Student Information

Full Name : Ahmet Dara VEFA
Id Number : 2237899

Answer 1

a.

Without assuming independence:
$P(Z = 0|X = 5, Y = 10) = P(Z = 0 \cap (X = 5, Y = 10))\ /\ P(X = 5, Y = 10) = 0.075/(0.075 + 0.050) = 0.6$

b.

$f(x, y, z)$ =joint distribution of x,y and z. Just do the summing:
$P_x(x) = \sum_y \sum_z f(x, y, z)$
$P_x(3) = \sum_y \sum_z f(3, y, z) = f(3, 10, 0)+f(3, 10, 1)+f(3, 20, 0)+f(3, 20, 1)+f(3, 30, 0)+f(3, 30, 1) = 0.025 + 0.025 + 0.03 + 0.02 + 0.05 + 0.15 = 0.3$
$P_x(5) = \sum_y \sum_z f(5, y, z) = 0.075 + 0.050 + 0.025 + 0.030 + 0.020 + 0.2 = 0.4$
$P_x(7) = \sum_y \sum_z f(7, y, z) = 0.04 + 0.06 + 0.02 + 0.050 + 0.025 + 0.1 = 0.3$
To check if the result is correct:
$P_x(3) + P_x(5) + P_x(7) = 1$, hence the result is correct

c.

$\mu = E(X) = \sum_x xP(x) = 3 * 0.3 + 5 * 0.4 + 7 * 0.3 = 5$
$\sigma^2 = Var(X) = \sum_x (x - \mu)^2 P(x) = 4P(3) + 0 * P(5) + 4P(7) = 2.4$

d.

$P(X|Z = 1) = P(X \cap Z = 1)\ /\ P(Z = 1)$
$P_z(1) = \sum_x \sum_y f(x, y, 1) = 0.685$(I won't show the sums etc.)
$P(X = 3 \cap Z = 1)\ /\ P(Z = 1) = 0.195/0.685 = 0.285$
$P(X = 5 \cap Z = 1)\ /\ P(Z = 1) = 0.280/0.685 = 0.408$
$P(X = 7 \cap Z = 1)\ /\ P(Z = 1) = 0.210/0.685 = 0.307$
(sums of the last 3 values are 1 so it checks)

e.

$\mu = E(X|Z=1) = \sum_x xP(x|z=1) = 3*0.285 + 5*0.408 + 7*0.307 = 5.044$
$\sigma^2 = Var(x) = E_x(x-\mu)^2 P(x) = 4.178 * P_x(3|Z=1) + 0.002 * P(5|Z=1) + 4.178 * P(7|Z=1) = 1.190 + 0.001 + 1.190 = 2.381$
"

Answer 2

a.

we know that $\Sigma_k P_X(k) = 1 \ and \ \Sigma_k P_Y(k) = 1$
$since \ k \geq 1:$
$C_1 \Sigma_{k=1}^{\infty} (1/2)^k = C_1(\frac{1-1/2^{\infty}}{1-1/2} - 1) = C_1(2-1) = C_1$
$\xrightarrow{using \ first \ equation} C_1 = 1$

$C_2 \Sigma_{k=1}^{\infty} (1/2)^k / k \xrightarrow{m=1/2} C_2 \Sigma_{k=1}^{\infty} \frac{m^{k+1}}{k+1} = C_2 h(m)$ take derivative
$\frac{dh(m)}{dm} = C_2 \Sigma_{k=0}^{\infty} m^k = \frac{1-m^{\infty}}{1-m} = \frac{1}{1-m}$ now take integral
$h(m) = \int \frac{1}{1-m} dm = -ln(1-m) \xrightarrow{m=1/2} -ln(1/2) = ln(2)$
using the second equation: $C_2 ln(2) = 1 \longrightarrow C_2 = 1/ln(2)$

b.

$P(X \ even) + P(x \ odd) = 1$
$P(x = 2m) + P(x = 2m+1) = 1$
$P(x = 2m) = \Sigma_{m=1}^{\infty} (1/2)^{2m} = \Sigma_{m=1}^{\infty} (1/4)^m = \frac{1-1/4^m}{1-1/4} - 1 = 1/3 = P(x \ even)$

c.

$P(x \ odd) = 1 - P(x \ even) = 2/3$
$P(X+Y=6|X \ odd) \xrightarrow{using \ bayes \ rule} P(X \ odd|X+Y=6)P(X+Y=6)/P(X \ odd)$
$X+Y=6 \Longrightarrow (x,y) = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$
Using the above cases $P(x \ odd|X+Y=6) = 0.672$
$P(X+Y=6) \xrightarrow{using \ above \ cases \ as \ basis} (1/ln2)(2^{-1}2^{-5}/5 + 2^{-2}2^{-4}/4 + 2^{-3}2^{-3}/3 + 2^{-4}2^{-2}/2 + 2^{-5}2^{-1}/1) = (1/ln2) * 2^{-6}(1/5 + 1/4 + 1/3 + 1/2 + 1) = 0.052$
now calculate the result:
$\frac{0.672*0.052}{2/3} = 0.0524$

d.

$(x,y) = \{(1,5),(2,4),(4,2),(5,1)\}$
$P(X=b_1, Y=b_2) \xrightarrow{using \ above \ cases \ as \ basis} (1/ln2)(2^{-1}2^{-5}/5 + 2^{-2}2^{-4}/4 + 2^{-4}2^{-2}/2 + 2^{-5}2^{-1}/1) =$

$(1/ln2)2^{-6}(1/5+1/4+1/2+1)=0.044$

Answer 3

a.

if $P_{x,y}=P(x)*P(y)\ \forall\ x,y$ they are independent(I won't show $P_{x,y}(a,b)=\sum_z f(a,b,z)$
$Px,y(3,10)=0.05=?=0.0825=0.3*0.275=P_x(3)*P_y(10)$ since it is not equal we don't need to show the rest. They are dependent.

b.

Let A be the probability that we get 5 when we toss a 6 sided die. A is independent of itself.
$P(A)=1/6$ so it is false.

c.

$P_{A,B}=?=P_A(a)*P_B(b)\ \forall\ a,b$
if $P(A)=0$
$P_{A,B}(0,b)=0\ and\ P(A)*P(B)=0$ so $P(A)=0\ checks$
if $P(A)=1$
$P_{A,B}(1,B)=P(B)\ and\ P(A)*P(B)=1*P(B)=P(B)$ so $P(A)=1\ checks$
so it is true

Answer 4

a.

$P(G=n-1+m|G>n-1)=P(G=n-1+m\cap G>n-1)\ /\ P(G>n-1)$
$P(G>n-1)=1-P(G\leq n-1)=1-\sum_{k=1}^{n-1}\xrightarrow{using\ geometric\ progression\ formula}1-p\frac{1-(1-p)^{n-1}}{1-(1-p)}=(1-p)^{n-1}$
$P(G=n-1+m\cap G>n-1)=P(G=n-1+m)$ since $(G=n-1+m\cap G>n-1)=(G=n-1+m)$
$P(G=n-1+m)=(1-p)^{n+m-2}p$
so $\frac{p(1-p)^{n+m-2}}{(1-p)^{n-1}}=?=(1-p)^{m-1}p\Longrightarrow(1-p)^{n-1}=?=(1-p)^{n-1}$ checks.

b.

I already showed it in 4a but here is the formulas:
$P(G\leq n)=\sum_{k=1}^{n}\xrightarrow{using\ geometric\ progression\ formula}p\frac{1-(1-p)^n}{1-(1-p)}=1-(1-p)^n$

c.

$P(65 \leq G \leq 75) = P(G \leq 75) - P(G \leq 64) = (1-(1-p)^{75}) - (1-(1-p)^{64}) = (1-3*(1/6)^3)^{75} - (1-3*(1/6)^3)^{64}$

Answer 5

a.

n=20,000 and p=1/10,000
$P(x=3) = \binom{20{,}000}{3}(1/10,000)^3(9,999/10,000)^{19{,}997}$

b.

Use Poisson to compute. $\lambda = np = 2$
$P_{Poission}(X=3) = F_{Poission}(3) - F_{Poission}(2) = 0.857 - 0.677 = 0.18$
I chose poission since $n \geq 30$ and $p \leq 0.05$. Poisson approximation is really close to real value(for big n small p)

Answer 6

I'll use $X = O$ for simplicity

a.

$E[X^n] = \sum_{k=0}^{\infty} k^n\ P(X=k) = \sum_{k=1}^{\infty} k^n e^{-\lambda}\frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} k^{n-1}e^{-\lambda}\frac{\lambda^k}{(k-1)!} = \lambda\sum_{k=1}^{\infty} k^{n-1}e^{-\lambda}\frac{\lambda^{k-1}}{(k-1)!} = \lambda\sum_{m=0}^{\infty}(m+1)^{n-1}e^{-\lambda}\frac{\lambda^m}{m!}$
$= \lambda E[(X+1)^{n-1}]$

b.

$E[X^3] = \lambda E[(X+1)^2] = \lambda E[X^2+2X+1] = \lambda E[X^2] + 2\lambda E[X] + \lambda$
$= \lambda^2(\lambda+1) + 2\lambda^2 + \lambda = \lambda^3 + 3\lambda^2 + \lambda.$

BONUS 1

BONUS 2

a.

b.

c.

d.