Discrete: Maths Olympiad

G V V Sharma*

1. For any natural number n, (n ≥ 3), let f(n) denote the number of non-congruent integer-sided triangles with perimeter n (e.g., f(3) = 1, f(4) = 0, f(7) = 2). Show that
   a) $f(1999) > f(1996)$
   b) f(2000) = f(1997).
2. Let R denote the set of all real numbers. Find all functions f : R → R satisfying the condition
$$f(x + y) = f(x)f(y)f(xy)$$
for all x, y in R.
3. Given any nine integers show that it is possible to choose, from among them, four integers a, b, c, d such that a + b - c - d is divisible by 20. Further show that such a selection is not possible if we start with eight integers instead of nine.
4. Suppose the n^2 numbers 1,2,3,.....n^2 are arranged to form an n by n array consisting of n rows and n columns such that the numbers in each row and each column are in increasing order. Denote by a_{jk} the number in j-th row and k-th column. Suppose b_j is the maximum possible number of entries that can occur as a_{ij}, $1 \le j \le n$. Prove that
$$b_1 + b_2 + b_3 + ...b_n \le \frac{n}{3}(n^2 - 3n + 5).$$
5. Do there exist 100 lines in the plane, no three of them concurrent, such that they intersect exactly in 2002 points?
6. Find all 7-digit numbers formed by using only the digits 5 and 7, and divisible by both 5 and 7.
7. In a lottery, tickets are given nine digit numbers using only the digits 1,2,3. They are also coloured red, blue or green in such a way that two tickets whose numbers differ in all the nine places get different colours. Suppose the ticket bearing the number 122222222 is red and that bearing the number 222222222 is green. Determine with proof the colour of the ticket bearing the number 123123123.
8. Let S denotes the set of all 6-tuples (a,b,c,d,e,f) of positive integers such that
$$a^2 + b^2 + c^2 + d^2 + e^2 + f^2.$$
Consider the set T=$\{abcdef : (a, b, c, d, e, f) \in S\}$ Find greatest common divisor of all the members of T.
9. Find all solutions f : R → R such that
$$f(x^2 + yf(z)) = xf(x) + zf(y)$$
for all x,y,z in R.
10. All possible 6-digit numbers, in each of which the digits occur in non-increasing order(from left to right e.g., 877550) are written as a sequence in increasing order. Find the 2005^{th} number in this sequence.
11. Let X denote the set of all triples (a, b, c) of integers. Define a function f: X → X by
$$f(a, b, c) = (a + b + c, ab + bc + ca, abc).$$

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in.

Find all triples (a, b, c) in X such that f(f(a, b, c)) = (a, b, c).

12. Some 46 squares are randomly chosen from 9×9 chess board and are coloured red. Show that there exists a 2×2 block of squares of which at least three are coloured red.
13. Let $\sigma = (a_1, a_2, a_3, ..., a_n)$ be a permutation of (1, 2, 3,....,n). A pair (a_i, a_j) is said to correspond to an inversion of σ, $i < j$ but $a_i > a_j$. (Example: In the permutation (2,4,5,3,1) there are 6 inversions corresponding to the pairs (2,1), (4,3),(4,1),(5,3),(5,1),(3,1).) How many permutations of (1,2,3,...,n), $(n \geq 3)$, have exactly two inversions?
14. All the points in the plane are coloured using three colours, Prove that there exists a triangle with vertices having the same colour such that either it is isosceles or its angles are in geometric progression.
15. Find all functions f: R → R such that

$$f(x+y)f(x-y) = (f(x) + f(y))^2 - 4x^2 f(y),$$

for all x,y ∈ R, R denotes the set of all real numbers.

16. Suppose five of the nine vertices of a regular nine-sided polygon are arbitarly chosen. Show that one can select four among these five such that they are the vertices of a trapezium.
17. Define a sequence $f_0(x), f_1(x), f_2(x)......$ of functions by

$$f_0(x) = 1, f_1(x) = x, (f_n(x))^2 - 1 = f_{n+1}(x)f_{n-1}(x), forn \geq 1.$$

Prove that each $f_n(x)$ is a polynomial with integer coefficients.

18. Let f: Z → Z be a function satisfying f(0) ≤ 0, f(1)= 0 and
    a) f(xy) + f(x)f(y) = f(x) + f(y)
    b) (f(x - y) - f(0))f(x)f(y) = 0

    for all x,y ∈ Z,simultaneously.
    a) Find the set of all possible values of the function f.
    b) If f(10) ≠ 0 and f(2) = 0,Find the set of all integers n such that f(n) ≠ 0.
19. Let n be a positive integer. Call a nonempty subset S of $\{1, 2, ...n\}$ good if the arithmeteic mean of the elements of S is also an integer. Further let t_n denote the number of good subsets $\{1, 2, ...n\}$. Prove that t_n and n are both odd or both even.
20. Let a,b be natural numbers with $ab > 2$. Suppose that the sum of their greatest common divisor and least common multiple is divisible by a+b. Prove that the quotient is at most (a+b)/4. When is this quotient exactly equal to (a+b)/4?
21. Let n be a natural number and X = 1, 2,......., n. For subsets A and B of X we define A△B to be the set of all those elements of X which belong to exactly one of A and B. Let F be a collection of subsets of X such that for any two distinct elements A and B in F the set A△B has at least two elements. Show that F has at most 2^{n-1} elements. Find all such collections F with 2^{n-1} elements.
22. Find all real functions f from R → R satisfying the relation

$$f(x^2 + yf(x)) = xf(x+y).$$

23. There are four basket ball players A, B, C, D. Initially, the ball is with A. The ball is always passed from one person to a different person. In how many ways can the ball come back to A after seven passes? (For example A → C → B → D → A → B → C → A and A → D → A → D → C → A → B → A are two ways in which the ball can come back to A after seven passes.)
24. Let N denote the set of all natural numbers. Define a function T : N → N by T(2k) = k and T(2k + 1) = 2k + 2. We write $T^2(n)$ = T(T(n)) and in general $T^k(n) = T^{k-1}(T(n))$ for any $k > 1$.
    a) Show that for each n ∈ N, there exists k such that T^k(n) = 1.
    b) For k ∈ N, let c_k denote the number of elements in the set $\{n : T^k(n) = 1\}$. Prove that $c_{k+2} = c_{k+1} + c_k$s, for $k \geq 1$.

25. Suppose 2016 points of the circumference of a circle are coloured red and the remaining points are coloured blue. Given any natural number $n \geq 3$, Prove that there is a regular n-sided polygon all of whose vertices are blue.