Quiz 1 Section B - Solutions

Discrete Structures Monsoon 2024, IIIT Hyderabad

1. [2 points] Let $L(x, y)$ be the predicate "the student x likes the course y", $H(x, y)$ the predicate "x works hard for course y", and $G(x, y)$ the predicate "x makes a good grade for course y". Then formalize:

   (a) Some students like course $y = c$.

   (b) Every student likes some course y.

   (c) There is a course every student likes.

   (d) Unless a student works hard for a certain course $y = c$, he won't make a good grade in that course.

   (e) Only students who like course $y = c$ work hard for $y = c$.

Solution: Given the predicates:

- $L(x, y)$: The student x likes the course y
- $H(x, y)$: The student x works hard for the course y
- $G(x, y)$: The student x makes a good grade for the course y

(a) Some students like course $y = c$.

Formalization:

$$\exists x \, L(x, c)$$

Explanation: This statement means that there is at least one student x who likes the course y where y is specifically c. The existential quantifier $\exists x$ asserts the existence of such a student.

(b) Every student likes some course y.

Formalization:

$$\forall x \, \exists y \, L(x, y)$$

Explanation: This statement means that for every student x, there exists at least one course y that the student likes. The universal quantifier $\forall x$ applies to all students, and the existential quantifier $\exists y$ indicates that each student likes some course.

(c) There is a course every student likes.

Formalization:

$$\exists y\, \forall x\, L(x, y)$$

Explanation: This statement means that there exists at least one course y such that every student x likes that course. The existential quantifier $\exists y$ asserts the existence of such a course, and the universal quantifier $\forall x$ ensures that every student likes this course.

(d) Unless a student works hard for a certain course $y = c$, he won't make a good grade in that course.

Formalization:

$$\forall x\, (\neg H(x, c) \to \neg G(x, c))$$

Explanation: This statement means that if a student x does not work hard for a specific course $y = c$, then the student will not make a good grade in that course. The implication $\neg H(x, c) \to \neg G(x, c)$ captures the idea that not working hard leads to not getting a good grade.

(e) Only students who like course $y = c$ work hard for $y = c$.

Formalization:

$$\forall x\, (H(x, c) \to L(x, c))$$

Explanation: This statement means that if a student x works hard for the course $y = c$, then the student must like that course. The implication $H(x, c) \to L(x, c)$ expresses that working hard for the course is only true for students who like the course.

2. [6 points] Prove or disprove whether the formula is a tautology or not, without using truth tables.

   (a) $(p \to q) \vee (q \to p)$

   (b) $((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$

   (c) $(p \to (q \to r)) \leftrightarrow ((p \wedge q) \to r)$

Solution:

(a) $(p \to q) \vee (q \to p)$

Proof: To show that $(p \to q) \vee (q \to p)$ is a tautology, we use logical equivalences:

1. Recall the implication equivalence:

$$p \to q \equiv \neg p \vee q$$

$$q \to p \equiv \neg q \vee p$$

2. Substitute these into the original formula:

$$(p \to q) \lor (q \to p) \equiv (\neg p \lor q) \lor (\neg q \lor p)$$

3. Use the associativity and commutativity of $\lor$:

$$(\neg p \lor q) \lor (\neg q \lor p) \equiv \neg p \lor q \lor \neg q \lor p$$

4. Apply the Tautology Law (a disjunction involving a variable and its negation is always true):

$$\neg p \lor p \text{ is always true (Tautology)}$$

$$q \lor \neg q \text{ is always true (Tautology)}$$

5. Therefore:

$$\neg p \lor p \lor q \lor \neg q \text{ is always true}$$

Thus, $(p \to q) \lor (q \to p)$ is a tautology.

(b) $((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$

Proof:

To show that $((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$ is a tautology, we use logical equivalences:

1. Use the implication equivalence:

$$A \to B \equiv \neg A \lor B$$

Thus:

$$((p \lor q) \land (\neg p \lor r)) \to (q \lor r) \equiv \neg((p \lor q) \land (\neg p \lor r)) \lor (q \lor r)$$

2. Apply De Morgan's Laws to $\neg((p \lor q) \land (\neg p \lor r))$:

$$\neg((p \lor q) \land (\neg p \lor r)) \equiv \neg(p \lor q) \lor \neg(\neg p \lor r)$$

$$\neg(p \lor q) \equiv \neg p \land \neg q \text{ (De Morgan's Law)}$$

$$\neg(\neg p \lor r) \equiv p \land \neg r \text{ (De Morgan's Law)}$$

$$\neg((p \lor q) \land (\neg p \lor r)) \equiv (\neg p \land \neg q) \lor (p \land \neg r)$$

3. Substitute into the formula:

$$(\neg p \land \neg q) \lor (p \land \neg r) \lor (q \lor r)$$

4. Use the Distributive Law to simplify:

$$(\neg p \land \neg q) \lor q \text{ simplifies to } \neg p \lor q \text{ (Absorption Law)}$$

$$(p \land \neg r) \lor r \text{ simplifies to } p \lor r \text{ (Absorption Law)}$$

Thus:

$$\neg p \lor q \lor p \lor r$$

5. Since $\neg p \lor p$ and $q \lor \neg q$ are always true:

$$\neg p \lor p \text{ is true (Tautology), and } q \lor \neg q \text{ is true (Tautology)}$$

Thus, $((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$ is a tautology.

(c) $(p \to (q \to r)) \leftrightarrow ((p \land q) \to r)$

Proof:

To show that $(p \to (q \to r)) \leftrightarrow ((p \land q) \to r)$ is a tautology, we use logical equivalences:

1. Recall the implications:

$$p \to (q \to r) \equiv \neg p \lor (q \to r)$$
$$q \to r \equiv \neg q \lor r$$
$$p \to (q \to r) \equiv \neg p \lor (\neg q \lor r)$$

2. Similarly:

$$(p \land q) \to r \equiv \neg(p \land q) \lor r$$
$$\neg(p \land q) \equiv \neg p \lor \neg q \text{ (De Morgan's Law)}$$
$$(p \land q) \to r \equiv (\neg p \lor \neg q) \lor r$$

3. Both expressions simplify to:

$$\neg p \lor \neg q \lor r$$

4. Since both sides of the bi-implication $(p \to (q \to r))$ and $((p \land q) \to r)$ simplify to the same expression $\neg p \lor \neg q \lor r$, they are logically equivalent. Therefore, their bi-implication is true in all cases.

5. Because the bi-implication is always true regardless of the truth values of p, q, and r, it is a tautology.

Thus, $(p \to (q \to r)) \leftrightarrow ((p \land q) \to r)$ is a tautology.

3. [3 points] Consider the following sequence of numbers $a_0, a_1, a_2 \ldots$ where $a_1 = 1$ and for all non negative integers m and n with $m \geq n$, we have

$$a_{m+n} + a_{m-n} = \frac{1}{2}(a_{2m} + a_{2n})$$

Prove that $a_n = n^2$ using **mathematical induction**.(Only proofs using induction will be considered).

Solution: We want to prove that for all non-negative integers n, the sequence $a_n = n^2$ satisfies the recurrence relation:

$$a_{m+n} + a_{m-n} = \frac{1}{2}(a_{2m} + a_{2n})$$

Base Case:

First, check the base case where $n = 0$.

Substitute $m = n$ into the recurrence relation:

$$a_{m+m} + a_{m-m} = \frac{1}{2}(a_{2 \cdot m} + a_{2 \cdot m})$$

$$a_{2\cdot m} + a_0 = \frac{1}{2}(a_{2\cdot m} + a_{2\cdot m})$$

$$a_0 = 0$$

Since $a_0 = 0^2 = 0$, the base case is satisfied.

Inductive Hypothesis:

Assume that for some integer $k \geq 0$, the statement $a_n = n^2$ holds for all non-negative integers n up to k. Specifically, assume:

$$a_n = n^2 \quad \text{for all } 0 \leq n \leq k$$

Substitute $m = k$ & $n = 0$ into the recurrence relation:

$$a_{k+0} + a_{k-0} = \frac{1}{2}(a_{2\cdot k} + a_{2\cdot 0})$$

$$2a_k = \frac{1}{2}(a_{2\cdot k} + a_0)$$

$$a_{2\cdot k} = 4a_k$$

Inductive Step:

We need to show that $a_{k+1} = (k+1)^2$

Substitute $m = k$ & $n = 1$ into the recurrence relation:

$$a_{k+1} + a_{k-1} = \frac{1}{2}(a_{2\cdot k} + a_{2\cdot 1})$$

Using the inductive hypothesis, we have:

$$a_{k-1} = (k-1)^2$$

$$a_2 = 2^2$$

$$a_{2k} = 4k^2$$

Substitute these:

$$a_{k+1} + (k-1)^2 = \frac{1}{2}(4k^2 + 2^2)$$

$$a_{k+1} + k^2 + 1 - 2k = 2k^2 + 2$$

$$a_{k+1} = k^2 + 1 + 2k$$

$$a_{k+1} = (k+1)^2$$

Hence Proved, By mathematical induction.

$\square$

4. [5 points] Prove or disprove

   1. Let n be a positive integer. Prove that if $3n^2 + 8$ is even, then n is even.

2. $(d \mid m) \wedge (d \mid n) \implies d \mid (k.m + l.n)$

Solution: Part 1: *Let n be a positive integer. Prove that if $3n^2 + 8$ is even, then n is even, using a proof by contraposition.*

Proof by Contraposition:

The contrapositive of the statement is: If n is odd, then $3n^2 + 8$ is odd.

Step 1: Assume n is odd.

Assume n is odd. Then n can be expressed as:

$$n = 2k + 1 \quad \text{for some integer } k.$$

Step 2: Substitute n into the expression $3n^2 + 8$.

Substituting $n = 2k + 1$ into the expression $3n^2 + 8$:

$$3n^2 + 8 = 3(2k + 1)^2 + 8$$

Expanding $(2k + 1)^2$:

$$(2k + 1)^2 = 4k^2 + 4k + 1$$

Substituting this back:

$$3n^2 + 8 = 3(4k^2 + 4k + 1) + 8 = 12k^2 + 12k + 3 + 8 = 12k^2 + 12k + 11$$

Step 3: Simplify the expression.

Notice that $12k^2 + 12k$ is even (since it is a multiple of 2), and adding 11 to it results in an odd number. Therefore:

$$3n^2 + 8 = \text{odd number}$$

This proves that if n is odd, then $3n^2 + 8$ is odd.

Step 4: Conclusion.

By contrapositive, which is logically equivalent to the original statement, we conclude that if $3n^2 + 8$ is even, then n must be even.

The reason we use the contrapositive is that sometimes it is easier to prove the contrapositive of a statement directly. Since the contrapositive is logically equivalent to the original proposition, proving the contrapositive is sufficient to prove the original statement.

$\square$

Part 2:

Prove that if $d \mid m$ and $d \mid n$, then $d \mid (k \cdot m + l \cdot n)$ for any integers k and l.

Proof:

Given:

$$d \mid m \quad \text{and} \quad d \mid n$$

This means:

$$m = d \cdot a \quad \text{for some integer } a$$

$$n = d \cdot b \quad \text{for some integer } b$$

We need to show that $d \mid (k \cdot m + l \cdot n)$ for any integers k and l.

Step 1: Substitute the expressions for m and n into $k \cdot m + l \cdot n$.

Substitute the expressions for m and n into $k \cdot m + l \cdot n$:

$$k \cdot m + l \cdot n = k \cdot (d \cdot a) + l \cdot (d \cdot b)$$

Step 2: Factor out d.

Factor out d:

$$k \cdot m + l \cdot n = d \cdot (k \cdot a + l \cdot b)$$

Step 3: Conclusion.

Since $k \cdot a + l \cdot b$ is an integer (because a and b are integers and k and l are integers), we have:

$$k \cdot m + l \cdot n = d \cdot (k \cdot a + l \cdot b)$$

This shows that $k \cdot m + l \cdot n$ is a multiple of d, which means:

$$d \mid (k \cdot m + l \cdot n)$$

Thus, we have proven that if $d \mid m$ and $d \mid n$, then $d \mid (k \cdot m + l \cdot n)$ for any integers k and l.

5. [4 points] Given two sets S and T, and $S - T = S \cap \bar{T}$, prove that $(S-T) \cup (T-S) = (S \cup T) - (S \cap T)$.

Solution:

Step 1: Express the LHS in terms of intersections and complements.

$$\text{LHS} = (S - T) \cup (T - S)$$

Using the definition of set difference:

$$S - T = S \cap \overline{T} \quad \text{and} \quad T - S = T \cap \overline{S}$$

So,

$$\text{LHS} = (S \cap \overline{T}) \cup (T \cap \overline{S})$$

Step 2: Expand the RHS using set theory identities.

The RHS is:

$$\text{RHS} = (S \cup T) - (S \cap T)$$

Using the definition of set difference:

$$(S \cup T) - (S \cap T) = (S \cup T) \cap \overline{(S \cap T)}$$

Using De Morgan's law, we know:

$$\overline{(S \cap T)} = \overline{S} \cup \overline{T}$$

So,

$$\text{RHS} = (S \cup T) \cap (\overline{S} \cup \overline{T})$$

Step 3: Distribute the intersection over the union on the RHS.

Using the distributive law:

$$(S \cup T) \cap (\overline{S} \cup \overline{T}) = \left[S \cap (\overline{S} \cup \overline{T})\right] \cup \left[T \cap (\overline{S} \cup \overline{T})\right]$$

Apply the distributive property inside the brackets:

$$S \cap (\overline{S} \cup \overline{T}) = (S \cap \overline{S}) \cup (S \cap \overline{T})$$

$$T \cap (\overline{S} \cup \overline{T}) = (T \cap \overline{S}) \cup (T \cap \overline{T})$$

Step 4: Simplify the expressions.

We know that $S \cap \overline{S} = \emptyset$ (a set and its complement are disjoint) and $T \cap \overline{T} = \emptyset$, so:

$$S \cap (\overline{S} \cup \overline{T}) = \emptyset \cup (S \cap \overline{T}) = S \cap \overline{T}$$

$$T \cap (\overline{S} \cup \overline{T}) = \emptyset \cup (T \cap \overline{S}) = T \cap \overline{S}$$

Thus, the RHS becomes:

$$\text{RHS} = (S \cap \overline{T}) \cup (T \cap \overline{S})$$

Step 5: Conclusion.

We observe that:

$$\text{LHS} = (S \cap \overline{T}) \cup (T \cap \overline{S}) = \text{RHS}$$

Therefore, we have proven that:

$$(S - T) \cup (T - S) = (S \cup T) - (S \cap T)$$