Particular case: Isotropic $\chi^{(3)}$ for counter- and co-rotating pumps

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1 Isotropic $\chi^{(3)}$ with Counter-Rotating Pumps: Probe, Sidebands, and Cascaded Back-Action in Tensor Notation

Fields and basis. Time convention $e^{-\mathrm{i}\omega t}$. Let $\{e_i^+, e_i^-, e_i^z\}$ be the circular basis with $e_i^+ e_i^+ = 0$, $e_i^- e_i^- = 0$, $e_i^+ e_i^- = 1$, and $(e_i^+)^* = e_i^-$. We take

$$E_i^{(1)} = E_1 e_i^+, \qquad E_i^{(2)} = E_2 e_i^-, \qquad E_i^{(s)} = E_+ e_i^+ + E_- e_i^-.$$

Define $\Omega_\pm := \omega_s \pm \Delta$ with $\Delta = \omega_1 - \omega_2$.

Isotropic $\chi^{(3)}$ decomposition. For any frequency tuple Ξ (explicitly indicated below),

$$\chi^{(3)}_{ijkl}(\Xi) = A(\Xi)\,\delta_{ij}\delta_{kl} \;+\; B(\Xi)\,\delta_{ik}\delta_{jl} \;+\; C(\Xi)\,\delta_{il}\delta_{jk}.$$

We will keep A, B, C *distinct for each tuple.*

1.1 Direct third-order polarization at ω_s

Keeping only terms linear in $E^{(s)}$ (SPM on the probe and XPM from the pumps),

$$P_i^{(3)}(\omega_s) = \frac{3}{4}\varepsilon_0 \sum_{m\in\{s,1,2\}} \left[A_s^{(m)}\,\delta_{ij}\delta_{kl} + B_s^{(m)}\,\delta_{ik}\delta_{jl} + C_s^{(m)}\,\delta_{il}\delta_{jk}\right] E_j^{(s)} E_k^{(m)} E_l^{(m)*}, \tag{1}$$

with shorthand

$$A_s^{(m)} := A\big(-\omega_s;\omega_s,\omega_m,-\omega_m\big),\quad B_s^{(m)} := B\big(-\omega_s;\omega_s,\omega_m,-\omega_m\big),\quad C_s^{(m)} := C\big(-\omega_s;\omega_s,\omega_m,-\omega_m\big).$$

Insert the pump/probe polarizations. Using $|E^{(1)}|^2 = |E_1|^2$, $|E^{(2)}|^2 = |E_2|^2$, $(E^{(s)}\cdot E^{(1)}) = E_-E_1$, $(E^{(s)}\cdot E^{(1)*}) = E_+E_1^*$, $(E^{(s)}\cdot E^{(2)}) = E_+E_2$, $(E^{(s)}\cdot E^{(2)*}) = E_-E_2^*$, and $E^{(s)}\cdot E^{(s)} = 2E_+E_-$, one finds

$$P_i^{(3)}(\omega_s) = P_+\, e_i^+ + P_-\, e_i^-,$$

with scalar envelopes

$$\begin{aligned} P_+ = \frac{3}{4}\varepsilon_0\Big[&(A_s^{(s)}+C_s^{(s)})\big(|E_+|^2+|E_-|^2\big)\,E_+ + 2B_s^{(s)}|E_-|^2E_+ \\ &+ A_s^{(1)}|E_1|^2E_+ + A_s^{(2)}|E_2|^2E_+ + B_s^{(1)}|E_1|^2E_+ + C_s^{(2)}|E_2|^2E_+\Big], \end{aligned} \tag{2}$$

$$\begin{aligned} P_- = \frac{3}{4}\varepsilon_0\Big[&(A_s^{(s)}+C_s^{(s)})\big(|E_+|^2+|E_-|^2\big)\,E_- + 2B_s^{(s)}|E_+|^2E_- \\ &+ A_s^{(2)}|E_2|^2E_- + A_s^{(1)}|E_1|^2E_- + C_s^{(1)}|E_1|^2E_- + B_s^{(2)}|E_2|^2E_-\Big]. \end{aligned} \tag{3}$$

Thus, the $\chi_{\text{eff}}^{(3)}$ in circular basis is diagonal, and the difference between the elements is (we omit nonlinear terms for probe, $E^{(s)}$):

$$\chi_{++} - \chi_{--} = |E_1|^2(B_s^{(1)} - C_s^{(1)}) - |E_2|^2(B_s^{(2)} - C_s^{(2)}) \tag{4}$$

So, we can conlude that if $|E_1| = |E_2|$ or $B_s^{(m)} = C_s^{(m)}$, there is no Faraday rotation.

1.2 Probe sidebands at $\Omega_\pm = \omega_s \pm \Delta$

From the isotropic sideband formula (with distinct coefficients for the *sideband-generation* tuples),

$$\mathbf{P}^{(3)}(\Omega_+) = \frac{3}{4}\varepsilon_0\Big[A_+^{\text{sb}}\,(E^{(1)}\cdot E^{(2)*})\,\mathbf{E}^{(s)} + B_+^{\text{sb}}\,(\mathbf{E}^{(s)}\cdot E^{(2)*})\,\mathbf{E}^{(1)} + C_+^{\text{sb}}\,(\mathbf{E}^{(s)}\cdot E^{(1)})\,\mathbf{E}^{(2)*}\Big], \tag{5}$$

$$\mathbf{P}^{(3)}(\Omega_-) = \frac{3}{4}\varepsilon_0\Big[A_-^{\text{sb}}\,(E^{(2)}\cdot E^{(1)*})\,\mathbf{E}^{(s)} + B_-^{\text{sb}}\,(\mathbf{E}^{(s)}\cdot E^{(1)*})\,\mathbf{E}^{(2)} + C_-^{\text{sb}}\,(\mathbf{E}^{(s)}\cdot E^{(2)})\,\mathbf{E}^{(1)*}\Big], \tag{6}$$

with

$$\begin{aligned} A_+^{\text{sb}} &:= A(-\Omega_+;\omega_s,\omega_1,-\omega_2),\ B_+^{\text{sb}} := B(-\Omega_+;\omega_s,\omega_1,-\omega_2),\ C_+^{\text{sb}} := C(-\Omega_+;\omega_s,\omega_1,-\omega_2), \\ A_-^{\text{sb}} &:= A(-\Omega_-;\omega_s,\omega_2,-\omega_1),\ B_-^{\text{sb}} := B(-\Omega_-;\omega_s,\omega_2,-\omega_1),\ C_-^{\text{sb}} := C(-\Omega_-;\omega_s,\omega_2,-\omega_1). \end{aligned}$$

Using $E^{(1)}\cdot E^{(2)*} = 0$, $(\mathbf{E}^{(s)}\cdot E^{(2)*}) = E_-E_2^*$, $(\mathbf{E}^{(s)}\cdot E^{(1)}) = E_-E_1$, and $\mathbf{E}^{(1)} \parallel e^+$, $\mathbf{E}^{(2)*} \parallel e^+$, both (5) contributions align with e^+:

$$\boxed{\mathbf{P}^{(3)}(\Omega_+) = \frac{3}{4}\varepsilon_0\,(B_+^{\text{sb}} + C_+^{\text{sb}})\,E_1E_2^*\,E_-\,\mathbf{e}^+.}$$

Similarly, using $(\mathbf{E}^{(s)}\cdot E^{(1)*}) = E_+E_1^*$, $(\mathbf{E}^{(s)}\cdot E^{(2)}) = E_+E_2$ and $\mathbf{E}^{(2)} \parallel e^-$, $\mathbf{E}^{(1)*} \parallel e^-$, both terms in (6) align with e^-:

$$\boxed{\mathbf{P}^{(3)}(\Omega_-) = \frac{3}{4}\varepsilon_0\,(B_-^{\text{sb}} + C_-^{\text{sb}})\,E_2E_1^*\,E_+\,\mathbf{e}^-.}$$

From sideband polarization to field. Let $\mathcal{G}_{ja}(\Omega)$ be the linear Green tensor. Define its circular projections

$$G_{++}(\Omega) := e_j^{(+)*}\,\mathcal{G}_{ja}(\Omega)\,e_a^{(+)}, \qquad G_{--}(\Omega) := e_j^{(-)*}\,\mathcal{G}_{ja}(\Omega)\,e_a^{(-)},$$

(i.e., $U^\dagger \mathcal{G} U$ and pick diagonal $\pm$ elements). Then the generated sideband *fields* are

$$\boxed{\mathbf{E}^{(s+)} = G_{++}(\Omega_+)\,\mathbf{P}^{(3)}(\Omega_+), \qquad \mathbf{E}^{(s-)} = G_{--}(\Omega_-)\,\mathbf{P}^{(3)}(\Omega_-).}$$

1.3 Cascaded back-mixing to ω_s

The sidebands mix with the *opposite* pump pair:

$$(\Omega_+) + \omega_2 - \omega_1 = \omega_s, \qquad (\Omega_-) + \omega_1 - \omega_2 = \omega_s.$$

The cascaded polarization has two arms, each with its own *mixing* coefficients

$$\begin{aligned} B_+^{\mathrm{mx}} &:= B(-\omega_s;\Omega_+,\omega_2,-\omega_1), & C_+^{\mathrm{mx}} &:= C(-\omega_s;\Omega_+,\omega_2,-\omega_1),\\ B_-^{\mathrm{mx}} &:= B(-\omega_s;\Omega_-,\omega_1,-\omega_2), & C_-^{\mathrm{mx}} &:= C(-\omega_s;\Omega_-,\omega_1,-\omega_2). \end{aligned}$$

Using the isotropic contraction rules, the surviving terms are

$$\begin{aligned} \mathbf{P}^{(3),\mathrm{casc}}(\omega_s) = \frac{3}{4}\varepsilon_0\Big\{ & B_+^{\mathrm{mx}}\,(\mathbf{E}^{(s+)}\!\cdot E^{(1)*})\,\mathbf{E}^{(2)} + C_+^{\mathrm{mx}}\,(\mathbf{E}^{(s+)}\!\cdot E^{(2)})\,\mathbf{E}^{(1)*}\\ & + B_-^{\mathrm{mx}}\,(\mathbf{E}^{(s-)}\!\cdot E^{(2)*})\,\mathbf{E}^{(1)} + C_-^{\mathrm{mx}}\,(\mathbf{E}^{(s-)}\!\cdot E^{(1)})\,\mathbf{E}^{(2)*}\Big\}. \end{aligned} \tag{7}$$

Since $\mathbf{E}^{(s+)} \parallel e^+$, $E^{(1)*} \parallel e^-$, $E^{(2)} \parallel e^-$ and $\mathbf{E}^{(s-)} \parallel e^-$, $E^{(2)*} \parallel e^+$, $E^{(1)} \parallel e^+$, all four dot products are unity and the A-channel vanishes. Substituting the sideband fields gives a diagonal correction in the circular basis:

$$\boxed{\begin{aligned} \mathbf{P}^{(3),\mathrm{casc}}(\omega_s) = \Big(\frac{3}{4}\varepsilon_0\Big)^2 |E_1|^2|E_2|^2 \Big[& (B_+^{\mathrm{mx}})(B_+^{\mathrm{sb}} + C_+^{\mathrm{sb}})\,G_{++}(\Omega_+)\,E_-\,\mathbf{e}^-\\ & + (B_-^{\mathrm{mx}})(B_-^{\mathrm{sb}} + C_-^{\mathrm{sb}})\,G_{--}(\Omega_-)\,E_+\,\mathbf{e}^+\Big]\\ + \Big(\frac{3}{4}\varepsilon_0\Big)^2 |E_1|^2|E_2|^2 \Big[& (C_+^{\mathrm{mx}})(B_+^{\mathrm{sb}} + C_+^{\mathrm{sb}})\,G_{++}(\Omega_+)\,E_-\,\mathbf{e}^-\\ & + (C_-^{\mathrm{mx}})(B_-^{\mathrm{sb}} + C_-^{\mathrm{sb}})\,G_{--}(\Omega_-)\,E_+\,\mathbf{e}^+\Big]. \end{aligned}}$$

Equivalently, grouping B and C on each arm:

$$\boxed{\begin{aligned} \mathbf{P}^{(3),\mathrm{casc}}(\omega_s) = \Big(\frac{3}{4}\varepsilon_0\Big)^2 |E_1|^2|E_2|^2 \Big[& \underbrace{(B_-^{\mathrm{mx}} + C_-^{\mathrm{mx}})(B_-^{\mathrm{sb}} + C_-^{\mathrm{sb}})\,G_{--}(\Omega_-)}_{\Pi_-(\Omega_-)}\,E_+\,\mathbf{e}^+\\ & + \underbrace{(B_+^{\mathrm{mx}} + C_+^{\mathrm{mx}})(B_+^{\mathrm{sb}} + C_+^{\mathrm{sb}})\,G_{++}(\Omega_+)}_{\Pi_+(\Omega_+)}\,E_-\,\mathbf{e}^-\Big]. \end{aligned}}$$

If we substitute the sideband fields (for TCMT), we get:

$$\mathbf{P}^{(3),\mathrm{casc}}(\omega_s) = \frac{3}{4}\varepsilon_0\,|E_1|^2|E_2|^2\Big[(B_-^{\mathrm{mx}} + C_-^{\mathrm{mx}})\,E_{s-}^{(-)}\,\mathbf{e}^+ + (B_+^{\mathrm{mx}} + C_+^{\mathrm{mx}})E_{s+}^{(+)}\,\mathbf{e}^-\Big]. \tag{8}$$

1.4 Total probe polarization at ω_s

Writing $\mathbf{P}(\omega_s) = P_+^{\rm tot}\,\mathbf{e}^+ + P_-^{\rm tot}\,\mathbf{e}^-$, we have

$$\boxed{\begin{aligned} P_+^{\rm tot} &= P_+ \;+\; \Big(\frac{3}{4}\varepsilon_0\Big)^2 |E_1|^2|E_2|^2\,\Pi_-(\Omega_-)\,E_+, \\ P_-^{\rm tot} &= P_- \;+\; \Big(\frac{3}{4}\varepsilon_0\Big)^2 |E_1|^2|E_2|^2\,\Pi_+(\Omega_+)\,E_-, \end{aligned}}$$

with $P_\pm$ from (2)–(3). The cascaded factors $\Pi_\pm$ explicitly carry the *distinct* coefficients from sideband generation ($B_\pm^{\rm sb}, C_\pm^{\rm sb}$), the back-mixing stage ($B_\pm^{\rm mx}, C_\pm^{\rm mx}$), and the circular projections of the linear Green tensor $G_{\pm\pm}(\Omega_\pm)$.

Remarks.

- No Kleinman symmetry was used; all A, B, C are evaluated at the specific frequency tuples indicated by the superscripts sb (sideband) and mx (mixing), and by the arm sign $+$ or $-$.
- The A-channel drops out of the cascaded step because $E^{(2)} \cdot E^{(1)*} = 0$ for counter-rotating pumps in the transverse circular basis.
- In gyrotropic or anisotropic backgrounds, keep the full 2×2 circular block of $\mathcal{G}$; here we used diagonal projections $G_{\pm\pm}$.

2 Co-rotating Pumps: Direct and Cascaded $P^{(3)}(\omega_s)$ in Isotropic Tensor Form

Setup and notation Time convention $e^{-\mathrm{i}\omega t}$. Circular basis $\{\mathbf{e}^{(+)}, \mathbf{e}^{(-)}, \mathbf{e}^{(z)}\}$ with $\mathbf{e}^{(+)}\cdot\mathbf{e}^{(-)} = 1$, $\mathbf{e}^{(\pm)}\cdot\mathbf{e}^{(\pm)} = 0$, and $(\mathbf{e}^{(+)})^* = \mathbf{e}^{(-)}$. Fields:

$$E_i^{(1)} = E_1\, e_i^{(+)}, \qquad E_i^{(2)} = E_2\, e_i^{(+)}, \qquad E_i^{(s)} = E_+\, e_i^{(+)} + E_-\, e_i^{(-)},$$

and $\Delta = \omega_1 - \omega_2$, $\Omega_\pm = \omega_s \pm \Delta$.

Isotropic third-order tensor (frequency-tuple dependent). For any frequency tuple Ξ,

$$\chi^{(3)}_{ijkl}(\Xi) = A(\Xi)\,\delta_{ij}\delta_{kl} + B(\Xi)\,\delta_{ik}\delta_{jl} + C(\Xi)\,\delta_{il}\delta_{jk}.$$

We keep A, B, C distinct for each Ξ.

2.1 Direct third-order polarization at ω_s

Start from

$$P_i^{(3)}(\omega_s) = \frac{3}{4}\varepsilon_0 \sum_{m\in\{s,1,2\}} \Big[A_s^{(m)}\delta_{ij}\delta_{kl} + B_s^{(m)}\delta_{ik}\delta_{jl} + C_s^{(m)}\delta_{il}\delta_{jk}\Big]\, E_j^{(s)} E_k^{(m)} E_l^{(m)*}, \tag{9}$$

with $A_s^{(m)} := A(-\omega_s; \omega_s, \omega_m, -\omega_m)$ and similarly $B_s^{(m)}, C_s^{(m)}$.

Write $P_i^{(3)}(\omega_s) = P_+ e_i^{(+)} + P_- e_i^{(-)}$. A straightforward contraction in the circular basis gives a diagonal result:

$$\boxed{\begin{aligned} P_+ &= \frac{3}{4}\varepsilon_0\Big[(A_s^{(s)} + C_s^{(s)})(|E_+|^2 + |E_-|^2)\,E_+ + 2B_s^{(s)}|E_-|^2\,E_+ \\ &\qquad\qquad + (A_s^{(1)} + B_s^{(1)})|E_1|^2\,E_+ + (A_s^{(2)} + B_s^{(2)})|E_2|^2\,E_+\Big], \\ P_- &= \frac{3}{4}\varepsilon_0\Big[(A_s^{(s)} + C_s^{(s)})(|E_+|^2 + |E_-|^2)\,E_- + 2B_s^{(s)}|E_+|^2\,E_- \\ &\qquad\qquad + (A_s^{(1)} + C_s^{(1)})|E_1|^2\,E_- + (A_s^{(2)} + C_s^{(2)})|E_2|^2\,E_-\Big]. \end{aligned}} \tag{10}$$

2.2 Probe sidebands at $\Omega_\pm = \omega_s \pm \Delta$

Using the isotropic contraction for triplets $(\omega_s, \omega_1, -\omega_2)$ and $(\omega_s, \omega_2, -\omega_1)$,

$$\mathbf{P}^{(3)}(\Omega_+) = \tfrac{3}{4}\varepsilon_0\Big[A_+^{\mathrm{sb}}\,(\mathbf{E}^{(1)}\cdot\mathbf{E}^{(2)*})\,\mathbf{E}^{(s)} + B_+^{\mathrm{sb}}\,(\mathbf{E}^{(s)}\cdot\mathbf{E}^{(2)*})\,\mathbf{E}^{(1)} + C_+^{\mathrm{sb}}\,(\mathbf{E}^{(s)}\cdot\mathbf{E}^{(1)})\,\mathbf{E}^{(2)*}\Big], \tag{11}$$

$$\mathbf{P}^{(3)}(\Omega_-) = \tfrac{3}{4}\varepsilon_0\Big[A_-^{\mathrm{sb}}\,(\mathbf{E}^{(2)}\cdot\mathbf{E}^{(1)*})\,\mathbf{E}^{(s)} + B_-^{\mathrm{sb}}\,(\mathbf{E}^{(s)}\cdot\mathbf{E}^{(1)*})\,\mathbf{E}^{(2)} + C_-^{\mathrm{sb}}\,(\mathbf{E}^{(s)}\cdot\mathbf{E}^{(2)})\,\mathbf{E}^{(1)*}\Big], \tag{12}$$

where, e.g., $A_+^{\mathrm{sb}} := A(-\Omega_+; \omega_s, \omega_1, -\omega_2)$, etc.

For co-rotating pumps $\mathbf{E}^{(1)} \parallel \mathbf{e}^{(+)}$, $\mathbf{E}^{(2)} \parallel \mathbf{e}^{(+)}$:

$$\mathbf{E}^{(1)}\cdot\mathbf{E}^{(2)*} = E_1E_2^*, \quad \mathbf{E}^{(s)}\cdot\mathbf{E}^{(2)*} = E_+E_2^*, \quad \mathbf{E}^{(s)}\cdot\mathbf{E}^{(1)} = E_-E_1.$$

Decomposing along $\mathbf{e}^{(\pm)}$ gives

$$\boxed{\mathbf{P}^{(3)}(\Omega_+) = \tfrac{3}{4}\varepsilon_0\,E_1E_2^*\Big[(A_+^{\mathrm{sb}} + B_+^{\mathrm{sb}})\,E_+\,\mathbf{e}^{(+)} + (A_+^{\mathrm{sb}} + C_+^{\mathrm{sb}})\,E_-\,\mathbf{e}^{(-)}\Big],} \tag{13}$$

$$\boxed{\mathbf{P}^{(3)}(\Omega_-) = \tfrac{3}{4}\varepsilon_0\,E_2E_1^*\Big[(A_-^{\mathrm{sb}} + B_-^{\mathrm{sb}})\,E_+\,\mathbf{e}^{(+)} + (A_-^{\mathrm{sb}} + C_-^{\mathrm{sb}})\,E_-\,\mathbf{e}^{(-)}\Big].} \tag{14}$$

From sideband polarization to field. Let the circular-basis Green block be

$$\mathbf{G}(\Omega) = \begin{pmatrix} G_{++}(\Omega) & G_{+-}(\Omega) \\ G_{-+}(\Omega) & G_{--}(\Omega) \end{pmatrix}, \quad G_{\alpha\beta}(\Omega) := e_j^{(\alpha)*}\,\mathcal{G}_{ja}(\Omega)\,e_a^{(\beta)}.$$

Define column vectors

$$\mathbf{S}^{(+)} = \begin{bmatrix} (A_+^{\mathrm{sb}} + B_+^{\mathrm{sb}})\,E_+ \\ (A_+^{\mathrm{sb}} + C_+^{\mathrm{sb}})\,E_- \end{bmatrix}, \qquad \mathbf{S}^{(-)} = \begin{bmatrix} (A_-^{\mathrm{sb}} + B_-^{\mathrm{sb}})\,E_+ \\ (A_-^{\mathrm{sb}} + C_-^{\mathrm{sb}})\,E_- \end{bmatrix}.$$

Then

$$\begin{bmatrix} E_+^{(s+)} \\ E_-^{(s+)} \end{bmatrix} = \tfrac{3}{4}\varepsilon_0\,(E_1E_2^*)\,\mathbf{G}(\Omega_+)\,\mathbf{S}^{(+)}, \qquad \begin{bmatrix} E_+^{(s-)} \\ E_-^{(s-)} \end{bmatrix} = \tfrac{3}{4}\varepsilon_0\,(E_2E_1^*)\,\mathbf{G}(\Omega_-)\,\mathbf{S}^{(-)}. \tag{15}$$

2.3 Cascaded back-mixing to ω_s

Frequencies and coefficients. The cascaded back-mixing paths satisfy $(\Omega_+) + \omega_2 - \omega_1 = \omega_s$ and $(\Omega_-) + \omega_1 - \omega_2 = \omega_s$. For the *mixing* stage define

$$\begin{aligned} A_+^{\mathrm{mx}} &:= A(-\omega_s; \Omega_+, \omega_2, -\omega_1), \quad B_+^{\mathrm{mx}} := B(-\omega_s; \Omega_+, \omega_2, -\omega_1), \quad C_+^{\mathrm{mx}} := C(-\omega_s; \Omega_+, \omega_2, -\omega_1), \\ A_-^{\mathrm{mx}} &:= A(-\omega_s; \Omega_-, \omega_1, -\omega_2), \quad B_-^{\mathrm{mx}} := B(-\omega_s; \Omega_-, \omega_1, -\omega_2), \quad C_-^{\mathrm{mx}} := C(-\omega_s; \Omega_-, \omega_1, -\omega_2). \end{aligned}$$

Upper arm (+). Using $E^{(2)} \parallel \mathbf{e}^{(+)}$, $E^{(1)*} \parallel \mathbf{e}^{(-)}$, the mixing at Ω_+ preserves the sideband's circular index within each channel, so the two channels (A, B) add on the + component and (A, C) add on the − component:

$$
\begin{aligned}
P^{(3),\mathrm{casc}}_{(+),+} &= \tfrac{3}{4}\varepsilon_0 \left(A^{\mathrm{mx}}_{+} + B^{\mathrm{mx}}_{+}\right) E_2 E_1^* \, E^{(s+)}_{+}, \\
P^{(3),\mathrm{casc}}_{(+),-} &= \tfrac{3}{4}\varepsilon_0 \left(A^{\mathrm{mx}}_{+} + C^{\mathrm{mx}}_{+}\right) E_2 E_1^* \, E^{(s+)}_{-}.
\end{aligned}
$$

Lower arm (−). Using $E^{(1)} \parallel \mathbf{e}^{(+)}$, $E^{(2)*} \parallel \mathbf{e}^{(-)}$, the mixing at Ω_- gives

$$
\begin{aligned}
P^{(3),\mathrm{casc}}_{(-),+} &= \tfrac{3}{4}\varepsilon_0 \left(A^{\mathrm{mx}}_{-} + B^{\mathrm{mx}}_{-}\right) E_1 E_2^* \, E^{(s-)}_{+}, \\
P^{(3),\mathrm{casc}}_{(-),-} &= \tfrac{3}{4}\varepsilon_0 \left(A^{\mathrm{mx}}_{-} + C^{\mathrm{mx}}_{-}\right) E_1 E_2^* \, E^{(s-)}_{-}.
\end{aligned}
$$

Collected result. Summing the two arms and projecting on $\{\mathbf{e}^{(+)}, \mathbf{e}^{(-)}\}$ gives

$$
\boxed{
\begin{aligned}
P^{(3),\mathrm{casc}}_{+} &= (A^{\mathrm{mx}}_{+} + B^{\mathrm{mx}}_{+}) \, E_2 E_1^* \, E^{(s+)}_{+} + (A^{\mathrm{mx}}_{-} + B^{\mathrm{mx}}_{-}) \, E_1 E_2^* \, E^{(s-)}_{+}, \\
P^{(3),\mathrm{casc}}_{-} &= (A^{\mathrm{mx}}_{+} + C^{\mathrm{mx}}_{+}) \, E_2 E_1^* \, E^{(s+)}_{-} + (A^{\mathrm{mx}}_{-} + C^{\mathrm{mx}}_{-}) \, E_1 E_2^* \, E^{(s-)}_{-}.
\end{aligned}
}
$$

Cascaded back-mixing with sideband fields included. From (15),

$$
\begin{bmatrix} E^{(s+)}_{+} \\ E^{(s+)}_{-} \end{bmatrix} = \tfrac{3}{4}\varepsilon_0 \, (E_1 E_2^*) \, \mathbf{G}(\Omega_+) \, \mathbf{S}^{(+)}, \qquad
\begin{bmatrix} E^{(s-)}_{+} \\ E^{(s-)}_{-} \end{bmatrix} = \tfrac{3}{4}\varepsilon_0 \, (E_2 E_1^*) \, \mathbf{G}(\Omega_-) \, \mathbf{S}^{(-)}.
$$

Using the collected mixing relations

$$
\begin{aligned}
P^{(3),\mathrm{casc}}_{+} &= (A^{\mathrm{mx}}_{+} + B^{\mathrm{mx}}_{+}) \, E_2 E_1^* \, E^{(s+)}_{+} + (A^{\mathrm{mx}}_{-} + B^{\mathrm{mx}}_{-}) \, E_1 E_2^* \, E^{(s-)}_{+}, \\
P^{(3),\mathrm{casc}}_{-} &= (A^{\mathrm{mx}}_{+} + C^{\mathrm{mx}}_{+}) \, E_2 E_1^* \, E^{(s+)}_{-} + (A^{\mathrm{mx}}_{-} + C^{\mathrm{mx}}_{-}) \, E_1 E_2^* \, E^{(s-)}_{-},
\end{aligned}
$$

we obtain the compact block form

$$
\boxed{
\begin{bmatrix} P^{\mathrm{casc}}_{+} \\ P^{\mathrm{casc}}_{-} \end{bmatrix} = \left(\tfrac{3}{4}\varepsilon_0\right)^2 |E_1|^2 |E_2|^2 \Big[\mathbf{D}^{(+)} \, \mathbf{G}(\Omega_+) \, \mathbf{S}^{(+)} + \mathbf{D}^{(-)} \, \mathbf{G}(\Omega_-) \, \mathbf{S}^{(-)} \Big],
}
$$

where

$$
\mathbf{S}^{(+)} = \begin{bmatrix} (A^{\mathrm{sb}}_{+} + B^{\mathrm{sb}}_{+}) \, E_+ \\ (A^{\mathrm{sb}}_{+} + C^{\mathrm{sb}}_{+}) \, E_- \end{bmatrix}, \qquad
\mathbf{S}^{(-)} = \begin{bmatrix} (A^{\mathrm{sb}}_{-} + B^{\mathrm{sb}}_{-}) \, E_+ \\ (A^{\mathrm{sb}}_{-} + C^{\mathrm{sb}}_{-}) \, E_- \end{bmatrix}
$$

and

$$
\mathbf{D}^{(+)} = \begin{pmatrix} A^{\mathrm{mx}}_{+} + B^{\mathrm{mx}}_{+} & 0 \\ 0 & A^{\mathrm{mx}}_{+} + C^{\mathrm{mx}}_{+} \end{pmatrix}, \qquad
\mathbf{D}^{(-)} = \begin{pmatrix} A^{\mathrm{mx}}_{-} + B^{\mathrm{mx}}_{-} & 0 \\ 0 & A^{\mathrm{mx}}_{-} + C^{\mathrm{mx}}_{-} \end{pmatrix}, \tag{16}
$$

Scalar expansions. Writing $\mathbf{G}(\Omega) = \begin{pmatrix} G_{++} & G_{+-} \\ G_{-+} & G_{--} \end{pmatrix}$,

$$\boxed{\begin{aligned}
P_+^{(3),\mathrm{casc}} &= \left(\tfrac{3}{4}\varepsilon_0\right)^2 |E_1|^2|E_2|^2 \Big[(A_+^{\mathrm{mx}} + B_+^{\mathrm{mx}}) \big(G_{++}(\Omega_+)\, S_1^{(+)} + G_{+-}(\Omega_+)\, S_2^{(+)} \big) \\
&\qquad\qquad + (A_-^{\mathrm{mx}} + B_-^{\mathrm{mx}}) \big(G_{++}(\Omega_-)\, S_1^{(-)} + G_{+-}(\Omega_-)\, S_2^{(-)} \big) \Big], \\
P_-^{(3),\mathrm{casc}} &= \left(\tfrac{3}{4}\varepsilon_0\right)^2 |E_1|^2|E_2|^2 \Big[(A_+^{\mathrm{mx}} + C_+^{\mathrm{mx}}) \big(G_{-+}(\Omega_+)\, S_1^{(+)} + G_{--}(\Omega_+)\, S_2^{(+)} \big) \\
&\qquad\qquad + (A_-^{\mathrm{mx}} + C_-^{\mathrm{mx}}) \big(G_{-+}(\Omega_-)\, S_1^{(-)} + G_{--}(\Omega_-)\, S_2^{(-)} \big) \Big],
\end{aligned}}$$

with

$$S_1^{(+)} = (A_+^{\mathrm{sb}} + B_+^{\mathrm{sb}})\, E_+, \quad S_2^{(+)} = (A_+^{\mathrm{sb}} + C_+^{\mathrm{sb}})\, E_-, \quad S_1^{(-)} = (A_-^{\mathrm{sb}} + B_-^{\mathrm{sb}})\, E_+, \quad S_2^{(-)} = (A_-^{\mathrm{sb}} + C_-^{\mathrm{sb}})\, E_-.$$

Notes. (i) The pump products combine as $(E_2E_1^*)(E_1E_2^*) = |E_1|^2|E_2|^2$, so all residual phase sensitivity resides in $\mathbf{G}(\Omega_\pm)$ and the probe $E_\pm$. (ii) Off-diagonal $G_{\pm\mp}$ terms are the channels that convert circular components and are the ones that ultimately drive polarization rotation in the probe.

2.4 Total probe polarization at ω_s

Let $P_\pm^{\mathrm{dir}}$ denote the direct terms from (10). The full result is

$$\boxed{\begin{bmatrix} P_+^{\mathrm{tot}} \\ P_-^{\mathrm{tot}} \end{bmatrix} = \begin{bmatrix} P_+^{\mathrm{dir}} \\ P_-^{\mathrm{dir}} \end{bmatrix} + \left(\tfrac{3}{4}\varepsilon_0\right)^2 \Big[\mathbf{D}^{(+)}\, \mathbf{G}(\Omega_+)\, \mathbf{S}^{(+)} + \mathbf{D}^{(-)}\, \mathbf{G}(\Omega_-)\, \mathbf{S}^{(-)} \Big].}$$

Remarks.

- No Kleinman symmetry was used; every A, B, C carries its own frequency tuple: $\{-\omega_s; \omega_s, \omega_m, -\omega_m\}$ for the direct term, $\{-\Omega_\pm; \omega_s, \omega_{1/2}, -\omega_{2/1}\}$ for sideband generation, and $\{-\omega_s; \Omega_\pm, \omega_{2/1}, -\omega_{1/2}\}$ for back-mixing.
- In homogeneous isotropic propagation, $\mathbf{G}(\Omega_\pm)$ reduces to a diagonal scalar times $\mathbf{I}_2$, simplifying the cascaded matrices.
- The direct term is strictly diagonal in $\{+,-\}$ (no mixing); any $+ \leftrightarrow -$ coupling at ω_s here originates from the cascaded pathway and/or off-diagonal $G_{\alpha\beta}$.