GATE 2021 EE.20

EE23BTECH11010 - VENKATESH BANDAWAR*

I. Laplace Transform

Laplace transform of integrals:
Let the function defined as $y(t) = \int_0^t f(u)du$ for all $t > 0$
Laplace transform of y(t) in t

$$\mathcal{L}(y(t)) = \int_0^\infty e^{-st} y(t)dt \tag{1}$$
$$= \int_0^\infty e^{-st} \int_0^t f(u)dudt \tag{2}$$
$$= \int_0^t f(u)du \left[-\frac{e^{-st}}{s}\right]_0^\infty + \int_0^\infty \frac{e^{-st}}{s} f(t)dt \tag{3}$$
$$= \frac{F(s)}{s} \tag{4}$$

II. RLC Low Pass Filter

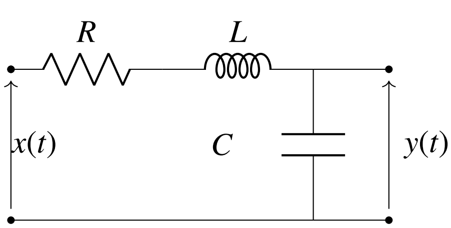

Fig. 1: RLC Low pass filter

Parameter	Description
Z_C	Reactance of Capacitor
Z_R	Reactance of Resistor
Z_L	Recatance of Inductor
$x(t) = u(t)$	Input Response
$y(t)$	Output across capacitor
ω_0	Angular resonant frequency

TABLE I: Input Parameters

$$Y(s) = I(s)Z_C \tag{5}$$
$$= \frac{X(s)}{Z_L + Z_R + Z_C} Z_C \tag{6}$$
$$H(s) = \frac{Y(s)}{X(s)} \tag{7}$$
$$= \frac{Z_C}{Z_L + Z_R + Z_C} \tag{8}$$
$$= \frac{\frac{1}{sC}}{sL + R + \frac{1}{sC}} \tag{9}$$
$$= \frac{1}{s^2LC + sRC + 1} \tag{10}$$
$$\implies H(s) = \omega_0^2 \frac{1}{(s - p_1)(s - p_2)} \tag{11}$$

where,

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{12}$$
$$p_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \tag{13}$$
$$= -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \tag{14}$$

where

$$\alpha = \frac{R}{2L} \tag{15}$$

Damping Factor is given by,

$$\zeta = \frac{\alpha}{\omega_0} \tag{16}$$
$$= \frac{R}{2}\sqrt{\frac{C}{L}} \tag{17}$$

ζ	Pole Location	Referred to as	Condition
$\zeta > 1$	Different locations on the negative real axis	Overdamped	$R > 2\sqrt{\frac{L}{C}}$
$\zeta = 1$	Coincide on the negative real axis	Critically Damped	$R = 2\sqrt{\frac{L}{C}}$
$\zeta < 1$	Complex Conjugate poles in the left half of s-plane	Underdamped	$R < 2\sqrt{\frac{L}{C}}$

TABLE II: Effect of Damping Coefficient ζ on system behaviour

1) Overdamped Response

$$Y(s) = X(s)H(s) \tag{18}$$
$$= \omega_0^2 \frac{1}{s(s-p_1)(s-p_2)} \tag{19}$$
$$= \frac{c_0}{s} + \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} \tag{20}$$

where,

$$c_0 = 1 \tag{21}$$
$$c_1 = \frac{p_2}{p_1 - p_2} \tag{22}$$
$$c_2 = \frac{p_1}{p_2 - p_1} \tag{23}$$

Taking inverse Laplace,

$$y(t) = c_0 + c_1 e^{p_1 t} + c_2 e^{p_2 t} \tag{24}$$
$$= \left(1 + \frac{p_2}{p_1 - p_2} e^{p_1 t} + \frac{p_1}{p_2 - p_1} e^{p_2 t}\right) u(t) \tag{25}$$

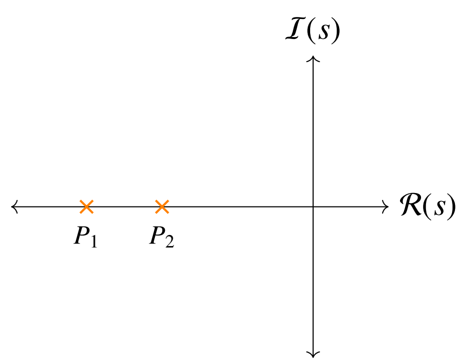

Fig. 2: s-Plane for Overdamped case

2) Critically Damped Response

$$Y(s) = X(s)H(s) \tag{26}$$
$$= \omega_0^2 \frac{1}{s(s-p)^2} \tag{27}$$
$$= \frac{c_0}{s} + \frac{c_1}{(s-p)^2} + \frac{c_2}{s-p} \tag{28}$$

where

$$c_0 = 1 \tag{29}$$
$$c_1 = p \tag{30}$$
$$c_2 = -1 \tag{31}$$

Taking Inverse Laplace,

$$y(t) = c_0 + (c_1 t + c_2)e^{pt} \tag{32}$$
$$= \left(1 + (pt - 1)e^{pt}\right) u(t) \tag{33}$$

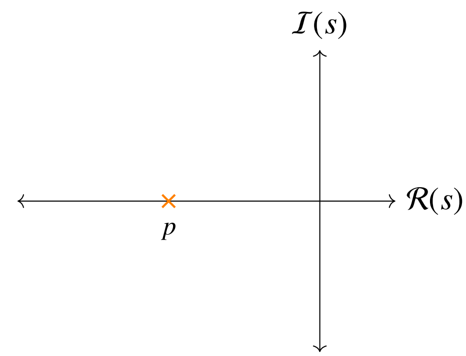

Fig. 3: s-Plane for Critically damped case

3) Underdamped Response

$$Y(s) = X(s)H(s) \tag{34}$$
$$= \omega_0^2 \frac{1}{s(s-p)(s-p^*)} \tag{35}$$
$$= \frac{c_0}{s} + \frac{c_1}{s-p} + \frac{c_2}{s-p^*} \tag{36}$$

where,

$$c_0 = 1 \tag{37}$$
$$c_1 = \frac{p^*}{p - p^*} \tag{38}$$
$$c_2 = \frac{p}{p^* - p} \tag{39}$$

Taking Inverse Laplace,

$$y(t) = c_0 + c_1 e^{pt} + c_2 e^{p^* t} \tag{40}$$
$$= 1 + \frac{|p|}{\omega_d} e^{-\sigma t} \frac{e^{j(\omega_d t + \varphi)} + e^{-j(\omega_d t + \varphi)}}{2} \tag{41}$$
$$= \left(1 + \frac{|p|}{\omega_d} e^{-\sigma t} \cos(\omega_d t + \varphi)\right) u(t) \tag{42}$$

where,

$$|p| = \sqrt{{\omega_d}^2 + \sigma^2} \tag{43}$$

$$\omega_d = \omega_0 \sqrt{1 - \zeta^2} \tag{44}$$

$$\sigma = \omega_0 \zeta \tag{45}$$

$$\varphi = \pi - \tan^{-1} \frac{\sigma}{\omega_d} \tag{46}$$

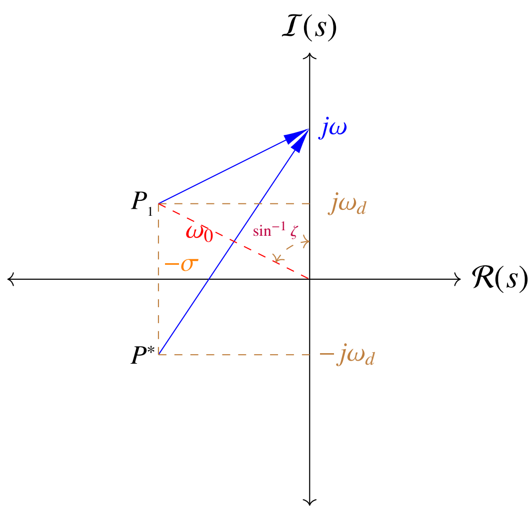

Fig. 4: s-Plane for Under damped case

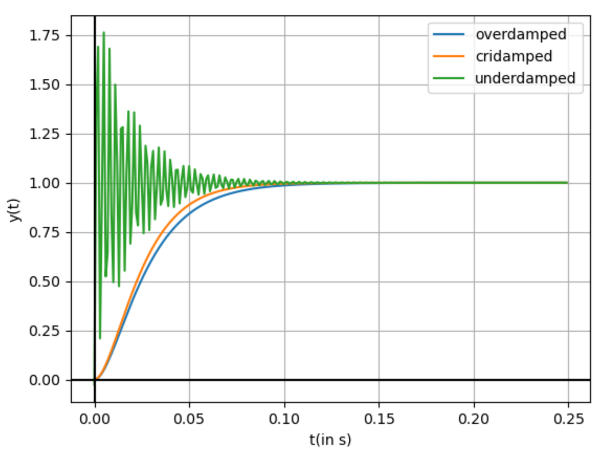

Fig. 5: Step response in all three cases

III. Frequency Response

The frequency response $y_{ss}(t)$ is defined as the steady state response to a sinusoidal input signal $x(t) = \sin \omega t$. It describes how well the filter can distinguish between different frequencies.

$$y_{ss}(t) = |H(j\omega)| \sin(\omega t + \angle H(j\omega)) \tag{47}$$

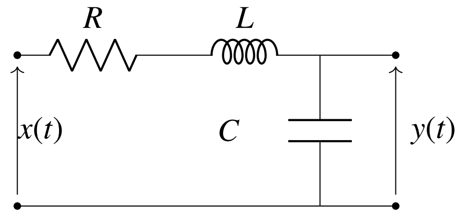

Fig. 6: RLC Low pass filter

1) Overdamped Case

$$H(s) = \omega_0^2 \frac{1}{(s - p_1)(s - p_2)} \tag{48}$$

where,

$$p_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \tag{49}$$

$$H(s) = |H(s)| \ e^{j\angle H(s)} \tag{50}$$

$$|H(s)| = \omega_0^2 \frac{1}{|s - p_1| |s - p_2|} \tag{51}$$

$$= \omega_0^2 \frac{1}{|j\omega - p_2| \ |j\omega - p_2|} \tag{52}$$

$$= \omega_0^2 \frac{1}{\sqrt{\omega^2 + {p_1}^2} \sqrt{\omega^2 + {p_2}^2}} \tag{53}$$

The magnitude of the transfer function expressed on a logarithmic scale:

$$|H_{dB}(\omega)| = 20 \log(\omega_0^2) - 20 \log \sqrt{\omega^2 + {p_1}^2} - 20 \log \sqrt{\omega^2 + {p_2}^2} \tag{54}$$

2) Critically damped case

$$H(s) = \omega_0^2 \frac{1}{(s - p)^2} \tag{55}$$

where,

$$p = \sqrt{\frac{1}{LC}} \tag{56}$$

$$H(s) = |H(s)| \ e^{j\angle H(s)} \tag{57}$$

$$|H(s)| = \omega_0^2 \frac{1}{|s - p|^2} \tag{58}$$

$$|H(j\omega)| = \omega_0^2 \frac{1}{|j\omega - p|^2} \tag{59}$$

$$= \omega_0^2 \frac{1}{\omega^2 + p^2} \tag{60}$$

The magnitude of the transfer function expressed on a logarithmic scale:

$$|H_{dB}(\omega)| = 20 \log(\omega_0^2) - 20 \log(\omega^2 + p^2) \tag{61}$$

3) Underdamped Case

$$H(s) = \omega_0^2 \frac{1}{(s-p)(s-p^*)} \tag{62}$$

where,

$$p,\ p^* = \omega_n \left(-\zeta \pm j\sqrt{1-\zeta^2}\right) \tag{63}$$

$$= -\sigma \pm j\omega_d \tag{64}$$

$$H(s) = |H(s)|\ e^{j\angle H(s)} \tag{65}$$

$$|H(s)| = \omega_0^2 \frac{1}{\sqrt{({\omega_n}^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \tag{66}$$

The magnitude of the transfer function expressed on a logarithmic scale:

$$|H_{dB}(\omega)| = 20\log(\omega_0^2) - 10\log(({\omega_n}^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2) \tag{67}$$

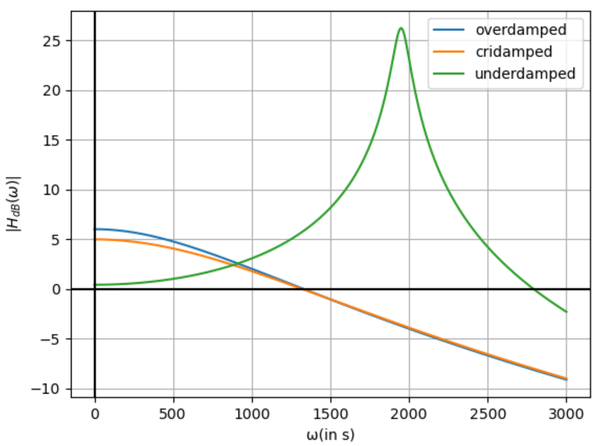

Fig. 7: Frequency response of all cases