THESIS

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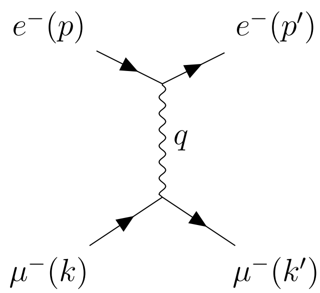
Ngày 16 tháng 10 năm 2024

1/ Electron-muon Processes

1.1/ $e^-\mu^- \to e^-\mu^-$



Dựa vào giản đồ, ta có biên độ tán xạ Feynman:

$$\begin{aligned} i\mathcal{M} &= \left[\overline{u}^{s2}(p')\left(-ie\gamma^\nu\right)u^{s1}(p)\right]\left(\frac{-ig_{\mu\nu}}{q^2}\right)\left[\overline{u}^{s4}(k')\left(-ie\gamma^\mu\right)u^{s3}(k)\right] \\ &= \frac{ie^2}{q^2}\left[\overline{u}^{s2}(p')\gamma^\nu u^{s1}(p)\right]\left[\overline{u}^{s4}(k')\gamma_\nu u^{s3}(k)\right] \qquad (1) \end{aligned}$$

Lấy liên hợp phức ta được:

$$-i\mathcal{M}^* = \frac{-ie^2}{q^2}\left[\overline{u}^{s3}(k)\gamma_\nu u^{s4}(k')\right]\left[\overline{u}^{s1}(p)\gamma^\nu u^{s2}(p')\right] \qquad (2)$$

Nhân (1) và (2) lại, ta được:

$$\begin{aligned} |\mathcal{M}|^2 &= \frac{e^4}{\left(q^2\right)^2}\left[\overline{u}^{s2}(p')\gamma^\nu u^{s1}(p)\right]\left[\overline{u}^{s4}(k')\gamma_\nu u(k)^{s3}\right]\left[\overline{u}^{s3}(k)\gamma_\mu u^{s4}(k')\right]\left[\overline{u}^{s1}(p)\gamma^\mu u^{s2}(p')\right] \\ &= \frac{e^4}{\left(q^2\right)^2}\left[\overline{u}^{s1}(p)\gamma^\mu u^{s2}(p')\right]\left[\overline{u}^{s2}(p')\gamma^\nu u^{s1}(p)\right]\left[\overline{u}^{s4}(k')\gamma_\nu u(k)^{s3}\right]\left[\overline{u}^{s3}(k)\gamma_\mu u^{s4}(k')\right] \end{aligned}$$

Áp dụng công thức:

$$\sum_s u_p^s\overline{u}_p^s = \not{p} + m$$

Đồng thời lấy trung bình spin, ta được:

$$\begin{aligned} \frac{1}{4}\sum_{spin}|\mathcal{M}|^2 &= \frac{1}{4}\sum_{s1,s4}\frac{e^4}{q^4}\left[\overline{u}^{s1}(p)\gamma^\mu\left(\not{p}'+m_e\right)\gamma^\nu u^{s1}(p)\right]\left[\overline{u}^{s4}(k')\gamma_\nu\left(\not{k}+m_\mu\right)\gamma_\mu u^{s4}(k')\right] \\ &= \sum_{s1,s4}\frac{e^4}{4q^4}\sum_{\alpha,\beta}\left(\overline{u}_\alpha^{s1}(p)\gamma^\mu_{\alpha\beta}\left(\not{p}'+m_e\right)_{\beta\alpha}\gamma^\nu_{\alpha\beta}u_\beta^{s1}(p)\right)\sum_{a,b}\left[\overline{u}_a^{s4}(k')\gamma_{\nu_{ab}}\left(\not{k}+m_\mu\right)_{ba}\gamma_{\mu_{ab}}u_b^{s4}(k')\right] \\ &= \sum_{s1,s4}\frac{e^4}{4q^4}\sum_{\alpha,\beta}\left[u_\beta^{s1}(p)\overline{u}_\alpha^{s1}(p)\gamma^\mu_{\alpha\beta}\left(\not{p}'+m_e\right)_{\beta\alpha}\gamma^\nu_{\alpha\beta}\right]\sum_{a,b}\left[u_b^{s4}(k')\overline{u}_a^{s4}(k')\gamma_{\nu_{ab}}\left(\not{k}+m_\mu\right)_{ba}\gamma_{\mu_{ab}}\right] \\ &= \frac{e^4}{4q^4}\mathrm{Tr}\left[\left(\not{p}+m_e\right)\gamma^\mu\left(\not{p}'+m_e\right)\gamma^\nu\right]\mathrm{Tr}\left[\left(\not{k}'+m_\mu\right)\gamma_\nu\left(\not{k}+m_\mu\right)\gamma_\mu\right] \end{aligned}$$

Để đơn giản, ta cho $m_e = m_\mu \to 0$. Phương trình trở thành:

$$\frac{1}{4}\sum_{spin}|\mathcal{M}|^2 = \frac{e^4}{4q^4}\text{Tr}\left[\not{p}\gamma^\mu\not{p}'\gamma^\nu\right]\text{Tr}\left[\not{k}'\gamma_\nu\not{k}\gamma_\mu\right] = \frac{1}{4}\text{Tr}\left[p_\rho\gamma^\rho\gamma^\mu p'_\sigma\gamma^\sigma\gamma^\nu\right]\left[k'^\rho\gamma_\rho\gamma_\nu k^\sigma\gamma_\sigma\gamma_\mu\right]$$

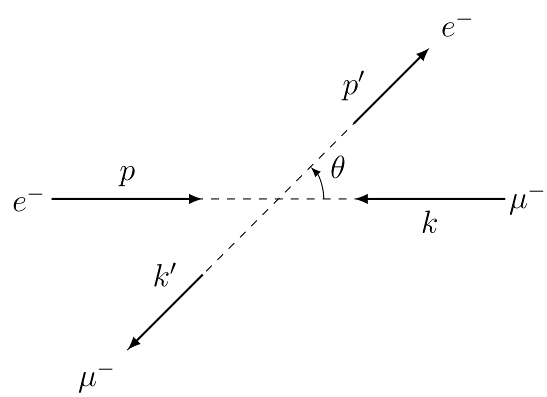
Tiếp tục, ta áp dụng công thức:

$$\text{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right] = 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)$$

Phương trình trở thành:

$$\begin{aligned}
\frac{1}{4}\sum_{spin}|\mathcal{M}|^2 &= \frac{e^4}{4q^4}4\left(p_\rho p'_\sigma\right)\left(g^{\rho\mu}g^{\sigma\nu} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\nu}g^{\mu\sigma}\right)4\left(k'^\rho k^\sigma\right)\left(g_{\rho\nu}g_{\sigma\mu} - g_{\rho\sigma}g_{\nu\mu} + g_{\rho\mu}g_{\nu\sigma}\right) \\
&= \frac{4e^4}{q^4}\left[p^\mu p'^\nu + p^\nu p'^\mu - g^{\mu\nu}(p\cdot p')\right]\left[k'_\nu k_\mu + k'_\mu k_\nu - g_{\mu\nu}(k'\cdot k)\right] \\
&= \frac{8e^4}{q^4}\left[(p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k)\right] \qquad (3)
\end{aligned}$$

Tính tiết diện tán xạ trong hệ quy chiếu khối tâm của 2 hạt tới



Với:

$$\begin{aligned}
p &= (E;\ 0;\ 0;\ p_z) \\
k &= (E;\ 0;\ 0;\ -p_z) \\
p' &= (E;\ 0;\ |p'|\sin\theta;\ |p'|\cos\theta) \\
k' &= (E;\ 0;\ -|k'|\sin\theta';\ -|k'|\cos\theta) \\
=>& |p'| = |k'| = E
\end{aligned}$$

Center of mass $(\vec{p}+\vec{k}) => S = (p+k)^2 = 4E^2$
Lại có $p^2 = E^2 - \vec{p_z}^2 = m_e^2 = 0$. Do đó, $|p_z| = E$
Từ đó suy ra các thành phần có dạng:

$$\begin{aligned}
p &= (E;\ 0;\ 0;\ E) \\
k &= (E;\ 0;\ 0;\ -E) \\
p' &= (E;\ 0;\ E\sin\theta;\ E\cos\theta)
\end{aligned}$$

$$k' = (E;\ 0;\ -E\sin\theta;\ -E\cos\theta)$$

Để tính tiết diện tán xạ, ta cần:

$$\begin{cases} p\cdot k = E^2 + E^2 = 2E^2 \\ p'\cdot k' = E^2 + E^2 = 2E^2 \\ p\cdot k' = E^2 + E^2\cos\theta = E^2(1+\cos\theta) \\ p'\cdot k = E^2 + E^2\cos\theta = E^2(1+\cos\theta) \\ q^2 = -2p\cdot p' = -2E^2(1-\cos\theta) \end{cases}$$

Thay vào (3), ta được:

$$\begin{aligned} \frac{1}{4}\sum_{spin}|\mathcal{M}|^2 &= \frac{8e^4}{4E^4(1-\cos\theta)^2}\left[2E^2 2E^2 + E^2(1+\cos\theta)E^2(1+\cos\theta)\right] \\ &= \frac{8e^4}{4E^4(1-\cos\theta)^2}E^4\left[4+(1+\cos\theta)^2\right] \\ &= \frac{2e^4}{(1-\cos\theta)^2}\left[4+(1+\cos\theta)^2\right] \end{aligned}$$

Thay vào công thức tính tiết diện tán xạ vi phân:

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2E^2}\frac{1}{4}\sum_{spin}|\mathcal{M}|^2 \\ &= \frac{1}{64\pi^2E^2}\frac{2e^2}{(1-\cos\theta)^2}\left[4+(1+\cos\theta)^2\right] \end{aligned}$$

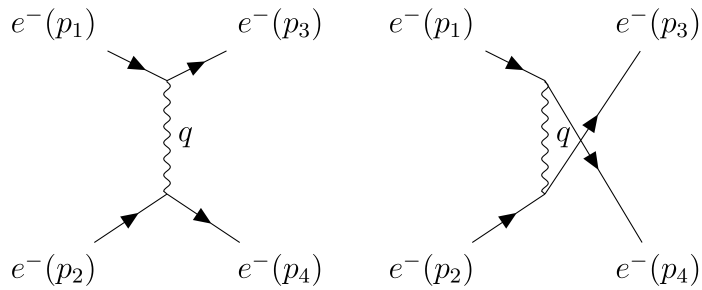
Đặt $\alpha = \dfrac{e^2}{4\pi}$, ta rút gọn biểu thức trên thành:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2E^2(1-\cos\theta)^2}\left[4+(1+\cos\theta)^2\right]$$

Cuối cùng, ta thu được tiết diện tán xạ có dạng:

$$\begin{aligned} \sigma_{total} = \int d\Omega\frac{d\sigma}{d\Omega} &= \int_0^{2\pi}\int_0^{\pi}\frac{\alpha^2}{2E^2(1-\cos\theta)^2}\left[4+(1+\cos\theta)^2\right]\sin\theta d\theta d\varphi \\ &= \frac{\alpha^2}{2E^2}2\pi\left[\int_{-1}^{1}\frac{4}{(1-\cos\theta)^2}d\cos\theta + \int_{-1}^{1}\frac{(1+\cos\theta)^2}{(1-\cos\theta)^2}d\cos\theta\right] \end{aligned}$$

1.2/ Møller Scattering



Kênh t Kênh u

Dựa vào giản đồ, ta có biên độ tán xạ Feynman:

$$
\begin{aligned}
i\mathcal{M} &= i(\mathcal{M}_t - \mathcal{M}_u) \\
&= \left[\overline{u}^{s3}(p_3)\left(-ie\gamma^{\nu}\right)u^{s1}(p_1)\right]\left(\frac{-ig_{\mu\nu}}{q^2}\right)\left[\overline{u}^{s4}(p_4)\left(-ie\gamma^{\mu}\right)u^{s2}(p_2)\right] \\
&- \left[\overline{u}^{s4}(p_4)\left(-ie\gamma^{\nu}\right)u^{s1}(p_1)\right]\left(\frac{-ig_{\mu\nu}}{q'^2}\right)\left[\overline{u}^{s3}(p_3)\left(-ie\gamma^{\mu}\right)u^{s2}(p_2)\right] \\
&= \frac{ie^2}{q^2}\left[\overline{u}^{s3}(p_3)\gamma^{\nu}u^{s1}(p_1)\right]\left[\overline{u}^{s4}(p_4)\gamma_{\nu}u^{s2}(p_2)\right] - \frac{ie^2}{q'^2}\left[\overline{u}^{s4}(p_4)\gamma^{\nu}u^{s1}(p_1)\right]\left[\overline{u}^{s3}(p_3)\gamma_{\nu}u^{s2}(p_2)\right] \\
&= ie^2\left\{\left[\overline{u}^{s3}(p_3)\gamma^{\nu}u^{s1}(p_1)\right]\left[\overline{u}^{s4}(p_4)\gamma_{\nu}u^{s2}(p_2)\right]\frac{1}{q^2} - \left[\overline{u}^{s4}(p_4)\gamma^{\nu}u^{s1}(p_1)\right]\left[\overline{u}^{s3}(p_3)\gamma_{\nu}u^{s2}(p_2)\right]\frac{1}{q'^2}\right\}
\end{aligned} \tag{4}
$$

Lấy liên hợp phức, ta được:

$$
-i\mathcal{M}^* = -ie^2\left\{\frac{1}{q^2}\left[\overline{u}^{s2}(p_2)\gamma_{\nu}u^{s4}(p_4)\right]\left[\overline{u}^{s1}(p_1)\gamma^{\nu}u^{s3}(p_3)\right] - \frac{1}{q'^2}\left[\overline{u}^{s2}(p_2)\gamma_{\nu}u^{s3}(p_3)\right]\left[\overline{u}^{s1}(p_1)\gamma^{\nu}u^{s4}(p_4)\right]\right\}
$$

Nhân (4) với liên hợp phức của nó, ta được:

$$
\begin{aligned}
|\mathcal{M}|^2 &= \frac{e^4}{q^4}\left[\overline{u}^{s3}(p_3)\gamma^{\nu}u^{s1}(p_1)\right]\left[\overline{u}^{s4}(p_4)\gamma_{\nu}u^{s2}(p_2)\right]\left[\overline{u}^{s2}(p_2)\gamma_{\mu}u^{s4}(p_4)\right]\left[\overline{u}^{s1}(p_1)\gamma^{\mu}u^{s3}(p_3)\right] \\
&+ \frac{e^4}{q'^4}\left[\overline{u}^{s4}(p_4)\gamma^{\nu}u^{s1}(p_1)\right]\left[\overline{u}^{s3}(p_3)\gamma_{\nu}u^{s2}(p_2)\right]\left[\overline{u}^{s2}(p_2)\gamma_{\mu}u^{s3}(p_3)\right]\left[\overline{u}^{s1}(p_1)\gamma^{\mu}u^{s4}(p_4)\right] \\
&- \frac{e^4}{q^2q'^2}\left[\overline{u}^{s3}(p_3)\gamma^{\nu}u^{s1}(p_1)\right]\left[\overline{u}^{s4}(p_4)\gamma_{\nu}u^{s2}(p_2)\right]\left[\overline{u}^{s2}(p_2)\gamma_{\mu}u^{s3}(p_3)\right]\left[\overline{u}^{s1}(p_1)\gamma^{\mu}u^{s4}(p_4)\right] \\
&- \frac{e^4}{q'^2q^2}\left[\overline{u}^{s4}(p_4)\gamma^{\nu}u^{s1}(p_1)\right]\left[\overline{u}^{s3}(p_3)\gamma_{\nu}u^{s2}(p_2)\right]\left[\overline{u}^{s2}(p_2)\gamma_{\mu}u^{s4}(p_4)\right]\left[\overline{u}^{s1}(p_1)\gamma^{\mu}u^{s3}(p_3)\right] \\
&= \frac{e^4}{q^4}\left[\overline{u}^{s1}(p_1)\gamma^{\mu}u^{s3}(p_3)\right]\left[\overline{u}^{s3}(p_3)\gamma^{\nu}u^{s1}(p_1)\right]\left[\overline{u}^{s4}(p_4)\gamma_{\nu}u^{s2}(p_2)\right]\left[\overline{u}^{s2}(p_2)\gamma_{\mu}u^{s4}(p_4)\right] \\
&+ \frac{e^4}{q'^4}\left[\overline{u}^{s1}(p_1)\gamma^{\mu}u^{s4}(p_4)\right]\left[\overline{u}^{s4}(p_4)\gamma^{\nu}u^{s1}(p_1)\right]\left[\overline{u}^{s3}(p_3)\gamma_{\nu}u^{s2}(p_2)\right]\left[\overline{u}^{s2}(p_2)\gamma_{\mu}u^{s3}(p_3)\right] \\
&- \frac{e^4}{q^2q'^2}\left[\overline{u}^{s3}(p_3)\gamma^{\nu}u^{s1}(p_1)\right]\left[\overline{u}^{s1}(p_1)\gamma^{\mu}u^{s4}(p_4)\right]\left[\overline{u}^{s4}(p_4)\gamma_{\nu}u^{s2}(p_2)\right]\left[\overline{u}^{s2}(p_2)\gamma_{\mu}u^{s3}(p_3)\right]
\end{aligned}
$$

$$- \frac{e^4}{q'^2 q^2} \left[\overline{u}^{s4}(p_4)\gamma^\nu u^{s1}(p_1)\right] \left[\overline{u}^{s1}(p_1)\gamma^\mu u^{s3}(p_3)\right] \left[\overline{u}^{s3}(p_3)\gamma_\nu u^{s2}(p_2)\right] \left[\overline{u}^{s2}(p_2)\gamma_\mu u^{s4}(p_4)\right]$$

Áp dụng công thức:

$$\sum_s u_p^s \overline{u}_p^s = \not{p} + m$$

Đồng thời lấy trung bình spin, ta được:

$$\begin{aligned}
\frac{1}{4}\sum_{spin}|\mathcal{M}|^2 &= \frac{1}{4}\sum_{s1,s4}\frac{e^4}{q^4}\left[\overline{u}^{s1}(p_1)\gamma^\mu\left(\not{p}_3+m_e\right)\gamma^\nu u^{s1}(p_1)\right]\left[\overline{u}^{s4}(p_4)\gamma_\nu\left(\not{p}_2+m_e\right)\gamma_\mu u^{s4}(p_4)\right] \\
&+ \frac{1}{4}\sum_{s1,s3}\frac{e^4}{q'^4}\left[\overline{u}^{s1}(p_1)\gamma^\mu\left(\not{p}_4+m_e\right)\gamma^\nu u^{s1}(p_1)\right]\left[\overline{u}^{s3}(p_3)\gamma_\nu\left(\not{p}_2+m_e\right)\gamma_\mu u^{s3}(p_3)\right] \\
&- \frac{1}{4}\sum_{s3}\frac{e^4}{q^2q'^2}\left[\overline{u}^{s3}(p_3)\gamma^\nu\left(\not{p}_1+m_e\right)\gamma^\mu\left(\not{p}_4+m_e\right)\gamma_\nu\left(\not{p}_2+m_e\right)\gamma_\mu u^{s3}(p_3)\right] \\
&- \frac{1}{4}\sum_{s3}\frac{e^4}{q'^2q^2}\left[\overline{u}^{s4}(p_4)\gamma^\nu\left(\not{p}_1+m_e\right)\gamma^\mu\left(\not{p}_3+m_e\right)\gamma_\nu\left(\not{p}_2+m_e\right)\gamma_\mu u^{s4}(p_4)\right] \\
&= \sum_{s1,s4}\frac{e^4}{4q^4}\sum_{\alpha,\beta}\left[\overline{u}_\alpha^{s1}(p_1)\gamma^\mu_{\alpha\beta}\left(\not{p}_3+m_e\right)_{\beta\alpha}\gamma^\nu_{\alpha\beta}u_\beta^{s1}(p_1)\right]\sum_{a,b}\left[\overline{u}_a^{s4}(p_4)\gamma_{\nu_{ab}}\left(\not{p}_2+m_e\right)_{ba}\gamma_{\mu_{ab}}u_b^{s4}(p_4)\right] \\
&+ \sum_{s1,s3}\frac{e^4}{4q'^4}\sum_{\alpha,\beta}\left[\overline{u}_\alpha^{s1}(p_1)\gamma^\mu_{\alpha\beta}\left(\not{p}_4+m_e\right)_{\beta\alpha}\gamma^\nu_{\alpha\beta}u_\beta^{s1}(p_1)\right]\sum_{a,b}\left[\overline{u}_a^{s3}(p_3)\gamma_{\nu_{ab}}\left(\not{p}_2+m_e\right)_{ba}\gamma_{\mu_{ab}}u_b^{s3}(p_3)\right] \\
&- \frac{1}{4}\sum_{s3}\frac{e^4}{q^2q'^2}\sum_{\alpha,\beta}\left[\overline{u}_\alpha^{s3}\gamma^\nu_{\alpha\beta}\left(\not{p}_1+m_e\right)_{\beta\alpha}\gamma^\mu_{\alpha\beta}\left(\not{p}_4+m_e\right)_{\beta\alpha}\gamma_{\nu_{\alpha\beta}}\left(\not{p}_2+m_e\right)_{\beta\alpha}\gamma_{\mu_{\alpha\beta}}u^{s3}(p_3)_\beta\right] \\
&- \frac{1}{4}\sum_{s3}\frac{e^4}{q'^2q^2}\sum_{\alpha,\beta}\left[\overline{u}^{s4}(p_4)_\alpha\gamma^\nu_{\alpha\beta}\left(\not{p}_1+m_e\right)_{\beta\alpha}\gamma^\mu_{\alpha\beta}\left(\not{p}_3+m_e\right)_{\beta\alpha}\gamma_{\nu_{\alpha\beta}}\left(\not{p}_2+m_e\right)_{\beta\alpha}\gamma_{\mu_{\alpha\beta}}u^{s4}(p_4)_\beta\right] \\
&= \sum_{s1,s4}\frac{e^4}{4q^4}\sum_{\alpha,\beta}\left[u_\beta^{s1}(p_1)\overline{u}_\alpha^{s1}(p_1)\gamma^\mu_{\alpha\beta}\left(\not{p}_3+m_e\right)_{\beta\alpha}\gamma^\nu_{\alpha\beta}\right]\sum_{a,b}\left[u_b^{s4}(p_4)\overline{u}_a^{s4}(p_4)\gamma_{\nu_{ab}}\left(\not{p}_2+m_e\right)_{ba}\gamma_{\mu_{ab}}\right] \\
&+ \sum_{s1,s3}\frac{e^4}{4q'^4}\sum_{\alpha,\beta}\left[u_\beta^{s1}(p_1)\overline{u}_\alpha^{s1}(p_1)\gamma^\mu_{\alpha\beta}\left(\not{p}_4+m_e\right)_{\beta\alpha}\gamma^\nu_{\alpha\beta}\right]\sum_{a,b}\left[u_b^{s3}(p_3)\overline{u}_a^{s3}(p_3)\gamma_{\nu_{ab}}\left(\not{p}_2+m_e\right)_{ba}\gamma_{\mu_{ab}}\right] \\
&- \frac{1}{4}\sum_{s3}\frac{e^4}{q^2q'^2}\sum_{\alpha,\beta}\left[u^{s3}(p_3)_\beta\overline{u}_\alpha^{s3}\gamma^\nu_{\alpha\beta}\left(\not{p}_1+m_e\right)_{\beta\alpha}\gamma^\mu_{\alpha\beta}\left(\not{p}_4+m_e\right)_{\beta\alpha}\gamma_{\nu_{\alpha\beta}}\left(\not{p}_2+m_e\right)_{\beta\alpha}\gamma_{\mu_{\alpha\beta}}\right] \\
&- \frac{1}{4}\sum_{s3}\frac{e^4}{q'^2q^2}\sum_{\alpha,\beta}\left[u^{s4}(p_4)_\beta\overline{u}^{s4}(p_4)_\alpha\gamma^\nu_{\alpha\beta}\left(\not{p}_1+m_e\right)_{\beta\alpha}\gamma^\mu_{\alpha\beta}\left(\not{p}_3+m_e\right)_{\beta\alpha}\gamma_{\nu_{\alpha\beta}}\left(\not{p}_2+m_e\right)_{\beta\alpha}\gamma_{\mu_{\alpha\beta}}\right] \\
&= \frac{e^4}{4q^4}\mathrm{Tr}\left[\left(\not{p}_1+m_e\right)\gamma^\mu\left(\not{p}_3+m_e\right)\gamma^\nu\right]\mathrm{Tr}\left[\left(\not{p}_4+m_e\right)\gamma_\nu\left(\not{p}_2+m_e\right)\gamma_\mu\right] \\
&+ \frac{e^4}{4q'^4}\mathrm{Tr}\left[\left(\not{p}_1+m_e\right)\gamma^\mu\left(\not{p}_4+m_e\right)\gamma^\nu\right]\mathrm{Tr}\left[\left(\not{p}_3+m_e\right)\gamma_\nu\left(\not{p}_2+m_e\right)\gamma_\mu\right] \\
&- \frac{e^4}{2q^2q'^2}\mathrm{Tr}\left[\left(\not{p}_3+m_e\right)\gamma^\nu\left(\not{p}_1+m_e\right)\gamma^\mu\left(\not{p}_4+m_e\right)\gamma_\nu\left(\not{p}_2+m_e\right)\gamma_\mu\right]
\end{aligned}$$

Ở đây, ta cho $m_e = 0$, phương trình còn lại:

$$
\begin{aligned}
\frac{1}{4}\sum_{spin}|\mathcal{M}|^2 &= \frac{e^4}{4q^4}\text{Tr}\left[\not{p}_1\gamma^\mu\not{p}_3\gamma^\nu\right]\text{Tr}\left[\not{p}_4\gamma_\nu\not{p}_2\gamma_\mu\right] + \frac{e^4}{4q'^4}\text{Tr}\left[\not{p}_1\gamma^\mu\not{p}_4\gamma^\nu\right]\text{Tr}\left[\not{p}_3\gamma_\nu\not{p}_2\gamma_\mu\right] \\
&\quad - \frac{e^4}{2q^2q'^2}\text{Tr}\left[\not{p}_3\gamma^\nu\not{p}_1\gamma^\mu\not{p}_4\gamma_\nu\not{p}_2\gamma_\mu\right] \\
&= \frac{e^4}{4q^4}\text{Tr}\left[p_{1\rho}\gamma^\rho\gamma^\mu p_{3\sigma}\gamma^\sigma\gamma^\nu\right]\left[p_4^\rho\gamma_\rho\gamma_\nu p_2^\sigma\gamma_\sigma\gamma_\mu\right] + \frac{e^4}{4q'^4}\text{Tr}\left[p_{1\rho}\gamma^\rho\gamma^\mu p_{4\sigma}\gamma^\sigma\gamma^\nu\right]\left[p_3^\rho\gamma_\rho\gamma_\nu p_2^\sigma\gamma_\sigma\gamma_\mu\right] \\
&\quad - \frac{e^4}{2q^2q'^2}\text{Tr}\left[p_{3\rho}\gamma^\rho\gamma^\nu p_{1\sigma}\gamma^\sigma\gamma^\mu p_{4\lambda}\gamma^\lambda\gamma_\nu p_{2\tau}\gamma^\tau\gamma_\mu\right]
\end{aligned}
$$

Tiếp tục, ta áp dụng công thức:

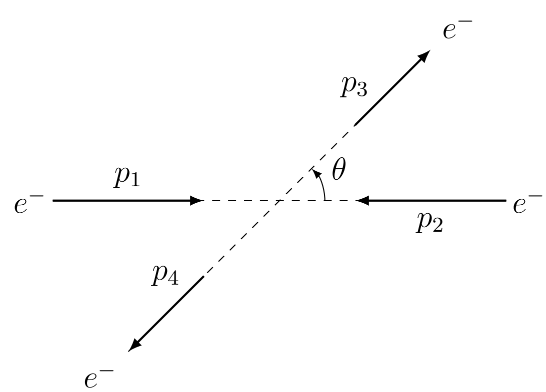
$$
\text{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right] = 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)
$$

$$
\text{Tr}\left[\gamma^\rho\gamma^\nu\gamma^\sigma\gamma^\mu\gamma^\lambda\gamma_\nu\gamma^\tau\gamma_\mu\right] = -2\text{Tr}\left[\gamma^\rho\gamma^\lambda\gamma^\mu\gamma^\sigma\gamma^\tau\gamma_\mu\right] = -8g^{\sigma\tau}\text{Tr}\left[\gamma^\rho\gamma^\lambda\right] = -32g^{\sigma\tau}g^{\rho\lambda}
$$

Phương trình trở thành:

$$
\begin{aligned}
\frac{1}{4}\sum_{spin}|\mathcal{M}|^2 &= \frac{e^4}{4q^4}\left(p_{1\rho}p_{3\sigma}\right)4\left(g^{\rho\mu}g^{\sigma\nu} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\nu}g^{\mu\sigma}\right)\left(p_4^\rho p_2^\sigma\right)4\left(g_{\rho\nu}g_{\sigma\mu} - g_{\rho\sigma}g_{\mu\nu} + g_{\rho\mu}g_{\nu\sigma}\right) \\
&\quad + \frac{e^4}{4q'^4}\left(p_{1\rho}p_{4\sigma}\right)4\left(g^{\rho\mu}g^{\sigma\nu} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\nu}g^{\mu\sigma}\right)\left(p_3^\rho p_2^\sigma\right)4\left(g_{\rho\nu}g_{\sigma\mu} - g_{\rho\sigma}g_{\mu\nu} + g_{\rho\mu}g_{\nu\sigma}\right) \\
&\quad + \frac{e^4}{2q^2q'^2}\left(p_{3\rho}p_{1\sigma}p_{4\lambda}p_{2\tau}\right)32g^{\sigma\tau}g^{\rho\lambda} \\
&= \frac{4e^4}{q^4}\left[p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - g^{\mu\nu}\left(p_1\cdot p_3\right)\right]\left[p_{4\nu}p_{2\mu} + p_{4\mu}p_{2\nu} - g_{\mu\nu}\left(p_4\cdot p_2\right)\right] \\
&\quad + \frac{4e^4}{q'^4}\left[p_1^\mu p_4^\nu + p_1^\nu p_4^\mu - g^{\mu\nu}\left(p_1\cdot p_4\right)\right]\left[p_{3\nu}p_{2\mu} + p_{3\mu}p_{2\nu} - g_{\mu\nu}\left(p_3\cdot p_2\right)\right] \\
&\quad + \frac{16e^4}{q^2q'^2}\left(p_1\cdot p_2\cdot\right)\left(p_3\cdot p_4\right) \\
&= \frac{8e^4}{q^4}\left[\left(p_1\cdot p_2\right)\left(p_3\cdot p_4\right) + \left(p_1\cdot p_4\right)\left(p_2\cdot p_3\right)\right] + \frac{8e^4}{q'^4}\left[\left(p_1\cdot p_2\right)\left(p_4\cdot p_3\right) + \left(p_1\cdot p_3\right)\left(p_2\cdot p_4\right)\right] \\
&\quad + \frac{16e^4}{q^2q'^2}\left(p_1\cdot p_2\cdot\right)\left(p_3\cdot p_4\right)
\end{aligned}
$$

Tính tiết diện tán xạ trong hệ quy chiếu khối tâm của 2 hạt tới



Với:

$$
\begin{aligned}
p_1 &= (E;\ 0;\ 0;\ p_z) \\
p_2 &= (E;\ 0;\ 0;\ -p_z) \\
p_3 &= (E;\ 0;\ |p_3|\sin\theta;\ |p_3|\cos\theta) \\
p_4 &= (E;\ 0;\ -|p_4|\sin\theta';\ -|p_4|\cos\theta) \\
=>|p_3| &= |p_4| = E
\end{aligned}
$$

Center of mass => $s = (p_1 + p_2)^2 = (p_3 + p_4)^2 = 4E^2$
Lại có $p^2 = E^2 - \vec{p_z}^2 = m_e^2 = 0$. Do đó, $|p_z| = E$
Từ đó suy ra các thành phần có dạng:

$$
\begin{aligned}
p_1 &= (E;\ 0;\ 0;\ E) \\
p_2 &= (E;\ 0;\ 0;\ -E) \\
p_3 &= (E;\ 0;\ E\sin\theta;\ E\cos\theta) \\
p_4 &= (E;\ 0;\ -E\sin\theta;\ -E\cos\theta)
\end{aligned}
$$

Để tính tiết diện tán xạ, ta cần:

$$
\begin{cases}
p_1 \cdot p_2 = E^2 + E^2 = 2E^2 \\
p_3 \cdot p_4 = E^2 + E^2 = 2E^2 \\
p_1 \cdot p_4 = E^2 + E^2\cos\theta = E^2(1+\cos\theta) \\
p_2 \cdot p_3 = E^2 + E^2\cos\theta = E^2(1+\cos\theta) \\
p_1 \cdot p_3 = E^2 - E^2\cos\theta = E^2(1-\cos\theta) \\
p_2 \cdot p_4 = E^2 - E^2\cos\theta = E^2(1-\cos\theta) \\
q^2 = (p_3 - p_1)^2 \to q^2 = p_3^2 + p_1^2 - 2p_1 \cdot p_3 = -2E^2(1-\cos\theta) \\
q'^2 = (p_4 - p_1)^2 \to q^2 = p_4^2 + p_1^2 - 2p_1 \cdot p_4 = -2E^2(1+\cos\theta)
\end{cases}
$$

Từ đó, biên độ tán xạ Feynman trở thành:

$$
\begin{aligned}
\frac{1}{4}\sum_{spin}|\mathcal{M}|^2 &= \frac{8e^4}{q^4}\left[(p_1\cdot p_2)^2 + (p_1\cdot p_4)^2\right] + \frac{8e^4}{q'^4}\left[(p_1\cdot p_2)^2 + (p_1\cdot p_3)^2\right] + \frac{16e^4}{q^2q'^2}(p_1\cdot p_2)^2 \\
&= \frac{8e^4}{4E^2(1-\cos\theta)^2}\left[4E^4 + E^4(1+\cos\theta)^2\right] + \frac{8e^4}{4E^2(1+\cos\theta)^2}\left[4E^4 + E^4(1-\cos\theta)^2\right] \\
&\quad + \frac{16e^4}{4E^4\left(1-\cos^2\theta\right)}4E^4 \\
&= \frac{8e^4}{4E^2(1-\cos\theta)^2}E^4\left[1 + (1+\cos\theta)^2\right] + \frac{8e^4}{4E^2(1+\cos\theta)^2}E^4\left[4 + (1-\cos\theta)^2\right] + \frac{16e^4}{\sin^2\theta} \\
&= \frac{2e^4}{(1-\cos\theta)^2}\left[4 + (1+\cos\theta)^2\right] + \frac{2e^4}{(1+\cos\theta)^2}\left[4 + (1-\cos\theta)^2\right] + \frac{16e^4}{\sin^2\theta}
\end{aligned}
$$

Thay vào công thức tính tiết diện tán xạ vi phân:

$$
\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2E^2}\frac{1}{4}\sum_{spin}|\mathcal{M}|^2
$$

$$= \frac{1}{64\pi^2 E^2}\left\{\frac{2e^4}{(1-\cos\theta)^2}\left[4+(1+\cos\theta)^2\right]+\frac{2e^4}{(1+\cos\theta)^2}\left[4+(1-\cos\theta)^2\right]+\frac{16e^4}{\sin^2\theta}\right\}$$
$$= \frac{e^4}{32\pi^2 E^2 (1-\cos\theta)^2}\left[4+(1+\cos\theta)^2\right]+\frac{e^4}{32\pi^2 E^2 (1+\cos\theta)^2}\left[4+(1-\cos\theta)^2\right]+\frac{e^4}{4\pi^2 E^2 \sin^2\theta}$$

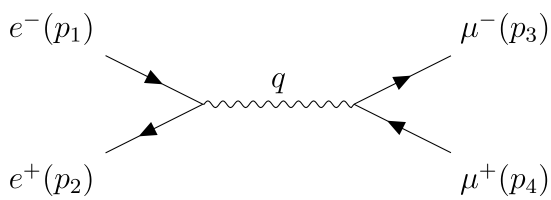
Đặt $\alpha = \dfrac{e^2}{4\pi}$ Tiết diện tán xạ vi phân được rút gọn thành:

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{\alpha^2}{2E^2}\left[\frac{4+(1+\cos\theta)^2}{(1-\cos\theta)^2}+\frac{4+(1-\cos\theta)^2}{(1+\cos\theta)^2}+\frac{8}{\sin^2\theta}\right] \\
&= \frac{\alpha^2}{2E^2}\left[4\left(\frac{1}{(1-\cos\theta)^2}+\frac{1}{(1+\cos\theta)^2}\right)+\frac{8}{\sin^2\theta}+\frac{(1+\cos\theta)^2}{(1-\cos\theta)^2}+\frac{(1-\cos\theta)^2}{(1+\cos\theta)^2}\right] \\
&= \frac{\alpha^2}{2E^2}\left[\frac{16}{\sin^4\theta}-\frac{8}{\sin^2\theta}+\frac{8}{\sin^2\theta}+\frac{(1+\cos\theta)^2}{(1-\cos\theta)^2}+\frac{(1-\cos\theta)^2}{(1+\cos\theta)^2}\right] \\
&= \frac{\alpha^2}{2E^2}\left[\frac{16}{\sin^4\theta}+\frac{(1+\cos\theta)^2}{(1-\cos\theta)^2}+\frac{(1-\cos\theta)^2}{(1+\cos\theta)^2}\right] \\
&= \frac{\alpha^2}{2E^2}\left[\frac{16}{\sin^4\theta}+\frac{16}{\sin^4\theta}-\frac{16}{\sin^2\theta}+2\right] \\
&= \frac{\alpha^2}{2E^2}\left[\frac{32}{\sin^4\theta}-\frac{16}{\sin^2\theta}+2\right] \\
&= \frac{\alpha^2}{E^2}\left[\frac{16}{\sin^4\theta}-\frac{8}{\sin^2\theta}+1\right]
\end{aligned}$$

Cuối cùng, ta thu được tiết diện tán xạ toàn phần có dạng:

$$\sigma_{total} = \int d\Omega \frac{d\sigma}{d\Omega} = \int_0^{2\pi}\int_0^{\pi}\frac{\alpha^2}{E^2}\left[\frac{16}{\sin^4\theta}-\frac{11}{\sin^2\theta}+1\right]\sin\theta d\theta d\varphi$$

1.3/ $e^+e^- \to \mu^+\mu^-$



Dựa vào giản đồ, ta có biên độ tán xạ Feynman:

$$\begin{aligned}
i\mathcal{M} &= \left[\overline{v}^{s2}(p_2)\left(ie\gamma^\mu\right)u^{s1}(p_1)\right]\left(\frac{-ig_{\mu\nu}}{q^2}\right)\left[\overline{v}^{s4}(p_4)\left(ie\gamma^\nu\right)u^{s3}(p_3)\right] \\
&= \frac{ie^2}{q^2}\left[\overline{v}^{s2}(p_2)\gamma^\mu u^{s1}(p_1)\right]\left[\overline{v}^{s4}(p_4)\gamma_\mu u^{s3}(p_3)\right] \qquad (5)
\end{aligned}$$

Lấy liên hợp phức, ta được:

$$-i\mathcal{M}^* = \frac{-ie^2}{q^2}\left[\overline{u}^{s3}(p_3)\gamma_\nu v^{s4}(p_4)\right]\left[\overline{u}^{s1}(p_1)\gamma^\nu v^{s2}(p_2)\right]$$

Nhân với (5), ta được:

$$\begin{aligned}|\mathcal{M}|^2 &= \frac{e^4}{q^4}\left[\overline{v}^{s2}(p_2)\gamma^\mu u^{s1}(p_1)\right]\left[\overline{v}^{s4}(p_4)\gamma_\mu u^{s3}(p_3)\right]\left[\overline{u}^{s3}(p_3)\gamma_\nu v^{s4}(p_4)\right]\left[\overline{u}^{s1}(p_1)\gamma^\nu v^{s2}(p_2)\right]\\ &= \frac{e^4}{q^4}\left[\overline{u}^{s1}(p_1)\gamma^\nu v^{s2}(p_2)\right]\left[\overline{v}^{s2}(p_2)\gamma^\mu u^{s1}(p_1)\right]\left[\overline{v}^{s4}(p_4)\gamma_\mu u^{s3}(p_3)\right]\left[\overline{u}^{s3}(p_3)\gamma_\nu v^{s4}(p_4)\right]\end{aligned}$$

Áp dụng công thức:

$$\sum_s u_p^s \overline{u}_p^s = \not{p} + m$$
$$\sum_s v_p^s \overline{v}_p^s = \not{p} - m$$

Lấy trung bình theo spin, ta được:

$$\begin{aligned}\frac{1}{4}\sum_{spin}|\mathcal{M}|^2 &= \sum_{s1,s4}\frac{e^4}{4q^4}\left[\overline{u}^{s1}(p_1)\gamma^\nu\left(\not{p}_2 - m_\mu\right)\gamma^\mu u^{s1}(p_1)\right]\left[\overline{v}^{s4}(p_4)\gamma_\nu\left(\not{p}_3 + m_e\right)\gamma_\mu v^{s4}(p_4)\right]\\ &= \sum_{s1,s4}\frac{e^4}{4q^4}\sum_{\alpha,\beta}\left[\overline{u}_\alpha^{s1}(p_1)\gamma^\nu_{\alpha\beta}\left(\not{p}_2 - m_\mu\right)_{\beta\alpha}\gamma^\mu_{\alpha\beta}u_\beta^{s1}(p_1)\right]\sum_{a,b}\left[\overline{v}_a^{s4}(p_4)\gamma_{\nu_{ab}}\left(\not{p}_3 + m_e\right)_{ba}\gamma_{\mu_{ab}}v_a^{s4}(p_4)\right]\\ &= \sum_{s1,s4}\frac{e^4}{4q^4}\sum_{\alpha,\beta}\left[u_\beta^{s1}(p_1)\overline{u}_\alpha^{s1}(p_1)\gamma^\nu_{\alpha\beta}\left(\not{p}_2 - m_\mu\right)_{\beta\alpha}\gamma^\mu_{\alpha\beta}\right]\sum_{a,b}\left[v_a^{s4}(p_4)\overline{v}_a^{s4}(p_4)\gamma_{\nu_{ab}}\left(\not{p}_3 + m_e\right)_{ba}\gamma_{\mu_{ab}}\right]\\ &= \frac{e^4}{4q^4}\mathrm{Tr}\left[\left(\not{p}_1 + m_e\right)\gamma^\nu\left(\not{p}_2 - m_\mu\right)\gamma^\mu\right]\mathrm{Tr}\left[\left(\not{p}_4 - m_\mu\right)\gamma_\nu\left(\not{p}_3 + m_e\right)\gamma_\mu\right]\end{aligned}$$

Chọn cho $m_e = 0$ và $m_\mu = 0$:

$$\begin{aligned}\frac{1}{4}\sum_{spin}|\mathcal{M}|^2 &= \frac{e^4}{4q^4}\mathrm{Tr}\left[\not{p}_1\gamma^\nu\not{p}_2\gamma^\mu\right]\mathrm{Tr}\left[\not{p}_4\gamma_\nu\not{p}_3\gamma_\mu\right]\\ &= \frac{e^4}{4q^4}\mathrm{Tr}\left[p_{1\rho}\gamma^\rho\gamma^\nu p_{2\sigma}\gamma^\sigma\gamma^\mu\right]\mathrm{Tr}\left[p_4^\rho\gamma_\rho\gamma_\nu p_3^\sigma\gamma_\sigma\gamma_\mu\right]\\ &= \frac{e^4}{4q^4}\left(p_{1\rho}p_{2\sigma}\right)\mathrm{Tr}\left[\gamma^\rho\gamma^\nu\gamma^\sigma\gamma^\mu\right]\left(p_4^\rho p_3^\sigma\right)\mathrm{Tr}\left[\gamma_\rho\gamma_\nu\gamma_\sigma\gamma_\mu\right]\end{aligned}$$

Tiếp tục, ta áp dụng công thức:

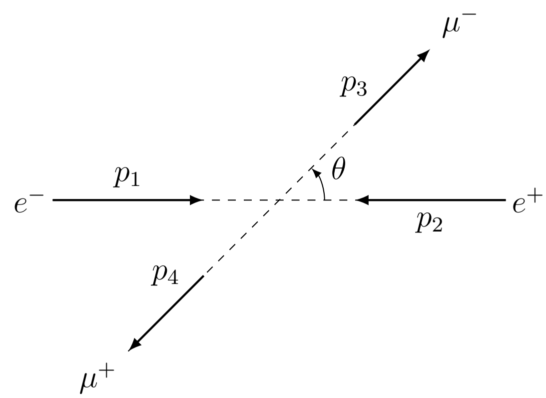
$$\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right] = 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)$$

Phương trình trở thành:

$$\frac{1}{4}\sum_{spin}|\mathcal{M}|^2 = \frac{e^4}{4q^4}\left(p_{1\rho}p_{2\sigma}\right)4\left(g^{\rho\nu}g^{\sigma\mu} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\mu}g^{\nu\sigma}\right)\left(p_4^\rho p_3^\sigma\right)4\left(g_{\rho\nu}g_{\sigma\mu} - g_{\rho\sigma}g_{\nu\mu} + g_{\rho\mu}g_{\nu\sigma}\right)$$

$$= \frac{4e^4}{q^4}\left[p_1^{\nu}p_2^{\mu} + p_1^{\mu}p_2^{\nu} - g^{\mu\nu}\left(p_1 \cdot p_2\right)\right]\left[p_{4\nu}p_{3\mu} + p_{4\mu}p_{3\nu} - g_{\mu\nu}\left(p_4 \cdot p_3\right)\right]$$
$$= \frac{8e^4}{q^4}\left[\left(p_1 \cdot p_4\right)\left(p_2 \cdot p_3\right) + \left(p_1 \cdot p_3\right)\left(p_2 \cdot p_4\right)\right]$$

Tính tiết diện tán xạ trong hệ quy chiếu khối tâm của 2 hạt tới



Với:

$$\begin{aligned}
p_1 &= (E;\ 0;\ 0;\ p_z) \\
p_2 &= (E;\ 0;\ 0;\ -p_z) \\
p_3 &= (E;\ 0;\ |p_3|\sin\theta;\ |p_3|\cos\theta) \\
p_4 &= (E;\ 0;\ -|p_4|\sin\theta';\ -|p_4|\cos\theta) \\
=>& |p_3| = |p_4| = E
\end{aligned}$$

Center of mass $=> s = (p_1 + p_2)^2 = (p_3 + p_4)^2 = 4E^2 = q^2$
Lại có $p^2 = E^2 - \vec{p_z}^2 = m_e^2 = 0$. Do đó, $|p_z| = E$
Từ đó suy ra các thành phần có dạng:

$$\begin{aligned}
p_1 &= (E;\ 0;\ 0;\ E) \\
p_2 &= (E;\ 0;\ 0;\ -E) \\
p_3 &= (E;\ 0;\ E\sin\theta;\ E\cos\theta) \\
p_4 &= (E;\ 0;\ -E\sin\theta;\ -E\cos\theta)
\end{aligned}$$

Để tính tiết diện tán xạ, ta cần:

$$\begin{cases}
p_1 \cdot p_2 = E^2 + E^2 = 2E^2 \\
p_3 \cdot p_4 = E^2 + E^2 = 2E^2 \\
p_1 \cdot p_4 = E^2 + E^2\cos\theta = E^2(1+\cos\theta) \\
p_2 \cdot p_3 = E^2 + E^2\cos\theta = E^2(1+\cos\theta) \\
p_1 \cdot p_3 = E^2 - E^2\cos\theta = E^2(1-\cos\theta) \\
p_2 \cdot p_4 = E^2 - E^2\cos\theta = E^2(1-\cos\theta) \\
q^2 = (p_3 - p_1)^2 \rightarrow q^2 = p_3^2 + p_1^2 - 2p_1 \cdot p_3 = -2E^2(1-\cos\theta) \\
q'^2 = (p_4 - p_1)^2 \rightarrow q^2 = p_4^2 + p_1^2 - 2p_1 \cdot p_4 = -2E^2(1+\cos\theta)
\end{cases}$$

Từ đó, biên độ tán xạ Feynman trở thành:

$$\begin{aligned}\frac{1}{4}\sum_{spin}|\mathcal{M}|^2 &= \frac{8e^4}{16E^4}\left[E^4\left(1+\cos\theta\right)^2 + E^4\left(1-\cos\theta\right)^2\right] \\ &= e^4\left(1+\cos^2\theta\right)\end{aligned}$$

Thay vào công thức tính tiết diện tán xạ vi phân:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2E^2}\frac{1}{4}\sum_{spin}|\mathcal{M}|^2 = \frac{e^4}{64\pi^2E^2}\left(1+\cos^2\theta\right)$$

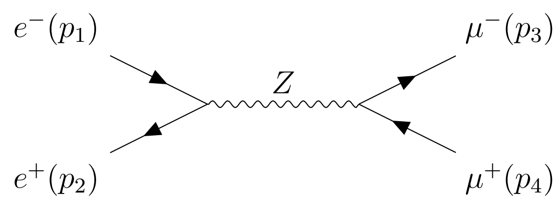
Đặt $\alpha = \dfrac{e^2}{4\pi}$, phương trình được tối giản thành:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2}\left(1+\cos^2\theta\right)$$

Tiết diện tán xạ toàn phần:

$$\begin{aligned}\sigma_{total} = \int d\Omega\frac{d\sigma}{d\Omega} &= \int_0^{2\pi}\int_0^{\pi}\frac{\alpha^2}{4E^2}\left(1+\cos^2\theta\right)\sin\theta d\theta d\varphi \\ &= \frac{\pi\alpha^2}{2E^2}\int_{-1}^{1}\left(1+\cos^2\theta\right)d\cos\theta = \frac{\pi\alpha^2}{2E^2}\left(\cos\theta+\frac{\cos^3\theta}{3}\right)_{-1}^{1} \\ &= \frac{4\pi\alpha^2}{3E^2}\end{aligned}$$

1.4/ $e^+e^- \to \mu^+\mu^-$ với Z boson là hàm truyền



Như ta biết, coupling của Z boson với e^+e^- và $\mu^+\mu-$ có dạng:

$$\frac{e}{s_wc_w}\gamma^\mu\left(\frac{1-\gamma^5}{2}-s_w^2\right)$$

Ta rút gọn nó như sau:

$$\frac{g_z}{2}\gamma^\mu\left(c_v-c_a\gamma^5\right)$$

Với:

$$g_z = \frac{e}{s_wc_w}$$

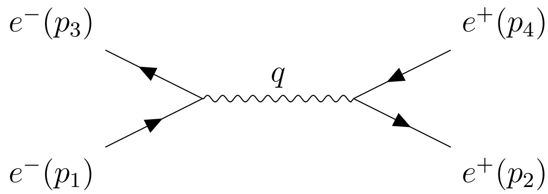
$$c_v \text{ (đối với electron và muon) } = I_3 - 2Qs_w^2 = -\frac{1}{2}+2s_w^2$$

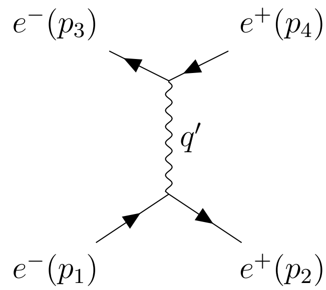
$$c_a \text{ (đối với electron và muon) } = I_3 = -\frac{1}{2}$$

Dựa vào giản đồ, ta viết được biểu thức cho biên độ tán xạ Feynman:

$$i\mathcal{M} = [..]$$

1.5/ Bhabha scattering





Dựa theo đồ thị, ta có biên độ tán xạ Feynman:

$$\begin{aligned}
i\mathcal{M} = i\left(\mathcal{M}_t - \mathcal{M}_s\right) &= \left[\bar{u}^{s3}(p_3)(-ie\gamma^\mu)u^{s1}(p_1)\right]\left(\frac{-ig_{\mu\nu}}{(p_1-p_3)^2}\right)\left[\bar{v}^{s2}(p_2)(-ie\gamma^\nu)v^{s4}(p_4)\right] \\
&\quad - \left[\bar{u}^{s3}(p_3)(-ie\gamma^\mu)v^{s4}(p_4)\right]\left(\frac{-ig_{\mu\nu}}{(p_1+p_2)^2}\right)\left[\bar{v}^{s2}(p_2)(-ie\gamma^\nu)u^{s1}(p_1)\right] \\
&= \frac{ie^2}{(p_1-p_3)^2}\left[\bar{u}^{s3}(p_3)\gamma^\mu u^{s1}(p_1)\right]\left[\bar{v}^{s2}(p_2)\gamma_\mu v^{s4}(p_4)\right] \\
&\quad - \frac{ie^2}{(p_1+p_2)^2}\left[\bar{u}^{s3}(p_3)\gamma^\mu v^{s4}(p_4)\right]\left[\bar{v}^{s2}(p_2)\gamma_\mu u^{s1}(p_1)\right]
\end{aligned}$$

Lấy liên hợp phức ta được:

$$-i\mathcal{M}^* = \frac{-ie^2}{(p_1-p_3)^2}\left[\bar{v}^{s4}(p_4)\gamma_\nu v^{s2}(p_2)\right]\left[\bar{u}^{s1}(p_1)\gamma^\nu u^{s3}(p_3)\right] + \frac{ie^2}{(p_1+p_2)^2}\left[\bar{u}^{s1}(p_1)\gamma_\nu v^{s2}(p_2)\right]\left[\bar{v}^{s4}(p_4)\gamma^\nu u^{s3}(p_3)\right]$$

Nhân lại, ta được:

$$\begin{aligned}
|\mathcal{M}|^2 &= \frac{e^4}{(p_1-p_3)^4}\left[\bar{u}^{s3}(p_3)\gamma^\mu u^{s1}(p_1)\right]\left[\bar{v}^{s2}(p_2)\gamma_\mu v^{s4}(p_4)\right]\left[\bar{v}^{s4}(p_4)\gamma_\nu v^{s2}(p_2)\right]\left[\bar{u}^{s1}(p_1)\gamma^\nu u^{s3}(p_3)\right] \\
&\quad - \frac{e^4}{(p_1-p_3)^2(p_1+p_2)^2}\left[\bar{u}^{s3}(p_3)\gamma^\mu u^{s1}(p_1)\right]\left[\bar{v}^{s2}(p_2)\gamma_\mu v^{s4}(p_4)\right]\left[\bar{u}^{s1}(p_1)\gamma_\nu v^{s2}(p_2)\right]\left[\bar{v}^{s4}(p_4)\gamma^\nu u^{s3}(p_3)\right] \\
&\quad - \frac{e^4}{(p_1-p_3)^2(p_1+p_2)^2}\left[\bar{u}^{s3}(p_3)\gamma^\mu v^{s4}(p_4)\right]\left[\bar{v}^{s2}(p_2)\gamma_\mu u^{s1}(p_1)\right]\left[\bar{v}^{s4}(p_4)\gamma_\nu v^{s2}(p_2)\right]\left[\bar{u}^{s1}(p_1)\gamma^\nu u^{s3}(p_3)\right] \\
&\quad + \frac{e^4}{(p_1+p_2)^4}\left[\bar{u}^{s3}(p_3)\gamma^\mu v^{s4}(p_4)\right]\left[\bar{v}^{s2}(p_2)\gamma_\mu u^{s1}(p_1)\right]\left[\bar{u}^{s1}(p_1)\gamma_\nu v^{s2}(p_2)\right]\left[\bar{v}^{s4}(p_4)\gamma^\nu u^{s3}(p_3)\right] \\
&= \frac{e^4}{(p_1-p_3)^4}\left[\bar{u}^{s1}(p_1)\gamma^\nu u^{s3}(p_3)\right]\left[\bar{u}^{s3}(p_3)\gamma^\mu u^{s1}(p_1)\right]\left[\bar{v}^{s2}(p_2)\gamma_\mu v^{s4}(p_4)\right]\left[\bar{v}^{s4}(p_4)\gamma_\nu v^{s2}(p_2)\right] \\
&\quad - \frac{e^4}{(p_1-p_3)^2(p_1+p_2)^2}\left[\bar{u}^{s3}(p_3)\gamma^\mu u^{s1}(p_1)\right]\left[\bar{u}^{s1}(p_1)\gamma_\nu v^{s2}(p_2)\right]\left[\bar{v}^{s2}(p_2)\gamma_\mu v^{s4}(p_4)\right]\left[\bar{v}^{s4}(p_4)\gamma^\nu u^{s3}(p_3)\right]
\end{aligned}$$

$$- \frac{e^4}{(p_1-p_3)^2(p_1+p_2)^2} \left[\bar{u}^{s3}(p_3)\gamma^\mu v^{s4}(p_4)\right] \left[\bar{v}^{s4}(p_4)\gamma_\nu v^{s2}(p_2)\right] \left[\bar{v}^{s2}(p_2)\gamma_\mu u^{s1}(p_1)\right] \left[\bar{u}^{s1}(p_1)\gamma^\nu u^{s3}(p_3)\right]$$
$$+ \frac{e^4}{(p_1+p_2)^4} \left[\bar{v}^{s4}(p_4)\gamma^\nu u^{s3}(p_3)\right] \left[\bar{u}^{s3}(p_3)\gamma^\mu v^{s4}(p_4)\right] \left[\bar{v}^{s2}(p_2)\gamma_\mu u^{s1}(p_1)\right] \left[\bar{u}^{s1}(p_1)\gamma_\nu v^{s2}(p_2)\right]$$

Áp dụng công thức:

$$\sum_s u_p^s \bar{u}_p^s = \not{p} + m$$
$$\sum_s v_p^s \bar{v}_p^s = \not{p} - m$$

Lấy trung bình theo spin, ta được:

$$\frac{1}{4}\sum_{spin} |\mathcal{M}|^2 = \sum_{s1,s2} \frac{e^4}{4(p_1-p_3)^4} \left[\bar{u}^{s1}(p_1)\gamma^\nu \left(\not{p}_3 + m_e\right)\gamma^\mu u^{s1}(p_1)\right] \left[\bar{v}^{s2}(p_2)\gamma_\mu \left(\not{p}_4 - m_e\right)\gamma_\nu v^{s2}(p_2)\right]$$
$$- \sum_{s3} \frac{e^4}{4(p_1-p_3)^2(p_1+p_2)^2} \left[\bar{u}^{s3}(p_3)\gamma^\mu \left(\not{p}_1 + m_e\right)\gamma_\nu \left(\not{p}_2 - m_e\right)\gamma_\mu \left(\not{p}_4 - m_e\right)\gamma^\nu u^{s3}(p_3)\right]$$
$$- \sum_{s3} \frac{e^4}{4(p_1-p_3)^2(p_1+p_2)^2} \left[\bar{u}^{s3}(p_3)\gamma^\mu \left(\not{p}_4 - m_e\right)\gamma_\nu \left(\not{p}_2 - m_e\right)\gamma_\mu \left(\not{p}_1 + m_e\right)\gamma^\nu u^{s3}(p_3)\right]$$
$$+ \sum_{s2,s4} \frac{e^4}{4(p_1+p_2)^4} \left[\bar{v}^{s4}(p_4)\gamma^\nu \left(\not{p}_3 + m_3\right)\gamma^\mu v^{s4}(p_4)\right] \left[\bar{v}^{s2}(p_2)\gamma_\mu \left(\not{p}_1 + m_e\right)\gamma_\nu v^{s2}(p_2)\right]$$
$$= \sum_{s1,s2} \frac{e^4}{4(p_1-p_3)^4} \sum_{\alpha,\beta} \left[\bar{u}^{s1}(p_1)_\alpha \gamma^\nu_{\alpha\beta} \left(\not{p}_3 + m_e\right)_{\beta\alpha} \gamma^\mu_{\alpha\beta} u^{s1}(p_1)_\beta\right]$$
$$\sum_{a,b} \left[\bar{v}^{s2}(p_2)_a \gamma_{\mu_{ab}} \left(\not{p}_4 - m_e\right)_{ba} \gamma_{\nu_{ab}} v^{s2}(p_2)_b\right]$$
$$- \sum_{s3} \frac{e^4}{4(p_1-p_3)^2(p_1+p_2)^2} \sum_{\alpha,\beta} \left[\bar{u}^{s3}(p_3)_\alpha \gamma^\mu_{\alpha\beta} \left(\not{p}_1 + m_e\right)_{\beta\alpha} \gamma_{\nu_{\alpha\beta}} \left(\not{p}_2 - m_e\right)_{\beta\alpha} \gamma_{\mu_{\alpha\beta}}\right.$$
$$\left.\left(\not{p}_4 - m_e\right)_{\beta\alpha} \gamma^\nu_{\alpha\beta} u^{s3}(p_3)_\beta\right]$$
$$- \sum_{s3} \frac{e^4}{4(p_1-p_3)^2(p_1+p_2)^2} \sum_{\alpha,\beta} \left[\bar{u}^{s3}(p_3)_\alpha \gamma^\mu_{\alpha\beta} \left(\not{p}_4 - m_e\right)_{\beta\alpha} \gamma_{\nu_{\alpha\beta}} \left(\not{p}_2 - m_e\right)_{\beta\alpha} \gamma_{\mu_{\alpha\beta}}\right.$$
$$\left.\left(\not{p}_1 + m_e\right)_{\beta\alpha} \gamma^\nu_{\alpha\beta} u^{s3}(p_3)_\beta\right]$$
$$+ \sum_{s2,s4} \frac{e^4}{4(p_1+p_2)^4} \sum_{\alpha,\beta} \left[\bar{v}^{s4}(p_4)_\alpha \gamma^\nu_{\alpha\beta} \left(\not{p}_3 + m_3\right)_{\beta\alpha} \gamma^\mu_{\alpha\beta} v^{s4}(p_4)_\beta\right]$$
$$\sum_{a,b} \left[\bar{v}^{s2}(p_2)_a \gamma_{\mu_{ab}} \left(\not{p}_1 + m_e\right)_{ba} \gamma_{\nu_{ab}} v^{s2}(p_2)_b\right]$$
$$= \sum_{s1,s2} \frac{e^4}{4(p_1-p_3)^4} \sum_{\alpha,\beta} \left[u^{s1}(p_1)_\beta \bar{u}^{s1}(p_1)_\alpha \gamma^\nu_{\alpha\beta} \left(\not{p}_3 + m_e\right)_{\beta\alpha} \gamma^\mu_{\alpha\beta}\right]$$
$$\sum_{a,b} \left[v^{s2}(p_2)_b \bar{v}^{s2}(p_2)_a \gamma_{\mu_{ab}} \left(\not{p}_4 - m_e\right)_{ba} \gamma_{\nu_{ab}}\right]$$

$$
-\sum_{s3}\frac{e^4}{4(p_1-p_3)^2(p_1+p_2)^2}\sum_{\alpha,\beta}\Big[u^{s3}(p_3)_\beta\bar{u}^{s3}(p_3)_\alpha\gamma^\mu_{\alpha\beta}\left(\not{p}_1+m_e\right)_{\beta\alpha}\gamma_{\nu_{\alpha\beta}}\left(\not{p}_2-m_e\right)_{\beta\alpha}
$$

$$
\gamma_{\mu_{\alpha\beta}}\left(\not{p}_4-m_e\right)_{\beta\alpha}\gamma^\nu_{\alpha\beta}\Big]
$$

$$
-\sum_{s3}\frac{e^4}{4(p_1-p_3)^2(p_1+p_2)^2}\sum_{\alpha,\beta}\Big[u^{s3}(p_3)_\beta\bar{u}^{s3}(p_3)_\alpha\gamma^\mu_{\alpha\beta}\left(\not{p}_4-m_e\right)_{\beta\alpha}\gamma_{\nu_{\alpha\beta}}\left(\not{p}_2-m_e\right)_{\beta\alpha}
$$

$$
\gamma_{\mu_{\alpha\beta}}\left(\not{p}_1+m_e\right)_{\beta\alpha}\gamma^\nu_{\alpha\beta}\Big]
$$

$$
+\sum_{s2,s4}\frac{e^4}{4(p_1+p_2)^4}\sum_{\alpha,\beta}\Big[v^{s4}(p_4)_\beta\bar{v}^{s4}(p_4)_\alpha\gamma^\nu_{\alpha\beta}\left(\not{p}_3+m_3\right)_{\beta\alpha}\gamma^\mu_{\alpha\beta}\Big]
$$

$$
\sum_{a,b}\Big[v^{s2}(p_2)_b\bar{v}^{s2}(p_2)_a\gamma_{\mu_{ab}}\left(\not{p}_1+m_e\right)_{ba}\gamma_{\nu_{ab}}\Big]
$$

$$
=\frac{e^4}{4(p_1-p_3)^4}\mathrm{Tr}\left[\left(\not{p}_1+m_e\right)\gamma^\nu\left(\not{p}_3+m_e\right)\gamma^\mu\right]\mathrm{Tr}\left[\left(\not{p}_2-m_e\right)\gamma_\mu\left(\not{p}_4-m_e\right)\gamma_\nu\right]
$$

$$
-\frac{e^4}{4(p_1-p_3)^2(p_1+p_2)^2}\mathrm{Tr}\left[\left(\not{p}_3+m_e\right)\gamma^\mu\left(\not{p}_1+m_e\right)\gamma_\nu\left(\not{p}_2-m_e\right)\gamma_\mu\left(\not{p}_4-m_e\right)\gamma^\nu\right]
$$

$$
-4(p_1-p_3)^2(p_1+p_2)^2\mathrm{Tr}\left[\left(\not{p}_3+m_e\right)\gamma^\mu\left(\not{p}_4-m_e\right)\gamma_\nu\left(\not{p}_2-m_e\right)\gamma_\mu\left(\not{p}_1+m_e\right)\gamma^\nu\right]
$$

$$
+\frac{e^4}{4(p_1+p_2)^4}\mathrm{Tr}\left[\left(\not{p}_4-m_e\right)\gamma^\nu\left(\not{p}_3+m_e\right)\gamma^\mu\right]\mathrm{Tr}\left[\left(\not{p}_2-m_e\right)\gamma_\mu\left(\not{p}_1+m_e\right)\gamma_\nu\right]
$$

Để $m_e=0$, biểu thức đơn giản thành:

$$
\begin{aligned}
\frac{1}{4}\sum_{\text{spin}}|\mathcal{M}|^2 &= \frac{e^4}{4(p_1-p_3)^4}\mathrm{Tr}\left[\not{p}_1\gamma^\nu\not{p}_3\gamma^\mu\right]\mathrm{Tr}\left[\not{p}_2\gamma_\mu\not{p}_4\gamma_\nu\right]\\
&\quad-\frac{e^4}{4(p_1-p_3)^2(p_1+p_2)^2}\mathrm{Tr}\left[\not{p}_3\gamma^\mu\not{p}_1\gamma_\nu\not{p}_2\gamma_\mu\not{p}_4\gamma^\nu\right]\\
&\quad-\frac{e^4}{4(p_1-p_3)^2(p_1+p_2)^2}\mathrm{Tr}\left[\not{p}_3\gamma^\mu\not{p}_4\gamma_\nu\not{p}_2\gamma_\mu\not{p}_1\gamma^\nu\right]\\
&\quad+\frac{e^4}{4(p_1+p_2)^4}\mathrm{Tr}\left[\not{p}_4\gamma^\nu\not{p}_3\gamma^\mu\right]\mathrm{Tr}\left[\not{p}_2\gamma_\mu\not{p}_1\gamma_\nu\right]\\
&=\frac{e^4}{4(p_1-p_3)^4}\mathrm{Tr}\left[p_{1\rho}\gamma^\rho\gamma^\nu p_{3\sigma}\gamma^\sigma\gamma^\mu\right]\mathrm{Tr}\left[p_2^\rho\gamma_\rho\gamma_\mu p_4^\sigma\gamma_\sigma\gamma_\nu\right]\\
&\quad-\frac{e^4}{4(p_1-p_3)^2(p_1+p_2)^2}\mathrm{Tr}\left[p_{3\rho}\gamma^\rho\gamma^\mu p_{1\sigma}\gamma^\sigma\gamma_\nu p_{2\lambda}\gamma^\lambda\gamma_\mu p_{4\tau}\gamma^\tau\gamma^\nu\right]\\
&\quad-\frac{e^4}{4(p_1-p_3)^2(p_1+p_2)^2}\mathrm{Tr}\left[p_{3\rho}\gamma^\rho\gamma^\mu p_{4\sigma}\gamma^\sigma\gamma_\nu p_{2\lambda}\gamma^\lambda\gamma_\mu p_{1\tau}\gamma^\tau\gamma^\nu\right]\\
&\quad+\frac{e^4}{4(p_1+p_2)^4}\mathrm{Tr}\left[p_{4\rho}\gamma^\rho\gamma^\nu p_{3\sigma}\gamma^\sigma\gamma^\mu\right]\mathrm{Tr}\left[p_2^\rho\gamma_\rho\gamma_\mu p_1^\sigma\gamma^\sigma\gamma_\nu\right]\\
&=\frac{e^4}{4(p_1-p_3)^4}\left(p_{1\rho}p_{3\sigma}\right)\mathrm{Tr}\left[\gamma^\rho\gamma^\nu\gamma^\sigma\gamma^\mu\right]\left(p_2^\rho p_4^\sigma\right)\mathrm{Tr}\left[\gamma_\rho\gamma_\mu\gamma_\sigma\gamma_\nu\right]\\
&\quad-\frac{e^4}{4(p_1-p_3)^2(p_1+p_2)^2}\left(p_{3\rho}p_{1\sigma}p_{2\lambda}p_{4\tau}\right)\mathrm{Tr}\left[\gamma^\rho\gamma^\mu\gamma^\sigma\gamma_\nu\gamma^\lambda\gamma_\mu\gamma^\tau\gamma^\nu\right]
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{e^4}{4(p_1-p_3)^2(p_1+p_2)^2}\left(p_{3\rho}p_{4\sigma}p_{2\lambda}p_{1\tau}\right)\mathrm{Tr}\left[\gamma^\rho\gamma^\mu\gamma^\sigma\gamma_\nu\gamma^\lambda\gamma_\mu\gamma^\tau\gamma^\nu\right] \\
&+ \frac{e^4}{4(p_1+p_2)^4}\left(p_{4\rho}p_{3\sigma}\right)\mathrm{Tr}\left[\gamma^\rho\gamma^\nu\gamma^\sigma\gamma^\mu\right]\left(p_2^\rho p_1^\sigma\right)\mathrm{Tr}\left[\gamma_\rho\gamma_\mu\gamma_\sigma\gamma_\nu\right]
\end{aligned}
$$

Tiếp tục, ta áp dụng công thức:

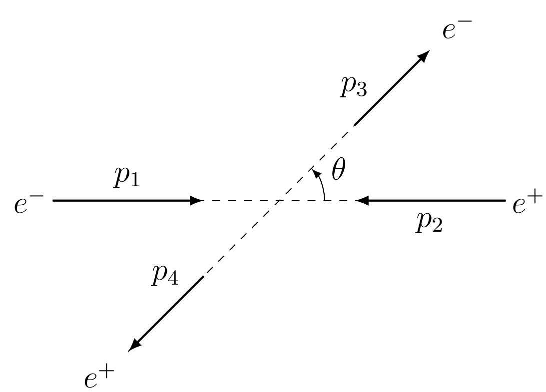
$$
\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right] = 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)
$$

$$
\mathrm{Tr}\left[\gamma^\rho\gamma^\mu\gamma^\sigma\gamma_\nu\gamma^\lambda\gamma_\mu\gamma^\tau\gamma^\nu\right] = \mathrm{Tr}\left[\gamma^\tau\gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma\gamma_\nu\gamma^\lambda\gamma_\mu\right] = -2\mathrm{Tr}\left[\gamma^\tau\gamma^\sigma\gamma^\mu\gamma^\rho\gamma^\lambda\gamma_\mu\right] = -8g^{\rho\lambda}\mathrm{Tr}\left[\gamma^\tau\gamma^\sigma\right] = -32g^{\rho\lambda}g^{\tau\sigma}
$$

Ta thu được:

$$
\begin{aligned}
\frac{1}{4}\sum_{spin}|\mathcal{M}|^2 &= \frac{e^4}{4(p_1-p_3)^4}\left(p_{1\rho}p_{3\sigma}\right)4\left(g^{\rho\nu}g^{\sigma\mu} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\mu}g^{\nu\sigma}\right)\left(p_2^\rho p_4^\sigma\right)4\left(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\sigma}g_{\mu\nu} + g_{\rho\nu}g_{\mu\sigma}\right) \\
&\quad + \frac{e^4}{4(p_1-p_3)^2(p_1+p_2)^2}\left(p_{3\rho}p_{1\sigma}p_{2\lambda}p_{4\tau}\right)32g^{\rho\lambda}g^{\tau\sigma} \\
&\quad + \frac{e^4}{4(p_1-p_3)^2(p_1+p_2)^2}\left(p_{3\rho}p_{4\sigma}p_{2\lambda}p_{1\tau}\right)32g^{\rho\lambda}g^{\tau\sigma} \\
&\quad + \frac{e^4}{4(p_1+p_2)^4}\left(p_{4\rho}p_{3\sigma}\right)4\left(g^{\rho\nu}g^{\sigma\mu} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\mu}g^{\nu\sigma}\right)\left(p_2^\rho p_1^\sigma\right)4\left(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\sigma}g_{\mu\nu} + g_{\rho\nu}g_{\mu\sigma}\right) \\
&= \frac{4e^4}{(p_1-p_3)^4}\left[p_1^\nu p_3^\mu + p_1^\mu p_3^\nu - g^{\mu\nu}(p_1\cdot p_3)\right]\left[p_{2\mu}p_{4\nu} + p_{2\nu}p_{4\mu} - g_{\mu\nu}\left(p_2\cdot p_4\right)\right] \\
&+ \frac{8e^4}{(p_1-p_3)^2(p_1+p_2)^2}\left(p_3\cdot p_2\right)\left(p_1\cdot p_4\right) + \frac{8e^4}{(p_1-p_3)^2(p_1+p_2)^2}\left(p_3\cdot p_2\right)\left(p_1\cdot p_4\right) \\
&+ \frac{4e^4}{(p_1+p_2)^4}\left[p_4^\nu p_3^\mu + p_4^\mu p_3^\nu - g^{\mu\nu}\left(p_4\cdot p_3\right)\right]\left[p_{2\mu}p_{1\nu} + p_{2\nu}p_{1\mu} - g_{\mu\nu}\left(p_2\cdot p_1\right)\right] \\
&= \frac{8e^4}{(p_1-p_3)^4}\left[\left(p_1\cdot p_4\right)\left(p_3\cdot p_2\right) + \left(p_1\cdot p_2\right)\left(p_3\cdot p_4\right)\right] \\
&+ \frac{8e^4}{(p_1-p_3)^2(p_1+p_2)^2}\left(p_3\cdot p_2\right)\left(p_1\cdot p_4\right) + \frac{8e^4}{(p_1-p_3)^2(p_1+p_2)^2}\left(p_3\cdot p_2\right)\left(p_1\cdot p_4\right) \\
&+ \frac{8e^4}{(p_1+p_2)^4}\left[\left(p_1\cdot p_4\right)\left(p_3\cdot p_2\right) + \left(p_1\cdot p_3\right)\left(p_2\cdot p_4\right)\right]
\end{aligned}
$$

Tính tiết diện tán xạ trong hệ quy chiếu khối tâm của 2 hạt tới



Với:

$$
\begin{aligned}
p_1 &= (E;\ 0;\ 0;\ p_z) \\
p_2 &= (E;\ 0;\ 0;\ -p_z) \\
p_3 &= (E;\ 0;\ |p_3|\sin\theta;\ |p_3|\cos\theta) \\
p_4 &= (E;\ 0;\ -|p_4|\sin\theta';\ -|p_4|\cos\theta) \\
=> &|p_3| = |p_4| = E
\end{aligned}
$$

Center of mass => $s = (p_1 + p_2)^2 = (p_3 + p_4)^2 = 4E^2$
Lại có $p^2 = E^2 - \vec{p_z}^2 = m_e^2 = 0$. Do đó, $|p_z| = E$
Từ đó suy ra các thành phần có dạng:

$$
\begin{aligned}
p_1 &= (E;\ 0;\ 0;\ E) \\
p_2 &= (E;\ 0;\ 0;\ -E) \\
p_3 &= (E;\ 0;\ E\sin\theta;\ E\cos\theta) \\
p_4 &= (E;\ 0;\ -E\sin\theta;\ -E\cos\theta)
\end{aligned}
$$

Để tính tiết diện tán xạ, ta cần:

$$
\begin{cases}
p_1 \cdot p_2 = p_3 \cdot p_4 = E^2 + E^2 = 2E^2 \\
p_1 \cdot p_4 = p_2 \cdot p_3 = E^2 + E^2\cos\theta = E^2(1+\cos\theta) \\
p_1 \cdot p_3 = p_2 \cdot p_4 = E^2 - E^2\cos\theta = E^2(1-\cos\theta) \\
(p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1 \cdot p_3 = -2p_1 \cdot p_3 = -2E^2(1-\cos\theta)
\end{cases}
$$

Thay vào biên độ tán xạ Feynman, ta được:

$$
\begin{aligned}
\frac{1}{4}\sum_{spin}|\mathcal{M}|^2 &= \frac{8e^4}{4E^4\left(1-\cos\theta\right)^2}\left[E^4\left(1+\cos\theta\right)^2 + 4E^4\right] + \frac{8e^4}{2E^2(\cos\theta - 1)4E^2}E^4(1+\cos\theta)^2 \\
&+ \frac{8e^4}{2E^2(\cos\theta - 1)4E^2}E^4(1+\cos\theta)^2 + \frac{8e^4}{16E^4}\left[E^4(1+\cos\theta)^2 + E^4(1-\cos\theta)^2\right] \\
&= \frac{2e^4}{(1-\cos\theta)^2}\left[(1+\cos\theta)^2 + 4\right] + \frac{2e^4}{(\cos\theta - 1)}(1+\cos\theta)^2 + e^4\left(1+\cos^2\theta\right)
\end{aligned}
$$

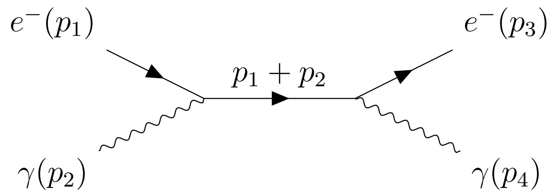
Thay vào công thức tính tiết diện tán xạ vi phân:

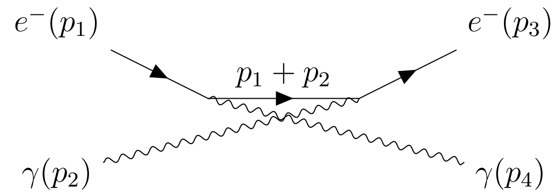
$$
\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2E^2}\frac{1}{4}\sum_{spin}|\mathcal{M}|^2 \\
&= \frac{1}{64\pi^2E^2}\left[\frac{2e^4}{(1-\cos\theta)^2}\left[(1+\cos\theta)^2 + 4\right] + \frac{2e^4}{(\cos\theta - 1)}(1+\cos\theta)^2 + e^4\left(1+\cos^2\theta\right)\right]
\end{aligned}
$$

Đặt $\alpha = \dfrac{e^2}{4\pi}$:

$$
\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2E^2}\left[\frac{(1+\cos\theta)^2 + 4}{(1-\cos\theta)^2} - \frac{(1+\cos\theta)^2}{(1-\cos\theta)} + \frac{(1+\cos^2\theta)}{2}\right]
$$

2/ Compton Scattering





Dựa theo đồ thị, ta có biên độ tán xạ Feynman:

$$
\begin{aligned}
i\mathcal{M} &= \left[\epsilon_{\mu}^{*,\lambda'}(p_4)(-ie\gamma^{\mu})\overline{u}^{s3}(p_3)\right] \frac{i\left(\not{p}_1 + \not{p}_2 + m_e\right)}{\left(p_1+p_2\right)^2 - m_e^2} \left[\epsilon_{\nu}^{\lambda}(p_2)(-ie\gamma^{\nu})u^{s1}(p_1)\right] \\
&\quad + \left[\epsilon_{\nu}^{\lambda}(p_2)(-ie\gamma^{\nu})\overline{u}^{s3}(p_3)\right] \frac{i\left(\not{p}_1 - \not{p}_4 + m_e\right)}{\left(p_1-p_4\right)^2 - m_e^2} \left[\epsilon_{\mu}^{*,\lambda'}(p_4)(-ie\gamma^{\mu})u^{s1}(p_1)\right] \\
&= -ie^2\epsilon_{\mu}^{*,\lambda'}(p_4)\epsilon_{\nu}^{\lambda}(p_2)\overline{u}^{s3}(p_3)\left[\frac{\gamma^{\mu}\left(\not{p}_1 + \not{p}_2 + m_e\right)\gamma^{\nu}}{\left(p_1+p_2\right)^2 - m_e^2} + \frac{\gamma^{\nu}\left(\not{p}_1 - \not{p}_4 + m_e\right)\gamma^{\mu}}{\left(p_1-p_4\right)^2 - m_e^2}\right]u^{s1}(p_1) \\
&= -ie^2\epsilon_{\mu}^{*,\lambda'}(p_4)\epsilon_{\nu}^{\lambda}(p_2)\overline{u}^{s3}(p_3)\left[\frac{\gamma^{\mu}\left(\not{p}_1 + \not{p}_2 + m_e\right)\gamma^{\nu}}{2p_1\cdot p_2} + \frac{\gamma^{\nu}\left(\not{p}_1 - \not{p}_4 + m_e\right)\gamma^{\mu}}{-2p_1\cdot p_4}\right]u^{s1}(p_1)
\end{aligned}
$$

Ta có:

$$
\begin{aligned}
\left(\not{p}_1 + m_e\right)\gamma^{\nu}u^{s1}(p_1) &= \left(2p_1^{\nu} - \gamma^{\nu}\not{p}_1 + \gamma^{\nu}m_e\right)u^{s1}(p_1) \\
&= 2p^{\nu}u^{s1}(p_1) - \gamma^{\nu}\left(\not{p}_1 - m_e\right)u^{s1}(p_1) \\
&= 2p^{\nu}u^{s1}(p_1)
\end{aligned}
$$

Do đó:

$$
\mathcal{M} = -e^2\epsilon_{\mu}^{*,\lambda'}(p_4)\epsilon_{\nu}^{\lambda}(p_2)\overline{u}^{s3}(p_3)\left[\frac{\gamma^{\mu}\not{p}_2\gamma^{\nu} + 2\gamma^{\mu}p_1^{\nu}}{2p_1\cdot p_2} + \frac{\gamma^{\nu}p_4\gamma^{\mu} - 2\gamma^{\nu}p_1^{\mu}}{2p_1\cdot p_4}\right]u^{s1}(p_1)
$$

Ta tìm được liên hợp phức:

$$
\mathcal{M}^* = -e^2\epsilon_{\alpha}^{*,\lambda'}(p_4)\epsilon_{\beta}^{\lambda}(p_2)\overline{u}^{s1}(p_1)\left[\frac{\gamma^{\alpha}\not{p}_2\gamma^{\beta} + 2\gamma^{\alpha}p^{\beta}}{2p_1\cdot p_2} + \frac{\gamma^{\beta}p_4\gamma^{\alpha} - 2\gamma^{\beta}p_1^{\alpha}}{2p_1\cdot p_4}\right]u^{s3}(p_3)
$$

Nhân lại, ta được:

$$
\begin{aligned}
\frac{1}{4}\sum_{s_1,s_3,\lambda,\lambda'}|\mathcal{M}|^2 = \frac{e^4}{4}&\left[\left\{\sum_{\lambda'}\epsilon_{\mu}^{*,\lambda'}(p_4)\epsilon_{\alpha}^{*,\lambda'}(p_4)\right\}\left\{\sum_{\lambda}\epsilon_{\nu}^{\lambda}(p_2)\epsilon_{\beta}^{\lambda}(p_2)\right\}\right] \\
&\times \mathrm{Tr}\left\{\left(\not{p}_3 + m_e\right)\left[\frac{\gamma^{\mu}\not{p}_2\gamma^{\nu} + 2\gamma^{\mu}p_1^{\nu}}{2p_1\cdot p_2} + \frac{\gamma^{\nu}\not{p}_4\gamma^{\mu} - 2\gamma^{\nu}p_1^{\mu}}{2p_1\cdot p_4}\right]\right. \\
&\times \left.\left(\not{p}_1 + m_e\right)\left[\frac{\gamma^{\alpha}\not{p}_2\gamma^{\beta} + 2\gamma^{\alpha}p_1^{\beta}}{2p_1\cdot p_2} + \frac{\gamma^{\beta}\not{p}_4\gamma^{\alpha} - 2\gamma^{\beta}p_1^{\alpha}}{2p_1\cdot p_4}\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{e^4}{4} g_{\mu\alpha} g_{\nu\beta} \\
&\times \mathrm{Tr}\left\{ (\not{p}_3 + m_e) \left[\frac{\gamma^\mu \not{p}_2 \gamma^\nu + 2\gamma^\mu p_1^\nu}{2p_1 \cdot p_2} + \frac{\gamma^\nu \not{p}_4 \gamma^\mu - 2\gamma^\nu p_1^\mu}{2p_1 \cdot p_4} \right] \right. \\
&\left. \times (\not{p}_1 + m_e) \left[\frac{\gamma^\alpha \not{p}_2 \gamma^\beta + 2\gamma^\alpha p_1^\beta}{2p_1 \cdot p_2} + \frac{\gamma^\beta \not{p}_4 \gamma^\alpha - 2\gamma^\beta p_1^\alpha}{2p_1 \cdot p_4} \right] \right\} \\
&= \frac{e^4}{4} \left[\frac{\mathrm{Tr}\left[\not{p}_3 \left(\gamma^\mu \not{p}_2 \gamma^\nu + 2\gamma^\mu p_1^\nu \right) \not{p}_1 \left(\gamma_\mu \not{p}_2 \gamma_\nu + 2\gamma_\mu p_{1\nu} \right) \right]}{(2p_1 \cdot p_2)^2} \right. \\
&+ \frac{\mathrm{Tr}\left[\not{p}_3 \left(\gamma^\mu \not{p}_2 \gamma^\nu + 2\gamma^\mu p_1^\nu \right) \not{p}_1 \left(\gamma_\nu \not{p}_4 \gamma_\mu - 2\gamma_\nu p_{1\mu} \right) \right]}{4 (p_1 \cdot p_2)(p_1 \cdot p_4)} \\
&+ \frac{\mathrm{Tr}\left[\not{p}_3 \left(\gamma^\nu \not{p}_4 \gamma^\mu - 2\gamma^\nu p_1^\mu \right) \not{p}_1 \left(\gamma_\mu \not{p}_2 \gamma_\nu + 2\gamma_\mu p_{1\nu} \right) \right]}{4 (p_1 \cdot p_4)(p_1 \cdot p_2)} \\
&\left. + \frac{\mathrm{Tr}\left[\not{p}_3 \left(\gamma^\nu \not{p}_4 \gamma^\mu - 2\gamma^\nu p_1^\mu \right) \not{p}_1 \left(\gamma_\nu \not{p}_4 \gamma_\mu - 2\gamma_\nu p_{1\mu} \right) \right]}{(2p_1 \cdot p_4)^2} \right]
\end{aligned}
$$

Xét số hạng đầu tiên:

$$
\begin{aligned}
&\frac{\mathrm{Tr}\left[\not{p}_3 \left(\gamma^\mu \not{p}_2 \gamma^\nu + 2\gamma^\mu p_1^\nu \right) \not{p}_1 \left(\gamma_\mu \not{p}_2 \gamma_\nu + 2\gamma_\mu p_{1\nu} \right) \right]}{(2p_1 \cdot p_2)^2} \\
=&\frac{\mathrm{Tr}\left[\not{p}_3 \gamma^\mu \not{p}_2 \gamma^\nu \not{p}_1 \gamma_\mu \not{p}_2 \gamma_\nu + 2\gamma^\mu \not{p}_3 \not{p}_1 \not{p}_1 \gamma_\mu \not{p}_2 + 2\gamma^\mu \not{p}_1 \not{p}_1 \not{p}_3 \gamma_\mu \not{p}_2 + 4p_1^2 \gamma^\mu \not{p}_1 \gamma_\mu \not{p}_3 \right]}{(2p_1 \cdot p_2)^2}
\end{aligned}
$$

Dùng tính chất của ma trận γ như đã dùng ở phần tán xạ Moller, ta rút gọn được:

$$
\frac{32m^2 (p_1 \cdot p_3 - p_2 \cdot p_4) + 32 (p_1 \cdot p_2)(p_3 \cdot p_2)}{(2p_1 \cdot p_2)^2}
$$

Có thể thấy hạng tử thứ 4 chỉ khác thứ nhất ở chỗ đổi p_2 thành $-p_4$. Vì vậy, ta có được hạng tử thứ 2 bằng:

$$
\frac{32m^2 (p_1 \cdot p_3 - p_2 \cdot p_4) + 32 (p_1 \cdot p_4)(p_3 \cdot p_4)}{(2p_1 \cdot p_4)^2}
$$

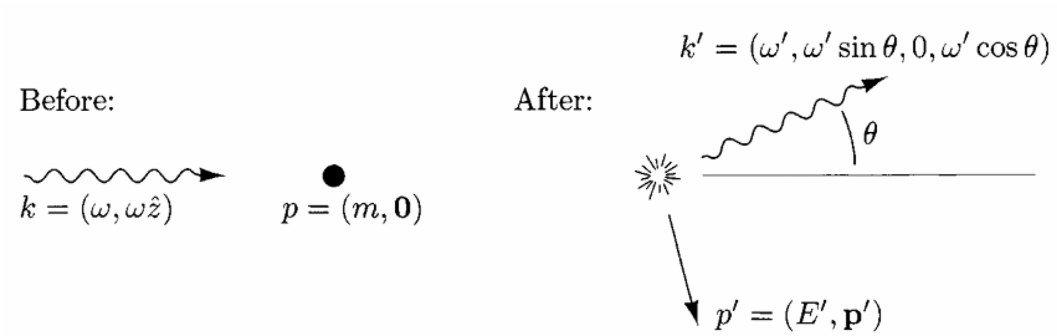
Làm điều tương tự, ta thu được:

$$
\frac{\mathrm{Tr}\left[\not{p}_3 \left(\gamma^\mu \not{p}_2 \gamma^\nu + 2\gamma^\mu p_1^\nu \right) \not{p}_1 \left(\gamma_\nu \not{p}_4 \gamma_\mu - 2\gamma_\nu p_{1\mu} \right) \right]}{4 (p_1 \cdot p_2)(p_1 \cdot p_4)} = \frac{-32 (p_1 \cdot p_3)^2 + (16m^2 + 32p_1 \cdot p_3)(p_2 \cdot p_1 - p_2 \cdot p_3)}{4 (p_1 \cdot p_2)(p_1 \cdot p_4)}
$$

$$
\frac{\mathrm{Tr}\left[\not{p}_3 \left(\gamma^\nu \not{p}_4 \gamma^\mu - 2\gamma^\nu p_1^\mu \right) \not{p}_1 \left(\gamma_\mu \not{p}_2 \gamma_\nu + 2\gamma_\mu p_{1\nu} \right) \right]}{4 (p_1 \cdot p_4)(p_1 \cdot p_2)} = \frac{-32 (p_1 \cdot p_3)^2 + (16m^2 + 32p_1 \cdot p_3)(-p_4 \cdot p_1 + p_4 \cdot p_3)}{4 (p_1 \cdot p_2)(p_1 \cdot p_4)}
$$

Như ta đã biết $p_1 \cdot p_2 = p_3 \cdot p_4$ và $p_1 \cdot p_4 = p_2 \cdot p_3$ do đó hạng tử thứ 2 và 3 bằng nhau.
Ngoài ra, do $p_1 \cdot p_3 = -m^2 + p_2 \cdot p_1 - p_2 \cdot p_3$. Ta có thể rút gọn biên độ Feynman thành:

$$
\frac{1}{4} \sum_{s1,s2,\lambda,\lambda'} |\mathcal{M}|^2 = 2e^4 \left[\left(\frac{p_1 \cdot p_4}{p_1 \cdot p_2} + \frac{p_1 \cdot p_2}{p_1 \cdot p_4} \right) + 2m^2 \left(\frac{1}{p_1 \cdot p_4} - \frac{1}{p_1 \cdot p_2} \right) + m^4 \left(\frac{1}{p_1 \cdot p_2} - \frac{1}{p_1 \cdot p_4} \right) \right]
$$



Tính tiết diện tán xạ trong hệ quy chiếu phòng thí nghiệm

Ở đây: $\begin{cases} p = p_1 \\ k = p_2 \\ p' = p_3 \\ k' = p_4 \end{cases}$

Từ hình, ta thu được:

$$\begin{cases} p_1 \cdot p_4 = -m\omega' \\ p_1 \cdot p_2 = -m\omega \\ (p_3)^2 = (p_1 + p_2 - p_4)^2 \to -m^2 = -m^2 + 2p_1 \cdot (p_2 - p_4) + (p_2 - p_4)^2 \to m(\omega - \omega') - \omega\omega'(1 - \cos\theta) = 0 \\ \to \dfrac{1}{\omega'} - \dfrac{1}{\omega} = \dfrac{1 - \cos\theta}{m} \to \omega' = \dfrac{\omega}{1 + \dfrac{\omega}{m}(1 - \cos\theta)} \end{cases}$$

Thay vào biên độ tán xạ Feynman vừa tính, ta thu được:

$$\frac{1}{4} \sum_{s1, s2, \lambda, \lambda'} |\mathcal{M}|^2 = 2e^4 \left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + 2m \left(\frac{-1}{\omega'} + \frac{1}{\omega} \right) + m^2 \left(\frac{-1}{\omega} + \frac{1}{\omega'} \right)^2 \right]$$

Vì đây là hệ quy chiếu khá khác so với hệ quy chiếu khối tâm hay xét. Do đó không gian pha của hệ quy chiếu này có dạng:

$$\begin{aligned} \int d\Pi_2 &= \int \frac{d^3p_4}{(2\pi)^3} \frac{1}{2\omega'} \frac{d^3p_3}{(2\pi)^3} \frac{1}{2E'} (2\pi)^4 \delta^{(4)}(k' + p' - k - p) \\ &= \int \frac{(\omega')^2 d\omega' d\Omega}{(2\pi)^3} \frac{1}{4\omega' E'} \times 2\pi\delta \left(\omega' + \sqrt{m^2 + \omega^2 + (\omega')^2 - 2\omega\omega' \cos\theta} - \omega - m \right) \\ &= \int \frac{d\cos\theta}{2\pi} \frac{\omega'}{4E'} \frac{1}{\left| 1 + \frac{\omega' - \omega\cos\theta}{E'} \right|} \\ &= \frac{1}{8\pi} \int d\cos\theta \frac{\omega'}{m + \omega(1 - \cos\theta)} \\ &= \frac{1}{8\pi} \int d\cos\theta \frac{(\omega')^2}{\omega m} \end{aligned}$$

Thay vào công thức, ta có được tiết diện tán xạ vi phân:

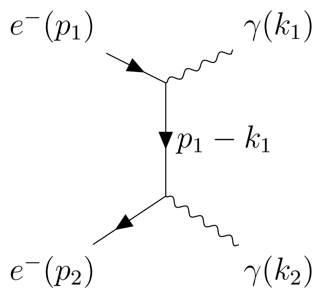
$$\frac{d\sigma}{d\cos\theta} = \frac{1}{4m\omega} \frac{1}{8\pi} \frac{(\omega')^2}{\omega m} \frac{1}{4} \sum_{s1, s2, \lambda, \lambda'} |\mathcal{M}|^2$$

$$= \frac{e^4}{16\pi m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega}+\frac{\omega}{\omega'}+2(\cos\theta-1)+(1-\cos\theta)^2\right]$$
$$= \frac{e^4}{16\pi m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega}+\frac{\omega}{\omega'}-\sin^2\theta\right]$$

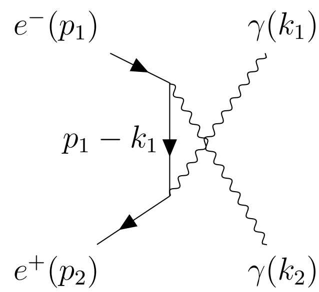
Thay $\alpha = \dfrac{e^2}{4\pi}$:

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega}+\frac{\omega}{\omega'}-\sin^2\theta\right]$$

3/ Annihilation Process



Kênh t



Kênh u

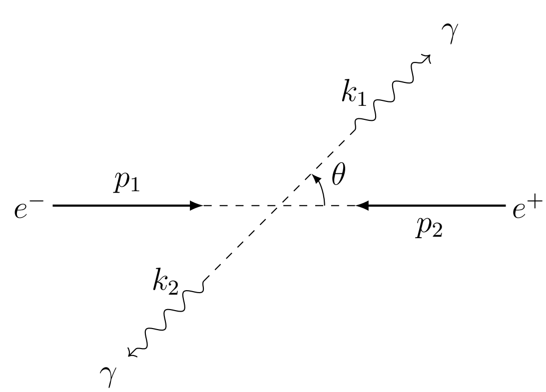
Quá trình này có liên hệ với tán xạ Compton thông qua tính đối xứng giao nhau; ta có thể lấy được biên độ chính xác từ biên độ Compton bằng cách thực hiện các phép thay thế:

$$\begin{cases} p_1 \to p_1 \\ p_3 \to -p_2 \\ p_2 \to -k_1 \\ p_4 \to k_2 \end{cases}$$

Ta thu được biên độ tán xạ Feynman:

$$\frac{1}{4}\sum_{spin}|\mathcal{M}|^2 = -2e^4\left[\frac{p_1\cdot k_2}{p_1\cdot k_1}+\frac{p_1\cdot k_1}{p_1\cdot k_2}+2m^2\left(\frac{1}{p_1\cdot k_1}+\frac{1}{p_1\cdot k_2}\right)-m^4\left(\frac{1}{p_1\cdot k_1}+\frac{1}{p_1\cdot k_2}\right)^2\right]$$

Xét trong hệ quy chiếu khối tâm:



Với:

$$\begin{cases} p_1 = (E,\ 0,\ 0,\ E) \\ p_2 = (E,\ 0,\ 0,\ -E) \\ k_1 = (E,\ E\sin\theta,\ 0,\ E\cos\theta) \\ k_2 = (E,\ -E\sin\theta,\ 0,\ -E\cos\theta) \end{cases}$$

Để tính tiết diện tán xạ, ta cần:

$$\begin{cases} p_1 \cdot k_2 = E^2 + E^2\cos\theta = E^2(1+\cos\theta) \\ p_1 \cdot k_1 = E^2 - E^2\cos\theta = E^2(1-\cos\theta) \end{cases}$$

Trong giới hạn năng lượng cao $E \gg m$, tiết diện tán xạ vi phân:

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2}\frac{1}{4E^2}\frac{E}{|p|}\frac{1}{4}\sum_{spin}|\mathcal{M}|^2 \\ &= \frac{e^4}{32\pi^2}\frac{1}{4E^2}\left(\frac{E^2(1+\cos\theta)}{E^2(1-\cos\theta)} + \frac{E^2(1-\cos\theta)}{E^2(1+\cos\theta)}\right) \\ &= \frac{e^4}{32\pi^2}\frac{1}{4E^2}\frac{(1+\cos\theta)^2 + (1-\cos\theta)^2}{1-\cos^2\theta} \\ &= \frac{e^4}{16\pi^2}\frac{1}{4E^2}\frac{1+\cos^2\theta}{\sin^2\theta} \end{aligned}$$

Đặt $\alpha = \dfrac{e^2}{4\pi}$:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2}\frac{1+\cos^2\theta}{\sin^2\theta}$$

Tiết diện tán xạ toàn phần:

$$\sigma_{total} = \int d\Omega \frac{d\sigma}{d\Omega} = \frac{2\pi\alpha}{4E^2}\int_{-1}^{1}\frac{1+\cos^2\theta}{\sin^2\theta}d\cos\theta = \frac{\pi\alpha}{E^2}\int_{0}^{1}\frac{1+\cos^2\theta}{\sin^2\theta}d\cos\theta$$