Ae4-ASM-106 - Stability and Analysis of Structures - 2023
Representative problems for lectures 1-10.

Instructions

1. Deadline: return your answers on or before Friday January 12, 2024 at 5:00 PM in BrightSpace.
2. You may consult and use directly the results in the class notes and any of the textbooks in the reference list. You can use any language/package for calculations (Python, matlab, etc). You may also use symbolic and numerical packages such as Mathematica and Maple. Additionally, you may consult general math reference books and/or software manuals if required. Clearly indicate all sources used, including software/codes.
3. Consultation with classmates is allowed, however, each student must submit his/her own work, which should reflect a thorough understanding of the assigned problems. Consultation should be limited to general questions. Do not solve entire problems with someone else (use your own judgement). Consultation with someone not enrolled in the current course is not allowed. If required, you may consult with me. You may arrange for an appointment via email.

Problems

1. Stress tensor. Suppose that at a given point in a structure the stress tensor has been computed (in a global Cartesian basis) as (components in MPa)

$$[\boldsymbol{\sigma}]_{\mathrm{e}} = \begin{bmatrix} 150 & 110 & 70 \\ 110 & 160 & 0 \\ 70 & 0 & -140 \end{bmatrix}_{\mathrm{e}}$$

(i) Compute the principal stresses and order them in descending order.

(ii) Compute the (normalized) principal stress directions.

(iii) The von Mises criterion indicates that the material remains elastic if the von Mises stress is below the yield stress, whereas the Tresca criterion indicates that the material remains elastic if the maximum shearing stress remains below half of the yield stress, i.e.,

$$\sigma_m = \sqrt{\frac{1}{2}\left((\sigma^{(1)} - \sigma^{(2)})^2 + (\sigma^{(1)} - \sigma^{(3)})^2 + (\sigma^{(2)} - \sigma^{(3)})^2\right)} \leq \sigma_y$$

$$\tau_{\max} = \frac{1}{2}\max\left\{|\sigma^{(1)} - \sigma^{(2)}|, |\sigma^{(1)} - \sigma^{(3)}|, |\sigma^{(2)} - \sigma^{(3)}|\right\} \leq \frac{1}{2}\sigma_y \ .$$

Suppose the material has a yield stress of $\sigma_y = 380$ MPa. Is the material in the elastic range or not?

2. Simple deformations. Compute the infinitesimal strain tensor ϵ_{ij} and the relative change in volume $\epsilon_{11} + \epsilon_{22} + \epsilon_{33}$ of an initially cubic volume element aligned with Cartesian axes x_1, x_2, x_3 subject to the following displacement fields. In each case, sketch the shape of the deformed cubic element.

(i) Simple extension:

$$u_1 = ax_1$$
$$u_2 = 0$$
$$u_3 = 0$$

where a is a positive constant.

(ii) Extension with lateral contraction:

$$u_1 = ax_1$$
$$u_2 = -bx_2$$
$$u_3 = -bx_3$$

where $a > b > 0$ are constants.

(iii) Uniform volumetric contraction:

$$u_1 = -ax_1$$
$$u_2 = -ax_2$$
$$u_3 = -ax_3$$

where a is a positive constant.

(iv) Simple shear in the $1, 2$-plane:

$$u_1 = 2ax_2$$
$$u_2 = 0$$
$$u_3 = 0$$

(v) Pure shear in the $1,2$-plane:

$$u_1 = ax_2$$
$$u_2 = ax_1$$
$$u_3 = 0$$

(vi) Pure shear:

$$u_1 = a(x_2 + x_3)$$
$$u_2 = a(x_1 + x_3)$$
$$u_3 = a(x_1 + x_2)$$

3. Change in length of a fiber. The displacement field of a solid is given by

$$\boldsymbol{u}(\boldsymbol{x}) = \sum_{i=1}^{3} u_i(\boldsymbol{x})\boldsymbol{e}_i = (x_1^2+20)10^{-4}\boldsymbol{e}_1+(2x_1x_2)10^{-3}\boldsymbol{e}_2+(x_3^2-x_1x_2)10^{-4}\boldsymbol{e}_3\ .$$

a) Consider two points, P and Q, that have coordinates $(2, 5, 7)$ and $(3, 8, 9)$ in the undeformed body (for accurate results, keep at least 6 significant digits).

   (i) Using the displacement field given above, determine the position vector (coordinates) of points P and Q in the *deformed* body.

   (ii) Using the coordinates of points P and Q in the *undeformed* body, compute the distance between P and Q *before* deformation.

   (iii) Using the coordinates of points P and Q in the deformed body, compute the distance between P and Q *after* deformation.

   (iv) From the results of (ii) and (iii), compute the average relative elongation of a fiber between points P and Q.

   (v) Compute the components of the infinitesimal strain tensor $\epsilon_{ij}(\boldsymbol{x})$. Evaluate the strain tensor at point P.

   (vi) Consider an *infinitesimal* fiber *at* point P aligned with the vector PQ. Compute the cartesian components n_i, $i = 1, 2, 3$ of the unit vector $\boldsymbol{n}$ in the direction of the line PQ. Use the formula developed in class to compute the *relative elongation* of the infinitesimal fiber at point P. Does your result coincide with the calculation done in (iv)? Why/why not?

b) Repeat the calculations from part (a) for the same point P but now consider a point Q' with the following coordinates in the undeformed body: $(2.13363, 5.40089, 7.26726)$. The line PQ' is parallel to the line PQ, but point Q' is closer to P. Repeat question (vi) of part (a) with this new data.

4. Anisotropic material. Six separate tests are performed on a material and the stress and corresponding strain tensors are measured, as indicated below. Previous tests showed that the following elastic coefficients are zero: $C_{14} = C_{15} = C_{16} = C_{24} = C_{25} = C_{26} = C_{34} = C_{35} = C_{36} = C_{45} = C_{46} = C_{56} = 0$ (using Voigt's notation). Based on these tests, determine the elastic coefficients C_{IJ} (provide your results in GPa). What type of symmetry has this material? (stress values indicated in the table are in MPa and the strains should be multiplied by 10^{-4}).

Tensor		test 1	test 2	test 3	test 4	test 5	test 6
σ_{11}	=	9	3	0.1	0	0	0
σ_{22}	=	32.1	0.1	1.5	0	0	0
σ_{33}	=	42.2	0.2	0.1	0	0	0
σ_{23}	=	0	0	0	0.2	0	0
σ_{13}	=	0	0	0	0	0.3	0
σ_{12}	=	0	0	0	0	0	0.4
ϵ_{11}	=	10	10	0	0	0	0
ϵ_{22}	=	2	0	10	0	0	0
ϵ_{33}	=	2	0	0	0	0	0
$2\epsilon_{23}$	=	0	0	0	2	0	0
$2\epsilon_{13}$	=	0	0	0	0	2	0
$2\epsilon_{12}$	=	0	0	0	0	0	2

5. Relations between material parameters for isotropic materials. Isotropic materials are characterized by only two material parameters. In practice, several parameters are used (Young's modulus E, Poisson's ratio ν, shear modulus μ, Lamé modulus λ and bulk modulus κ). To establish the relations between these parameters we can proceed as follows:

(i) Consider a homogeneous, isotropic, linearly elastic material subject to a state of uniaxial tension, i.e., the stress tensor is

$$[\sigma_{ij}] = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where σ is the axial stress. For this stress state, use the strain-stress relation to compute the corresponding strain tensor, i.e., compute ϵ_{ij} from

$$\epsilon_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sum_{k=1}^{3}\sigma_{kk}\right] ,$$

(ii) With the strain tensor computed in (i), use now the stress-strain relation to compute the corresponding stress tensor, i.e.,

$$\sigma_{ij} = \lambda\delta_{ij}\sum_{k=1}^{3}\epsilon_{ij} + 2\mu\epsilon_{ij} .$$

(iii) Use the fact that the stress tensor computed in (ii) corresponds to a state of unxaxial tension to express E and ν as functions of λ and μ and, conversely, to express λ and μ as functions of E and ν. Compute λ and μ for a steel with $E = 200$ GPa and $\nu = 0.3$.

(iv) For the bulk modulus, we proceed as follows: consider a general deformation with stress tensor σ_{ij} and a corresponding strain tensor ϵ_{ij}. The bulk modulus κ is defined such that

$$(\sigma_{11} + \sigma_{22} + \sigma_{33}) = 3\kappa(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) .$$

Use the stress-strain relation to express the bulk modulus κ a function of λ and μ. Repeat the procedure but this time using the strain-stress relation to obtain κ as a function of E and ν.

6. Axial deformations; isotropic material. Consider a homogeneous, linearly elastic isotropic steel with Young's modulus E and Poisson's ratio ν. A rectangular sample of the material is subject to the following tests:

(i) Simple extension. The displacement field is

$$u_1 = \epsilon x_1$$
$$u_2 = 0$$
$$u_3 = 0$$

where ϵ is the axial strain. Compute the strain tensor. Starting from the strain-stress relation

$$\epsilon_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sum_{k=1}^{3}\sigma_{kk}\right] ,$$

compute the corresponding stress tensor (alternatively, you may use the stress-strain relation and express λ and μ in terms of E and ν using the results from question 1). For $E = 200$ GPa and $\nu = 0.3$, plot the axial stress σ_{11} as a function of the axial strain ϵ_{11}.

(ii) Uniaxial tension. The stress tensor is

$$[\sigma_{ij}] = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where σ is the axial stress. Compute the corresponding strain tensor. Plot the axial stress σ_{11} as a function of the axial strain ϵ_{11}. Comment on the differences between the slopes of this curve and the one obtained in part (i). Provide (in words) a brief physical interpretation of this difference.

7. Euler-Bernoulli beam under compressive load. [Note: For this exercise, you may use results from the class notes. You should indicate the meaning of the symbols and quantities that you require for this problem and indicate in words the steps to obtain each formula.]

Consider a homogeneous linearly elastic beam subjected to an axial compressive force F applied on both ends along its centerline. Do not consider lateral deflections in this problem.

(i) Write down the total potential energy of the beam and formulate the axial loading problem as a minimization problem.

(ii) Consider Ritz' method for the axial deformation and choose a single function to approximate the axial displacement. Does this function need to be zero at the end points? Why/why not? Use Ritz's method to solve the problem (clearly indicate all the steps in your solution method).

(iii) From the potential energy, formulate the principle of virtual work for the axial loading problem.

(iv) Using the same function as for Ritz' method, use Galerkin's method to solve the problem (clearly indicate all the steps in your solution method). Compare the solutions from Galerkin and Ritz's methods.

(v) From the principle of virtual work, obtain the strong formulation for the axial loading problem. Indicate the possible boundary conditions at both ends. What type of boundary conditions were used in the formulations above?

(vi) Solve the problem again using direct integration. Is this solution the same or different than the solutions above (minimum potential energy, principle of virtual work)?

8. Simply-supported Kirchhoff-Love plate with uniform lateral load. [Note: For this exercise, you may use results from the class notes. You should indicate the meaning of the symbols and quantities that you require for this problem and indicate in words the steps to obtain each formula.]

Consider a homogeneous, isotropic linearly elastic plate of dimensions $a \times b \times h$ subjected to a uniform lateral load q_0 (i.e., pressure-like distributed force).

(i) Complete the missing steps in the class notes regarding the solution to this problem using Galerkin's method (use the same single term approximation for the lateral displacement as used for Ritz' method). Is the solution from Galerkin's method the same as from Ritz' method or not?

(ii) Consider the solution given in the lecture notes in terms of Fourier series. *Choose your own values* for the plate dimensions, material properties and loading (clearly indicate all the values used, including units). Compute the maximum lateral deflection of the plate using the single term Galerkin solution from (i) and the Fourier series solutions using 1, 2 and up to 3 terms (you can consider more if you want). What can you say about the error in the computed displacement in terms of the number of terms used? What about Galerkin's solution?