A Way of Seeing Primes

Observations on Doubled Intervals, and a Structural Path to Cramér

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We found ourselves wondering: what if primes appear at the precise moments when they must rather than being scattered unpredictably?

1. A Different Way of Looking

Mathematicians have long asked: how big can gaps between primes get? Cramér, Granville, and others have pursued this question with remarkable depth.

We found ourselves drawn to a different question: when must new primes appear? This shift led us to look at intervals differently, as a prime-defined generative process.

The ancient Greeks sometimes contrasted two kinds of time: Χρόνος (the uniform) and Καιρός (the rhythmic). The gift of this perspective was learning to sense Καιρός , the right moment, rather than only counting Χρόνος. What if we organized primes by their own natural joints rather than our decimal grid?

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1.1 Methodology

This work combines rigorous proof with large-scale computational verification. We are explicit throughout about the status of each claim:

Proven: The Index Bound Theorem, the Recursive Bootstrap, the Residue Framework, and the Fulcrum Bound Theorem (conditional) are established analytically using classical results (Rosser-Dusart bounds, modular arithmetic).

Observed: The Balance Property and its deviation formula are verified computationally but not proven from first principles.

Conjectured: The Safe Gap Conjecture is tested against data but is unproven.

Computational scope: Primary analysis covers $n = 68{,}000{,}000$ consecutive prime pairs. Metadata records both $p_n = 1{,}358{,}208{,}601$ (used in the threshold formula) and $p_{n+1} = 1{,}358{,}208{,}653$ (which defines the final interval endpoint). Multi-scale verification at 10M, 20M, 40M, and 68M intervals is used to track threshold behavior across scales.

Left/Right definition: For interval i with fulcrum F_i, we count a prime p as LEFT if $p < F_i$, and RIGHT if $p \geq F_i$. Thus F_i, if prime, is counted on the right.

Pooled statistic: Balance statistics are pooled across all primes found in all analyzed intervals. Let L be the total number of primes in a doubled interval with $p < F_i$, and R the total number of primes with $p \geq F_i$. We report left_fraction = L/(L+R), right_fraction = R/(L+R), and deviation_from_50 = left_fraction − 0.5.

Index convention: Intervals are defined for $i \geq 1$, but computational runs typically start at $i = 2$ (so both p_i and p_{i+1} are odd), avoiding the special edge case involving $p_1 = 2$.

Candidate set: In each interval $I_i = [2p_i, 2p_{i+1})$ we test only odd candidates $n \in \{2p_i + 1, 2p_i + 3, \ldots, 2p_{i+1} - 1\}$ for primality.

We believe transparency about what is proven versus observed strengthens rather than weakens the work. The patterns we observe are striking; explaining *why* they hold remains open.

1.2 Overview

Our investigation began with a simple observation about consecutive primes that led us to define a quantity we call the *fulcrum* (Section 2). The fulcrum reveals a surprising balance in how primes distribute (Section 3). Reframing primes as generators rather than survivors (Section 4) led us to formalize *doubled intervals* as natural search zones (Section 5). The Index Bound Theorem (Section 6) shows these zones are tractable; the Residue Framework (Section 7) shows how small primes constrain them. These tools bootstrap all primes from {2, 3} (Section 8) and suggest a threshold conjecture for when intervals must contain primes (Section 9.1–9.4). Finally, we show that if our conjectures hold, they structurally imply Cramér's bound on prime gaps — offering a potential route to one of number theory's major open problems (Section 9.5).

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2. The Fulcrum

2.1 Definition

For consecutive primes p_i and p_{i+1}, we define the **fulcrum** of the i-th interval as:

$$F_i = p_i + p_{i+1} + 1$$

The fulcrum has a natural geometric interpretation. The midpoint between p_i and p_{i+1} is $m = (p_i + p_{i+1})/2$. Doubled, this gives $2m = p_i + p_{i+1}$, which is even. Adding 1 shifts it to odd:

$$F_i = 2m + 1$$

The fulcrum is the odd projection of the midpoint into the doubled interval. The "+1" is necessary because the sum of two odd primes is even, and only odd numbers greater than 2 can be prime. For $\mathbf{i \geq 2}$, the fulcrum lies strictly inside $\mathbf{I_i}$; the case $\mathbf{i = 1}$ is a boundary exception because $\mathbf{F_1 = 6}$ equals the excluded endpoint of **[4,6)**.

Within the interval $[2p_i, 2p_{i+1})$, the fulcrum sits at relative position $(g_i + 1)/(2g_i) = 1/2 + 1/(2g_i)$, where $g_i = p_{i+1} - p_i$ is the prime gap. For twin primes ($g = 2$), the fulcrum sits at 75% of the interval; for larger gaps, it approaches 50%.

2.2 Prior Work and Context

The expression $p_n + p_{n+1} + 1$ for consecutive primes has been studied in several contexts. The sequence of primes of this form is catalogued in OEIS A092738. The theoretical framework for understanding when such expressions yield primes derives from Hardy and Littlewood's work on prime k-tuples [4].

Crucially, Lemke Oliver and Soundararajan [6] demonstrated that consecutive primes exhibit unexpected biases in their distribution across residue classes. These biases directly affect the primality rate of expressions like F_i.

2.3 Arithmetic Structure

Empirically, fulcrums are prime more often than a "typical" odd number of comparable size: in the 68M dataset, **11.71%** of intervals have a prime fulcrum (metadata), which is several times larger than the baseline prime probability near these magnitudes ($\approx 1/\ln x$).

Parity. For $i \geq 2$, both p_i and p_{i+1} are odd, so their sum is even and F_i is odd. Thus F_i avoids the most common obstruction to primality.

Divisibility by the bounding primes. The fulcrum is never divisible by p_{i+1} (algebraically guaranteed for $i \geq 2$), and rarely divisible by p_i.

Mod 3 and mod 6 structure. All primes greater than 3 satisfy $p \equiv \pm 1 \pmod 6$. When consecutive primes are both $\equiv 1 \pmod 6$, their sum is $\equiv 2 \pmod 6$, so $F \equiv 3 \pmod 6$ and is divisible by 3.

2.4 The Perfect Square Pattern

When the fulcrum $F = k^2$ is a perfect square, we have $p_i + p_{i+1} = k^2 - 1 = (k-1)(k+1)$. Two consecutive primes sum to a product of two consecutive integers. Remarkably, k is itself often prime.

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3. The Balance Property

3.1 The Discovery

The most surprising finding concerns how primes distribute around the fulcrum.

Geometrically, the fulcrum sits *above* the midpoint of the interval - at 75% for twin primes, 62.5% for gap-4, and so on. If primes were uniformly distributed, we would expect more primes on the left side (which has more room) than the right.

But empirically they split almost exactly 50-50.

We verified this at multiple scales, observing that the balance *improves* as n increases:

Scale	Left Fraction	Deviation from 50%
10M	0.499910	0.0090%
20M	0.499923	0.0077%
40M	0.499941	0.0059%
68M	0.499972	0.0028%

The deviation shrinks monotonically, suggesting convergence to perfect balance. At 68 million intervals, 131 million primes split with only ~7,500 excess on one side, a deviation of just 0.0028%.

The fulcrum acts as a natural center of mass. Each interval does its own thing - some empty, some skewed left, some skewed right - yet when aggregated, the global totals equalize with remarkable precision.

3.2 The Empirical Deviation Formula

The deviation from 50-50 follows a striking pattern. **Empirically, we observe** that:

$$\textit{Deviation from } 50\% \approx -1/(2g)$$

where g is the gap size. **This is an empirical observation, not a proven theorem.**

The data matches this formula with remarkable precision:

Gap	Geometric Left	Observed Left	Deviation
2	75.0%	50.05%	−24.95%
4	62.5%	49.99%	−12.51%
6	58.3%	50.01%	−8.33%

Here "observed left" is computed by pooling across all intervals with the same gap g: observed_left(g) = (total primes with p < F_i in gap-g intervals) / (total primes in gap-g intervals), and the deviation is observed_left(g) − 0.5.

The predicted deviation −1/(2g) matches observed values: for gap 2, predicted −0.250 vs observed −0.2495; for gap 6, predicted −0.0833 vs observed −0.0833. *Why this formula holds remains an open question.*

3.3 Predictive Power

The fulcrum's *type* is associated with the interval's prime richness (illustrative sample).

Table 3.3 (diagnostic sample; 100k intervals): Fulcrum type vs. interval richness (i = 2 to 100,000).

Fulcrum Type	Count	% of Total	Avg Primes	Avg Gap	Empty Intervals	% Empty
Prime	18,324	18.32%	2.53	12.8	0	0.00%
Square	131	0.13%	1.56	10.2	45	34.35%
Other composite	81,544	81.54%	1.76	13.0	24,907	30.54%

This breakdown conditions on fulcrum type and is not the same statistic as the global pooled balance reported in Section 3.1.

In this 100k diagnostic run, prime-fulcrum intervals have the highest average prime count (2.53) despite similar average gaps across types (12.8–13.0). This suggests that fulcrum primality is associated with greater prime richness through arithmetic structure, not merely through wider intervals.

Note that for i ≥ 2, the fulcrum $F_i = p_i + p_{i+1} + 1$lies strictly inside the interval $I_i = [2p_i, 2p_{i+1})$. Therefore, if $F_i$is prime then $I_i$is automatically non-empty (it contains F_i). The nontrivial signal is that prime-fulcrum intervals have a substantially higher *average* prime count, not merely the guaranteed minimum of 1.

For reference, full 68M harvest metadata reports: prime fulcrums = 7,960,450 (11.71% of intervals); square fulcrums = 2,389. The prime-fulcrum rate is higher in the 100k diagnostic range than at 68M because the first 10^5 prime gaps lie at much smaller magnitudes, where small-modulus effects are stronger; the diagnostic table is used for profiling rather than estimating asymptotic rates.

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4. Generators, Not Survivors

The patterns in Section 3: the Balance Property, the deviation formula, the predictive power of fulcrum type suggest that primes are not randomly scattered but structurally constrained. This invites a philosophical reframe: what if primes are not survivors but generators?

We noticed something elementary, yet striking: from {2, 3} alone, all primes can be generated.

Start with generators {2, 3}. These generate composites: 4, 6, 8, 9, 12... The smallest positive integer greater than 1 that they cannot generate is 5. Therefore 5 must be prime. Continue: {2, 3, 5} cannot generate 7, so 7 is prime. And so on.

But where exactly should we look for these births? The doubled intervals provide the answer. Each interval $I_i = [2p_i, 2p_{i+1})$ is a bounded search zone: any prime born there must be odd, must survive division by all small primes, and must appear before the next doubled prime boundary. The Index Bound Theorem (Section 6) will show that we need only a finite toolkit — the first √(2i) primes — to verify every birth in interval i. The intervals are not arbitrary containers; they are the natural zones where the logic of generation forces new primes into existence.

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5. The Doubled Intervals

Having established that primes are generators rather than survivors, we now formalize the search zones where new primes must appear. Let p_i denote the i-th prime. We define the i-th doubled-prime interval as:

$$I_i = [2p_i, 2p_{i+1})$$

These intervals partition all integers ≥ 4. Each has length $2g_i$ where g_i is the i-th prime gap. Some intervals contain many primes. Some contain none. We asked: what determines which?

The factor of 2 is not arbitrary. It emerges from our organizing principle (doubling consecutive primes) and it reappears throughout the framework: in the interval boundaries, in the Index Bound (√(2i)), and in the verification requirements.

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6. The Index Bound Theorem

The question 'what determines which intervals contain primes?' requires a verification method. The following theorem shows that verification is surprisingly tractable.

Theorem (Index Bound). *For any composite number n in interval $I_i = [2p_i, 2p_{i+1})$, there exists a prime divisor p_j with $j \leq \sqrt{(2i)}$.*

Status: PROVEN. This theorem is rigorously established using Rosser-Dusart bounds on prime indices.

What this means: to determine whether interval I_i contains primes, we need only the first √(2i) primes. The interval's index governs the verification complexity, not the size of the numbers involved.

Practical corollary (interval-local verification). Fix i and consider the interval I_i = [2p_i, 2p_{i+1}). Let the "toolkit primes" be the first floor(sqrt(2i)) primes: p_1, p_2, …, p_floor(sqrt(2i)). By the Index Bound Theorem, every composite integer in I_i is divisible by at least one of these toolkit primes. Therefore, to detect compositeness inside I_i, it is sufficient to test divisibility only by this finite prefix of the prime sequence. Equivalently, if an integer in I_i is not divisible by any of these primes, it cannot be composite and therefore must be prime.

Proof

The proof proceeds through three lemmas.

Lemma 1 (Elementary). *Every composite number n has a prime factor $p \leq \sqrt{n}$.*

If n = ab with 1 < a ≤ b, then a ≤ √n. The smallest prime factor of n divides a, hence is at most √n. ■

Lemma 2 (Interval Bound). *For any $n \in I_i$, we have $n < 2p_{i+1}$.*

Immediate from the half-open interval definition. ■

Lemma 3 (Index-Value Bound). If pj ≤ √(2pi+1), then j ≤ √(2i) for all i ≥ 1.

This lemma converts a bound on prime values into a bound on prime indices. We prove the contrapositive: if j > √(2i), then pj > √(2pi+1).

Proof of Lemma 3

We use explicit bounds for primes from Dusart [2] (stated here in the forms used in this proof):

(1) Lower bound: $p_k > k(\ln k + \ln \ln k - 1)$ for $k \geq 2$.

(2) Upper bound: $p_k < k(\ln k + \ln \ln k)$ for $k \geq 6$.

In Part B we will use a simpler lower bound $p_k > k(\ln k - 1)$, which follows from (1) since $\ln \ln k > 0$ for $k \geq 3$.

Part A: Computational Verification ($i \leq 10{,}000$).

For each i, compute

$j_{max} = \max\{ j : p_j \leq \sqrt{2p_i+1} \}$

and verify $j_{max} \leq \sqrt{2i}$. The bound holds for all $i \leq 10{,}000$.

Part B: Explicit Bounds ($i > 10{,}000$).

Assume $j > \sqrt{2i}$. Define

$j_0 = \text{floor}(\sqrt{2i}) + 1$,

so $j_0 > \sqrt{2i}$ and therefore $j_0^2 > 2i$.

Since $j \geq j_0$ and primes are increasing, $p_j \geq p_{j_0}$. It suffices to show:

$(p_{j_0})^2 > 2p_i+1$.

Step 1: Lower bound for $(p_{j_0})^2$.

Using the conservative bound $p_k > k(\ln k - 1)$, we obtain:

$(p_{j_0})^2 > j_0^2 (\ln j_0 - 1)^2$.

Because $j_0 > \sqrt{2i}$, we have $j_0^2 > 2i$ and $\ln j_0 \geq \ln(\sqrt{2i}) = (1/2) \ln(2i)$.

Therefore:

$(p_{j_0})^2 > 2i * ((1/2) \ln(2i) - 1)^2$. (A)

Step 2: Upper bound for $2p_{i+1}$.

Using Dusart's upper bound (valid since $i > 10{,}000 \Rightarrow i+1 \geq 6$):

$2p_{i+1} < 2(i+1)(\ln(i+1) + \ln \ln(i+1))$. (B)

Step 3: Compare (A) and (B) for $i > 10{,}000$.

For $i > 10{,}000$ we have $i+1 \leq (1 + 10^{-4})i$. Also $i+1 \leq 2i$, and since ln and ln ln are increasing in this range:

$\ln(i+1) + \ln \ln(i+1) \leq \ln(2i) + \ln \ln(2i)$.

So from (B):

$2p_{i+1} < 2i(1 + 10^{-4})(\ln(2i) + \ln \ln(2i))$. (C)

Thus it is enough to show that for $i > 10{,}000$:

$((1/2) \ln(2i) - 1)^2 > (1 + 10^{-4})(\ln(2i) + \ln \ln(2i))$. (D)

Let $x = \ln(2i)$. For $i > 10{,}000$ we have $x > \ln(20{,}000) \approx 9.903$. Define:

$g(x) = ((1/2)x - 1)^2 - (1 + 10^{-4})(x + \ln x)$.

Then

$g'(x) = (1/2)x - 2 - (1 + 10^{-4})/x$.

For $x \geq 9.903$, $g'(x) > 0$, so $g(x)$ is increasing on this range. It therefore suffices to check $x_0 = \ln(20{,}000) \approx 9.903$:

$((1/2)x_0 - 1)^2 \approx (3.9515)^2 \approx 15.61$,

$(1 + 10^{-4})(x_0 + \ln x_0) \approx 1.0001(9.903 + 2.293) \approx 12.20$,

so $g(x_0) > 0$. Hence $g(x) > 0$ for all $x \geq x_0$, proving (D) for all $i > 10{,}000$.

Combining (A) with (C) and (D), we conclude $(p_{j0})^2 > 2p_{i+1}$, so $p_{j0} > \sqrt{2p_{i+1}}$, and therefore $p_j \geq p_{j0} > \sqrt{2p_{i+1}}$.

This completes the contrapositive, and thus proves the lemma. ■

Main Theorem Proof: Let n be composite and $n \in I_i$. By Lemma 1, n has a prime divisor p_j

with $p_j \leq \sqrt{n}$. By Lemma 2, $n < 2p_i+1$, hence $\sqrt{n} < \sqrt{(2p_i+1)}$. Therefore $p_j \leq \sqrt{n} < \sqrt{(2p_i+1)}$, so in particular $p_j \leq \sqrt{(2p_i+1)}$. Applying Lemma 3 gives $j \leq \sqrt{(2i)}$. ■

The Index Bound tells us *which* primes suffice for verification. But how do these primes actually constrain candidates? The Residue Framework answers this, after which we return to the Index Bound's deepest consequence: self-sustaining recursion.

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7. The Residue Framework

The Index Bound tells us which primes matter for verification. The Residue Framework tells us how they interact.

For any interval $I_i = [2p_i, 2p_{i+1})$, the candidates for primality are $2p_i + k$ where k is odd. Each small prime q creates constraints on which candidates it can divide.

Danger Zone Criterion: $q \mid (2p_i + k) \Leftrightarrow p_i \equiv (q - k)/2 \pmod q$

Proof. Starting from $q \mid (2p_i + k)$, we have $2p_i + k \equiv 0 \pmod q$. Rearranging: $2p_i \equiv -k \pmod q$. Since q is an odd prime, 2 has a multiplicative inverse modulo q. Multiplying both sides by 2^{-1}: $p_i \equiv -k \cdot 2^{-1} \pmod q$. Since $(q - k)/2 \equiv -k \cdot 2^{-1} \pmod q$, we obtain $p_i \equiv (q - k)/2 \pmod q$. ■

This is structure, not randomness. The composites appear where they must; the primes appear where the danger zones leave gaps.

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8. The Recursive Bootstrap

Corollary (Self-Sustaining Recursion). *Starting from {2, 3}, processing intervals in order i = 1, 2, 3, ... generates all primes, with each step having sufficient known primes.*

Status: PROVEN (follows directly from Index Bound Theorem).

Proof. At step i, verification requires at most $\sqrt{(2i)}$ primes (by the Index Bound). After completing steps 1 through $i-1$, we have discovered approximately i primes. We need $i \geq \sqrt{(2i)}$, which holds for all $i \geq 2$. For i = 1, 2, the base case {2, 3} suffices directly. ■

This means the entire infinite sequence of primes is implicit in {2, 3} plus the verification rule.

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9. The Safe Gap Conjecture

The Recursive Bootstrap guarantees that primes can be found interval by interval. But it does not tell us which intervals will contain primes and which will be empty. We now turn to this question, offering an empirical conjecture about the threshold that separates fertile intervals from barren ones.

9.1 The Conjecture

We arrive at our central empirical observation, offered as a **conjecture** for others to examine.

Conjecture (Safe Gap). *For the i-th prime p_i with gap $g_i = p_{i+1} - p_i$:*

$$\text{If } g_i \geq (1/\pi) \cdot [\ln(2) \cdot \ln(2i) \cdot \ln(p_i) + 2 \cdot \ln(i)],$$

then the interval $I_i = [2p_i, 2p_{i+1})$ contains at least one prime.

Status: CONJECTURE. Supported by frontier evidence at 68M (and by multi-scale tracking at 10M, 20M, 40M, 68M), but full per-interval verification of the implication remains to be completed.

9.2 Motivation for the Form

The conjecture's formula is not arbitrary. *Each component reflects known structure*. This is reasoning, not proof, but it explains why this particular form emerges:

- **The term $\ln(p_i)$** reflects the Prime Number Theorem, primes thin logarithmically. The "cost" of finding a prime grows as ln(p).
- **The term ln(2i)** connects to the Index Bound: the factor of 2 reflects our doubled intervals. Verification complexity scales with $\sqrt{(2i)}$, and ln(2i) captures this growth.
- **The product form** respects the multiplicative nature of prime interactions. Sieving is multiplicative; thresholds should be too.
- **The constant ln(2)/π** ≈ 0.2206 emerges from the interaction of binary structure (ln 2 from sieving out evens) and cyclic structure (π from residue class cycles). It differs from $C_2/3 \approx 0.2201$ (where C_2 is the Hardy-Littlewood twin prime constant) by only 0.26%.
- **Asymptotically**, $\ln(2i) \cdot \ln(p_i)$ grows like $\ln^2(p)$, matching Cramér's scale for maximum gaps. This is not coincidence: if the conjecture is true, it would provide structural support for Cramér's bound.

The sample-size correction term (2/π) · ln(i) accounts for cumulative probability across many intervals. Even if each interval has tiny probability of being empty, with millions of intervals, rare events occur. This term ensures the threshold rises fast enough to keep expected empty intervals below 1.

Caveat: This reasoning motivates the formula's structure but does not constitute a proof. The connections to PNT, Index Bound, and Cramér are suggestive, not deductive. We offer this reasoning to help others see the logic and to invite scrutiny of where it might fail.

9.3 Verification at 68 Million Intervals

At n = 68,000,000 intervals, no empty interval occurs for prime gaps $g \geq 98$. The largest gap among empty intervals is 96. The corrected threshold T(68M) = 98.38 exceeds this by a margin of 2.38.

Verification note: This confirms the frontier condition max_gap_among_empty_intervals < T(n), which is consistent with the conjecture at scale n, but it is not yet a full per-interval verification of the implication "if g_i ≥ T(i), then I_i is non-empty".

The transition zone is remarkably sharp:

Gap	Intervals	Empty	Status
88	41,616	1	
90	83,676	1	
92	28,160	1	
94	25,660	1	
96	44,937	1	← max empty
98	23,564	0	← threshold
100	24,725	0	

The gap-96 outlier. The single empty interval at gap 96 is I_{26,235,002}, defined by consecutive primes p = 497,575,847 and p' = 497,575,943. The interval [995,151,694, 995,151,886) contains no primes. It's a verified prime desert of 192 consecutive composites. This lone outlier (1 of 44,937 gap-96 intervals, or 0.002%) persists from n = 40M onward.

Multi-scale tracking. The threshold T(n) tracks the emptiness frontier across scales:

n	T(n)	Max Empty Gap	Margin	Status
10M	80.75	72	+8.75	✓
20M	86.93	80	+6.93	✓
40M	93.33	96	−2.67	outlier exceeds
68M	98.38	96	+2.38	✓

At 40M, the gap-96 outlier briefly exceeds T(n), but by 68M the threshold catches up. The correction term (2/π)·ln(n) is essential: without it, the original threshold fails at ALL tested scales.

9.3.1 The Correction Term

The formula includes a correction term (2/π)·ln(i) that accounts for cumulative probability across many intervals. Its necessity is empirically demonstrated:

Scale	T(n) corrected	T(n) original	Max Empty	Original fails?
10M	80.75	70.49	72	Yes (by 1.51)
20M	86.93	76.23	80	Yes (by 3.77)

Scale	T(n) corrected	T(n) original	Max Empty	Original fails?
40M	93.33	82.19	96	Yes (by 13.81)
68M	98.38	86.90	96	Yes (by 9.10)

Without the correction, the threshold falls below the observed maximum empty gap at every scale tested. The correction grows from ~10.3 at 10M to ~11.5 at 68M, modest but essential.

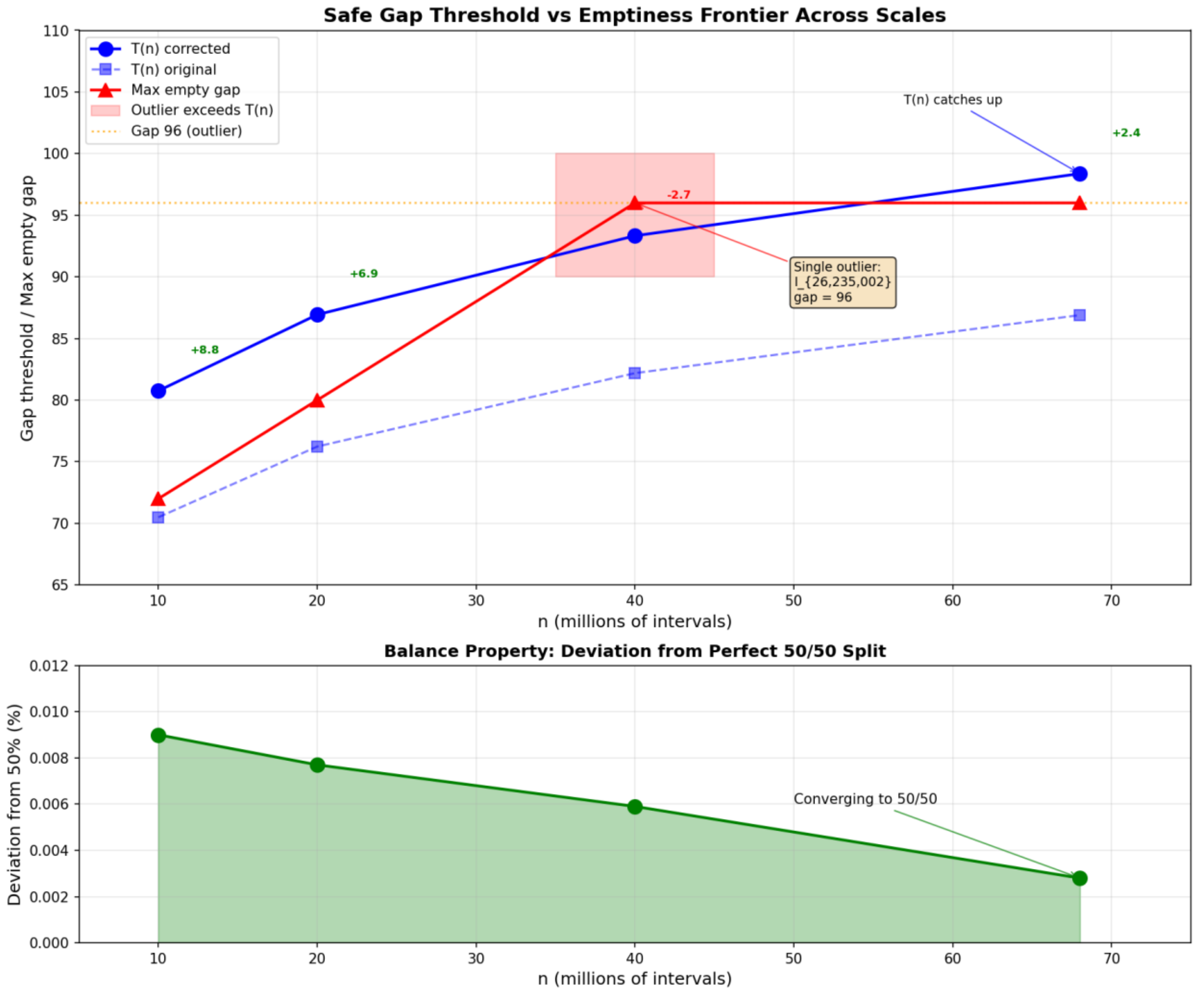

Figure 1: Multi-scale analysis of Safe Gap threshold versus emptiness frontier. Top panel shows T(n) corrected (solid blue) tracking the maximum gap among empty intervals (red), with the 40M outlier region highlighted. Bottom panel shows Balance Property deviation converging toward zero.

9.4 Empty Interval Streaks and the Fulcrum Bound

Recall that each interval has a fulcrum $F_i = p_i + p_{i+1} + 1$, sitting just above the midpoint (Section 2). We now show that fulcrums bound the gaps that traverse empty streaks. Some doubled intervals contain no primes. When consecutive intervals are all empty, they form a *streak* and a prime gap must traverse the entire streak.

Theorem (Fulcrum Bound). *Let q be the last prime in interval I_a and r be the first prime in interval I_b, with empty intervals between them. IF $q \geq F_a$ and $r \leq F_b$, THEN: gap = $r - q \leq F_b - F_a$.*

Status: THE THEOREM IS PROVEN; THE CONDITIONS ARE EMPIRICAL.

Proof. Pure algebra. If $q \geq F_a$ and $r \leq F_b$, then $r - q \leq F_b - F_a$. ■

Important caveat: The theorem is conditional. The algebra is proven, but whether the conditions (q ≥ F_a and r ≤ F_b) hold is empirical. In our 68M dataset, both conditions hold in 63.9% of streak-bounding cases (among streaks with complete boundary info). The theorem tells us: IF primes land in certain positions, THEN gaps are bounded. Whether primes land in those positions is an observed property, not yet explained.

Empirical data at 68M intervals:

- Fulcrum conditions (q ≥ F_a AND r ≤ F_b) hold in 63.9% of streak-bounding cases
- Maximum (r − q)/(ln m)^2 observed, where m = (q + r)/2: 73.9% of Cramér
- When conditions fail, gaps can exceed (F_b − F_a), but remain below Cramér in the tested range

The Fulcrum Bound provides a deterministic upper bound when its conditions hold; empirically, these conditions hold in the majority of cases.

9.5 Toward Cramér: A Structural Path

The Safe Gap Conjecture and the 2× Rule, if true, would structurally imply Cramér's famous conjecture on prime gaps. We present this argument not as a proof, but as a potential path forward.

The Setup

Our doubled intervals partition the integers: $[2p_i, 2p_{i+1}) \cup [2p_{i+1}, 2p_{i+2}) \cup \cdots$forms a contiguous region. While each interval can be analyzed independently (via the Index Bound), bounding prime gaps requires understanding how far one can travel across consecutive intervals before necessarily encountering a prime.

The 2× Rule

From Section 9.4 and Appendix A.4, we observe that consecutive empty intervals form streaks whose total span is bounded. Empirically:

$$\text{Maximum streak span} \leq 2 \times \text{Safe Gap threshold}$$

At 68M intervals, the observed maximum is 1.89× the threshold — essentially twice the safe gap. We call this the **2× Rule**.

The Traversal Argument

Consider the worst-case construction for a prime gap at scale $2p$:

Step	Description	Maximum Size
1	Begin just after a prime (at the end of some non-empty interval)	0
2	Traverse a maximum streak of consecutive empty intervals	$\approx \frac{2}{3}\ln^2(2p)$
3	The streak terminates (by Safe Gap, intervals above threshold are non-empty)	—
4	Worst case: the prime appears at the far end of this interval	$\approx \frac{1}{3}\ln^2(2p)$
Total	**Maximum possible prime gap**	$\approx \ln^2(2p)$

Since $\ln^2(2p) = (\ln\ 2 + \ln\ p)^2 \to \ln^2(p)$asymptotically, this recovers Cramér's bound.

The Numerology

The arithmetic is not coincidental:

- Safe Gap threshold $T \approx \frac{1}{3}\ln^2(2p)$— sets the "unit"
- Maximum streak span $\approx 2T$— the 2× Rule
- Maximum prime gap = streak + interval $\approx 2T + T = 3T \approx \ln^2(2p)$

Three independent quantities (threshold, streak bound, Cramér's bound) lock together in ratio 1 : 2 : 3.

Two Paths to the Same Peak

Cramér's 1936 conjecture was based on probabilistic heuristics — treating primes as random with density $1/\ln(p)$. Our approach is entirely structural — built from the doubled-interval partition, the Index Bound, and the Residue Framework.

Two independent paths — probabilistic and structural — converging on the same $\ln^2(p)$scaling suggests that Cramér's conjecture captures something real about prime distribution.

Status and Caveats

This argument is conditional:

- **Requires:** Safe Gap Conjecture (Section 9.1) — currently empirical, not proven
- **Requires:** 2× Rule holding asymptotically — observed at 68M, not proven
- **Conclusion:** IF both hold, THEN Cramér's bound follows structurally

We do not claim to have proven Cramér's conjecture. We claim to have identified a structural route: **prove Safe Gap, and Cramér may follow as a corollary.**

Whether this route is passable — whether Safe Gap can be proven from the Residue Framework or other tools — remains open. We offer it to those better equipped for the climb.

* * *

10. Summary: What We Share

We have shared a way of looking at primes not as survivors of a sieve, but as generators that appear when necessity demands. The doubled-prime intervals provide an organizational framework in which certain patterns become visible.

We are explicit about the status of each claim.

What We Proved

• **The Index Bound Theorem:** Interval i requires only $\sqrt{(2i)}$ primes for verification. (Section 6)
• **The Residue Framework:** The Danger Zone Criterion follows from modular arithmetic. (Section 7)
• **The Recursive Bootstrap:** All primes derive from {2, 3} via the Index Bound. (Section 8)
• **The Fulcrum Bound Theorem:** IF the fulcrum conditions hold, THEN gaps are bounded. (Section 9.4)

What We Observed (Empirically Verified, Not Proven)

- **Sharpness:** The bound is achieved at $i = 2$; critical cases are perfect squares of primes.
- **The Balance Property:** Primes split ~50-50 around the fulcrum, with deviation shrinking monotonically from ≈0.009% at 10M to ≈0.003% at 68M, suggesting asymptotic convergence to perfect balance.
- **The Deviation Formula:** Deviation from 50% $\approx -1/(2g)$. Matches data precisely; no proof from first principles.
- **Fulcrum Conditions:** The conditions for the Fulcrum Bound Theorem hold in 63.9% of streak-bounding cases at 68M scale (among streaks with complete boundary info).
- **Predictive Power:** Prime-fulcrum intervals show higher average prime counts than other fulcrum types; their non-emptiness is automatic when the fulcrum itself is prime ($i \geq 2$). (Appendix A, sample scope).
- **Multi-scale threshold tracking:** T(n) stays above the emptiness frontier at 3 of 4 tested scales, with a single outlier interval causing temporary exceedance at 40M before T(n) catches up.

What We Conjecture

- **The Safe Gap Conjecture:** $\mathbf{T(i) = (1/\pi)[\ln(2)\cdot\ln(2i)\cdot\ln(p_i) + 2\cdot\ln(i)]}$. In tested ranges, **T(n)** tracks the emptiness frontier strongly; it is **above the frontier at 10M, 20M, and 68M**, and is **temporarily exceeded at 40M** by a single persistent gap-96 empty interval before the margin turns positive again at 68M. The correction term $\mathbf{(2/\pi)\cdot\ln(i)}$ is essential: without it, the threshold fails at all tested scales.

- **Structural path to Cramér:** If proven, our conjecture would constrain maximum prime gaps. Our threshold is ~22% of Cramér's bound at 68M scale.

Perhaps the deepest reminder from this exploration is that how we look shapes what we see. The primes have always been there, arranged by the logic of multiplication. We offer one lens. Others may find better ones.

Perhaps this is what it means to see primes in Καιρός rather than Χρόνος: not counting uniformly, but recognizing the moments when structure demands a new generator.

We looked, we saw something, we thought about it. Here is our best interpretation. We could be wrong. We hope it is useful.

* * *

Acknowledgments

This work was developed in collaboration with Claude (Anthropic) and ChatGPT 5.2 as AI assistants. The collaboration was genuine: core observations and intuitions are mine; articulation, challenge, and refinement emerged through dialogue. The framework appeared in a paper notebook but took this shape over a month of iterative conversation, coding, observation and interpretation. Much faster and differently than would have been possible alone. All mathematical claims were verified computationally; any errors remain the authors' responsibility.

I thank my mathematics teachers in Bulgaria, Nikola Ginkov, Maria Denkova, Georgi Geninsky, Maxim Yordanov, Daria Marinova and Snezhina Nedyalkova, who shaped my courage to wonder. In my opinion in today's age teachers become more important than ever as they are the ones who first guide us into both rigor and imagination.

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Appendix A: Empirical Data

This appendix provides detailed empirical data supporting the observations in the main text. All data derives from analysis of 68 million consecutive prime pairs unless otherwise noted.

A.1 The Balance Property by Gap Size

The Balance Property (primes splitting ~50-50 around the fulcrum despite asymmetric geometry) varies systematically with gap size:

Gap	Geometric Left	Observed Left	Deviation	−1/(2g)
2	75.0%	50.05%	−24.95%	−25.00%
4	62.5%	49.99%	−12.51%	−12.50%
6	58.3%	50.01%	−8.32%	−8.33%
8	56.25%	50.00%	−6.25%	−6.25%
12	54.2%	50.01%	−4.17%	−4.17%

The empirical deviation formula −1/(2g) matches observations to within 0.05% across all gap sizes tested.

A.2 Empty Intervals by Gap Size

Empty intervals, those containing no primes, concentrate dramatically in small gaps:

Gap	Total Intervals	Empty	Empty %
2 (twins)	1,270	807	63.5%
4	1,263	607	48.1%
6	2,012	530	26.3%
12	1,008	56	5.6%
≥14	varies	few	<3%

For twin prime intervals, only four numbers exist: $2p_i$, $2p_i+1$, $2p_i+2$, and $2p_i+3$. Two are even, leaving only two odd candidates. If both are composite, the interval is empty.

Note: The table above is computed on a 100,000-interval illustrative sample to show the concentration of emptiness in small gaps. Full 68M by-gap emptiness counts are available in the dataset JSON outputs.

A.3 Safe Gap Threshold Verification

Definition (Stable zero gap): At a fixed scale n, we call a gap value g “stable zero” if no interval of gap ≥ g is empty within the analyzed range (equivalently: beyond g, empties never reappear).

The Safe Gap Conjecture predicts a threshold above which all doubled intervals contain at least one prime. Multi-scale verification:

n	p_n	T(n)	⌈T(n)⌉	Max Empty	Stable Zero	Margin
10M	179,424,673	80.75	81	72	74	+8.75
20M	373,587,883	86.93	87	80	82	+6.93
40M	776,531,401	93.33	94	96	98	−2.67
68M	1,358,208,601	98.38	99	96	98	+2.38

The frontier (max empty gap) grows with scale, and T(n) tracks it with positive margin at 3 of 4 scales. At 40M, a single outlier interval (I_{26,235,002} with gap 96) temporarily exceeds the threshold.

The outlier: Interval I_{26,235,002} is defined by consecutive primes 497,575,847 and 497,575,943. The interval [995,151,694, 995,151,886) contains no primes, verified computationally. This is 1 of 44,937 gap-96 intervals (0.002%).

Correction term necessity:

n	T_original	T_corrected	Max Empty	Original fails by
10M	70.49	80.75	72	1.51
20M	76.23	86.93	80	3.77
40M	82.19	93.33	96	13.81
68M	86.90	98.38	96	9.10

Without the correction term (2/π)·ln(n), the threshold fails at all scales.

Cramér comparison at 68M:

- Cramér bound $(\ln p)^2 \approx 442$
- Max gap (all intervals): 288 (65.1% of Cramér)
- Max gap (empty intervals): 96 (21.7% of Cramér)
- Our threshold T(n): 98.4 (22.2% of Cramér)

A.4 Consecutive Empty Interval Streaks

When consecutive intervals are all empty, they form streaks. From 100,000 intervals:

Streak Length	Count	Max Span	Avg Span
1 (isolated)	14,695	32	5.5
2	3,540	42	11.1
3	773	46	16.9
4	167	46	22.7
5	24	54	31.7
6	8	60	34.0
7	2	68	60.0
8	1	52	52.0

Maximum streak span observed: 68 (at streak length 7). This is approximately 1.89× the safe gap threshold of 36, consistent with the "2× rule" discussed in the main text.

The 2× Rule: The maximum streak span observed (68 at streak length 7) is approximately 1.89× the safe gap threshold of 36 at that scale. This ratio — streak span bounded by roughly twice the threshold — appears stable across the tested range and is essential to the traversal argument connecting our framework to Cramér's conjecture (Section 9.5).

A.5 Fulcrum Type Distribution

Scope note: Unless otherwise stated, Appendix A.5 reports **early-range diagnostics (first 10,000 intervals)**; Section 3.3 reports a **100,000-interval diagnostic table**; full-scale **68M** fulcrum counts are reported only as metadata (not as full tables) in the current draft.

Among the first 10,000 intervals:

• Prime fulcrums: 22.8% (compare to ~8-12% for random odd numbers of similar magnitude)
• Square fulcrums: 0.15% (15 instances; in 73% of these, √F is itself prime)
• Protected fulcrums (not divisible by p_i or p_{i+1}): 99.9%

Among non-prime fulcrums, F−2 is prime in 24.2% of cases while F+2 is prime in only 11.3%. This asymmetry arises from mod 3 structure: F+2 is divisible by 3 in 56.6% of cases.

A.6 The Base 12 Revelation

The doubled interval lengths show peaks at 12, 24, 36, 48... with exponential decay (each peak ~58% of the previous). In base 10, these values seem arbitrary, why 12? Why not 10 or 15?

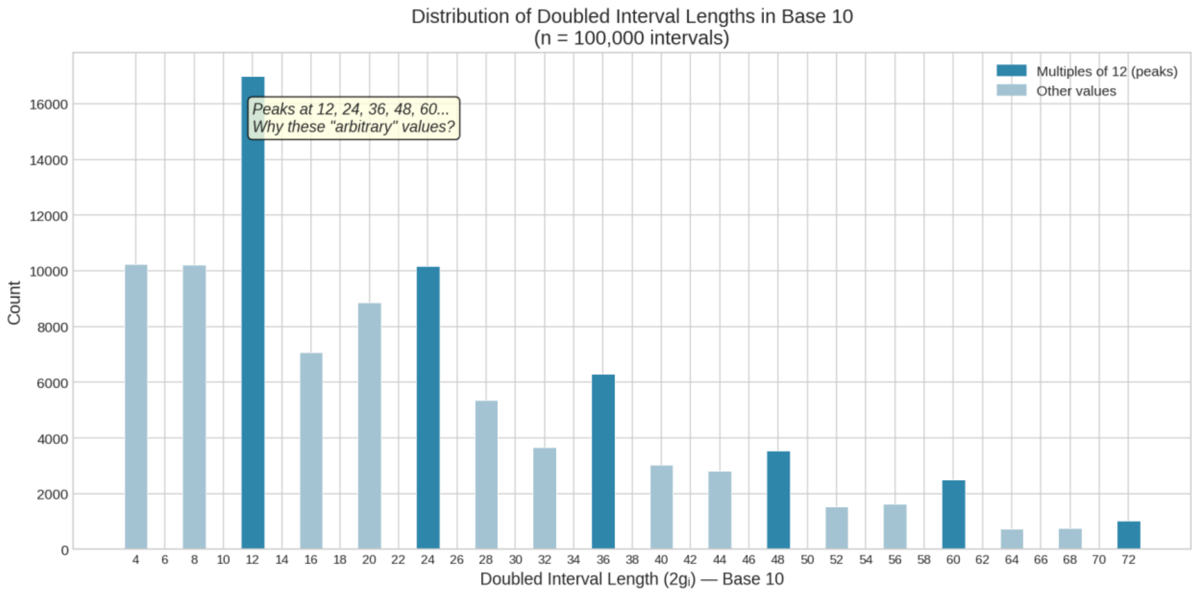

Figure A1: Distribution of doubled interval lengths in base 10. Peaks at 12, 24, 36... appear arbitrary.

The answer becomes obvious when we change representation. In base 12 (dozenal), these same values are 10, 20, 30, 40... The "mysterious" peaks are simply the round numbers.

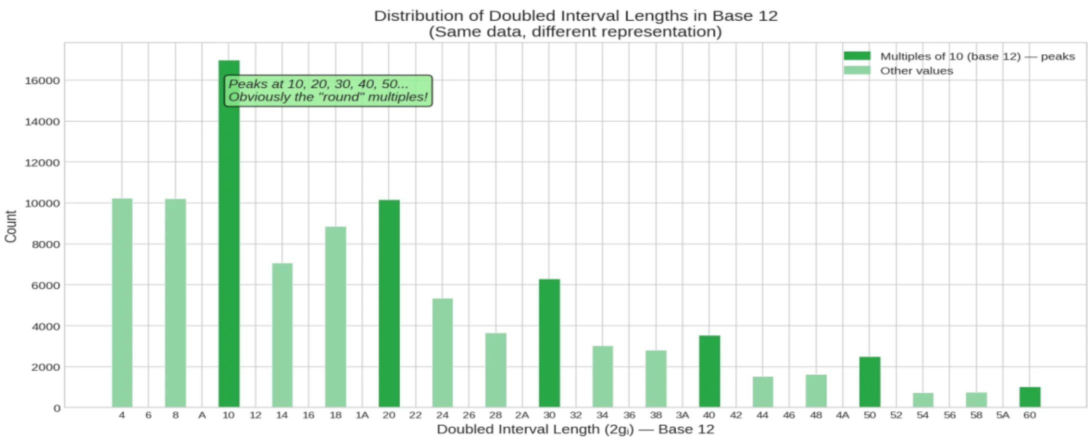

Figure A2: The same data in base 12. Peaks at 10, 20, 30... are obviously the round numbers.

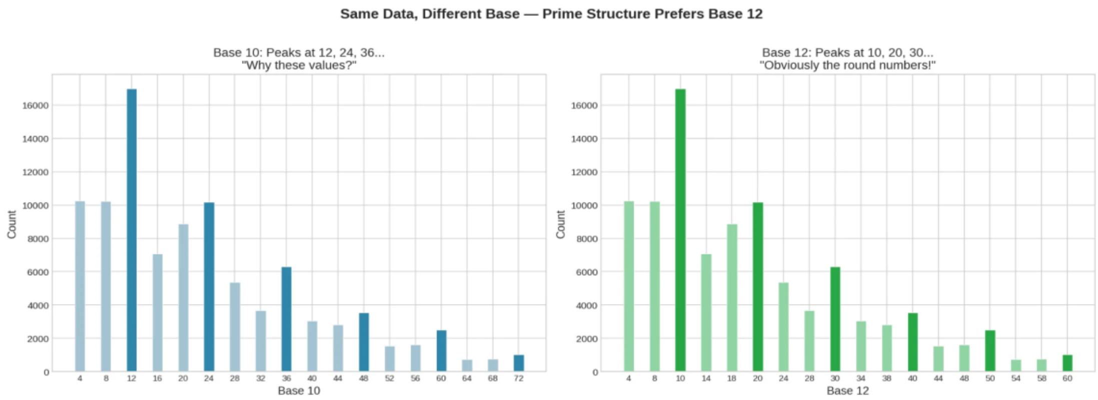

Figure A3: Same data, different base - prime structure prefers base 12.

Why base 12? All primes > 3 satisfy p ≡ ±1 (mod 6), meaning prime gaps are always multiples of 2 or 6. Since 6 divides 12, the primes' natural rhythm aligns with base 12, not base 10.

The mathematics doesn't change, but perception is transformed. This is perhaps the deepest reminder of our investigation: **how we look shapes what we see.** The primes have always preferred base 12. We just weren't looking in the right representation.

Appendix B: Reproducibility

All computations in this paper are deterministic and reproducible.

B.1 Computational Environment

- Language: Python 3.x with standard library only
- Prime generation: Sieve of Eratosthenes
- Primary analysis: 68,000,000 consecutive prime pairs
- Extended analysis: 100,000,000 intervals for some verifications
- Largest prime analyzed: 1,358,208,653 (at i = 68M)

B.2 Code Availability

The following scripts are available as supplementary material:
prime_harvest_v3_1.py — Core interval analysis with streak extraction, balance statistics, and three-pass verification (samples, by-gap totals, streak invariants). Memory-efficient implementation using bisect.
safe_gap_analysis.py — Safe Gap threshold testing with exact p_n from metadata, aggregate consistency check, and transition zone analysis.
fulcrum_analysis.py — Fulcrum condition analysis using pre-computed streaks (instant execution, no prime regeneration).
multiscale_frontier.py — Multi-harvest comparison of emptiness frontiers across scales, with correction term analysis.
find_gap96_empty.py — Utility to locate specific empty intervals by gap.
visualize_frontier.py — Generates charts comparing T(n) vs frontier.

B.3 Verification

Independent verification is encouraged. The Index Bound Theorem can be checked analytically or computationally for any range of intervals. The empirical observations (Balance Property, Safe Gap threshold, streak statistics) can be reproduced by running the provided code.

Any errors in computation or interpretation remain the authors' responsibility.

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