Problem Setup

Continuous time CWH Equations:

$$\ddot{x} = 3\omega^2 x + 2\omega\dot{y} + u_1$$
$$\ddot{y} = -2\omega\dot{x} + u_2,$$

where $\omega := \sqrt{\mu/R_0^3}$. For a spacecraft in LEO, I used $R_0 = R_e + 415$ km as the orbital radius, where $R_e = 6378.1$ km is the radius of the Earth. To write this system in state space form, let $x := [x, \dot{x}, y, \dot{y}]^\intercal \in \mathbb{R}^4$, so that we have the LTI system $\dot{x} = Ax + Bu$, where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 2\omega & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

To discretize the system, let $N = 20$ time steps with a time step $\Delta t = 0.5$, so that $T = N\Delta t = 10$. Assume a zero-order holder (ZOH) on the control and first order approximation, i.e. $A_d = I_4 + \Delta t A$ and $B_d = \Delta t B$. Adding noise to the system gives $x_{k+1} = A_d x_k + B_d u_k + G w_k$, where

$$A_d = \begin{bmatrix} 1 & \Delta t & 0 & 0 \\ 3\Delta t\omega^2 & 1 & 2\Delta t\omega & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -2\Delta t\omega & 0 & 1 \end{bmatrix}, \qquad B_d = \begin{bmatrix} 0 & 0 \\ \Delta t & 0 \\ 0 & 0 \\ 0 & \Delta t \end{bmatrix}, \qquad G = \mathrm{diag}(10^{-4}, 10^{-4}, 5\cdot 10^{-8}, 5\cdot 10^{-8})$$

Boundary Conditions:

$$x_0 = \begin{bmatrix} -1.5 \\ -1.5 \\ 0.1 \\ 0.1 \end{bmatrix}, \ x_f = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \Sigma_0 = 10^{-2} \cdot \text{diag}(0.1, 0.1, 0.01, 0.01), \ \Sigma_f = 0.5\Sigma_0$$

The chance constraints are formulated as $\mathcal{X} = \bigcap_{j=1}^{2}\{x : \alpha_j^\intercal x \leq \beta_j\}$, where

$$\alpha_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \ \alpha_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ \beta_1 = \beta_2 = 0.5$$

This corresponds to a constraint space of a triangle, given by the intersection of $x - y \leq 0.5$ and $x + y \leq 0.5$. Lastly, the cost is

$$J = \sum_{k=0}^{N-1} x_k^\intercal Q x_k + u_k^\intercal R u_k,$$

where

$$Q = \text{diag}(10, 10, 1, 1), \ R = \text{diag}(10^3, 10^3)$$

This is the end [1].

References

[1] M. Ono and B. C. Williams, "Iterative risk allocation: A new approach to robust model predictive control with a joint chance constraint," in *47th IEEE Conference on Decision and Control*, Cancun, Mexico, 2008, pp. 3427–3432.