2D Barnett-Lothe integral formalism

- The Green operator is obtained as follows from the Green's function,

$$4\Gamma_{ijkl}(r,\theta) := G^{(2)}_{ik,jl}(r,\theta) + G^{(2)}_{il,jk}(r,\theta) + G^{(2)}_{jk,il}(r,\theta) + G^{(2)}_{jl,ik}(r,\theta)$$

- Irrespectively of the material symmetry, 2D Green's functions are a by-product of the Barnett-Lothe (1973) integral formalism. We have

$$2\mathbf{G}(r,\theta) = -\frac{1}{\pi}\ln(r)\mathbf{H}(\pi) - \mathbf{S}(\theta)\cdot\mathbf{H}(\pi) - \mathbf{H}(\theta)\cdot\mathbf{S}^T(\pi)$$

where $\mathbf{S}(\theta) = \frac{1}{\pi}\int_0^\theta \mathbf{N}^1(\psi)\mathrm{d}\psi$ and $\mathbf{H}(\theta) = \frac{1}{\pi}\int_0^\theta \mathbf{N}^2(\psi)\mathrm{d}\psi$ are incomplete Barnett-Lothe integrals with integrands readily computable for every symmetry.

- To evaluate Γ_{ijkl}, we only need those integrands and the complete integrals $\mathbf{S}(\pi)$ and $\mathbf{H}(\pi)$, which we evaluate numerically.
- We derive the following recurrence relations:

$$2\pi G^{(n)}_{ij,k_1\ldots k_n}(r,\theta) = (-r)^{-n} h^n_{ijk_1\ldots k_n}(\theta)$$

$$h^n_{ijk_1\ldots k_n}(\theta) = (n-1)h^{n-1}_{ijk_1\ldots k_{n-1}}(\theta)n_{k_n}(\theta) - \partial_\theta[h^{n-1}_{ijk_1\ldots k_{n-1}}(\theta)]m_{k_n}(\theta) \quad \text{for } n \geq 2$$

$$\partial^k_\theta[h^n_{ijk_1\ldots k_n}(\theta)] = \sum_{s=0}^{k}\binom{k}{s}\left\{(n-1)\partial^{k-s}_\theta[h^{n-1}_{ijk_1\ldots k_{n-1}}(\theta)]\partial^s_\theta[n_{k_n}(\theta)] - \partial^{k-s+1}_\theta[h^{n-1}_{ijk_1\ldots k_{n-1}}(\theta)]\partial^{s+1}_\theta[n_{k_n}(\theta)]\right\}$$

$$h^1_{ijk_1}(\theta) = H_{ij}n_{k_1}(\theta) + [N^1_{is}(\theta)H_{sj} + N^2_{is}(\theta)S_{js}]m_{k_1}(\theta)$$

$$\partial^k_\theta[h^1_{ijk_1}(\theta)] = H_{ij}\partial^k_\theta[n_{k_1}(\theta)] + \sum_{s=0}^{k}\binom{k}{s}\left\{H_{lj}\partial^{k-s}_\theta[N^1_{il}(\theta)] + S_{jl}\partial^{k-s}_\theta[N^2_{il}(\theta)]\right\}\partial^s_\theta[m_{k_1}(\theta)]$$

Requires evaluation of $\partial^k_\theta[N^1_{il}(\theta)]$ and $\partial^k_\theta[N^2_{il}(\theta)]$ for $k = 0,\ldots,n-1$

2D Anisotropy

- Polar representation of 2D anisotropic stiffnesses, see Vannucci (2016)

$$
\begin{aligned}
L_{1111} &= T_0 + 2T_1 + R_0\cos(4\Phi_0) + 4R_1\cos(2\Phi_1)\\
L_{1112} &= R_0\sin(4\Phi_0) + 2R_1\sin(2\Phi_1)\\
L_{1122} &= -T_0 + 2T_1 - R_0\cos(4\Phi_0)\\
L_{1212} &= T_0 - R_0\cos(4\Phi_0)\\
L_{2212} &= -R_0\sin(4\Phi_0) + 2R_1\sin(2\Phi_1)\\
L_{2222} &= T_0 + 2T_1 + R_0\cos(4\Phi_0) - 4R_1\cos(2\Phi_1)
\end{aligned}
$$

$T_0,\ T_1$: Isotropic polar invariants

$R_0,\ R_1,\ \Phi_0-\Phi_1$: Anisotropic polar invariants

Substitute Φ_j by $\Phi_j-\theta$ for counter clockwise positive passive rotation

Conditions for positive strain energy

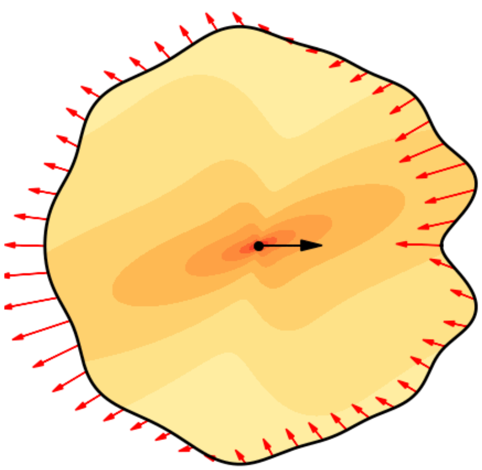
$$
\begin{aligned}
T_0 - R_0 &> 0,\\
T_1(T_0^2-R_0^2) - 2R_1^2\{T_0 - R_0\cos[4(\Phi_0-\Phi_1)]\} &> 0,\\
R_0 &\geq 0,\\
R_1 &\geq 0.
\end{aligned}
$$

$$\boxed{R_0R_1^2\sin[4(\Phi_0-\Phi_1)] \neq 0}$$

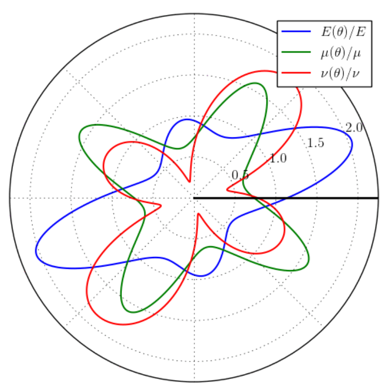
$$\boxed{R_0R_1^2\sin[4(\Phi_0-\Phi_1)] = 0 \implies \text{Symmetry}}$$

Validation

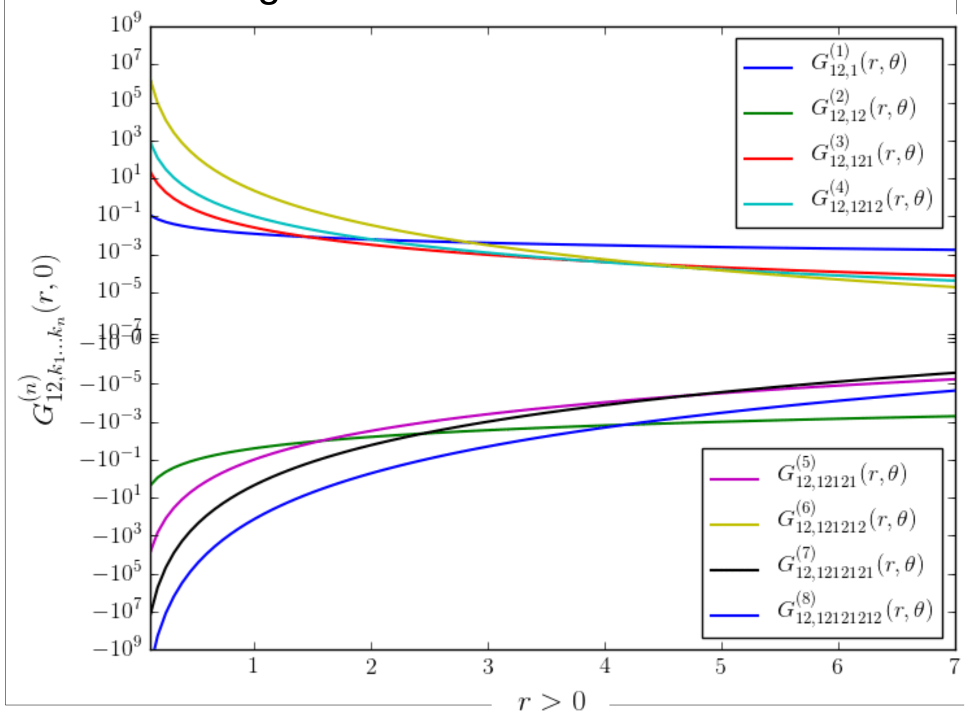
Equilibrated traction fields on random curves



Polar diagram of generalized moduli

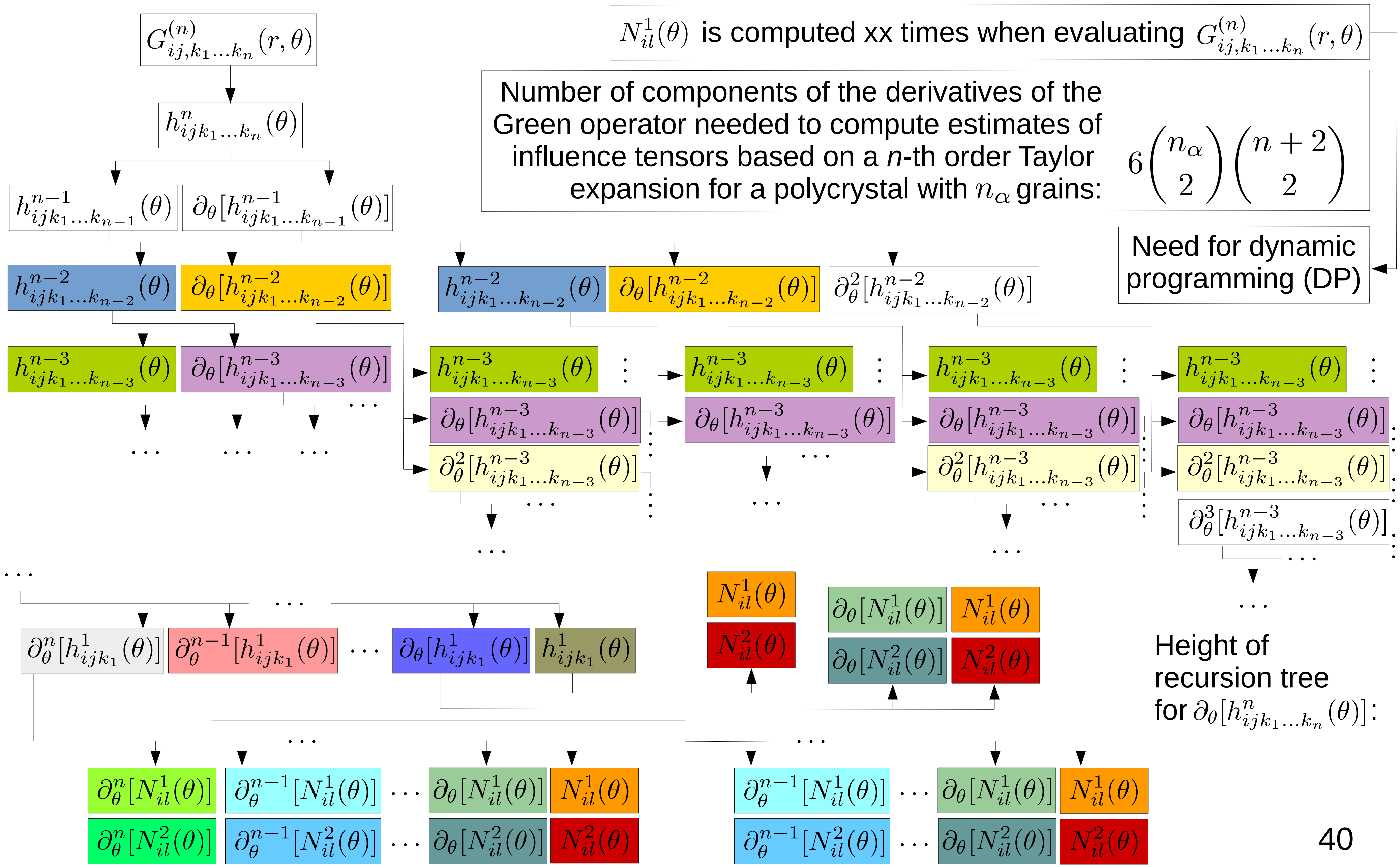


Computed components of some gradients of the Green's function



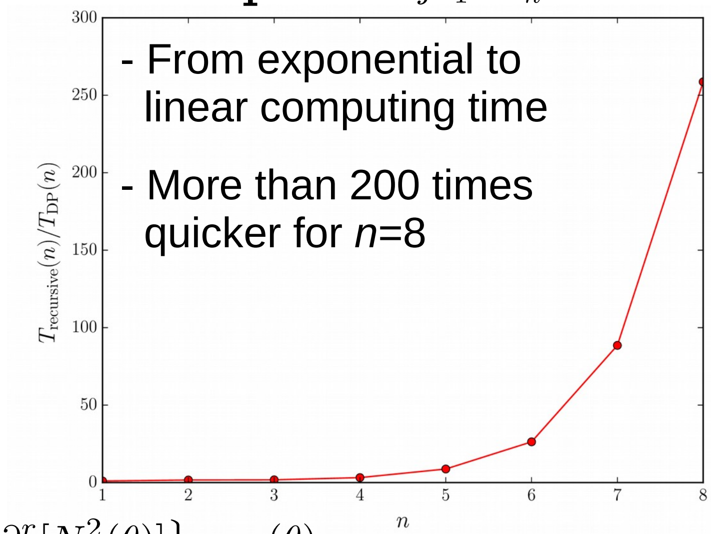
Drawback of a simple recursive implementation

- Computing the n-th derivative of an anisotropic Green's function at a location (r, θ) leads up to the following recurrence tree:



A bottom-up DP algorithm

- We derive the following bottom-up DP algorithm to compute $h^n_{ijk_1\ldots k_n}(\theta)$:



$$
\begin{aligned}
&\text{def } h^n_{ijk_1\ldots k_n}(\theta):\\
&\quad \texttt{d0hk} := \text{zeros}(n)\\
&\quad \text{for } k \in [1,n]:\\
&\qquad \text{for } rr \in [0,n-k]:\\
&\qquad\quad r = n-k-rr\\
&\qquad\quad \text{for } s \in [0,r]:\\
&\qquad\qquad \text{if } (s==0):\\
&\qquad\qquad\quad \text{if } (k==1):\\
&\qquad\qquad\qquad \texttt{d0hk}[r+k-1] = H_{ij}\partial^r_\theta[n_{k_1}(\theta)] + \left\{H_{lj}\partial^r_\theta[N^1_{il}(\theta)] + S_{jl}\partial^r_\theta[N^2_{il}(\theta)]\right\} m_{k_1}(\theta)\\
&\qquad\qquad\quad \text{else}:\\
&\qquad\qquad\qquad \texttt{d0hk}[r+k-1] = (k-1)\texttt{d0hk}[r+k-2]n_{k_k}(\theta) - \texttt{d0hk}[r+k-1]\partial^1_\theta[n_{k_k}(\theta)]\\
&\qquad\qquad \text{else}:\\
&\qquad\qquad\quad \text{if } (k==1):\\
&\qquad\qquad\qquad \texttt{d0hk}[r+k-1] += \binom{r}{s}\left\{H_{lj}\partial^{r-s}_\theta[N^1_{il}(\theta)] + S_{jl}\partial^{r-s}_\theta[N^2_{il}(\theta)]\right\}\partial^s_\theta[m_{k_1}(\theta)]\\
&\qquad\qquad\quad \text{else}:\\
&\qquad\qquad\qquad \texttt{d0hk}[r+k-1] += \binom{r}{s}\left\{(k-1)\texttt{d0hk}[r-s+k-2]\partial^s_\theta[n_{k_k}(\theta)] - \texttt{d0hk}[r-s+k-1]\partial^{s+1}_\theta[n_{k_k}(\theta)]\right\}\\
&\qquad \texttt{\#At this stage, } r \in [0,n-k] \implies \texttt{d0hk}[r+k-1] = \partial^r_\theta[h^k_{ijk_1\ldots k_k}(\theta)]\\
&\quad \texttt{\#At this stage, } k \in [1,n] \implies \texttt{d0hk}[k-1] = h^k_{ijk_1\ldots k_k}(\theta)\\
&\quad \text{return } \texttt{d0hk}[n-1]
\end{aligned}
$$