Assignment 6

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***Abstract*—This document explains the the concept of finding the angle between the two straight lines from given second degree equation**

Download all latex-tikz codes from

https://github.com/venkateshelangovan/IIT−Hyderabad−Assignments/tree/master/Assignment6_Matrix_Theory

1 Problem

Prove that the equation $12x^2+7xy-10y^2+13x+45y-35=0$ represents two straight lines and find the angle between them

2 Pair of staraight lines

The general second order equation is given by ,

$$ax^2+2bxy+cy^2+2dx+2ey+f=0 \tag{2.0.1}$$

the above equation 2.0.1 can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x}+2\mathbf{u}^T\mathbf{x}+f=0 \tag{2.0.2}$$

where

$$\mathbf{V}=\mathbf{V}^T=\begin{pmatrix}a & b\\ b & c\end{pmatrix} \tag{2.0.3}$$

$$\mathbf{u}=\begin{pmatrix}d\\ e\end{pmatrix} \tag{2.0.4}$$

the above equation 2.0.2 represents a pair of straight lines if

$$\begin{vmatrix}\mathbf{V} & \mathbf{u}\\ \mathbf{u}^T & f\end{vmatrix}=0 \tag{2.0.5}$$

3 Solution

Given,

$$12x^2+7xy-10y^2+13x+45y-35=0 \tag{3.0.1}$$

The above equation 3.0.1 can be expressed as shown in equations 2.0.2, 2.0.3, 2.0.4

$$\mathbf{x}^T\begin{pmatrix}12 & \frac{7}{2}\\ \frac{7}{2} & -10\end{pmatrix}\mathbf{x}+2\begin{pmatrix}\frac{13}{2} & \frac{45}{2}\end{pmatrix}\mathbf{x}-35=0 \tag{3.0.2}$$

Comparing equation 3.0.2 with 2.0.2 we get

$$\mathbf{V}=\mathbf{V}^T=\begin{pmatrix}12 & \frac{7}{2}\\ \frac{7}{2} & -10\end{pmatrix} \tag{3.0.3}$$

$$\mathbf{u}=\begin{pmatrix}\frac{13}{2}\\ \frac{45}{2}\end{pmatrix} \tag{3.0.4}$$

$$f=-35 \tag{3.0.5}$$

Substituting the above equations 3.0.3, 3.0.4, 3.0.5 in LHS of equation 2.0.5 to verify the given equation is pair of straight lines

$$\delta=\begin{vmatrix}12 & \frac{7}{2} & \frac{13}{2}\\ \frac{7}{2} & -10 & \frac{45}{2}\\ \frac{13}{2} & \frac{45}{2} & -35\end{vmatrix} \tag{3.0.6}$$

Expanding the above determinant , we get

$$\delta=0 \tag{3.0.7}$$

Since equation 2.0.5 is satisfied, we could say that the given equation 3.0.1 represents two straight lines

From equation 3.0.1

Consider ,

$$12x^2+7xy-10y^2 \tag{3.0.8}$$

$$\implies 12x^2+15xy-8xy-10y^2 \tag{3.0.9}$$

$$\implies 3x(4x+5y)-2y(4x+5y) \tag{3.0.10}$$

$$\implies (3x-2y)(4x+5y) \tag{3.0.11}$$

where

$$\mathbf{V}=\begin{pmatrix}12 & \frac{7}{2}\\ \frac{7}{2} & -10\end{pmatrix} \tag{3.0.12}$$

The characteristic equation of $\mathbf{V}$ is obtained by evaluating the determinant

$$\left|\lambda\mathbf{I}-V\right|=0 \tag{3.0.13}$$

$$\begin{vmatrix}\lambda-12 & \frac{7}{2}\\ \frac{7}{2} & \lambda+10\end{vmatrix}=0 \tag{3.0.14}$$

$$\implies 4\lambda^2-8\lambda-529=0 \tag{3.0.15}$$

The eigenvalues are the roots of equation 3.0.15 is given by

$$\lambda_1 = 1 - \frac{\sqrt{153}}{2} \tag{3.0.16}$$

$$\lambda_2 = 1 + \frac{\sqrt{153}}{2} \tag{3.0.17}$$

The eigenvector p is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{3.0.18}$$

$$\implies (\lambda\mathbf{I} - \mathbf{V})\mathbf{p} = 0 \tag{3.0.19}$$

For $\lambda_1 = 1 - \frac{\sqrt{153}}{2}$,

$$(\lambda_1\mathbf{I} - \mathbf{V}) = \begin{pmatrix} -11 - \frac{\sqrt{153}}{2} & \frac{7}{2} \\ \frac{7}{2} & 11 - \frac{\sqrt{153}}{2} \end{pmatrix} \tag{3.0.20}$$

By row reduction ,

$$\begin{pmatrix} -11 - \frac{\sqrt{153}}{2} & \frac{7}{2} \\ \frac{7}{2} & 11 - \frac{\sqrt{153}}{2} \end{pmatrix} \tag{3.0.21}$$

$$\xleftrightarrow{R_2 = R_2 - \left(\frac{7}{2\left(-11 - \frac{\sqrt{153}}{2}\right)}\right)R_1} \begin{pmatrix} -11 - \frac{\sqrt{153}}{2} & \frac{7}{2} \\ 0 & 0 \end{pmatrix} \tag{3.0.22}$$

$$\xleftrightarrow{R_1 = \frac{R_1}{\left(-11 - \frac{\sqrt{153}}{2}\right)}} \begin{pmatrix} 1 & \frac{7}{2\left(-11 - \frac{\sqrt{153}}{2}\right)} \\ 0 & 0 \end{pmatrix} \tag{3.0.23}$$

Subsituting equation 3.0.32 in equation 3.0.19 we get

$$\begin{pmatrix} 1 & \frac{7}{2\left(-11 - \frac{\sqrt{153}}{2}\right)} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.0.24}$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

Let $v_2 = t$

$$v_1 = \frac{-7t}{2\left(-11 - \frac{\sqrt{153}}{2}\right)} \tag{3.0.25}$$

On solving further we get

$$v_1 = \frac{(\sqrt{153} - 22)t}{7} \tag{3.0.26}$$

Eigen vector $\mathbf{p_1}$ is given by

$$\mathbf{p_1} = \begin{pmatrix} \frac{(\sqrt{153} - 22)t}{7} \\ t \end{pmatrix} \tag{3.0.27}$$

Let $t = 1$, we get

$$\mathbf{p_1} = \begin{pmatrix} \frac{\sqrt{153} - 22}{7} \\ 1 \end{pmatrix} \tag{3.0.28}$$

For $\lambda_2 = 1 + \frac{\sqrt{153}}{2}$,

$$(\lambda_2\mathbf{I} - \mathbf{V}) = \begin{pmatrix} -11 + \frac{\sqrt{153}}{2} & \frac{7}{2} \\ \frac{7}{2} & 11 + \frac{\sqrt{153}}{2} \end{pmatrix} \tag{3.0.29}$$

By row reduction ,

$$\begin{pmatrix} -11 + \frac{\sqrt{153}}{2} & \frac{7}{2} \\ \frac{7}{2} & 11 + \frac{\sqrt{153}}{2} \end{pmatrix} \tag{3.0.30}$$

$$\xleftrightarrow{R_2 = R_2 - \left(\frac{7}{2\left(-11 + \frac{\sqrt{153}}{2}\right)}\right)R_1} \begin{pmatrix} -11 + \frac{\sqrt{153}}{2} & \frac{7}{2} \\ 0 & 0 \end{pmatrix} \tag{3.0.31}$$

$$\xleftrightarrow{R_1 = \frac{R_1}{\left(-11 + \frac{\sqrt{153}}{2}\right)}} \begin{pmatrix} 1 & \frac{7}{2\left(-11 + \frac{\sqrt{153}}{2}\right)} \\ 0 & 0 \end{pmatrix} \tag{3.0.32}$$

Subsituting equation 3.0.32 in equation 3.0.19 we get

$$\begin{pmatrix} 1 & \frac{7}{2\left(-11 + \frac{\sqrt{153}}{2}\right)} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.0.33}$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

Let $v_2 = t$

$$v_1 = \frac{-7t}{2\left(-11 + \frac{\sqrt{153}}{2}\right)} \tag{3.0.34}$$

On solving further we get

$$v_1 = \frac{-t(\sqrt{153} + 22)}{7} \tag{3.0.35}$$

Eigen vector $\mathbf{p_2}$ is given by

$$\mathbf{p_2} = \begin{pmatrix} \frac{-t(\sqrt{153} + 22)}{7} \\ t \end{pmatrix} \tag{3.0.36}$$

Let $t = 1$, we get

$$\mathbf{p_2} = \begin{pmatrix} \frac{-(\sqrt{153} + 22)}{7} \\ 1 \end{pmatrix} \tag{3.0.37}$$

By eigen decompostion $\mathbf{V}$ can be represented by

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{3.0.38}$$

where

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} \tag{3.0.39}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{3.0.40}$$

Substituting equations 3.0.28, 3.0.37 in equation 3.0.39 we get

$$\mathbf{P} = \begin{pmatrix} \frac{\sqrt{153}-22}{7} & \frac{-(\sqrt{153}+22)}{7} \\ 1 & 1 \end{pmatrix} \tag{3.0.41}$$

Substituting equations 3.0.16, 3.0.17 in 3.0.40 we get

$$\mathbf{D} = \begin{pmatrix} 1 - \frac{\sqrt{153}}{2} & 0 \\ 0 & 1 + \frac{\sqrt{153}}{2} \end{pmatrix} \tag{3.0.42}$$

Substitute the equations 3.0.12, 3.0.41, 3.0.42 in equation 3.0.38 we get

$$\begin{pmatrix} 12 & \frac{7}{2} \\ \frac{7}{2} & -10 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{153}-22}{7} & \frac{-(\sqrt{153}+22)}{7} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 - \frac{\sqrt{153}}{2} & 0 \\ 0 & 1 + \frac{\sqrt{153}}{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{153}-22}{7} & 1 \\ \frac{-(\sqrt{153}+22)}{7} & 1 \end{pmatrix} \tag{3.0.43}$$

Therefore equation 3.0.1 can be modified as

$$(3x - 2y + l)(4x + 5y + m) = 0 \tag{3.0.44}$$

Expanding the above equation 3.0.44 we get

$$12x^2 + 7xy - 10y^2 + (3m + 4l)x + (-2m + 5l)y + lm = 0 \tag{3.0.45}$$

Equating x and y co-efficients of the equations 3.0.1 and 3.0.45 , we get ,

$$3m + 4l = 13 \tag{3.0.46}$$

$$-2m + 5l = 45 \tag{3.0.47}$$

Solving equations 3.0.46, 3.0.47 we get ,

$$l = 7 \tag{3.0.48}$$

$$m = -5 \tag{3.0.49}$$

Substituting the equations 3.0.48, 3.0.49 in 3.0.44 we get

$$(3x - 2y + 7)(4x + 5y - 5) = 0 \tag{3.0.50}$$

The above equation 3.0.50 represents two straights and straight line equation is given by

$$3x - 2y + 7 = 0 \tag{3.0.51}$$

$$4x + 5y - 5 = 0 \tag{3.0.52}$$

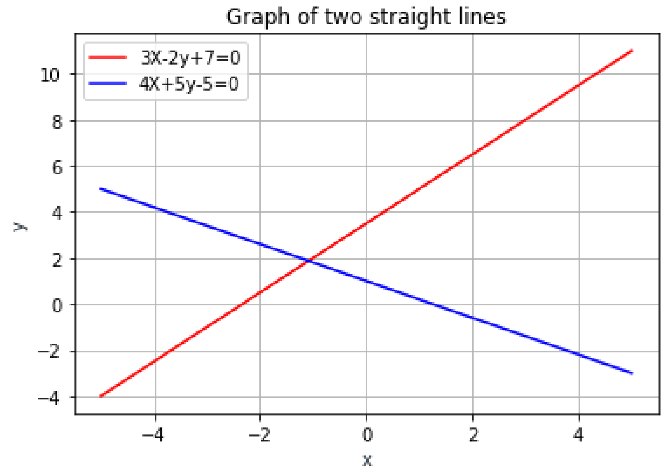

Fig. 1: Pair of straight lines

The above figure 1 represents the pair of straight lines

4 Angle between the straight lines

From equation 3.0.51

Normal Vector $\mathbf{n_1}$ is given by

$$\mathbf{n_1} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{4.0.1}$$

From equation 3.0.52

Normal Vector $\mathbf{n_2}$ is given by

$$\mathbf{n_2} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \tag{4.0.2}$$

Angle between the two straight lines is given by

$$\theta = \cos^{-1}\left(\frac{\mathbf{n_1}^T \mathbf{n_2}}{||\mathbf{n_1}||\,||\mathbf{n_2}||}\right) \tag{4.0.3}$$

$$\mathbf{n_1}^T \mathbf{n_2} = \begin{pmatrix} 3 & -2 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = 2 \tag{4.0.4}$$

$$||\mathbf{n_1}|| = \sqrt{3^2 + (-2)^2} = \sqrt{13} \tag{4.0.5}$$

$$||\mathbf{n_2}|| = \sqrt{4^2 + 5^2} = \sqrt{41} \tag{4.0.6}$$

Substituting equations 4.0.4, 4.0.5 ,4.0.6 in equation 4.0.3, we get

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{13}\sqrt{41}}\right) \tag{4.0.7}$$

$$\theta = 85^\circ \tag{4.0.8}$$

***Result* :**
Angle between the two straight line is given by

$$\theta = 85^\circ \tag{4.0.9}$$