Assignment 14

Venkatesh E
AI20MTECH14005

***Abstract*—This document explains the proof such that if A is an $m \times n$ matrix and B is an $n \times m$ matrix and $n < m$,then AB is not invertible**

Download all latex-tikz codes from

https://github.com/venkateshelangovan/IIT−Hyderabad−Assignments/tree/master/Assignment14_Matrix_Theory

1 Problem

Prove that if $\mathbf{A}$ is an $m \times n$ matrix, $\mathbf{B}$ is an $n \times m$ matrix and $n < m$,then $\mathbf{AB}$ is not invertible

2 Rank of a Matrix

2.1 Definition

The rank of a matrix is defined as

1. The maximum number of linearly independent column vectors in the matrix or

2. The maximum number of linearly independent row vectors in the matrix.

2.2 To prove Row Rank=Column Rank

Consider the matrix $\mathbf{A}$ of order $m \times n$,

The row rank is the maximum number of linearly independent rows in the matrix $\mathbf{A}$

$$RowRank(\mathbf{A}) \leq m \tag{2.2.1}$$

The column rank is the maximum number of linearly independent column in the matrix $\mathbf{A}$

$$ColumnRank(\mathbf{A}) \leq n \tag{2.2.2}$$

A matrix $\mathbf{P_n}$ is a permutation matrix of order $n \times n$ if and only if it is obtained from $n \times n$ Identity matrix $\mathbf{I_n}$ by performing one or more interchanges of the rows and columns of $\mathbf{I_n}$.

One of the 3×3 permutation matrix is given by ,

$$\mathbf{P_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.2.3}$$

Let $\mathbf{P_n}$ be an $n \times n$ permuation matrix,

$$\mathbf{AP_n} = \begin{pmatrix} \mathbf{X} & \mathbf{W} \end{pmatrix} \tag{2.2.4}$$

where columns of $\mathbf{X}$ are the d pivot columns of $\mathbf{A}$.

Every column of $\mathbf{W}$ is a linear combination of the columns of $\mathbf{X}$, so there is a matrix $\mathbf{K}$ such that,

$$\mathbf{W} = \mathbf{XK} \tag{2.2.5}$$

where the columns of $\mathbf{K}$ contain the coefficients of each of those linear combinations.

Substituting the equation (2.2.5) in equation (2.2.4),we get

$$\mathbf{AP_n} = \begin{pmatrix} \mathbf{X} & \mathbf{XK} \end{pmatrix} \tag{2.2.6}$$

$$\implies \mathbf{AP_n} = \mathbf{X}\begin{pmatrix} \mathbf{I_d} & \mathbf{K} \end{pmatrix} \tag{2.2.7}$$

where $\mathbf{I_d}$ represents the $d \times d$ identity matrix

Transforming the matrix $\mathbf{A}$ into reduced row echelon form, we get ,

$$\mathbf{B} = \mathbf{EA} \tag{2.2.8}$$

where $\mathbf{E}$ is the product of elementary matrices and $\mathbf{B}$ is the reduced row echelon form of $\mathbf{A}$

Multiplying $\mathbf{E}$ to the equation (2.2.7) on both sides,

$$\mathbf{EAP_n} = \mathbf{EX}\begin{pmatrix} \mathbf{I_d} & \mathbf{K} \end{pmatrix} \tag{2.2.9}$$

$$\implies \mathbf{BP_n} = \mathbf{EX}\begin{pmatrix} \mathbf{I_d} & \mathbf{K} \end{pmatrix} \tag{2.2.10}$$

Where,

$$\mathbf{EX} = \begin{pmatrix} \mathbf{I_d} \\ 0 \end{pmatrix} \tag{2.2.11}$$

Substituting the equation (2.2.11) in equation (2.2.10),

$$\mathbf{BP_n} = \begin{pmatrix} \mathbf{I_d} & \mathbf{K} \\ 0 & 0 \end{pmatrix} \tag{2.2.12}$$

Here ,we could see that the nonzero d rows of the reduced row echelon form with the same permutation on the columns as we did for $\mathbf{A}$.Therefore we could say that,

$$\begin{pmatrix}\mathbf{I_d} & \mathbf{K}\end{pmatrix} = \mathbf{YP_n} \tag{2.2.13}$$

Substituting the equation (2.2.13) in equation (2.2.7),we get ,

$$\mathbf{AP_n} = \mathbf{XYP_n} \tag{2.2.14}$$

Here $\mathbf{P_n}$ is a permutation matrix and it is invertible.This implies,

$$\mathbf{A} = \mathbf{XY} \tag{2.2.15}$$

where matrix $\mathbf{X}$ is of order $m \times d$ and matrix $\mathbf{Y}$ is of order $d \times n$

Consider the example ,

$$\mathbf{A} = \begin{pmatrix}4 & 8 & 9\\ 2 & 4 & 7\end{pmatrix} \tag{2.2.16}$$

$$\mathbf{AP_3} = \begin{pmatrix}4 & 9 & 8\\ 2 & 7 & 4\end{pmatrix} \tag{2.2.17}$$

Let,

$$\mathbf{X} = \begin{pmatrix}4 & 9\\ 2 & 7\end{pmatrix} \tag{2.2.18}$$

$$\mathbf{W} = \mathbf{XK} = \begin{pmatrix}8\\ 4\end{pmatrix} \tag{2.2.19}$$

$$\mathbf{I_2} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{2.2.20}$$

$$\mathbf{AP_3} = \begin{pmatrix}\mathbf{X} & \mathbf{W}\end{pmatrix} \tag{2.2.21}$$

$$\mathbf{AP_3} = \begin{pmatrix}\mathbf{X} & \mathbf{XK}\end{pmatrix} \tag{2.2.22}$$

$$\mathbf{AP_3} = \mathbf{X}\begin{pmatrix}\mathbf{I_2} & \mathbf{K}\end{pmatrix} \tag{2.2.23}$$

From equation (2.2.19),

$$\mathbf{K} = \mathbf{X}^{-1}\mathbf{W} \tag{2.2.24}$$

$$\mathbf{K} = \begin{pmatrix}2\\ 0\end{pmatrix} \tag{2.2.25}$$

Let $\mathbf{B}$ be reduced row echelon form of matrix $\mathbf{A}$,

$$\mathbf{B} = \begin{pmatrix}1 & 2 & 0\\ 0 & 0 & 1\end{pmatrix} \tag{2.2.26}$$

$$\mathbf{BP_n} = \begin{pmatrix}1 & 0 & 2\\ 0 & 1 & 0\end{pmatrix} \tag{2.2.27}$$

Let,

$$\begin{pmatrix}\mathbf{I_2} & \mathbf{K}\end{pmatrix} = \mathbf{YP_n} \tag{2.2.28}$$

$$\begin{pmatrix}1 & 0 & 2\\ 0 & 1 & 0\end{pmatrix} = \mathbf{Y}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} \tag{2.2.29}$$

$$\implies \mathbf{Y} = \begin{pmatrix}1 & 2 & 0\\ 0 & 0 & 1\end{pmatrix} \tag{2.2.30}$$

From the above equations (2.2.16),(2.2.18) and (2.2.30), it can be seen that ,

$$\mathbf{A} = \mathbf{XY}$$

Now for the matrix $\mathbf{A}$ of order $m \times n$,

Every row of $\mathbf{A}$ is the linear combination of the rows of $\mathbf{Y}$

$$rowspace(\mathbf{A}) \subsetneq span(\{\mathbf{Y_1}, \mathbf{Y_2}, \dots, \mathbf{Y_d}\}) \tag{2.2.31}$$

where $\mathbf{Y_1}, \mathbf{Y_2}, \dots, \mathbf{Y_d}$ were the rows of $\mathbf{Y}$

The dimension of the row space is atmost d

Every column of $\mathbf{A}$ is the linear combination of the columns of $\mathbf{X}$

$$columnspace(\mathbf{A}) \subsetneq span(\{\mathbf{X_1}, \mathbf{X_2}, \dots, \mathbf{X_d}\}) \tag{2.2.32}$$

where $\mathbf{X_1}, \mathbf{X_2}, \dots, \mathbf{X_d}$ were the columns of $\mathbf{X}$

The dimension of the column space is atmost d.

Let $\mathbf{A}$ is of the order $m \times n$. If the dimension of the row space of $\mathbf{A}$ is r(Row Rank),then $\mathbf{A} = \mathbf{XY}$ for some $m \times r$ matrix $\mathbf{X}$ and $r \times m$ matrix $\mathbf{Y}$

Let $\mathbf{Y_1}, \mathbf{Y_2}, \dots, \mathbf{Y_r}$ be a basis for the row space of $\mathbf{A}$ and let $\mathbf{A_1}, \mathbf{A_2}, \dots, \mathbf{A_m}$ be the rows of $\mathbf{A}$.Then,

$$\begin{aligned}
\mathbf{A_1} &= x_{11}\mathbf{Y_1} + x_{12}\mathbf{Y_2} + \dots + x_{1r}\mathbf{Y_r}\\
\mathbf{A_2} &= x_{21}\mathbf{Y_1} + x_{22}\mathbf{Y_2} + \dots + x_{2r}\mathbf{Y_r}\\
&\vdots\\
\mathbf{A_m} &= x_{m1}\mathbf{Y_1} + x_{m2}\mathbf{Y_2} + \dots + x_{mr}\mathbf{Y_r}
\end{aligned}$$

Thus $\mathbf{A} = \mathbf{XY}$ where $\mathbf{Y}$ is the matrix with rows $\mathbf{Y_1}, \dots, \mathbf{Y_r}$ and $\mathbf{X}$ is the matrix of coefficients $\mathbf{X}(i, j) = x_{ij}$.

$$ColumnRank \leq RowRank \tag{2.2.33}$$

Let $\mathbf{X_1}, \mathbf{X_2}, \dots, \mathbf{X_r}$ be a basis for the column space of

$\mathbf{A}$ and let $\mathbf{A_1}, \mathbf{A_2}, \ldots, \mathbf{A_n}$ be the columns of $\mathbf{A}$.Then,

$$\mathbf{A_1} = \mathbf{X_1}y_{11} + \mathbf{X_2}y_{21} + \cdots + \mathbf{X_r}y_{r1}$$
$$\mathbf{A_2} = \mathbf{X_1}y_{12} + \mathbf{X_2}y_{22} + \cdots + \mathbf{X_r}y_{r2}$$
$$\vdots$$
$$\mathbf{A_n} = \mathbf{X_1}y_{1n} + \mathbf{X_2}y_{2n} + \cdots + \mathbf{X_r}y_{rn}$$

Thus $\mathbf{A} = \mathbf{XY}$ where $\mathbf{X}$ is the matrix with columns $\mathbf{X_1}, \ldots, \mathbf{X_r}$ and $\mathbf{Y}$ is the matrix of coefficients $\mathbf{Y}(i, j) = y_{ij}$.

$$RowRank \leq ColumnRank \tag{2.2.34}$$

From equations (2.2.33) and (2.2.34),we get ,

$$RowRank = ColumnRank \tag{2.2.35}$$

From the rank definition and the above equations (2.2.1),(2.2.2) and (2.2.35),

$$Rank(\mathbf{A}) \leq min(m, n) \tag{2.2.36}$$

2.3 Properties

For a matrix $\mathbf{A}$ of order $m \times n$,

(a) If m is less than n, then the rank of the matrix will be atmost m.

(b) If m is greater than n, then the rank of the matrix will be atmost n.

3 Proof

Given , Matrix $\mathbf{A}$ is of order $m \times n$ and Matrix $\mathbf{B}$ is of order $n \times m$

$$n < m \tag{3.0.1}$$

From equation (2.2.36),since given $n < m$,

$$Rank(\mathbf{A}) \leq n \tag{3.0.2}$$
$$Rank(\mathbf{B}) \leq n \tag{3.0.3}$$

$\mathbf{A}^T$ will be of order $n \times m$

From equation (2.2.36),

$$Rank(\mathbf{A}^T) \leq min(n, m) \tag{3.0.4}$$
$$\implies Rank(\mathbf{A}^T) = Rank(\mathbf{A}) \tag{3.0.5}$$

The maximum possible rank of $\mathbf{A}$ and $\mathbf{B}$ is given by

$$Rank(\mathbf{A}) = n \tag{3.0.6}$$
$$Rank(\mathbf{B}) = n \tag{3.0.7}$$

$\mathbf{AB}$ will be of order $m \times m$

Consider a vector $\mathbf{v}$,

$$\mathbf{v} \in col((\mathbf{AB})) \tag{3.0.8}$$
$$\mathbf{v} = (\mathbf{AB})\mathbf{x} \tag{3.0.9}$$
$$\mathbf{v} = \mathbf{A}(\mathbf{Bx}) \tag{3.0.10}$$
$$\mathbf{v} \in col((\mathbf{A})) \tag{3.0.11}$$

From equations (3.0.8) and (3.0.11),we could say that for every vector in the column space of $\mathbf{AB}$ the same vector will be there in column space of $\mathbf{A}$ aswell.

$$Rank(\mathbf{AB}) \leq Rank(\mathbf{A}) \tag{3.0.12}$$

From equation (3.0.5),

$$Rank(\mathbf{AB}) = Rank((\mathbf{AB})^T) \tag{3.0.13}$$
$$\implies Rank((\mathbf{AB})^T) = Rank(\mathbf{B}^T\mathbf{A}^T) \tag{3.0.14}$$

From equation (3.0.12),

$$\implies Rank(\mathbf{B}^T\mathbf{A}^T) \leq Rank(\mathbf{B}^T) \tag{3.0.15}$$
$$\implies Rank(\mathbf{B}^T\mathbf{A}^T) \leq Rank(\mathbf{B}) \tag{3.0.16}$$
$$\implies Rank(\mathbf{AB}) \leq Rank(\mathbf{B}) \tag{3.0.17}$$

From the equations (3.0.12) and (3.0.17),

$$Rank(\mathbf{AB}) \leq min(Rank(\mathbf{A}), Rank(\mathbf{B})) \tag{3.0.18}$$
$$Rank(\mathbf{AB}) \leq n \tag{3.0.19}$$

The maximum possible rank of $\mathbf{AB}$ of order $m \times m$ is given by

$$Rank(\mathbf{AB}) = n < m \tag{3.0.20}$$

From (3.0.20) we could say that $\mathbf{AB}$ does not have a full rank and if the matrix does not have a full rank then it is not invertible.Hence, $\mathbf{AB}$ is not invertible.

Hence Proved