Assignment 10

Venkatesh E
AI20MTECH14005

***Abstract*—This document explains the concept of finding the closest points on the lines using SVD provided the given lines are not intersecting each other**

Download all latex-tikz codes from

https://github.com/venkateshelangovan/IIT−Hyderabad−Assignments/tree/master/Assignment10_Matrix_Theory

1 Problem

Check whether the given line equations intersect. If they didn't intersect find the closest points on the lines

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \tag{1.0.1}$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{1.0.2}$$

2 Solution

Given

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \tag{2.0.1}$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{2.0.2}$$

The above equations (2.0.1), (2.0.4) are in the form

$$L_1 : \quad \mathbf{x} = \mathbf{a_1} + \lambda_1 \mathbf{b_1} \tag{2.0.3}$$

$$L_2 : \quad \mathbf{x} = \mathbf{a_2} + \lambda_2 \mathbf{b_2} \tag{2.0.4}$$

Here ,

$$\mathbf{a_1} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} \qquad \mathbf{a_2} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} \tag{2.0.5}$$

$$\mathbf{b_1} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \qquad \mathbf{b_2} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{2.0.6}$$

Now let us assume the lines L_1 and L_2 are intersecting at a point. Therefore ,

$$\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{2.0.7}$$

$$\lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} \tag{2.0.8}$$

$$\begin{pmatrix} 3 & -1 \\ 2 & -2 \\ 6 & -2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} \tag{2.0.9}$$

The augumented matrix of (2.0.9) is given by

$$\left(\begin{array}{cc|c} 3 & -1 & 5 \\ 2 & -2 & -1 \\ 6 & -2 & -1 \end{array}\right) \tag{2.0.10}$$

$$\left(\begin{array}{cc|c} 3 & -1 & 5 \\ 2 & -2 & -1 \\ 6 & -2 & -1 \end{array}\right) \xleftrightarrow{R_2 = R_2 - \frac{2}{3}R_1} \left(\begin{array}{cc|c} 3 & -1 & 5 \\ 0 & -\frac{4}{3} & -\frac{13}{3} \\ 6 & -2 & -1 \end{array}\right) \tag{2.0.11}$$

$$\left(\begin{array}{cc|c} 3 & -1 & 5 \\ 0 & -\frac{4}{3} & -\frac{13}{3} \\ 6 & -2 & -1 \end{array}\right) \xleftrightarrow{R_3 = R_3 - 2R_1} \left(\begin{array}{cc|c} 3 & -1 & 5 \\ 0 & -\frac{4}{3} & -\frac{13}{3} \\ 0 & 0 & -11 \end{array}\right) \tag{2.0.12}$$

Since the rank of augmented matrix will be 3. We can say that lines do not intersect.Hence our assumptions is wrong

Equation (2.0.9) can be expressed as

$$\mathbf{Mx} = \mathbf{b} \tag{2.0.13}$$

By singular value decomposition $\mathbf{M}$ can be expressed as

$$\mathbf{M} = \mathbf{USV}^T \tag{2.0.14}$$

Where the columns of $\mathbf{V}$ are the eigenvectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigenvectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 49 & -19 \\ -19 & 9 \end{pmatrix} \tag{2.0.15}$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 10 & 8 & 20 \\ 8 & 8 & 16 \\ 20 & 16 & 40 \end{pmatrix} \tag{2.0.16}$$

2.1 To get **V** *and* **S**

The characteristic equation of $\mathbf{M}^T\mathbf{M}$ is obtained by evaluating the determinant

$$\begin{vmatrix} 49-\lambda & -19 \\ -19 & 9-\lambda \end{vmatrix} = 0 \tag{2.1.1}$$

$$\implies \lambda^2 - 58\lambda + 80 = 0 \tag{2.1.2}$$

The eigenvalues are the roots of equation 2.1.2 is given by

$$\lambda_{11} = 29 + \sqrt{761} \tag{2.1.3}$$

$$\lambda_{12} = 29 - \sqrt{761} \tag{2.1.4}$$

The eigen vectors comes out to be ,

$$\mathbf{u_{11}} = \begin{pmatrix} \frac{-20-\sqrt{761}}{19} \\ 1 \end{pmatrix}, \mathbf{u_{12}} = \begin{pmatrix} \frac{-20+\sqrt{761}}{19} \\ 1 \end{pmatrix} \tag{2.1.5}$$

Normalising the eigen vectors,

$$l_{11} = \sqrt{\left(\frac{-20-\sqrt{761}}{19}\right)^2 + 1^2} \tag{2.1.6}$$

$$\implies l_{11} = \frac{\sqrt{1522+40\sqrt{761}}}{19} \tag{2.1.7}$$

$$\mathbf{u_{11}} = \begin{pmatrix} \frac{-20-\sqrt{761}}{\sqrt{1522+40\sqrt{761}}} \\ \frac{19}{\sqrt{1522+40\sqrt{761}}} \end{pmatrix} \tag{2.1.8}$$

$$l_{12} = \sqrt{\left(\frac{-20+\sqrt{761}}{19}\right)^2 + 1^2} \tag{2.1.9}$$

$$\implies l_{12} = \frac{\sqrt{1522-40\sqrt{761}}}{19} \tag{2.1.10}$$

$$\mathbf{u_{12}} = \begin{pmatrix} \frac{-20+\sqrt{761}}{\sqrt{1522-40\sqrt{761}}} \\ \frac{19}{\sqrt{1522-40\sqrt{761}}} \end{pmatrix} \tag{2.1.11}$$

$$\mathbf{V} = \begin{pmatrix} \frac{-20-\sqrt{761}}{\sqrt{1522+40\sqrt{761}}} & \frac{-20+\sqrt{761}}{\sqrt{1522-40\sqrt{761}}} \\ \frac{19}{\sqrt{1522+40\sqrt{761}}} & \frac{19}{\sqrt{1522-40\sqrt{761}}} \end{pmatrix} \tag{2.1.12}$$

S is given by

$$\mathbf{S} = \begin{pmatrix} \sqrt{29+\sqrt{761}} & 0 \\ 0 & \sqrt{29-\sqrt{761}} \\ 0 & 0 \end{pmatrix} \tag{2.1.13}$$

2.2 To get **U**

The characteristic equation of $\mathbf{M}\mathbf{M}^T$ is obtained by evaluating the determinant

$$\begin{vmatrix} 10-\lambda & 8 & 20 \\ 8 & 8-\lambda & 16 \\ 20 & 16 & 40-\lambda \end{vmatrix} = 0 \tag{2.2.1}$$

$$\implies \lambda^3 - 58\lambda^2 + 80\lambda = 0 \tag{2.2.2}$$

The eigenvalues are the roots of equation 2.2.2 is given by

$$\lambda_{21} = 29 + \sqrt{761} \tag{2.2.3}$$

$$\lambda_{22} = 29 - \sqrt{761} \tag{2.2.4}$$

$$\lambda_{23} = 0 \tag{2.2.5}$$

The eigen vectors comes out to be ,

$$\mathbf{u_{21}} = \begin{pmatrix} \frac{-1}{2} \\ \frac{-\sqrt{761}+21}{16} \\ -1 \end{pmatrix}, \mathbf{u_{22}} = \begin{pmatrix} \frac{1}{2} \\ \frac{-\sqrt{761}-21}{16} \\ 1 \end{pmatrix}, \mathbf{u_{23}} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \tag{2.2.6}$$

Normalising the eigen vectors,

$$l_{21} = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{21-\sqrt{761}}{16}\right)^2 + (-1)^2} \tag{2.2.7}$$

$$\implies l_{21} = \frac{\sqrt{1522-42\sqrt{761}}}{16} \tag{2.2.8}$$

$$\mathbf{u_{21}} = \begin{pmatrix} \frac{-8}{\sqrt{1522-42\sqrt{761}}} \\ \frac{21-\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} \\ \frac{-16}{\sqrt{1522-42\sqrt{761}}} \end{pmatrix} \tag{2.2.9}$$

$$l_{22} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{-21 - \sqrt{761}}{16}\right)^2 + 1^2} \tag{2.2.10}$$

$$\implies l_{22} = \frac{\sqrt{1522 + 42\sqrt{761}}}{16} \tag{2.2.11}$$

$$\mathbf{u_{22}} = \begin{pmatrix} \frac{8}{\sqrt{1522+42\sqrt{761}}} \\ \frac{-21-\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} \\ \frac{16}{\sqrt{1522+42\sqrt{761}}} \end{pmatrix} \tag{2.2.12}$$

$$l_{23} = \sqrt{(-2)^2 + 1^2} = \sqrt{5} \tag{2.2.13}$$

$$\mathbf{u_{23}} = \begin{pmatrix} \frac{-2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \end{pmatrix} \tag{2.2.14}$$

$$\mathbf{U} = \begin{pmatrix} \frac{-8}{\sqrt{1522-42\sqrt{761}}} & \frac{8}{\sqrt{1522+42\sqrt{761}}} & \frac{-2}{\sqrt{5}} \\ \frac{21-\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} & \frac{-21-\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} & 0 \\ \frac{-16}{\sqrt{1522-42\sqrt{761}}} & \frac{16}{\sqrt{1522+42\sqrt{761}}} & \frac{1}{\sqrt{5}} \end{pmatrix} \tag{2.2.15}$$

2.3 To get $\mathbf{x}$

From equation (2.0.14) we rewrite $\mathbf{M}$ as follows,

$$\begin{pmatrix} 3 & -1 \\ 2 & -2 \\ 6 & -2 \end{pmatrix} = \begin{pmatrix} \frac{-8}{\sqrt{1522-42\sqrt{761}}} & \frac{8}{\sqrt{1522+42\sqrt{761}}} & \frac{-2}{\sqrt{5}} \\ \frac{21-\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} & \frac{-21-\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} & 0 \\ \frac{-16}{\sqrt{1522-42\sqrt{761}}} & \frac{16}{\sqrt{1522+42\sqrt{761}}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{29 + \sqrt{761}} & 0 \\ 0 & \sqrt{29 - \sqrt{761}} \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{-20-\sqrt{761}}{\sqrt{1522+40\sqrt{761}}} & \frac{-20+\sqrt{761}}{\sqrt{1522-40\sqrt{761}}} \\ \frac{19}{\sqrt{1522+40\sqrt{761}}} & \frac{19}{\sqrt{1522-40\sqrt{761}}} \end{pmatrix}^T \tag{2.3.1}$$

By substituting the equation (2.0.14) in equation (2.0.13) we get

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{2.3.2}$$

$$\implies \mathbf{x} = \mathbf{VS_+U^Tb} \tag{2.3.3}$$

Where $\mathbf{S}_+$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{29+\sqrt{761}}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{29-\sqrt{761}}} & 0 \end{pmatrix} \tag{2.3.4}$$

From (2.3.3) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{\sqrt{761}-45}{\sqrt{1522-42\sqrt{761}}} \\ \frac{45+\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} \\ -\frac{11}{\sqrt{5}} \end{pmatrix} \tag{2.3.5}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{761\sqrt{15}-761-45\sqrt{11415}+45\sqrt{761}}{10654} \\ \frac{45\sqrt{11415}+45\sqrt{761}+761\sqrt{15}+761}{10654} \end{pmatrix} \tag{2.3.6}$$

$$\mathbf{x} = \mathbf{VS}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{11}{20} \\ \frac{21}{20} \end{pmatrix} \tag{2.3.7}$$

2.4 Verification of $\mathbf{x}$

Verifying the solution of (2.3.7) using,

$$\mathbf{M}^T\mathbf{Mx} = \mathbf{M}^T\mathbf{b} \tag{2.4.1}$$

Evaluating the R.H.S in (2.4.1) we get,

$$\mathbf{M}^T\mathbf{Mx} = \begin{pmatrix} 7 \\ -1 \end{pmatrix} \tag{2.4.2}$$

$$\implies \begin{pmatrix} 49 & -19 \\ -19 & 9 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 7 \\ -1 \end{pmatrix} \tag{2.4.3}$$

Solving the augmented matrix of (2.4.3) we get,

$$\begin{pmatrix} 49 & -19 & 7 \\ -19 & 9 & -1 \end{pmatrix} \xleftrightarrow{R_2 = R_2 + \frac{19}{49}R_1} \begin{pmatrix} 49 & -19 & 7 \\ 0 & \frac{80}{49} & \frac{12}{7} \end{pmatrix} \tag{2.4.4}$$

$$\xleftrightarrow{R_1 = \frac{1}{49}R_1} \begin{pmatrix} 1 & \frac{-19}{49} & \frac{7}{49} \\ 0 & \frac{80}{49} & \frac{12}{7} \end{pmatrix} \tag{2.4.5}$$

$$\xleftrightarrow{R_2 = \frac{80}{49}R_2} \begin{pmatrix} 1 & \frac{-19}{49} & \frac{7}{49} \\ 0 & 1 & \frac{21}{20} \end{pmatrix} \tag{2.4.6}$$

$$\xleftrightarrow{R_1 = R_1 + \frac{19}{49}R_2} \begin{pmatrix} 1 & 0 & \frac{11}{20} \\ 0 & 1 & \frac{21}{20} \end{pmatrix} \tag{2.4.7}$$

Hence, Solution of (2.4.1) is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{11}{20} \\ \frac{21}{20} \end{pmatrix} \tag{2.4.8}$$

Comparing results of $\mathbf{x}$ from (2.3.7) and (2.4.8) we conclude that the solution is verified.