Assignment 8

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***Abstract*—This document explains the concept of finding the conics representation from the given second degree equations**

Download all latex-tikz codes from

https://github.com/venkateshelangovan/IIT−Hyderabad−Assignments/tree/master/Assignment8_Matrix_Theory

1 Problem

What conics do the following equation represent? When possible, find the centres and also their equations referred to the centre

$$12x^2 - 23xy + 10y^2 - 25x + 26y = 14 \tag{1.0.1}$$

2 Solution

The given equation (1.0.1) can be expressed as

$$\mathbf{x}^T \begin{pmatrix} 12 & \frac{-23}{2} \\ \frac{-23}{2} & 10 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} \frac{-25}{2} & 13 \end{pmatrix} \mathbf{x} - 14 = 0 \tag{2.0.1}$$

where

$$\mathbf{V} = \begin{pmatrix} 12 & \frac{-23}{2} \\ \frac{-23}{2} & 10 \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{u} = \begin{pmatrix} \frac{-25}{2} \\ 13 \end{pmatrix} \tag{2.0.3}$$

$$f = -14 \tag{2.0.4}$$

$$\det(\mathbf{V}) = \begin{vmatrix} 12 & \frac{-23}{2} \\ \frac{-23}{2} & 10 \end{vmatrix} \tag{2.0.5}$$

$$\implies \det(\mathbf{V}) = \frac{-49}{4} \tag{2.0.6}$$

$$\implies \det(\mathbf{V}) < 0 \tag{2.0.7}$$

Since $\det(\mathbf{V}) < 0$ the given equation (2.0.1) represents the hyperbola The characteristic equation of $\mathbf{V}$ is obtained by evaluating the determinant

$$\left| V - \lambda\mathbf{I} \right| = 0 \tag{2.0.8}$$

$$\begin{vmatrix} 12-\lambda & \frac{-23}{2} \\ \frac{-23}{2} & 10-\lambda \end{vmatrix} = 0 \tag{2.0.9}$$

$$\implies 4\lambda^2 - 88\lambda - 49 = 0 \tag{2.0.10}$$

The eigenvalues are the roots of equation 2.0.10 is given by

$$\lambda_1 = \frac{22 + \sqrt{533}}{2} \tag{2.0.11}$$

$$\lambda_2 = \frac{22 - \sqrt{533}}{2} \tag{2.0.12}$$

The eigenvector p is defined as

$$\mathbf{Vp} = \lambda\mathbf{p} \tag{2.0.13}$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \tag{2.0.14}$$

For $\lambda_1 = \frac{22-\sqrt{533}}{2}$,

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} \frac{\sqrt{553}+2}{2} & \frac{-23}{2} \\ \frac{-23}{2} & \frac{\sqrt{533}-2}{2} \end{pmatrix} \tag{2.0.15}$$

By row reduction ,

$$\begin{pmatrix} \frac{\sqrt{533}+2}{2} & \frac{-23}{2} \\ \frac{-23}{2} & \frac{\sqrt{533}-2}{2} \end{pmatrix} \tag{2.0.16}$$

$$\xleftrightarrow{R_1 = \frac{R_1}{\left(\frac{\sqrt{533}+2}{2}\right)}} \begin{pmatrix} 1 & \frac{2-\sqrt{533}}{23} \\ \frac{-23}{2} & \frac{\sqrt{533}-2}{2} \end{pmatrix} \tag{2.0.17}$$

$$\xleftrightarrow{R_2 = R_2 + \frac{23}{2}R_1} \begin{pmatrix} 1 & \frac{2-\sqrt{533}}{23} \\ 0 & 0 \end{pmatrix} \tag{2.0.18}$$

Subsituting equation 2.0.18 in equation 2.0.14 we get

$$\begin{pmatrix} 1 & \frac{2-\sqrt{533}}{23} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.19}$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

Let $v_2 = t$

$$v_1 = \frac{-t(2-\sqrt{533})}{23} \tag{2.0.20}$$

Eigen vector $\mathbf{p_1}$ is given by

$$\mathbf{p_1} = \begin{pmatrix} \frac{-t(2-\sqrt{533})}{23} \\ t \end{pmatrix} \tag{2.0.21}$$

Let $t = 1$, we get

$$\mathbf{p_1} = \begin{pmatrix} \frac{\sqrt{533}-2}{23} \\ 1 \end{pmatrix} \tag{2.0.22}$$

For $\lambda_2 = \frac{22+\sqrt{533}}{2}$,

$$(\mathbf{V} - \lambda_2\mathbf{I}) = \begin{pmatrix} \frac{2-\sqrt{553}}{2} & \frac{-23}{2} \\ \frac{-23}{2} & \frac{-2-\sqrt{533}}{2} \end{pmatrix} \tag{2.0.23}$$

By row reduction ,

$$\begin{pmatrix} \frac{2-\sqrt{533}}{2} & \frac{-23}{2} \\ \frac{-23}{2} & \frac{-2-\sqrt{533}}{2} \end{pmatrix} \tag{2.0.24}$$

$$\xleftrightarrow{R_1 = \frac{R_1}{\left(\frac{2-\sqrt{533}}{2}\right)}} \begin{pmatrix} 1 & \frac{2+\sqrt{533}}{23} \\ \frac{-23}{2} & \frac{-2-\sqrt{533}}{2} \end{pmatrix} \tag{2.0.25}$$

$$\xleftrightarrow{R_2 = R_2 + \frac{23}{2}R_1} \begin{pmatrix} 1 & \frac{2-\sqrt{533}}{23} \\ 0 & 0 \end{pmatrix} \tag{2.0.26}$$

Subsituting equation 2.0.26 in equation 2.0.14 we get

$$\begin{pmatrix} 1 & \frac{2+\sqrt{533}}{23} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.27}$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

Let $v_2 = t$

$$v_1 = \frac{-t(2+\sqrt{533})}{23} \tag{2.0.28}$$

Eigen vector $\mathbf{p_2}$ is given by

$$\mathbf{p_2} = \begin{pmatrix} \frac{-t(2+\sqrt{533})}{23} \\ t \end{pmatrix} \tag{2.0.29}$$

Let $t = 1$, we get

$$\mathbf{p_2} = \begin{pmatrix} \frac{-\sqrt{533}-2}{23} \\ 1 \end{pmatrix} \tag{2.0.30}$$

By eigen decompostion $\mathbf{V}$ can be represented by

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{2.0.31}$$

where

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} \tag{2.0.32}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{2.0.33}$$

Substituting equations 2.0.22, 2.0.30 in equation 2.0.32 we get

$$\mathbf{P} = \begin{pmatrix} \frac{\sqrt{533}-2}{23} & \frac{-\sqrt{533}-2}{23} \\ 1 & 1 \end{pmatrix} \tag{2.0.34}$$

Substituting equations 2.0.11, 2.0.12 in 2.0.33 we get

$$\mathbf{D} = \begin{pmatrix} \frac{22-\sqrt{533}}{2} & 0 \\ 0 & \frac{22+\sqrt{533}}{2} \end{pmatrix} \tag{2.0.35}$$

Centre of the hyperbola is given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{2.0.36}$$

$$\implies \mathbf{c} = -\begin{pmatrix} \frac{-40}{49} & \frac{-46}{49} \\ \frac{-46}{49} & \frac{-48}{49} \end{pmatrix} \begin{pmatrix} \frac{-25}{2} \\ 13 \end{pmatrix} \tag{2.0.37}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{40}{49} & \frac{46}{49} \\ \frac{46}{49} & \frac{48}{49} \end{pmatrix} \begin{pmatrix} \frac{-25}{2} \\ 13 \end{pmatrix} \tag{2.0.38}$$

$$\implies \mathbf{c} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.0.39}$$

Since,

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 26 > 0 \tag{2.0.40}$$

there isn't a need to swap axes

In hyperbola,

$$axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} \\ \sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} \end{cases} \tag{2.0.41}$$

From above equations we can say that,

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \frac{2\sqrt{13}}{\sqrt{22+\sqrt{533}}} \tag{2.0.42}$$

$$\sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} = \frac{2\sqrt{13}}{\sqrt{\sqrt{533}-22}} \tag{2.0.43}$$

Now (2.0.1) can be written as,

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \tag{2.0.44}$$

where ,

$$\mathbf{y} = \mathbf{P}^T(\mathbf{x} - \mathbf{c}) \tag{2.0.45}$$

To get $\mathbf{y}$,

$$\mathbf{y} = \mathbf{P}^T\mathbf{x} - \mathbf{P}^T\mathbf{c} \tag{2.0.46}$$

$$\mathbf{y} = \begin{pmatrix} \frac{\sqrt{533}-2}{23} & 1 \\ \frac{-\sqrt{533}-2}{23} & 1 \end{pmatrix}\mathbf{x} - \begin{pmatrix} \frac{\sqrt{533}-2}{23} & 1 \\ \frac{-\sqrt{533}-2}{23} & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.0.47}$$

$$\mathbf{y} = \begin{pmatrix} \frac{\sqrt{533}-2}{23} & 1 \\ \frac{-\sqrt{533}-2}{23} & 1 \end{pmatrix}\mathbf{x} - \begin{pmatrix} \frac{2(\sqrt{533}-2)}{23} + 1 \\ \frac{2(-\sqrt{533}-2)}{23} + 1 \end{pmatrix} \tag{2.0.48}$$

Subsituting the eqauations (2.0.40), (2.0.35) in equation (2.0.44)

$$\mathbf{y}^T \begin{pmatrix} \frac{22+\sqrt{533}}{2} & 0 \\ 0 & \frac{22-\sqrt{533}}{2} \end{pmatrix}\mathbf{y} - 26 = 0 \tag{2.0.49}$$

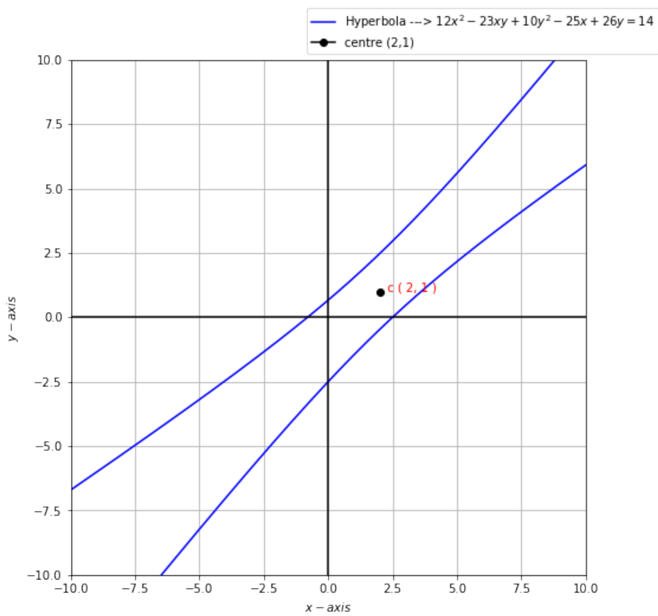

Fig. 1: Hyperbola when origin is shifted

The figure 1 verifies the given equation (2.0.1) as hyperbola with centre $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$