Quiz 1 Section A - Solutions

Discrete Structures Monsoon 2024, IIIT Hyderabad

1. [2 points] Determine which of the following Students : Rudransh, Pramod, Chinmay, or Bipasha is guilty of lying (there can be multiple liars). These are their testimonies

   - Rudransh said: *"If Pramod is saying the truth then so is Chinmay. Similarly, the converse is true as well."*.
   - Pramod said: *"If Bipasha is saying the truth then Rudransh is as well."*.
   - Chinmay said: *"Neither Bipasha nor Pramod are saying the truth"*.
   - Bipasha said: *"It is not the case that both Rudransh and Chinmay are lying."*.

   There is one witness, Nikhita, who is guaranteed to speak the truth. According to her: "*If Bipasha is saying the truth then so is Chinmay*". Figure out who is saying the truth and who isn't. Note that it may not be possible to arrive at a conclusion on some of the testimonies.

Solution: Let us denote Rudransh, Pramod, Chinmay, Bipasha stating the truth as R, P, C, and B respectively. We have the following statements:

- Rudransh: $P \iff C$
- Pramod: $B \Rightarrow R$
- Chinmay: $\neg B \wedge \neg P$
- Bipasha: $R \vee C$

We also have one confirmed true statement: $B \Rightarrow C$.

Start with assuming that Chinmay is stating the truth. That is, truth value of C is T. Then, according to their statement, the truth values of B and P are both F. But, looking at Bipasha's statement, we see that it is a true statement since the truth value of C is T which is a contradiction. So, the **truth value of C is always F.**

Now, to satisfy the confirmed true statement $B \Rightarrow C$, given that C is False, we require B **to also be False.**

Given that B is False, observe their statement. For it to be false, we require R **to be false as well**.

Similarly, observing P's statement - $B \Rightarrow R$, this is always True given that B is false. So, the **truth value of P is True**.

Therefore, Pramod saying the truth and all others are lying is the only possible scenario. (Other statements that weren't utilized may be verified to be consistent in this system).

2. [6 points] Decide whether the following formulas are equivalent. In cases where your answer is "not equivalent," you must give an explanation.

   (a) $\exists x(Q(x) \wedge P(x))$ and $\exists x Q(x) \wedge \exists x P(x)$
   (b) $\exists x(Q(x) \vee P(x))$ and $\exists x Q(x) \vee \exists x P(x)$
   (c) $\exists x(Q(x) \rightarrow P(x))$ and $\exists x Q(x) \rightarrow \exists x P(x)$

Solution:

(a) Not equivalent.
Say, $Q(x)$ is true for some $x = a$ and false everywhere else. Let $P(x) \equiv \neg Q(x)$. Now, LHS is false but RHS is true.

(b) Equivalent.

(c) Not equivalent.
Let $Q(x)$ is true for $x = a$ and false everywhere else. $P(x)$ is false always. Now, LHS is true for $x \neq a$. But, RHS is false.

3. [5 points] Prove or disprove

   1. For any integer n, the number $n^3 - n$ is even.
   2. For all positive integers k, m, n, where m and n are co-prime,

$$k \mid (m \cdot n) \implies (k \mid m) \vee (k \mid n),$$

   where $a \mid b$ means "a divides b".

Solution:

1. Direct Proof

To prove $n^3 - n$ is even, we need to show that $n^3 - n$ is divisible by 2.

$$n^3 - n = n(n^2 - 1)$$

$n^2 - 1$ can be factored as:

$$n^2 - 1 = (n - 1)(n + 1)$$

$$n^3 - n = n(n - 1)(n + 1)$$

The product $n(n-1)(n+1)$ is the product of three consecutive integers. In any three consecutive integers, at least one of them is even.

Need to validate this statement by taking n even and odd separately and mention which term is odd/ even.

Get the form y = 2m [where m = ...]

Hence, the product $n(n - 1)(n + 1)$ is even, meaning $n^3 - n$ is even.

2. Proof by Cases

We can also prove this by considering the parity (evenness or oddness) of n.

Case 1: n is even

If n is even, then $n = 2k$ for some integer k. Substituting:

$$n^3 - n = (2k)^3 - 2k = 8k^3 - 2k = 2(4k^3 - k)$$

Get the form y = 2m [where m = $4k^3 - k$] therefore $n^3 - n$ is even.

Case 2: n is odd

If n is odd, then $n = 2k + 1$ for some integer k. Substituting into the expression:

$$n^3 - n = (2k + 1)^3 - (2k + 1)$$

Expanding $(2k + 1)^3$:

$$(2k + 1)^3 = 8k^3 + 12k^2 + 6k + 1$$

So, [get this expression correct else 0 for this case]

$$n^3 - n = (8k^3 + 12k^2 + 6k + 1) - (2k + 1) = 8k^3 + 12k^2 + 4k$$

Get the form y = 2m [where m = ...]
This expression is even since it is divisible by 2.
In both cases, $n^3 - n$ is even.

3. Proof by Induction

Let's use mathematical induction to prove $n^3 - n$ is even for all integers n.

Base Case: $n = 1$

For $n = 1$:

$$1^3 - 1 = 0$$

0 is even, so the base case holds.

Inductive Step

Assume $n^3 - n$ is even for some integer $n = k$, i.e., $k^3 - k$ is even.
Now consider $n = k + 1$:

$$(k + 1)^3 - (k + 1)$$

Expanding $(k + 1)^3$:

$$(k + 1)^3 = k^3 + 3k^2 + 3k + 1$$

So,

$$(k+1)^3 - (k+1) = (k^3 + 3k^2 + 3k + 1) - (k+1) = k^3 - k + 3k^2 + 3k$$

Notice that:

$$(k+1)^3 - (k+1) = (k^3 - k) + 3k(k+1)$$

By the induction hypothesis, $k^3 - k$ is even, and $3k(k+1)$ is also even because $k(k+1)$ is the product of two consecutive integers, so one of them is even.

Thus, $(k+1)^3 - (k+1)$ is even.

By induction, $n^3 - n$ is even for all integers n.

4. Proof by Contradiction

Assume, for contradiction, that $n^3 - n$ is odd for some integer n. If $n^3 - n$ is odd, then n^3 and n must have different parities (one is even, the other is odd). However, for any integer n, both n and n^3 have the same parity (if n is even, n^3 is even; if n is odd, n^3 is odd).

Thus, the assumption that $n^3 - n$ is odd leads to a contradiction.

Hence, $n^3 - n$ must be even.

Solution: This is not true, take $k = 10$, $m = 2$ and $n = 5$. k divides mn but neither of m or n.

4. [3 points] Prove that for all non-negative integers n, 10 divides $n^5 - n$ using **mathematical induction** (Only proofs using induction would be considered).

Solution: Base Case : It is very easy to see that this holds for $n = 1$ and $n = 0$, since $n^5 - n$ evaluates to 0 for both of them.

Inductive Hypothesis : We shall use **strong induction**. Notice that

$$\begin{aligned}
(n+1)^5 - n &= ((n-1)+2)^5 - ((n-1)+2) \\
&= (n-1)^5 + 5 \cdot 2 \cdot (n-1)^4 + 10 \cdot 2^2 \cdot (n-1)^3 + 10 \cdot 2^3 \cdot (n-1)^2 + 5 \cdot 2^4 \cdot (n-1) + 2^5 \\
&\quad - (n-1) - 2 \\
&= \underbrace{(n-1)^5 - (n-1)}_{t_1} + \underbrace{10(\dots)}_{t_2} + \underbrace{30}_{t_3}
\end{aligned}$$

Notice that t_1 is divisible by virtue of the inductive hypothesis on $P(n-1)$, and t_2 and t_3 are obviously divisible since they are multiples of 10. Hence we managed to show that $P(n+1)$ is also true.

An important thing to note is that if you are using $P(n-1)$ being true to prove that $P(n+1)$ is true, then you should show for both base cases $n = 0$ and $n = 1$. This is because with a particular base case you would only be able to show that it is true for n with same odd-even parity as the base case (Base case $n = 1$ only shows that it works for odd n)

5. [4 points] Suppose that A, B, and C are sets such that $(A - C) \cup (C - A) = (B - C) \cup (C - B)$. Must it be the case that $A = B$?

Solution: Proof by contradiction: Assume $A \neq B$. Then, without loss of generality, there exists an element $x \in A$ such that $x \notin B$.

Now, if $x \in C$, then, $x \notin (A - C) \cup (C - A)$ but, $x \in (B - C) \cup (C - B)$.

On the other hand, if $x \notin C$, then, $x \in (A - C) \cup (C - A)$ but, $x \notin (B - C) \cup (C - B)$.

In both cases, we get LHS $\neq$ RHS. So, it must be the case that $A = B$.