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A SPHERICAL MONTE CARLO APPROACH FOR CALCULATING VALUE-AT-RISK AND EXPECTED SHORTFALL IN FINANCIAL RISK MANAGEMENT

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ABSTRACT

Accurate and efficient calculation for expected values is challenging in finance as well as various disciplines. In general, expected values can be written as high-dimensional integrals. Monte Carlo simulation is an indispensable tool for calculating them, but it is notoriously known for its slow convergence. For spherical distributions, this paper proposes a variance reduction technique and investigates its applications in finance. By using polar transformation, the expected value is written as an integral, and the innermost integral is with respect to the radius and the outermost integral is with respect to the unit sphere. The spherical Monte Carlo estimator is the average of function values of some random points generated by lattice. We consider Value-at-Risk and expected shortfall calculation under heavy-tailed distributions and demonstrate the superiority of the proposed method via numerical studies in terms of variance, computation time, and efficiency.

1 INTRODUCTION

In finance, numerous quantities are related to expected values of a target function for a high-dimensional random vector, which can be written as high-dimensional integrals in general. Calculating high-dimensional integrals is challenging and of critical importance in various disciplines. For instance, the Value-at-Risk (VaR) in risk management is the quantile of a portfolio's loss distribution. Its evaluation requires to calculate the probability that the portfolio's loss distribution exceeding a given threshold.

The Monte Carlo method is an indispensable tool for calculating high-dimensional integrals because it can avoid the curse of dimensionality. Although Monte Carlo method is flexible and widely applicable for calculating high-dimensional integrals, the length for the confidence interval of the Monte Carlo estimator is inversely proportional to the the square root of the sample size. Hence, Monte Carlo method is notoriously known for its slow convergence.

Another approach replies on providing an alternative unbiased estimator with less variance than the original Monte Carlo estimator. This type of approach refers to the variance reduction technique. Useful variance reduction techniques, such as antithetic variates, control variables, moment matching methods, importance sampling, among others, for applications in finance are summarized in Boyle, Broadie, and Glasserman (1997) and Glasserman (2003).

However, variance reduction techniques are either problem dependent or the amount of variance reduction is dissatisfying. Consequently, additional sophisticated analysis is required. To overcome these problems, this paper proposes a unified framework of spherical Monte Carlo approach that is widely applicable without additional mathematical treatments, yet the amount of reduced variance is drastic in many cases. In addition, it is straightforward to combine the proposed spherical Monte Carlo estimator with existing variance reduction techniques, such as conditioning and importance sampling. As a result, additional variance reduction can be obtained.

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To illustrate the proposed method, we consider Value-at-Risk (VaR) and expected shortfall calculation in financial risk management under heavy-tailed distributions. Specifically, this paper focuses on Monte Carlo simulation for calculating VaR (Duffie and Pan 1997). That is, the portfolio's profit and loss is assumed to follow a certain distribution or a stochastic process, and VaR is the quantile of the distribution.

Here, a critical step relates to rare event simulation, and recent studies focus on importance sampling. The basic idea of importance sampling is to select a proper sampling distribution and adjust the sample by importance sampling weight. For example, Glasserman, Heidelberger, and Shahabuddin (2000), Glasserman, Heidelberger, and Shahabuddin (2002), and Fuh et al. (2011) propose exponential tilting measure for multivariate normal and Student-*t* distributions and search suitable tilting parameters based on various criteria. Expected shortfall is another risk measure that receives more attention recently. It is a coherent measure, but its calculation remains challenging (Artzner et al. 1999).

Spherical Monte Carlo method utilizes the geometric feature of spherical distributions, and an integration rule is given by randomly rotating a predetermined set of well-located points. For example, Monahan and Genz (1997) use randomized extended simplex design for the spherical integral and Simpson weights for the radial integral. Deák (1980) and Deák (2000) generate the predetermined set of point by lattice for calculating normal probabilities. Teng, Kang, and Fuh (2015) find an upper bound for the variance the Monte Carlo estimator and suggest a solution for the set of point on the unit sphere which is related to the maximum kissing number in sphere packings. A survey on spherical Monte Carlo method for normal and Student-*t* probabilities can be found in Genz and Bretz (2009).

Spherical distributions include a large class of distribution for random vectors and are indispensable tools in financial applications as exemplified as follows. The spherical distributions extend the standard multivariate normal distribution to multivariate Student-*t* distributions, Laplace distributions, Bessel distributions, Exponential Power distributions, and Logistic distributions, among others (Fang, Kotz, and Ng 1989), and can be generalized to elliptical distributions using Affine transformation.

Spherical distributions are widely used in Markowitz mean-variance approach for portfolio optimization (Campbell, Lo, and MacKinlay 1997). In addition, elliptical copulas are defined via elliptical distributions, and are widely used for pricing complex collateralized debt obligations (Li 2000) and modeling dependent time series data (Patton 2012). For further information, please refer to McNeil, Frey, and Embrechts (2005).

The essential idea of the spherical Monte Carlo approach is to rewrite the high-dimensional integral in Cartesian coordinate to spherical coordinate, so that the innermost integral is with respect to the radius whereas the outermost integral is with respect to the unit sphere. Then, the outermost integral is written using a random orthogonal group which is the unique left-invariant probability measure on the orthogonal group consisting of all $d \times d$ squared matrix. Then, we select a set of points on the unit sphere generated by lattice, multiply them by random radii. The average function values on the resulting points is the proposed spherical Monte Carlo estimator.

The rest of this paper is organized as follows. Section 2 presents the spherical Monte Carlo approach. Section 3 discusses issues that arise in practical implementations. Section 4 conducts numerical studies for VaR calculation under heavy-tailed distribution. Section 5 provide simulation comparisons for calculating expected shortfall. The last section concludes.

2 THE SPHERICAL MONTE CARLO APPROACH

We are interested in calculating the following high-dimensional integral which can be expressed as the expected value of a function $\wp$ with a random vector X,

$$m = \int_{\Re^d} \wp(x) f(x) dx = E[\wp(X)], \tag{1}$$

where $\wp(\cdot)$ is a real-value function from $\Re^d$, and X has a spherical distribution with pdf $f(x)$.

Recall that a random vector has a spherical distribution if it is invariant under rotations, i.e.,

$$VX \stackrel{d}{=} X,$$

for every orthogonal matrix $V \in \Re^{d\times d}$, where $\stackrel{d}{=}$ denotes equality in distribution. Here, V is an orthogonal $d\times d$ matrix that satisfies $VV' = V'V = I_d$ with I_d being the identity matrix of size d. The plum denotes vector or matrix transpose.

It is known that X has a spherical distribution, if and only if it has the stochastic representation

$$X \stackrel{d}{=} RU, \tag{2}$$

where U is uniformly distributed on the unit sphere $S^{d-1} = \{s \in \Re^d : s's = 1\}$, and $R \geq 0$ is a non-negative random variable and is independent of U. For example, when $R \sim \chi_d$, the chi-distribution with degree of freedom d, then X yields the multivariate normal distribution.

Based on the stochastic representation (2) for a spherical distribution, an alternative way to simulate a sample of X is to simulate a unit vector u uniformly rotating on the unit sphere and an independent radius r from the desired distribution, and multiple the above two to obtain a sample.

Eq. (2) yields $m = E[\wp(X)] = E[\wp(R,U)]$. In an integral form,

$$m = E[\wp(R,U)] = \int_{S^{d-1}}\int_{\Re}\frac{1}{\mathrm{Area}(S^{d-1})}\wp(r,u)g(r)drdu,$$

where $g(r)$ is the density function of r. We set the innermost integral to be with respect to the radius, because it is then one-dimensional and can be simplified in some cases. For example, the innermost integral can be calculated explicitly, or by other deterministic numerical integration, like quadrature method.

Now, denote the innermost integral by

$$h(u) = \int_{\Re}\wp(r,u)g(r)dr. \tag{3}$$

Suppose that $h(\cdot)$ is continuous. For a continuous function $h(u)$ on S^{d-1}, and any unit vector $v \in S^{d-1}$, we have

$$\frac{1}{\mathrm{Area}(S^{d-1})}\int_{S^{d-1}}h(u)du = \int_{O(d)}h(Tv)dT, \tag{4}$$

where $T \sim U(O(d))$, and dT is the unique left-invariant probability measure on $O(d)$. Here, $O(d)$ is the orthogonal group consisting of all $d\times d$ matrices D satisfying $DD' = I$ (Stewart 1980, Conway 1990). Recall that group is a set of numbers plus an operation, that is closed, commutative, and inverse element, and unitary element exist.

An alternative stochastic representation for $U \sim U(S^{d-1})$ is

$$U \stackrel{d}{=} Tv, \tag{5}$$

where $T \sim U(O(d))$. This identity means that if we fix a unit vector and let a random orthogonal matrix act on it, the resulting vector is uniformly distributed on S^{d-1}.

Eq. (5) allows a different approach to generate a sample uniformly distributed on S^{d-1} using a random orthogonal group T. One standard approach to generate a random orthogonal group T is to first generate a $d\times d$ matrix, where each entry is a sample of independent standard normal random variable. Then, apply the Gram-Schmit method to this matrix. The resulting matrix is a sample of random orthogonal matrix.

Let V denote a set of point on the unit sphere. A spherical Monte Carlo estimator using V is

$$\hat{m}_V = \frac{1}{|V|}\sum_{v\in V}\wp(r_v, Tv), \tag{6}$$

where $r_v \stackrel{i.i.d.}{\sim} \chi_d$, $T \sim U(O(d))$, and $r_v \perp T$. The spherical Monte Carlo estimator incorporates a set of point on the unit sphere.

3 ISSUES IN PRACTICAL IMPLEMENTATIONS

This section begins with discussions about the selection of points on the unit sphere to construct V, gives measure to compare various Monte Carlo estimators, and gives the software and system specs used for implementing simulation studies.

3.1 How to Select The Point on The Unit Sphere?

A practical question arises in the selection of V. One simple way to construct evenly located points on a unit sphere is through lattice construction. Given a basis of R^d, a lattice L is defined as the set of all integral linear combinations of the basis. Clearly, the origin point belongs to L. To obtain V from a lattice L, we simply collect all the shortest vectors and normalize them to unit vectors, and denote it by V_L.

Let the standard basis be $e_1, \cdots, e_d$ of $\mathbb{R}^d$. We outline several well-known lattices in the following.

- $\mathbb{Z}_d$: $V_{\mathbb{Z}_d} = \{\pm e_1, \cdots, \pm e_d\}$, $|V_{\mathbb{Z}_d}| = 2d$.
- A_d: V_{A_d} consists of all permutations of $\frac{1}{\sqrt{2}}(1, -1, 0, \cdots, 0) \in \mathbb{R}^{d+1}$, $|V_{A_d}| = d(d+1)$.
- D_d: V_{D_d} consists of all permutations of $\frac{1}{\sqrt{2}}(\pm 1, \pm 1, 0, \cdots, 0)$, $|V_{D_d}| = 2d(d-1)$.

Figure 1 depicts two lattices and associated V_L in two dimensions. The left panel dots a lattice, $\mathscr{Z}^2$, which is generated by the standard basis $e_1 = (1,0)$ and $e_2 = (0,1)$. The right panel dots a lattice A_2, which is generate by the basis $v_1 = (\sqrt{3}/2, 1/2)$ and $v_2(0,1)$.

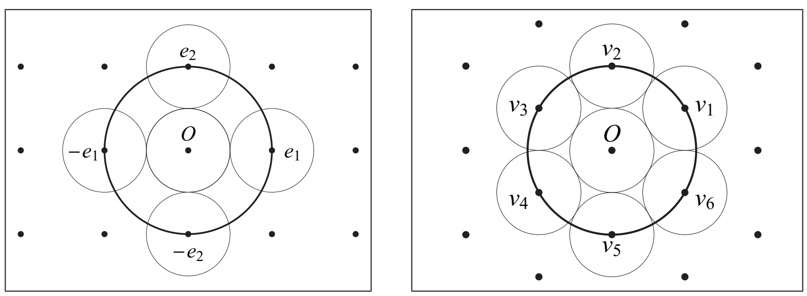

Figure 1: Left panel: Lattice $\mathbb{Z}^2$ and the corresponding $V_{\mathbb{Z}^2}$. Right panel: Lattice A_2 and the corresponding $V_{\mathscr{A}_2}$. Source: Teng, Kang, and Fuh (2015).

For example, in calculating multivariate probabilities, Deák (2000) consider using the points in the unit sphere generated by the lattice D_d. Teng, Kang, and Fuh (2015) suggest to construct V via the maximal kissing number problem in sphere packings. For sphere packing in geometry, a kissing number is defined as the number of non-overlapping unit spheres that can be arranged so that that they each touch another given unit sphere.

Although it is shown that in calculating multivariate normal probabilities, using points in the unit sphere through the maximal kissing number in sphere packing produces least variance among all spherical Monte Carlo estimators, constructing V via the maximal kissing number problem is not generic and can not be obtained automatically. For this reason, we focus on building V via lattice construction with the lattice D_d.

3.2 Measures for Comparisons

To compare the efficiency of various estimator, we first report variance. Variance is a direct measure to compare efficiency of various estimators.

However, implementing the spherical Monte Carlo estimator requires longer time to obtain a sample because it needs the generation of a random orthogonal matrix and calculating the function value of every point in V.

To investigate the total computation time required for a Monte Carlo estimator, let l be a small predetermined precision level. The required minimum sample size in Monte Carlo simulation for the i-th Monte Carlo estimator is

$$N_i \geq \mathrm{Var}(\hat{m}_i)/l^2.$$

Let $\mathscr{T}_i$ denote the total computational time required to run an Monte Carlo estimator to achieve the predetermined precision level l,

$$\mathscr{T}_i = N_i T_i,$$

where T_i is the computation time of implementing one sample of the i-th Monte Carlo estimator.

By simple algebra, the ratio of total computation times between the i-th and j-th Monte Carlo estimators is

$$\frac{\mathscr{T}_i}{\mathscr{T}_j} = \frac{\mathrm{Var}(\hat{m}_i)T_i}{\mathrm{Var}(\hat{m}_j)T_j}.$$

This indicates that the a comparison of total computational times between two Monte Carlo estimators is closely related to the value of product of the variance of an estimator and the running time of implementing one of implementing one sample.

To simplify presentations, we follow LÉcuyer (1994) to report the efficiency number, which is defined as

$$\text{efficiency} = \frac{1}{\text{Variance} \times \text{Computation time}}. \tag{7}$$

Then, it is clear that the ratio of two total computation times between the i-th and j-th Monte Carlo estimators equals

$$\frac{\mathscr{T}_i}{\mathscr{T}_j} = \frac{\text{Efficiency}_j}{\text{Efficiency}_i}.$$

Now, a Monte Carlo estimator with higher efficiency number than the other implies that it requires less total computation time to achieve the same precision level. As a result, we defined that a Monte Carlo estimator is more efficient if it has higher efficiency number.

We note that all the simulation results were programmed using Matlab version 7.11 in a personal laptop (Intel Core i7 CPU M640 with Ram 8.00 GB).

4 CALCULATING VALUE-AT-RISK UNDER HEAVY-TAILED DISTRIBUTIONS

This section is organized as follows. Section 4.1 reviews the calculation of Value-at-Risk under heavy-tailed distributions. We extend the spherical Monte Carlo approach with importance sampling in Section 4.3 and conditioning in 4.4. Section 4.5 summarizes simulation results and concludes. Section 5 provide simulation results for calculating another important risk measure, the expected shortfall.

4.1 Review on The Calculation of Value-at-Risk

Although the term VaR entered the financial lexicon until the early 1990s, the origins of VaR measures can be traced back to an informal capital requirements test by the New York Stock Exchange (NYSE) to member US securities firms around 1922 (Holton 2003).

The Value-at-Risk (VaR) is a critical measure for financial risk management that measures the potential loss in the value of a portfolio. VaR is the market risk measure prescribed in Basel Accord II and III. The Basel accords require financial institutions to retain capital in line with their credit, market and operational risks, and the calculation of the capital is closely related to VaR.

In addition to being used for computing regulatory capital, VaR is used in three categories in financial applications: risk management, financial control, and financial reporting. VaR is also used in non-financial applications (McNeil, Frey, and Embrechts 2005).

For calculating VaR, the exponential smoothing (also known as variance covariance or delta-normal method) and historical simulation are two of the most widespread approaches. The former applies exponentially declined weights to past returns to estimate conditional volatility and invokes a normal assumption to obtain the VaR, whereas the latter estimates VaR directly from the data without making any assumption about a distribution of the portfolio value. Extensions to these methods include a hybrid method in Boudoukh, Richardson, and Whitelaw (1998), for example.

Now, we focus on the Monte Carlo simulation for calculating VaR. Let *PnL* be the random variable to denote the portfolio's profit and loss. Let $F(y) = P(PnL \leq y)$ be the cumulative distribution function of L. Given a confidence level $\alpha \in (0,1)$, VaR of the underlying portfolio's loss at the confidence α, denoted by v_α, is the smallest number such that the probability that the underlying *PnL* exceeds v_α is at least α.

Mathematically speaking, the VaR is the α-quantile satisfying,

$$v_\alpha = \inf\{y : F(y) \geq \alpha\}.$$

A key step in calculating VaR is the calculation of probabilities that *PnL* exceeds a threshold. Once these probabilities are calculated, the VaR can be inverted by standard numerical procedures.

Following Glasserman, Heidelberger, and Shahabuddin (2002) and Fuh et al. (2011) closely, suppose that the portfolio value $V(t,S)$ at time t is exposed to d underlying risk factors $S = (S_1, \ldots, S_d)'$. Let ΔS denote the change in S from the current time t to the end of the horizon time $t + \Delta t$. *PnL* is approximated by the delta-gamma method,

$$L = V(t,S) - V(t + \Delta t, S) \approx a_0 + a'\Delta S + \Delta S' A \Delta S, \tag{8}$$

where $a_0 = -\partial V(t,S)/\partial t$ is a scalar, $a = (a_1, \ldots, a_d)'$ is a d-variate vector with $a_i = -\partial V(t,S)/\partial S_i$, and $A = [A_{ij}]$ is a $d \times d$ matrix with $A_{ij} = -\partial^2 V(t,S)/(2\partial S_i \partial S_j)$. Here, all derivatives are evaluated at the initial point (t,S). In practice, parameters a_0, a, and A are given as known values.

For ease of notations, we denote $Q = a'\Delta S + \Delta S' A \Delta S$ and focus on the calculation of the probability that Q exceeds a given threshold q, i.e.,

$$m = P(Q > q). \tag{9}$$

Assume ΔS is a d-variate multivariate Student-t distribution with degrees of freedom $\nu > 0$ and a positive definite covariance matrix Σ. Then, ΔS has the stochastic representation as

$$\Delta S \stackrel{d}{=} \frac{\xi}{\sqrt{Y/\nu}},$$

where $\xi \sim N_d(0, \Sigma)$, $Y \sim \chi^2_\nu$ (a chi-squared distribution with ν degrees of freedom), and, ξ and Y are independent.

In the following, we rewrite the elliptical distribution ΔS as an affine transformation of a spherical distribution X. Let C be the square root of the positive definite covariance matrix Σ, such that $\Sigma = C'C$ and $C'AC = \Lambda$ is diagonalized with diagonal element $\lambda_1, \ldots, \lambda_d$. Then, ΔS has the stochastic representation as CX,

$$\Delta S \stackrel{d}{=} CX,$$

where $X \sim t_{d,\nu}$ is the standardized d-variate Student-t distribution with ν degrees of freedom. Recall that X also has the stochastic representation as

$$X \stackrel{d}{=} Z\sqrt{\nu/Y},$$

where $Z \sim N_d(0, I_d)$, $Y \sim \chi^2_\nu$, and Z is independent of Y. Standard algebra gives

$$Q = a'\Delta S + \Delta S' A \Delta S = a'CX + (CX)'A(CX) = b'X + X'\Lambda X,$$

where $b = a'C$.

The crude Monte Carlo estimator of m in Eq. (9) is

$$\hat{m}_0 = I_{\{b'X + X'\Lambda X > q\}}(Z,Y), \tag{10}$$

where $X = Z\sqrt{\nu/Y}$, $Y \sim \chi^2_\nu$, $Z \sim N_d(0, I_d)$, and Y and Z are independent. Here, $I_{\{A\}}(\cdot)$ is the indicator function with support set A.

4.2 The Spherical Monte Carlo Estimator

The spherical Monte Carlo estimator is

$$\hat{m}_V = \frac{1}{|V|}\sum_{v\in V} I_{\{b'X_v + X_v'\Lambda X_v > q\}}(R_v, T, Y_v), \tag{11}$$

where $X_v = R_v T v\sqrt{\nu/Y_v}$, $R_v \sim \chi_d$, $T \sim U(O(d))$, $Y_v \sim \chi^2_\nu$, for $v \in V$. Here, R_v, T, and Y_v are mutually independent.

4.3 Importance Sampling

Importance sampling seeks a change of probability measure and is useful for rare event simulation. Its implementation relies on the statistics of interest and the support set, and is problem dependent. Both Glasserman, Heidelberger, and Shahabuddin (2002) and Fuh et al. (2011) consider VaR calculation under heavy-tailed distributions. Although the importance sampling estimator in Fuh et al. (2011) is more efficient than that in Glasserman, Heidelberger, and Shahabuddin (2002) for moderate event simulation, searching the optimal tilting parameters needs to calculate several complicated and high-dimensional integrals. For these reasons, we follow the importance sampling estimator in Glasserman, Heidelberger, and Shahabuddin (2000).

In the following, we outline the importance sampling estimator in Glasserman, Heidelberger, and Shahabuddin (2002) but omit tedious calculation. Set

$$T_X = \frac{(b'X + X'\Lambda X - q)Y}{\nu}. \tag{12}$$

Standard algebra gives the cumulant function of T_X,

$$\psi(\theta) = \log(E[\exp^{T_X\theta}]) = -\frac{1}{2}\left(\nu\log(1-2\alpha(\theta)) + \sum_{i=1}^{d}\log(1-2\theta\lambda_i)\right), \tag{13}$$

where

$$\alpha(\theta) = -\frac{q\theta}{\nu} + \frac{1}{2\nu}\sum_{i=1}^{d}\frac{\theta^2 b_i^2}{1-2\theta\lambda_i}. \tag{14}$$

Let $\Gamma_{a,b}$ denote the Gamma distribution with shape parameter a and scale parameter b. The importance sampling estimator is

$$\hat{m}_\theta = I_{\{b'X + X'\Lambda X > q\}}(Z,Y)\mathrm{e}^{-T_X\theta + \psi(\theta)}, \tag{15}$$

where $X = Z\sqrt{\nu/Y}$, $Y \sim \Gamma\left(\frac{\nu}{2}, \frac{2}{1-\alpha(\theta)}\right)$, and $Z \sim N_d(\mu(\theta), \Sigma^2(\theta))$. Here, the mean vector $\mu(\theta) = (\mu_1(\theta), \ldots, \mu_d(\theta))'$ and the $d \times d$ diagonal matrix $\Sigma(\theta) = [\Sigma_{ij}(\theta)]$ are given as follows,

$$\mu_j(\theta) = \frac{b_j\theta}{1-2\theta\lambda_j}, \tag{16}$$

$$\Sigma^2_{jj}(\theta) = \frac{1}{1-2\theta\lambda_j}, \tag{17}$$

for $j = 1,\ldots,d$. θ is chosen to minimize an upper bound of the variance of the importance sampling estimator, and is consequently the root of $\psi'(\theta) = 0$, i.e.,

$$\frac{-q + \sum_{i=1}^{d} \frac{\theta b_i^2 (1-\theta\lambda_i)}{(1-2\theta\lambda_i)^2}}{1-2\alpha(\theta)} + \sum_{i=1}^{d} \frac{\lambda_i}{1-2\theta\lambda_i} = 0. \tag{18}$$

Let $\Gamma(\theta)$ be the lower triangular matrix satisfying $\Gamma(\theta)'\Gamma(\theta) = \Sigma(\theta)$, i.e., the Cholesky decomposition of $\Sigma(\theta)$. Because $Z \sim N_d(\mu(\theta),\Sigma(\theta))$ for $\hat{m}_\theta$ in (15), we replace Z with $\mu(\theta) + R_v\Gamma(\theta)T_v$ for $v \in V$ to obtain the spherical Monte Carlo estimator with importance sampling,

$$\hat{m}_{V,\theta} = \frac{1}{|V|}\sum_{v} I_{\{b'X_v + X_v'\Lambda X_v > 0\}}(R_v, T, Y_v)\exp^{-T_{X_v}\theta + \psi(\theta)}, \tag{19}$$

where $X_v = (\mu(\theta) + R_v\Gamma(\theta)Tv)\sqrt{\nu/Y_v}$, $T_{X_v} = (X_v'b + X_v'\Lambda X_v - q)Y_v/\nu$, $R_v \sim \chi_d$, $T \sim U(O(d))$, $Y_v \sim \Gamma_{\frac{\nu}{2},\frac{2}{1-\alpha(\theta)}}$ for $v \in V$. Here, T, R_v, and Y_v are mutually independent.

4.4 Conditioning

Recall that when $X \sim t_{d,\nu}$, we have the following stochastic representations:

$$X \overset{d}{=} Z\sqrt{\nu/Y} \overset{d}{=} RU\sqrt{\nu/Y},$$

where $Z \sim N_d(0, I_d)$, $R \sim \chi_d$, $U \sim U(S^{d-1})$, and $Y \sim \chi^2_\nu$. Now, define

$$F = R\sqrt{\nu/Y}. \tag{20}$$

Then, the support set for $\hat{m}_0$ in Eq. (10) equals a union of two disjoint intervals,

$$\begin{aligned}\{X : b'X + X'\Lambda X > q\} &= \{F : b'(FU) + (FU)'\Lambda(UF) - q > 0\}\\ &= \{F : F^2(U'\Lambda U) + F(b'U) - q > 0\}\\ &= \{F : F < \alpha(U), F > \beta(U)\},\end{aligned}$$

where

$$a(u) = \frac{-(b'u) - \sqrt{(b'u)^2 + 4(u'\Lambda u)q}}{2u'\Lambda u} \text{ and } b(u) = \frac{-(b'u) + \sqrt{(b'u)^2 + 4(u'\Lambda u)q}}{2u'\Lambda u} \tag{21}$$

are roots of the quadratic equation $F^2(u'\Lambda u) + F(b'u) - q = 0$.

Let $\mathscr{F}_{d_1,d_2}$ and $\mathscr{F}_{d_1,d_2}(\cdot)$ denote the F distribution with parameters d_1 and d_2 and its cdf, respectively. By the definition of F in (20), we obtain

$$P(F < a(u)) = P\left(R\sqrt{\nu/Y} < a(u)\right) = P\left(\sqrt{\frac{R^2/d}{Y/\nu}}\sqrt{d} < a(u)\right) = P(\mathscr{F}_{d,\nu} < a(u)^2/d) = \mathscr{F}_{d,\nu}(a(u)^2/d),$$

where the last equality holds because of $\frac{R^2/d}{Y/\nu} \sim \mathscr{F}_{d,\nu}$. Similarly, we have

$$P(F > b(u)) = P(\mathscr{F}_{d,\nu} > b(u)^2/d) = 1 - \mathscr{F}_{d,\nu}(b(u)^2/d).$$

As a result, the conditioning estimator is

$$\hat{m}^* = \mathscr{F}_{d,\nu}(a(U)^2/d) + 1 - \mathscr{F}_{d,\nu}(b(U)^2/d),$$

where $U \sim U(S^{d-1})$, and $a(\cdot)$ and $b(\cdot)$ are given in Eq. (21). It is straightforward to obtain spherical Monte Carlo estimator with conditioning as

$$\hat{m}_V^* = \frac{1}{|V|} \sum_{v \in V} \left(\mathscr{F}_{d,\nu} \left(a(Tv)^2/d \right) + 1 - \mathscr{F}_{d,\nu} \left(b(Tv)^2/d \right) \right), \tag{22}$$

where $T \sim U(O(d))$.

4.5 Simulation Results

Figure 2 compares the variance and efficiency number for calculating probabilities of loss exceeding a given threshold in 15 dimensions for five estimators: the crude Monte Carlo estimator $\hat{m}_0$ in Eq. (10), importance sampling estimator $\hat{m}_\theta$ in Eq. (15), spherical Monte Carlo estimator $\hat{m}_V$ in Eq. (11), spherical Monte Carlo estimator with importance sampling $\hat{m}_{V,\theta}$ in Eq. (19), and spherical Monte Carlo estimator with conditioning $\hat{m}_V^*$ in Eq. (22).

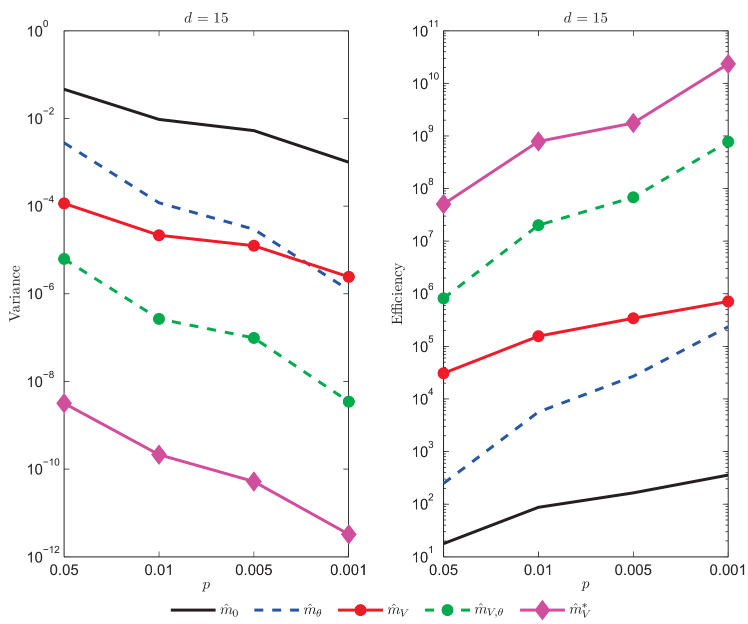

Figure 2: Value-at-Risk calculation under heavy-tailed distributions. Settings: $\nu = 3$, $b_i = 0.1 + i/100$, $\lambda_i = 0.05i$ for $i = 1, \ldots, 15$. Threshold = [52.58, 161.61, 259.3, 762.8], and $p = [0.05, 0.01, 0.005, 0.001]$.

In terms of variance, $\hat{m}_V^*$ is the most competitive, followed by $\hat{m}_{V,\theta}$, $\hat{m}_V$, $\hat{m}_\theta$, and $\hat{m}_0$. It is interesting that spherical Monte Carlo estimators in general produce less variance. This is because that the set V constructed via the lattice D_d allows to generate random samples more evenly on the space, so that a good amount of variance reduction is obtained.

Recall that antithetic variates use two opposite points against the original point to form an estimator. The spherical Monte Carlo estimator essentially generalizes the idea of antithetic variates, in the sense that antithetic variates can be regarded as a special case of the spherical Monte Carlo estimator where V contains just two points, $V = \{e_1, -e_1\}$.

Although $\hat{m}_V$ and $\hat{m}_\theta$ are comparable, a deeper investigation finds that $\hat{m}_\theta$ is more efficient than $\hat{m}_V$ in calculating smaller probabilities. A combination of importance sampling and spherical Monte Carlo leads to $\hat{m}_{V,\theta}$, which outperforms both $\hat{m}_V$ and $\hat{m}_\theta$.

The variance reduction of $\hat{m}_V^*$ over $\hat{m}_V$ is justified by the Raw-Blackwell theorem, yet the amount of variance reduction is promising, i.e., to about a multiple of 10^4. Indeed, $\hat{m}_V^*$ is the most competitive estimator, because it only involves the generation of unit vector but not the radius, and the amount of variance is drastic as illustrated via simulation.

In terms of efficiency, similar conclusion can be drawn: $\hat{m}_V^*$ remains dominant, followed by $\hat{m}_{V,\theta}$, $\hat{m}_V$, $\hat{m}_\theta$, and $\hat{m}_0$. The improvement in efficiency of $\hat{m}_V^*$ over $\hat{m}_0$ is dramatic, to about a multiple of 10^7.

The above simulation results highlight the merit of adopting the idea of spherical Monte Carlo approach for rare event simulation. First, $\hat{m}_V$ produces less variance than $\hat{m}_0$. Second, it is simple to combine the idea of spherical Monte Carlo with existing variance reduction techniques, such as importance sampling in $\hat{m}_{V,\theta}$ and conditioning in $\hat{m}_V^*$. These extended spherical estimators yield additional variance reduction.

5 EXPECTED SHORTFALL

Expected shortfall is another risk measure receiving more attention recently. Expected shortfall is also known as conditional VaR (cVaR), Average VaR (AVAR), expected tail loss (ETL), or tail VaR. Expected shortfall is a coherent measure of financial portfolio risk, and is defined as the expected loss of portfolio value given that a loss is occurring at or below the α-quantile (Artzner et al. 1999).

Following the distribution assumptions on the portfolio loss in Section 4, expected shortfall is mathematically

$$\text{Expected shortfall} = E\left[a_0 + a'\Delta S + \Delta S' A \Delta S \,|a_0 + a'\Delta S + \Delta S' A \Delta S > v_\alpha\right].$$

Recall that $Q = a'\Delta S + \Delta S' A \Delta S$. For simplicity sake, we focus on the calculation of

$$m = E\left[Q I_{\{Q>q\}}\right], \tag{23}$$

with q being a given threshold. Then, expected shortfall can be calculated as m in Eq. (23) divided by $P(Q > q)$, which can be calculated efficiently by various spherical Monte Carlo approaches provided in Section 4.

Because it is not clear how to apply existing variance reduction techniques in simulating expected shortfall, we simply consider the crude Monte Carlo estimator and the proposed spherical Monte Carlo estimator. Now, the crude Monte Carlo estimator of m in Eq. (23) is

$$\hat{m}_0 = (b'X + X'\Lambda X > q) I_{b'X + X'\Lambda X > q}(Z, Y), \tag{24}$$

where $X = Z\sqrt{\nu/Y}$, $Y \sim \chi^2_\nu$, $Z \sim N_d(0, I_d)$, and Y and Z are independent.

The spherical Monte Carlo estimator of m in Eq. (23) is

$$\hat{m}_V = \frac{1}{|V|} \sum_{v \in V} \left(b'X_v + X_v'\Lambda X_v\right) I_{\{b'X_v + X_v'\Lambda X_v > q\}}(R_v, T, Y_v), \tag{25}$$

where $X_v = R_v T v \sqrt{\nu/Y_v}$, $R_v \sim \chi_d$, $T \sim U(O(d))$, $Y_v \sim \chi^2_\nu$, for $v \in V$. Here, R_v, T, and Y_v are mutually independent.

Figure 3 compares variance and efficiency number between $\hat{m}_0$ in Eq. (24) and $\hat{m}_V$ in Eq. (25) at various parameter settings. It is shown clearly that the spherical Monte Carlo estimator $\hat{m}_V$ enjoys less variance and than the crude Monte Carlo estimator $\hat{m}_0$ to about a multiple of 10^3. In terms of efficiency number, the improvement of $\hat{m}_V$ over $\hat{m}_0$ is also drastic, to about a multiple of 10^3.

6 CONCLUDING REMARKS

Spherical Monte Carlo approaches utilize the symmetric property of spherical distributions, and have been applied to Bayesian inference and multivariate normal probability calculation. However, the applications of spherical Monte Carlo approaches in finance remains unexplored. To bridge the gap in literature, this paper

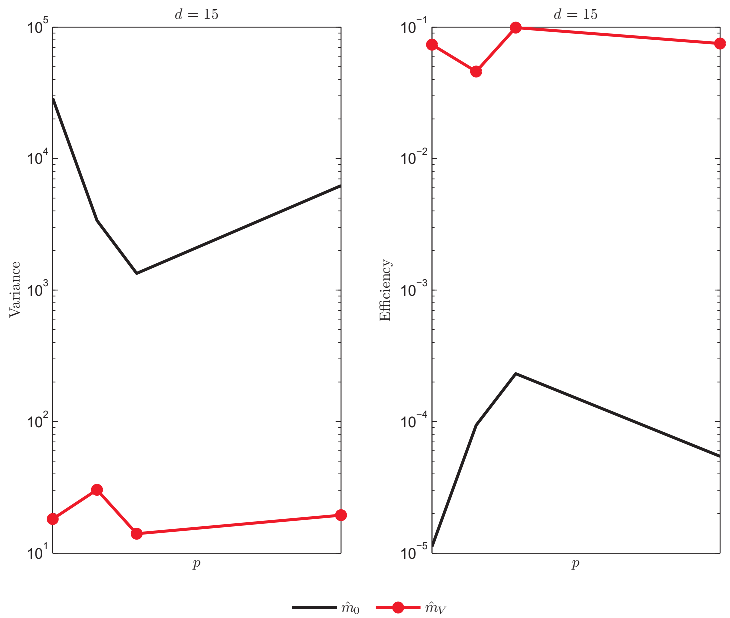

Figure 3: Expected shortfall calculation under heavy-tailed distributions. Settings: $\nu = 3$, $b_i = 0.1 + i/100$, $\lambda_i = 0.05i$ for $i = 1, \ldots, 15$. Threshold = [52.58, 161.61, 259.3, 762.8], and $p = [0.05, 0.01, 0.005, 0.001]$.

proposes a spherical Monte Carlo approach and demonstrates its substantial improvements in computation efficiency via VaR and expected shortfall calculation under heavy-tailed distributions.

The core research question in this paper is on the calculation of high-dimensional integrals, which remains challenging and timely in various scientific disciplines. Through this line of research, extensions including a construction of a spherical integration rule using group representation to reduce the cost in generating a random orthogonal matrix, will be studied in separate papers. Spherical integration rules using unequal weight is worthy of future studies.

Another category of future extension is to apply the proposed spherical Monte Carlo method to the use of extreme value theory (such as generalize Pareto distribution) for simulating risk factors. Last but not the least, it is interesting to investigate the applicability of spherical Monte Carlo methods in various fields, such as hypothetical testing in statistics and critical values of complicated statistics in radar detection.

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