Assignment 7

Venkatesh E
AI20MTECH14005

***Abstract*—This document explains the concept of affline transformation of equations when the origin is moved to the point**

Download all latex-tikz codes from

https://github.com/venkateshelangovan/IIT−Hyderabad−Assignments/tree/master/Assignment7_Matrix_Theory

1 Problem

What does the equation

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\mathbf{x} - \begin{pmatrix}4 & 6\end{pmatrix}\mathbf{x} - 6 = 0 \tag{1.0.1}$$

become when the origin is moved to the point $\begin{pmatrix}2\\ -3\end{pmatrix}$?

2 Solution

Given,

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\mathbf{x} + 2\begin{pmatrix}-2 & -3\end{pmatrix}\mathbf{x} - 6 = 0 \tag{2.0.1}$$

where

$$\mathbf{V} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{2.0.2}$$

$$\mathbf{u} = \begin{pmatrix}-2\\ -3\end{pmatrix} \tag{2.0.3}$$

Origin which is moved to the point is given by

$$\mathbf{c} = \begin{pmatrix}2\\ -3\end{pmatrix} \tag{2.0.4}$$

The above equation (2.0.1) can be modified as

$$(\mathbf{x}+\mathbf{c})^T\mathbf{V}(\mathbf{x}+\mathbf{c}) + 2\mathbf{u}^T(\mathbf{x}+\mathbf{c}) - 6 = 0 \tag{2.0.5}$$

From equation (2.0.5) consider,

$$\implies (\mathbf{x}+\mathbf{c})^T\mathbf{V}(\mathbf{x}+\mathbf{c}) \tag{2.0.6}$$

$$\implies \mathbf{x}^T\mathbf{V}\mathbf{x} + \mathbf{c}^T\mathbf{V}\mathbf{x} + \mathbf{x}^T\mathbf{V}\mathbf{c} + \mathbf{c}^T\mathbf{V}\mathbf{c} \tag{2.0.7}$$

we know that

$$\mathbf{x}^T\mathbf{V}\mathbf{c} = \mathbf{c}^T\mathbf{V}\mathbf{x} \tag{2.0.8}$$

Substituting equation (2.0.8) in equation (2.0.7)

$$\implies \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{c}^T\mathbf{V}\mathbf{x} + \mathbf{c}^T\mathbf{V}\mathbf{c} \tag{2.0.9}$$

$$\mathbf{c}^T\mathbf{V}\mathbf{x} = \begin{pmatrix}2 & -3\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\mathbf{x} = \begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} \tag{2.0.10}$$

$$\mathbf{c}^T\mathbf{V}\mathbf{c} = \begin{pmatrix}2 & -3\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}2\\ -3\end{pmatrix} = -5 \tag{2.0.11}$$

Substituting the equations (2.0.10), (2.0.11) in equation (2.0.9) we get

$$\implies \mathbf{x}^T\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\mathbf{x} + 2\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} - 5 \tag{2.0.12}$$

From equation (2.0.5) consider,

$$\implies 2\mathbf{u}^T(\mathbf{x}+\mathbf{c}) \tag{2.0.13}$$

$$\implies 2\begin{pmatrix}-2 & -3\end{pmatrix}\mathbf{x} + 2\begin{pmatrix}-2 & -3\end{pmatrix}\begin{pmatrix}2\\ -3\end{pmatrix} \tag{2.0.14}$$

$$\implies -2\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} + 10 \tag{2.0.15}$$

Substituting equations (2.0.12), (2.0.15) in equation (2.0.5) we get

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\mathbf{x} + 2\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} - 2\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} + 10 - 11 = 0 \tag{2.0.16}$$

$$\implies \mathbf{x}^T\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\mathbf{x} - 1 = 0 \tag{2.0.17}$$

Given equation (2.0.1) is modified to equation (2.0.17) when the origin is moved to the point $\begin{pmatrix}2\\ -3\end{pmatrix}$ which is verified through the plot 1

From equations (2.0.17) ,(2.0.1) $\mathbf{V}$ doesn't change

$$\det(\mathbf{V}) = -1 \tag{2.0.18}$$

Since $\det(\mathbf{V}) < 0$ the given equation represents the hyperbola

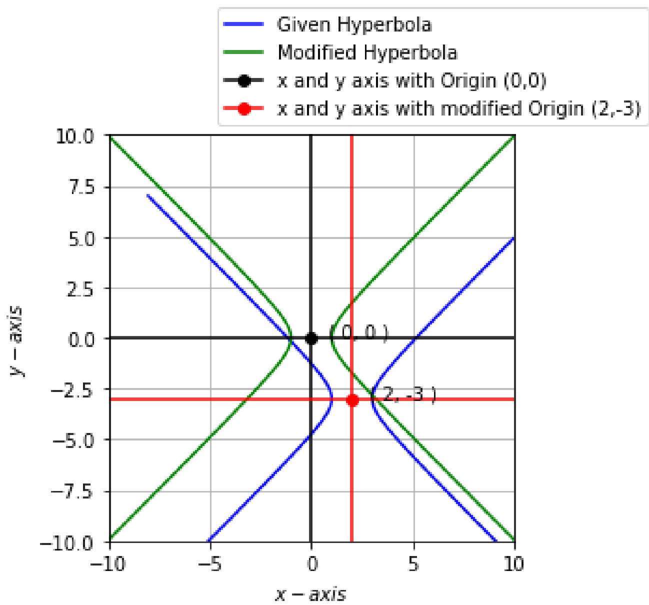

Fig. 1: Hyperbola when origin is shifted

The plot 1 verifies the given hyperbola

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 4 & 6 \end{pmatrix} \mathbf{x} - 6 = 0 \tag{2.0.19}$$

The above hyperbola was plotted with respect to origin with

centre,

$$\mathbf{c_1} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \tag{2.0.20}$$

vertices,

$$\mathbf{v_{11}} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} \tag{2.0.21}$$

$$\mathbf{v_{12}} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \tag{2.0.22}$$

The given hyperbola equation (2.0.1) which is modified to the hyperbola

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} - 1 = 0 \tag{2.0.23}$$

centre,

$$\mathbf{c_2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.24}$$

vertices

$$\mathbf{v_{21}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.25}$$

$$\mathbf{v_{22}} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{2.0.26}$$

when the origin is moved to the point

$$\mathbf{o} \begin{pmatrix} 2 \\ -3 \end{pmatrix} \tag{2.0.27}$$