Assignment 15

Venkatesh E
AI20MTECH14005

***Abstract*—This document explains the concept of linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^2$**

Download all latex-tikz codes from

https://github.com/venkateshelangovan/IIT−Hyderabad−Assignments/tree/master/Assignment15_Matrix_Theory

1 Problem

Is there a linear transformation $\mathbf{T}$ from $\mathbb{R}^3$ into $\mathbb{R}^2$ such that,

$$\mathbf{T}\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \tag{1.0.1}$$

$$\mathbf{T}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} \tag{1.0.2}$$

2 Linear Transformation

A linear transformation is a function $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^m$ which satisfies:

1. $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$\mathbf{T}\left(\mathbf{x} + \mathbf{y}\right) = \mathbf{T}\left(\mathbf{x}\right) + \mathbf{T}\left(\mathbf{y}\right) \tag{2.0.1}$$

2. $\forall \mathbf{x} \in \mathbb{R}^n$ and c $\in \mathbb{R}$,

$$\mathbf{T}\left(c\mathbf{x}\right) = c\mathbf{T}\left(\mathbf{x}\right) \tag{2.0.2}$$

2.1 Matrix of the Linear Transformation

Let, $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and $\mathbf{x} \in \mathbb{R}^n$ is given by ,

$$\mathbf{x} = \begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix} \tag{2.1.1}$$

$$= x_1\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} + x_2\begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix} + \cdots + x_n\begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix} \tag{2.1.2}$$

Let $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}$ be the standard basis of $\mathbb{R}^n$ and the equation (2.1.2) can be rewritten as,

$$\mathbf{x} = \sum_{i=1}^{n} x_i\mathbf{e_i} \tag{2.1.3}$$

$$\mathbf{T}\left(\mathbf{x}\right) = \sum_{i=1}^{n} x_i\mathbf{T}\left(\mathbf{e_i}\right) \tag{2.1.4}$$

$$= \left(\mathbf{T}\left(\mathbf{e_1}\right) \quad \mathbf{T}\left(\mathbf{e_2}\right) \quad \ldots \quad \mathbf{T}\left(\mathbf{e_n}\right)\right)\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix} \tag{2.1.5}$$

$$\mathbf{T}\left(\mathbf{x}\right) = \mathbf{A}\mathbf{x} \tag{2.1.6}$$

Where,

$$\mathbf{A} = \left(\mathbf{T}\left(\mathbf{e_1}\right) \quad \mathbf{T}\left(\mathbf{e_n}\right) \quad \ldots \quad \mathbf{T}\left(\mathbf{e_n}\right)\right) \tag{2.1.7}$$

If $\mathbf{T}$ is any linear transformation which maps $\mathbb{R}^n \to \mathbb{R}^m$ there is always an $m \times n$ matrix $\mathbf{A}$ with the property that

$$\mathbf{T}\left(\mathbf{x}\right) = \mathbf{A}\mathbf{x}$$

where , $\mathbf{x} \in \mathbb{R}^n$

3 Solution

Let,

$$\mathbf{v} = \begin{pmatrix}1\\-1\\1\end{pmatrix} \tag{3.0.1}$$

$$\mathbf{u} = \begin{pmatrix}1\\1\\1\end{pmatrix} \tag{3.0.2}$$

Given,

$$\mathbf{T}\left(\mathbf{v}\right) = \begin{pmatrix}1\\0\end{pmatrix} \tag{3.0.3}$$

$$\mathbf{T}\left(\mathbf{u}\right) = \begin{pmatrix}0\\1\end{pmatrix} \tag{3.0.4}$$

Let the standard basis vectors is denoted as,

$$\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{3.0.5}$$

$$\mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{3.0.6}$$

$$\mathbf{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{3.0.7}$$

Let, $\mathbf{T} : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation. Then the function $\mathbf{T}$ is just matrix-vector multiplication $\mathbf{T}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ for some matrix $\mathbf{A}$ as shown in equation (2.1.6)

Matrix $\mathbf{A}$ of order 2×3 is given by,

$$\mathbf{A} = \begin{pmatrix} \mathbf{T}\left(\mathbf{e_1}\right) & \mathbf{T}\left(\mathbf{e_2}\right) & \mathbf{T}\left(\mathbf{e_3}\right) \end{pmatrix} \tag{3.0.8}$$

Consider the vector $\mathbf{b} \in \mathbb{R}^3$ which is the linear combinations of the vectors $\mathbf{v}$ and $\mathbf{u}$.

For $x_1, x_2 \in \mathbb{R}$,

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{3.0.9}$$

$$\mathbf{T}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = x_1 \mathbf{T}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + x_2 \mathbf{T}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{3.0.10}$$

To find x_1, x_2, we solve the linear system,$\mathbf{M}\mathbf{x} = \mathbf{b}$ where $\mathbf{M}$ is the 3×2 matrix obtained by stacking the given vectors $\mathbf{v}$ and $\mathbf{u}$ as columns

$$\mathbf{M} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix} \tag{3.0.11}$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{3.0.12}$$

The augumented matrix of the equation (3.0.12) is given by ,

$$\left(\begin{array}{cc|c} 1 & 1 & b_1 \\ -1 & 1 & b_2 \\ 1 & 1 & b_3 \end{array}\right) \tag{3.0.13}$$

By row reducing the above equation (3.0.13),

$$\left(\begin{array}{cc|c} 1 & 1 & b_1 \\ -1 & 1 & b_2 \\ 1 & 1 & b_3 \end{array}\right) \xleftrightarrow{R_2 = R_2 + R_1} \left(\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 2 & b_2 + b_1 \\ 1 & 1 & b_3 \end{array}\right) \tag{3.0.14}$$

$$\left(\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 2 & b_2 + b_1 \\ 1 & 1 & b_3 \end{array}\right) \xleftrightarrow{R_3 = R_3 - R_1} \left(\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 2 & b_2 + b_1 \\ 0 & 0 & b_3 - b_1 \end{array}\right) \tag{3.0.15}$$

$$\left(\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 2 & b_2 + b_1 \\ 0 & 0 & b_3 - b_1 \end{array}\right) \xleftrightarrow{R_2 = \frac{R_2}{2}} \left(\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 1 & \frac{b_2 + b_1}{2} \\ 0 & 0 & b_3 - b_1 \end{array}\right) \tag{3.0.16}$$

$$\left(\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 1 & \frac{b_2 + b_1}{2} \\ 0 & 0 & b_3 - b_1 \end{array}\right) \xleftrightarrow{R_1 = R_1 - R_2} \left(\begin{array}{cc|c} 1 & 0 & \frac{b_1 - b_2}{2} \\ 0 & 1 & \frac{b_2 + b_1}{2} \\ 0 & 0 & b_3 - b_1 \end{array}\right) \tag{3.0.17}$$

Now equation (3.0.12) can be written as,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{b_1 - b_2}{2} \\ \frac{b_2 + b_1}{2} \\ b_3 - b_1 \end{pmatrix} \tag{3.0.18}$$

Solving the above equation we get ,

$$x_1 = \frac{b_1 - b_2}{2} \tag{3.0.19}$$

$$x_2 = \frac{b_1 + b_2}{2} \tag{3.0.20}$$

Substituting the above equations (3.0.19),(3.0.20) in equation (3.0.10), we get,

$$\mathbf{T}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \left(\frac{b_1 - b_2}{2}\right) \mathbf{T}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \left(\frac{b_1 + b_2}{2}\right) \mathbf{T}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{3.0.21}$$

Substituting the equations (1.0.1) and (1.0.2) in equation (3.0.21) we get,

$$\mathbf{T}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \left(\frac{b_1 - b_2}{2}\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left(\frac{b_1 + b_2}{2}\right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.0.22}$$

$$\mathbf{T}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} \frac{b_1 - b_2}{2} \\ \frac{b_1 + b_2}{2} \end{pmatrix} \tag{3.0.23}$$

Using the above equation (3.0.23) we compute,

$$\mathbf{T}\left(\mathbf{e_1}\right) = \mathbf{T}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix} \tag{3.0.24}$$

$$\mathbf{T}\left(\mathbf{e_2}\right) = \mathbf{T}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}\frac{-1}{2}\\\frac{1}{2}\end{pmatrix} \tag{3.0.25}$$

$$\mathbf{T}\left(\mathbf{e_3}\right) = \mathbf{T}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \tag{3.0.26}$$

Substituting the equations (3.0.24),(3.0.25) and (3.0.26) in equation (3.0.8) we get,

$$\mathbf{A} = \begin{pmatrix}\frac{1}{2} & \frac{-1}{2} & 0\\\frac{1}{2} & \frac{1}{2} & 0\end{pmatrix} \tag{3.0.27}$$

$$\implies \mathbf{A} = \frac{1}{2}\begin{pmatrix}1 & -1 & 0\\1 & 1 & 0\end{pmatrix} \tag{3.0.28}$$

Therefore from the above matrix $\mathbf{A}$ we can say that there is a linear transformation $\mathbf{T}$ from $\mathbb{R}^3$ into $\mathbb{R}^2$ which satisfies the given conditions (1.0.1) and (1.0.2).