Numerical Linear Algebra for Computational Science and Information Engineering
CITHN2006

Final Exam

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Problem 1 (9 pts)

Let $A \in \mathbb{R}^{n\times n}$ be a non-singular matrix. We are interested in finding a right-approximate inverse of A, that is, $M^{-1} \in \mathbb{R}^{n\times n}$ such that $I_n - AM^{-1}$ is small in some sense.

a. Show that $(X, Y) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \mapsto \operatorname{tr}(X^T Y)$ is an inner-product over $\mathbb{R}^{n\times n}$. (3 pts)

b. Show that $X \in \mathbb{R}^{n\times n} \mapsto \|X\|_F$ is a norm induced by the inner-product, i.e., $(X, X) = \|X\|_F^2$. (1 pt)

c. Consider the procedure described as follows:

$$\text{Find } \; M_1^{-1} \in M_0^{-1} + \operatorname{span}\{G\} \; \text{ such that } \; R_1 := I_n - AM_1^{-1} \perp A\operatorname{span}\{G\} \tag{1}$$

where $M_0^{-1} \in \mathbb{R}^{n\times n}$ is an initial right-approximate inverse of A, and $G \in \mathbb{R}^{n\times n}$ is a search direction. Assume $AG \neq 0_{n\times n}$, and find $\alpha \in \mathbb{R}$ such that

$$M_1^{-1} = M_0^{-1} + \alpha G.$$

For the sake of brevity, introduce $R_0 := I_n - AM_0^{-1}$. (2 pts)

d. Show that M_1^{-1} is given by Eq. (1) if and only if

$$M_1^{-1} = \arg \min_{M^{-1} \in M_0^{-1} + \operatorname{span}\{G\}} \|I_n - AM^{-1}\|_F.$$

Hint: Use the orthogonal projection theorem. That is, for any $\mathcal{S} \subset \mathbb{R}^{n\times n}$ and $X \in \mathbb{R}^{n\times n}$, there exists a unique $Y \in \mathcal{S}$ such that $\|X - Y\|_F = \min_{Z\in\mathcal{S}} \|X - Z\|_F$ if and only if $X - Y \perp \mathcal{S}$. (3 pts)

Problem 2 (3 pts)

Consider a tall-and-skinny matrix $A \in \mathbb{R}^{m\times n}$, i.e., such that $m \gg n$, and let $A = QR$ be the thin QR factorization of A, where $Q \in \mathbb{R}^{m\times n}$ is orthogonal, and $R \in \mathbb{R}^{n\times n}$ is upper-triangular.

a. Write down the algorithm of the CholeskyQR method to compute the QR factorization of A. (2 pts)

b. Is this algorithm stable? Explain why. (1 pt)

Problem 3 (3 pts)

Let $A = \begin{bmatrix} 2 & 7 & 2 & 0 \\ 0 & 3 & 6 & 0 \\ 2 & 7 & 2 & 4 \\ 0 & 0 & 5 & 0 \end{bmatrix}$, and answer the following questions with proper explanations:

a. Is A singular? (1 pt)

b. Does A admit an LU decomposition *without pivoting*? (2 pts)

Problem 4 (5 pts)

For the matrices

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

a. Find the Rayleigh-Ritz pairs of A with respect to range(V). (3 pts)

b. Assemble the reduced eigenvalue problem to solve in order to find the harmonic Ritz values of A with respect to range(V) for a shift $\sigma = 1$. (2 pts)

Problem 5 (9 pts)

Let $A = \begin{bmatrix} 1 & 4 & 5 & 6 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$, answer the following questions, and provide proper explanations:

a. What is the spectrum of A? (1 pt)

b. Is A singular? (1 pt)

c. Is A defective? (1 pt)

d. Is A diagonalizable? (1 pt)

e. Is A normal? (1 pt)

f. What is the conditioning number of the smallest eigenvalue of A? (3 pts)

g. What is the conditioning number of each eigenvalue of $B := A + A^T$? (1 pt)

Problem 6 (3 pts)

Complete the flowchart in Fig. 1 with the correct names of the methods covered in class:

- Conjugate gradient (CG),
- Minimal residual (MINRES),
- SYMMLQ,
- General minimal residual (GMRES),
- Quasi-minimal residual (QMR),
- Bi-conjugate gradient stabilized (Bi-CGSTAB)
- Conjugate gradient squared (CGS).

All the methods must be placed. Some boxes contain more than one method.

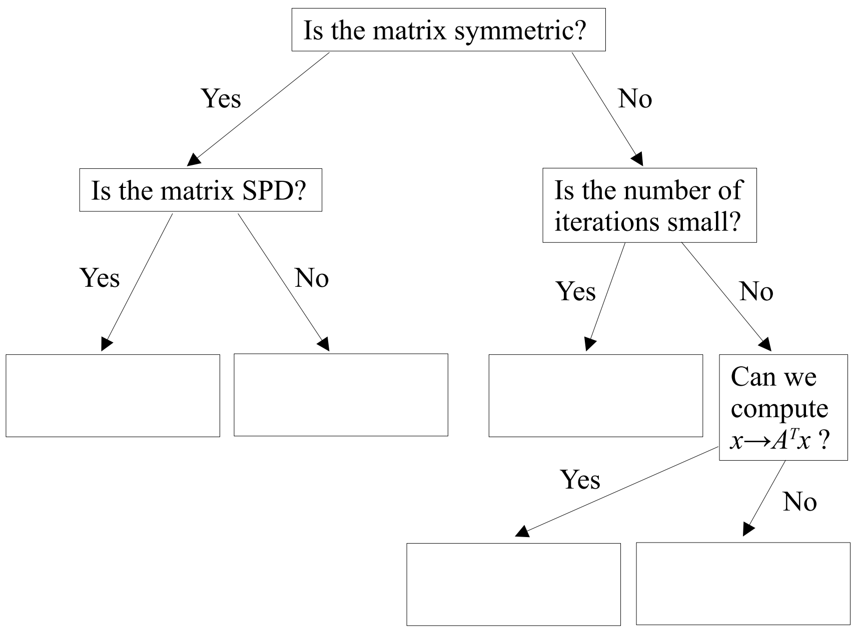

Figure 1: Flowchart of Krylov subspace-based linear iterative solvers.