TCMT for Faraday-Rotation FWM in Isotropic $\chi^{(3)}$

Contents

1 Maxwell wave equation in time domain and setup

We work in nonmagnetic media $\mu = \mu_0$ and split polarization into a linear part (absorbed into ϵ) and a nonlinear part $\mathbf{P}_{\mathrm{NL}}$. The *time-domain* electric-field wave equation is

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}, t) + \mu_0 \epsilon(\mathbf{r})\, \partial_t^2 \mathbf{E}(\mathbf{r}, t) = -\,\mu_0\, \partial_t^2 \mathbf{P}_{\mathrm{NL}}(\mathbf{r}, t), \tag{1}$$

with the usual outgoing-radiation boundary conditions. For frequency components $e^{-\mathrm{i}\omega t}$ this becomes

$$\nabla \times \mu_0^{-1} \nabla \times \mathbf{E}(\mathbf{r},\omega) - \frac{\omega^2}{c^2}\,\epsilon(\mathbf{r},\omega)\mathbf{E}(\mathbf{r},\omega) = \omega^2 \mu_0\, \mathbf{P}_{\mathrm{NL}}(\mathbf{r},\omega). \tag{2}$$

Fields and carriers. We consider five carriers with slowly varying envelopes (SVEA): pumps at $\omega_1 = \omega_p + \Delta/2$ and $\omega_2 = \omega_p - \Delta/2$, probe at ω_s, and probe-generated sidebands at $\Omega_\pm = \omega_s \pm \Delta$. Circular basis $\{\mathbf{e}^{(+)}, \mathbf{e}^{(-)}\}$ with $\mathbf{e}^{(+)}\cdot\mathbf{e}^{(-)} = 1$, $\mathbf{e}^{(\pm)}\cdot\mathbf{e}^{(\pm)} = 0$.

Polarizations. Counter-rotating pumps and probe:

$$\mathbf{E}^{(1)} = E_1\,\mathbf{e}^{(+)}, \quad \mathbf{E}^{(2)} = E_2\,\mathbf{e}^{(-)}, \quad \mathbf{E}^{(s)} = E_+\,\mathbf{e}^{(+)} + E_-\,\mathbf{e}^{(-)},$$

with $|E_\pm| \ll |E_{1,2}|$. We keep only terms linear in the probe envelopes.

2 Eigenmodes and normalization

Let $\{\mathbf{u}_\mu(\mathbf{r}), \omega_\mu\}$ denote (quasi-normal) cavity eigenmodes solving the source-free frequency-domain problem

$$\nabla \times \mu_0^{-1} \nabla \times \mathbf{u}_\mu(\mathbf{r}) = \frac{\omega_\mu^2}{c^2}\,\epsilon(\mathbf{r},\omega_\mu)\,\mathbf{u}_\mu(\mathbf{r}), \tag{3}$$

with outgoing-radiation boundary conditions. We expand the field near each carrier as

$$\mathbf{E}_\mu(\mathbf{r},t) = x_\mu(t)\,\mathbf{u}_\mu(\mathbf{r})\,e^{-\mathrm{i}\omega_\mu t} + \text{c.c.}$$

We adopt the energy normalization

$$U_\mu = \frac{1}{4}\int \Big[\epsilon\,\mathbf{u}_\mu\cdot\mathbf{u}_\mu^* + \mu_0\,\mathbf{h}_\mu\cdot\mathbf{h}_\mu^*\Big]\,dV = 1, \tag{4}$$

so that $|x_\mu|^2$ equals the energy stored in mode μ. *If $U_\mu \neq 1$ is used, then **all** nonlinear overlap coefficients below must be divided by U_μ accordingly.*

Modes and ports. Linear radiative/absorptive losses are captured by total decay rates κ_μ; coupling to external ports p is described by rates $\kappa_{\mu,p}$ with $\kappa_\mu = \kappa_{\mu,\mathrm{i}} + \sum_p \kappa_{\mu,p}$.

3 Isotropic $\chi^{(3)}$ tensor bookkeeping

For any frequency tuple Ξ we use the Maker–Terhune decomposition

$$\chi^{(3)}_{ijkl}(\Xi) = A(\Xi)\,\delta_{ij}\delta_{kl} + B(\Xi)\,\delta_{ik}\delta_{jl} + C(\Xi)\,\delta_{il}\delta_{jk}, \tag{5}$$

with distinct $\{A, B, C\}$ for each tuple; no Kleinman symmetry is assumed. Contractions are performed in the circular basis.

4 Counter-rotating pumps

4.1 Direct third-order polarization at ω_s

Keeping only terms linear in probe,

$$P^{(3)}_i(\omega_s) = \frac{3}{4}\varepsilon_0 \sum_{m\in\{s,1,2\}} \Big[A^{(m)}_s \delta_{ij}\delta_{kl} + B^{(m)}_s \delta_{ik}\delta_{jl} + C^{(m)}_s \delta_{il}\delta_{jk}\Big] E^{(s)}_j E^{(m)}_k E^{(m)*}_l, \tag{6}$$

where $A^{(m)}_s := A(-\omega_s; \omega_s, \omega_m, -\omega_m)$ and similarly for $B^{(m)}_s, C^{(m)}_s$. In $\{\mathbf{e}^{(+)}, \mathbf{e}^{(-)}\}$ this is diagonal:

$$\mathbf{P}^{(3)}(\omega_s) = P_+\,\mathbf{e}^{(+)} + P_-\,\mathbf{e}^{(-)},$$

with P_+ containing $(A^{(1)}_s + B^{(1)}_s)|E_1|^2 E_+$ and $(A^{(2)}_s + C^{(2)}_s)|E_2|^2 E_+$, and P_- the $(+) \leftrightarrow (-)$ counterparts. Thus the *direct* Kerr action is circular-diagonal.

4.2 Probe sidebands at $\Omega_\pm$

With $\Omega_\pm = \omega_s \pm \Delta$ and counter-rotating pumps one finds

$$\mathbf{P}^{(3)}(\Omega_+) = \frac{3}{4}\varepsilon_0\,(B^{\mathrm{sb}}_+ + C^{\mathrm{sb}}_+)\,E_1 E_2^*\,E_-\,\mathbf{e}^{(+)}, \tag{7}$$

$$\mathbf{P}^{(3)}(\Omega_-) = \frac{3}{4}\varepsilon_0\,(B^{\mathrm{sb}}_- + C^{\mathrm{sb}}_-)\,E_2 E_1^*\,E_+\,\mathbf{e}^{(-)}, \tag{8}$$

where, e.g., $B^{\mathrm{sb}}_+ := B(-\Omega_+; \omega_s, \omega_1, -\omega_2)$.

4.3 TCMT from Maxwell projection

Projecting (2) onto $\mathbf{u}^*_\nu$ and using (4) gives, under SVEA and RWA,

$$\dot{x}_\nu = (\mathrm{i}\Delta_\nu - \tfrac{\kappa_\nu}{2})x_\nu + \sum_p \sqrt{\kappa_{\nu,p}}\,s_{\nu p,\mathrm{in}} + F_\nu, \qquad F_\nu = \mathrm{i}\frac{\omega_\nu}{2U_\nu}\int \mathbf{u}^*_\nu(\mathbf{r})\cdot\mathbf{P}_{\mathrm{NL}}(\mathbf{r}, \omega_\nu)\,dV. \tag{9}$$

We now instantiate (9) for the five carriers with envelopes

$$\{A_1 \leftrightarrow \omega_1,\ A_2 \leftrightarrow \omega_2,\ a_\pm \leftrightarrow \omega_s,\ b_+ \leftrightarrow \Omega_+,\ b_- \leftrightarrow \Omega_-\}.$$

4.4 Pumps

$$\dot{A}_1 = (\mathrm{i}\Delta_1 - \tfrac{\kappa_1}{2})A_1 + \sum_p \sqrt{\kappa_{1,p}}\, s_{1p,\mathrm{in}}, \qquad \dot{A}_2 = (\mathrm{i}\Delta_2 - \tfrac{\kappa_2}{2})A_2 + \sum_p \sqrt{\kappa_{2,p}}\, s_{2p,\mathrm{in}}. \tag{10}$$

4.5 Probe at ω_s (direct Kerr)

Projecting the direct $\mathbf{P}^{(3)}(\omega_s)$ onto the probe mode profiles $\mathbf{u}_{s\pm}$ yields

$$\dot{a}_+ = \left[\mathrm{i}\Delta_s - \tfrac{\kappa_s}{2}\right]a_+ + \mathrm{i}\alpha_1|A_1|^2 a_+ + \mathrm{i}\alpha_2|A_2|^2 a_+ + \sum_p \sqrt{\kappa_{s,p}^{(+)}}\, s_{sp,\mathrm{in}}^{(+)}, \tag{11}$$

$$\dot{a}_- = \left[\mathrm{i}\Delta_s - \tfrac{\kappa_s}{2}\right]a_- + \mathrm{i}\tilde{\alpha}_1|A_1|^2 a_- + \mathrm{i}\tilde{\alpha}_2|A_2|^2 a_- + \sum_p \sqrt{\kappa_{s,p}^{(-)}}\, s_{sp,\mathrm{in}}^{(-)}, \tag{12}$$

with overlap coefficients (for $U_{s\pm} = U_{1,2} = 1$)

$$\alpha_1 = \frac{3\omega_s\varepsilon_0}{8}\int (A_s^{(1)} + B_s^{(1)})\, |\mathbf{u}_{s+}|^2\, |\mathbf{u}_{1+}|^2\, dV, \quad \alpha_2 = \frac{3\omega_s\varepsilon_0}{8}\int (A_s^{(2)} + C_s^{(2)})\, |\mathbf{u}_{s+}|^2\, |\mathbf{u}_{2-}|^2\, dV, \tag{13}$$

$$\tilde{\alpha}_1 = \frac{3\omega_s\varepsilon_0}{8}\int (A_s^{(1)} + C_s^{(1)})\, |\mathbf{u}_{s-}|^2\, |\mathbf{u}_{1+}|^2\, dV, \quad \tilde{\alpha}_2 = \frac{3\omega_s\varepsilon_0}{8}\int (A_s^{(2)} + B_s^{(2)})\, |\mathbf{u}_{s-}|^2\, |\mathbf{u}_{2-}|^2\, dV. \tag{14}$$

If $U_\mu \neq 1$, divide each coefficient by the corresponding U factors from (9).

4.6 Sidebands at $\Omega_\pm$ (generation)

Using $\mathbf{P}^{(3)}(\Omega_\pm)$ and projecting onto $\mathbf{u}_{b\pm}$,

$$\dot{b}_+ = \left(\mathrm{i}\Delta_{b+} - \tfrac{\kappa_{b+}}{2}\right)b_+ + \mathrm{i}\,\zeta_+\,(A_1 A_2^*)\,a_-, \tag{15}$$

$$\dot{b}_- = \left(\mathrm{i}\Delta_{b-} - \tfrac{\kappa_{b-}}{2}\right)b_- + \mathrm{i}\,\zeta_-\,(A_2 A_1^*)\,a_+, \tag{16}$$

with

$$\zeta_+ = \frac{3\Omega_+\varepsilon_0}{8}\int (B_+^{\mathrm{sb}} + C_+^{\mathrm{sb}})\,(\mathbf{u}_{b+}^* \cdot \mathbf{u}_{1+})\,(\mathbf{u}_{2-}^* \cdot \mathbf{u}_{s-})\,dV, \tag{17}$$

$$\zeta_- = \frac{3\Omega_-\varepsilon_0}{8}\int (B_-^{\mathrm{sb}} + C_-^{\mathrm{sb}})\,(\mathbf{u}_{b-}^* \cdot \mathbf{u}_{2-})\,(\mathbf{u}_{1+}^* \cdot \mathbf{u}_{s+})\,dV. \tag{18}$$

4.7 Cascaded back-mixing into the probe

The sidebands mix with the opposite pump pair to return to ω_s, producing *off-diagonal* probe drives:

$$\dot{a}_+ = \cdots + \mathrm{i}\,\eta_-\,b_-, \qquad\qquad \dot{a}_- = \cdots + \mathrm{i}\,\eta_+\,b_+, \tag{19}$$

with

$$\eta_+ = \frac{3\omega_s\varepsilon_0}{8}\int (B_+^{\mathrm{mx}}+C_+^{\mathrm{mx}})\,(\mathbf{u}_{s-}^*\cdot\mathbf{u}_{2-})\,(\mathbf{u}_{1+}^*\cdot\mathbf{u}_{b+})\,dV, \tag{20}$$

$$\eta_- = \frac{3\omega_s\varepsilon_0}{8}\int (B_-^{\mathrm{mx}}+C_-^{\mathrm{mx}})\,(\mathbf{u}_{s+}^*\cdot\mathbf{u}_{1+})\,(\mathbf{u}_{2-}^*\cdot\mathbf{u}_{b-})\,dV. \tag{21}$$

In the counter-rotating isotropic case the A-channel cancels in the cascade.

4.8 Adiabatic elimination of sidebands and effective coupling

If sidebands are fast ($|\Delta_{b\pm} - \mathrm{i}\kappa_{b\pm}/2|$ large), set $\dot{b}_\pm \simeq 0$:

$$b_+ \simeq \frac{\zeta_+}{\frac{\kappa_{b+}}{2} - \mathrm{i}\Delta_{b+}}(A_1A_2^*)\,a_-, \qquad b_- \simeq \frac{\zeta_-}{\frac{\kappa_{b-}}{2} - \mathrm{i}\Delta_{b-}}(A_2A_1^*)\,a_+.$$

Then the probe obeys

$$\boxed{\begin{aligned} \dot{a}_+ &= \Big[\mathrm{i}(\Delta_s + \Phi_+) - \tfrac{\kappa_s}{2}\Big]a_+ + \mathrm{i}\,g_{\mathrm{eff}}\,a_+,\\ \dot{a}_- &= \Big[\mathrm{i}(\Delta_s + \Phi_-) - \tfrac{\kappa_s}{2}\Big]a_- + \mathrm{i}\,g_{\mathrm{eff}}^*\,a_-, \end{aligned}} \tag{22}$$

with direct self-phase shifts $\Phi_+ = \alpha_1|A_1|^2 + \alpha_2|A_2|^2$ and $\Phi_- = \tilde{\alpha}_1|A_1|^2 + \tilde{\alpha}_2|A_2|^2$, and cascaded *mixing* strength

$$\boxed{g_{\mathrm{eff}} = \eta_+\frac{\zeta_+}{\frac{\kappa_{b+}}{2} - \mathrm{i}\Delta_{b+}}(A_1A_2^*) = \eta_-^*\frac{\zeta_-^*}{\frac{\kappa_{b-}}{2} + \mathrm{i}\Delta_{b-}}(A_1A_2^*).} \tag{23}$$

Polarization rotation of the probe arises from g_{eff} (off-diagonal coupling) and any circular-birefringent difference $\Phi_+ - \Phi_-$.

4.9 Ports and observables

For port p coupled to mode μ, the usual input–output relation is

$$s_{p,\mathrm{out}} = \sum_{p'} C_{pp'}\,s_{p',\mathrm{in}} + \sqrt{\kappa_{\mu,p}}\,x_\mu,$$

with unitary direct scattering matrix C fixed by energy conservation and time reversal. The transmitted/reflected Jones vector of the probe follows from $a_\pm$ and yields the rotation angle.

Notes. (i) All overlap coefficients inherit the frequency-tuple labels of A, B, C; no Kleinman symmetry is assumed. (ii) In homogeneous isotropic propagation, the dyadic Green tensor reduces to a diagonal scalar, which is consistent with the single-mode $b_\pm$ description used here. (iii) If $U_\mu \neq 1$, remember to divide each nonlinear overlap by U_μ as indicated below (9).

5 Co-rotating Pumps: Direct and Cascaded $\chi^{(3)}$ Paths in TCMT

We now take co-rotating pumps,

$$\mathbf{E}^{(1)} = E_1\,\mathbf{e}^{(+)}, \qquad \mathbf{E}^{(2)} = E_2\,\mathbf{e}^{(+)}, \qquad \Delta = \omega_1 - \omega_2, \quad \Omega_\pm = \omega_s \pm \Delta,$$

and the probe $\mathbf{E}^{(s)} = E_+\,\mathbf{e}^{(+)} + E_-\,\mathbf{e}^{(-)}$ with $|E_\pm| \ll |E_{1,2}|$. The isotropic tensor is kept in Maker–Terhune form with tuple-dependent A, B, C and no Kleinman symmetry.

Direct Kerr at ω_s. As in the counter-rotating case, the direct third-order polarization is diagonal in $\{\mathbf{e}^{(+)}, \mathbf{e}^{(-)}\}$:

$$\mathbf{P}^{(3)}(\omega_s) = P_+^{\mathrm{dir}}\,\mathbf{e}^{(+)} + P_-^{\mathrm{dir}}\,\mathbf{e}^{(-)},$$

with

$$P_+^{\mathrm{dir}} = \tfrac{3}{4}\varepsilon_0\Big[(A_s^{(s)}+C_s^{(s)})(|E_+|^2+|E_-|^2)E_+ + 2B_s^{(s)}|E_-|^2E_+ + \tag{24}$$
$$(A_s^{(1)}+B_s^{(1)})|E_1|^2E_+ + (A_s^{(2)}+B_s^{(2)})|E_2|^2E_+\Big],$$
$$P_-^{\mathrm{dir}} = \tfrac{3}{4}\varepsilon_0\Big[(A_s^{(s)}+C_s^{(s)})(|E_+|^2+|E_-|^2)E_- + 2B_s^{(s)}|E_+|^2E_- + \tag{25}$$
$$(A_s^{(1)}+C_s^{(1)})|E_1|^2E_- + (A_s^{(2)}+C_s^{(2)})|E_2|^2E_-\Big],$$

where $A_s^{(m)} := A(-\omega_s; \omega_s, \omega_m, -\omega_m)$ etc. (As before, with modal energy normalization $U_\mu \neq 1$, divide each nonlinear overlap coefficient by U_μ.)

Sideband generation at $\Omega_\pm$. For co-rotating pumps, both circular components of the sideband polarization survive:

$$\mathbf{P}^{(3)}(\Omega_+) = \tfrac{3}{4}\varepsilon_0\,E_1E_2^*\Big[(A_+^{\mathrm{sb}}+B_+^{\mathrm{sb}})E_+\,\mathbf{e}^{(+)} + (A_+^{\mathrm{sb}}+C_+^{\mathrm{sb}})E_-\,\mathbf{e}^{(-)}\Big], \tag{26}$$
$$\mathbf{P}^{(3)}(\Omega_-) = \tfrac{3}{4}\varepsilon_0\,E_2E_1^*\Big[(A_-^{\mathrm{sb}}+B_-^{\mathrm{sb}})E_+\,\mathbf{e}^{(+)} + (A_-^{\mathrm{sb}}+C_-^{\mathrm{sb}})E_-\,\mathbf{e}^{(-)}\Big], \tag{27}$$

with, e.g., $A_+^{\mathrm{sb}} := A(-\Omega_+; \omega_s, \omega_1, -\omega_2)$.

Sideband vectorization and compact TCMT We group the two circular polarizations of each arm into vectors

$$\boldsymbol{b}_+ = \begin{bmatrix} b_{+,+} \\ b_{+,-} \end{bmatrix} \leftrightarrow \Omega_+, \qquad \boldsymbol{b}_- = \begin{bmatrix} b_{-,+} \\ b_{-,-} \end{bmatrix} \leftrightarrow \Omega_-, \qquad \boldsymbol{a} = \begin{bmatrix} a_+ \\ a_- \end{bmatrix} \leftrightarrow \omega_s.$$

Here, $b_{\sigma,\tau}$ denotes the sideband at arm $\sigma \in \{+,-\}$ with circular polarization $\tau \in \{+,-\}$. We also allow a general *linear* 2×2 sideband block for each arm,

$$\mathbf{K}_{b\pm} \equiv \begin{bmatrix} \frac{\kappa_{b\pm,+}}{2} - \mathrm{i}\,\Delta_{b\pm,+} & -\mathrm{i}\,\Lambda_{b\pm} \\ -\mathrm{i}\,\Lambda^*_{b\pm} & \frac{\kappa_{b\pm,-}}{2} - \mathrm{i}\,\Delta_{b\pm,-} \end{bmatrix},$$

which captures distinct linewidths/detunings for $+$ and $-$ and any linear cross-polarization mixing ($\Lambda_{b\pm} = 0$ for an isotropic cavity without linear $+\leftrightarrow-$ coupling).

Sideband generation (vector form). Projecting the isotropic $\chi^{(3)}$ sideband polarizations onto the sideband modes gives

$$\dot{\boldsymbol{b}}_+ \;=\; -\mathbf{K}_{b+}\,\boldsymbol{b}_+ + \mathrm{i}\,(A_1A_2^*)\,\mathbf{Z}^{(+)}\,\boldsymbol{a}, \qquad \dot{\boldsymbol{b}}_- \;=\; -\mathbf{K}_{b-}\,\boldsymbol{b}_- + \mathrm{i}\,(A_2A_1^*)\,\mathbf{Z}^{(-)}\,\boldsymbol{a}, \tag{28}$$

with 2×2 *generation* matrices

$$\mathbf{Z}^{(+)} = \begin{bmatrix} \zeta^{(+)}_{++} & \zeta^{(+)}_{+-} \\ \zeta^{(+)}_{-+} & \zeta^{(+)}_{--} \end{bmatrix}, \qquad \mathbf{Z}^{(-)} = \begin{bmatrix} \zeta^{(-)}_{++} & \zeta^{(-)}_{+-} \\ \zeta^{(-)}_{-+} & \zeta^{(-)}_{--} \end{bmatrix}.$$

For *co-rotating* pumps in an isotropic medium the only nonzero entries are the *diagonals*

$$\zeta^{(+)}_{++} = \frac{3\Omega_+\varepsilon_0}{8U_{b+}} \int (A^{\mathrm{sb}}_+ + B^{\mathrm{sb}}_+)\,(\mathbf{u}^*_{b+,+}\cdot\mathbf{u}_{1+})(\mathbf{u}^*_{2+}\cdot\mathbf{u}_{s+})\,dV, \tag{29}$$

$$\zeta^{(+)}_{--} = \frac{3\Omega_+\varepsilon_0}{8U_{b+}} \int (A^{\mathrm{sb}}_+ + C^{\mathrm{sb}}_+)\,(\mathbf{u}^*_{b+,-}\cdot\mathbf{u}_{1+})(\mathbf{u}^*_{2+}\cdot\mathbf{u}_{s-})\,dV, \tag{30}$$

(and $\zeta^{(+)}_{+-} = \zeta^{(+)}_{-+} = 0$); similarly at Ω_- with the tuples $(-\Omega_-; \omega_s, \omega_2, -\omega_1)$. For *counter-rotating* pumps, $\mathbf{Z}^{(+)}$ and $\mathbf{Z}^{(-)}$ are diagonal but with only one nonzero element each: $b_{+,+} \leftarrow a_-$ and $b_{-,-} \leftarrow a_+$, exactly matching the contractions in your earlier note.

Back-mixing into the probe (vector form). Projecting the cascaded polarization onto the probe modes yields

$$\dot{\boldsymbol{a}} \;=\; \Big[\mathrm{i}(\Delta_s\mathbf{I}+\boldsymbol{\Phi})-\tfrac{\kappa_s}{2}\mathbf{I}\Big]\boldsymbol{a} + \mathrm{i}\,(A_2A_1^*)\,\mathbf{M}^{(+)}\,\boldsymbol{b}_+ + \mathrm{i}\,(A_1A_2^*)\,\mathbf{M}^{(-)}\,\boldsymbol{b}_- + (\text{probe inputs}), \tag{31}$$

with $\boldsymbol{\Phi} = \mathrm{diag}(\Phi_+, \Phi_-)$ the direct (diagonal) Kerr shifts and 2×2 *mixing* matrices

$$\mathbf{M}^{(+)} = \begin{bmatrix} \eta^{(+)}_{+,+} & \eta^{(+)}_{+,-} \\ \eta^{(+)}_{-,+} & \eta^{(+)}_{-,-} \end{bmatrix}, \qquad \mathbf{M}^{(-)} = \begin{bmatrix} \eta^{(-)}_{+,+} & \eta^{(-)}_{+,-} \\ \eta^{(-)}_{-,+} & \eta^{(-)}_{-,-} \end{bmatrix}.$$

In the *co-rotating, isotropic* case these are again diagonal,

$$\eta^{(+)}_{+,+} = \frac{3\omega_s\varepsilon_0}{8U_{s+}}\int (A^{\mathrm{mx}}_+ + B^{\mathrm{mx}}_+)\,(\mathbf{u}^*_{s+}\cdot\mathbf{u}_{2+})(\mathbf{u}^*_{1+}\cdot\mathbf{u}_{b+,+})\,dV, \tag{32}$$

$$\eta^{(+)}_{-,-} = \frac{3\omega_s\varepsilon_0}{8U_{s-}}\int (A^{\mathrm{mx}}_+ + C^{\mathrm{mx}}_+)\,(\mathbf{u}^*_{s-}\cdot\mathbf{u}_{2+})(\mathbf{u}^*_{1+}\cdot\mathbf{u}_{b+,-})\,dV, \tag{33}$$

(and $\eta^{(+)}_{+,-} = \eta^{(+)}_{-,+} = 0$); similarly for $\mathbf{M}^{(-)}$ with tuples $(-\omega_s; \Omega_-, \omega_1, -\omega_2)$. If the cavity exhibits linear circular mixing at the sideband (nonzero $\Lambda_{b\pm}$), $\mathbf{K}_{b\pm}$ accounts for it and the diagonality of $\mathbf{M}^{(\pm)}$ and $\mathbf{Z}^{(\pm)}$ still holds for an isotropic $\chi^{(3)}$.

Adiabatic elimination and effective probe coupling. When $\mathbf{K}_{b\pm}$ are "fast" (sidebands off-resonant or strongly damped), set $\dot{\boldsymbol{b}}_\pm \simeq \mathbf{0}$ in (28):

$$\boldsymbol{b}_+ \;\simeq\; \mathrm{i}\,(A_1A_2^*)\,\mathbf{K}_{b+}^{-1}\,\mathbf{Z}^{(+)}\,\boldsymbol{a}, \qquad \boldsymbol{b}_- \;\simeq\; \mathrm{i}\,(A_2A_1^*)\,\mathbf{K}_{b-}^{-1}\,\mathbf{Z}^{(-)}\,\boldsymbol{a}.$$

Substituting into (31) gives

$$\dot{\boldsymbol{a}} = \Big[\mathrm{i}(\Delta_s\mathbf{I}+\boldsymbol{\Phi}) - \tfrac{\kappa_s}{2}\mathbf{I}\Big]\boldsymbol{a} \;+\; \mathrm{i}\,\mathbf{G}_{\mathrm{eff}}\,\boldsymbol{a}, \tag{34}$$

$$\mathbf{G}_{\mathrm{eff}} = \mathbf{M}^{(+)}\mathbf{K}_{b+}^{-1}\mathbf{Z}^{(+)}|A_1|^2|A_2|^2 \;+\; \mathbf{M}^{(-)}\mathbf{K}_{b-}^{-1}\mathbf{Z}^{(-)}|A_1|^2|A_2|^2. \tag{35}$$

In isotropic co-rotating pumps, $\mathbf{Z}^{(\pm)}$ and $\mathbf{M}^{(\pm)}$ are diagonal, so $\mathbf{G}_{\mathrm{eff}}$ acquires off-diagonal terms *only* through the linear sideband block $\mathbf{K}_{b\pm}^{-1}$ (i.e., via $\Lambda_{b\pm} \neq 0$) or through asymmetry between the two arms. In counter-rotating pumps, only a single element in each $\mathbf{Z}^{(\pm)}$ is nonzero, reproducing the simpler two-equation form in the main text.

A Ports, Transmission/Reflection, and Polarization Rotation Angle

Input–output relations

For a mode μ coupled to one (left) and/or two (left/right) external waveguide ports with rates $\kappa_{\mu,L}, \kappa_{\mu,R}$, the standard input–output relations are

$$s_{L,\text{out}} = c_{LL}\, s_{L,\text{in}} + c_{LR}\, s_{R,\text{in}} + \sqrt{\kappa_{\mu,L}}\, x_\mu, \tag{36}$$

$$s_{R,\text{out}} = c_{RL}\, s_{L,\text{in}} + c_{RR}\, s_{R,\text{in}} + \sqrt{\kappa_{\mu,R}}\, x_\mu, \tag{37}$$

with a unitary direct-scattering matrix $C = [c_{pq}]$ fixed by energy conservation and time-reversal. For a single-ended cavity (only left port driven), $s_{R,\text{in}} = 0$ and $c_{LL} = -1$ (conventionally).

For the probe, one uses the two circular components $(+, -)$:

$$\boldsymbol{s}_\text{out} = C\, \boldsymbol{s}_\text{in} + \begin{bmatrix} \sqrt{\kappa_{s,L}^{(+)}} & 0 \\ 0 & \sqrt{\kappa_{s,L}^{(-)}} \end{bmatrix} \begin{bmatrix} a_+ \\ a_- \end{bmatrix} \quad \text{(left port),} \qquad \text{and analogously for the right port.}$$

Transmission and reflection (single-ended example)

Driving from the left with circular components $\boldsymbol{s}_\text{in} = [s_{\text{in},+}, s_{\text{in},-}]^T$, the transmitted Jones vector is

$$\boldsymbol{t}(\omega_s) = \frac{\boldsymbol{s}_{R,\text{out}}}{\boldsymbol{s}_\text{in}} = C_{R:}\, \mathbf{e}_L + \begin{bmatrix} \sqrt{\kappa_{s,R}^{(+)}} & 0 \\ 0 & \sqrt{\kappa_{s,R}^{(-)}} \end{bmatrix} \left(-\mathbf{M}^{-1}\right) \begin{bmatrix} \sqrt{\kappa_{s,L}^{(+)}} & 0 \\ 0 & \sqrt{\kappa_{s,L}^{(-)}} \end{bmatrix},$$

where $\mathbf{e}_L = [1, 0]^T$, $C_{R:}$ is the row of C for the right port, and

$$\mathbf{M} = \left[\tfrac{\kappa_s}{2} - \mathrm{i}(\Delta_s \mathbf{I} + \boldsymbol{\Phi})\right] - \mathrm{i}\, \boldsymbol{G}_\text{eff}$$

is the 2×2 dynamical matrix of the probe at steady state (with pumps fixed). The reflection matrix $\boldsymbol{r}(\omega_s)$ follows by replacing the right-port couplings by the left ones in the second factor and using C_{LL} in the direct term.

Polarization rotation angle from circular components

Let the transmitted probe Jones vector in the circular basis be $\boldsymbol{E}^{(\text{out})} = [E_+^{(\text{out})}, E_-^{(\text{out})}]^T$. Define the complex ratio

$$\rho = \frac{E_+^{(\text{out})}}{E_-^{(\text{out})}}.$$

The polarization ellipse satisfies

$$\text{rotation angle} \quad \psi = \tfrac{1}{2}\arg(\rho), \qquad \text{ellipticity angle} \quad \chi = \tfrac{1}{2}\arcsin\left(\frac{|E_+^{(\mathrm{out})}|^2 - |E_-^{(\mathrm{out})}|^2}{|E_+^{(\mathrm{out})}|^2 + |E_-^{(\mathrm{out})}|^2}\right).$$

Equivalently, one may compute Stokes parameters

$$S_0 = |E_+|^2 + |E_-|^2, \;\; S_1 = 2\,\mathrm{Re}(E_+E_-^*), \;\; S_2 = 2\,\mathrm{Im}(E_+E_-^*), \;\; S_3 = |E_+|^2 - |E_-|^2,$$

then $\psi = \frac{1}{2}\arctan 2(S_2, S_1)$ and $\chi = \frac{1}{2}\arcsin(S_3/S_0)$. For small rotation (nearly linear, $|E_+| \simeq |E_-|$), $\psi \simeq \frac{1}{2}\,\mathrm{Im}[\ln(\rho)]$.