We would like to show how to write $\cos(\theta)$ as a nested radical. First, we consider the case where $\theta = \frac{a\pi}{2^n}$ for some $a, n \in \mathbb{Z}$ with a odd and $n > 1$. Consider the half-angle formula $\cos(\frac{\tau}{2}) = \pm\sqrt{\frac{1+\cos(\tau)}{2}} = \pm\frac{\sqrt{2+2\cos(\tau)}}{2}$. Substituting $\theta = 2\tau$, we obtain the formula

$$\cos(\theta) = \pm\frac{\sqrt{2+2\cos(2\theta)}}{2}$$

where the $\pm$ is determined by what quadrant θ is in. We can apply this formula recursively, for example

$$\cos(\theta) = \pm\frac{\sqrt{2+2\cos(2\theta)}}{2} = \pm\frac{\sqrt{2\pm\sqrt{2+2\cos(4\theta)}}}{2} = \pm\frac{\sqrt{2\pm\sqrt{2\pm\sqrt{2+2\cos(8\theta)}}}}{2} = \dots$$

After $n-1$ iterations of this, we will arrive at

$$\cos\theta = \pm\frac{\sqrt{2\pm\sqrt{2\pm\cdots\pm\sqrt{2+2\cos(2^{n-1}\theta)}}}}{2}$$

But $2^{n-1}\theta = \frac{a\pi}{2}$, so $\cos(2^{n-1}\theta) = 0$. Thus the above equation gives us an exact value for $\cos(\theta)$ as a nested radical.

Algorithmically, one can compute $\cos(\theta)$ as follows. Let

$$\mathrm{sign}(x) = \frac{|x|}{x}.$$

Then $\mathrm{sign}(\cos 2^{k-1}\theta)$ determines the kth $\pm$ in the above expression. One need simply compute $\mathrm{sign}(\cos(2^{k-1}\theta))$ for each $2 \leqslant k \leqslant n$ and plug in the appropriate $\pm$ in the above equation. For example, $\mathrm{sign}(\cos(\frac{3\pi}{16})) = 1$, $\mathrm{sign}(\cos(\frac{3\pi}{8})) = 1$, and $\mathrm{sign}(\cos(\frac{3\pi}{4})) = -1$, so $\cos(\frac{3\pi}{16}) = \frac{+\sqrt{2+\sqrt{2-\sqrt{2}}}}{2}$.

Now let θ be such that θ is not of the form $\frac{a\pi}{2^n}$. We claim that

$$\cos(\theta) = \text{sign}(\cos(\theta))\frac{\sqrt{2 + \text{sign}(\cos(2\theta))\sqrt{2 + \text{sign}(\cos(2^2\theta))\sqrt{2 + \dots}}}}{2}.$$

Indeed, let a_i be the nesting sequence

$a_1 = \text{sign}(\cos(\theta))\frac{\sqrt{2}}{2}$,

$a_2 = \text{sign}(\cos(\theta))\frac{\sqrt{2+\text{sign}(\cos(2\theta))\sqrt{2}}}{2}$,

$a_3 = \text{sign}(\cos(\theta))\frac{\sqrt{2+\text{sign}(\cos(2\theta))\sqrt{2+\text{sign}(\cos(4\theta))\sqrt{2}}}}{2}$,

$\vdots$

Since θ is not of the form $\frac{a\pi}{2^n}$, it must be true that for each i there exists a c_i such that $\theta \in (\frac{c_i\pi}{2^i}, \frac{(c_i+1)\pi}{2^i})$. Furthermore, each such interval uniquely determines $\text{sign}(\cos(2^{j-1})\theta)$ for all $j <= i$. Let $\theta_i = \frac{(2c_i+1)\pi}{2^{i+1}}$. Then $\theta_i \in (\frac{c_i\pi}{2^i}, \frac{(c_i+1)\pi}{2^i})$ and so it follows that $\text{sign}(\cos(2^{j-1}\theta)) = \text{sign}(\cos(2^{j-1}\theta_i))$ for all $j \leqslant i$. We conclude that $a_i = \cos(\theta_i)$. By continuity of $\cos(x)$, it suffices to show that $\lim_{i\to\infty}\theta_i = \theta$. But $|\theta - \theta_i| < \frac{1}{2^{i+1}}$. The result follows. $\square$