Assignment 19

Venkatesh E
AI20MTECH14005

Download latex-tikz codes from

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https://github.com/venkateshelangovan/IIT−Hyderabad−Assignments/tree/master/
    Assignment19_Matrix_Theory
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1 Problem

Let $\mathbf{M} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 4 \\ -2 & 1 & -4 \end{pmatrix}$. Given that 1 is an eigenvalue of $\mathbf{M}$, then which of the following are correct ?

1) The minimal polynomial of $\mathbf{M}$ is $(x-1)(x+4)$
2) The minimal polynomial of $\mathbf{M}$ is $(x-1)^2(x+4)$
3) $\mathbf{M}$ is not diagonalizable
4) $\mathbf{M}^{-1} = \frac{1}{4}(\mathbf{M}+3\mathbf{I})$

2 Solution

Given	$$\mathbf{M} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 4 \\ -2 & 1 & -4 \end{pmatrix} \quad (2.0.1)$$ One of the eigenvalue of $\mathbf{M}$ is 1
Solution	Let the eigenvalues of matrix $\mathbf{M}$ of order 3×3 be $\lambda_1, \lambda_2, \lambda_3$ From given , let $\lambda_1 = 1$. We know that sum of the eigenvalues of matrix is Trace of the matrix and product of eigenvalues of matrix is Determinant of the matrix. Trace of the square matrix(Tr($\mathbf{M}$)) is the sum of the elements in the main diagonal of $\mathbf{M}$. $$Tr(\mathbf{M}) = 1+1-4 \quad (2.0.2)$$ $$\implies Tr(\mathbf{M}) = -2 \quad (2.0.3)$$ $$\implies \lambda_1+\lambda_2+\lambda_3 = -2 \quad (2.0.4)$$ $$\implies \lambda_2+\lambda_3 = -3 \quad (2.0.5)$$ $$\implies \lambda_2 = -3-\lambda_3 \quad (2.0.6)$$ By row reducing the matrix $\mathbf{M}$, we get , $$\mathbf{M} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & -\frac{4}{3} \end{pmatrix} \quad (2.0.7)$$

	$$Det(\mathbf{M}) = 1\left(3\left(-\frac{4}{3}\right)\right) = -4 \quad (2.0.8)$$ $$\implies \lambda_1\lambda_2\lambda_3 = -4 \quad (2.0.9)$$ $$\implies \lambda_2\lambda_3 = -4 \quad (2.0.10)$$ Solving equations (2.0.6) and (2.0.10) one of the possibilities we get, $$\lambda_1 = 1 \quad (2.0.11)$$ $$\lambda_2 = 1 \quad (2.0.12)$$ $$\lambda_3 = -4 \quad (2.0.13)$$
	Using the eigenvalues the characteristic polynomial of matrix $\mathbf{M}$ is given by, $$c(x) = x^3 + 2x^2 - 7x + 4 = 0 \quad (2.0.14)$$ The Cayley Hamilton Theorem states that every square matrix satisfies its own characteristic equation. Using the above theorem, the equation (2.0.14) can be written as, $$\mathbf{M}^3 + 2\mathbf{M}^2 - 7\mathbf{M} + 4\mathbf{I} = 0 \quad (2.0.15)$$ $$\mathbf{M}^2 + 2\mathbf{M} - 7\mathbf{I} + 4\mathbf{M}^{-1} = 0 \quad (2.0.16)$$ $$\implies \mathbf{M}^{-1} = -\frac{1}{4}(\mathbf{M}^2 + 2\mathbf{M} - 7\mathbf{I}) \quad (2.0.17)$$
Statement 1	The minimal polynomial of $\mathbf{M}$ is $(x-1)(x+4)$
	If (x-1)(x+4) is a minimal polynomial of M then, $$(\mathbf{M}-\mathbf{I})(\mathbf{M}+4\mathbf{I}) = \mathbf{0}_{3\times3} \quad (2.0.18)$$ But, $$(\mathbf{M}-\mathbf{I})(\mathbf{M}+4\mathbf{I}) = \begin{pmatrix} -4 & -4 & -4 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \neq \mathbf{0}_{3\times3} \quad (2.0.19)$$ **False Statement**
Statement 2	The minimal polynomial of $\mathbf{M}$ is $(x-1)^2(x+4)$
	Let m(x) be the minimal polynomial $$m(x) = (x-1)^2(x+4) \quad (2.0.20)$$ $$= x^3 + 2x^2 - 7x + 4 \quad (2.0.21)$$ $$= c(x)$$ In this case both minimal polynomial and characteristic polynomial were same. Therefore wecould say that equation (2.0.20) is the minimal polynomial of $\mathbf{M}$ as it satisfies equation (2.0.15) by Cayley Hamilton Theorem. **True Statement**
Statement 3	$\mathbf{M}$ is not diagonalizable.
	$\mathbf{M}$ is diagonalizable if and only if its minimal polynomial contains only linear factors.

	From equation (2.0.20) we could see that one of the factor of minimal polynomial is repeated and it is not a linear factor. Therefore, Matrix **M** is not diagonalizable. **True Statement**
Statement 4	$$\mathbf{M}^{-1} = \frac{1}{4}(\mathbf{M} + 3\mathbf{I}) \quad (2.0.22)$$
	Comparing equation (2.0.17) and (2.0.22) we could say that the given statement is **False Statement**.

TABLE 1: Explanation