

GATE 2021 EE.20

EE23BTECH11010 - VENKATESH BANDAWAR*

I. LAPLACE TRANSFORM

Laplace transform of integrals:

Let the function defined as $y(t) = \int_0^t f(u)du$ for all $t > 0$

Laplace transform of $y(t)$ in t

$$\mathcal{L}(y(t)) = \int_0^\infty e^{-st} y(t) dt \quad (1)$$

$$= \int_0^\infty e^{-st} \int_0^t f(u) du dt \quad (2)$$

$$= \int_0^\infty f(u) du \left[-\frac{e^{-st}}{s} \right]_0^\infty + \int_0^\infty \frac{e^{-st}}{s} f(t) dt \quad (3)$$

$$= \frac{F(s)}{s} \quad (4)$$

where,

$$Y(s) = I(s)Z_C \quad (5)$$

$$= \frac{X(s)}{Z_L + Z_R + Z_C} Z_C \quad (6)$$

$$H(s) = \frac{Y(s)}{X(s)} \quad (7)$$

$$= \frac{Z_C}{Z_L + Z_R + Z_C} \quad (8)$$

$$= \frac{\frac{1}{sC}}{sL + R + \frac{1}{sC}} \quad (9)$$

$$= \frac{1}{s^2 LC + sRC + 1} \quad (10)$$

$$\Rightarrow H(s) = \omega_0^2 \frac{1}{(s - p_1)(s - p_2)} \quad (11)$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad (12)$$

$$p_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \quad (13)$$

$$= -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \quad (14)$$

where

$$\alpha = \frac{R}{2L} \quad (15)$$

Damping Factor is given by,

$$\zeta = \frac{\alpha}{\omega_0} \quad (16)$$

$$= \frac{R}{2} \sqrt{\frac{C}{L}} \quad (17)$$

II. RLC Low PASS FILTER

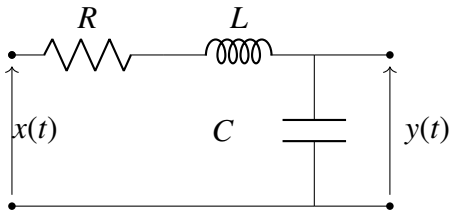


Fig. 1: RLC Low pass filter

Parameter	Description
Z_C	Reactance of Capacitor
Z_R	Reactance of Resistor
Z_L	Reactance of Inductor
$x(t) = u(t)$	Input Response
$y(t)$	Output across capacitor
ω_0	Angular resonant frequency

TABLE I: Input Parameters

ζ	Pole Location	Referred to as	Condition
$\zeta > 1$	Different locations on the negative real axis	Overdamped	$R > 2\sqrt{\frac{L}{C}}$
$\zeta = 1$	Coincide on the negative real axis	Critically Damped	$R = 2\sqrt{\frac{L}{C}}$
$\zeta < 1$	Complex Conjugate poles in the left half of s-plane	Underdamped	$R < 2\sqrt{\frac{L}{C}}$

TABLE II: Effect of Damping Coefficient ζ on system behaviour

1) Overdamped Response

$$Y(s) = X(s)H(s) \quad (18)$$

$$= \omega_0^2 \frac{1}{s(s-p_1)(s-p_2)} \quad (19)$$

$$= \frac{c_0}{s} + \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} \quad (20)$$

where,

$$c_0 = 1 \quad (21)$$

$$c_1 = \frac{p_2}{p_1 - p_2} \quad (22)$$

$$c_2 = \frac{p_1}{p_2 - p_1} \quad (23)$$

Taking inverse Laplace,

$$y(t) = c_0 + c_1 e^{p_1 t} + c_2 e^{p_2 t} \quad (24)$$

$$= \left(1 + \frac{p_2}{p_1 - p_2} e^{p_1 t} + \frac{p_1}{p_2 - p_1} e^{p_2 t} \right) u(t) \quad (25)$$

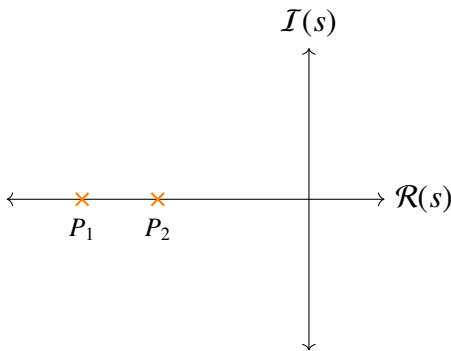


Fig. 2: s-Plane for Overdamped case

2) Critically Damped Response

$$Y(s) = X(s)H(s) \quad (26)$$

$$= \omega_0^2 \frac{1}{s(s-p)^2} \quad (27)$$

$$= \frac{c_0}{s} + \frac{c_1}{(s-p)^2} + \frac{c_2}{s-p} \quad (28)$$

where

$$c_0 = 1 \quad (29)$$

$$c_1 = p \quad (30)$$

$$c_2 = -1 \quad (31)$$

Taking Inverse Laplace,

$$y(t) = c_0 + (c_1 t + c_2) e^{pt} \quad (32)$$

$$= (1 + (pt - 1) e^{pt}) u(t) \quad (33)$$

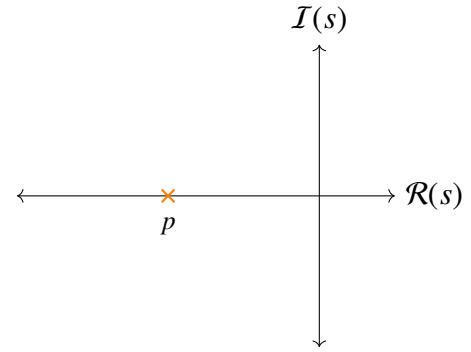


Fig. 3: s-Plane for Critically damped case

3) Underdamped Response

$$Y(s) = X(s)H(s) \quad (34)$$

$$= \omega_0^2 \frac{1}{s(s-p)(s-p^*)} \quad (35)$$

$$= \frac{c_0}{s} + \frac{c_1}{s-p} + \frac{c_2}{s-p^*} \quad (36)$$

where,

$$c_0 = 1 \quad (37)$$

$$c_1 = \frac{p^*}{p - p^*} \quad (38)$$

$$c_2 = \frac{p}{p^* - p} \quad (39)$$

Taking Inverse Laplace,

$$y(t) = c_0 + c_1 e^{pt} + c_2 e^{p^* t} \quad (40)$$

$$= 1 + \frac{|p|}{\omega_d} e^{-\sigma t} \frac{e^{j(\omega_d t + \varphi)} + e^{-j(\omega_d t + \varphi)}}{2} \quad (41)$$

$$= \left(1 + \frac{|p|}{\omega_d} e^{-\sigma t} \cos(\omega_d t + \varphi) \right) u(t) \quad (42)$$

where,

$$|p| = \sqrt{\omega_d^2 + \sigma^2} \quad (43)$$

$$\omega_d = \omega_0 \sqrt{1 - \zeta^2} \quad (44)$$

$$\sigma = \omega_0 \zeta \quad (45)$$

$$\varphi = \pi - \tan^{-1} \frac{\sigma}{\omega_d} \quad (46)$$

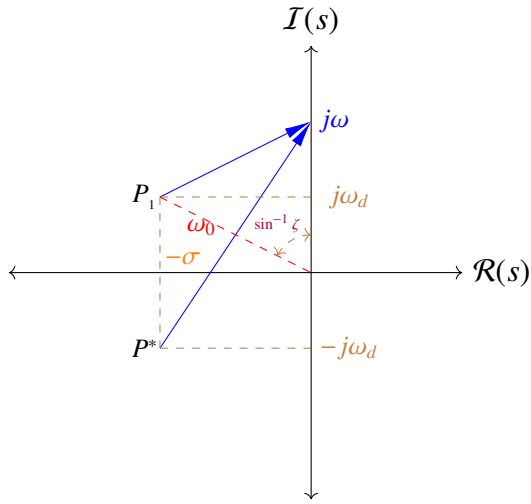


Fig. 4: s-Plane for Under damped case

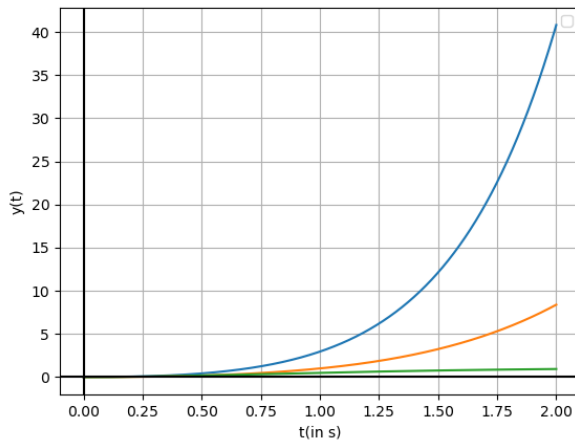


Fig. 5: Plot of $y(t)$ in all three cases