CENG 384 - Signals and Systems for Computer Engineers
Spring 2018-2019
Written Assignment 3

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1. (a) x[n] is a periodic signal with period N=4. To find Fourier series coefficients we use the formula $a_k = 1/N \ \sum_{n=<N>} x[n]e^{-jkw_0n}$ where N = 4 and $w_0 = 2\pi/T = \pi/2$

$$a_0 = 1/4 \sum_{n=<N>} x[n]e^{-j0\pi/2n} = 1/4$$

$$a_1 = 1/4 \sum_{n=<N>} x[n]e^{-jw_0n} = 1/4(e^{-j(\pi/2)} + 2e^{-2j(\pi/2)} + 3e^{-3j(\pi/2)}) = -1/2$$
$$a_2 = 1/4 \sum_{n=<N>} x[n]e^{-j2w_0n} = 1/4(e^{-j2(\pi/2)} + 2e^{-4j(\pi/2)} + 3e^{-6j(\pi/2)}) = 0$$
$$a_3 = 1/4 \sum_{n=<N>} x[n]e^{-j3w_0n} = 1/4(e^{-j3(\pi/2)} + 2e^{-6j(\pi/2)} + 3e^{-9j(\pi/2)}) = -1/2$$

Since our signal is periodic $a_n = a_{n+N}$.

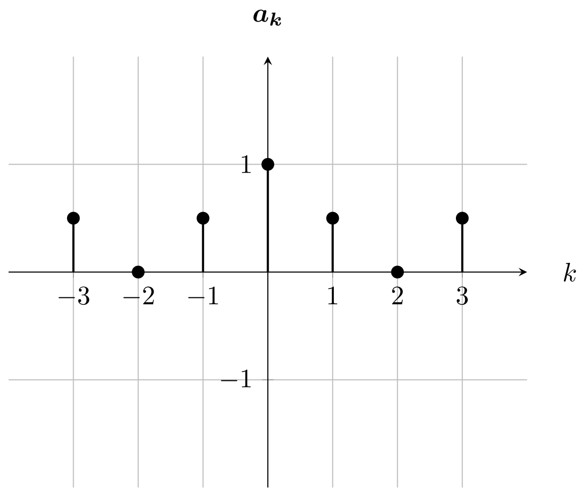

Figure 1: k vs. $|a_k|$.

(b) (i)

$$y[n] = x[n] - \sum_{k=-\infty}^{\infty} \delta[n - 3 + 4k]$$

(ii) To find the coefficient, it is enough to find a_0, a_1, a_2, a_3. Again using the formula in the part a;

$$a_0 = 1/4 \sum_{n=<N>} y[n]e^{-j0\pi/2n} = 3/4$$
$$a_1 = 1/4 \sum_{n=<N>} y[n]e^{-j\pi/2n} = -1/4(j + 2)$$
$$a_2 = 1/4 \sum_{n=<N>} y[n]e^{-j2\pi/2n} = 1/4$$
$$a_3 = 1/4 \sum_{n=<N>} y[n]e^{-j0\pi/2n} = 1/4(j - 2)$$

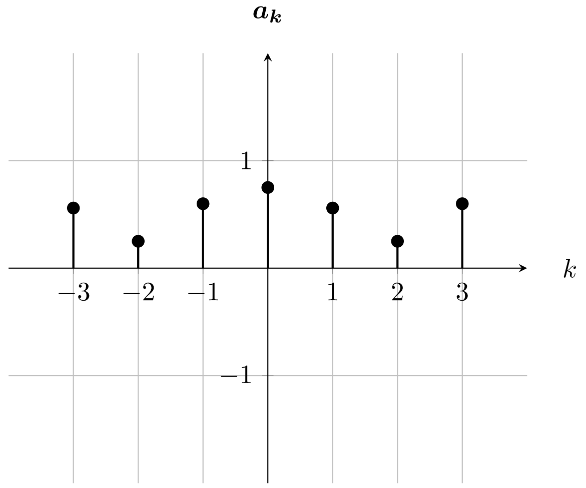

Figure 2: k vs. $|a_k|$.

2. Using A:
   Since period is 4
   $w_0 = 2\pi/4 = \pi/2$
   and $x[n] = x[n+4]$

   Using B and previous findings:
   $2 \times \Sigma_{k=0}^{3} x[k] = 8$(include 0 in the interval to make the calculations a bit easier)
   hence $x[0] + x[1] + x[2] + x[3] = 4$

   Using C and previous findings:
   $a_{-3} = a_1$
   $a_3 = a_{11}$
   $a_{-3} = a_{15}^* \longrightarrow a_1 = a_3^*$
   $|a_1 - a_{11}| = |a_1 - a_3| = 1$

   Using D and previous findings:
   We know that one of the a_0 a_1 a_2 a_3 is 0;

   Using E and previous findings:
   First convert complex exponentials to cos and sin
   $e^{-j\pi k/2} = cos(-\pi k/2) + jsin(-\pi k/2)$
   $e^{-j3\pi k/2} = cos(-3\pi k/2) + jsin(-3\pi k/2)$
   We can convert terms with $(3\pi k/2)$ to terms with $(\pi k/2)$ by removing $2\pi k$ from each term :
   $sin(3\pi k/2) = -sin(\pi k/2)$
   $cos(3\pi k/2) = cos(\pi k/2)$
   Now find the sum of complex exponentials given in e:
   $e^{-j\pi k/2} + e^{-j3\pi k/2} = cos(\pi k/2) - jsin(\pi k/2) + cos(\pi k/2) + jsin(\pi k/2) = 2cos(\pi k/2)$
   Now we have:
   $\Sigma_{k=0}^{3}(x[k]2cos(\pi k/2)) = 4$, open the sum(k=1 & k=3 terms are 0 because of cos):
   $2x[0] - 2x[2] = 4 \Longrightarrow x[0] - x[2] = 2$

   Now lets find a_k's for k=0,1,2,3:
   $a_k = \frac{1}{N}\Sigma_{n=<N>} x[n](e^{-jkw_0n}) \xrightarrow{in\ our\ case(N=4,\ w_0=\pi/2)} a_k = \frac{1}{4}\Sigma_{n=0}^{3} x[n](e^{-jkn\pi/2})$
   So
   $a_0 = 1/4(x[0] + x[1] + x[2] + x[3]) \xrightarrow{using\ part\ B} = 1/4 \times 4 = 1$

   Since a_1 and a_3 are conjugate complex numbers let $a_1 = x + jy \Longrightarrow a_3 = x - yj$
   $|a_1 - a_3| = |2yj| = 1$(using part C)
   Since a_1 and a_3 have imaginary parts they can't be 0, and since a_0 is not 0, a_2 must be zero:
   $a_2 = 1/4(x[0] - x[1] + x[2] - x[3]) = 0$

   Now we need to find a_1 or a_3:
   $a_1 = 1/4(x[0] + x[1]e^{-jw_0} + x[2]e^{-2jw_0} + x[3]e^{-3jw_0}) \xrightarrow{convert\ to\ sin\&cos\ form\ with\ w_0=\pi/2} 1/4(x[0] + x[1](cos(\pi/2) - jsin(\pi/2)) + x[2](cos(2\pi/2) - jsin(2\pi/2)) + x[3](cos(3\pi/2) - jsin(3\pi/2)))$, simplify:
   $a_1 = 1/4(x[0] - jx[1] - x[2] + jx[3])$

Since a_3 is conjugate of a_1:
$a_3 = 1/4(x[0] + jx[1] - x[2] - jx[3])$

Now we have to find x[0],x[1],x[2],x[3]:

In part C we found that:
$|a_1 - a_3| = 1$, put the values in:
$|(1/4)(2j)(x[3] - x[1])| = 1$, simplify:
$|j(x[3] - x[1])| = |x[3] - x[1]| = 2$

We can get two relations of x[n] from $a_0 + -a_2$ since we found their values:
$a_0 - a_2 = 1/4(2x[1] + 2x[3]) = 1 \Longrightarrow x[1] + x[3] = 2$
$a_0 + a_2 = 1/4(2x[0] + 2x[2]) = 1 \Longrightarrow x[0] + x[2] = 2$
We found $x[0] - x[2] = 2$ in part E, using this equation and previous 2 equations we find:
$x[0] = 2$
$x[2] = 0$
We found $|x[3] - x[1]| = 2$ previously, lets assume x[3]¿x[1] and take it out, then we can find:
$x[1] = 0$
$x[3] = 2$(note that x[3] and x[1] will change depending on our assumption)
Now we have found all the values we can draw(x[0]=2,x[1]=0,x[2]=0,x[3]=2)

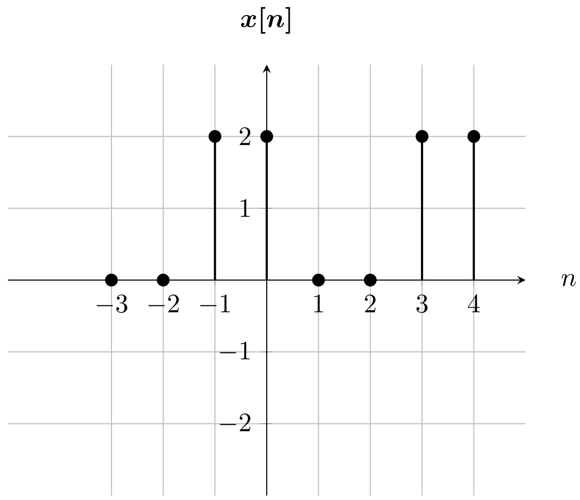

Figure 3: n vs. $x[n]$.

3. it is given that:
$x(t) = h(t) * (x(t) + r(t))$

Apply Fourier Transform:
$X(jw) = H(jw)(X(jw) + R(jw)) \longleftarrow$ since convolution is equal to multiplication in fourier transform
$X(jw) = H(jw)X(jw) + H(jw)R(jw)$, it is given that $R(jw) = 0$ $when$ $|w| \leq K2\pi/T$
$X(jw) = H(jw)X(jw) \Longrightarrow H(jw) = 1$ $when$ $|w| \leq K2\pi/T$

Apply inverse F.T to find h(t):
$h(t) = \frac{1}{2\pi}\int_{\infty}^{\infty} H(jw)e^{jwt}dw = \frac{1}{2\pi}\int_{-K2\pi/T}^{K2\pi/T} e^{jwt}dw$, take it out of the integral and we get:
$h(t) = \frac{1}{2\pi}(\frac{e^{jwt}}{jt}|_{w=-K2\pi/T}^{K2\pi/T}) = \frac{1}{2\pi}(\frac{e^{jtK2\pi/T}}{jt} - \frac{e^{-jtK2\pi/T}}{jt})$, convert to sin& cos form:
$h(t) = \frac{1}{\pi t}sin(\frac{K2\pi t}{T})$

4. (a) The differential equation of the block diagram is;

$$y(t) = \int[\int -6y(t) + x(t)dt] - 5y(t) + 4x(t)dt$$
$$y'(t) = [\int -6y(t) + x(t)dt] - 5y(t) + 4x(t)$$
$$y''(t) = -6y(t) + x(t) - 5y'(t) + 4x'(t)$$

We are asked to find frequency response of the system above. The input is e^{jwt} the frequency response H(jw). Hence, $y(t) = H(jw)e^{jwt}$. By substituting necessary values;

$$y''(t) = -6y(t) + x(t) - 5y'(t) + 4x'(t)$$

$$(jw)^2 e^{jwt} H(jw) = -6e^{jwt} H(jw) + e^{jwt} - 5(jw)e^{jwt} H(jw) + 4jwe^{jwt}$$

$$H(jw)e^{jwt}((jw)^2 + 6 + 5jw) = e^{jwt}(1 + 4jw)$$

$$H(jw) = \frac{(1+4jw)}{(jw)^2+5jw+6}$$

(b) To find the impulse response of the system we need to take inverse Fourier transform of the impulse response.

$$H(jw) = \frac{(1+4jw)}{(jw)^2+5jw+6}$$

$$H(jw) = \frac{A}{(jw+2)} + \frac{B}{(jw+3)}$$

$$Ajw + Bjw = 4jw$$

$$3A + 2B = 1$$

Equations implies that A = -7 and B = 11, hence $H(jw) = \frac{-7}{(jw+2)} + \frac{11}{(jw+3)}$ .Take inverse Fourier by using the table we get, $h(t) = 11e^{-3t}u(t) - 7e^{-2t}u(t)$.

(c) To find y(t) when input is $x(t) = 1/4e^{-t/4}u(t)$. First we can find Fourier transform of x(t):

$$\text{Table says: } e^{|a|t}u(t) \longrightarrow \frac{1}{|a|+jw}$$

$$\text{So : } X(jw) = 1/4 \times \frac{1}{1/4+jw}$$

We can find Y(jw) by using the property : $Y(jw) = H(jw) \times X(jw)$.

$$Y(jw) = (\frac{-7}{(jw+2)} + \frac{11}{(jw+3)}) \times (1/4 \times \frac{1}{1/4+jw})$$

$$Y(jw) = \frac{1}{(jw+2)(jw+3)}$$

Finally by taking inverse Fourier of Y(jw) we can find y(t).

$$Y(jw) = \frac{1}{(jw+2)(jw+3)}$$

$$Y(jw) = \frac{-1}{(jw+2)} + \frac{1}{(jw+3)}$$

$$\text{Using table : } y(t) = e^{-3t}u(t) - e^{-2t}u(t)$$

$$y(t) = (e^{-3t} - e^{-2t})u(t)$$