45)\ (123)\ (132) \\
Z_4^{4,2} &: \ (24)(56)\ (13)(56)\ (24)(56)\ (13)(56)\ (1234)\ (1432)
\end{aligned}
$$

2.5. Exercise. Prove that the quandles given in 2.4 are not indecomposable.

2.6. The **enveloping group** of G is the group

$$G_X = \langle x_i : x_i x_j = x_{i \triangleright j} x_i \text{ for all } i, j \in X \rangle.$$

The enveloping group has the following **universal property**: For any group G and any map $f : X \to G$ satisfying $f(x \triangleright y) = f(x)f(y)f(x)^{-1}$ there exists a unique group homomorphism $g : G_X \to G$ such that $f = g \circ \partial$, where $\partial : X \to G_X$, $i \mapsto x_i$.

2.7. Example. Let

$$T = \langle z \rangle \times \langle x_1, x_2, x_3, x_4 : x_i x_j = x_{\varphi_i(j)} x_i, \quad i, j \in \{1, 2, 3, 4\} \rangle,$$

where $\varphi_1 = (243)$, $\varphi_2 = (134)$, $\varphi_3 = (142)$ and $\varphi_4 = (123)$, and for $n \geqslant 2$ let

$$\Gamma_n = \langle g, h, \epsilon : hg = \epsilon gh,\ g\epsilon = \epsilon^{-1} g,\ h\epsilon = \epsilon h,\ \epsilon^n = 1 \rangle.$$

The enveloping groups of the quandles of 2.4 are listed in the following table:

Quandle	$Z_2^{2,2}$	$Z_3^{3,1}$	$Z_3^{3,2}$	$Z_4^{4,2}$	$Z_T^{4,1}$
Enveloping group	Γ_2	Γ_3	Γ_3	Γ_4	T

2.8. Let X be a quandle and $V = \mathbb{K}X$ be the vector space (over the field $\mathbb{K}$) with basis $\{x : x \in X\}$. Let $q\colon X \times X \to \mathbb{K}^\times$ be a map and consider the map $c \in \mathbf{GL}(V \otimes V)$ given by $c(x \otimes y) = q(x, y)(x \triangleright y) \otimes x$. Then (V, c) is a braided vector space if and only if q is a **2-cocycle** of X, i.e.

$$q(x, y \triangleright z) q(y, z) = q(x \triangleright y, x \triangleright z) q(x, z)$$

for all $x, y, z \in X$. The braided vector space (V, c) is said to be of type (X, q).

2.9. Example. Let $X = (123)^{\mathbb{A}_4}$ be the quandle associated with the conjugacy class of (123) in the alternating group $\mathbb{A}_4$. The map $q\colon X \times X \to \mathbb{C}$ given by

(2.9.1)

	(243)	(123)	(134)	(142)
(243)	ω	ω	ω	ω
(123)	ω	ω	$-\omega$	$-\omega$
(134)	ω	$-\omega$	ω	$-\omega$
(142)	ω	$-\omega$	$-\omega$	ω

where $\omega \in \mathbb{C}$ is a primitive n-th root of 1, is a 2-cocycle of X.

2.10. Example. Let $n \geqslant 3$ and $X_n = (12)^{\mathbb{S}_n}$. The map $\chi\colon X \times X \to \mathbb{C}$ given by

$$\chi(\sigma, \tau) = \begin{cases} 1 & \text{if } \sigma(i) < \sigma(j), \\ -1 & \text{otherwise}, \end{cases}$$

where $\tau = (ij)$, $i < j$, is a 2-cocycle of X_n.

3. Fomin–Kirillov algebras

In [**?**], Fomin and Kirillov introduced a family of quadratic algebras and proved that these algebras contain a commutative subalgebra isomorphic to the cohomology ring of the flag manifold. The problems and conjectures listed in this section are known, see for example [**?**], [**?**] and [**?**].

3.1. For an integer $n \geqslant 3$ denote by $\mathcal{E}_n$ the $\mathbb{C}$-algebra (of type A_{n-1}) with generators $x_{(ij)}$, where $1 \leqslant i < j \leqslant n$, and relations

$$\begin{aligned}
&x_{(ij)}^2 = 0, && \text{for } 1 \leqslant i < j \leqslant n,\\
&x_{(ij)}x_{(jk)} = x_{(jk)}x_{(ik)} + x_{(ik)}x_{(ij)}, && \text{for } 1 \leqslant i < j < k \leqslant n,\\
&x_{(jk)}x_{(ij)} = x_{(ik)}x_{(jk)} + x_{(ij)}x_{(ik)}, && \text{for } 1 \leqslant i < j < k \leqslant n,\\
&x_{(ij)}x_{(kl)} = x_{(kl)}x_{(ij)}, && \text{for any discinct } i, j, k, l.
\end{aligned}$$

The algebras $\mathcal{E}_n$ are graded by $\deg(x_{(ij)}) = 1$. Write

$$\mathcal{E}_n = (\mathcal{E}_n)_0 \oplus (\mathcal{E}_n)_1 \oplus (\mathcal{E}_n)_2 \oplus \cdots,$$

where $(\mathcal{E}_n)_0 = \mathbb{C}$ and $(\mathcal{E}_n)_k$ denotes the homogeneous component of degree k. Then one defines the **Hilbert series** of $\mathcal{E}_n$ as

$$\mathcal{H}_n(t) = \sum_{k=0}^{\infty} (\dim(\mathcal{E}_n)_k) t^k.$$

It is natural to ask whether $\mathcal{E}_n$ is finite-dimensional or not. It is known that $\mathcal{E}_n$ is finite-dimensional for $n \leqslant 5$.

3.2. Problem. Compute the Hilbert series of $\mathcal{E}_n$.

3.3. Example. The algebra $\mathcal{E}_3$ has dimension 12. The Hilbert series $\mathcal{H}_3(t)$ of $\mathcal{E}_3$ is a polynomial of degree 4: $\mathcal{H}_3(t) = (2)_t^2(3)_t$, where

$$(k)_t = 1 + t + \cdots + t^{k-1}.$$

3.4. Example. Computer calculations yield $\dim \mathcal{E}_4 = 576$. The Hilbert series $\mathcal{H}_4(t)$ of $\mathcal{E}_4$ is a polynomial of degree 12: $\mathcal{H}_4(t) = (2)_t^2(3)_t^2(4)_t^2$.

3.5. Example. Computer calculations yield $\dim \mathcal{E}_5 = 8294400$. The Hilbert series $\mathcal{H}_5(t)$ of $\mathcal{E}_5$ is a polynomial of degree 40: $\mathcal{H}_5(t) = (4)_t^4(5)_t^2(6)_t^4$.

3.6. Example. The Hilbert series $\mathcal{H}_6(t)$ of $\mathcal{E}_6$ cannot be written as a product of t-numbers. Further,

$$\mathcal{H}_6(t) = 1 + 15t + 125t^2 + 765t^3 + 3831t^4 + 16605t^5 + 64432t^6 + \cdots$$

3.7. Conjecture. $\dim \mathcal{E}_n = \infty$ for all $n \geqslant 6$.

3.8. Conjecture. $\dim(\mathcal{E}_n)_k \sim \left(\binom{\binom{n}{2}}{k}\right)$.

3.9. Let us explain the connection between Fomin–Kirillov algebras and Nichols algebras. Let V_n be the vector space with basis

$$\{v_{(ij)} : 1 \leqslant i < j \leqslant n\}$$

and consider the map $c \in \mathbf{GL}(V_n \otimes V_n)$ defined by

$$c(v_\sigma \otimes v_\tau) = \chi(\sigma,\tau) v_{\sigma\tau\sigma^{-1}} \otimes v_\sigma, \qquad \chi(\sigma,\tau) = \begin{cases} 1 & \text{if } \sigma(i) < \sigma(j), \\ -1 & \text{otherwise,} \end{cases}$$

where σ and τ are transpositions, and $\tau = (ij)$ with $i < j$. Since (V_n, c) is a braided vector space, it is possible to consider the Nichols algebra $\mathfrak{B}(V_n)$. One has a surjective homomorphism of algebras $\mathcal{E}_n \to \mathfrak{B}(V_n)$, see for example [**?**]. It is known that $\mathfrak{B}(V_n) = \mathcal{E}_n$ if $3 \leqslant n \leqslant 5$; this was proved by Milinski and Schneider for $n \leqslant 4$, and by Graña for $n = 5$. Compare the braided vector space V_n with Example 2.10.

3.10. Conjectures.

(1) $\mathfrak{B}(V_n)$ is quadratic for all n.
(2) $\mathcal{E}_n \simeq \mathfrak{B}(V_n)$ for all n.
(3) $\dim \mathfrak{B}(V_n) = \infty$ for all $n \geqslant 6$.

3.11. Remark. The numbers 4, 12 and 40 appear also as the numbers of indecomposable representations of the preprojective algebra of a quiver of type A, and in the cluster algebra structure of the coordinate ring of the the maximal unipotent subgroup N of $\mathbf{SL}_n(\mathbb{C})$. See [**?**], [**?**] and [**?**].

4. Nichols algebras of indecomposable modules

Let us review the list of known examples of finite-dimensional Nichols algebras over *indecomposable* braided vector spaces of group type.

4.1. Examples. Let $X = (123)^{\mathbb{A}_4}$ and consider the 2-cocycle of X given by (2.9.1).

(1) Assume that $\omega = -1$. Then the complex Nichols algebra $\mathfrak{B}(X, q)$ of the braided vector space of type (X, q) has dimension 72 and its Hilbert series is $\mathcal{H}(t) = (2)_t^2(3)_t(6)_t$.

(2) Assume that ω is a primitive cubic root of 1. Then the complex Nichols algebra $\mathfrak{B}(X, \omega q)$ of the braided vector space of type $(X, \omega q)$ has dimension 5184 and its Hilbert series is $\mathcal{H}(t) = (6)_t^4(2)_{t^2}^2$. This Nichols algebra can be presented with four generators, four relations in degree two, four relations in degree three and one in degree six. This example appeared in [**?**].

4.2. It is known that many Nichols algebras over *indecomposable* braided vector spaces are infinite-dimensional, see for example [**?, ?**] and [**?**]. Only few examples of finite-dimensional Nichols algebras over *indecomposable* braided vector spaces of group type are known. The complete list appears in Table 2. For these examples we refer to [**?**] and the references therein.

Table 2. Finite-dimensional Nichols algebras.

rank	dimension	Hilbert series	remarks
3	12	$(2)_t^2(3)_t$	
3	432	$(3)_t(4)_t(6)_t(6)_{t^2}$	$\mathrm{char}\mathbb{K} = 2$
4	36	$(2)_t^2(3)_t^2$	$\mathrm{char}\mathbb{K} = 2$
4	72	$(2)_t^2(3)_t(6)_t$	$\mathrm{char}\mathbb{K} \neq 2$
4	5184	$(6)_t^4(2)_{t^2}^2$	
6	576	$(2)_t^2(3)_t^2(4)_t^2$	
6	576	$(2)_t^2(3)_t^2(4)_t^2$	
6	576	$(2)_t^2(3)_t^2(4)_t^2$	
5	1280	$(4)_t^4(5)_t$	
5	1280	$(4)_t^4(5)_t$	
7	326592	$(6)_t^6(7)_t$	
7	326592	$(6)_t^6(7)_t$	
10	8294400	$(4)_t^4(5)_t^2(6)_t^4$	
10	8294400	$(4)_t^4(5)_t^2(6)_t^4$	

4.3. One common feature of the algebras appearing in Table 2 is the factorization of the Hilbert series as

$$\prod_{i=1}^{k_1}(a_i)_t \prod_{i=1}^{k_2}(b_i)_{t^2} \tag{4.3.1}$$

for some $k_1, k_2 \geqslant 0$, $a_1, \dots, a_{k_1}, b_1, \dots, b_{k_2} \geqslant 2$, and where

$$(a)_{t^b} = 1 + t^b + t^{2b} + \cdots + t^{(a-1)b}$$

for all $a, b \geqslant 1$. This particular factorization was the starting point of [**?**], [**?**] and [**?**].

4.4. Conjecture. All finite-dimensional Nichols algebras of group type over an absolutely simple Yetter-Drinfeld module have a Hilbert series of the form (4.3.1)[1]. Any such Nichols algebra is one of those listed in Table 2.

[1]The conjecture was posed in October 2012 at the Oberwolfach mini-workshop "Nichols algebras and Weyl groupoids". For a proof or counterexample you will receive a bottle of wine!

5. Nichols algebras of decomposable modules I: two summands

Now we review the classification of finite-dimensional Nichols algebras over *decomposable* braided vector spaces of group type in the case of two simple summands. This classification was obtained in collaboration with I. Heckenberger, see [**?, ?, ?, ?**].

5.1. Let G be a group and let $V = \oplus_{g\in G} V_g$ be a finite-dimensional Yetter-Drinfeld module over G. The **support** of V is

$$\operatorname{supp} V = \{g \in G : V_g \neq 0\}.$$

5.2. To avoid a discussion of group representations depending on the field and since field extensions of Nichols algebras are well understood, in general it is more promising to extend the field appropriately before studying a Nichols algebra.

Let G be a group and $V \in {}^G_G\mathcal{YD}$. We say that a V is **absolutely simple** if $V \neq 0$ and if for any field extension $\mathbb{L}$ of $\mathbb{K}$ the only Yetter-Drinfeld submodules of $\mathbb{L} \otimes_{\mathbb{K}} V$ over $\mathbb{L}G$ are $\{0\}$ and $\mathbb{L} \otimes_{\mathbb{K}} V$. Absolutely simple Yetter-Drinfeld modules over G are parametrized by pairs (g^G, ρ), where g^G is a conjugacy class of G and $\rho : \mathbb{K}G^g \to \operatorname{End}(W)$ is an absolutely irreducible representation of the centralizer G^g.

5.3. Theorem. *Let* G *be a non-abelian group. Let* V *and* W *be two absolutely simple Yetter-Drinfeld modules over* G *such that* G *is generated by the support* $\operatorname{supp}(V \oplus W)$ *of* $V \oplus W$. *Assume that* $\dim \mathfrak{B}(V \oplus W) < \infty$. *If* $c_{W,V}c_{V,W} \neq \mathrm{id}_{V\otimes W}$, *then* G *is an epimorphic image of*

$$T = \langle z\rangle \times \langle x_1, x_2, x_3, x_4 : x_ix_j = x_{\varphi_i(j)}x_i, \quad i,j \in \{1,2,3,4\}\rangle,$$

where $\varphi_1 = (243)$, $\varphi_2 = (134)$, $\varphi_3 = (142)$ *and* $\varphi_4 = (123)$, *or an epimorphic image of*

$$\Gamma_n = \langle g, h, \epsilon : hg = \epsilon gh,\ g\epsilon = \epsilon^{-1}g,\ h\epsilon = \epsilon h,\ \epsilon^n = 1\rangle$$

for some $n \in \{2,3,4\}$.

5.4. Theorem 5.3 has deep consequences. After some work one obtains the list of all V and W in ${}^G_G\mathcal{YD}$ such that $\mathfrak{B}(V \oplus W)$ is finite-dimensional. Moreover, one obtains the Hilbert series of each $\mathfrak{B}(V \oplus W)$.

Theorem. Let G be a non-abelian group and V and W be two absolutely simple Yetter-Drinfeld modules over G such that G is generated by the support of $V \oplus W$. Assume that $(\mathrm{id} - c_{W,V}c_{V,W})(V \otimes W)$ is non-zero and that $\dim \mathfrak{B}(V \oplus W) < \infty$. Then $\mathfrak{B}(V \oplus W)$ is one of the Nichols algebras of Table 3.

5.5. Let us show an example of one of the algebras appearing in the context of Theorem 5.4. Let G be a non-abelian epimorphic image of the group T and suppose that G has elements $z, x_1, \dots, x_4 \in G$ such that $[z, x_i] = 1$ for all i and $x_ix_j = x_{\varphi_i(j)}x_i$ for all i, j. Let $V, W \in {}^G_G\mathcal{YD}$. Assume that $V \simeq M(z, \rho)$, where ρ is a character of the centralizer $G^z = G$, and $W = M(x_1, \sigma)$, where σ is a character of $G^{x_1} = \langle x_1, x_2x_3, z\rangle$ with $\sigma(x_1) = -1$ and $\sigma(x_2x_3) = 1$. Let $v \in V_z \setminus \{0\}$. Then $\{v\}$ is basis of V. The action of G on V is given by

$$zv = \rho(z)v, \quad x_iv = \rho(x_1)v \quad \text{for all } i \in \{1,2,3,4\}.$$

Let $w_1 \in W_{x_1}$ such that $w_1 \neq 0$. Then the vectors

$$w_1,\ w_2 := -x_4w_1,\ w_3 := -x_2w_1,\ w_4 := -x_3w_1$$

form a basis of W. The degrees of these vectors are x_1, x_2, x_3 and x_4, respectively. The action of G on W is given by the following table:

W	w_1	w_2	w_3	w_4
x_1	$-w_1$	$-w_4$	$-w_2$	$-w_3$
x_2	$-w_3$	$-w_2$	$-w_4$	$-w_1$
x_3	$-w_4$	$-w_1$	$-w_3$	$-w_2$
x_4	$-w_2$	$-w_3$	$-w_1$	$-w_4$
z	$\sigma(z)w_1$	$\sigma(z)w_2$	$\sigma(z)w_3$	$\sigma(z)w_4$

TABLE 3. Nichols algebras with finite root system of rank two over a field $\mathbb{K}$.

rank	group	dimension	char$\mathbb{K}$	support
4	Γ_2	64		$Z_2^{2,2}$
4	Γ_2	1296	3	$Z_2^{2,2}$
4	Γ_3	10368	$\neq 2,3$	$Z_3^{3,1}$
4	Γ_3	5184	2	$Z_3^{3,1}$
4	Γ_3	1152	3	$Z_3^{3,1}$
4	Γ_3	2239488	2	$Z_3^{3,1}$
5	Γ_3	10368	$\neq 2,3$	$Z_3^{3,2}$
5	Γ_3	5184	2	$Z_3^{3,2}$
5	Γ_3	1152	3	$Z_3^{3,2}$
5	Γ_3	2304		$Z_3^{3,2}$
5	Γ_3	2304		$Z_3^{3,1}$
5	Γ_3	2239488	2	$Z_3^{3,2}$
5	T	80621568	$\neq 2$	$Z_T^{4,1}$
5	T	1259712	2	$Z_T^{4,1}$
6	Γ_4	262144	$\neq 2$	$Z_4^{4,2}$
6	Γ_4	65536	2	$Z_4^{4,2}$

Assume further that

$$(\rho(x_1)\sigma(z))^2 - \rho(x_1)\sigma(z) + 1 = 0, \quad \rho(x_1z)\sigma(z) = 1.$$

Then $\mathfrak{B}(V \oplus W)$ is finite-dimensional. If char$\mathbb{K} \neq 2$, then

$$\mathcal{H}(t_1,t_2) = (6)_{t_1}(6)_{t_1t_2^3}(6)_{t_1^2t_2^3}(2)^2_{t_2}(3)_{t_2}(6)_{t_2}(2)^2_{t_1t_2}(3)_{t_1t_2}(6)_{t_1t_2}(2)^2_{t_1t_2^2}(3)_{t_1t_2^2}(6)_{t_1t_2^2}$$

and $\dim \mathfrak{B}(V \oplus W) = 6^3\, 72^3 = 80621568$, and if char$\mathbb{K} = 2$, then

$$\mathcal{H}(t_1,t_2) = (3)_{t_1}(3)_{t_1t_2^3}(3)_{t_1^2t_2^3}(2)^2_{t_2}(3)^2_{t_2}(2)^2_{t_1t_2}(3)^2_{t_1t_2}(2)^2_{t_1t_2^2}(3)^2_{t_1t_2^2}$$

and $\dim \mathfrak{B}(V \oplus W) = 3^3\, 36^3 = 1259712$.

Sketch of the proof of Theorem 5.3.

5.6. To simplify the presentation let us assume that $\mathbb{K} = \mathbb{C}$. The assumptions of Theorem 5.3 are:

- G is a non-abelian group,
- $V = \oplus_{x \in g^G} V_x$ and $W = \oplus_{y \in h^G} W_y$ are finite-dimensional simple Yetter-Drinfeld modules over G,
- G is generated by $g^G \cup h^G$,
- $\mathfrak{B}(V \oplus W)$ is finite-dimensional, and
- $c_{W,V}c_{V,W} \neq \mathrm{id}_{V \otimes W}$.

5.7. We split the proof into several steps.
The first steps use deep results of Andruskiewitsch, Heckenberger and Schneider, and the classification of finite Weyl groupoids of rank two of Cuntz and Heckenberger.

5.8. CLAIM. The pair (V,W) admits all reflections and its Weyl groupoid is finite. Indeed, since $\dim \mathfrak{B}(V \oplus W) < \infty$, the claim follows from [**?**, Cor. 3.18 and Prop. 3.23].

5.9. CLAIM. In the Weyl groupoid $\mathcal{W}(V, W)$ there exists an object with an indecomposable Cartan matrix of finite-type:

$$\begin{pmatrix} 2 & -c_1 \\ -c_2 & 2 \end{pmatrix} \quad \text{where} \quad 1 \leqslant c_1 c_2 \leqslant 3.$$

This is a consequence of the classification of Heckenberger and Cuntz of finite Weyl groupoids of rank two in terms of continued fractions [**?**].

5.10. So we may assume that our object (V, W) satisfies

$$(\mathrm{ad}V)(W) \neq 0, \quad (\mathrm{ad}V)^2(W) = 0, \quad (\mathrm{ad}W)^4(V) = 0. \tag{5.10.1}$$

Then we classify which $\mathrm{supp}(V \oplus W)$ can appear. Since $\mathrm{supp}(V \oplus W)$ generates G, it follows that G is an epimorphic image of the enveloping group of the quandle $\mathrm{supp}(V \oplus W)$. (This is one of the most important steps in the proof.)

5.11. Now the following proposition gives a complete description of $\mathrm{supp}(V \oplus W)$ under our assumptions.

PROPOSITION. *Let G be a non-abelian group, and V and W be two Yetter-Drinfeld modules over G. Assume that*

(1) G *is generated by the supports of* V *and* W*,*
(2) $\mathrm{supp}V$ *and* $\mathrm{supp}W$ *are conjugacy classes,*
(3) $(\mathrm{ad}V)^2(W) = 0$,
(4) $(\mathrm{ad}W)^4(V) = 0$.

If $(\mathrm{id} - c_{W,V}c_{V,W})(V \otimes W) \neq 0$, *then* $\mathrm{supp}(V \oplus W)$ *is isomorphic to one of the quandles:*

$$Z_T^{4,1}, Z_2^{2,2}, Z_3^{3,1}, Z_3^{3,2}, Z_4^{4,2},$$

and G *is an epimorphic image of the corresponding enveloping groups* T, Γ_2, Γ_3, Γ_4, *respectively.*

Sketch of the proof of Proposition 5.11.

5.12. LEMMA. [**?**, Thm. 1.1] *Let* $\varphi_0 = 0$, $X_0^{V,W} = W$, *and*

$$\begin{aligned} \varphi_m &= \mathrm{id} - c_{V^{\otimes(m-1)} \otimes W, V}\, c_{V, V^{\otimes(m-1)} \otimes W} + (\mathrm{id} \otimes \varphi_{m-1}) c_{1,2}, \\ X_m^{V,W} &= \varphi_m (V \otimes X_{m-1}) \end{aligned}$$

for all $m \geqslant 1$. *Then* $(\mathrm{ad}V)^n(W) \simeq X_n^{V,W}$ *for all* $n \in \mathbb{N}_0$.

5.13. LEMMA. *Let* $m \in \mathbb{N}$. *Assume that a the conjugacy class* g^G *is an indecomposable quandle ,* $(\mathrm{ad}V)^m(W) \neq 0$ *and* $(\mathrm{ad}V)^{m+1}(W) = 0$. *Then* $|g^G| \leqslant 2m$.

5.14. Remember that our pair (V, W) satisfies

$$(\mathrm{ad}V)(W) \neq 0, \quad (\mathrm{ad}V)^2(W) = 0, \quad (\mathrm{ad}W)^4(V) = 0.$$

To prove Proposition 5.11 we need to consider two cases.

First we assume that the classes g^G and h^G commute, i.e. $xy = yx$ for all $x \in g^G$ and $y \in h^G$. Using the assumption $G = \langle g^G \cup h^G \rangle$, we obtain that the classes g^G and h^G are indecomposable quandles. Then Lemma 5.13 implies that $|g^G| \leqslant 2$ and $|h^G| \leqslant 6$. Since g^G is indecomposable, $g^G = \{g\}$.

5.15. LEMMA. *Let* $r_1, r_2, r_3, r_4 \in g^G$ *and* $s \in h^G$. *Assume that* g^G *and* h^G *commute,* $0 \neq (r_3, r_4, s) \in (\mathrm{ad}V)^2(W)$, *and*

$$r_2 \notin \{r_3, r_4, r_4^{-1} \triangleright r_3\}, \quad r_2 \triangleright r_4 \neq r_4, \tag{5.15.1}$$

$$r_1 \notin \{r_2 \triangleright r_3, r_2, r_4, r_4^{-1} \triangleright r_2, r_4^{-1} \triangleright r_3\}, \quad r_1 \triangleright r_4 \neq r_4. \tag{5.15.2}$$

Then $(\mathrm{ad}V)^4(W) \neq 0$.

5.16. With Lemma 5.15 we obtain that $|h^G| \leqslant 4$. Now since the classification of indecomposable quandles of small size is known, we conclude that $g^G = \{g\}$ and h^G is (isomorphic to) one of the following quandles $(12)^{\mathbb{S}_3}$ or $(123)^{\mathbb{A}_4}$. Therefore $g^G \cup h^G = \{g\} \cup (12)^{\mathbb{S}_3}$ is one of the quandles:

$$Z_3^{3,1} = \{g\} \cup (12)^{\mathbb{S}_3} \text{ or } Z_T^{4,1} = \{g\} \cup (123)^{\mathbb{A}_4}. \tag{5.16.1}$$

5.17. Lemma. *Now assume that* g^G *and* h^G *do not commute. Assume further that* $(\operatorname{ad} V)^2(W) = 0$. *The following hold.*

(1) *Let* $s \in h^G$. *Then there exist* $r_1, r_2 \in g^G$ *such that* $\varphi_s|_{g^G} = (r_1 r_2)$.
(2) g^G *is commutative and* $h^G \neq g^G$.

5.18. So we know that g^G is commutative. There are two cases to consider.
If h^G is indecomposable, then $|h^G| \leqslant 6$ by Lemma 5.13 and (since the classification of small indecomposable quandles [or more precisely quandles] is known) we obtain en explicit description of the quandles h^G. Then Lemma 5.17 and some computations yield the list of possibilities for $g^G \cup h^G$. This list can drastically be reduced using the following lemma.

5.19. Lemma. *Let* $r_1, r_2, r_3 \in g^G$ *and* $s \in h^G$. *Assume that:*

(1) $r_2 \triangleright r_3 \neq r_3$,
(2) $r_1 \notin \{r_3 r_2 \triangleright r_3, r_3 \triangleright r_2, r_3, s^{-1} \triangleright r_3, s^{-1} \triangleright r_2\}$,
(3) $s \triangleright r_2, s \triangleright r_3 \notin \{r_2, r_3\}$,
(4) $r_1 \triangleright s \neq s$ *or* $r_1 \triangleright r_3 \neq r_3$.

Then $(\operatorname{ad} V)^4(W) \neq 0$.

5.20. After using Lemma 5.19 we obtain only one possibility for $g^G \cup h^G$. This is the quandle $Z_3^{3,2}$ described by the following permutations:

$$Z_3^{3,2} : (23)(45)\ (13)(45)\ (12)(45)\ (123)\ (132)$$

5.21. Suppose now that h^G is decomposable. Using Lemma 5.19 and after many computations one proves that $g^G \cup h^G$ is one the quandles:

$$Z_2^{2,2}: \ (24)\ (13)\ (24)\ (13)$$
$$Z_4^{2,2}: \ (24)(56)\ (13)(56)\ (24)(56)\ (13)(56)\ (1234)\ (1432)$$

At the end we have only five quandles!
Since G is generated by $g^G \cup h^G$, the group G is an epimorphic image of the enveloping group of the quandle $g^G \cup h^G$. The enveloping groups of our five quandles are given in the following table:

Quandle	$Z_2^{2,2}$	$Z_3^{3,1}$	$Z_3^{3,2}$	$Z_4^{4,2}$	$Z_T^{4,1}$
Enveloping group	Γ_2	Γ_3	Γ_3	Γ_4	T

6. Nichols algebras of decomposable modules II: the general case

Let $\theta \in \mathbb{N}$ with $\theta \geqslant 3$. Let $\mathbb{K}$ be a field, G be a group and $M_1, \dots, M_\theta$ be absolutely simple Yetter-Drinfeld modules over G. Let us review the classification of finite-dimensional Nichols algebras $\mathfrak{B}(M_1 \oplus \cdots \oplus M_\theta)$, see [**?**].

6.1. We say that the tuple $M = (M_1, \dots, M_\theta)$ of finite-dimensional absolutely simple Yetter-Drinfeld modules over G has a **skeleton** if

(1) for all $1 \leqslant i \leqslant \theta$ there exist $s_i \in \operatorname{supp} M_i$ and a character σ_i of G^{s_i} such that $M_i \simeq M(s_i, \sigma_i)$, and
(2) M is absolutely plain, and for all $1 \leqslant i < j \leqslant \theta$ with $a_{ij} \neq 0$ at least one of a_{ij}, a_{ji} is equal to -1, where $A = (a_{ij})_{1 \leqslant i,j \leqslant \theta}$ is the Cartan matrix of M.

In this case a **skeleton** of M is a partially oriented partially labeled loopless graph with θ vertices satisfying the following properties.

(1) For all $1 \leqslant i \leqslant \theta$, the i-th vertex is symbolized by $|\mathrm{supp} M_i| = \dim M_i$ points.
(2) For all $i, j \in \{1, \dots, \theta\}$ with $i \neq j$ there are $a_{ij} a_{ji}$ edges between the i-th and j-th vertex. The edge is oriented towards j if and only if $a_{ij} = -1$, $a_{ji} < -1$.
(3) Let $1 \leqslant i < j \leqslant \theta$ with $a_{ij} < 0$. Then the edges between the i-th and j-th vertex are continuous lines if $\mathrm{supp} M_i$ and $\mathrm{supp} M_j$ commute, and dashed lines otherwise.
(4) If the i-th vertex has a label $p \in \mathbb{K}^\times$, then $\sigma_i(s_i) = p$.
(5) If the edges between the i-th and j-th vertex have a label $p \in \mathbb{K}^\times$, then $s_i \in G^{s_j}$, $s_j \in G^{s_i}$, and $\sigma_i(s_j)\sigma_j(s_i) = p$.

6.2. It is clear from the definition that two skeletons can at most differ by labels of vertices and edges. In what follows we will usually omit all labels which can be obtained from the other data of the skeleton.

6.3. A skeleton is called **connected** if the underlying graph is connected. A connected skeleton with at least three vertices is said to be of **finite type** if it appears in Figure 6.3.1.

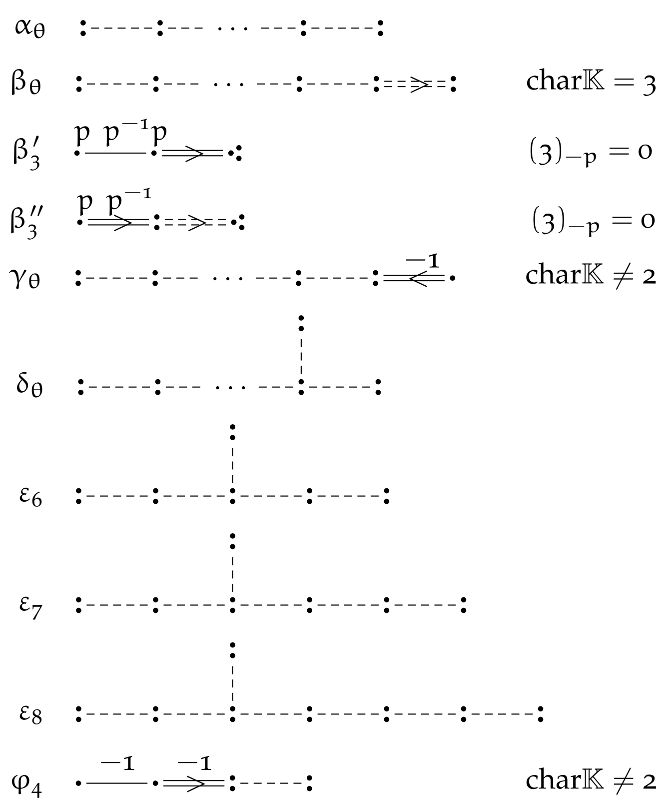

FIGURE 6.3.1. Skeletons of finite type with at least three vertices

6.4. We say that a tuple $M = (M_1, \dots, M_\theta)$ of Yetter-Drinfeld modules over G is **braid-indecomposable** if there is no decomposition

$$M_1 \oplus \cdots \oplus M_\theta = M' \oplus M''$$

in ${}^G_G\mathcal{YD}$ with $M' \neq 0$, $M'' \neq 0$ and $(\mathrm{id} - c^2)(M' \otimes M'') = 0$.

6.5. THEOREM. *Let $\theta \geqslant 3$, G be a non-abelian group and $M = (M_1, \dots, M_\theta)$ be a tuple of finite-dimensional absolutely simple Yetter-Drinfeld modules over G. Assume that M is braid-indecomposable. Then $\mathfrak{B}(M_1 \oplus \cdots \oplus M_\theta)$ is finite-dimensional if and only if M has a skeleton of finite-type.*

6.6. Theorem 6.5 gives the dimensions of $\mathfrak{B}(M_1 \oplus \cdots \oplus M_\theta)$. The structure of the M_i can be read off from the skeletons of Figure 6.3.1. Let us show an example. In the case where M has a simply-laced skeleton of finite type (i.e. of type ADE, or with a skeleton of type α_n, δ_n, ϵ_6, ϵ_7 or ϵ_8) or the Nichols algebra $\mathfrak{B}(M)$ is finite-dimensional and its Hilbert series is

$$\mathcal{H}(t) = \prod_{\alpha \in \Delta_+} (1 + t^{\alpha})^2,$$

where Δ_+ denotes the set of positive root of the root system associated with the Weyl grpoupoid $\mathcal{W}(M)$. The dimensions of these Nichols algebras are listed in Table 4.

Table 4. Nichols algebras with root system of type ADE.

root system	A_θ	D_θ	E_6	E_7	E_8
$\dim \mathcal{B}(M)$	$4^{\theta(\theta+1)/2}$	$4^{\theta(\theta-1)}$	4^{36}	4^{63}	4^{120}
skeleton	α_θ	δ_θ	ε_6	ε_7	ε_8

Departamento de Matemática – FCEN, Universidad de Buenos Aires, Pab. I – Ciudad Universitaria (1428) Buenos Aires – Argentina
E-mail address: `lvendramin@dm.uba.ar`