Assignment 6

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***Abstract*—This document explains the the concept of finding the angle between the two straight lines from given second degree equation**

Download all latex-tikz codes from

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https://github.com/venkateshelangovan/IIT-Hyderabad-Assignments/tree/master/Assignment6_Matrix_Theory
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1 Problem

Prove that the equation $12x^2+7xy-10y^2+13x+45y-35=0$ represents two straight lines and find the angle between them

2 Pair of staraight lines

The general second order equation is given by ,

$$ax^2+2bxy+cy^2+2dx+2ey+f=0 \tag{2.0.1}$$

the above equation (2.0.1) can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x}+2\mathbf{u}^T\mathbf{x}+f=0 \tag{2.0.2}$$

where

$$\mathbf{V}=\mathbf{V}^T=\begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{u}=\begin{pmatrix} d \\ e \end{pmatrix} \tag{2.0.4}$$

the above equation (2.0.2) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix}=0 \tag{2.0.5}$$

3 Solution

Given,

$$12x^2+7xy-10y^2+13x+45y-35=0 \tag{3.0.1}$$

The above equation (3.0.1) can be expressed as shown in equations (2.0.2), (2.0.3), (2.0.4)

$$\mathbf{x}^T\begin{pmatrix} 12 & \frac{7}{2} \\ \frac{7}{2} & -10 \end{pmatrix}\mathbf{x}+2\begin{pmatrix} \frac{13}{2} & \frac{45}{2} \end{pmatrix}\mathbf{x}-35=0 \tag{3.0.2}$$

Comparing equation (3.0.2) with (2.0.2) we get

$$\mathbf{V}=\mathbf{V}^T=\begin{pmatrix} 12 & \frac{7}{2} \\ \frac{7}{2} & -10 \end{pmatrix} \tag{3.0.3}$$

$$\mathbf{u}=\begin{pmatrix} \frac{13}{2} \\ \frac{45}{2} \end{pmatrix} \tag{3.0.4}$$

$$f=-35 \tag{3.0.5}$$

Substituting the above equations (3.0.3), (3.0.4), (3.0.5) in LHS of equation (2.0.5) to verify the given equation is pair of straight lines

$$\delta=\begin{vmatrix} 12 & \frac{7}{2} & \frac{13}{2} \\ \frac{7}{2} & -10 & \frac{45}{2} \\ \frac{13}{2} & \frac{45}{2} & -35 \end{vmatrix} \tag{3.0.6}$$

Expanding the above determinant , we get

$$\delta=0 \tag{3.0.7}$$

Since equation (2.0.5) is satisfied, we could say that the given equation (3.0.1) represents two straight lines

$$\det(V)=\begin{vmatrix} 12 & \frac{7}{2} \\ \frac{7}{2} & -10 \end{vmatrix}<0 \tag{3.0.8}$$

Since $\det(V)<0$ we could say two intersecting lines are obtained

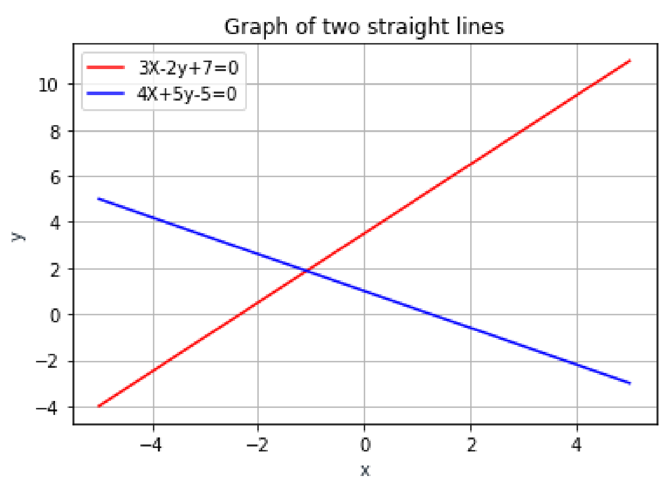

Fig. 1: Pair of straight lines

Let the pair of straight lines in vector form is given by

$$\mathbf{n_1}^T\mathbf{x} = c_1 \tag{3.0.9}$$
$$\mathbf{n_2}^T\mathbf{x} = c_2 \tag{3.0.10}$$

Equating their product with (3.0.2)

$$(\mathbf{n_1}^T\mathbf{x} - c_1)(\mathbf{n_2}^T\mathbf{x} - c_2) = \mathbf{x}^T\begin{pmatrix}12 & \frac{7}{2}\\ \frac{7}{2} & -10\end{pmatrix}\mathbf{x}$$
$$+2\begin{pmatrix}\frac{13}{2} & \frac{45}{2}\end{pmatrix}\mathbf{x} - 35 \tag{3.0.11}$$

$$\mathbf{n_1} * \mathbf{n_2} = \{12, 7, -10\} \tag{3.0.12}$$
$$c_2\mathbf{n_1} + c_1\mathbf{n_2} = -2\begin{pmatrix}\frac{13}{2}\\ \frac{45}{2}\end{pmatrix} \tag{3.0.13}$$
$$c_1c_2 = -35 \tag{3.0.14}$$

The slopes of the lines are given by the roots of the polynomial

$$cm^2 + 2bm + a = 0 \tag{3.0.15}$$
$$\implies m_i = \frac{-b \pm \sqrt{-\det(V)}}{c} \tag{3.0.16}$$
$$\mathbf{n_i} = k\begin{pmatrix}-m_i\\ 1\end{pmatrix} \tag{3.0.17}$$

Substituting the given data in above equations (3.0.15) we get,

$$-10m^2 + 7m + 12 = 0 \tag{3.0.18}$$
$$\implies m_i = \frac{\frac{-7}{2} \pm \sqrt{-(\frac{-529}{4})}}{-10} \tag{3.0.19}$$

Solving equation (3.0.19) we get ,

$$m_1 = -\frac{4}{5} \tag{3.0.20}$$
$$m_2 = \frac{3}{2} \tag{3.0.21}$$
$$\mathbf{n_1} = k_1\begin{pmatrix}\frac{4}{5}\\ 1\end{pmatrix} \tag{3.0.22}$$
$$\mathbf{n_2} = k_2\begin{pmatrix}-\frac{3}{2}\\ 1\end{pmatrix} \tag{3.0.23}$$

Substituting equations (3.0.22), (3.0.23) in equation (3.0.12) we get

$$k_1k_2 = -10 \tag{3.0.24}$$

Possible combinations of (k_1, k_2) are (10,-1), (-1,10), (5,-2), (-2,5)

Lets assume $k_1 = 5$, $k_2 = -2$, we get

$$\mathbf{n_1} = \begin{pmatrix}4\\ 5\end{pmatrix} \tag{3.0.25}$$
$$\mathbf{n_2} = \begin{pmatrix}3\\ -2\end{pmatrix} \tag{3.0.26}$$

From equation (3.0.13) we get

$$\begin{pmatrix}\mathbf{n_1} & \mathbf{n_2}\end{pmatrix}\begin{pmatrix}c_2\\ c_1\end{pmatrix} = -2\begin{pmatrix}\frac{13}{2}\\ \frac{45}{2}\end{pmatrix} \tag{3.0.27}$$
$$\begin{pmatrix}4 & 3\\ 5 & -2\end{pmatrix}\begin{pmatrix}c_2\\ c_1\end{pmatrix} = -2\begin{pmatrix}\frac{13}{2}\\ \frac{45}{2}\end{pmatrix} \tag{3.0.28}$$
$$4c_2 + 3c_1 = -13 \tag{3.0.29}$$
$$5c_2 - 2c_1 = -45 \tag{3.0.30}$$

Solving equations (3.0.29) ,(3.0.30) we get

$$c_1 = 5 \tag{3.0.31}$$
$$c_2 = -7 \tag{3.0.32}$$

Equations (3.0.9), (3.0.10) can be modified as

$$\begin{pmatrix}4 & 5\end{pmatrix}\mathbf{x} = 5 \tag{3.0.33}$$
$$\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} = -7 \tag{3.0.34}$$

4 Angle between the straight lines

Angle between the two straight lines is given by

$$\theta = \cos^{-1}\left(\frac{\mathbf{n_1}^T\mathbf{n_2}}{\|\mathbf{n_1}\|\,\|\mathbf{n_2}\|}\right) \tag{4.0.1}$$
$$\mathbf{n_1}^T\mathbf{n_2} = \begin{pmatrix}4 & 5\end{pmatrix}\begin{pmatrix}3\\ -2\end{pmatrix} = 2 \tag{4.0.2}$$
$$\|\mathbf{n_1}\| = \sqrt{4^2 + 5^2} = \sqrt{41} \tag{4.0.3}$$
$$\|\mathbf{n_2}\| = \sqrt{3^2 + (-2)^2} = \sqrt{13} \tag{4.0.4}$$

Substituting equations (4.0.2), (4.0.3) ,(4.0.4) in equation (4.0.1), we get

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{41}\sqrt{13}}\right) \tag{4.0.5}$$
$$\theta = 85° \tag{4.0.6}$$

***Result* :**
Angle between the two straight line is given by

$$\theta = 85° \tag{4.0.7}$$