Probe, Sidebands, and Cascaded Back-Action at ω_s in the General $\chi^{(3)}_{ijkl}$ Tensor Formalism

Conventions and fields

Time convention $e^{-i\omega t}$. The (analytic) electric field is written as

$$E_i(t) = \frac{1}{2}\sum_q \Big(E_i^{(q)} e^{-i\omega_q t} + E_i^{(q)*} e^{+i\omega_q t}\Big), \tag{1}$$

with a weak probe at ω_s and two pumps at ω_1, ω_2 (the latter will later be specialized to $\omega_{1,2} = \omega_p \pm \Delta/2$ only to name their beat Δ; none of the derivations below assumes isotropy or a polarization basis). We use Einstein summation over repeated Cartesian indices $i, j, k, l, \ldots$.

Third-order polarization (frequency domain). For any target frequency Ω,

$$P_i^{(3)}(\Omega) = \varepsilon_0\, \chi^{(3)}_{ijkl}\big(-\Omega; \omega_\alpha, \omega_\beta, \omega_\gamma\big)\, E_j^{(\alpha)} E_k^{(\beta)} E_l^{(\gamma)}\ \delta(\omega_\alpha + \omega_\beta + \omega_\gamma - \Omega). \tag{2}$$

It is convenient to use the *intrinsic-permutation symmetrization* over the last three slots (indices/frequencies):

$$\bar{\chi}^{(3)}_{ijkl}\big(-\Omega; \omega_a, \omega_b, \omega_c\big) := \frac{1}{3!}\sum_{\pi\in S_3} \chi^{(3)}_{i\,\pi(jkl)}\big(-\Omega; \pi[\omega_a, \omega_b, \omega_c]\big). \tag{3}$$

When extracting Ω components from an analytic field (half-field decomposition), the usual degeneracy factor 3/4 appears for the fundamental-like channels considered below.

A. Direct third-order polarization at $\boldsymbol{\omega_s}$

Retain only terms *linear in the probe* $E^{(s)}$. The contributing triplets are $(\omega_s, \omega_m, -\omega_m)$ with $m \in \{s, 1, 2\}$. Hence,

$$P_i^{(3)}(\omega_s) = \frac{3}{4}\,\varepsilon_0 \sum_{m\in\{s,1,2\}} \bar{\chi}^{(3)}_{ijkl}\big(-\omega_s; \omega_s, \omega_m, -\omega_m\big)\, E_j^{(s)}\, E_k^{(m)} E_l^{(m)*}. \tag{4}$$

Equation (4) contains probe SPM ($m = s$) and XPM from each pump ($m = 1, 2$). No basis or material symmetry has been assumed.

B. Probe sidebands at $\boldsymbol{\omega_s \pm \Delta}$

Let $\Delta_{mn} := \omega_m - \omega_n$. The sideband frequencies that are *linear in the probe* are

$$\Omega^{(+)}_{mn} = \omega_s + \Delta_{mn} = \omega_s + \omega_m - \omega_n, \qquad \Omega^{(-)}_{mn} = \omega_s - \Delta_{mn} = \omega_s - \omega_m + \omega_n. \tag{5}$$

From (2) with the contributing triplets $(\omega_s, \omega_m, -\omega_n)$ and permutations, we obtain

$$P_i^{(3)}\big(\Omega^{(+)}_{mn}\big) = \frac{3}{4}\,\varepsilon_0\, \bar{\chi}^{(3)}_{ijkl}\big(-\Omega^{(+)}_{mn}; \omega_s, \omega_m, -\omega_n\big)\, E_j^{(s)}\, E_k^{(m)} E_l^{(n)*}, \tag{6}$$

$$P_i^{(3)}\big(\Omega^{(-)}_{mn}\big) = \frac{3}{4}\,\varepsilon_0\, \bar{\chi}^{(3)}_{ijkl}\big(-\Omega^{(-)}_{mn}; \omega_s, -\omega_m, \omega_n\big)\, E_j^{(s)}\, E_k^{(m)*} E_l^{(n)}. \tag{7}$$

Equations (6)–(7) are fully tensorial and basis free.

C. Linear propagation from polarization to sideband fields

Let $\mathcal{G}_{ja}(\Omega)$ denote the *linear* Green tensor at frequency Ω that maps polarization to field (this includes local material response and propagation/phase matching). Then the sideband fields generated by (6)–(7) are

$$E_j^{(s+)}\big(\Omega_{mn}^{(+)}\big) = \mathcal{G}_{ja}\big(\Omega_{mn}^{(+)}\big)\ P_a^{(3)}\big(\Omega_{mn}^{(+)}\big), \tag{8}$$

$$E_j^{(s-)}\big(\Omega_{mn}^{(-)}\big) = \mathcal{G}_{ja}\big(\Omega_{mn}^{(-)}\big)\ P_a^{(3)}\big(\Omega_{mn}^{(-)}\big). \tag{9}$$

D. Cascaded back-action to $\boldsymbol{\omega_s}$

The generated sidebands can mix with the *opposite* pump pair to return energy to ω_s because

$$\big(\omega_s + \omega_m - \omega_n\big) + \omega_n - \omega_m = \omega_s, \qquad \big(\omega_s - \omega_m + \omega_n\big) + \omega_m - \omega_n = \omega_s.$$

Therefore, the cascaded third-order polarization (again linear in $E^{(s)}$) is

$$P_i^{(3),\mathrm{casc}}(\omega_s) = \frac{3}{4}\,\varepsilon_0 \sum_{m,n} \Big[\bar{\chi}_{ijkl}^{(3)}\big(-\omega_s; \Omega_{mn}^{(+)}, \omega_n, -\omega_m\big)\ E_j^{(s+)}\big(\Omega_{mn}^{(+)}\big)\ E_k^{(n)} E_l^{(m)*}$$
$$+\ \bar{\chi}_{ijkl}^{(3)}\big(-\omega_s; \Omega_{mn}^{(-)}, \omega_m, -\omega_n\big)\ E_j^{(s-)}\big(\Omega_{mn}^{(-)}\big)\ E_k^{(m)} E_l^{(n)*} \Big]. \tag{10}$$

Substituting (8)–(9) and then (6)–(7) yields a compact *fifth-order* effective response acting on the probe:

$$P_i^{(3),\mathrm{casc}}(\omega_s) = \Big(\frac{3}{4}\,\varepsilon_0\Big)^2 \sum_{m,n} \Bigg\{ \underbrace{\bar{\chi}_{ijkl}^{(3)}\big(-\omega_s; \Omega_{mn}^{(+)}, \omega_n, -\omega_m\big)\, \mathcal{G}_{ja}\big(\Omega_{mn}^{(+)}\big)\, \bar{\chi}_{abcd}^{(3)}\big(-\Omega_{mn}^{(+)}; \omega_s, \omega_m, -\omega_n\big)}_{\mathcal{K}_{ibklcd}^{(+)}(\omega_s;\, \Omega_{mn}^{(+)})} \tag{11}$$

$$E_b^{(s)}\ E_c^{(m)} E_d^{(n)*}\ E_k^{(n)} E_l^{(m)*}$$
$$+\ \underbrace{\bar{\chi}_{ijkl}^{(3)}\big(-\omega_s; \Omega_{mn}^{(-)}, \omega_m, -\omega_n\big)\, \mathcal{G}_{ja}\big(\Omega_{mn}^{(-)}\big)\, \bar{\chi}_{abcd}^{(3)}\big(-\Omega_{mn}^{(-)}; \omega_s, -\omega_m, \omega_n\big)}_{\mathcal{K}_{ibklcd}^{(-)}(\omega_s;\, \Omega_{mn}^{(-)})} \tag{12}$$

$$E_b^{(s)}\ E_c^{(m)*} E_d^{(n)}\ E_k^{(m)} E_l^{(n)*} \Bigg\}. \tag{13}$$

It is natural to introduce pump dyadics $M_{kl}^{(mn)} := E_k^{(n)} E_l^{(m)*}$ and probe vectors $E_b^{(s)}$ to exhibit the effective rank-2 response felt by the probe:

$$P_i^{(3),\mathrm{casc}}(\omega_s) = \Theta_{ib}^{(5)}(\omega_s)\ E_b^{(s)}, \tag{14}$$

$$\Theta_{ib}^{(5)}(\omega_s) = \Big(\frac{3}{4}\,\varepsilon_0\Big)^2 \sum_{m,n} \Big[\mathcal{K}_{ibklcd}^{(+)}(\omega_s; \Omega_{mn}^{(+)})\, M_{kl}^{(mn)}\ M_{cd}^{(mn)} + \mathcal{K}_{ibklcd}^{(-)}(\omega_s; \Omega_{mn}^{(-)})\, M_{kl}^{(nm)}\ M_{cd}^{(nm)}\Big]. \tag{15}$$

Equation (15) is a *general* (basis-free) expression for the cascaded, pump-intensity-quadratic correction at ω_s produced by $\chi^{(3)}$ sideband generation plus linear propagation plus $\chi^{(3)}$ back-mixing.

E. Total probe polarization at $\boldsymbol{\omega_s}$

Combining the direct third-order response (4) and the cascaded contribution (15) gives

$$\begin{aligned} P_i(\omega_s) &= P_i^{(3)}(\omega_s) \;+\; P_i^{(3),\mathrm{casc}}(\omega_s) \\ &= \frac{3}{4}\,\varepsilon_0 \sum_{m\in\{s,1,2\}} \bar{\chi}^{(3)}_{ijkl}\big(-\omega_s;\omega_s,\omega_m,-\omega_m\big)\,E_j^{(s)}\,E_k^{(m)}E_l^{(m)*} \;+\; \Theta^{(5)}_{ib}(\omega_s)\,E_b^{(s)}. \end{aligned} \tag{16}$$

The first term is the standard SPM/XPM tensor response (order $\chi^{(3)}E_s|E|^2$); the second is an *effective fifth-order* correction (order $\chi^{(3)}\mathcal{G}\chi^{(3)}\,E_s|E_1|^2|E_2|^2$) whose size and sign are governed by the full tensor contractions and the linear Green tensor at the sideband frequencies.

Remarks.

- No assumptions about isotropy, Kleinman symmetry, or polarization basis were made. All frequency arguments are explicit.
- The sideband routes $\Omega^{(\pm)}_{mn}$ may include $m=n$ (collapsing to the fundamental and reproducing the direct term) or distinct pumps $m\neq n$ (true probe satellites at $\omega_s \pm \Delta_{mn}$).
- A concrete propagation model (bulk plane waves, guided modes, etc.) specifies $\mathcal{G}_{ja}(\Omega)$ and thus exposes explicit phase-mismatch denominators inside $\Theta^{(5)}_{ib}(\omega_s)$.