

Numerical Linear Algebra for Computational Science and Information Engineering

Lecture 07 Orthogonalization and Least-Squares Problems

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Winter 2025-26



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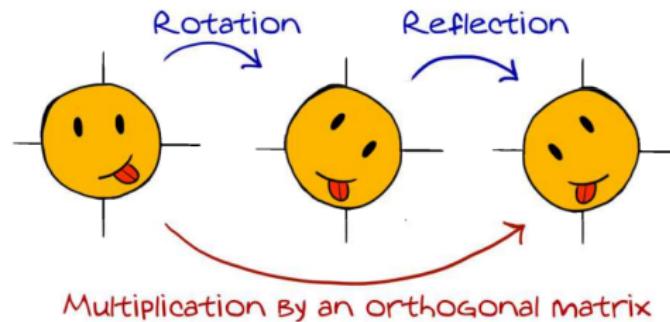
QR factorization

Section 4 in Darve & Wootters (2021)

QR factorization

- ▶ A QR factorization decomposes a matrix as the product of an orthogonal matrix Q with an upper-triangular matrix R .
- ▶ Recall that a matrix Q is orthogonal if $Q^T Q = I$.
- ▶ If a matrix is orthogonal, then $\|Qx\|_2 = \|x\|_2$ for all x , i.e., Q doesn't change the length of vectors.

The operations that do not change the length of vectors are **rotations** and **reflections**, so an orthogonal matrix can be thought of as a map that combines a rotation with a reflection.



Darve, E., & Wootters, M. (2021). Numerical linear algebra with Julia. Society for Industrial and Applied Mathematics.

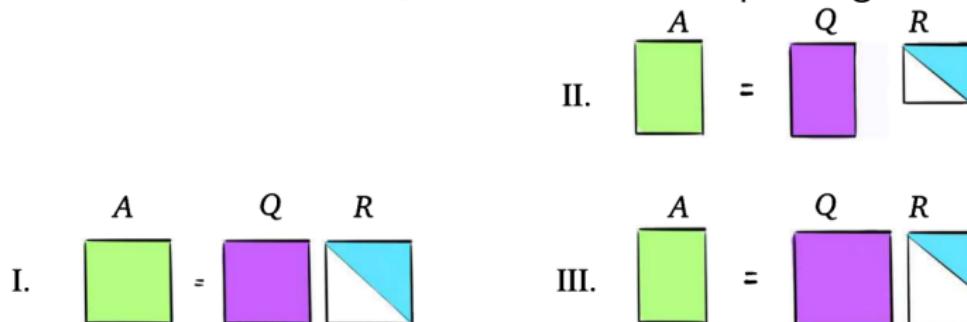
QR factorization, cont'd

QR factorization

- Let A be a real $m \times n$ matrix with $m \geq n$. Then, there is an orthogonal matrix Q and an upper-triangular matrix R such that $A = QR$. This is called the **QR factorization**.
- When A is complex, there is still a factorization $A = QR$, but Q is unitary, i.e., $Q^H Q = I$.

For the rest this lecture, we assume A is real, but QR factorizations do exist for complex matrices.

- There are different forms of QR factorizations, depending on the shape of A :



Applications of the QR factorization

The QR factorization has several applications in numerical linear algebra:

- ① It can be used to solve least-squares problems, i.e., problems of the form $\arg \min_x \|Ax - b\|_2$ where A is tall and skinny.
- ② It is used as part of eigen- and singular value algorithms for small dense matrices.
- ③ It is also used in iterative methods to solve linear systems and compute eigenvalues, such as in Krylov methods.

QR factorization and least-squares problems

To see why the QR factorization can be useful, let's look briefly at the least-squares problem:

- ▶ Let $A \in \mathbb{R}^{m \times n}$ with $m > n$. We want to find x such that Ax is closest to b in Euclidean distance. That is

$$x^* = \arg \min_x \|Ax - b\|_2.$$

To do this, we use the QR factorization, with a square Q , i.e., case III from slide #2:

$$\|Ax - b\|_2 = \|Q^T(Ax - b)\|_2 = \|Q^T(QRx - b)\|_2 = \|Rx - Q^Tb\|_2$$

where we used the fact that for any vector y , $\|Q^T y\|_2 = \|y\|_2$ because

$$\|Q^T y\|_2^2 = (Q^T y)^T (Q^T y) = y^T Q Q^T y = y^T (Q^T Q)^T y = y^T I^T y = y^T y = \|y\|_2^2.$$

As it turns out, it is easier to find x that minimizes $\|Rx - Q^T b\|_2$ than it is to minimize $\|Ax - b\|_2$.

Householder reflections

Section 4.1 in Darve & Wootters (2021)

Householder reflections

- ▶ Householder reflections are one of the most reliable methods to compute a QR factorization with a square Q , i.e., cases I and III.
- ▶ That is, we ask the question, does there exist a matrix Q s.t. $Q^T A = R$.
- ▶ Our goal is thus to create zero entries below the diagonal. Starting by the first column, we have:

Let's focus on this column

$$Q_1^T \quad A = \text{[Diagram showing a green rectangle with its first column circled in red]}$$

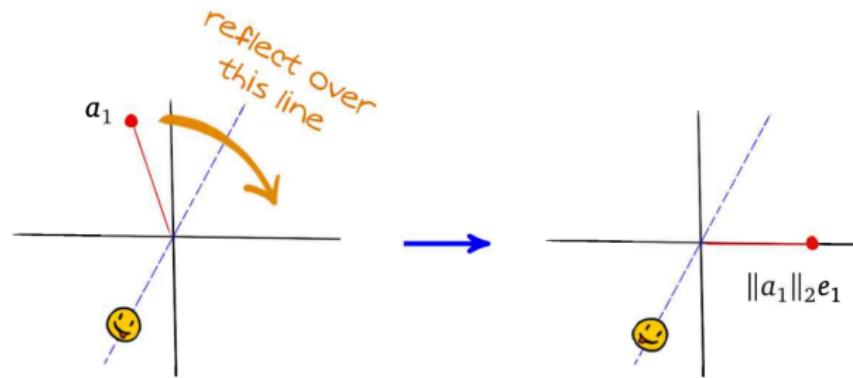
- ▶ We need to apply an orthogonal transformation Q_1^T to transform the first column of A into a vector in the direction of e_1 .

Let's write $A = [a_1 | \dots | a_n]$. Then, since Q_1^T does not change the norm of a_1 , we should have:

$$Q_1^T \quad a_1 = \pm \|a_1\|_2 e_1$$

Householder reflections, cont'd₁

- ▶ A logical choice for Q_1^T would be a rotation that maps a_1 parallel to e_1 . However, rotations in high dimensions are not so easy to set up. Thus, we'll instead choose Q_1^T to be a **reflection** that maps a_1 parallel to e_1 :



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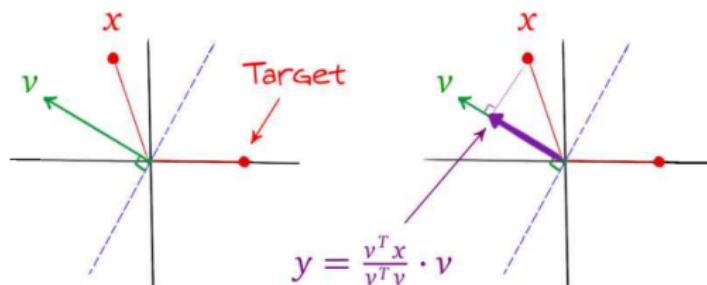
Now that we have an idea of what the reflection should be doing, we need to figure out its mathematical formula.

Householder reflections, cont'd₂

- Let us consider reflections in general. A reflection is defined by a vector:

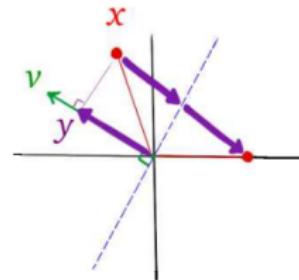


Given v , we can reason geometrically about what a reflection is:



Say we want to reflect x with respect to the hyperplane orthogonal to v

Consider the projection y of x onto v



Then the target is $x - 2y = \left(I - \frac{2vv^T}{v^Tv}\right)x$

Householder reflections, cont'd₃

Reflection

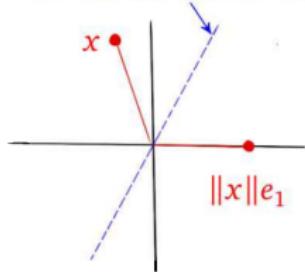
Let P be the matrix which represents a reflection over the hyperplane orthogonal to some vector v . Then P is given by

$$P = I - \beta vv^T \text{ where } \beta = \frac{2}{v^T v}.$$

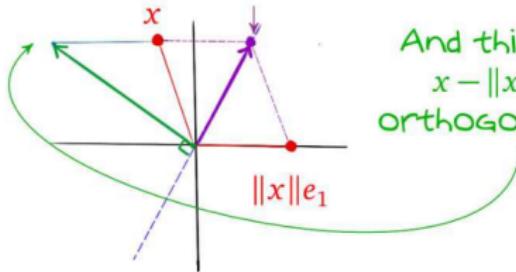
- Now we need to pick v to arrive at a Householder reflection, i.e., to get a transformation from x to $\|x\|_2 e_1$.

The following geometric argument shows that $v = x - \|x\|_2 e_1$ will work

What is this line?



This point is $x + \|x\|_2 e_1$.



And this point,
 $x - \|x\|_2 e_1$, is
orthogonal to it.

Householder reflections, cont'd₃

Let $v = x - \|x\|_2 e_1$, then we see that

$$\begin{aligned} Px &= \left(I - 2 \frac{vv^T}{v^T v} \right) x \\ &= \left(I - 2 \frac{(x - \|x\|_2 e_1)(x - \|x\|_2 e_1)^T}{(x - \|x\|_2 e_1)^T (x - \|x\|_2 e_1)} \right) x \\ &= \left(I - 2 \frac{(x - \|x\|_2 e_1)(x - \|x\|_2 e_1)^T}{2(\|x\|_2^2 - 2\|x\|_2 x_1)} \right) x \\ &= x - \frac{(x - \|x\|_2 e_1)(\|x\|_2^2 - \|x\|_2 x_1)}{(\|x\|_2^2 - 2\|x\|_2 x_1)} \\ &= x - (x - \|x\|_2 e_1) \\ &= \|x\|_2 e_1 \end{aligned}$$

so that, indeed, a reflection over the hyperplane orthogonal to $v = x - \|x\|_2 e_1$, is a Householder reflection of x .

Iterating Householder reflections

- ▶ Now that we know how to operate a first Householder reflection from a_1 to $\|a_1\|_2 e_1$, we can apply a series of Householder transformations to progressively reduce A to an upper-triangular form.

We proceed by first zeroing entries in the first column, then in the second column, and so on.

In the end, for $A \in \mathbb{R}^{m \times n}$ with $m \geq n$, we have

$$Q_{n-1}^T \dots Q_1^T A = R,$$

which is equivalent to

$$A = Q_1 \dots Q_{n-1} R = QR$$

where $Q = Q_1 \dots Q_{n-1} \in \mathbb{R}^{m \times m}$, and R has zeros in the $m - n$ rows if $m > n$.

- ▶ In practice, when reducing A to R , the matrix P is never formed explicitly, instead we compute $PA = A - \beta v(v^T A)$ which carries a cost $\mathcal{O}(2mn)$, instead of $\mathcal{O}(m^2n)$ when P is assembled and applied, i.e., as P is dense.

Givens rotations

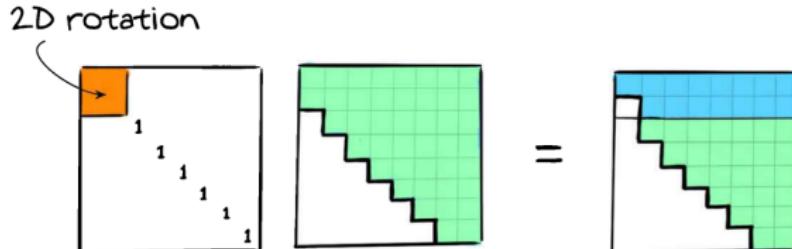
Section 4.2 in Darve & Wootters (2021)

Givens rotations

- When the matrix A is upper Hessenberg, i.e., where $a_{ij} = 0$ for all $i > j + 1$, most of the subdiagonal components are already zero, and using Householder transformations in this situation is a bit of an overkill:

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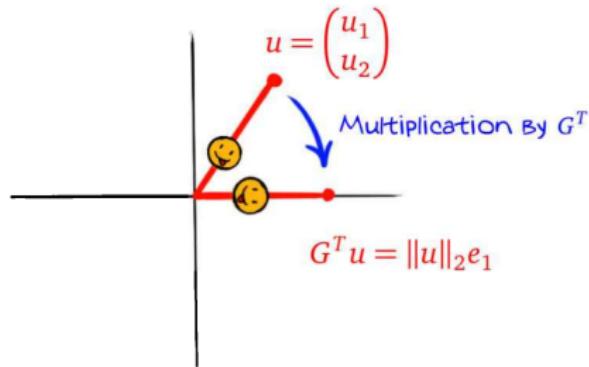
- On every column of A , only one entry needs to be zeroed, so that 2D rotations, which are easy to set up, can be deployed for the job:



Darve, E., & Wootters, M. (2021). Numerical linear algebra with Julia. Society for Industrial and Applied Mathematics.

Givens rotations, cont'd₁

- ▶ Zeroing a single subdiagonal entry can be reduced to considering a 2D vector $u = (u_1, u_2)$ and finding a rotation G^T such that the vector u becomes aligned with e_1 :



Darve, E., & Wootters, M. (2021). Numerical linear algebra with Julia. Society for Industrial and Applied Mathematics.

With some algebra, we find:

Givens rotation

A **Givens rotation** which rotates $u = (u_1, u_2)^T$ to $\|u\|_2 e_1$ is the 2×2 matrix defined by

$$G^T = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}, \quad c = \frac{u_1}{\|u\|_2}, \quad s = -\frac{u_2}{\|u\|_2}.$$

Givens rotations, cont'd₂

- ▶ For an upper Hessenberg matrix A of size $m \times n$, we can compute its QR factorization using a sequence of Givens rotations.
- ▶ The algorithm is as follows:
 1. For each column $j = 1, \dots, n - 1$:
 2. Construct a Givens rotation matrix G^T that zeros $a_{j+1,j}$
 3. Apply G^T to rows j and $j + 1$ of A

CholeskyQR

CholeskyQR factorization

- ▶ The **CholeskyQR** algorithm builds an economic factorization (i.e., case III) of $A \in \mathbb{R}^{m \times n}$, typically for $n \ll m$.
- ▶ It proceeds by first obtaining the R factor through a Cholesky factorization of the Gram matrix $A^T A$, then retrieving the Q factor by forward substitution:

The algorithm is as follows:

1. $X := A^T A$ // BLAS 3
2. Find the upper triangular R s.t. $R^T R = X$ // Cholesky factorization
3. $Q := AR^{-1}$ // triangular solves

- ▶ CholeskyQR reaches higher arithmetic intensity and requires less synchronizations than Householder QR:
 - ⇒ favored for distributed implementations.
- ▶ But, unlike Householder QR, CholeskyQR suffers from instability, with

$$\text{LOO}(Q) \in \mathcal{O}(u \cdot \kappa(A)).$$

Reorthogonalized variants

- ▶ The lack of stability of an orthogonalization procedure can be partly remedied by repetition. The repetition of CholeskyQR is referred to as **CholeskyQR2**:

1. $(Q_1, R_1) \leftarrow \text{CholeskyQR}(A)$
2. $(Q, R_2) \leftarrow \text{CholeskyQR}(Q_1)$
3. $R := R_2 R_1$

Repeating CholeskyQR significantly improves orthogonality, yielding $\text{LOO}(Q) \in \mathcal{O}(u)$ under the condition that $\kappa(A) \in \mathcal{O}(u^{-1/2})$.

- ▶ Another way to improve the stability of CholeskyQR is to shift the Gram matrix, decreasing its condition number, thus improving the stability of the Cholesky factorization. The resulting **Shifted CholeskyQR** is given by:

1. $X := A^T A$
2. $s := 11(mn + n(n+1))u\|A\|_2^2$ // calculate shift
3. $X := X + sI_n$ // shift Gram matrix
4. Find the upper triangular R s.t. $R^T R = X$ // Cholesky factorization
5. $Q := AR^{-1}$ // triangular solves

which ensures $\kappa(Q) \in \mathcal{O}(u^{-1/2})$ as long as $\kappa(A) \in \mathcal{O}(u^{-1})$.

Reorthogonalized variants, cont'd

- ▶ Therefore, Shifted CholeskyQR can be used as a preconditioner to CholeskyQR2. This procedure is referred to as **Shifted CholeskyQR3**:

1. $(Q_1, R_1) \leftarrow \text{Shifted CholeskyQR}(A)$
2. $(Q, R_2) \leftarrow \text{CholeskyQR2}(Q_1)$
3. $R := R_2 R_1$

which yields $\text{LOO}(Q) \in \mathcal{O}(u)$ as long as $\kappa(A) \in \mathcal{O}(u^{-1})$.

Yamamoto, Y., Nakatsukasa, Y., Yanagisawa, Y., & Fukaya, T. (2015). Roundoff error analysis of the CholeskyQR2 algorithm. *Electron. Trans. Numer. Anal.*, 44(01), 306–326.

Fukaya, T., Kannan, R., Nakatsukasa, Y., Yamamoto, Y., & Yanagisawa, Y. (2020). Shifted Cholesky QR for computing the QR factorization of ill-conditioned matrices. *SIAM Journal on Scientific Computing*, 42(1), A477–A503.

Tall-and-skinny QR

Tall-and-skinny QR (TSQR)

- ▶ Householder QR is unconditionally stable but memory-bound, and CholeskyQR variants, although they achieve high arithmetic intensity, offer limited stability.
- ▶ Tall-and-skinny (TSQR) algorithms offer both unconditional stability and high arithmetic intensity.

TSQR is particularly relevant when only the upper triangular factor R is needed.

- ▶ The key idea of TSQR is to **partition the matrix A into blocks** and compute QR factorizations hierarchically.

Tall-and-skinny QR (TSQR), cont'd₁

- ▶ For a matrix $A \in \mathbb{R}^{m \times n}$ with $m \gg n$, we partition A into p blocks:

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_p \end{bmatrix}$$

where each $A_i \in \mathbb{R}^{(m/p) \times n}$.

- ▶ We then compute the QR factorization of each block independently:

$$A_i = Q_i R_i, \quad i = 1, \dots, p.$$

- ▶ This gives us:

$$A = \begin{bmatrix} Q_1 R_1 \\ Q_2 R_2 \\ \vdots \\ Q_p R_p \end{bmatrix} = \begin{bmatrix} Q_1 & & & & R_1 \\ & Q_2 & & & R_2 \\ & & \ddots & & \vdots \\ & & & Q_p & R_p \end{bmatrix}.$$

Tall-and-skinny QR (TSQR), cont'd₂

- ▶ Next, we need to compute the QR factorization of the stacked R matrices:

$$\begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_p \end{bmatrix} = \tilde{Q}\tilde{R}$$

where $\tilde{Q} \in \mathbb{R}^{(pn) \times n}$ and $\tilde{R} \in \mathbb{R}^{n \times n}$.

- ▶ The final QR factorization is then:

$$A = \begin{bmatrix} Q_1 & & & \\ & Q_2 & & \\ & & \ddots & \\ & & & Q_p \end{bmatrix} \tilde{Q}\tilde{R} = Q\tilde{R}$$

where $Q = \text{diag}(Q_1, Q_2, \dots, Q_p)\tilde{Q}$.

- ▶ **Key advantage:** Each block can be processed independently, making TSQR highly parallelizable.

Tall-and-skinny QR (TSQR), cont'd₃

- ▶ The TSQR algorithm is as follows:
 1. Partition A into p blocks: $A = [A_1^T, A_2^T, \dots, A_p^T]^T$
 2. Parallelizable step: Compute $(Q_i, R_i) = \text{QR}(A_i)$ for $i = 1, \dots, p$
 3. Stack the R factors: $\tilde{A} = [R_1^T, R_2^T, \dots, R_p^T]^T$
 4. Compute $(\tilde{Q}, \tilde{R}) = \text{QR}(\tilde{A})$
 5. If only R is needed, return \tilde{R} . Otherwise, $Q = \text{diag}(Q_1, \dots, Q_p)\tilde{Q}$
- ▶ The loss of orthogonality of TSQR is such that $\text{LOO}(Q) \in \mathcal{O}(u)$ irrespective of A , i.e., TSQR is unconditionally stable.
- ▶ TSQR combines the best of both worlds: numerical stability of Householder QR with high arithmetic intensity and parallelizability.
⇒ TSQR favored for high-performance implementations.

Gram-Schmidt procedures

Section 4.3 in Darve & Wootters (2021)

Gram-Schmidt procedures

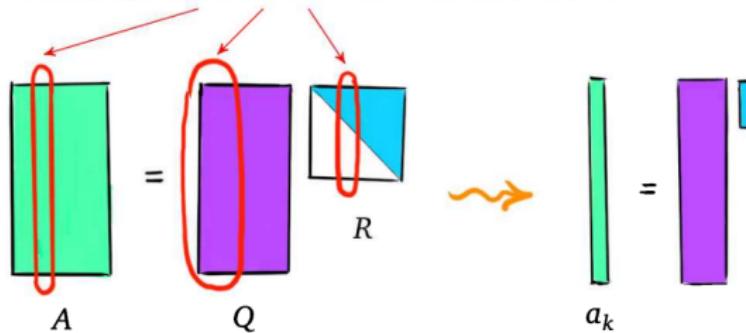
- ▶ Householder reflections and Givens rotations produce a square matrix $Q \in \mathbb{R}^{m \times m}$, even when $A \in \mathbb{R}^{m \times n}$ with $m > n$, i.e., case III.
On the other hand, **Gram-Schmidt** procedures will **produce a rectangular, tall-and-skinny matrix** $Q \in \mathbb{R}^{m \times n}$, i.e., like in case II.
- ▶ Another peculiarity of **Gram-Schmidt** procedures is that they **work column-by-column**, i.e., to compute q_i in $Q = [q_1, \dots, q_n]$, you only need access to a_i from $A = [a_1, \dots, a_n]$ and q_1, \dots, q_{k-1} .
This feature of the Gram-Schmidt procedures is particularly **useful in Krylov methods** where A is not available all at once, and the new column a_i to orthogonalize is only available after performing a full iteration of computations.
- ▶ The first k columns q_1, \dots, q_k formed by Gram-Schmidt procedure in Q are an **orthonormal basis** of the subspace spanned by a_1, \dots, a_k .

Gram-Schmidt procedures, cont'd₁

- ▶ Visualizing the column $a_k = QR_{:,k}$, and the fact that $R_{:,k}$ has k non-zero entries followed by $m - k$ zeros on the subdiagonal, we can write

$$a_k = Q_{:,1:k} R_{1:k,k}:$$

Consider just this part of the equation



a.k.a., a_k lies in the span of the first k columns of Q , and the k th column of R gives the coefficients.

Darve, E., & Wootters, M. (2021). Numerical linear algebra with Julia. Society for Industrial and Applied Mathematics.

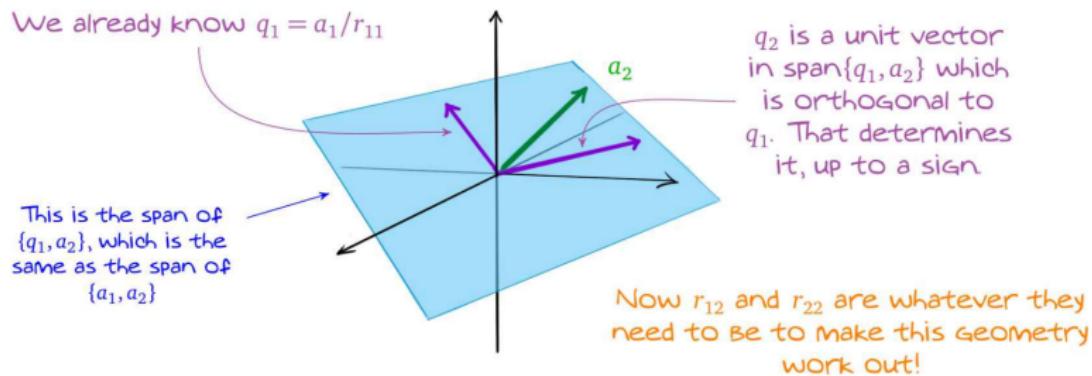
That is, a_k is formed by linear combination of q_1, \dots, q_k :

$$a_k = r_{1k}q_1 + \cdots + r_{kk}q_k.$$

- ▶ Thus, instead of searching for the matrix Q that makes $Q^T A$ upper triangular, we are rather going to search for the upper triangular matrix R such that every a_k is given by linear combination of q_1, \dots, q_k .

Gram-Schmidt procedures, cont'd₂

- ▶ First, since we have $a_1 = r_{11}q_1$, and q_1 has unit norm, we set $r_{11} = \|a_1\|_2$ and $q_1 = a_1/r_{11}$.
- ▶ Then, we continue iteratively, i.e., $a_2 = r_{12}q_1 + r_{22}q_2$ so that q_2 is a unit vector in $\text{span}\{q_1, a_2\} = \text{span}\{a_1, a_2\}$, orthogonal to q_1 :



Darve, E., & Wootters, M. (2021). Numerical linear algebra with Julia. Society for Industrial and Applied Mathematics.

Then r_{12} and r_{22} are found to close the system.

Classical Gram-Schmidt (CGS) procedure

- More formally, for each $1 \leq k \leq n$, we write

$$a_k = \sum_{i=1}^k r_{ik} q_i = r_{kk} q_k + \sum_{i=1}^{k-1} r_{ik} q_i$$

Assuming we already know q_j and r_{ij} for all $j < k$ and $i \leq j$, we can then use this formula to find expressions for the r_{ik} 's and q_k .

First, multiplying by q_i^T and invoking the orthonormality of the basis given by q_1, \dots, q_k , we get

$$q_i^T a_k = \sum_{j=1}^k r_{jk} q_i^T q_j = r_{ik} \implies r_{ik} = q_i^T a_k \text{ for } i < k.$$

Next, to find r_{kk} , we have $q_k r_{kk} = a_k - \sum_{i=1}^{k-1} r_{ik} q_i$ where q_k has unit norm so that

$$r_{kk} = \left\| a_k - \sum_{i=1}^{k-1} r_{ik} q_i \right\|_2.$$

Classical Gram-Schmidt (CGS) procedure, cont'd₁

Note that r_{kk} could also be chosen to be negative. However, it is standard to let R have positive components on the diagonal.

Finally, we have

$$q_k = \frac{1}{r_{kk}} \left(a_k - \sum_{i=1}^{k-1} r_{ik} q_i \right).$$

This procedure is referred to as the **classical Gram-Schmidt** algorithm. We see indeed that, in order to compute q_k , you need access to a_k and q_1, \dots, q_{k-1} .

Classical Gram-Schmidt (CGS) procedure, cont'd₂

When we get the k th column of A :

The first $k-1$ columns, q_1, \dots, q_{k-1} , have already been orthogonalized.

STEP 1: Set the k th column of R



This matrix has rows that are q_1, \dots, q_{k-1} .

STEP 2: Update a_k

A diagram showing the update of a_k . It starts with a yellow rectangle labeled a_k with a purple arrow pointing to its rightmost column. Below it is a yellow rectangle with a purple arrow pointing to its rightmost column, followed by a minus sign and a purple rectangle labeled R' . Below that is another yellow rectangle with a purple arrow pointing to its rightmost column, followed by a minus sign and a purple rectangle labeled R' . An equals sign follows. To the right, there is a large bracketed expression: $= \left(\begin{array}{c|c} \text{yellow square} & \text{purple rectangle} \\ \hline \text{yellow square} & \text{purple rectangle} \end{array} - R' \right) a_k$.

This matrix projects a_k onto the space orthogonal to q_1, \dots, q_{k-1} .

STEP 3: Renormalize a_k and set r_{kk} appropriately.

Now we've turned a_k into $q_k \perp q_1, \dots, q_{k-1}$! Move on to the next column.

Classical Gram-Schmidt (CGS) procedure, cont'd₃

► So the CGS algorithm is implemented as follows:

1. $r_{11} := \|a_1\|_2; q_1 := a_1/r_{11}$
2. For each $k = 2, \dots, n$:
 3. $R_{1:k-1,k} := Q_{:,1:k-1}^T a_k // \text{BLAS 2}$
 4. $q_k := a_k - Q_{:,1:k-1} R_{1:k-1,k} // \text{BLAS 2}$
 5. $r_{kk} := \|q_k\|_2; q_k := q_k/r_{kk}$

so that CGS relies on two BLAS 2 calls per iteration.

Similarly, we can write

1. $r_{11} := \|a_1\|_2; q_1 := a_1/r_{11}$
2. For each $k = 2, \dots, n$:
 3. $q_k := \Pi_{k-1} a_k$
 4. $r_{kk} := \|q_k\|_2; q_k := q_k/r_{kk}$

where $\Pi_{k-1} := I_m - Q_{:,1:k-1} Q_{:,1:k-1}^T$ is an **orthogonal projector** onto the subspace $\text{range}(Q_{:,1:k-1})^\perp$ so, indeed, q_k is made orthogonal to the previously formed vectors q_1, \dots, q_{k-1} .

Instability of CGS

- ▶ CGS is known to **not** being very **stable**.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{bmatrix}.$$

- ▶ For example, consider the matrix

If we assume ε^2 is smaller than the unit roundoff u , then the Q matrix

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ \varepsilon & -1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}.$$

Then we see that q_2 and q_3 are far from being orthogonal as we have
 $q_2^T q_3 = 1/2$.

- ▶ Numerical stability is measured with respect to the loss of orthogonality, LOO, defined by $\text{LOO}(Q) := \|I_m - Q^T Q\|_2$.
- ▶ With CGS, the LOO depends on A , i.e., $\text{LOO} \in \mathcal{O}(u \cdot \kappa^{n-1}(A))$, irrespective of $\kappa(A)$, although this bound is not sharp.

Re-orthogonalization, CGS2

- An alternative is to orthogonalize twice by CGS, leading to CGS2:

- | | |
|--|---|
| 1. $r_{11} := \ a_1\ _2; q_1 := a_1/r_{11}$ | 1. $r_{11} := \ a_1\ _2; q_1 := a_1/r_{11}$ |
| 2. For each $k = 2, \dots, n$: | 2. For each $k = 2, \dots, n$: |
| 3. $q_k := \Pi_{k-1} a_k$ | 3. $R_{1:k-1,k} := Q_{:,1:k-1}^T a_k$ |
| 4. $q_k := \Pi_{k-1} q_k$ | 4. $q_k := a_k - Q_{:,1:k-1} R_{1:k-1,k}$ |
| 5. $r_{kk} := \ q_k\ _2; q_k := q_k/r_{kk}$ | 5. $S_{1:k-1} := Q_{:,1:k-1}^T q_k$ |
| where $\Pi_{k-1} := I_m - Q_{:,1:k-1} Q_{:,1:k-1}^T$. | 6. $q_k := q_k - Q_{:,1:k-1} S_{1:k-1}$ |
| | 7. $r_{kk} := \ q_k\ _2; q_k := q_k/r_{kk}$ |

- The loss of orthogonality becomes $\text{LOO}(Q) \in \mathcal{O}(u)$ under the assumption that $\kappa(A) \in \mathcal{O}(u^{-1})$.
- However, CGS2 requires $4mn^2$ FLOPs instead of $2mn^2$ for CGS.

Modified Gram-Schmidt, MGS

- Another alternative to CGS, referred to as modified Gram-Schmidt (MGS), is obtained by letting $\Pi_{k-1} := (I_m - q_{:,k-1}q_{:,k-1}^T) \dots (I_m - q_{:,1}q_{:,1}^T)$ in

1. $r_{11} := \|a_1\|_2; q_1 := a_1/r_{11}$
2. For each $k = 2, \dots, n$:
3. $q_k := \Pi_{k-1}a_k$
5. $r_{kk} := \|q_k\|_2; q_k := q_k/r_{kk}$

Assuming perfect arithmetic, this is equivalent to CGS, but it relies on BLAS 1 instead BLAS 2 operations:

1. $r_{11} := \|a_1\|_2; q_1 := a_1/r_{11}$
2. For each $k = 2, \dots, n$:
3. $q_k := a_k$
4. For each $\ell = 1, \dots, k-1$:
 3. $r_{\ell k} := q_\ell^T q_k // \text{BLAS 1}$
 4. $q_k := q_k - r_{\ell k}q_\ell // \text{BLAS 1}$
5. $r_{kk} := \|q_k\|_2; q_k := q_k/r_{kk}$

Modified Gram-Schmidt, MGS, cont'd

- ▶ The loss of orthogonality of MGS is $\text{LOO}(Q) \in \mathcal{O}(u \cdot \kappa(A))$, irrespective of $\kappa(A)$, so that it is more stable than CGS.

$$\begin{bmatrix} 1 & 1 & 1 \\ \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{bmatrix}$$

- ▶ Considering once again the matrix $A =$

$$\begin{bmatrix} 1 & 0 & 0 \\ \varepsilon & -1/\sqrt{2} & -1/\sqrt{6} \\ 0 & 1/\sqrt{2} & -1/\sqrt{6} \\ 0 & 0 & \sqrt{2}/\sqrt{3} \end{bmatrix}.$$

MGS yields a Q matrix $A =$ Contrarily to CGS, we see that q_2 and q_3 are exactly orthogonal, i.e., $q_2^T q_3 = 0$, and q_1 is nearly orthogonal to q_2 and q_3 , with $|q_1^T q_2| = \varepsilon/\sqrt{2}$ and $|q_1^T q_3| = \varepsilon/\sqrt{6}$.

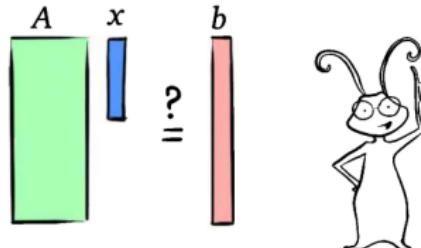
- ▶ In practice, MGS and CGS2 are used instead of CGS.
- ▶ MGS is often preferred by default, but CGS2 is more stable and reaches higher arithmetic intensity.

Least-squares problems

Section 4.4 in Darve & Wootters (2021)

Geometry of least-squares

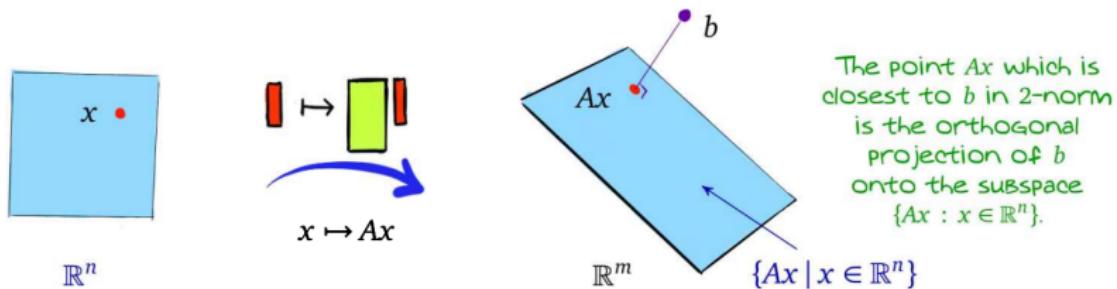
- One of the applications of the QR decomposition is the solving of least-squares problems $\arg \min_x \|Ax - b\|_2$:



Since A is tall and skinny, there are more equations than unknowns! So there may not be some x so that $Ax = b$. The least-squares problem is to find the x so that Ax is closest to b , in 2-norm.

with a tall-and-skinny matrix $A \in \mathbb{R}^{m \times n}$ and a vector $b \in \mathbb{R}^n$.

- To minimize the 2-norm from a point b to a subspace $\{Ax, x \in \mathbb{R}^n\}$, we can just do an orthogonal projection:



The point Ax which is closest to b in 2-norm is the orthogonal projection of b onto the subspace $\{Ax : x \in \mathbb{R}^n\}$.

Method of normal equations

- As per property of orthogonal projections, the x that minimizes $\|Ax - b\|_2$ has an error $e := Ax - b$ which is orthogonal to the range of A . This can be written as

$$A^T(Ax - b) = 0. \quad (1)$$

Assuming A is full-rank, this equation can be used to solve for x by a method called **normal equations**.

Eq. (1) may also be derived from calculus, namely, the optimal x which minimizes the cost function

$$f(x) = \|Ax - b\|_2^2 = (Ax - b)^T(Ax - b) = x^T A^T A x - 2x^T A^T b + b^T b$$

is obtained for $\nabla f(x) = 0$ where

$$\nabla f(x) = 2A^T A x - 2A^T b,$$

which equivalently yields Eq. (1).

Method of normal equations, cont'd

- ▶ Assuming A is full-rank, $A^T A$ is SPD so that we may compute its Cholesky factorization and solve for x in $A^T A x = A^T b$.

Normal equations

Finding the solution x to the least-squares problem $\arg \min \|Ax - b\|_2$ by solving the system $A^T A x = A^T b$ is called the method of **normal equations**.

- ▶ Since the condition number of $A^T A$ is the square of that of A , the method of normal equations can run into issues when A is poorly conditioned.
- ▶ For cases where A is poorly conditioned, the QR factorization can be used to yield a more accurate computation of the solution x to the least-squares problem.

QR factorization for least-squares problems

- ▶ The origin of the method of normal equations stems from saying that the error $Ax - b$ is orthogonal to the range of A .

But if we know a QR factorization $A = QR$ where $Q \in \mathbb{R}^{m \times n}$, then the range of A is the same as the range of Q .

The orthogonality condition can then be re-stated as

$$Q^T(Ax - b) = 0. \quad (2)$$

Since Q is orthogonal, it is necessarily well-conditioned, and the conditioning problem of the method of normal equations can be avoided.

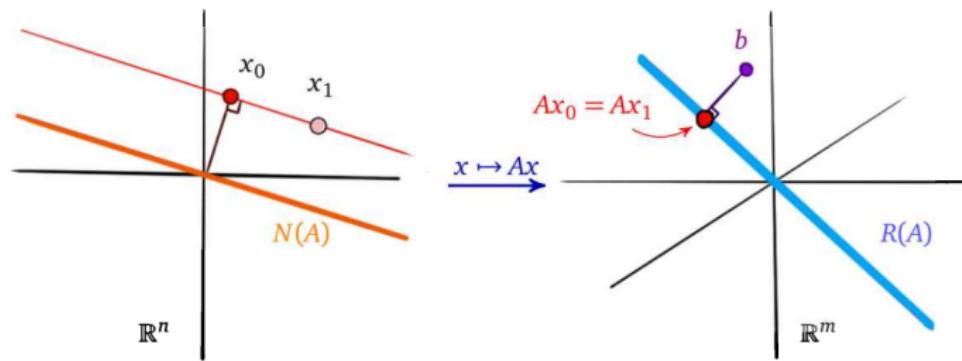
Since $A = QR$, due to the orthogonality of Q , we have $Q^T A = R$ so that Eq. (2) becomes

$$Rx = Q^T b$$

where R is non-singular as long as A is non-singular, so that there exists a unique solution x .

Case of rank-deficient A

- ▶ If A is rank-deficient, the null space of A is non-trivial. Then, for some x that minimizes $\|Ax - b\|_2$, there are infinitely many $\delta x \in \text{null}(A)$ such that $A(x + \delta x) = Ax$. Hence, the solution to the least-squares problem is not unique.
- ▶ In case of non-uniqueness of solution, one can search for the unique x_0 which minimizes both $\|Ax - b\|_2$ and $\|x\|_2$:



Darve, E., & Wootters, M. (2021). Numerical linear algebra with Julia. Society for Industrial and Applied Mathematics.

We can see the x_0 we are after is orthogonal to the null space of A , while any other solution x_1 is of the form $x_0 + \delta x$.

SVD method for solving least-squares with rank-deficient A

- Let $A \in \mathbb{R}^{m \times n}$ be of rank $r < n < m$ have an SVD given by $A = U\Sigma V^T$ with $U \in \mathbb{R}^{m \times m}$, $V \in \mathbb{R}^{n \times n}$ and $\Sigma \in \mathbb{R}^{m \times n}$ where Σ has zeros from row $r+1$ to m .

Then we can ignore the columns of U and V that correspond to zeros in Σ to create the thin SVD $A = \tilde{U}\tilde{\Sigma}\tilde{V}^T$:

$$A = U \Sigma V^T = \tilde{U} \tilde{\Sigma} \tilde{V}^T$$

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- Now, $Ax = 0$ if and only if $\tilde{V}^T x = 0$, which means that the null space of A is the same as that of \tilde{V}^T , i.e., $\text{null}(A) = \text{null}(\tilde{V}^T)$.

SVD method for solving least-squares with rank-deficient A

- We know that any solution x to the least-squares problem satisfies

$$A^T A x = A^T b$$

$$(\tilde{U} \tilde{\Sigma} \tilde{V}^T)^T \tilde{U} \tilde{\Sigma} \tilde{V}^T x = (\tilde{U} \tilde{\Sigma} \tilde{V}^T)^T b$$

$$\tilde{V} \tilde{\Sigma}^T \tilde{\Sigma} \tilde{V}^T x = \tilde{V} \tilde{\Sigma}^T \tilde{U}^T b$$

$$\tilde{\Sigma} \tilde{V}^T x = \tilde{U}^T b$$

where $r < n$ so that $\tilde{\Sigma} \tilde{V}^T$ is not full-column-rank and this equation admits infinitely many solutions.

- However, we can find one solution as follows.

First, let's solve the system $\tilde{\Sigma} \omega = \tilde{U}^T b$ for $\omega \in \mathbb{R}^r$. This gives

$$\omega_i = \frac{\tilde{u}_i^T b}{\tilde{\sigma}_{ii}}$$

where $\tilde{U} = [\tilde{u}_1, \dots, \tilde{u}_r]$ and $\tilde{\Sigma} = \text{diag}(\tilde{\sigma}_{11}, \dots, \tilde{\sigma}_{rr})$.

SVD method for solving least-squares with rank-deficient A

Then, since $\omega = \tilde{\Sigma}^{-1}\tilde{U}^T b$, we have

$$\tilde{\Sigma}\tilde{V}^T(\tilde{V}\omega) = \tilde{\Sigma}\tilde{V}^T\tilde{V}\tilde{\Sigma}^{-1}\tilde{U}^T b = \tilde{U}^T b$$

so that $x_0 := \tilde{V}\omega$ is solution of $\tilde{\Sigma}\tilde{V}^T x_0 = \tilde{U}^T b$ and thus, as explained before, it is also solution of the least-squares problem.

Note that $x_0 := \tilde{V}\omega$ is the solution with smallest norm.

To see this, we need to show $x_0 \perp \text{null}(A)$. First, let

$$\text{null}(\tilde{V}^T) = \{y \in \mathbb{R}^n, \tilde{V}^T y = 0\}$$

and consider that for each $y \in \text{null}(\tilde{V}^T)$, we have

$$x_0^T y = (\tilde{V}\omega)^T y = \omega \tilde{V}^T y = 0$$

so that $x_0 \perp \text{null}(\tilde{V}^T)$.

But since $\text{null}(\tilde{V}^T) = \text{null}(A)$, we have that $x_0 \perp \text{null}(A)$.

LSQR

Iterative solve of normal equations

- ▶ We saw that the least-squares solution x_* of $\min_{x \in \mathbb{R}^n} \|Ax - b\|_2$ for $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ with $m > n$ is such that $A^T A x_* = A^T b$, i.e., x_* is solution of the normal equation.
- ▶ For very large matrices, the cost of computing a QR factorization by Householder QR, or even by CholeskyQR, can be prohibitive. When the matrix is sparse, computing a QR factorization is generally an overkill.
- ▶ In future lectures, we will look into iterative methods to solve linear systems of the form $Bx = b$ with a square matrix $B \in \mathbb{R}^{n \times n}$.

In particular, if A is full-rank, one can use the conjugate gradient algorithm to solve $A^T A x_* = b$.

However, in practice, for cases where A is ill-conditioned, this approach can suffer from significantly delayed convergence.

- ▶ **LSQR** is an algorithm proposed by Paige and Sanders (1982) which, in case of exact arithmetic, reproduces the iterates of the conjugate gradient algorithm applied to the normal equation but, in practice, is **more reliable**.

Paige, C. C., & Saunders, M. A. (1982). LSQR: An algorithm for sparse linear equations and sparse least squares. ACM Transactions on Mathematical Software (TOMS), 8(1), 43-71.

Bidiagonalization

- Bidiagonalization is a procedure proposed by Golub and Kahan (1965) which reduces any general matrix $A \in \mathbb{R}^{m \times n}$ into lower bidiagonal form. Let $x_0 \in \mathbb{R}^n$ be an initial approximation of x_* with residual $r_0 := b - Ax_0$. Starting the bidiagonalization procedure with r_0 goes as follows:

$$\left. \begin{array}{l} \beta_1 u_1 = r_0, \alpha_1 v_1 = A^T u_1 \\ \beta_{i+1} u_{i+1} = A v_i - \alpha_i u_i \\ \alpha_{i+1} v_{i+1} = A^T u_{i+1} - \beta_{i+1} v_i \end{array} \right\} \text{for } i = 1, 2, \dots$$

where the scalars $\alpha_i \geq 0$ and $\beta_i \geq 0$ are chosen so that $\|u_i\|_2 = \|v_i\|_2 = 1$. Let $U_k := [u_1, \dots, u_k]$, $V_k := [v_1, \dots, v_k]$,

$$B_k := \begin{bmatrix} \alpha_1 & & & & \\ \beta_2 & \alpha_2 & & & \\ & \ddots & \ddots & & \\ & & & \beta_k & \alpha_k \end{bmatrix} \quad \text{and} \quad \underline{B}_k := \begin{bmatrix} \alpha_1 & & & & \\ \beta_2 & \alpha_2 & & & \\ & \ddots & \ddots & & \\ & & & \beta_k & \alpha_k \\ & & & & \beta_{k+1} \end{bmatrix}.$$

Golub, G., & Kahan, W. (1965). Calculating the singular values and pseudo-inverse of a matrix. Journal of the Society for Industrial and Applied Mathematics, Series B: Numerical Analysis, 2(2), 205-224.

Paige, C. C. (1974). Bidiagonalization of matrices and solution of linear equations. SIAM Journal on Numerical Analysis, 11(1), 197-209.

Bidiagonalization, cont'd

Then, we have

$$U_{k+1}(\beta_1 e_1) = r_0 \quad (e_1 := I_{k+1}[:, 1])$$

$$AV_k = U_{k+1}\underline{B}_k = U_k B_k + \beta_{k+1} u_{k+1} e_k^T \quad (e_k := I_k[:, k])$$

$$A^T U_{k+1} = V_k \underline{B}_k^T + \alpha_{k+1} v_{k+1} e_{k+1}^T. \quad (e_{k+1} := I_{k+1}[:, k+1])$$

In exact arithmetic, we have $U_k^T U_k = I_k$ and $V_k^T V_k = I_k$.

Clearly, $U_{k+1}^T AV_k = \underline{B}_k$, $U_k^T AV_k = B_k$, $\alpha_k = u_k^T Av_k$ and $\beta_k = u_k^T Av_{k-1}$.

Moreover, the columns of U_k and V_k are orthonormal bases of Krylov subspaces of AA^T and $A^T A$, respectively, i.e.,

$$\text{range}(U_k) = \mathcal{K}_k(AA^T, u_1) = \text{span} \left\{ u_1, AA^T u_1, \dots, (AA^T)^{k-1} u_1 \right\},$$

$$\text{range}(V_k) = \mathcal{K}_k(A^T A, v_1) = \text{span} \left\{ v_1, A^T Av_1, \dots, (A^T A)^{k-1} v_1 \right\}.$$

Bidiagonalization is an orthogonal equivalence transformation which plays a key role in the iterative solve of singular value decompositions.

Golub, G., & Kahan, W. (1965). Calculating the singular values and pseudo-inverse of a matrix. Journal of the Society for Industrial and Applied Mathematics, Series B: Numerical Analysis, 2(2), 205-224.

Paige, C. C. (1974). Bidiagonalization of matrices and solution of linear equations. SIAM Journal on Numerical Analysis, 11(1), 197-209.

LSQR

- ▶ LSQR defines a sequence of iterates x_1, x_2, \dots, x_k which approximate the solution x_* of $\min_{x \in \mathbb{R}^n} \|b - Ax\|_2$ by

$$x_k \in x_0 + \mathcal{K}_k(A^T A, v_1).$$

That is, we search for $x_k = x_0 + V_k y_k$ such that

$$\begin{aligned} x_k &= \arg \min_{x \in x_0 + \text{range}(V_k)} \|b - Ax\|_2 \implies y_k = \arg \min_{y \in \mathbb{R}^k} \|b - A(x_0 + V_k y)\|_2 \\ &= \arg \min_{y \in \mathbb{R}^k} \|r_0 - U_{k+1} \underline{B}_k y\|_2 \\ &= \arg \min_{y \in \mathbb{R}^k} \|U_{k+1}(\beta_1 e_1) - U_{k+1} \underline{B}_k y\|_2 \\ &= \arg \min_{y \in \mathbb{R}^k} \|\beta_1 e_1 - \underline{B}_k y\|_2. \end{aligned}$$

In exact arithmetic, the LSQR iterates exhibit monotonic decrease of residual norm, i.e., $\|r_{k+1}\|_2 \leq \|r_k\|_2$.

LSQR, cont'd

- A basic implementation of the LSQR algorithm goes as follows:

1. $r_0 := b - Ax_0$
2. $u_1 := r_0, \beta_1 := \|u_1\|_2, u_1 := u_1/\beta_1$
3. $v_1 := A^T u_1, \alpha_1 := \|v_1\|_2, v_1 := v_1/\alpha_1$
4. for $i = 1, 2, \dots$
5. $u_{i+1} := Av_i - \alpha_i u_i, \beta_{i+1} := \|u_{i+1}\|_2, u_{i+1} := u_{i+1}/\beta_{i+1}$
6. $v_{i+1} := A^T u_{i+1} - \beta_{i+1} v_i, \alpha_{i+1} := \|v_{i+1}\|_2, v_{i+1} := v_{i+1}/\alpha_{i+1}$
7. $y_i := \arg \min_{y \in \mathbb{R}^i} \|\beta_1 e_1 - \underline{B}_i y\|_2$

When convergence is achieved, the iterate $x_i := x_0 + V_i y_i$ is formed.

Convergence monitoring is reliant on the evaluation of $\|r_i\|_2$ and $\|A^T r_i\|_2$.

Note that we have

$$\begin{aligned}\|r_i\|_2 &= \|b - A(x_0 + V_i y_i)\|_2 = \|\underline{t}_i\|_2 \quad \text{where } \underline{t}_i := \beta_1 e_1 - \underline{B}_i y_i \\ \|A^T r_i\|_2 &= \|A^T U_{i+1} \underline{t}_i\|_2 = \|(V_i \underline{B}_i^T + \alpha_{i+1} v_{i+1} e_{i+1}^T) \underline{t}_i\|_2 = \|\underline{B}_{i+1}^T \underline{t}_i\|_2\end{aligned}$$

so that convergence can be monitored without forming either x_i nor r_i .

Homework problems

Homework problem

Turn in **your own** solution to the following problem:

Pb. 17 Let $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

- (a) Find a QR decomposition of A applying a Gram-Schmidt procedure with a pen and paper.
- (b) Find the least-squares problem solution $x_* = \arg \min_x \|Ax - b\|_2$ making use of the QR factorization.

Practice session

Practice session

- 1 Implement and test and time your implementation of Householder QR.
- 2 Implement CholeskyQR, CholeskyQR2 and Shifted CholeskyQR3.
- 3 Compare the runtime, loss of orthogonality and residual obtained with each CholeskyQR implementation for multiple tall-and-skinny matrices with different conditioning numbers.
- 4 Implement CGS, CGS2 and MGS procedures.
- 5 Compare the runtime, loss of orthogonality and residual obtained with each Gram-Schmidt implementation for multiple tall-and-skinny matrices with different conditioning numbers.
- 6 Implement the Golub-Kahan bidiagonalization procedure. Then, compare $U_{k+1}^T A V_k$ to \underline{B}_k , and check the loss of orthogonality of U_{k+1} and V_k . What do you observe?
- 7 Implement LSQR using the following stopping criterion:

$$\frac{\|A^T r_i\|_2}{\|A\|_F \|r_i\|_2} < \varepsilon_{tol}.$$

Apply your algorithm to large sparse least-squares problems.