Numerical Linear Algebra for Computational Science and Information Engineering
CITHN2006

Final Exam

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Problem 1 (4 pts)

Let $P \in \mathbb{R}^{n\times n}$ be any orthogonal projector, i.e., $P^T = P$. We want to show that P has a unit 2-norm, i.e., $\|P\|_2 = 1$.

a. Show that $\|x\|^2 = \|Px\|_2^2 + \|(I_n - P)x\|_2^2$ and $\|Px\|_2 \leq \|x\|_2 \,\forall\, x \in \mathbb{R}^n$. (2 pt)

b. Using the result of a. and the consistency of the 2-norms, show that $\|P\|_2 \geq 1$. (1 pt)

c. Using the result of a. with the definition of the matrix 2-norm, show that $\|P\|_2 \leq 1$. (1 pt)

Problem 2 (2 pts)

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix}.$$

a. Is A normal, or not? Explain. (1 pt)

b. What is the conditioning number of solving for the (simple) smallest eigenvalue of A? (1 pt)

Problem 3 (5 pts)

Assume A is a non-singular matrix and $\sigma \notin \mathrm{Sp}(A)$ is a given shift where $\mathrm{Sp}(A)$ denotes the spectrum of A.

a. Show that

$$\mathrm{Sp}(A) = \left\{\sigma + \frac{1}{\vartheta},\, \vartheta \in \mathrm{Sp}\left((A - \sigma I_n)^{-1}\right)\right\}$$

and

$$\min_{\lambda \in \mathrm{Sp}(A)} |\sigma - \lambda| = \sigma + \left(\max_{\vartheta \in \mathrm{Sp}((A-\sigma I_n)^{-1})} |\vartheta|\right)^{-1}.$$

(2 pts)

b. What is the shift-and-invert map $x \mapsto (A - \sigma I_n)^{-1}x$ used for when computing eigenvalues? Explain why. (2 pts)

c. Is it true or not that the harmonic Ritz vectors of A with respect to a search space $\mathcal{S}$ near a shift σ are obtained by applying a Rayleigh-Ritz procedure to the shift-and-invert operator $(A - \sigma I_n)^{-1}$ with respect to $(A - \sigma I_n)\mathcal{S}$ followed by one power iteration. (1 pt)

Problem 6 (3 pts)

Complete the flowchart in Fig. 1 with the correct names of the methods covered in class:

- Conjugate gradient (CG),
- Minimal residual (MINRES),
- SYMMLQ,
- General minimal residual (GMRES),
- Quasi-minimal residual (QMR),
- Bi-conjugate gradient stabilized (Bi-CGSTAB)
- Conjugate gradient squared (CGS).

All the methods must be placed. Some boxes contain more than one method.

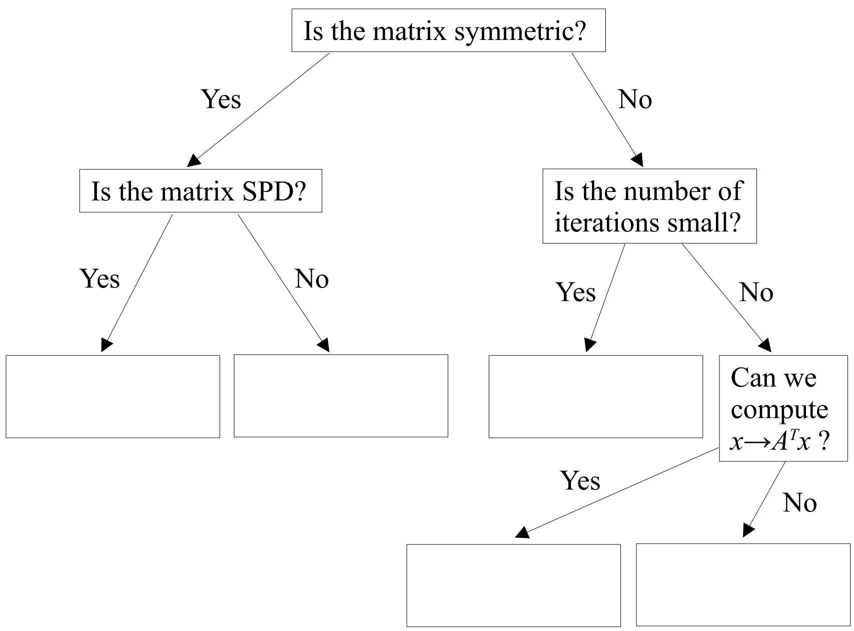

Figure 1: Flowchart of Krylov subspace-based linear iterative solvers.