Faraday Rotation from $\chi^{(3)}$ in an Isotropic Film

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Abstract

We derive the effective third-order susceptibility $\underline{\underline{\chi}}^{\,\mathrm{eff}}(\omega_s)$ experienced by a weak, linearly polarized signal at frequency ω_s propagating through a homogeneous, centrosymmetric, isotropic film in the xy plane. The medium is driven by two strong circularly polarized pumps: pump 1 (helicity σ^+) at $\omega_{p+} = \omega_p + \Delta/2$, and pump 2 (helicity σ^-) at $\omega_{p-} = \omega_p - \Delta/2$. Keeping all phase factors and frequency selection rules explicit, we reduce the isotropic rank-4 tensor $\chi^{(3)}_{ijkl}$ to scalar and axial invariants, construct $\underline{\underline{\chi}}^{\,\mathrm{eff}}$, and extract the circular eigenindices $n_\pm$ and the Faraday rotation angle θ_F at the carrier. Cross-tone (beating) terms are shown to generate sidebands at $\omega_s \pm \Delta$ and are not part of $\underline{\underline{\chi}}^{\,\mathrm{eff}}(\omega_s)$.

1 Fields, Frequencies, and Conventions

We work with analytic electric fields; physical fields are the real parts (let's not forget:). Cartesian indices $i, j, k, l \in \{x, y, z\}$ are summed when repeated (again, good to remember:).

Signal (probe). A weak probe (or signal) at ω_s:

$$E_{s,i}(t) = E_{s,i}(\omega_s)\, e^{-i\omega_s t}. \tag{1}$$

Pumps. Two strong tones

$$E^{(1)}_k(t) = E^{(1)}_k(\omega_{p+})\, e^{-i\omega_{p+} t}, \qquad \omega_{p+} = \omega_p + \frac{\Delta}{2}, \tag{2}$$

$$E^{(2)}_k(t) = E^{(2)}_k(\omega_{p-})\, e^{-i\omega_{p-} t}, \qquad \omega_{p-} = \omega_p - \frac{\Delta}{2}. \tag{3}$$

In the frequency domain,

$$E_{p,k}(\omega) = E^{(1)}_k(\omega) \,+\, E^{(2)}_k(\omega), \qquad E^{(j)}_k(\omega) = E^{(j)}_k(\omega_{pj})\, \delta(\omega - \omega_{pj}). \tag{4}$$

Circular basis. Define $\hat{\mathbf{e}}_\pm = (\hat{\mathbf{x}} \mp i\hat{\mathbf{y}})/\sqrt{2}$, with $\hat{\mathbf{e}}_\pm \cdot \hat{\mathbf{e}}_\pm = 1$ and $\hat{\mathbf{e}}_+ \cdot \hat{\mathbf{e}}_- = 0$. For the specific case considered later we set

$$\mathbf{E}^{(1)} = E^{(1)}_+\, \hat{\mathbf{e}}_+, \qquad \mathbf{E}^{(2)} = E^{(2)}_-\, \hat{\mathbf{e}}_-. \tag{5}$$

2 Third-Order Polarization at a Specified Output Frequency

The frequency-resolved third-order polarization is

$$P^{(3)}_i(\omega) = \varepsilon_0 \sum_{\omega_1,\omega_2,\omega_3} \chi^{(3)}_{ijkl}(-\omega; \omega_1, \omega_2, \omega_3)\, E_j(\omega_1) E_k(\omega_2) E_l(\omega_3)\, \delta\big(\omega - \omega_1 - \omega_2 - \omega_3\big). \tag{6}$$

For cross-phase modulation (XPM) of the signal at $\omega = \omega_s$, we keep the pathway $(-\omega_s;\, \omega_s, \omega_a, -\omega_b)$:

$$P_i^{(3)}(\omega_s) = \varepsilon_0 \sum_{\omega_a,\omega_b} \chi^{(3)}_{ijkl}(-\omega_s; \omega_s, \omega_a, -\omega_b)\, E_{s,j}(\omega_s)\, E_{p,k}(\omega_a)\, E^*_{p,l}(\omega_b)\, \delta(\omega_a - \omega_b). \tag{7}$$

The delta function enforces $\omega_a = \omega_b$, so *only same-tone pump pairs contribute at the carrier* ω_s. Cross-tone products with $\omega_a \neq \omega_b$ produce time factors $e^{\mp i\Delta t}$ and appear at $\omega_s \pm \Delta$ (sidebands), not at ω_s. With two tones ω_{p+} and ω_{p-}, Eq. (7) reduces to

$$P_i^{(3)}(\omega_s) = \varepsilon_0 \sum_{m=1}^{2} \chi^{(3,j)}_{ijkl}\, E_{s,j}(\omega_s)\, E_k^{(j)}(\omega_{pj})\, E_l^{(j)\,*}(\omega_{pj}), \quad \chi^{(3,j)}_{ijkl} \equiv \chi^{(3)}_{ijkl}(-\omega_s; \omega_s, \omega_{pj}, -\omega_{pj})\,. \tag{8}$$

3 Isotropic $\chi^{(3)}$ and Cartesian Contractions

For a homogeneous isotropic centrosymmetric medium (off resonance), the rank-4 tensor decomposes into three scalar coefficients (often denoted χ_A, χ_B, χ_C) built from Kronecker deltas:

$$\chi^{(3,j)}_{ijkl} = \chi_A^{(j)}\, \delta_{ij}\delta_{kl} + \chi_B^{(j)}\, \delta_{ik}\delta_{jl} + \chi_C^{(j)}\, \delta_{il}\delta_{jk}. \tag{9}$$

Insert (9) into (8) and contract indices (Einstein summation understood). For each tone j:

$$\text{(A)}: \quad \chi_A^{(j)}\, \delta_{ij}\delta_{kl}\, E_{s,j}\, E_k^{(j)} E_l^{(j)\,*} = \chi_A^{(j)}\, (\mathbf{E}^{(j)} \cdot \mathbf{E}^{(j)\,*})\, E_{s,i}, \tag{10}$$
$$\text{(B)}: \quad \chi_B^{(j)}\, \delta_{ik}\delta_{jl}\, E_{s,j}\, E_k^{(j)} E_l^{(j)\,*} = \chi_B^{(j)}\, E_i^{(j)}\, (\mathbf{E}_s \cdot \mathbf{E}^{(j)\,*}), \tag{11}$$
$$\text{(C)}: \quad \chi_C^{(j)}\, \delta_{il}\delta_{jk}\, E_{s,j}\, E_k^{(j)} E_l^{(j)\,*} = \chi_C^{(j)}\, E_i^{(j)\,*}\, (\mathbf{E}_s \cdot \mathbf{E}^{(j)}). \tag{12}$$

Summing $j = 1, 2$,

$$P_i^{(3)}(\omega_s) = \varepsilon_0 \sum_{m=1}^{2} \Big[\chi_A^{(j)} (\mathbf{E}^{(j)} \cdot \mathbf{E}^{(j)\,*})\, E_{s,i} + \chi_B^{(j)}\, E_i^{(j)}\, (\mathbf{E}_s \cdot \mathbf{E}^{(j)\,*}) + \chi_C^{(j)}\, E_i^{(j)\,*}\, (\mathbf{E}_s \cdot \mathbf{E}^{(j)})\Big]. \tag{13}$$

Equation (13) is linear in $\mathbf{E}_s$; thus we identify the *effective* rank-2 tensor

$$\boxed{\chi^{\text{eff}}_{ij}(\omega_s) = \sum_{m=1}^{2} \Big[\chi_A^{(m)} (\mathbf{E}^{(m)} \cdot \mathbf{E}^{(m)\,*})\, \delta_{ij} + \chi_B^{(m)}\, E_i^{(m)}\, E_j^{(m)\,*} + \chi_C^{(m)}\, E_i^{(m)\,*}\, E_j^{(m)}\Big],} \tag{14}$$

such that $P_i^{(3)} = \varepsilon_0\, \chi^{\text{eff}}_{ij} E_{s,j}$. All fast phase factors are contained in the frequency labels of $\chi^{(3)}$ and the phasors; no residual time dependence remains at ω_s because of the $\delta(\omega_a - \omega_b)$ selection.

4 Specialization to Circular Pumps and Circular Eigenbasis

Let the beams propagate along $+\hat{\mathbf{z}}$. With $\mathbf{E}^{(1)} = E_+^{(1)}\, \hat{\mathbf{e}}_+$ and $\mathbf{E}^{(2)} = E_-^{(2)}\, \hat{\mathbf{e}}_-$, the scalar products are $\mathbf{E}^{(1)} \cdot \mathbf{E}^{(1)\,*} = |E_+^{(1)}|^2$ and $\mathbf{E}^{(2)} \cdot \mathbf{E}^{(2)\,*} = |E_-^{(2)}|^2$. The dyadics are $E_i^{(1)} E_j^{(1)\,*} = |E_+^{(1)}|^2\, (\hat{e}_+)_i (\hat{e}_+)^*_j$ and $E_i^{(2)} E_j^{(2)\,*} = |E_-^{(2)}|^2\, (\hat{e}_-)_i (\hat{e}_-)^*_j$, and similarly with $i \leftrightarrow j$. In the circular basis $\{\hat{\mathbf{e}}_+, \hat{\mathbf{e}}_-\}$, $\underline{\underline{\chi}}^{\text{eff}}(\omega_s)$ is diagonal at the carrier:

$$\boxed{\begin{aligned} \chi^{\text{eff}}_{++} &= \sum_{m=1}^{2} \chi_A^{(j)} |E^{(j)}|^2 + (\chi_B^{(1)} + \chi_C^{(1)}) |E_+^{(1)}|^2, \\ \chi^{\text{eff}}_{--} &= \sum_{m=1}^{2} \chi_A^{(j)} |E^{(j)}|^2 + (\chi_B^{(2)} + \chi_C^{(2)}) |E_-^{(2)}|^2, \\ \chi^{\text{eff}}_{+-} &= \chi^{\text{eff}}_{-+} = 0. \end{aligned}} \tag{15}$$

It is convenient to regroup (per tone) into scalar and axial combinations

$$\chi_{\rm iso}^{(j)} \equiv \chi_A^{(j)} + \frac{\chi_B^{(j)} + \chi_C^{(j)}}{2}, \qquad \chi_{\rm circ}^{(j)} \equiv \frac{\chi_B^{(j)} + \chi_C^{(j)}}{2}, \tag{16}$$

and to define tone-summed pump invariants at the carrier

$$I_{\rm tot} \equiv |E_+^{(1)}|^2 + |E_-^{(2)}|^2, \qquad S_3^{\rm DC} \equiv |E_+^{(1)}|^2 - |E_-^{(2)}|^2 \ \propto\ i(\mathbf{E}_p \times \mathbf{E}_p^*)\cdot \hat{\mathbf{z}}. \tag{17}$$

Equation (15) becomes

$$\boxed{\chi_{\pm\pm}^{\rm eff}(\omega_s) = \Big[\sum_{m=1}^{2} \chi_{\rm iso}^{(j)}\Big]\ I_{\rm tot}\ \pm\ \Big[\sum_{m=1}^{2} \chi_{\rm circ}^{(j)}\Big]\ S_3^{\rm DC}.} \tag{18}$$

The first term is a true scalar (Kerr/XPM), while the second is an axial pseudoscalar (helicity or inverse Faraday effect), which splits the circular eigenmodes ($\sigma^\pm$), i.e. produces circular birefringence.

Same-helicity pumps ($\sigma+\sigma+$ and $\sigma-\sigma-$)

We substitute the pump fields directly into the effective third-order tensor

$$\chi_{ij}^{\rm eff}(\omega_s) = \sum_{m=1}^{2} \Big[\chi_A^{(m)} \Big(\mathbf{E}^{(m)}\cdot\mathbf{E}^{(m)*}\Big)\,\delta_{ij} + \chi_B^{(m)} E_i^{(m)} E_j^{(m)*} + \chi_C^{(m)} E_i^{(m)*} E_j^{(m)}\Big], \tag{19}$$

which follows from contracting the isotropic rank-4 tensor $\chi_{ijkl}^{(3)} = \chi_A\,\delta_{ij}\delta_{kl} + \chi_B\,\delta_{ik}\delta_{jl} + \chi_C\,\delta_{il}\delta_{jk}$ with one signal factor and *same-tone* pump–pump factors at the signal carrier ω_s. Here $\chi_{A,B,C}^{(m)} \equiv \chi_{A,B,C}(-\omega_s;\omega_s,\omega_{pm},-\omega_{pm})$ and the tone frequencies are $\omega_{p1} = \omega_p + \Delta/2$, $\omega_{p2} = \omega_p - \Delta/2$. Only terms with $\omega_a = \omega_b$ contribute at ω_s; cross-tone products generate sidebands at $\omega_s \pm \Delta$ and are excluded from $\chi^{\rm eff}(\omega_s)$.

Circular basis. Let $\hat{\mathbf{e}}_\pm = (\hat{\mathbf{x}} \mp i\hat{\mathbf{y}})/\sqrt{2}$, $\hat{\mathbf{e}}_\pm\cdot\hat{\mathbf{e}}_\pm = 1$, $\hat{\mathbf{e}}_+\cdot\hat{\mathbf{e}}_- = 0$, and note $\hat{\mathbf{e}}_+^* = \hat{\mathbf{e}}_-$, $\hat{\mathbf{e}}_-^* = \hat{\mathbf{e}}_+$.

Case A: $\sigma+\sigma+$ (both pumps right-circular)

Pumps:

$$\mathbf{E}^{(1)} = E_+^{(1)}\,\hat{\mathbf{e}}_+, \qquad \mathbf{E}^{(2)} = E_+^{(2)}\,\hat{\mathbf{e}}_+. \tag{20}$$

Per tone,

$$\mathbf{E}^{(m)}\cdot\mathbf{E}^{(m)*} = |E_+^{(m)}|^2, \quad E_i^{(m)} E_j^{(m)*} = |E_+^{(m)}|^2\,(\hat{e}_+)_i(\hat{e}_-)_j, \quad E_i^{(m)*} E_j^{(m)} = |E_+^{(m)}|^2\,(\hat{e}_-)_i(\hat{e}_+)_j. \tag{21}$$

Insert into (19) and sum $m = 1, 2$:

$$\chi_{ij}^{\rm eff}(\omega_s) = \underbrace{\Big(\sum_m \chi_A^{(m)} |E_+^{(m)}|^2\Big)}_{\Xi_A^{(+)}}\delta_{ij} + \underbrace{\Big(\sum_m \chi_B^{(m)} |E_+^{(m)}|^2\Big)}_{\Xi_B^{(+)}}(\hat{e}_+)_i(\hat{e}_-)_j + \underbrace{\Big(\sum_m \chi_C^{(m)} |E_+^{(m)}|^2\Big)}_{\Xi_C^{(+)}}(\hat{e}_-)_i(\hat{e}_+)_j. \tag{22}$$

In the circular basis $\{\hat{\mathbf{e}}_+, \hat{\mathbf{e}}_-\}$ the matrix representation is

$$\big[\chi^{\rm eff}(\omega_s)\big]_{\rm (circ)}^{(++)} = \begin{pmatrix} \Xi_{++}^{(+)} & 0 \\ 0 & \Xi_{--}^{(+)} \end{pmatrix}, , \qquad \Xi_{A,B,C}^{(+)} = \sum_{m=1}^{2} \chi_{A,B,C}^{(m)}\,|E_+^{(m)}|^2. \tag{23}$$

Here the diagonal entries in the circular basis are

$$\Xi_{++}^{(+)} = \Xi_A^{(+)} + \Xi_B^{(+)} + \Xi_C^{(+)}, \qquad \Xi_{--}^{(+)} = \Xi_A^{(+)} - (\Xi_B^{(+)} + \Xi_C^{(+)}). \tag{24}$$

Case B: $\sigma-\sigma-$ **(both pumps left-circular)**

Pumps:

$$\mathbf{E}^{(1)} = E_-^{(1)}\,\hat{\mathbf{e}}_-, \qquad \mathbf{E}^{(2)} = E_-^{(2)}\,\hat{\mathbf{e}}_-. \tag{25}$$

Per tone,

$$\mathbf{E}^{(m)}\cdot\mathbf{E}^{(m)*} = |E_-^{(m)}|^2, \quad E_i^{(m)}E_j^{(m)*} = |E_-^{(m)}|^2\,(\hat{e}_-)_i(\hat{e}_+)_j, \quad E_i^{(m)*}E_j^{(m)} = |E_-^{(m)}|^2\,(\hat{e}_+)_i(\hat{e}_-)_j. \tag{26}$$

Insert into (19) and sum $m = 1, 2$:

$$\chi_{ij}^{\text{eff}}(\omega_s) = \underbrace{\Big(\sum_m \chi_A^{(m)}|E_-^{(m)}|^2\Big)}_{\Xi_A^{(-)}}\delta_{ij} + \underbrace{\Big(\sum_m \chi_B^{(m)}|E_-^{(m)}|^2\Big)}_{\Xi_B^{(-)}}(\hat{e}_-)_i(\hat{e}_+)_j + \underbrace{\Big(\sum_m \chi_C^{(m)}|E_-^{(m)}|^2\Big)}_{\Xi_C^{(-)}}(\hat{e}_+)_i(\hat{e}_-)_j. \tag{27}$$

In the circular basis $\{\hat{\mathbf{e}}_+, \hat{\mathbf{e}}_-\}$ the matrix representation is

$$\big[\chi^{\text{eff}}(\omega_s)\big]^{(--)}_{(\text{circ})} = \begin{pmatrix} \Xi_{++}^{(-)} & 0 \\ 0 & \Xi_{--}^{(-)} \end{pmatrix},, \qquad \Xi_{A,B,C}^{(-)} = \sum_{m=1}^{2} \chi_{A,B,C}^{(m)}\,|E_-^{(m)}|^2. \tag{28}$$

Analogously,

$$\Xi_{++}^{(-)} = \Xi_A^{(-)} + \Xi_B^{(-)} + \Xi_C^{(-)}, \qquad \Xi_{--}^{(-)} = \Xi_A^{(-)} - (\Xi_B^{(-)} + \Xi_C^{(-)}). \tag{29}$$

Remarks. (i) In an axially symmetric (same-helicity) pump configuration the circular basis diagonalizes the tensor, so $\chi_{+-}^{\text{eff}} = \chi_{-+}^{\text{eff}} = 0$ at the carrier; off-diagonal terms vanish. (ii) Any axial, time-reversal-odd (Faraday-active) contribution appears as a *difference of circular eigenvalues* in the circular basis; it is tied to the pump helicity density $i\,\mathbf{E}\times\mathbf{E}^*$ and produces circular birefringence ($n_+ \neq n_-$) and a carrier Faraday rotation $\theta_F = \frac{k_0 L}{2}(n_+ - n_-)$.

Units and intensity conversion. We use $|E|^2 = 2I/(n\,\varepsilon_0 c)$ to connect intensities to field amplitudes. Accordingly, the Kerr index coefficients obey $n_{2,\cdot} = \frac{3}{4n_0 n_p\,\varepsilon_0 c}\,\mathrm{Re}[\chi_\cdot^{(3)}]$ under isotropy (Kleinman), with n_0 the probe index and n_p the pump index at their respective frequencies.

5 Eigenindices and Faraday Rotation

Treat the pump-dressed probe response as effectively linear at ω_s:

$$D_i(\omega_s) = \varepsilon_0\Big[\varepsilon_{ij}^{(1)}(\omega_s) + \chi_{ij}^{\text{eff}}(\omega_s)\Big]E_{s,j}(\omega_s). \tag{30}$$

For small nonlinearity,

$$n_\pm^2(\omega_s) \simeq n_0^2(\omega_s) + \mathrm{Re}\,\chi_{\pm\pm}^{\text{eff}}(\omega_s) \quad \Rightarrow \quad \delta n_\pm \simeq \frac{\mathrm{Re}\,\chi_{\pm\pm}^{\text{eff}}}{2\,n_0}. \tag{31}$$

Converting field to intensity via $I = (n_0\varepsilon_0 c/2)|E|^2$, it is convenient to define Kerr coefficients (SI units)

$$n_{2,\{\text{iso,circ}\}} = \frac{3}{4\,n_0\,n_0'\,\varepsilon_0 c}\,\mathrm{Re}\Big[\sum_j \chi_{\{\text{iso,circ}\}}^{(j)}(-\omega_s;\omega_s,\omega_{pj},-\omega_{pj})\Big], \tag{32}$$

so that

$$\boxed{n_\pm(\omega_s) = n_0(\omega_s) \;+\; n_{2,\text{iso}}\,I_{\text{tot}} \;\pm\; n_{2,\text{circ}}\,S_3^{\text{DC}}.} \tag{33}$$

A linearly polarized probe (equal $\sigma^{\pm}$ superposition) accumulates a relative phase

$$\Delta\phi = k_0 L\,[n_+ - n_-] = 2k_0 L\, n_{2,\mathrm{circ}}\, S_3^{\mathrm{DC}}, \qquad k_0 = \omega_s/c, \tag{34}$$

so the Faraday rotation (carrier component) is

$$\boxed{\theta_F^{(\mathrm{carrier})} = \frac{\Delta\phi}{2} = \frac{k_0 L}{2}\,[n_+ - n_-] = k_0 L\, n_{2,\mathrm{circ}}\, S_3^{\mathrm{DC}}.} \tag{35}$$

If the two counter-helicity pumps have equal intensities, $S_3^{\mathrm{DC}} = 0 \Rightarrow \theta_F^{(\mathrm{carrier})} = 0$. Flipping which helicity is stronger flips the sign of S_3^{DC} and hence of θ_F (inverse Faraday effect sign).

6 Cross-Tone (Beat) Terms and Sidebands

Keeping cross-tone contractions in Eq. (6) produces terms $\propto E^{(1)}E^{(2)*}e^{\mp i\Delta t}$ that contribute to $\mathbf{P}^{(3)}(\omega_s \pm \Delta)$. Physically, these modulate the probe phase/polarization at Δ and generate sidebands; they are *excluded* from the carrier effective tensor by the selection rule $\delta(\omega_a - \omega_b)$ in Eq. (7). Time-resolved polarimetry with bandwidth $\gtrsim \Delta$ can observe an oscillatory rotation, but its spectral home is the sidebands, not ω_s.

7 Sideband-dressed $\underline{\underline{\chi}}^{\mathrm{eff}}(\omega_s)$ and Faraday rotation

Set-up and selection rules. We consider two strong tones at $\omega_{p\pm} = \omega_p \pm \Delta/2$ (Sec. 1) and a weak probe at ω_s. Cross-tone contractions of $\chi^{(3)}_{ijkl}$ with the pumps generate probe *sidebands* at $\omega_s \pm \Delta$, cf. Sec. "Cross-Tone (Beat) Terms." Retaining only the first sidebands and working to linear order in the probe, the probe spectral amplitudes $E_s(\omega_s)$, $E_{s,\pm} \equiv E_s(\omega_s \pm \Delta)$ obey a 3×3 coupled system. Eliminating $E_{s,\pm}$ yields a *sideband-dressed* effective tensor at the carrier ω_s. Below we give the explicit algebra in our isotropic-tensor notation.

Coupled system and elimination. In frequency space the wave equation can be written schematically as $\mathcal{D}(\omega)\,E_i(\omega) = (\omega^2/\varepsilon_0 c^2)\,P_i^{(3)}(\omega)$. Near ω_s we define lumped complex detunings

$$\mathcal{D}_0 \equiv \mathcal{D}(\omega_s), \qquad \mathcal{D}_\pm \equiv \mathcal{D}(\omega_s \pm \Delta) = \Gamma_s \mp i\Delta, \tag{36}$$

where $\Gamma_s > 0$ is the probe's linear loss (phenomenological).

Write the cross-tone third-order polarization pieces that connect the probe to its sidebands in the compact *tensor-contracted* form (keeping only terms linear in $\mathbf{E}_s$):

$$P_i^{(3)}(\omega_s) \supset \varepsilon_0 \Big[\, \mathcal{M}_{ij}^{(-)}\, E_{s,-,j} + \mathcal{M}_{ij}^{(+)}\, E_{s,+,j} \Big], \tag{37}$$

$$P_i^{(3)}(\omega_s \pm \Delta) \supset \varepsilon_0 \Big[\, \widetilde{\mathcal{M}}_{ij}^{(\pm)}\, E_{s,j} \Big], \tag{38}$$

with the *mixing tensors* (from the isotropic decomposition (9))

$$\mathcal{M}_{ij}^{(\pm)} = \sum_{m=A,B,C} \chi^{(m)}_{(\mp 1)}(\omega_s; \omega_{p+}, -\omega_{p-}, \omega_s \pm \Delta)\ \mathcal{T}_{ij}^{(m)}\big[\mathbf{E}^{(1)}, \mathbf{E}^{(2)}\big], \tag{39}$$

$$\widetilde{\mathcal{M}}_{ij}^{(\pm)} = \sum_{m=A,B,C} \chi^{(m)}_{(\pm 1)}(\omega_s \pm \Delta; \omega_{p+}, -\omega_{p-}, \omega_s)\ \mathcal{T}_{ij}^{(m)}\big[\mathbf{E}^{(1)}, \mathbf{E}^{(2)}\big], \tag{40}$$

where $\chi^{(m)}_{(\pm 1)}$ denote the *Floquet blocks* of the third-order kernel indexed by the modulation harmonic ± 1, and $\mathcal{T}_{ij}^{(A,B,C)}$ are the contractions produced by the three isotropic invariants:

$$\mathcal{T}_{ij}^{(A)} = (\mathbf{E}^{(1)} \cdot \mathbf{E}^{(2)*})\,\delta_{ij}, \quad \mathcal{T}_{ij}^{(B)} = E_i^{(1)} E_j^{(2)*}, \quad \mathcal{T}_{ij}^{(C)} = E_i^{(1)*} E_j^{(2)}. \tag{41}$$

Collect the three probe unknowns as $\mathcal{E} \equiv \big(E_{s,i}, E_{s,+,i}, E_{s,-,i}\big)^{\mathrm{T}}$. To first order one obtains

$$\begin{pmatrix} \mathcal{D}_0\,\delta_{ij} & -\mathcal{K}^{(+)}_{ij} & -\mathcal{K}^{(-)}_{ij} \\ -\widetilde{\mathcal{K}}^{(+)}_{ij} & \mathcal{D}_+\,\delta_{ij} & 0 \\ -\widetilde{\mathcal{K}}^{(-)}_{ij} & 0 & \mathcal{D}_-\,\delta_{ij} \end{pmatrix} \begin{pmatrix} E_{s,j} \\ E_{s,+,j} \\ E_{s,-,j} \end{pmatrix} = \begin{pmatrix} S_{s,i} \\ 0 \\ 0 \end{pmatrix}, \qquad \mathcal{K}^{(\pm)}_{ij} \equiv \frac{\omega_s^2}{c^2}\,\mathcal{M}^{(\pm)}_{ij}, \quad \widetilde{\mathcal{K}}^{(\pm)}_{ij} \equiv \frac{(\omega_s \pm \Delta)^2}{c^2}\,\widetilde{\mathcal{M}}^{(\pm)}_{ij}. \tag{42}$$

Here $S_{s,i}$ is the external probe drive at ω_s (port/boundary excitation). Assuming $||\mathcal{K}^{(\pm)}||, ||\widetilde{\mathcal{K}}^{(\pm)}|| \ll |\mathcal{D}_\pm|$ (weak sideband excitation), solve the lower rows for the sidebands and substitute back:

$$E_{s,\pm,i} \approx \big(\mathcal{D}_\pm^{-1}\,\widetilde{\mathcal{K}}^{(\pm)}\big)_{ij} E_{s,j}. \tag{43}$$

The resulting *sideband-dressed* constitutive relation at ω_s is

$$P^{(3)}_i(\omega_s) = \varepsilon_0 \Big[\chi^{\mathrm{eff}}_{ij}(\omega_s) + \delta\chi^{(+)}_{ij}(\omega_s) + \delta\chi^{(-)}_{ij}(\omega_s)\Big] E_{s,j}, \tag{44}$$

with the corrections

$$\boxed{\delta\chi^{(\pm)}_{ij}(\omega_s) = \frac{c^2}{\omega_s^2}\,\big(\mathcal{K}^{(\pm)}\,\mathcal{D}_\pm^{-1}\,\widetilde{\mathcal{K}}^{(\pm)}\big)_{ij} = \frac{1}{\mathcal{D}_\pm}\,\mathcal{M}^{(\pm)}_{ik}\,\widetilde{\mathcal{M}}^{(\pm)}_{kj} \quad \propto \quad \frac{|E^{(1)}|^2|E^{(2)}|^2}{\Gamma_s \mp i\Delta}.} \tag{45}$$

Equations (40)–(45) are the sought *same-notation* expression for the sideband-dressed effective tensor at the carrier.

Circular basis: diagonal elements. Projecting (45) onto the circular eigenbasis $\{\hat{\mathbf{e}}_\pm\}$ and assuming collinear propagation (our Sec. "Specialization to Circular Pumps"), the effective diagonal elements become

$$\boxed{\chi^{\mathrm{eff,SB}}_{\pm\pm}(\omega_s) = \chi^{\mathrm{eff}}_{\pm\pm}(\omega_s) + \big[\delta\chi^{(+)}_{\pm\pm}(\omega_s) + \delta\chi^{(-)}_{\pm\pm}(\omega_s)\big], \qquad \delta\chi^{(\pm)}_{\alpha\alpha} = \hat{e}_{\alpha,i}\,\delta\chi^{(\pm)}_{ij}\,\hat{e}^{*}_{\alpha,j} \;\; (\alpha = \pm).} \tag{46}$$

Cases: $\sigma^+\sigma^-$ and $\sigma^+\sigma^+$ pumps. Let $\mathbf{E}^{(1)} = E^{(1)}_+\hat{\mathbf{e}}_+$ and $\mathbf{E}^{(2)} = E^{(2)}_-\hat{\mathbf{e}}_-$ for the $\sigma^+\sigma^-$ case, and $\mathbf{E}^{(1)} = E^{(1)}_+\hat{\mathbf{e}}_+$, $\mathbf{E}^{(2)} = E^{(2)}_+\hat{\mathbf{e}}_+$ for the $\sigma^+\sigma^+$ case (the $\sigma^-\sigma^-$ case follows by $+ \leftrightarrow -$). Using the isotropic contractions:

$$\sigma^+\sigma^-: \qquad \mathcal{T}^{(A)}_{ij} = E^{(1)}_+ E^{(2)}_-\,\delta_{ij}, \quad \mathcal{T}^{(B)}_{ij} = E^{(1)}_+ E^{(2)}_-\,(\hat{e}_+)_i(\hat{e}_-)_j, \quad \mathcal{T}^{(C)}_{ij} = E^{(1)}_+ E^{(2)}_-\,(\hat{e}_-)_i(\hat{e}_+)_j, \tag{47}$$

$$\sigma^+\sigma^+: \qquad \mathcal{T}^{(A)}_{ij} = E^{(1)}_+ E^{(2)}_+\,\delta_{ij}, \quad \mathcal{T}^{(B)}_{ij} = E^{(1)}_+ E^{(2)}_+\,(\hat{e}_+)_i(\hat{e}_+)_j, \quad \mathcal{T}^{(C)}_{ij} = E^{(1)}_+ E^{(2)}_+\,(\hat{e}_-)_i(\hat{e}_-)_j. \tag{48}$$

Substituting into (40)–(46) gives compact diagonal results:

$$\boxed{\sigma^+\sigma^-: \quad \begin{aligned} \delta\chi^{(\pm)}_{++} &= \frac{|E^{(1)}_+ E^{(2)}_-|^2}{\mathcal{D}_\pm}\Big[\chi^{(+)}_{(\mp1),B} + \chi^{(+)}_{(\mp1),C}\Big]\Big[\chi^{(+)}_{(\pm1),B} + \chi^{(+)}_{(\pm1),C}\Big], \\ \delta\chi^{(\pm)}_{--} &= \frac{|E^{(1)}_+ E^{(2)}_-|^2}{\mathcal{D}_\pm}\Big[\chi^{(-)}_{(\mp1),B} + \chi^{(-)}_{(\mp1),C}\Big]\Big[\chi^{(-)}_{(\pm1),B} + \chi^{(-)}_{(\pm1),C}\Big], \end{aligned}} \tag{49}$$

$$\boxed{\sigma^+\sigma^+: \quad \begin{aligned} \delta\chi^{(\pm)}_{++} &= \frac{|E^{(1)}_+ E^{(2)}_+|^2}{\mathcal{D}_\pm}\Big[\chi^{(+)}_{(\mp1),A} + \chi^{(+)}_{(\mp1),B} + \chi^{(+)}_{(\mp1),C}\Big]\Big[\chi^{(+)}_{(\pm1),A} + \chi^{(+)}_{(\pm1),B} + \chi^{(+)}_{(\pm1),C}\Big], \\ \delta\chi^{(\pm)}_{--} &= \frac{|E^{(1)}_+ E^{(2)}_+|^2}{\mathcal{D}_\pm}\Big[\chi^{(-)}_{(\mp1),A} - (\chi^{(-)}_{(\mp1),B} + \chi^{(-)}_{(\mp1),C})\Big]\Big[\chi^{(-)}_{(\pm1),A} - (\chi^{(-)}_{(\pm1),B} + \chi^{(-)}_{(\pm1),C})\Big]. \end{aligned}} \tag{50}$$

Here $\chi^{(\pm)}_{(\cdot),m}$ are the scalar amplitudes multiplying the isotropic A, B, C contractions for the Floquet-harmonic blocks that couple the indicated circular component. (The precise microscopic values follow from the density-matrix calculation used elsewhere in this file.)

Faraday rotation at the carrier. For small nonlinearities, $n_\pm(\omega_s) \simeq 1 + \frac{1}{2}\mathrm{Re}\,\chi_{\pm\pm}(\omega_s)$. A linearly polarized probe thus acquires the rotation

$$\boxed{\theta_F(\omega_s) = \frac{k_0 L}{4}\,\mathrm{Re}\Big\{\big[\chi^{\mathrm{eff}}_{++} - \chi^{\mathrm{eff}}_{--}\big] + \big[\delta\chi^{(+)}_{++} - \delta\chi^{(+)}_{--}\big] + \big[\delta\chi^{(-)}_{++} - \delta\chi^{(-)}_{--}\big]\Big\}_{\omega_s}.} \tag{51}$$

Discussion. (i) For $\sigma^+\sigma^-$ pumps with equal intensities and an isotropic, parity-symmetric medium, the stationary DC contribution satisfies $\chi^{\mathrm{eff}}_{++} = \chi^{\mathrm{eff}}_{--}$ (Sec. "Specialization to Circular Pumps"), so θ_F stems solely from the sideband loops in (49) and is *dispersive*, $\propto \mathrm{Re}(1/\mathcal{D}_\pm)$; it vanishes for $\Delta \to \infty$ and is maximal near $|\Delta| \sim \Gamma_s$.
(ii) For $\sigma^+\sigma^+$ pumps (or $\sigma^-\sigma^-$) the stationary term already yields $\chi^{\mathrm{eff}}_{++} \neq \chi^{\mathrm{eff}}_{--}$ (Sec. "Same-helicity pumps"), producing a nonzero θ_F even without sidebands. Equations (50)–(51) quantify the additional sideband dressing; the sign of $\mathrm{Re}\,[\delta\chi_{++} - \delta\chi_{--}]$ flips with Δ and with pump helicity.

The expressions above are fully consistent with the isotropic-tensor contraction in Eq. (9) and the circular-basis diagonalization in Eq. (18), while incorporating the cross-tone sidebands through the elimination procedure (42)–(45).

8 Time-resolved Faraday rotation

8.1 Fields, basis, and isotropic contractions

Probe (weak). A monochromatic probe at ω_s with Jones vector $\mathbf{E}_s$:

$$\mathbf{E}_s(t) = \tfrac{1}{2}\big[\mathbf{E}_s\, e^{-i\omega_s t} + \text{c.c.}\big], \qquad \|\mathbf{E}_s\| \ll \|\mathbf{E}^{(1,2)}\|.$$

Pumps (two strong tones, beat at Δ). Two pumps at

$$\omega_{p\pm} = \omega_p \pm \Delta/2, \qquad \mathbf{E}^{(1)}(t) = \tfrac{1}{2}\Big[\mathbf{E}^{(1)} e^{-i\omega_{p+} t} + \text{c.c.}\Big], \quad \mathbf{E}^{(2)}(t) = \tfrac{1}{2}\Big[\mathbf{E}^{(2)} e^{-i\omega_{p-} t} + \text{c.c.}\Big].$$

Their product carries a time modulation at Δ, which couples $\omega_s \leftrightarrow \omega_s \pm \Delta$ (Floquet picture).

Circular basis. We use $\{\hat{\mathbf{e}}_+, \hat{\mathbf{e}}_-\}$ for propagation along $+\hat{z}$, with $\mathbf{E}_s = E_{s,+}\hat{\mathbf{e}}_+ + E_{s,-}\hat{\mathbf{e}}_-$.

8.2 Temporal coupled-mode theory (TCMT) with sidebands and explicit $g_\pm, \kappa_\pm$

Define slowly varying envelopes per circular component $\alpha \in \{+, -\}$: $a_{0,\alpha}(t)$ at ω_s, and $a_{\pm,\alpha}(t)$ at $\omega_s \pm \Delta$. Let $\delta_{0,\alpha}$ and $\gamma_{0,\alpha}$ be the carrier detuning and decay, and $\delta_\pm = \delta_0 \pm \Delta$, $\gamma_\pm$ the sideband detuning/decay. The input–output coupling is $\sqrt{2\gamma_{e,\alpha}}\, s_{\mathrm{in},\alpha}(t)$.

The TCMT equations are

$$\boxed{\begin{aligned}
\dot{a}_{0,\alpha} &= \big(-\gamma_{0,\alpha} - i\delta_{0,\alpha}\big)a_{0,\alpha} + i\,\kappa^{(\alpha)}_+\, a_{+,\alpha} + i\,\kappa^{(\alpha)}_-\, a_{-,\alpha} + \sqrt{2\gamma_{e,\alpha}}\, s_{\mathrm{in},\alpha}(t),\\
\dot{a}_{+,\alpha} &= \big(-\gamma_+ - i\delta_+\big)a_{+,\alpha} + i\,g^{(\alpha)}_+\, a_{0,\alpha},\\
\dot{a}_{-,\alpha} &= \big(-\gamma_- - i\delta_-\big)a_{-,\alpha} + i\,g^{(\alpha)}_-\, a_{0,\alpha}.
\end{aligned}} \tag{52}$$

The *mixing coefficients* are circular projections of (40):

$$\kappa^{(\alpha)}_\pm = \hat{e}_{\alpha,i}\,\mathcal{M}^{(\pm)}_{ij}\,\hat{e}^*_{\alpha,j}, \qquad g^{(\alpha)}_\pm = \hat{e}_{\alpha,i}\,\widetilde{\mathcal{M}}^{(\pm)}_{ij}\,\hat{e}^*_{\alpha,j}.$$

Explicitly (list the nonzero scalars only):

$$
\begin{aligned}
\underline{\sigma^+\sigma^-:}\quad & g_+^{(+)} = \big[\chi^{(+)}_{(+1),B} + \chi^{(+)}_{(+1),C}\big] E_+^{(1)} E_-^{(2)}, && \kappa_+^{(+)} = \big[\chi^{(+)}_{(-1),B} + \chi^{(+)}_{(-1),C}\big] E_+^{(1)} E_-^{(2)},\\
& g_-^{(+)} = \big[\chi^{(+)}_{(-1),B} + \chi^{(+)}_{(-1),C}\big] E_+^{(1)} E_-^{(2)}, && \kappa_-^{(+)} = \big[\chi^{(+)}_{(+1),B} + \chi^{(+)}_{(+1),C}\big] E_+^{(1)} E_-^{(2)},\\
& g_+^{(-)} = \big[\chi^{(-)}_{(+1),B} + \chi^{(-)}_{(+1),C}\big] E_+^{(1)} E_-^{(2)}, && \kappa_+^{(-)} = \big[\chi^{(-)}_{(-1),B} + \chi^{(-)}_{(-1),C}\big] E_+^{(1)} E_-^{(2)},\\
& g_-^{(-)} = \big[\chi^{(-)}_{(-1),B} + \chi^{(-)}_{(-1),C}\big] E_+^{(1)} E_-^{(2)}, && \kappa_-^{(-)} = \big[\chi^{(-)}_{(+1),B} + \chi^{(-)}_{(+1),C}\big] E_+^{(1)} E_-^{(2)};
\end{aligned}
$$

$$
\begin{aligned}
\underline{\sigma^+\sigma^+:}\quad & g_+^{(+)} = \big[\chi^{(+)}_{(+1),A} + \chi^{(+)}_{(+1),B} + \chi^{(+)}_{(+1),C}\big] E_+^{(1)} E_+^{(2)}, && \kappa_+^{(+)} = \big[\chi^{(+)}_{(-1),A} + \chi^{(+)}_{(-1),B} + \chi^{(+)}_{(-1),C}\big] E_+^{(1)} E_+^{(2)},\\
& g_-^{(+)} = \big[\chi^{(+)}_{(-1),A} + \chi^{(+)}_{(-1),B} + \chi^{(+)}_{(-1),C}\big] E_+^{(1)} E_+^{(2)}, && \kappa_-^{(+)} = \big[\chi^{(+)}_{(+1),A} + \chi^{(+)}_{(+1),B} + \chi^{(+)}_{(+1),C}\big] E_+^{(1)} E_+^{(2)},\\
& g_+^{(-)} = \big[\chi^{(-)}_{(+1),A} - (\chi^{(-)}_{(+1),B} + \chi^{(-)}_{(+1),C})\big] E_+^{(1)} E_+^{(2)}, && \kappa_+^{(-)} = \big[\chi^{(-)}_{(-1),A} - (\chi^{(-)}_{(-1),B} + \chi^{(-)}_{(-1),C})\big] E_+^{(1)} E_+^{(2)},\\
& g_-^{(-)} = \big[\chi^{(-)}_{(-1),A} - (\chi^{(-)}_{(-1),B} + \chi^{(-)}_{(-1),C})\big] E_+^{(1)} E_+^{(2)}, && \kappa_-^{(-)} = \big[\chi^{(-)}_{(+1),A} - (\chi^{(-)}_{(+1),B} + \chi^{(-)}_{(+1),C})\big] E_+^{(1)} E_+^{(2)}.
\end{aligned}
$$

8.3 Solution and time-resolved Faraday rotation

For steady input $s_{\text{in},\alpha}(t) = s_{\text{in},\alpha}$ and weak mixing, set $\dot a_{\pm,\alpha} \approx 0$:

$$a_{\pm,\alpha} \approx \frac{i\, g_\pm^{(\alpha)}}{\gamma_\pm + i\delta_\pm}\, a_{0,\alpha}, \qquad \delta_\pm = \delta_0 \pm \Delta.$$

The carrier then obeys

$$\dot a_{0,\alpha} = \Big(-\gamma_{0,\alpha} - i\delta_{0,\alpha}\Big) a_{0,\alpha} + i\Sigma_\alpha\, a_{0,\alpha} + \sqrt{2\gamma_{e,\alpha}}\, s_{\text{in},\alpha}, \qquad \Sigma_\alpha \equiv \frac{\kappa_+^{(\alpha)} g_+^{(\alpha)}}{\gamma_+ + i\delta_+} + \frac{\kappa_-^{(\alpha)} g_-^{(\alpha)}}{\gamma_- + i\delta_-}.$$

Steady state:

$$a_{0,\alpha} = \frac{\sqrt{2\gamma_{e,\alpha}}\, s_{\text{in},\alpha}}{\gamma_{0,\alpha} + i\delta_{0,\alpha} - i\Sigma_\alpha}.$$

The detected probe (carrier+sidebands) is

$$\mathbf{E}(t) \approx \sum_{\alpha=\pm} \Big[a_{0,\alpha}\, e^{-i\omega_s t} + a_{+,\alpha}\, e^{-i(\omega_s+\Delta)t} + a_{-,\alpha}\, e^{-i(\omega_s-\Delta)t}\Big] \hat{\mathbf{e}}_\alpha.$$

Define small sideband ratios $r_\pm^{(\alpha)} \equiv a_{\pm,\alpha}/a_{0,\alpha} = i\, g_\pm^{(\alpha)}/(\gamma_\pm + i\delta_\pm)$. To first order in $r_\pm^{(\alpha)}$, the instantaneous polarization angle (Faraday rotation) is

$$\boxed{\theta(t) = \underbrace{\frac{k_0 L}{4}\, \mathrm{Re}\big[\chi^{\text{eff,SB}}_{++}(\omega_s) - \chi^{\text{eff,SB}}_{--}(\omega_s)\big]}_{\theta_0 \text{ (static, carrier)}} + \underbrace{\frac{1}{2}\big|R\big| \cos\big(\Delta t + \arg R\big)}_{\text{oscillation at } \Delta}, \qquad R \equiv r_+^{(+)} + r_-^{(+)} - \big(r_+^{(-)} + r_-^{(-)}\big).} \tag{53}$$

The static piece θ_0 reflects circular birefringence of the *carrier* alone; the oscillatory part arises from carrier–sideband interference and vanishes if the sidebands are spectrally filtered out. For $\sigma^+\sigma^-$ with equal intensities in an isotropic medium, the stationary carrier contribution often satisfies $\chi^{\text{eff}}_{++} = \chi^{\text{eff}}_{--}$, so $\theta_0 \approx 0$ and rotation is dominantly the Δ-oscillation. For $\sigma^+\sigma^+$, $\theta_0 \neq 0$ already without sidebands, which then dress it and add the Δ term.

8.4 TCMT at the carrier for circular polarizations (cavity form)

We now specialize to a resonant geometry (single cavity resonance at ω_0) and write a temporal coupled–mode theory (TCMT) [7] for the *carrier-frequency* probe envelopes in the circular basis.

Let $a_{0,+}(t)$ and $a_{0,-}(t)$ denote the intracavity envelopes of the probe's $\sigma^{\pm}$ components at the carrier $\omega_s \simeq \omega_0$; the external drives (ports) are $s_{\mathrm{in},\pm}(t)$ with the standard input–output relation $s_{\mathrm{out},\pm} = s_{\mathrm{in},\pm} - \sqrt{2\gamma_e}\, a_{0,\pm}$. Two strong pumps at $\omega_{p\pm} = \omega_p \pm \Delta/2$ and opposite helicities (σ^+, σ^-) bias the cavity and produce a time-periodic mixing at Δ between the circular probe modes (the engine behind the *optically driven Faraday effect* [6]).

Time-periodic TCMT (lab frame). Under the instantaneous Kerr assumption and after projecting the $\chi^{(3)}$ tensor onto the circular basis (our isotropic A, B, C reduction), the *carrier* envelopes obey

$$\boxed{\begin{aligned} \dot a_{0,+} &= \big(-\gamma_0 - i\delta_0\big)\, a_{0,+} \;+\; i\, D\, e^{+i\Delta t}\, a_{0,-} \;+\; \sqrt{2\gamma_e}\, s_{\mathrm{in},+}(t), \\ \dot a_{0,-} &= \big(-\gamma_0 - i\delta_0\big)\, a_{0,-} \;+\; i\, D^* e^{-i\Delta t}\, a_{0,+} \;+\; \sqrt{2\gamma_e}\, s_{\mathrm{in},-}(t), \end{aligned}} \tag{54}$$

where γ_0 is the intrinsic decay (half-linewidth), $\delta_0 \equiv \omega_s - \omega_0$ is the probe–cavity detuning, and D is a pump-controlled complex coupling proportional to the appropriate $\chi^{(3)}$ Floquet block(s) contracted with the pump Jones vectors (e.g., for $\sigma^+\sigma^-$ pumps $D \propto \big[\chi^{(+)}_{(+1),B} + \chi^{(+)}_{(+1),C}\big] E^{(1)}_+ E^{(2)*}_-$). Equation (54) is the TCMT analogue of the time-periodic polarization coupling used in Ref. [6] (their $e^{\pm i\Delta t}$ mixing).

Static coupling via a rotating frame. Define a slow rotating frame $b_\pm(t) \equiv a_{0,\pm}(t)\, e^{\mp i\Delta t/2}$. Then Eqs. (54) become

$$\boxed{\begin{aligned} \dot b_+ &= \Big(-\gamma_0 - i(\delta_0 - \tfrac{\Delta}{2})\Big)\, b_+ \;+\; i\, D\, b_- \;+\; \sqrt{2\gamma_e}\, s_{\mathrm{in},+}(t)\, e^{-i\Delta t/2}, \\ \dot b_- &= \Big(-\gamma_0 - i(\delta_0 + \tfrac{\Delta}{2})\Big)\, b_- \;+\; i\, D^*\, b_+ \;+\; \sqrt{2\gamma_e}\, s_{\mathrm{in},-}(t)\, e^{+i\Delta t/2}. \end{aligned}} \tag{55}$$

For quasi-CW probing near the cavity line, the exponential factors on the drives are slowly varying constants, so the coupling becomes *static*. This 2×2 system is convenient for closed-form steady-state solutions and matches the effective-Faraday cavity treatment of [6].

Closed-form steady state and rotation. With $\dot b_\pm = 0$ and constant inputs, write

$$\begin{bmatrix} b_+ \\ b_- \end{bmatrix} = \underbrace{\begin{bmatrix} \gamma_0 + i(\delta_0 - \frac{\Delta}{2}) & -iD \\ -iD^* & \gamma_0 + i(\delta_0 + \frac{\Delta}{2}) \end{bmatrix}^{-1}}_{\mathbf{G}(\delta_0, \Delta, D)} \begin{bmatrix} \sqrt{2\gamma_e}\, s_{\mathrm{in},+} \\ \sqrt{2\gamma_e}\, s_{\mathrm{in},-} \end{bmatrix}. \tag{56}$$

Transform back to the lab frame via $a_{0,\pm} = b_\pm e^{\pm i\Delta t/2}$ and use $s_{\mathrm{out},\pm} = s_{\mathrm{in},\pm} - \sqrt{2\gamma_e}\, a_{0,\pm}$ to obtain the port fields. The *instantaneous* polarization rotation is

$$\boxed{\theta(t) = \tfrac{1}{2}\big[\arg a_{0,+}(t) - \arg a_{0,-}(t)\big].} \tag{57}$$

Near a cavity line, the observable rotation is essentially *static* (the $e^{\pm i\Delta t/2}$ factors cancel in the Stokes parameters over the cavity lifetime), and one may define

$$\boxed{\theta_0 = \tfrac{1}{2} \arg\!\left(\frac{b_+}{b_-}\right) = \tfrac{1}{2} \arg\!\left(\frac{\gamma_0 + i(\delta_0 - \frac{\Delta}{2})}{\gamma_0 + i(\delta_0 + \frac{\Delta}{2})} \;+\; \frac{|D|^2}{\big[\gamma_0 + i(\delta_0 + \frac{\Delta}{2})\big]\big[\gamma_0 + i(\delta_0 - \frac{\Delta}{2})\big]}\right).} \tag{58}$$

Equations (54)–(58) are the cavity counterpart of our sideband-elimination picture: the Lorentzian dispersive factor $1/(\Gamma_s \mp i\Delta)$ appearing in the bulk propagation is here replaced by the cavity susceptibilities $1/(\gamma_0 + i(\delta_0 \mp \Delta/2))$, while the mixing amplitude D contains the same $\chi^{(3)}$ contractions with the pump helicities as in our isotropic A, B, C expansion. This reproduces the *optically driven effective Faraday effect* from Ref. [6] within our notation, and fits naturally in the TCMT framework [7].

Of course, there can be typos and stupid mistakes, but the whole logic is correct.

9 References

References

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