Name: Venkat Subramanian   Class: S02

Assignment 1 Optional Questions

Question 7

Q7) $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ $B = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

Identity matrix

$$AB = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}+\frac{1}{2} & -\frac{1}{2}+\frac{1}{2} \\ -\frac{1}{2}+\frac{1}{2} & \frac{1}{2}+\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$BA = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}+\frac{1}{2} & \frac{1}{2}-\frac{1}{2} \\ \frac{1}{2}-\frac{1}{2} & \frac{1}{2}+\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$AB = BA = I$ so $B = A^{-1}$

Question 8

Q8a) $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$

$A \cdot A^{-1} = I$

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x & y \\ z & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2x+z & 2y+a \\ -x & -y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$2x+z=1$ $\quad 2y+a=0$

$x=0$ $\quad -y=1,\ y=-1$

$z=1$ $\quad \therefore a=2$

$\therefore A^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$

$(A^{-1})^T = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$

Q8b) $A^T = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$

$(A^T)^{-1} \cdot (A^T) = I$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2a+b & -a \\ 2c+d & -c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$2a+b=1,\ -a=0 \to a=0$

$2c+d=0,\ -c=1 \to c=-1$

$d=2,\ b=1$

$(A^T)^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$

$\therefore (A^{-1})^T = (A^T)^{-1}$

Question 9

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix}$$

let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nn} \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

$\therefore \; Ax = b$

Question 10

Q10) $g(x) = a^T x$ for linear need to satisfy homogenity and additivity

a is $1 \times n$ and x is $1 \times n$ so $a^T x$ will give a scalar value

if we assume $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$bx = \begin{pmatrix} bx_1 \\ \vdots \\ bx_n \end{pmatrix}$

$$g(bx) = a^T \begin{pmatrix} bx_1 \\ bx_2 \\ \vdots \\ bx_n \end{pmatrix}$$
$$= a^T \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} . b$$
$$= g(x) . b$$

If $g(x) = a^T x$

and $g(y) = a^T y$

$$g(x+y) = a^T(x+y)$$
$$= a^T \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix} = (a_1 \; a_2 \; a_3 \cdots a_n) \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

$$g(x+y) = (a_1 \; a_2 \; a_3 \cdots a_n) \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

$$= a_1(x_1 + y_1) + a_2(x_2 + y_2) + \cdots + a_n(x_n + y_n)$$

$$= a_1x_1 + a_1y_1 + a_2x_2 + a_2y_2 + \cdots + a_nx_n + a_ny_n$$

$$= a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n + a_1y_1 + a_2y_2 + a_3y_3 + \cdots + a_ny_n$$

$$= a^T(x) + a^T(y)$$

$$= g(x) + g(y) \quad \therefore \text{additive.}$$

Since the additive and homogenous property is held, the function g is a linear function

Question 10b

Q10b) let $y(x)$ be $g(x)+b$

$\therefore \quad y(x) = a^T x + b$

$y(c(x)) = a^T(c(x)) + b$

$= a^T x \cdot c + b$

$c\, y(x) = c[a^T x + b]$

$a^T x . c + b \neq c[a^T x + b]$ Therefore not linear

Question 10c

Sorry Professor, I have no idea how to start this question. Hope you can shed some light, thank you

Question 12

Let X_{ij} be the amount of currency i to be converted to currency j, $\forall$ i = 1,…, N and $\forall$ j = 1,…,N

Since currency 1, 2,…, N-1 can all be converted to currency N, the amount of currency N can be rewritten as

$$\max(X_{1N}r_{1N} + X_{2N}r_{2N} + X_{3N}r_{3N} + \cdots + X_{N-1N}r_{N-1N}) = \max \sum_{i=1}^{N-1} X_{iN}r_{iN}$$

Subject to:

$$\sum_{j=1}^{N} X_{ij} \leq u_i$$

$$\sum_{i=1}^{N} \sum_{j=1}^{N} X_{ij} \leq B$$

Since the maximum rate is 1, the maximum number of notes that the person can hold at any point in time will be B.