Assignment 13

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***Abstract*—This document explains the concept of finding the solution to the system AX=Y**

Download all latex-tikz codes from

https://github.com/venkateshelangovan/IIT−Hyderabad−Assignments/tree/master/Assignment13_Matrix_Theory

1 Problem

Let

$$\mathbf{A} = \begin{pmatrix} 3 & -6 & 2 & -1 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \end{pmatrix} \tag{1.0.1}$$

For which (y_1, y_2, y_3, y_4) does the system of equations $\mathbf{AX} = \mathbf{Y}$ have a solution ?

2 Solution

Given ,

$$\mathbf{AX} = \mathbf{Y} \tag{2.0.1}$$

$$\begin{pmatrix} 3 & -6 & 2 & -1 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \end{pmatrix} \mathbf{X} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \tag{2.0.2}$$

Now we try to find the matrix $\mathbf{B}$ such that $\mathbf{BA}$ gives the row echelon form of matrix $\mathbf{A}$ Here,$\mathbf{B}$ is given by ,

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3} & 1 & 0 & 0 \\ -\frac{2}{7} & -\frac{3}{7} & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{BA} = \begin{pmatrix} 3 & -6 & 2 & -1 \\ 0 & 0 & \frac{7}{3} & \frac{7}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.0.4}$$

Therefore, rank of matrix $\mathbf{A}$ is 2 Now $\mathbf{B}$ is expressed in terms of two block matrices

$$\mathbf{B} = \begin{pmatrix} \mathbf{B_1} \\ \mathbf{B_2} \end{pmatrix} \tag{2.0.5}$$

$$\mathbf{B_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3} & 1 & 0 & 0 \end{pmatrix} \tag{2.0.6}$$

$$\mathbf{B_2} = \begin{pmatrix} -\frac{2}{7} & -\frac{3}{7} & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 \end{pmatrix} \tag{2.0.7}$$

Multiplying matrix $\mathbf{B}$ to both sides on the equation (2.0.1), we get ,

$$\begin{pmatrix} \mathbf{B_1} \\ \mathbf{B_2} \end{pmatrix} \mathbf{AX} = \begin{pmatrix} \mathbf{B_1} \\ \mathbf{B_2} \end{pmatrix} \mathbf{Y} \tag{2.0.8}$$

We know that , matrix $\mathbf{A}$ is of rank 2 The augumented matrix of (2.0.8) is given by

$$\left(\begin{array}{c|c} \mathbf{B_1A} & \mathbf{B_1Y} \\ \mathbf{B_2A} & \mathbf{B_2Y} \end{array}\right) \tag{2.0.9}$$

$$\mathbf{B_1A} = \begin{pmatrix} 3 & -6 & 2 & -1 \\ 0 & 0 & \frac{7}{3} & \frac{7}{3} \end{pmatrix} \tag{2.0.10}$$

$$\mathbf{B_2A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.0.11}$$

Since $\mathbf{B_2A}$ is zero matrix and for the given system $\mathbf{AX} = \mathbf{Y}$ to have a solution,

$$\mathbf{B_2Y} = 0 \tag{2.0.12}$$

$$\begin{pmatrix} -\frac{2}{7} & -\frac{3}{7} & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = 0 \tag{2.0.13}$$

The augumented matrix of (2.0.13) is given by,

$$\left(\begin{array}{cccc|c} -\frac{2}{7} & -\frac{3}{7} & 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 & 0 \end{array}\right) \tag{2.0.14}$$

By row reduction technique,

$$\xleftrightarrow{R_1=-\frac{7}{2}R_1}\left(\begin{array}{cccc|c}1 & \frac{3}{2} & -\frac{7}{2} & 0 & 0\\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 & 0\end{array}\right) \tag{2.0.15}$$

$$\xleftrightarrow{R_2=2R_2}\left(\begin{array}{cccc|c}1 & \frac{3}{2} & -\frac{7}{2} & 0 & 0\\ 0 & 1 & -3 & 2 & 0\end{array}\right) \tag{2.0.16}$$

$$\xleftrightarrow{R_1=R_1-\frac{3}{2}R_2}\left(\begin{array}{cccc|c}1 & 0 & 1 & -3 & 0\\ 0 & 1 & -3 & 2 & 0\end{array}\right) \tag{2.0.17}$$

Equation (2.0.13) can be modified as ,

$$\begin{pmatrix}1 & 0 & 1 & -3\\ 0 & 1 & -3 & 2\end{pmatrix}\begin{pmatrix}y_1\\ y_2\\ y_3\\ y_4\end{pmatrix} = 0 \tag{2.0.18}$$

Here y_3 and y_4 are free variables

If $y_3 = a$ and $y_4 = b$,then the solution to the system of equation $\mathbf{AX} = \mathbf{Y}$ is given by,

$$\begin{pmatrix}y_1\\ y_2\\ y_3\\ y_4\end{pmatrix} = a\begin{pmatrix}-1\\ 3\\ 1\\ 0\end{pmatrix} + b\begin{pmatrix}3\\ -2\\ 0\\ 1\end{pmatrix} \tag{2.0.19}$$

One of the solution when $a = 1$ and $b = 2$ is given by ,

$$\begin{pmatrix}y_1\\ y_2\\ y_3\\ y_4\end{pmatrix} = \begin{pmatrix}-1\\ 3\\ 1\\ 0\end{pmatrix} + 2\begin{pmatrix}3\\ -2\\ 0\\ 1\end{pmatrix} \tag{2.0.20}$$