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On-line VWAP Trading Strategies

Cheng-Der Fuh[1], Huei-Wen Teng[2], and Ren-Her Wang[3]

[1]Graduate Institute of Statistics, National Central University, Jhongli and Institute of Statistical Science, Academia Sinica, Taipei, Taiwan
[2]Department of Statistics, Pennsylvania State University, Pennsylvania, USA
[3]Department of Finance, National Taiwan University, Taipei, Taiwan

Abstract: VWAP stands for volume-weighted average price during a certain trading period, and a VWAP trade refers a trade that uses VWAP as a benchmark. This article provides on-line execution strategies for a VWAP trade. We first propose a simulation-based statistical price-volume model that enables the VWAP to be reformulated as a combination of two Brownian motions. Then we introduce the dynamic programming algorithm as the comparison of our trading strategies. Since the dynamic programming algorithm is model dependent and computationally intensive, we present some simple alternative strategies for VWAP trading. Among these, we first propose a modified cross-boundary strategy which can be shown as an asymptotic approximation of the dynamic programming strategy. Next, we introduce a relative rank strategy that ignores the actual stock price and considers only the stock price's relative rank. Our simulation results show that the (modified) cross-boundary strategy is better for stock prices with negative drift, whereas the relative rank strategy performs well for stock prices with positive drift. As a result, when a trader faces a long trading horizon with multi-periods and variate drifts, a hybrid algorithm based on the intra-day drift trend is proposed. Finally, to demonstrate the robustness of the trading strategies relative to the price-volume model, and evaluate the effect of VWAP trading, we present an empirical study of the trading strategies on the top 20 liquidity stocks on the Taiwan Stock Exchange Corporation.

Keywords: Boundary crossing; Dynamic programming; Price-volume model; Technical analysis; Trading strategy.

Subject Classifications: 62L10; 91B60; 91B24.

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Recommended by A. G. Tartakovsky
Address correspondence to Cheng-Der Fuh, Institute of Statistical Science, Academia Sinica, 128, Academia Rd. Sec. 2, Taipei 115, Taiwan; E-mail: stcheng@stat.sinica.edu.tw

1. INTRODUCTION

One of the most common activities in modern financial markets is known as volume-weighted average price (VWAP) trading. Informally, the VWAP of a stock over a specified market time period is simply the average price paid per share during that period. In VWAP trading, one attempts to buy or sell a fixed number of shares at a price that closely tracks the VWAP. Berkowitz et al. (1988) regards the VWAP benchmark as a good approximation of the price for a passive trader. Its computational simplicity is a major advantage, especially in markets where detailed trade level data are difficult or expensive to obtain. Although the volume might affect the value of VWAP, for instance, when there is a large volume after the selling of a VWAP trade, VWAP benchmarks are prevalent in United States, Japan, and continental Europe. The reader is referred to Madhavan (2002) for various definitions of VWAP.

The newest method of achieving VWAP is to use automated trading strategies to participate proportionately throughout the trading. This strategy is called a *fixed-schedule VWAP algorithm*. Fixed-schedule VWAP algorithms, or slight variants of them, are surprisingly common in practice. The problem with such strategies based on time patterns is that the volume pattern on any given day can depart significantly from the historical average. Kakade et al. (2004) demonstrates that such algorithms can perform considerably worse in terms of the worst case competitive ratio and provides an on-line model for the VWAP trading using the competitive ratio analysis. For the competitive ratio analysis of the on-line algorithm, the reader is referred to Borodin and El-Yaniv (1998) and El-Yaniv et al. (2001) for details. This observation motivates us to apply on-line algorithmic techniques to a realistic investment problem.

We first consider the relation between stock price and volume. Note that in empirical finance [cf. Karpoff, 1987 and Gallant et al., 1992], the role of volume in the market has been a long standing subject of interest. Blume et al. (1994) investigates the informational role of volume and its applicability for technical analysis. Then they develop a new equilibrium model in which aggregate supply is fixed and traders receive signals with differing quality, and show that volume provides information on information quality that cannot be deduced from the price statistic. The authors also illustrate how volume, information precision and price relate. Related literature on price-volume relation can be found in Admati and Pfleiderer (1988), Jain and Joh (1988), Andersen (1996), Konishi (2002), and references therein.

Motivated by the computational demands of VWAP trading, this article provides a simulation-based statistical price-volume model that is analogous to Blume, Easley, and O'Hara's model. Moreover, based on our proposed price-volume model, VWAP can be reformulated as a moving target, which can be seen in Section 2, as a combination of two Brownian motions. Hence, we provide a method to analyze the VWAP trading strategy via the optimal portfolio selection scheme. For example, see Shepp (1969), Merton (1969), Merton (1971), and Liu and Loewenstein (2002) for dynamic programming analysis, see Browne (1999) for optimal portfolio selection with stochastic benchmark, and see Lo et al. (2000) for a statistical method on technical analysis.

However, unlike the portfolio selection problem mentioned above, VWAP trading requires the trader to sell a fixed amount of one specific stock within a finite trading horizon. This constraint requires a new formulation. To simplify the problem caused by liquidity and computation, we break up the trading period into smaller trading units, such as trading days for instance. At the end of each trading unit, a fixed proportion of the volume is forced to be sold. This implies that the strategies considered in this article are based on each trading unit only. Now in each trading unit, we introduce the dynamic programming (backward induction) algorithm as the comparison of our trading strategies. Since the dynamic programming algorithm is model dependent and computationally intensive, we present some simple alternative strategies for VWAP trading. We first propose a modified cross-boundary strategy, which can be shown as an asymptotic approximation of the dynamic programming strategy. Next, we present a relative rank strategy that ignores the actual stock price and considers only the stock price's relative rank.

Based on the criteria of winning probability and expected revenue on VWAP trading, our simulation results show that the (modified) cross-boundary strategy is better for stock prices with negative drift; whereas the relative rank strategy performs well for stock prices with positive drift. As a result, when a trader faces a long trading horizon with multi-periods and variate drifts, a hybrid algorithm based on the intra-day drift trend is proposed. Similar phenomena can be found in Graversen et al. (2000) and Du Toit and Peskir (2007), which consider optimal prediction problems in Brownian motion.

The remainder of this article is organized as follows. In Section 2, we give a formal definition of VWAP and propose a simulation-based statistical price-volume model. Section 3 introduces the dynamic programming algorithm, proposes a modified cross-boundary strategy and presents a relative rank strategy. Simulation results are reported in Section 4, in which we also propose a hybrid strategy that combines the modified cross-boundary strategy and the relative rank strategy. As an illustration of our method, to demonstrate the robustness of the trading strategies relative to the price-volume model, and evaluate the effect of VWAP trading, an empirical study of the trading strategies on the top 20 liquidity stocks from the Taiwan Stock Exchange Corporation (TSEC) is given in Section 5. Conclusions are given in Section 6. Asymptotic optimality of the cross boundary strategy is given in the appendix.

2. VWAP AND PRICE-VOLUME MODEL SETTING

Consider the case that in a trading period from time 0 to time T, a trader would like to sell a certain number of shares and tries to get the average purchase price as close as possible to, with a goal of a little bit better than, the market VWAP. Let $\{\Omega, \mathcal{F}, \mathcal{F}_t, P\}$ be a filtered probability space that satisfies the usual conditions, and the stock price is defined as a random variable in the probability space Ω. Let T be the number of trading days, and n be the number of monitors within one trading day. Let S_0^t denote the initial stock price at the tth trading day and let S_i^t denote the stock price at the ith monitor on the tth trading day for $i = 1, \ldots, n$ and $t = 1, \ldots, T$. Let m_i^t and c_i^t be the market volume and the volume the trader sells, respectively, at the ith monitor on the tth day.

In the price-volume trading model, the stock price and the market volume are observed in pairs (S_i^t, m_i^t). Let $m = \sum_{t=1}^{T} \sum_{i=1}^{n} m_i^t$ and $c = \sum_{t=1}^{T} \sum_{i=1}^{n} c_i^t$. Denote Γ as the strategy to allocate c_i^t for VWAP trading. The market VWAP and the VWAP of the strategy Γ can be defined, respectively, as

$$\mathrm{VWAP} = \frac{\sum_{i=1}^{T}\sum_{i=1}^{n} S_i^t m_i^t}{m}, \tag{2.1}$$

$$\mathrm{VWAP}_\Gamma = \frac{\sum_{i=1}^{T}\sum_{i=1}^{n} S_i^t c_i^t}{c}. \tag{2.2}$$

The two criteria of winning rate (WR) (or winning probability) and expected revenue (ER) can be defined as

$$\mathrm{WR} = P(\mathrm{VWAP}_\Gamma - \mathrm{VWAP} \geq 0), \quad \mathrm{ER} = E(\mathrm{VWAP}_\Gamma - \mathrm{VWAP}). \tag{2.3}$$

An accurate estimate of the volume distribution, such as the evaluations of the value of VWAP trading in model (2.1) and (2.2), is a key element for a successful automated participation strategy. Blume et al. (1994) develops a Walrasian equilibrium model for the price and trading volume. The idea of their setting is to consider a repeated asset market in which agents can trade a risk-free and a risky asset. They assume that all trade is between the agents; and there is no exogenous supply of any asset. Each agent maximizes a negative exponential utility function.

The assets' eventual value is given by a normal random variable ψ, with mean ψ_0 and variance $1/\rho_0$. All traders initially have $N(\psi_0, 1/\rho_0)$, as their common prior on the asset value. Denote N as the total number of traders. Suppose there are two groups of traders, with $N_1 = \alpha N$ traders in group 1 (the informed group) and $N_2 = (1-\alpha)N$ in group 2 (the uninformed group), where $0 < \alpha < 1$ is a proportional rate (assume N_1 and N_2 are integers). The traders in each group receive signals from a common distribution, but the distributions are different for the two groups. Formally, each informed trader i in group 1, $i = 1, \ldots, N_1$, receives a signal at date t of $y_t^i = \psi + \omega_t + e_t^i$ where ω_t is a common error term with distribution $N(0, l/\rho_\omega)$. The e_t^i represents an idiosyncratic error with distribution $N(0, 1/\rho_t^1)$. Similarly, trader i in group 2, $i = N_1 + 1, \ldots, N$, receives signal $y_t^i = \psi + \omega_t + \varepsilon_i^t$, where each $\varepsilon_t^i \backsim N(0, 1/\rho^2)$. Note that ρ^2 is fixed (and known) to reduce the complexity.

The precision of group 1's signals (ρ_t^1) are random variables. All parameters other than the ρ_t^1's are known to all traders, but each ρ_t^1 is known only to traders in group 1. To make it easier to write asset demands, note that for traders in group 1 each signal y_t^i is distributed $N(\psi, 1/\rho_t^{s1})$ where $\rho_t^{s1} = \rho_\omega\rho_t^1/(\rho_\omega + \rho_t^1)$. Similarly, for traders in group 2 each y_t^i is distributed $N(\psi, 1/\rho^{s2})$ where $\rho^{s2} = \rho_\omega\rho^2/(\rho_\omega + \rho^2)$. Conditional on ω_t, each y_t^i is distributed $N(\theta_t, 1/\rho_t^1)$ for traders in group 1 and $N(\theta_t, 1/\rho^e)$ for traders in group 2, where $\theta_t = \psi + \omega_t$. Therefore, by the strong law of large numbers, the mean signal in each group, $\bar{y}_t^1 := \frac{1}{N_1}\sum_{i=1}^{N_1} y_i^t$ and $\bar{y}_t^2 := \frac{1}{N-N_1}\sum_{i=N_1+1}^{N} y_i^t$, converges almost surely to θ_t as $N \to \infty$.

Using Bayesian analysis, the argument for revelation of information through market statistics can be constructed inductively. That is, suppose that market statistics through period $t-1$ are revealing. Then upon entering trading period t, the traders' common prior is a normal random variable with mean $\bar{\theta}_{t-1} = [\rho_\omega \sum_{\tau=1}^{t-1}\theta_\tau + \rho_0\psi_0]/[(t-1)\rho_\omega + \rho_0]$ and variance $(\hat{\rho}_{t-1})^{-1} = [(t-1)\rho_\omega + \rho_0]^{-1}$. Assume that traders have myopic, or naive, demands so that each trader chooses his/her demand to maximize expected utility on a period by period basis.

Denoting the price of the risky asset by p_t, the t period demand for the risky asset for each trader i in group 1 is given by $\hat{\rho}_{t-1}(\bar{\theta}_{t-1} - p_t) + \rho_1^{s1}(y_t^i - p_t)$ and by $\hat{\rho}_{t-1}(\bar{\theta}_{t-1} - p_t) + \rho_1^{s2}(y_t^i - p_t)$ for each trader i in group 2.

For a large economy, the equilibrium period t stock price is given by

$$p_t = \frac{\hat{\rho}_{t-1}\bar{\theta}_{t-1} + (\alpha\rho_t^{s1} + (1-\alpha)\rho^{s2})\theta_t}{\hat{\rho}_{t-1} + \alpha\rho_t^{s1} + (1-\alpha)\rho^{s2}}. \tag{2.4}$$

Per capita volume in period t is

$$\begin{aligned} V_t = \frac{\mu}{2}\big[2(x_t^1)^{-1}\phi(\delta_t^1 x_t^1) + \delta_t^1(\Phi(\delta_t^1 x_t^1) - \Phi(-\delta_t^1 x_t^1))\big] \\ + \frac{1-\mu}{2}\big[2(x^2)^{-1}\phi(\delta_t^2 x^2) + \delta_t^2(\Phi(\delta_t^2 x^2) - \Phi(-\delta_t^2 x^2))\big], \end{aligned} \tag{2.5}$$

where $x_t^1 = \left(\frac{(\rho_t^{s1})^2}{\rho_t^1} + \frac{(\rho_{t-1}^{s1})^2}{\rho_{t-1}^1}\right)^{-1/2}$, $x^2 = (\frac{2(\rho^{s2})^2}{\rho^2})^{-1/2}$, $\delta_t^1 = \hat{\rho}_{t-1}(\bar{\theta}_{t-1} - p_t) + \rho_t^{s1}(\theta_t - p_t) - \hat{\rho}_{t-2}(\bar{\theta}_{t-2} - p_{t-1}) - \rho_{t-1}^{s1}(\theta_{t-1} - p_{t-1})$, and $\delta_t^2 = \hat{\rho}_{t-1}(\bar{\theta}_{t-1} - p_t) + \rho^{s2}(\theta_t - p_t) - \hat{\rho}_{t-2}(\bar{\theta}_{t-2} - p_{t-1}) - \rho^{s2}(\theta_{t-1} - p_{t-1})$.

Note that in the equilibrium model given by Blume et al. (1994), the estimation procedure of the unknown parameters needs to use private information that is difficult to obtain in practice. In the following, we provide a simulation-based statistical model that is analogous to the price-volume model (2.4) and (2.5).

To start with, we assume that the stock price follows a discrete geometric Brownian motion (GBM). That is, $\{S_i^t\}$ are generated by

$$S_i^t = S_{i-1}^t e^{(\mu-\sigma^2/2)\Delta + \sigma\sqrt{\Delta}\omega_i^t}, \tag{2.6}$$

where μ denotes the drift, σ denotes the volatility, Δ denotes the discrete time period, and w_i^t are independent and identically distributed (i.i.d.) $N(0, 1)$ random variables, for $i = 1, \ldots, n$ and $t = 1, \ldots, T$.

Regarding the volume model, two key factors will be considered in our setting. First, model (2.4) and (2.5) reveals a remarkable similarity to the findings of empirical researchers, as reported by Karpoff (1987), that a V-shape has been found by virtually all empirical investigators of the price-volume relation in equity markets. Second, the relation between absolute price changes and volume has been established for both equity and futures markets. In more recent work, Gallant et al. (1992) uses time series data to demonstrate the V-shape pattern between price and volume, and also reports that the dispersion of the distribution of the price changes increases uniformly with volume.

By making use of these two empirical phenomena, we propose a volume model as follows. Denote $r_i^t = S_i^t/S_{i-1}^t$, and let ε_i^t be a sequence of i.i.d. random noises with $N(0, 1)$ distribution. Define

$$m_i^t = \frac{1}{1 + e^{b_0 + b_1|r_i^t - 1| + b_2\varepsilon_i^t}}, \tag{2.7}$$

where b_0, b_1 and b_2 are unknown parameters, which can be estimated by the ordinary least square method using transformed data.

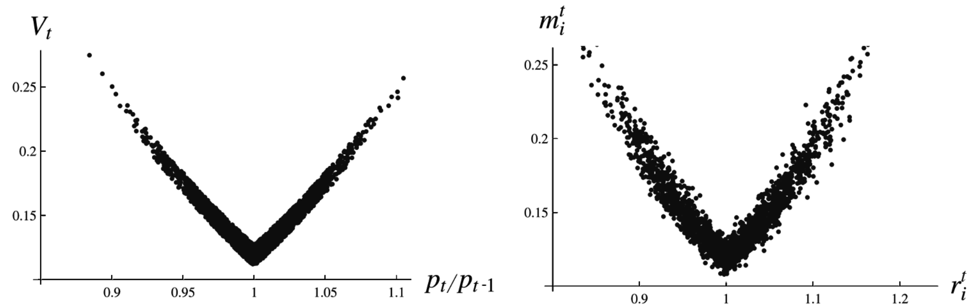
Note that the proposed price-volume model (2.6) and (2.7) is analogous to the equilibrium model (2.4) and (2.5). In model (2.4) and (2.5), the stock price is a time series model with a linear combination of normal random variables, and the volume V_t is a nonlinear model, via normal densities and normal cumulative distribution functions, of the stock price change in terms of δ_t^1 and δ_t^2. A simple calculation in (2.4) leads to

$$p_t = \frac{[(t-1)\rho_\omega + \rho_0]\bar{\theta}_{t-1} + (\alpha\rho_t^{s1} + (1-\alpha)\rho^{s2})(\psi + \omega_t)}{C} = C_1 + C_2 t + C_3\omega_t, \quad (2.8)$$

where $C = \hat{\rho}_{t-1} + \alpha\rho_t^{s1} + (1-\alpha)\rho^{s2}$, $C_1 = \frac{1}{C}[(-\rho_\omega + \rho_0)\bar{\theta}_{t-1} + (\alpha\rho_t^{s1} + (1-\alpha)\rho^{s2})\psi]$, $C_2 = \frac{1}{C}(\rho_\omega\bar{\theta}_{t-1})$, and $C_3 = \frac{1}{C}(\alpha\rho_t^{s1} + (1-\alpha)\rho^{s2})$. Note that in (2.8) p_t is a Bachelier's Brownian motion model.

For our proposed model (2.6) and (2.7), the stock price is a time series model such that the log return forms a sequence of i.i.d. normal random variables, a discrete time version of the celebrated Samuelson's geometric Brownian motion model. To simplify the volume model, we assume that the log transformation of volume m_i^t is a linear model of the stock price change $|r_i^t - 1|$. A similar idea can be found in Jain and Joh (1988), in which volume is a linear model of the stock change. Here, we observe that model (2.7) is a specific generalized linear model (GLM), in which each outcome of the dependent variable is assumed to be generated from an exponential function of the stock change. The relationship between Blume, Easley, and O'Hara's model and our proposed model is that in (2.4) and (2.5), they use p_t and V_t directly to build up the models; whereas in (2.6) and (2.7), we build up the model based on the log transformations of the stock return and volume. Since we only need to use public information to estimate the unknown parameters in (2.7), obtaining simulation data from (2.6) and (2.7) is much easier than that from (2.4) and (2.5).

The price-volume model (2.6) and (2.7) also reveals the empirical phenomenon of the V-shape. Figure 1 presents the price-volume relationship of model (2.4) and (2.5), and model (2.6) and (2.7) with given parameters. Here, we transform the price to return on the horizontal axis and set $p_{t-1} = 1$. The figure is from model (2.4) and (2.5), with parameter values $\psi_0 = \psi = 1$, $\rho_0 = \rho_\omega = 2$, $\mu = 0.5$, $\rho^2 = 0$ and $\rho_1^1 = 1/120$ on the left side. The figure is from model (2.6) and (2.7) with



(a) (b)

Figure 1. Given different parameter values, these two figures are constructed by drawing 2000 pairs data from their distributions and calculating the resulting equilibrium. The figure is plotted from model (2.4) and (2.5) on the left side, and from model (2.6) and (2.7) on the right side.

parameter values $b_0 = 1$, $b_1 = -10$, and $b_2 = 0.1$ on the right side. In the rest of this article, we will investigate the trading performance with different combinations of the parameters μ and σ in stock model (2.6). And we will use the parameter values $b_0 = 2$, $b_1 = -10$ and $b_2 = 0.1$, to estimate the volume model (2.7) using its stock price return.

3. VWAP TRADING STRATEGIES

When we use VWAP as a benchmark, the major impact on strategy has to do with the trading horizon. Fixed-time-period VWAP benchmarks encourage traders to spread their trades out over time to avoid the risk of trading at prices that are at the extreme for the whole period. In this section, we consider on-line VWAP trading strategies that require the trader to sell a fixed amount of one specific stock within a finite trading horizon. To simplify the problems of liquidity and computation, we break up the trading period into smaller trading units, such as trading days for instance. At the end of each trading unit, a fixed proportion of the volume is forced to be sold. Therefore, the strategies in this section are based on each trading unit only, and we consider the simple case of selling out the fixed proportion of the volume at a monitor time within each trading unit.

To describe our algorithms, we assume the price-volume model (2.6) and (2.7) is stationary within each trading unit. For simplicity, we also apply the celebrated Cox-Ross-Rubinstein (CRR) model, given by Cox et al. (1979), to approximate the geometric Brownian motion as follows: for given $0 < d < 1 < u$, and $0 < q < 1$, let

$$S_{i+1}^t = \begin{cases} S_i^t u & \text{with probability } q, \\ S_i^t d & \text{with probability } (1-q). \end{cases} \tag{3.1}$$

Next, we briefly describe how to extract the parameters (q, u, d) in the CRR model (3.1) from the parameters (μ, σ) in model (2.6). Taking n steps in each trading day (one day equals 1/250 year), and write $\Delta = 1/(250n)$. Write $U = \ln u$ and $D = \ln d$. Define $\hat{\mu} = (\mu - \sigma^2/2)\Delta$ and $\hat{\sigma} = \sigma\sqrt{\Delta}$. For each fixed $t = 1, \ldots, T$, we have after n steps that $E[\ln(S_n^t/S_0^t)] = (qU + (1-q)D)n = \hat{\mu}n$, $Var[\ln(S_n^t/S_0^t)] = q(1-q)(U-D)^2 n = \hat{\sigma}^2 n$. Hence, we can set

$$U = \sqrt{\sigma^2\Delta + (\mu - \sigma^2/2)^2\Delta^2}, \quad D = -U, \quad q = \frac{1}{2} + \frac{(\mu - \sigma^2/2)\Delta}{2U}. \tag{3.2}$$

The following algorithm is optimal based on the criterion of maximizing $E(\mathrm{VWAP}_\Gamma)$, and will be used as the benchmark strategy for all simulation studies below.

Algorithm 3.1. *The model-dependent "dynamic programming (DP)" follows that in Boyce* (1970).

Consider the following bond-selling problem: a corporation must repay \$10 million in bank loans in 3 months, and it wishes to sell bonds to repay the loan. However, the company's economists predict that in 3 months bond prices will be lower (interest rates higher). Should the corporation issue the bonds now, wait a month or two, or wait the full 3 months? For this bond selling problem, Boyce (1970) introduces a random version of Shepp's urn scheme, and proposes a dynamic programming procedure to compute the player's expected value at the stopping time when he or she uses an optimal strategy.

In principle, the optimal trading strategy can be drawn via dynamic programming. However, the dynamic programming algorithm is model dependent and a computationally intensive method, and is difficult to apply in practice. Therefore, some simple and practical alternative algorithms are needed. Among these, we first propose a cross-boundary strategy, which can be viewed as an asymptotically optimal strategy of the dynamic programming algorithm. For practical reasons, in the remainder of this article, we only consider the case of $\sigma \leq 1$ (note that $\sigma = 0.25$ in our empirical estimation for the stocks data set from the TSEC), and study the algorithm for variate μ. Note that the optimal strategy depends on $\sigma < 1$ or $\sigma > 1$, and the reader is referred to Boyce (1970) and Griffeath and Snell (1974) for details.

Algorithm 3.2. *The model-independent "cross-boundary (CB) strategy" is to sell the stock if the stock price touches the barrier. That is, for each $t = 1, \ldots, T$, and $i = 1, \ldots, n-1$, let*

$$c_i^t = \begin{cases} c/T & \text{for the first } i \text{ such that } S_i^t \leq H^t, \\ 0 & \text{otherwise}, \end{cases}$$

and c_n^t satisfies $c_1^t + \cdots + c_n^t = c/T$, where $H^t = S_0^t d^k$ and $d = e^{-\sigma\sqrt{\Delta}}$. When $n = 100$, $k = 3$.

We remark that the CB strategy given in Algorithm 3.2 is motivated by "the k in the hole drawing policy" in Chen et al. (2005), where the optimal k depends on n. It is shown in Chen et al. (2005) that the optimal choice of k is $k_n = O(n^{1/3})$. In addition, a list of optimal k given a collection of n's is provided.

Note that in Algorithm 3.2, the value of the probability q defined in (3.2) is taken over the whole parameter space. Therefore, the algorithm is independent of μ and σ. Chen et al. (2005) assumes that q is uniformly distributed in [0,1]. However, empirical evidence from intra-day TSEC data shows that the estimated upward probability q seem to be uniformly distributed in [0.4, 0.6] or follows a truncated normal $[\Phi(\frac{-\mu}{\sigma}), \Phi(\frac{1-\mu}{\sigma})]$.

To connect our CB strategy with the k in the hole drawing policy, the following theorem provides the asymptotic optimality of the cross-boundary algorithm with respect to the dynamic programming algorithm. Its proof is given in the appendix. Before presenting this theorem, we need some notations. Suppose there are n monitor times in each trading unit, and let k be the monitor time when a strategy is applied. Denote $E(n, k)$ as the expected value of using the dynamic programming strategy, and $H(n, k)$ as the expected value of using the cross-boundary strategy.

Theorem 3.1. *In the CRR model, assume q has a prior distribution that has a nondegenerate continuous density function $g(q)$ over the interval* [0, 1]. *Then $H(n, k)$ is asymptotically optimal in the sense that for $k_n = O(n^{1/3})$, as $n \to \infty$.*

$$E(n, k_n) = H(n, k_n) + o(n). \tag{3.3}$$

Note that Theorem 3.1 covers two interesting cases: uniformly distributed on $[a, b]$ for $0 \leq a < b \leq 1$, and truncated normal $N(\mu, \sigma)$ in the region of $[\Phi(\frac{-\mu}{\sigma}), \Phi(\frac{1-\mu}{\sigma})]$.

We also note that Algorithm 3.2 only has a lower boundary. The lower barrier H is set first and the stock is sold whenever the observed price reaches the boundary. Here H depends on n. Another important factor in VWAP trading is that we want to maximize $P(\mathrm{VWAP}_\Gamma - \mathrm{VWAP} \geq 0)$. Therefore we may use VWAP as an upper boundary. To set the upper boundary, the VWAP before the dth trading day, and the VWAP before the jth monitor on the dth trading day are defined, respectively, as

$$\mathrm{VWAP}^d = \frac{\sum_{t=1}^{d-1}\sum_{i=1}^{n} S_i^t m_i^t}{\sum_{t=1}^{d-1}\sum_{i=1}^{n} m_i^t}, \quad \mathrm{VWAP}_j^d = \frac{\sum_{t=1}^{d-1}\sum_{i=1}^{n} S_i^t m_i^t + \sum_{i=1}^{j-1} S_i^d m_i^d}{\sum_{t=1}^{d-1}\sum_{i=1}^{n} m_i^t + \sum_{i=1}^{j-1} m_i^d}.$$

Based on these two observations, we propose a two sided barriers strategy as follows.

Algorithm 3.3. *The model-independent "modified cross-boundary (MCB) strategy" is to sell the stock if the stock price touches the two-sided barrier. That is, for each* $t = 1, \ldots, T$, *and* $i = 1, \ldots, n-1$,

$$c_i^t = \begin{cases} c/T & \text{for the first } i \text{ such that } S_i^t \notin MH^t, \\ 0 & \text{otherwise}, \end{cases}$$

and c_n^t *satisfies* $c_1^t + \cdots + c_n^t = c/T$, *where* $MH^t = (S_0^t d^k, VWAP^k)$, *and* $d = e^{-\sigma\sqrt{\Delta}}$. *For* $n = 100$, $k = 4$.

Note that k is chosen to be 3 in Algorithm 3.2, based on the property of asymptotic optimality; whereas it is chosen to be 4 in Algorithm 3.3, based on our simulation study. Both Algorithms 3.2 and 3.3 are conservative algorithms, and only good for stock prices with negative drift (see Section 4 for details). Since we consider selling stock in which the lower boundary appears as a conservative algorithm, there is no theoretical upper boundary in our setting, which is necessary for positive drift. To remedy this shortcoming, we introduce the following algorithm, which is suitable for VWAP trading when the stock price has a positive drift.

In the following, we consider an assumption that the permutations of the absolute ranks of stock prices across monitor points are equally likely and propose the relative rank strategy in Algorithm 3.4. Let $S_{t_1}, \ldots, S_{t_n}$ be n stock prices observed sequentially, and denote $X_1, \ldots, X_n$ as the absolute rank of these stock prices. We rank the highest stock price as 1 and rank the lowest stock price as n. Therefore, $X_1, \ldots, X_n$ will be a permutation of $\{1, \ldots, n\}$. At time i, we only observe $S_{t_1}, \ldots, S_{t_i}$. Let Y_i denote the relative rank of the stock price at time i when the stock prices are observable up to time i. Hence, Y_i can be calculated by 1 plus the number of terms in $\{X_1, \ldots, X_{i-1}\}$ which are less than X_i.

Assume the $n!$ permutations of $X_1, \ldots, X_n$ are equally likely for the stock prices, then $P(Y_i = j) = 1/i$ for $j = 1, \ldots, i$ and $i = 1, \ldots, n$. Let $\binom{n}{k} = n!/k!(n-k)!$. The probability that the absolute rank of the stock price at time i is k for given $Y_i = j$ is $P(X_i = k | Y_i = j) = \frac{\binom{k-1}{j-1}\binom{n-k}{i-j}}{\binom{n}{i}}$. It follows that the expected absolute rank at time i for given $Y_i = j$ is $E(X_i \mid Y_i = j) = \frac{n+1}{i+1} j$.

For any stopping rule τ, the expected absolute rank EX_τ of the stock price selected is $EX_\tau = E(\frac{n+1}{\tau+1}Y_\tau)$. The goal is to find a stopping time τ that minimizes EX_τ, and this can be done by using backward induction. That is, for $i = 1, \ldots, n$, define $c_i = c_i(n) =$ minimum expected absolute rank selected, if the stopping rule τ satisfies $\tau \geq i+1$. Now let

$$c_{n-1} = E\left(\frac{n+1}{n+1}Y_n\right) = \frac{n+1}{2}, \tag{3.4}$$

and for $i = n-1, \ldots, 1$, let

$$c_{t-1} = E\left(\min\left(\frac{n+1}{i+1}Y_i, c_i\right)\right) = \frac{1}{i}\sum_{j=1}^{i}\min\left(\frac{n+1}{i+1}j, c_i\right). \tag{3.5}$$

Equations (3.4) and (3.5) allow us to compute the values $c_{n-1}, c_{n-2}, \ldots, c_1, c_0$ successively, and obtain an implicit expression of the optimal stopping rule. Let $s_n = n$, and $s_i = \left[\frac{i+1}{n+1}c_i\right]$, where $[a]$ denotes the greatest integer which is less than or equal to a. Then (3.5) equals

$$\begin{aligned} c_{i-1} &= \frac{1}{i}\left\{\frac{n+1}{i+1}(1+2+\cdots+s_i) + (i-s_i)c_i\right\} \\ &= \frac{1}{i}\left\{\frac{n+1}{i+1}\frac{s_i(s_i+1)}{2} + (i-s_i)c_i\right\}. \end{aligned} \tag{3.6}$$

The optimal stopping rule τ^* is defined as

$$\tau^* = \inf\{i \geq 1 : Y_i \leq s_i\}. \tag{3.7}$$

Algorithm 3.4. *The model-independent "relative rank (RR) strategy" is to sell the stock at time τ^* defined in* (3.7).

Note that in Algorithm 3.4 we do not need to know the whole probability structure since we only care about the rank of stock prices. A similar idea can be found in the secretary problem by Chow et al. (1964).

4. SIMULATION STUDIES

In this section, we first investigate the VWAP trading performances of the DP, CB, MCB, and RR strategies given in Algorithms 3.1 to 3.4 based on the price-volume model (2.6) and (2.7). Second, we propose the hybrid strategy as a simplified alternative to the DP strategy for practical purposes.

We consider stock prices with negative drift ($\mu = -0.76$), zero drift ($\mu = 0$), and positive drift ($\mu = 1.61$). The volatility σ is fixed at 0.25 to match our empirical estimation for the stock price data set from TSEC. For each trading day, we equally divide it into 100 monitor times. In addition, the boundary for the CB strategy is S_0d^3, and the boundary for the MCB strategy is S_0d^4. Tables 1 and 2 summarize WR and ER of the VWAP trading performances using these four strategies for a single trading day ($T = 1$) and for multiple trading days, respectively. The Monte Carlo sample size is 10,000. Standard errors of the simulation are recorded in parenthesis.

Table 1. Performances of VWAP trading for a single trading day

	DP	RR	CB	MCB
$\mu = -0.76$				
WR	0.5729* (0.0049)	0.5479 (0.0050)	0.4850 (0.0050)	0.5667 (0.0050)
ER	0.1442* (0.0089)	−0.0481 (0.0076)	0.0291 (0.0075)	0.1503 (0.0088)
EWIN	0.7462 (0.0074)	0.5035 (0.0043)	0.6413 (0.0074)	0.7562 (0.0074)
ELOSE	−0.6635 (0.0081)	−0.7164 (0.0086)	−0.5475 (0.0055)	−0.6423 (0.0078)
$\mu = 0$				
WR	0.5777* (0.0049)	0.5678 (0.0050)	0.4565 (0.0050)	0.4989 (0.0050)
ER	0.0041* (0.0073)	−0.0012 (0.0070)	−0.0152 (0.0077)	−0.0018 (0.0086)
EWIN	0.5084 (0.0044)	0.4822 (0.0042)	0.6551 (0.0078)	0.6838 (0.0073)
ELOSE	−0.6857 (0.0082)	−0.6362 (0.0081)	−0.5783 (0.0056)	−0.6844 (0.0074)
$\mu = 1.61$				
WR	0.6362* (0.0048)	0.5704 (0.0050)	0.4752 (0.0050)	0.3635 (0.0048)
ER	0.3117* (0.0088)	0.0347 (0.0060)	0.0308 (0.0089)	−0.2995 (0.0087)
EWIN	0.8413 (0.0069)	0.4468 (0.0039)	0.7712 (0.0089)	0.2998 (0.0053)
ELOSE	−0.6145 (0.0081)	−0.5124 (0.0067)	−0.6396 (0.0063)	−0.4065 (0.0052)

**The best strategy in terms of WR or ER.*

Table 2. Performances of VWAP trading for multiple trading days

	T	DP	RR	CB	MCB
$\mu = -0.76$					
WR	2	0.5943 (0.0049)	0.5117 (0.0050)	0.5008 (0.0050)	0.5964 (0.0049)
	4	0.6279 (0.0048)	0.4797 (0.0050)	0.5143 (0.0050)	0.6295 (0.0048)
	8	0.6773 (0.0047)	0.4409 (0.0050)	0.5371 (0.0050)	0.6569 (0.0047)
	15	0.7375 (0.0044)	0.4094 (0.0049)	0.5537 (0.0050)	0.6746 (0.0047)
ER	2	0.1424 (0.0063)	−0.0431 (0.0053)	0.0273 (0.0053)	0.1321 (0.0060)
	4	0.1401 (0.0044)	−0.0472 (0.0038)	0.0273 (0.0038)	0.1157 (0.0040)
	8	0.1387 (0.0031)	−0.0495 (0.0026)	0.0294 (0.0026)	0.0983 (0.0027)
	15	0.1362 (0.0022)	−0.0480 (0.0019)	0.0258 (0.0019)	0.0794 (0.0019)
$\mu = 0$					
WR	2	0.5325 (0.0050)	0.5350 (0.0050)	0.4736 (0.0050)	0.5133 (0.0050)
	4	0.5226 (0.0050)	0.5234 (0.0050)	0.4838 (0.0050)	0.5215 (0.0050)
	8	0.5093 (0.0050)	0.5088 (0.0050)	0.4929 (0.0050)	0.5144 (0.0050)
	15	0.5072 (0.0050)	0.5114 (0.0050)	0.4912 (0.0050)	0.5216 (0.0050)
ER	2	0.0018 (0.0052)	−0.0076 (0.0049)	-0.0076 (0.0055)	0.0048 (0.0060)
	4	−0.0020 (0.0037)	−0.0088 (0.0035)	−0.0029 (0.0039)	0.0017 (0.0042)
	8	−0.0010 (0.0026)	−0.0041 (0.0025)	−0.0004 (0.0027)	−0.0000 (0.0029)
	15	0.0000 (0.0019)	−0.0008 (0.0018)	−0.0017 (0.0020)	−0.0004 (0.0021)
$\mu = 1.61$					
WR	2	0.6935 (0.0046)	0.5713 (0.0049)	0.4946 (0.0050)	0.3270 (0.0047)
	4	0.7517 (0.0043)	0.5767 (0.0049)	0.5102 (0.0050)	0.2795 (0.0045)
	8	0.8312 (0.0037)	0.5914 (0.0049)	0.5209 (0.0050)	0.2049 (0.0040)
	15	0.9056 (0.0029)	0.6155 (0.0049)	0.5219 (0.0050)	0.1309 (0.0034)
ER	2	0.3136 (0.0063)	0.0391 (0.0043)	0.0262 (0.0062)	−0.2912 (0.0062)
	4	0.3114 (0.0045)	0.0383 (0.0030)	0.0255 (0.0045)	−0.2852 (0.0046)
	8	0.3163 (0.0033)	0.0393 (0.0022)	0.0220 (0.0032)	−0.2916 (0.0035)
	15	0.3219 (0.0025)	0.0421 (0.0016)	0.0208 (0.0024)	−0.3024 (0.0027)

Table 1 shows that the DP strategy outperforms all other three strategies in terms of WR and ER. Excluding the DP strategy and in terms of WR, the MCB strategy outperforms the RR and CB strategies for stock prices with negative drift, and the RR strategy outperforms the CB and MCB strategies for stock prices with zero and positive drift. Excluding the DP strategy and in terms of ER, the MCB strategy outperforms the CB and RR strategies for stock prices with negative drift, and the RR and CB strategies outperform the MCB strategy for stock prices with positive drift. However, the RR, CB, and MCB strategies produce similar ER for stock prices with zero drift. For more detailed analysis of this phenomenon, we split ER into expected win (EWIN) and expected loss (ELOSS). EWIN is the expected revenue conditioned on the trading result of the strategy being greater than or equal to VWAP, and ELOSS is the expected revenue conditioned on the trading result of the strategy being less than VWAP.

Table 2 shows that the DP strategy outperforms other three strategies in terms of WR and ER. Excluding the DP strategy and in terms of WR and ER, the RR strategy outperforms the CB and MCB strategies for stock prices with positive drift, whereas the MCB strategy outperforms the CB and RR strategies for stock prices with negative drift. By making use of the above simulation results and the fact that the drifts μ are different for each time period in practice, we introduce a hybrid strategy which combines the RR and MCB strategies according to the sign of the drift.

Algorithm 4.1. *The model-dependent "hybrid strategy" is*

(1) *using the relative rank strategy if the stock price has positive drift in the period*;
(2) *using the modified cross-boundary strategy if the stock price has negative drift in the period.*

Although the hybrid strategy is model dependent, it depends only on the sign of the drift. We can estimate the drift trend by using a suitable statistical method, and there is no need to estimate an accurate value of μ. Therefore, the computation for the hybrid strategy is much simpler than for the DP strategy. Table 3 summarizes the performance of the DP, MCB, RR, and hybrid strategies, for a longer horizon with different μ in each period. There are two cases for the hybrid strategy: with perfect information, and with lower accuracy (30% error) of estimating the drift μ. Note that in Table 3, even in the case of the hybrid strategy

Table 3. WR and ER for the DP, RR, MCB, and hybrid strategies for a longer horizon and μ is random from uniform $(-1, 1)$ in each period

T	DP	RR	MCB	Hybrid	Hybrid (30% error)
WR					
10	0.6244 (0.0128)	0.5322 (0.0143)	0.5472 (0.0131)	0.5832 (0.0133)	0.5723 (0.0139)
20	0.6388 (0.0103)	0.5410 (0.0122)	0.5478 (0.0117)	0.6153 (0.0118)	0.5971 (0.0121)
ER					
10	0.0781 (0.0033)	−0.0094 (0.0046)	0.0116 (0.0037)	0.0413 (0.0035)	0.0215 (0.0038)
20	0.0677 (0.0029)	0.0032 (0.0033)	0.0206 (0.0031)	0.0631 (0.0030)	0.0258 (0.0032)

with 30% error, its performance is still better than using only a single strategy (the RR strategy or the MCB strategy).

Graversen et al. (2000) and Du Toit and Peskir (2007) study the following *optimal prediction problem*: assume the stock price follows a Brownian motion, and the goal is to find an optimal stopping time that minimizes the mean square error loss of the stopping time with respect to the maximum of the Brownian motion. And they propose boundary-crossing strategies for zero and non-zero drifts, respectively. When the drift is zero or negative, it is a one-sided curved boundary; whereas when the drift is positive, it is a two-sided curved boundary. Although the model assumption and criterion are different in these two cases, Algorithm 4.1 has an analogy to the optimal strategies proposed by Graversen et al. (2000) and Du Toit and Peskir (2007). In the case of positive drift, Algorithm 4.1 uses a relative rank strategy that can be viewed as a discrete time analogue (the secretary problem) of the continuous time optimal prediction problem. In the case of negative drift, Algorithm 4.1 uses a simple one-sided flat boundary other than an one-sided curved boundary. A good summary of the optimal prediction problem can be found in Chapter VIII of Peskir and Shiryaev (2006).

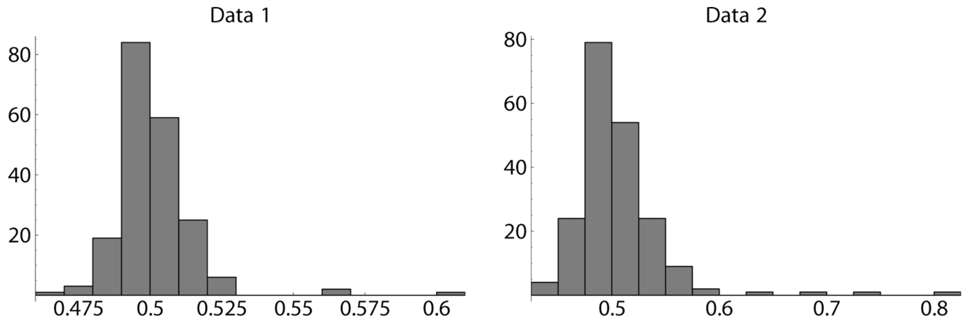
5. EMPIRICAL ANALYSIS

To illustrate our method, demonstrate the robustness of the trading strategies relative to the price-volume model, and evaluate the effect of VWAP trading, in this section, we estimate the unknown parameter q, compare the performance of VWAP and the MCB strategy, and evaluate the outcome of TSEC based on the VWAP trading strategy for the Taiwan stock exchange. Since the evaluation of VWAP trading is based on a real data set and the MCB strategy is model independent, we do not use the price-volume model (2.6) and (2.7) in this comparison study.

We first estimate the unknown parameter q in the CRR model based on an empirical data set. The data set is taken from the top 20 liquidity stocks based on intra-day data from the Taiwan Economic Journal (TEJ). We split the data into two data sets. Data set 1 includes the time period from March 3 to March 31, 2003, and Data set 2 includes the time period from September 1 to September 30, 2003. Base on the CRR model, the estimate $\hat{q}$ of q is defined as

$$\hat{q} = \frac{\text{No. of upwards } + 0.5(\text{No. of no move})}{\text{No. of monitors}}. \tag{5.1}$$

Figure 2 reports the histogram of $\hat{q}$, which shows that the range of $\hat{q}$ is between 0.45 and 0.625. The histogram was drawn using the top 20 stocks in Table 5, and applying formula (5.1) to calculate the probability of upward movement for each stock on each trading day. For each stock price S, we use the tick size to compute $\hat{u}$, i.e., $\hat{u} = (S + \text{tick})/S$. Tick size is 0.01 when the price is equal to or less than 5, 0.05 when the price is between 5 and 15, 0.1 when the price is between 15 and 50, 0.5 when the price is between 50 and 150, 1.0 when the price is between 150 and 1000, and 5.0 when the price is equal to or greater than 1000. The pair $(\hat{q}, \hat{u})$ in the CRR model can be transformed to $(\hat{\mu}, \hat{\sigma})$ in the GBM model. Summary statistics of $\hat{q}$ in the CRR model, and $\hat{\mu}$ and $\hat{\sigma}$ are given in Table 4. Here $\hat{\mu}$ and $\hat{\sigma}$ are defined as (3.2) and (3.2).



(a) (b)

Figure 2. Estimated q from data sets 1 and 2.

Table 4. Summary statistics for data sets 1 and 2

	Data 1		Data 2	
	Mean	Std	Mean	Std
		CRR		
$\hat{q}$	0.5018	0.0135	0.5070	0.0415
		GBM		
$\hat{\mu}$	0.1711	1.0675	0.5837	3.2817
$\hat{\sigma}$	0.2499	0.0004	0.2491	0.0049

Table 5 reports the comparison of VWAP and the MCB strategy in a data set that includes the top 20 liquidity stocks based on intra-day data from the TEJ. The time period is from January 2 to December 30, 2003. In Table 5, we include the stock ID, number of trades (number of monitors), mean, standard deviation (SD), minimum volume (min vol), and four trading strategies. VWAP(k) is defined as including the CB strategy to sell additional k thousand shares. That is, VWAP(k) is an adjusted VWAP for additional k thousand shares. VWAP(0) is the market VWAP without including any shares from the strategy. The underlined numbers in the MCB column represents when the performance of the MCB strategy is better than VWAP. Note that although the MCB strategy is not always dominant over VWAP, the winning rate is about 60%. Also note that the difference between VWAP(0) and VWAP(5) is in thousandth, and this conforms to the assumption of unaffected price for any trading behavior.

6. CONCLUSIONS

Although VWAP trading is relatively straightforward in concept, its implementation can be difficult. This article studies VWAP trading and proposes a simulation-based statistical price-volume model that virtually depicts the V-shape of the price and volume by using public information. The model is simple and useful for parameter estimation and makes it easy to simulate new data.

Table 5. Empirical VWAP of the top 20 number of trading volume stocks in TSEC

Stock ID	Number of trades	Mean	SD	Min vol.	CB	MCB	VWAP (0)	VWAP (5)
1313	397	14.289	0.182	1	14.65	14.65	14.301	14.301
1326	397	37.435	0.428	1	36.40	37.70	37.527	37.527
2002	392	19.549	0.078	1	19.60	19.60	19.582	19.582
2023	398	23.562	0.343	1	24.30	24.30	23.679	23.679
2201	399	41.621	0.649	2	39.60	42.60	41.828	41.827
2204	377	65.826	1.208	1	67.50	67.50	65.764	65.764
2303	399	21.261	0.081	1	20.90	21.10	21.246	21.246
2324	397	36.352	0.174	1	35.80	36.60	36.375	36.375
2330	398	42.910	0.184	1	42.80	42.40	42.853	42.853
2408	396	21.044	0.110	1	20.80	20.80	21.028	21.028
2409	399	21.295	0.123	3	21.20	21.20	21.330	21.330
2801	397	17.059	0.201	1	17.10	17.10	17.104	17.104
2807	397	11.999	0.148	5	12.00	12.00	12.040	12.040
2837	399	14.893	0.157	2	14.20	14.90	14.876	14.876
2880	398	26.568	0.534	1	25.00	27.00	26.631	26.631
2883	398	13.685	0.093	3	13.30	13.65	13.704	13.704
2886	399	17.370	0.298	1	17.80	17.80	17.494	17.494
2891	399	29.251	0.498	1	27.90	29.70	29.477	29.477
3024	399	47.290	0.311	1	47.60	47.20	47.281	47.282
9904	398	30.723	0.459	1	29.60	31.30	30.904	30.904

In VWAP trading, alternative strategies have their own advantages and disadvantages in different situations. For on-line VWAP trading, in this article, we introduce a dynamic programming strategy to capture the stock price uncertainty. Since the dynamic programming strategy is model dependent and a computationally intensive method, we also propose the (modified) cross-boundary strategy and introduce a relative rank strategy to approximate the VWAP trading benchmark.

Simulation results are given to compare these strategies. An interesting property is that there is a tradeoff between the winning rate and the expected revenue when the stock price has a negative drift among the model independent strategies. As a result, the drift trend in each trading horizon affects the choice of the strategy. Therefore, we suggest a hybrid strategy for variate drifts in multi-period trading horizons. To demonstrate the robustness of the trading strategies relative to the price-volume model, we also compare the trading strategies based on a real data set.

In practice, VWAP can be regarded as a stochastic benchmark. This article provides a first step analysis of the VWAP trading strategies. It is also in the framework of optimal prediction problem studied by Graversen et al. (2000), and Du Toit and Peskir (2007). Theoretical properties of price-volume modeling and trading strategies deserve further study.

APPENDIX: PROOFS

Motivated by studying a bond-selling problem, Boyce (1970) formulates a random version of Shepp's urn scheme [cf. Shepp (1969)]: a player is given an urn with

n balls, α of these balls have value +1 and $n-\alpha$ have value -1. The player is allowed to draw balls randomly, without replacement, until he or she wants to stop. Here n is given, whereas α is known only up to a number selected randomly from $\{0, 1, 2, \ldots, n\}$. The player wishes to maximize the expected value of the sum of the balls drawn. Abuse the notation a little bit, let q be the probability of getting $+1$ in this setting. Along this line, when q has a uniform distribution over the interval [0,1], Chen et al. (2005) shows that Algorithm 3.2 (a random coin tossing problem in their setting) is equivalent to the random version of Shepp's urn scheme problem. Furthermore, they prove the asymptotic optimality of the cross-boundary Algorithm 3.2. Theorem 3.1 generalizes the result to the case when q has a continuous density $g(q)$ over the interval [0,1]. Other than exact computation as that in Chen et al. (2005), the proof is based on mean value theorem with suitable approximations.

To prove Theorem 3.1, we will use the same notations as in Section 3. Recall that $E(n,k)$ is the expected value of using the optimal strategy (dynamic programming), and $H(n,k)$ is the expected value of using the CB strategy. Let $W(n,k)$ be the expected value of the following "k in the hole drawing policy," in the sense that we continue drawing balls until the number of "—1" balls drawn is k more than the number of "+1" balls drawn.

Proof of Theorem 3.1. To get the proof, we first show that for any $1 \le k \le n$, $W(n,k) = H(n,k)$.

For each $j = 0, 1, \ldots, n$, let $W(n,k,j)$ be the expected value at the stopping time when the player uses the "k in the hole drawing policy" and the urn contains j balls of value +1 and $n-j$ balls of value -1. When $0 \le j \le k \le n$, $j = 0, 1, \ldots, n$, let $H(n,k,j)$ be the expected value at the stopping time when the CB strategy is used, assuming that there are j upward values Su in the CRR model. Let A_j, be the event of $\{\alpha = j\}$, and denote $P(A_j)$ as the probability of getting A_j. It is easy to see that $W(n,k) = \sum_{j=0}^{n} P(A_j)W(n,k,j)$ and $H(n,k) = \sum_{j=0}^{n} P(A_j)H(n,k,j)$.

By using an argument similar to Lemma 2 in Chen et al. (2005), we have $H(n,k,j) = W(n,k,j)$. Therefore, $W(n,k) = \sum_{j=0}^{n} P(A_j)W(n,k,j) = \sum_{j=0}^{n} P(A_j)$ $H(n,k,j) = H(n,k)$.

Next, we will show that for any given n, the "k in the hole drawing policy" is asymptotically optimal if we choose the best k. By using the result on page 154 of Chen et al. (2005), we have $k_n = \min\{k \mid 1 \le k \le n, W(n,k) = \max\{W(n,j) \mid 1 \le j \le n\}\} = O(n^{1/3})$. That is, to complete the proof, we need to show that as $n \to \infty$

$$E(n,k_n) = W(n,k_n) + o(n). \tag{6.1}$$

To prove (6.1) holds, we first show that there exist $c_j \ge 0$ such that

$$P(A_j) = \frac{c_j}{n+1} \quad \text{for } j = 0, 1, \ldots, n, \text{ with } \sum_{j=0}^{n} \frac{c_j}{n+1} = 1. \tag{6.2}$$

To prove (6.2) holds, we first note that $P(A_j) = \binom{n}{j}\int_0^1 q^j(1-q)^{(n-j)}g(q)dq$. If the support of the continuous density $g(q)$ is in [0,1], then by the mean value theorem, there exists $\min g(q) < c_j < \max g(q)$ such that $\binom{n}{j}\int_0^1 q^j$ $(1-q)^{(n-j)}g(q)dq = c_j\binom{n}{j}\int_0^1 q^j(1-q)^{(n-j)}dq = \frac{c_j}{n+1}$. This implies (6.2) holds. Since $\sum_{j=0}^{n}\binom{n}{j}\int_0^1 q^j(1-q)^{(n-j)}g(q)dq = 1$, the second term in (6.2) holds.

Note that when q is uniform in $U(a, b)$, then $0 < c_j < \frac{1}{b-a}$. When q is truncated normal $N(\mu, \sigma)$ in the region of $[\Phi(\frac{-\mu}{\sigma}), \Phi(\frac{1-\mu}{\sigma})]$ then $\min\left(\frac{1}{C\sqrt{\pi}\sigma}e^{-\frac{\mu^2}{2\sigma^2}}, \frac{1}{C\sqrt{\pi}\sigma}e^{-\frac{(1-\mu)^2}{2\sigma^2}}\right) < c_j < 1$, with $C = \Phi(\frac{1-\mu}{\sigma}) - \Phi(\frac{-\mu}{\sigma})$.

Now assume the player starts with an urn with n balls and a known a of "+1" balls, let $V(n, \alpha)$ be the player's expected score at the stopping time when he or she uses the optimal drawing policy. It is easy to see $E(n, 0) \le \frac{1}{n+1}\sum_{j=0}^{n} c_j V(n, j)$. If $n = 2m$, by a theorem of Shepp [cf. Boyce (1970)], $V(2m, j) \le V(2m, m)$ for all $j = 0, 1, \ldots, m$ and $V(2m, j) \le 2j - 2m + V(2m, m)$ for all $j = m + 1, \ldots, 2m$. Therefore, by $\sum_{j=0}^{n} c_j = n + 1$, we obtain

$$\begin{aligned} E(n, 0) &\le \frac{1}{n+1}\sum_{j=0}^{n} c_j V(n, j) \le \frac{1}{n+1}\sum_{j=0}^{m} c_j V(2m, m) \\ &\quad + \frac{1}{n+1}\sum_{j=m+1}^{2m} c_j(2j - 2m + V(2m, m)) \\ &= \frac{1}{n+1}\sum_{j=m+1}^{2m} c_j(2j - 2m) + V(2m, m) := G(n). \end{aligned} \tag{6.3}$$

On the other hand, by an expression of $W(n, k_n, j)$ on page 154 of Chen et al. (2005), we have

$$\begin{aligned} W(n, k_n) &= \frac{1}{n+1}\sum_{j=0}^{n} c_j W(n, k_n, j) \\ &= \frac{1}{n+1}\sum_{j=0}^{\frac{n-k_n-2}{2}} c_j(-k_n) + \frac{1}{n+1}\sum_{j=\frac{n-k_n}{2}}^{n} c_j(2j - n) \\ &\quad - \frac{1}{n+1}\sum_{j=\frac{n-k_n}{2}}^{n-k_n} c_j(2j - n + k_n)\frac{\binom{n}{k_n+j}}{\binom{n}{j}}. \end{aligned} \tag{6.4}$$

By (6.3) and (6.4), we obtain

$$\begin{aligned} G(n) - W(n, k_n) &= V(2m, m) + \frac{1}{n+1}\sum_{j=0}^{\frac{n-k_n-2}{2}} c_j k_n - \frac{1}{n+1}\sum_{j=\frac{n-k_n}{2}}^{\frac{n}{2}} c_j(2_j - n) \\ &\quad + \frac{1}{n+1}\sum_{j=\frac{n-k_n}{2}}^{n-k_n} c_j(2j - n + k_n)\frac{\binom{n}{k_n+j}}{\binom{n}{j}}. \end{aligned} \tag{6.5}$$

Note that the first term of (6.5) is $V(2m, m) \approx \sqrt{m} = \sqrt{n/2} = o(n)$. Since $\sum_{j=0}^{n} c_j = n + 1$, the second term of (6.5) is $\frac{1}{n+1}\sum_{j=0}^{\frac{n-k_n-2}{2}} c_j k_n \le \frac{1}{n+1}\sum_{j=0}^{n} c_j k_n = k_n = O(n^{1/3}) = o(n)$. Let $C_k = \max\{c_j : j = \frac{n-k_n}{2}, \ldots, \frac{n}{2}\} > 0$ be a constant. Then the third term of (6.5) is $-\frac{1}{n+1}\sum_{j=\frac{n-k_n}{2}}^{\frac{n}{2}} c_j(2j - n) \le -\frac{C_k}{n+1}(\frac{k_n}{2} + 1)(-\frac{k_n}{2}) = \frac{C_k}{n+1}(\frac{k_n}{2} + 1)\frac{k_n}{2} = O(n^{-1/3}) = o(n)$.

By Theorem 5 of Chen et al. (2005), we have $\frac{1}{n^2}\sum_{j=\frac{n-k_n}{2}}^{n-k_n} c_j(2j-n+k_n)\frac{\binom{n}{k_n+j}}{\binom{n}{j}} \approx \frac{a_k}{(k_n+1)(k_n+2)}$, where a_k is a constant and $\frac{1}{8} < a_k < 1$. Let $C'_k = \max\{c_j : j = \frac{n-k_n}{2}, \ldots, n-k_n\} > 0$ be a constant. Therefore, the last term of (6.5) is

$$\begin{aligned}
\frac{1}{n+1}\sum_{j=\frac{n-k_n}{2}}^{n-k_n} c_j(2j-n+k_n)\frac{\binom{n}{k_n+j}}{\binom{n}{j}} &= \frac{n^2}{n+1}\frac{1}{n^2}\sum_{j=\frac{n-k_n}{2}}^{n-k_n} c_j(2j-n+k_n)\frac{\binom{n}{k_n+j}}{\binom{n}{j}} \\
&\approx \frac{n^2}{n+1}\frac{C'_k a_k}{(k_n+1)(k_n+2)} \\
&\approx C'_k a_k n^{1/3} = o(n). \qquad (6.6)
\end{aligned}$$

The proof for the case when n is odd is similar. Therefore (6.5) is satisfied. The rest of the proof of (6.1) follows that in Chen et al. (2005), and is omitted. □

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