Assignment 11

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***Abstract*—This document explains the proof that each subfield of the field of complex number contains every rational number**

Download all latex-tikz codes from

https://github.com/venkateshelangovan/IIT−Hyderabad−Assignments/tree/master/Assignment11_Matrix_Theory

1 Problem

Prove that each subfield of the field of complex number contains every rational number

2 Formal Defintion

2.1 Complex Numbers

A complex number is a number that can be expressed in the form $a + bi$, where a and b are real numbers, and i represents the imaginary unit, satisfying the equation $i^2 = -1$.The set of complex numbers is denoted by $\mathbb{C}$

$$\mathbb{C} = \{(a, b) : a, b \in \mathbb{R}\} \tag{2.1.1}$$

2.2 Rational Numbers

A number in the form $\frac{p}{q}$, where both p and q(non-zero) are integers, is called a rational number.The set of rational numbers is dentoed by $\mathbb{Q}$

3 Proof

Let $\mathbb{Q}$ be the set of rational numbers.

$$\mathbb{Q} = \left\{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z}_{\neq 0}\right\} \tag{3.0.1}$$

Let $\mathbb{C}$ be the field of complex numbers and given $\mathbb{F}$ be the subfield of field of complex numbers $\mathbb{C}$ Since $\mathbb{F}$ is the subfield , we could say that

$$0 \in \mathbb{F} \tag{3.0.2}$$

$$1 \in \mathbb{F} \tag{3.0.3}$$

3.1 Closed under addition

Here $\mathbb{F}$ is closed under addition since it is subfield

$$1 + 1 = 2 \in \mathbb{F} \tag{3.1.1}$$

$$1 + 1 + 1 = 3 \in \mathbb{F} \tag{3.1.2}$$

$$\vdots$$

$$1 + 1 + \cdots + 1(\text{p times}) = p \in \mathbb{F} \tag{3.1.3}$$

$$1 + 1 + \cdots + 1(\text{q times}) = q \in \mathbb{F} \tag{3.1.4}$$

By using the above property we could say that zero and other positive integers belongs to $\mathbb{F}$.Since p and q are integers we say,

$$p \in \mathbb{Z} \tag{3.1.5}$$

$$q \in \mathbb{Z} \tag{3.1.6}$$

3.2 Additive Inverse

Let x be the positive integer belong $\mathbb{F}$ and by additive inverse we could say,

$$\forall x \in \mathbb{F} \tag{3.2.1}$$

$$(-x) \in \mathbb{F} \tag{3.2.2}$$

Therefore field $\mathbb{F}$ contains every integers. Let n be a integer then,

$$n \in \mathbb{Z} \implies n \in \mathbb{F} \tag{3.2.3}$$

$$\mathbb{Z} \subseteq \mathbb{F} \tag{3.2.4}$$

Where $\mathbb{Z}$ is subset of $\mathbb{F}$

3.3 Multiplicative Inverse

Every element except zero in the subfield $\mathbb{F}$ has an multiplicative inverse. From equation (3.1.4), since $q \in \mathbb{F}$ we could say ,

$$\frac{1}{q} \in \mathbb{F} \quad \text{and } q \neq 0 \tag{3.3.1}$$

3.4 Closed under multiplication

Also, $\mathbb{F}$ is closed under multiplication and thus, from equation (3.1.3) and (3.3.1) we get ,

$$p \cdot \frac{1}{q} \in \mathbb{F} \tag{3.4.1}$$

$$\implies \frac{p}{q} \in \mathbb{F} \tag{3.4.2}$$

where , $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_{\neq 0}$ (from equation (3.1.6) and (3.3.1))

3.5 Conclusion

From (3.0.1) and (3.4.2) we could say ,

$$\mathbb{Q} \subseteq \mathbb{F} \tag{3.5.1}$$

From equation (3.5.1) we could say that each sub-field of the field of complex number contains every rational number

Hence Proved