Finance Stoch (2011) 15: 141–181
DOI 10.1007/s00780-010-0137-5

Unbiased and efficient Greeks of financial options

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Received: 24 July 2008 / Accepted: 26 August 2009 / Published online: 3 September 2010

Abstract The price of a derivative security equals the discounted expected payoff of the security under a suitable measure, and Greeks are price sensitivities with respect to parameters of interest. When closed-form formulas do not exist, Monte Carlo simulation has proved very useful for computing the prices and Greeks of derivative securities. Although finite difference with resimulation is the standard method for estimating Greeks, it is in general biased and suffers from erratic behavior when the payoff function is discontinuous. Direct methods, such as the pathwise method and the likelihood ratio method, are proposed to differentiate the price formulas directly and hence produce unbiased Greeks (Broadie and Glasserman, Manag. Sci. 42:269–285, 1996). The pathwise method differentiates the payoff function, whereas the likelihood ratio method differentiates the densities. When both methods apply, the pathwise method generally enjoys lower variances, but it requires the payoff function to be Lipschitz-continuous. Similarly to the pathwise method, our method differentiates the payoff function but lifts the Lipschitz-continuity requirements on the payoff function. We build a new but simple mathematical formulation so that formulas of Greeks for a broad class of derivative securities can be derived systematically. We then present an importance sampling method to estimate the Greeks. These formulas

Earlier versions of the paper were presented at the Midwest Finance Association 57th Annual Meeting, San Antonio, Texas, February 29, 2008, and at the Eighth International Conference on Monte Carlo and Quasi-Monte Carlo Methods in Scientific Computing, Montreal, Canada, July 10, 2008. The first author was supported in part by the National Science Council of Taiwan under Grant 97-2221-E-002-096-MY3 and Excellent Research Projects of National Taiwan University under Grant 98R0062-05.

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are the first in the literature. Numerical experiments show that our method gives unbiased Greeks for several popular multi-asset options (also called rainbow options) and a path-dependent option.

Keywords Option pricing · Rainbow options · Path-dependent options · Monte Carlo simulation · Greeks · Importance sampling

Mathematics Subject Classification (2000) 65C05 · 65D25 · 65L12

JEL Classification C02 · C15 · G13

1 Introduction

An option is a financial instrument whose payoff is based on other underlying assets such as stocks, indices, currencies, volatilities, commodities, bonds, mortgages, other derivatives, temperatures, and countless others. Besides an option's price, its Greeks are equally important. Greeks are the price's sensitivities with respect to certain parameters of interest such as the underlying asset's price, volatility, maturity, and interest rate. Although an option's price can often be observed in the market, this is not the case for its Greeks, which must be calculated. As Greeks are important for measuring and managing risk as well as executing dynamic trading strategies, how to calculate them efficiently and accurately is of critical importance both in theory and in practice [4].

Let us use C to denote an option's price, S the underlying asset's spot price, σ the volatility, T the time to maturity, and r the risk-free interest rate. Common Greeks are delta ($\Delta = \partial C/\partial S$), gamma ($\Gamma = \partial^2 C/\partial S^2$), vega ($\Lambda = \partial C/\partial \sigma$), theta ($\Theta = \partial C/\partial T$), and rho ($\rho = \partial C/\partial r$). For multi-asset options (also called rainbow options), cross-gammas $\Gamma_{ij} = \partial^2 C/\partial S_i \partial S_j$, where S_i and S_j are different underlying assets, are also important. Note that some Greeks are first-order partial derivatives, whereas others such as gamma and cross-gamma are second-order ones.

Easy-to-calculate closed-form formulas are rare for rainbow options; as a result, numerical methods are indispensable for calculating their prices and Greeks. See Table 1 for a variety of rainbow options in the literature. Deterministic numerical methods for rainbow options such as numerical integration and PDE-related methods suffer from the curse of dimensionality in that the computational complexity grows exponentially with the dimension. They are hence computationally infeasible. A few methods deal with the curse of dimensionality for a certain type of rainbow options, such as the closed-form approximation formulas of Kirk [16] and Carmona and Durrleman [5] for spread options and the method of Hörfelt [12] for options on the kth ranked asset. However, these methods are quite restrictive in their domains of applicability and their Greeks are biased in general. As a result, Monte Carlo simulation has proved to be the most important general-purpose numerical scheme for pricing rainbow options [2, 10].

Simulation methods for estimating the Greeks fall into two broad categories: methods that involve resimulation and those that do not. The first category, based on

Table 1 A sampling of rainbow options

Type	Payoff
Margrabe option [21]	$\max(S_1(T) - S_2(T), 0)$
Better-off option [23, 32]	$\max(S_1(T), \ldots, S_n(T))$
Worse-off option [23, 32]	$\min(S_1(T), \ldots, S_n(T))$
Binary maximum option*	$\mathbf{1}_{\{\max(S_1(T),\ldots,S_n(T))>K\}}$
Maximum option*[15, 30]	$\max(\max(S_1(T), \ldots, S_n(T)) - K, 0)$
Minimum option [15, 30]	$\max(\min(S_1(T), \ldots, S_n(T)) - K, 0)$
Spread option* [5, 24]	$\max(S_2(T) - S_1(T) - K, 0)$
Basket average option [13]	$\max(\frac{S_1(T)+\cdots+S_n(T)}{n} - K, 0)$
Multi-strike option [23, 32]	$\max(S_1(T) - K_1, \ldots, S_n(T) - K_n, 0)$
Pyramid rainbow option [23, 32]	$\max(\|S_1(T) - K_1\| + \cdots + \|S_n(T) - K_n\| - K)$
Madonna rainbow option [23, 32]	$\max(\sqrt{(S_1(T) - K_1)^2 + \cdots + (S_n(T) - K_n)^2} - K, 0)$

An asterisked rainbow option is one whose Greeks will be derived in the paper. The Greeks of all these rainbow options listed above except basket and Madonna options can be obtained directly by our method. For basket and Madonna options, our method is applicable after proper changes of variables in their payoff functions. See [23, 32] for more complete lists of rainbow options

Table 2 FD formulas for deltas, gammas and cross-gammas

Forward FD schemes	
Δ_i	$e^{-rT}[E[\wp(S_i + h)] - E[\wp(S_i)]]/h$
Γ_{ii}	$e^{-rT}[E[\wp(S_i + 2h)] - 2E[\wp(S_i + h)] + E[\wp(S_i)]]/h^2$
Γ_{ij}	$e^{-rT}[E[(S_i + h, S_j + h)] - E[(S_i + h, S_j)] - E[\wp(S_i, S_j + h)] + E[\wp(S_i, S_j)]]/h^2$
Central FD schemes	
Δ_i	$e^{-rT}[E[\wp(S_i + h/2)] - E[\wp(S_i - h/2)]]/h$
Γ_{ii}	$e^{-rT}[E[\wp(S_i + h)] - 2E[\wp(S_i)] + E[\wp(S_i - h)]]/h^2$
Γ_{ij}	$e^{-rT}[E[\wp(S_i + h/2, S_j + h/2)] - E[\wp(S_i + h/2, S_j - h/2)] - E[\wp(S_i - h/2, S_j + h/2)] + E[\wp(S_i - h/2, S_j - h/2)]]/h^2$

$\wp$ is the option payoff function, $\wp(S_i + h)$ denotes the payoff when the initial underlying asset's price S_i changes to $S_i + h$, and so on. The formulas assume the same form if the parameter of interest is not the stock price. For a complete list of formulas with error rates, see Tavella [31, p. 69]

finite-difference (FD) approximations, is easy to understand and implement. Let θ denote the parameter of interest. In the so-called forward FD method, for example, the first-order Greek is approximated by $[C(\theta + h) - C(\theta)]/h$, and the second-order Greek such as gamma by $[C(\theta + 2h) - 2C(\theta + h) + C(\theta)]/h^2$. Here, h denotes the perturbed size throughout the paper and must be suitably small to avoid bias due to higher-order terms. Note that resimulation is required because each of $C(\theta)$, $C(\theta - h)$, $C(\theta + h)$, and $C(\theta + 2h)$ has to be established by simulation. A variant is the more accurate but also more costly central FD method. For this method, the first-order Greek becomes $[C(\theta + h/2) - C(\theta - h/2)]/h$, and the second-order Greek becomes $[C(\theta + h) - 2C(\theta) - C(\theta - h)]/h^2$. Table 2 lists related formulas.

Although FD approximations are straightforward, they have one severe weakness in that deciding on the right h is difficult. If h is too large, the Greek estimate would be biased because of the nonlinearity of $C(\theta)$. When $C(\theta)$ is differentiable, FD approximations should be expected to converge to the true value with h small enough. However, this is not the case numerically for simulation methods. In fact, if h is too small, the variation between the original price $C(\theta)$ and the perturbed prices $C(\theta \pm h)$ makes the Greek estimates unstable. Although using common random numbers in resimulation can reduce the estimation error, the above observations remain valid even if variance reduction techniques or stratified sampling are employed [14, 33]. Higher-order Greek estimates are in most cases numerically unstable. That higher-order partial derivatives are estimated at a slower rate of convergence is sometimes referred to as the curse of differentiation.

Methods in the second category produce unbiased estimates but are more elaborate. With the direct method of Broadie and Glasserman [4], the information from a single simulation is used to estimate multiple Greeks besides the option's price. The direct method does not rely on resimulation in calculating the Greeks. Popular direct methods include the pathwise method and the likelihood ratio method. The pathwise method differentiates each simulated outcome with respect to the parameters of interest. The likelihood ratio method, on the other hand, differentiates the probability density function rather than the outcome.

Although the domains of applicability of the pathwise method and the likelihood ratio method overlap, no method dominates the other. When both apply, the pathwise method generally enjoys lower variances. Unfortunately, the applicability of the pathwise method is limited by the requirement of Lipschitz-continuity in the payoff function, which is needed for convergence guarantees. Although the likelihood ratio method can differentiate discontinuous payoff functions, it needs the explicit formula of the density function. Fournié et al. [8] apply integration-by-parts formulas to derive the Greeks using Malliavin calculus. This approach avoids differentiating a discontinuous payoff function and deriving the densities for the underlying securities. Greeks are the expectations of the product of the payoff function and the Malliavin weighting function. Benhamou [1] further bridges the Malliavin weighting method and the likelihood ratio method. It is shown that the likelihood ratio is the weighting function with the smallest total variance.

Several approaches have been proposed to extend the pathwise method to lift the requirement of Lipschitz-continuity. For example, the conditional Monte Carlo method is a very general scheme to smooth the discontinuous integrand by conditioning on some random variables [9]. However, the conditional Monte Carlo method is problem-dependent and may be difficult to implement, particularly for higher-order Greeks. In contrast, we provide a new mathematical formulation so that the Greeks can be derived systematically as long as the payoff function belongs to the class $\mathcal{C}$ defined later. Recently, Liu and Hong extend the pathwise method to options with discontinuous payoff functions [18]. However, our Greek estimates are unbiased and efficient, while their kernel estimates are biased and may suffer from slow convergence rates.

We now briefly summarize our approach as follows. An option's price equals the discounted expected value of its payoff function under the risk-neutral probability measure [11]. The option value is therefore a discounted integral whose integrand is the product of the payoff and a probability density function. Recall that Greeks are the partial derivatives of the price with respect to the parameter of interest. Our method is applicable whenever the payoff function belongs to a class we name $\mathcal{C}$. $\mathcal{C}$ is roughly a family of payoff functions that can be written as a sum of products of differentiable functions and indicator functions with the right kind of support. A formal definition will be given later. For a payoff function from $\mathcal{C}$, we (1) prove that expectation (equivalently, integration) and differentiation can be interchanged, (2) provide the "differentiation" of the indicator functions, and (3) guarantee the validity of the product rule. As a result, as long as an option's payoff function belongs to $\mathcal{C}$, its Greeks can be derived systematically and calculated without bias. The formulas for the Greeks of two rainbow options—the spread option and the maximum option—are given to illustrate the broad applicability of our method. We concentrate on deltas and gammas as other Greeks can be treated similarly.

Our method has several advantages over other schemes. First, it addresses the requirement of Lipschitz-continuity for the payoff function by the pathwise method. This condition has severely restricted the applicability of the pathwise method. For example, digital options cannot be handled by the pathwise method as their payoff function is an indicator function. But they pose no problems for our method. Second, our method gives provably unbiased estimates by doing away with FD. Third, our method is more efficient than FD approximations as resimulation is avoided. This feature is particularly beneficial for higher-order Greeks as we shall see later. Moreover, numerical results show that our method produces Greek estimates with lower standard errors than the likelihood ratio method. Last but not the least, our method is easy to implement and its application is almost mechanical compared with other available methods, such as the conditional Monte Carlo method and the likelihood ratio method. Subsequent sections will establish all these claims.

The paper is organized as follows. In Sect. 2, the mathematics necessary for the derivation of the Greeks are established. Fundamental theorems to differentiate an integration whose integrand is a product of a differentiable function and several indicator functions are presented in Sect. 3. In Sect. 4, formulas of the Greeks for several popular rainbow and path-dependent options are given. The paper ends with numerical results and conclusions in Sects. 5 and 6, respectively. The appendices contain proofs for several technical results stated in the main text.

2 Preliminaries

Let $\mathbf{x} = (x_1, \ldots, x_n)^t$, where the superscript "t" stands for matrix transpose throughout the paper. The n-variate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ is expressed as $N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. When $\boldsymbol{\Sigma}$ is positive definite and $\mathbf{x} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, the probability density function (pdf) of $\mathbf{x}$ is

$$f(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{\mathrm{e}^{-(\mathbf{x}-\boldsymbol{\mu})^t \boldsymbol{\Sigma}^{-1} (\mathbf{x}-\boldsymbol{\mu})/2}}{\sqrt{(2\pi)^n |\boldsymbol{\Sigma}|}}.$$

Above, $|\boldsymbol{\Sigma}|$ denotes the determinant of $\boldsymbol{\Sigma}$. This paper uses n to denote the number of underlying assets. Let θ denote a parameter of interest throughout this paper. The indicator function $\mathbf{1}_A(\mathbf{x})$ is a function from $\Re^n$ to $\{1, 0\}$ with a subset $A \subseteq \Re^n$ such that

$$\mathbf{1}_A(\mathbf{x}) = \begin{cases} 1, & \text{for } \mathbf{x} \in A, \\ 0, & \text{otherwise.} \end{cases}$$

To simplify the presentation, we may simply write down the predicate defining the set A instead of using the set-theoretical notation as above. The set A is also called the support of the indicator function.

Let $\wp$ denote the payoff function of an option. For the Black–Scholes model, the option's price is given by $C = \mathrm{e}^{-rT} E[\wp]$, where E denotes the expectation operator under the risk-neutral measure throughout this paper [11]. When the expectation $E[\wp]$ is intractable to calculate deterministically, we estimate the expected value $\mathrm{e}^{-rT} E[\wp]$ by

$$\frac{1}{N} \mathrm{e}^{-rT} \sum_{n=1}^{N} \wp\big(\omega^{(n)}\big),$$

where the N sampled paths $\omega^{(1)}, \ldots, \omega^{(n)}$ are drawn from a proper distribution. This is the standard Monte Carlo method [10]. The strong law of large numbers guarantees that the estimated number will converge to $\mathrm{e}^{-rT} E[\wp]$ with probability one when the sample size N is large enough under very weak regularity conditions.

For a rainbow option in the Black–Scholes model, the price of each underlying asset follows a log-normal diffusion process in the risk-neutral economy,

$$dS_i = rS_i\, dt + \sigma_i S_i\, dZ_i, \quad \text{for } i = 1, \ldots, n,$$

where

$$\begin{aligned} S_i &= \text{the current price of asset } i, \\ r &= \text{the risk-free interest rate}, \\ \sigma_i &= \text{the volatility of asset } i, \\ dZ_i &= \text{the Wiener process}, \\ \rho_{ij} &= \text{the instantaneous correlation between } dZ_i \text{ and } dZ_j. \end{aligned}$$

Let $S_i(T)$ denote the price of asset i at maturity and $\mathbf{S}_T$ denote the asset prices at maturity, i.e., $\mathbf{S}_T = (S_1(T), \ldots, S_n(T))$. By the log-normality of $\mathbf{S}_T$,

$$S_i(T) = S_i \mathrm{e}^{(r-\sigma_i^2/2)T + \sigma_i \sqrt{T} x_i}, \qquad \mathbf{x} \sim N_n(\mathbf{0}, \boldsymbol{\Sigma}), \quad i = 1, \ldots, n, \tag{2.1}$$

where the correlation matrix is defined as $\boldsymbol{\Sigma} = [\rho_{ij}]$ with ρ_{ij} being the instantaneous correlation between dZ_i and dZ_j, and $\mathbf{x}$ is the underlying randomness driving the transition from the initial stock prices to the stock prices at maturity. Note that $S_i(T)$

increases monotonically with x_i. As $\wp$ depends on $\mathbf{S}_T$, which in turn depends on $\mathbf{x}$, the rainbow option's price is the integral

$$C = \mathrm{e}^{-rT} E\big[\wp(\mathbf{S}_T)\big] = \mathrm{e}^{-rT} \int_{\Re^n} \wp(\mathbf{S}_T)\, f(\mathbf{x}; \mathbf{0}, \Sigma)\, d\mathbf{x}. \tag{2.2}$$

Note that the payoff function $\wp$ of a rainbow option depends on $\mathbf{S}_T$ and recall that θ is the parameter of interest. Since $\mathbf{S}_T$ depends on θ and a normally distributed random term $\mathbf{x}$, we may use $\wp(\theta, \mathbf{x})$ for $\wp$ to make the dependency explicit. Equation (2.2) shows that an option's price can be written as an integral of the form

$$C = \mathrm{e}^{-rT} E[\wp] = \mathrm{e}^{-rT} \int_{\Re^n} \wp(\theta, \mathbf{x}) f(\mathbf{x}; \mathbf{0}, \boldsymbol{\Sigma})\, d\mathbf{x}. \tag{2.3}$$

Note that (2.3) integrates over the domain of $\mathbf{x}$.

For single-asset path-dependent options, we generate the stock prices at discrete monitored dates $\{t_0, t_1, t_2, \ldots, t_m\}$ in equal time intervals $\Delta t = t_j - t_{j-1} = T/m$ for $j = 1, \ldots, m$. (Generalization to unequal time intervals is straightforward.) The underlying asset's initial price is $S(t_0) = S$, and the underlying asset's price at maturity is $S(t_m) = S(T)$. Let $\mathbf{S}$ denote a price path $(S(t_1), \ldots, S(t_m))$ generated by

$$S(t_{i+1}) = S(t_i)\, \mathrm{e}^{(r-\sigma^2/2)\Delta t + \sigma\sqrt{\Delta t}x_i}, \qquad \mathbf{x} \sim N_m(\mathbf{0}, \mathbf{I}_m). \tag{2.4}$$

Above, $\mathbf{I}_m$ is the $m \times m$ identity matrix, whose entries are ones in the diagonal and zeros elsewhere. Since $\wp$ depends on $\mathbf{S}$, which in turn depends on $\mathbf{x}$, the path-dependent option's price equals

$$C = \mathrm{e}^{-rT} E\big[\wp(\mathbf{S})\big] = \mathrm{e}^{-rT} \int_{\Re^m} \wp(\mathbf{S}) f(\mathbf{x}; \mathbf{0}, \mathbf{I}_m)\, d\mathbf{x},$$

again, an integral.

A first-order Greek with respect to θ equals

$$\frac{\partial}{\partial\theta} \mathrm{e}^{-rT} E[\wp].$$

Our methodology will depend on the validity of the interchange of the order of expectation and differentiation, i.e.,

$$\frac{\partial}{\partial\theta}\, \mathrm{e}^{-rT} E[\wp] = \mathrm{e}^{-rT} E\left[\frac{\partial\wp}{\partial\theta}\right]. \tag{2.5}$$

If (2.5) holds, the right-hand side equals the desired Greek, which is again an expectation and can often be estimated without bias by the standard Monte Carlo method. Equally importantly, if (2.5) holds, then there is no need for FD or resimulation. Under the assumption that (2.5) holds, a second-order Greek with respect to θ equals

$$\frac{\partial^2}{\partial\theta^2} \mathrm{e}^{-rT} E[\wp] = \frac{\partial}{\partial\theta}\, \mathrm{e}^{-rT} E\left[\frac{\partial\wp}{\partial\theta}\right]. \tag{2.6}$$

If it is again valid to interchange the order of expectation and differentiation in the right-hand side of (2.6), we have

$$\frac{\partial^2}{\partial\theta^2}\mathrm{e}^{-rT}E[\wp]=\mathrm{e}^{-rT}E\left[\frac{\partial^2\wp}{\partial\theta^2}\right]$$

by (2.5). In this case, a gamma equals the delta of a delta. Exactly the same argument can be repeated for cross-gammas and higher-order Greeks.

Broadie and Glasserman [4] establish a set of conditions on $\wp$ for (2.5) to hold, which yields the pathwise method. Loosely speaking, the order of differentiation and expectation can be interchanged for Lipschitz-continuous payoff functions. The pathwise method, however, may not be applicable when the payoff function is not Lipschitz-continuous. Indeed, many payoff functions are not Lipschitz-continuous. The payoff function of a digital option, for example, is not Lipschitz-continuous. The pathwise method is even less applicable to calculating the gammas [4]. For example, the European call option's price is

$$\mathrm{e}^{-rT}\int_{\Re}\big(S(T)-K\big)\mathbf{1}_{\{S(T)>K\}}(x)\,f\big(x;0,\sigma^2\big)\,dx,$$

where $S(T)=S\mathrm{e}^{(r-\sigma^2/2)T+\sigma\sqrt{T}x}$ with $x\sim N(0,\sigma^2)$. Since the integrand above is Lipschitz-continuous with respect to S, its delta can be derived by the pathwise method as

$$\mathrm{e}^{-rT}\int_{\Re}\mathrm{e}^{(r-\sigma^2/2)T+\sigma\sqrt{T}x}\mathbf{1}_{\{S(T)>K\}}(x)\,f\big(x;0,\sigma^2\big)\,dx.$$

As the integrand above is no longer Lipschitz-continuous, the pathwise method cannot be used to derive the gamma of the call option without undergoing some modifications. In this case, Greeks with the pathwise method do not yield to a simple, unified treatment.

One salient feature of an option's payoff is that a positive cash flow occurs only when the underlying asset's prices meet certain conditions, which is the reason it is called a contingent claim. For example, a vanilla call option pays off only when the stock price at maturity is higher than the strike price K. As a result, its payoff function is

$$\max\big(S(T)-K,0\big)=\big(S(T)-K\big)\times\mathbf{1}_{\{S(T)-K>0\}}(x),$$

which is a product of a differentiable function and an indicator function. Recall that $S(T)=S\mathrm{e}^{(r-\sigma^2/2)T+\sigma\sqrt{T}x}$, where $x\sim N(0,1)$. With $g(\theta,x):=S(T)-K$, where θ is the parameter of interest, the support $\{S(T)-K>0\}$ of the indicator function becomes $\{g(\theta,x)>0\}$. The options studied in this paper will undergo similar transformations.

We now formalize the above-mentioned transformation. Given any function $h(\theta,\mathbf{x})$ with pdf $f(\mathbf{x})$ for $\mathbf{x}$, let $h_\theta(\theta,\mathbf{x})$ denote the partial differentiation of $h(\theta,\mathbf{x})$ with respect to θ. Recall that x_k is the kth component of $\mathbf{x}$. We define a class of functions called $\mathcal{H}_k$ such that $h(\theta,\mathbf{x})\in\mathcal{H}_k$ if

1. $h(\theta, \mathbf{x})$ is θ-differentiable;
2. $\int |h(\theta, \mathbf{x})| f(\mathbf{x})\, d\mathbf{x} < \infty$ and $\int |h_\theta(\theta, \mathbf{x})| f(\mathbf{x})\, d\mathbf{x} < \infty$;
3. $h(\theta, \mathbf{x}) f(\mathbf{x})$ and $h_\theta(\theta, \mathbf{x}) f(\mathbf{x})$ are uniformly continuous with respect to θ and x_k on a compact set.

When $n = 1$, we simply write $h(\theta, x) \in \mathcal{H}$ as $\mathbf{x}$ is 1-dimensional.

Let $\mathbf{x}_k = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n)^t$, which denotes $\mathbf{x}$ with the kth component removed. We now define another class of functions named $\mathcal{G}_k$ such that $g(\theta, \mathbf{x}) \in \mathcal{G}_k$ if the following properties are satisfied:

1. $g(\theta, \mathbf{x})$ is θ-differentiable and $\partial g(\theta, \mathbf{x})/\partial\theta$ is continuous in θ;
2. $g(\theta, \mathbf{x})$ is x_k-differentiable and $\partial g(\theta, \mathbf{x})/\partial x_k$ is continuous in x_k;
3. $g(\theta, \mathbf{x})$ is strictly monotone in x_k;
4. there exists a point for x_k depending on θ and $\mathbf{x}_k$, written as $\chi(\theta, \mathbf{x}_k)$, such that

$$g(\theta, \mathbf{x})\big|_{x_k=\chi(\theta,\mathbf{x}_k)} := g\big(\theta, x_1, \dots, x_{k-1}, \chi(\theta, \mathbf{x}_k), x_{k+1}, \dots, x_n\big) = 0.$$

When $n = 1$, we write $g(\theta, x) \in \mathcal{G}$ as $\mathbf{x}$ is 1-dimensional and we replace $\chi(\theta, \mathbf{x}_k)$ with $\chi(\theta)$ because $\mathbf{x}_k$ is 0-dimensional. In general, by "$g_j(\theta, \mathbf{x}) \in \mathcal{G}_k$ for $j \in \mathcal{B}$, where $\mathcal{B}$ is a finite set of natural numbers", we mean that for each $j \in \mathcal{B}$, $g_j(\theta, \mathbf{x})$ belongs to $\mathcal{G}_k$ with corresponding function $\chi_j(\theta, \mathbf{x}_k)$ such that $g_j(\theta, \mathbf{x})|_{x_k=\chi_j(\theta,\mathbf{x}_k)} = 0$. We remark that the fourth assumption above does not require the existence of a closed-form formula for $\chi(\theta, \mathbf{x}_k)$. In addition, these four assumptions ensure that the partial differentiation of $\chi(\theta, \mathbf{x}_k)$ with respect to θ can be calculated alternatively as

$$\frac{\partial\chi(\theta, \mathbf{x}_k)}{\partial\theta} = -\left[\frac{\partial g(\theta, \mathbf{x})/\partial\theta}{\partial g(\theta, \mathbf{x})/\partial x_k}\right]_{x_k=\chi(\theta,\mathbf{x}_k)}$$

by the implicit function theorem. As will be shown later in Theorems 3.1–3.3, the term $\partial g(\theta, x)/\partial x_k$ in the denominator is nonzero because of the third property that $g(\theta, \mathbf{x})$ is strictly monotone in x_k.

We are ready to define the desired class $\mathcal{C}$ of payoff functions. In the simplest case, we say $\wp(\theta, \mathbf{x}) \in \mathcal{C}$ if $\wp(\theta, \mathbf{x})$ can be written as

$$\wp(\theta, \mathbf{x}) = h(\theta, \mathbf{x}) \prod_{j\in\mathcal{B}} \mathbf{1}_{\{g_j(\theta,\mathbf{x})>0\}}(\mathbf{x}), \tag{2.7}$$

where

1. $h(\theta, \mathbf{x}) \in \mathcal{H}_k$ with pdf $f(\mathbf{x})$ for $f(\mathbf{x})$;
2. $g_j(\theta, \mathbf{x}) \in \mathcal{G}_k$ for $j \in \mathcal{B}$, a finite set of natural numbers;
3. for $j \in \mathcal{B}$,

$$\int_{\Re^n} \left| h(\theta, \mathbf{x}) \frac{\partial g_j(\theta, \mathbf{x})/\partial\theta}{\partial g_j(\theta, \mathbf{x})/\partial x_k} \right| f(\mathbf{x})\, d\mathbf{x} < \infty.$$

We remove the subscript j from $g_j(\theta, \mathbf{x})$ when $\mathcal{B}$ is a singleton. More generally, $\wp(\theta, \mathbf{x}) \in \mathcal{C}$ if $\wp(\theta, \mathbf{x})$ is a sum such that each summand can be written like the right-hand side of (2.7). Roughly speaking, a payoff function is in $\mathcal{C}$ if it is a sum of products of a differentiable function and several indicator functions with the right kind of support. Although we may later refer to a payoff function being in $\mathcal{C}$, it is merely a loose—albeit convenient—expression because the definition of $\mathcal{C}$ involves not only the payoff function but also the distribution of $\mathbf{x}$ as well. For an option with a payoff function $\wp(\theta, \mathbf{x}) \in \mathcal{C}$, we shall show how to differentiate its price with respect to θ in the next section.

We remark that Lipschitz-continuity appears to be a subset of $\mathcal{C}$ under most circumstances of practical importance. More specifically, the domains of Lipschitz-continuity and $\mathcal{C}$ overlap, but do not have any inclusion relation. Lipschitz-continuity imposes stronger conditions than $\mathcal{C}$ in that it requires a function to be continuous. On the other hand, $\mathcal{C}$ imposes stronger conditions than Lipschitz-continuity in that it allows a function to be nondifferentiable on a set containing only a finite number of points, whereas Lipschitz-continuity allows a function to be nondifferentiable on a set with measure zero. Recall that a set containing only a finite number of points clearly has measure zero.

Although the focus of this paper is on Greeks under the Black–Scholes model, our method is applicable to different underlying models. For instance, if the underlying asset price follows a general diffusion process $dS_t = \mu(t, S_t) + \sigma(t, S_t)dZ_t$ where dZ_t is the Wiener process, then S may need to be simulated using a discretization scheme, e.g., the Euler scheme

$$S_{t_{i+1}} = S_{t_i} + \mu(t_i, S_{t_i})\Delta t + \sigma(t_i, S_{t_i})\sqrt{\Delta t}\, x_{i+1}, \tag{2.8}$$

for $i = 0, 1, \ldots, n-1$. Replacing $\mu(t_i, S_{t_i})\Delta t$ by μ_i and $\sigma(t_i, S_{t_i})\sqrt{\Delta t}$ by σ_i for brevity, we can write (2.8) as $S_{t_{i+1}} = S_{t_i} + \mu_i + \sigma_i x_{i+1}$. By the recursive formula, we have that

$$S_{t_n} = S_{t_{n-1}} + \mu_{n-1} + \sigma_{n-1}x_n = S_0 + \mu_0 + \cdots + \mu_{n-1} + \sigma_0 x_1 + \cdots + \sigma_{n-1}x_n$$

is strictly increasing in x_n. For illustration, consider a European call option having payoff function $\wp(\theta, \mathbf{x}) = h(\theta, \mathbf{x})\mathbf{1}_{\{g(\theta,\mathbf{x})\}}(\mathbf{x})$, where $h(\theta, \mathbf{x}) = S(T) - K = S_{t_n} - K$ and $g(\theta, \mathbf{x}) = h(\theta, \mathbf{x})$ with $\mathbf{x} = (x_1, \ldots, x_n)$ having pdf $f(\mathbf{x})$. Clearly, $g(\theta, \mathbf{x})$ is strictly monotone in x_n; hence, as long as (1) $h(\theta, \mathbf{x})$ belongs in $\mathcal{H}_n$, (2) $g(\theta, \mathbf{x})$ belongs to $\mathcal{G}_n$, and (3)

$$\int_{\Re^n} \left| h(\theta, \mathbf{x}) \frac{\partial g(\theta, \mathbf{x})/\partial\theta}{\partial g(\theta, \mathbf{x})/\partial x_n} \right| f(\mathbf{x})\, d\mathbf{x} < \infty,$$

the payoff function $\wp(\theta, \mathbf{x})$ belongs to $\mathcal{C}$. This example shows that our method is applicable to a general diffusion process, such as the CIR model and the CEV model, as they are special cases of the general diffusion process [6, 7, 27]. To be sure, our method is also applicable to path-dependent options if their payoff functions lie in $\mathcal{C}$.

3 Fundamental theorems

In Theorem 3.1, we show how to differentiate an integral whose integrand is a product of a function $h(\theta, x) \in \mathcal{H}$, a pdf $f(x)$ for x, and indicator functions whose supports are of the form $\{g_j(\theta, x) > 0\}$, where $g_j(\theta, x) \in \mathcal{G}$ for $j \in \mathcal{B}$, a finite set of natural numbers.

Theorem 3.1 *Suppose $h(\theta, x) \in \mathcal{H}$ with pdf $f(x)$ for x and $g_j(\theta, x) \in \mathcal{G}$ for $j \in \mathcal{B}$, a finite set of natural numbers. Then*

$$\begin{aligned}
&\frac{\partial}{\partial \theta} \int_{\Re} h(\theta, x) \prod_{j \in \mathcal{B}} \mathbf{1}_{\{g_j(\theta,x)>0\}}(x) f(x)\, dx \\
&= \int_{\Re} h_\theta(\theta, x) \prod_{j \in \mathcal{B}} \mathbf{1}_{\{g_j(\theta,x)>0\}}(x) f(x)\, dx \\
&\quad + \sum_{\ell \in \mathcal{B}} \left[h(\theta, x) J_\ell(\theta, x) \prod_{j \in \mathcal{B}\setminus\{\ell\}} \mathbf{1}_{\{g_j(\theta,x)>0\}}(x) f(x) \right]_{x=\chi_\ell(\theta)},
\end{aligned} \tag{3.1}$$

where

$$J_\ell(\theta, x) = \operatorname{sign}\left(\frac{\partial g_\ell(\theta, x)}{\partial x_k} \right) \frac{\partial g_\ell(\theta, x)/\partial \theta}{\partial g_\ell(\theta, x)/\partial x} \quad \text{for } \ell \in \mathcal{B}. \tag{3.2}$$

Proof See Appendix A. □

In plain language, Theorem 3.1 (1) guarantees the validity to interchange the order of differentiation and integration, (2) provides the differentiation of an indicator function with a support of the form $\{g_j(\theta, x) > 0\}$ when $g_j(\theta, x) \in \mathcal{G}$, and (3) establishes the product rule for differentiating a product of functions. Theorem 3.2 generalizes Theorem 3.1 to higher dimensions.

Theorem 3.2 *Suppose $h(\theta, \mathbf{x}) \in \mathcal{H}_k$ with pdf $f(\mathbf{x})$ for $\mathbf{x}$ and $g_j(\theta, \mathbf{x}) \in \mathcal{G}_k$ for $j \in \mathcal{B}$, a finite set of natural numbers. If*

$$\int_{\Re^n} \left| h(\theta, \mathbf{x}) \frac{\partial g_j(\theta, \mathbf{x})/\partial \theta}{\partial g_j(\theta, \mathbf{x})/\partial x_k} \right| f(\mathbf{x})\, d\mathbf{x} < \infty \quad \text{for } j \in \mathcal{B},$$

then

$$\begin{aligned}
&\frac{\partial}{\partial \theta} \int_{\Re^n} h(\theta, \mathbf{x}) \prod_{j \in \mathcal{B}} \mathbf{1}_{\{g_j(\theta,\mathbf{x})>0\}}(\mathbf{x}) f(\mathbf{x})\, d\mathbf{x} \\
&= \int_{\Re^n} h_\theta(\theta, \mathbf{x}) \prod_{j \in \mathcal{B}} \mathbf{1}_{\{g_j(\theta,\mathbf{x})>0\}}(\mathbf{x}) f(\mathbf{x})\, d\mathbf{x} \\
&\quad + \sum_{\ell \in \mathcal{B}} \int_{\Re^{n-1}} \left[h(\theta, \mathbf{x}) J_\ell(\theta, \mathbf{x}) \prod_{j \in \mathcal{B}\setminus\{\ell\}} \mathbf{1}_{\{g_j(\theta,\mathbf{x})>0\}}(\mathbf{x}) f(\mathbf{x}) \right]_{x_k=\chi_\ell(\theta,\mathbf{x}_k)} d\mathbf{x}_k,
\end{aligned} \tag{3.3}$$

where

$$J_\ell(\theta, \mathbf{x}) = \operatorname{sign}\left(\frac{\partial g_\ell(\theta, \mathbf{x})}{\partial x_k} \right) \frac{\partial g_\ell(\theta, \mathbf{x})/\partial \theta}{\partial g_\ell(\theta, \mathbf{x})/\partial x_k} \quad \text{for } \ell \in \mathcal{B}. \tag{3.4}$$

Proof See Appendix B. □

Differentiating an n-dimensional integral in Theorem 3.2 yields an n-dimensional integral and several $(n-1)$-dimensional integrals with the component x_k removed (see (3.3)). Since the n-dimensional integral equals

$$\int_{\Re^n} h_\theta(\theta, \mathbf{x}) \prod_{j\in\mathcal{B}} \mathbf{1}_{\{g_j(\theta,\mathbf{x})>0\}}(\mathbf{x}) f(\mathbf{x})\, d\mathbf{x} = E\left[h_\theta(\theta, \mathbf{x}) \prod_{j\in\mathcal{B}} \mathbf{1}_{\{g_j(\theta,\mathbf{x})>0\}}(\mathbf{x})\right], \quad (3.5)$$

it can in principle be estimated using the Monte Carlo method. Estimating each

$$\int_{\Re^{n-1}} \left[h(\theta, \mathbf{x}) J_\ell(\theta, \mathbf{x}) \prod_{j\in\mathcal{B}\setminus\{\ell\}} \mathbf{1}_{\{g_j(\theta,\mathbf{x})>0\}}(\mathbf{x}) f(\mathbf{x})\right]_{x_k=\chi_\ell(\theta,\mathbf{x}_k)} d\mathbf{x}_k \quad (3.6)$$

in (3.3) can be a major challenge for the following reasons, however. The crude Monte Carlo method estimates (3.6) as follows:

1. Draw N sample paths $\mathbf{x}_k^{(1)}, \ldots, \mathbf{x}_k^{(N)}$ uniformly over $\Re^{n-1}$.
2. Approximate (3.6) by $\frac{1}{N}\sum_{n=1}^N p_1(\mathbf{x}_k^{(n)})$, where

$$p_1(\mathbf{x}_k) := \left[h(\theta, \mathbf{x}) J_\ell(\theta, \mathbf{x}) \prod_{j\in\mathcal{B}\setminus\{\ell\}} \mathbf{1}_{\{g_j(\theta,\mathbf{x})>0\}}(\mathbf{x}) f(\mathbf{x})\right]_{x_k=\chi_\ell(\theta,\mathbf{x}_k)}.$$

Two difficulties arise with the above procedure. For step one, it is difficult to draw $\mathbf{x}_k$ uniformly over $\Re^{n-1}$ directly, which is unbounded. For step two, the estimate may be inefficient in that it may have a very large variance, which demands more replications. To overcome these difficulties, we next provide an alternative form of (3.6) using a distribution for $\mathbf{x}_k$ with pdf $q(\mathbf{x}_k)$. When $q(\mathbf{x}_k)$ has support $\Re^{n-1}$, (3.6) equals

$$\begin{aligned}
&\int_{\Re^{n-1}} \left[h(\theta, \mathbf{x}) J_\ell(\theta, \mathbf{x}) \prod_{j\in\mathcal{B}\setminus\{\ell\}} \mathbf{1}_{\{g_j(\theta,\mathbf{x})>0\}}(\mathbf{x}) \frac{f(\mathbf{x})}{q(\mathbf{x}_k)}\right]_{x_k=\chi_\ell(\theta,\mathbf{x}_k)} q(\mathbf{x}_k)\, d\mathbf{x}_k \\
&= E_q\left[\left[h(\theta, \mathbf{x}) J_\ell(\theta, \mathbf{x}) \prod_{j\in\mathcal{B}\setminus\{\ell\}} \mathbf{1}_{\{g_j(\theta,\mathbf{x})>0\}}(\mathbf{x}) \frac{f(\mathbf{x})}{q(\mathbf{x}_k)}\right]_{x_k=\chi_\ell(\theta,\mathbf{x}_k)}\right] \\
&= E_q\left[\left[h(\theta, \mathbf{x}) J_\ell(\theta, \mathbf{x}) \prod_{j\in\mathcal{B}\setminus\{\ell\}} \mathbf{1}_{\{g_j(\theta,\mathbf{x})>0\}}(\mathbf{x})\right]_{x_k=\chi_\ell(\theta,\mathbf{x}_k)} \tilde{\eta}(\theta, \mathbf{x}_k)\right], \quad (3.7)
\end{aligned}$$

where

$$\tilde{\eta}(\theta, \mathbf{x}_k) = \frac{f(\mathbf{x})|_{x_k=\chi_\ell(\theta,\mathbf{x}_k)}}{q(\mathbf{x}_k)}$$

and $\mathbf{x}_k$ has pdf $q(\mathbf{x})$ under E_q. Mathematically speaking, the identification of (3.6) with (3.7) is a change of measure, and it forms the theoretical basis of the importance sampling method in the following sense: To estimate (3.6), we sample $\mathbf{x}_k$ from an easy-to-sample sampling distribution with pdf $q(\mathbf{x}_k)$. We now estimate (3.6) using (3.7) as follows:

1. Draw N sample paths $\mathbf{x}_k^{(1)}, \ldots, \mathbf{x}_k^{(N)}$ from the sampling distribution $q(\mathbf{x}_k)$.
2. Calculate the importance weight $\tilde{\eta}^{(n)}$ for $n = 1, \ldots, N$ by plugging $\mathbf{x}_k^{(n)}$ into the ratio

$$\frac{f(\mathbf{x})|_{x_k=\chi_\ell(\theta,\mathbf{x}_k)}}{q(\mathbf{x}_k)}.$$

3. Approximate (3.6) by $\frac{1}{N}\sum_{n=1}^{N} \tilde{\eta}^{(n)} p_2(\mathbf{x}_k^{(n)})$, where

$$p_2(\mathbf{x}_k) := \left[h(\theta, \mathbf{x}) J_\ell(\theta, \mathbf{x}) \prod_{j \in \mathcal{B} \setminus \{\ell\}} \mathbf{1}_{\{g_j(\theta,\mathbf{x})>0\}}(\mathbf{x}) \right]_{x_k=\chi_\ell(\theta,\mathbf{x}_k)}.$$

As for which $q(\mathbf{x}_k)$ to use, a good candidate is one that is close to the shape of the integrand of (3.6). However, finding $q(\mathbf{x}_k)$ close to the integrand of (3.6) requires sophisticated analysis and can be extremely difficult, especially in high-dimensional cases [19]. For simplicity, we suggest the normal distribution as the sampling distribution. For illustration purposes, Theorem 4.2 presents the Greek formulas of the maximum option using the standard normal as the sampling distribution, whereas Theorem 5.1 uses the shifted normal as the sampling distribution. As will be shown in Sect. 5.2, using the shifted normal as the sampling distribution can improve the estimate of the Greeks for deep out-of-the-money options in terms of accuracy and standard error, whereas the standard normal works well for at-the-money and in-the-money options. Briefly speaking, using the normal distribution as the sampling distribution is easy and convenient; furthermore, it produces excellent Greek estimates as will be shown later.

Now, we are ready to present the way to calculate the first-order Greeks under the Black–Scholes model.

Theorem 3.3 *Consider an option under the Black–Scholes model. Suppose the payoff function $\wp(\theta, \mathbf{x})$ belongs in $\mathcal{C}$ with $\mathbf{x} \sim N_n(\mathbf{0}, \boldsymbol{\Sigma})$, i.e.,*

$$\wp(\theta, \mathbf{x}) = h(\theta, \mathbf{x}) \prod_{j \in \mathcal{B}} \mathbf{1}_{\{g_j(\theta,\mathbf{x})>0\}}(\mathbf{x})$$

for some functions $h(\theta, \mathbf{x}) \in \mathcal{H}_k$ and $g_j(\theta, \mathbf{x}) \in \mathcal{G}_k$ for $j \in \mathcal{B}$, a finite set of natural numbers. Moreover, suppose that $q(\mathbf{x}_k)$ is a pdf for $\mathbf{x}_k$ with support $\Re^{n-1}$. Then

$$\begin{aligned}
&\frac{\partial}{\partial\theta} E\big[\wp(\theta, \mathbf{x})\big] \\
&\quad = E\left[h_\theta(\theta, \mathbf{x}) \prod_{j \in \mathcal{B}} \mathbf{1}_{\{g_j(\theta,\mathbf{x})>0\}}(\mathbf{x}) \right] \\
&\qquad + \sum_{\ell \in \mathcal{B}} E_q\left[\left[h(\theta, \mathbf{x}) J_\ell(\theta, \mathbf{x}) \prod_{j \in \mathcal{B} \setminus \{\ell\}} \mathbf{1}_{\{g_j(\theta,\mathbf{x})>0\}}(\mathbf{x}) \right]_{x_k=\chi_\ell(\theta,\mathbf{x}_k)} \tilde{\eta}_\ell(\theta, \mathbf{x}_k) \right],
\end{aligned} \tag{3.8}$$

where

$$\tilde{\eta}_\ell(\theta, \mathbf{x}_k) = \frac{f(\mathbf{x}; \mathbf{0}, \mathbf{\Sigma})|_{x_k=\chi_\ell(\theta,\mathbf{x}_k)}}{q(\mathbf{x}_k)},$$

$J_\ell(\theta, \mathbf{x})$ *equals* (3.4), *and* $\mathbf{x}_k$ *has pdf* $q(\mathbf{x}_k)$ *under* E_q.

Proof Because

$$E\big[\wp(\theta, \mathbf{x})\big] = \int_{\Re^n} h(\theta, \mathbf{x}) \prod_{j\in\mathcal{B}} \mathbf{1}_{\{g_j(\theta,\mathbf{x})>0\}}(\mathbf{x}) f(\mathbf{x}; \mathbf{0}, \mathbf{\Sigma})\, d\mathbf{x},$$

$\partial E[\wp(\theta, \mathbf{x})]/\partial\theta$ equals the right-hand side of (3.3) except that $\mathbf{x}$ has pdf $f(\mathbf{x}; \mathbf{0}, \mathbf{\Sigma})$ by Theorem 3.2. Applying (3.5) and (3.7), (3.8) is proved. □

Now, first-order Greeks can be derived if the payoff function $\wp$ belongs to $\mathcal{C}$. To derive Greeks like delta, vega, and theta, simply apply the corresponding theorem with $\theta = S$, $\theta = \sigma$, and $\theta = T$, respectively. To derive rho, apply the corresponding theorem with $\theta = r$ via

$$\frac{\partial}{\partial r} \mathrm{e}^{-rT} E[\wp] = -T\mathrm{e}^{-rT} E[\wp] + \mathrm{e}^{-rT} \frac{\partial}{\partial r} E[\wp].$$

Alternatively, it is well known that

$$\Theta + r\sum_{i=1}^{n} S_i \Delta_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{ij}\sigma_i\sigma_j S_i S_j \Gamma_{ij} = r\mathcal{C},$$

which provides a numerical approach to deriving theta using deltas, gammas, and cross-gammas. See Cox et al. [7] and Boyle et al. [3] for details. Note that the right-hand side of (3.3) is a formula for first-order Greeks. When a closed-form formula for $\chi(\theta, \mathbf{x})$ exists, we simply plug $\chi(\theta, \mathbf{x}_k)$ into x_k. If a closed-form formula does not exist, we can plug into x_k a numerical solution using, say, the Newton–Raphson method. As a result, we only require the existence of $\chi(\theta, \mathbf{x}_k)$ when calculating first-order Greeks.

The second-order Greek is obtained by applying our theorems again to the corresponding first-order Greek. A closed-form formula for $\chi_\ell(\theta, \mathbf{x}_k)$ is required to calculate the second-order Greeks for the most general case. However, (3.3) can be much simplified before applying differentiation to obtain second-order Greeks when its second term reduces to zero (as in the deltas of spread options and maximum options shown later). When this is the case, we only need to differentiate

$$\int_{R^n} h_\theta(\theta, \mathbf{x}) \prod_{j\in\mathcal{B}} \mathbf{1}_{\{g_j(\theta,\mathbf{x})>0\}}(\mathbf{x}) f(\mathbf{x})\, d\mathbf{x} \tag{3.9}$$

of (3.3), and this reduces to the case of deriving first-order Greeks and is solved by applying Theorem 3.2 again. In this case, the existence of $\chi_\ell(\theta, \mathbf{x}_k)$ is sufficient (i.e., closed-form formulas for $\chi_\ell(\theta, \mathbf{x}_k)$ are not needed). As a remark, when differentiating

(3.9) with respect to θ, we need to check if (1) $h_\theta(\theta,\mathbf{x})$ belongs to $\mathcal{H}_k$, (2) $g_j(\theta,\mathbf{x})$ belongs to $\mathcal{G}_k$ for $j \in \mathcal{B}$, and (3)

$$\int_{\Re^n} \left| h_\theta(\theta,\mathbf{x}) \frac{\partial g_j(\theta,\mathbf{x})/\partial\theta}{\partial g_j(\theta,\mathbf{x})/\partial x_k} \right| f(\mathbf{x})\, d\mathbf{x} < \infty \quad \text{for } j \in \mathcal{B}.$$

Higher-order Greeks can be obtained recursively using Theorem 3.2.

4 Greeks of rainbow options and path-dependent options

The term "rainbow option" originates from Rubinstein [25], who describes a rainbow option as a combination of a variety of assets much as a rainbow is a combination of a variety of colors. There is an abundance of rainbow options in the literature (see Table 1). In the following, we show how to derive the unbiased Greeks of three rainbow options and two path-dependent options using our methodology.

4.1 Greeks of spread options

In this section, we derive Δ_1, Γ_{11} and Γ_{12} for the spread option. The payoff function of the spread option is

$$\wp = \max\big(S_2(T) - S_1(T) - K, 0\big) = \big(S_2(T) - S_1(T) - K\big)\mathbf{1}_{\{S_2(T)-S_1(T)-K>0\}}(\mathbf{S}_T).$$

Spread options are options written on the difference between the values of two stocks or two indices. They are used in many markets, such as the fixed-income markets, the currency markets, the commodity markets, and especially the energy markets. These options are popular because they are designed to mitigate adverse movements between two market variables. There is extensive literature on pricing the spread options; however, calculating their Greeks is still challenging because no closed-form formulas exist for their prices [5].

In the fixed-income markets and the currency markets, spread options are based on the difference between two interest rates or swap rates. In the commodity markets, spread options are based on the difference between the prices of the same commodity at two different locations or at two different points in time, as well as between the prices of different grades of the same commodity. At the New York Mercantile Exchange, spread options are traded on the difference between heating oil and crude oil, as well as between gasoline and crude oil. These spreads are known as crack spreads, hence options on these spreads are known as crack spread options. Another example in the energy market involves options on the price difference between oil and electricity which are known as spark spread options [5].

Using (2.1) to generate $\mathbf{S}_T$ and defining

$$g(\theta,\mathbf{x}) = h(\theta,\mathbf{x}) = S_2\, \mathrm{e}^{(r-\sigma_2^2/2)T+\sigma_2\sqrt{T}x_2} - S_1 \mathrm{e}^{(r-\sigma_1^2/2)T+\sigma_1\sqrt{T}x_1} - K,$$

we rewrite the payoff function as $\wp(\theta,\mathbf{x}) = h(\theta,\mathbf{x})\mathbf{1}_{\{g(\theta,\mathbf{x})>0\}}(\mathbf{x})$. We proceed to prove that the payoff function belongs to $\mathcal{C}$. Note that $h(\theta,\mathbf{x}) \in \mathcal{H}_2$ with pdf

$f(\mathbf{x};\mathbf{0},\mathbf{\Sigma})$ for $\mathbf{x}$ and $g(\theta,\mathbf{x})$ is increasing in θ and x_2. Let

$$\chi(\theta,\mathbf{x}_2)=\frac{\log\frac{S_1 \mathrm{e}^{(r-\sigma_1^2/2)T+\sigma_1\sqrt{T}x_1}+K}{S_2}-(r-\sigma_2^2/2)T}{\sigma_2\sqrt{T}}. \tag{4.1}$$

It is easy to verify that $g(\theta,\mathbf{x})|_{x_2=\chi(\theta,\mathbf{x}_2)}=0$; hence $g(\theta,\mathbf{x})\in\mathcal{G}_2$. We conclude that $\wp(\theta,\mathbf{x})\in\mathcal{C}$.

Theorem 4.1 *The delta, gamma and cross-gamma of the spread option equal*

$$\begin{aligned}
\varDelta_1&=\mathrm{e}^{-rT}E\big[-S_1(T)\mathbf{1}_{\{S_2(T)-S_1(T)-K>0\}}\big(\mathbf{S}(T)\big)\big]/S_1,\\
\varGamma_{11}&=\mathrm{e}^{-rT}E_q\big[\mathrm{e}^{2((r-\sigma_1^2/2)T+\sigma_1\sqrt{T}x_1)-((r-\sigma_2^2/2)T+\sigma_2\sqrt{T}\chi(\theta,\mathbf{x}_2))}\tilde{\eta}(\theta,\mathbf{x}_2)\big]/(S_2\sigma_2\sqrt{T}),\\
\varGamma_{12}&=-\mathrm{e}^{-rT}E_q\big[\mathrm{e}^{(r-\sigma_1^2/2)T+\sigma_1\sqrt{T}x_1}\tilde{\eta}(\theta,\mathbf{x}_2)\big]/(S_2\sigma_2\sqrt{T}),
\end{aligned}$$

respectively, where $\chi(\theta,\mathbf{x}_2)$ *is defined in* (4.1), $\tilde{\eta}(\theta,\mathbf{x}_2)$ *is defined in* (9.1), *and* $\mathbf{x}_2$ *has pdf* $q(\mathbf{x}_k)=f(\mathbf{x}_2;0,1)$ *under* E_q.

Proof See Appendix C. □

4.2 Greeks of maximum options

We now derive the Greeks of the maximum option. Let $\mathcal{B}=\{1,\dots,n\}$ throughout this section. By conditioning on which asset ends up as the maximum terminal price, the payoff function can be decomposed into

$$\begin{aligned}
\wp&=\max\big(\max\big(S_1(T),\dots,S_n(T)\big)-K,0\big)\\
&=\sum_{i\in\mathcal{B}}\big(S_i(T)-K\big)\mathbf{1}_{\{S_i(T)>K\}}(\mathbf{S}_T)\prod_{j\in\mathcal{B}\backslash\{i\}}\mathbf{1}_{\{S_i(T)>S_j(T)\}}(\mathbf{S}_T).
\end{aligned}$$

Maximum options depend on the maximum stock price at maturity, and minimum options depend on the minimum stock price. Stulz [30] derives formulas for maximum and minimum options on two risky assets and Johnson [15] generalizes these formulas to options on several assets. Maximum and minimum options are commonly used in compensation plans, risk-sharing contracts, collateralized loans and secured debts, indexed wages and option bonds [30].

Using (2.1) to generate $\mathbf{S}_T$ and defining

$$h_i(\theta,\mathbf{x})=S_i\mathrm{e}^{(r-\sigma_i^2/2)T+\sigma_i\sqrt{T}x_i}-K\quad\text{for } i\in\mathcal{B}, \tag{4.2}$$

$$g_{ii}(\theta,\mathbf{x})=S_i\mathrm{e}^{(r-\sigma_i^2/2)T+\sigma_i\sqrt{T}x_i}-K\quad\text{for } i\in\mathcal{B}, \tag{4.3}$$

$$g_{ij}(\theta,\mathbf{x})=S_i\mathrm{e}^{(r-\sigma_i^2/2)T+\sigma_i\sqrt{T}x_i}-S_j\mathrm{e}^{(r-\sigma_j^2/2)T+\sigma_i\sqrt{T}x_j}\quad\text{for } i\in\mathcal{B},\ j\in\mathcal{B}\backslash\{i\}, \tag{4.4}$$

the payoff function can be rewritten as

$$\wp(\theta, \mathbf{x}) = \sum_{i\in\mathcal{B}} h_i(\theta, \mathbf{x}) \prod_{j\in\mathcal{B}} \mathbf{1}_{\{g_{ij}(\theta,\mathbf{x})>0\}}(\mathbf{x}). \tag{4.5}$$

We proceed to prove that the payoff function belongs to $\mathcal{C}$. Note $h_i(\theta, \mathbf{x}) \in \mathcal{H}_i$ with pdf $f(\mathbf{x}; \mathbf{0}, \mathbf{\Sigma})$ for $\mathbf{x}$ and $g_{ij}(\theta, \mathbf{x})$ is strictly monotone in θ and in x_i. Let

$$\chi_{ii}(\theta, \mathbf{x}_i) = \frac{\log K/S_i - (r - \sigma_i^2/2)T}{\sigma_i\sqrt{T}} \quad \text{for } i \in \mathcal{B}, \tag{4.6}$$

$$\chi_{ij}(\theta, \mathbf{x}_i) = \frac{\log \frac{S_j e^{(r-\sigma_j^2/2)T+\sigma_j\sqrt{T}x_j}}{S_i} - (r - \sigma_i^2/2)T}{\sigma_i\sqrt{T}}$$
$$\text{for } i \in \mathcal{B},\ j \in \mathcal{B}\backslash\{i\}. \tag{4.7}$$

It is easy to verify that $g_{ij}(\theta, \mathbf{x})|_{x_i=\chi_{ij}(\theta,\mathbf{x}_i)} = 0$; hence $g_{ij}(\theta, \mathbf{x}) \in \mathcal{G}_i$. We conclude that $\wp(\theta, \mathbf{x}) \in \mathcal{C}$.

Theorem 4.2 *The delta, gamma and cross-gamma of the maximum option on n assets are*

$$\begin{aligned}
\Delta_i &= e^{-rT} E\bigg[e^{(r-\sigma_i^2/2)T+\sigma_i\sqrt{T}x_i} \mathbf{1}_{\{S_i e^{(r-\sigma_i^2/2)T+\sigma_i\sqrt{T}x_i} > K\}}(\mathbf{x}) \\
&\quad \times \prod_{j\in\mathcal{B}\backslash\{i\}} \mathbf{1}_{\{S_i e^{(r-\sigma_i^2/2)T+\sigma_i\sqrt{T}x_i} > S_j e^{(r-\sigma_j^2/2)T+\sigma_j\sqrt{T}x_j}\}}(\mathbf{x})\bigg], \\
\Gamma_{ii} &= e^{-rT}\bigg\{E_q\bigg[K \prod_{j\in\mathcal{B}\backslash\{i\}} \mathbf{1}_{\{S_j e^{(r-\sigma_j^2/2)T+\sigma_j\sqrt{T}x_j} < K\}}(\mathbf{x}_i)\tilde{\eta}_{ii}(\theta, \mathbf{x}_i)\bigg] \\
&\quad + \sum_{\ell\in\mathcal{B}\backslash\{i\}} E_q\bigg[S_\ell e^{(r-\sigma_\ell^2/2)T+\sigma_\ell\sqrt{T}x_\ell} \mathbf{1}_{\{S_\ell e^{(r-\sigma_\ell^2/2)T+\sigma_\ell\sqrt{T}x_l} > K\}}(\mathbf{x}_i) \\
&\quad \times \prod_{j\in\mathcal{B}\backslash\{\ell,i\}} \mathbf{1}_{\{S_\ell e^{(r-\sigma_l^2/2)T+\sigma_\ell\sqrt{T}x_\ell} > S_j e^{(r-\sigma_j^2/2)T+\sigma_j\sqrt{T}x_j}\}}(\mathbf{x}_i)\tilde{\eta}_{i\ell}(\theta, \mathbf{x}_i)\bigg] \\
&\quad \Big/ \big(S_i^2\sigma_i\sqrt{T}\big)\bigg\}, \\
\Gamma_{ij} &= -e^{-rT} E_q\bigg[e^{(r-\sigma_j^2/2)T+\sigma_j\sqrt{T}x_j} \mathbf{1}_{\{S_j e^{(r-\sigma_j^2/2)T+\sigma_j\sqrt{T}x_j} > K\}}(\mathbf{x}_i) \\
&\quad \times \prod_{\ell\in\mathcal{B}\backslash\{j,i\}} \mathbf{1}_{\{S_j e^{(r-\sigma_j^2/2)T+\sigma_j\sqrt{T}x_j} > S_\ell e^{(r-\sigma_\ell^2/2)T+\sigma_\ell\sqrt{T}x_\ell}\}}(\mathbf{x}_i)\tilde{\eta}_{ij}(\theta, \mathbf{x}_i)\bigg] \Big/ \big(S_i\sigma_i\sqrt{T}\big),
\end{aligned}$$

where $\tilde{\eta}_{ij}(\theta, \mathbf{x}_i)$ for $i, j \in \mathcal{B}$ is defined in (10.6), and $\mathbf{x}_i$ has pdf $f(\mathbf{x}_i; \mathbf{0}, \mathbf{I}_{n-1})$ under E_q for $i \in \mathcal{B}$.

Proof See Appendix D. □

4.3 Greeks of binary maximum options

To illustrate the applicability of our approach for an option with a non-Lipschitz-continuous payoff function, we consider the binary maximum option with the payoff function $\wp = \mathbf{1}_{\{\max(S_1(T),\ldots,S_n(T))>K\}}$. Define $g_{ii}(\theta,\mathbf{x})$ for $i \in \mathcal{B}$ as in (4.3) and $g_{ij}(\theta,\mathbf{x})$ for $i \in \mathcal{B}$, $j \in \mathcal{B}\backslash\{i\}$ as in (4.4). The payoff function can be written as

$$\wp(\theta,\mathbf{x}) = \sum_{i\in\mathcal{B}}\prod_{j\in\mathcal{B}} \mathbf{1}_{\{g_{ij}(\theta,\mathbf{x})>0\}}(\mathbf{x}).$$

Define $\chi_{ii}(\theta,\mathbf{x}_i)$ for $i \in \mathcal{B}$ as in (4.6) and $\chi_{ij}(\theta,\mathbf{x}_i)$ for $i \in \mathcal{B}$, $j \in \mathcal{B}\backslash\{i\}$ as in (4.7). Following similar arguments as in Sect. 4.2, we conclude that $g_{ij}(\theta,\mathbf{x}) \in \mathcal{G}_i$ for $i, j \in \mathcal{B}$ and $\wp(\theta,\mathbf{x}) \in \mathcal{C}$. Let $\mathbf{v_i}$ be an $n \times 1$ vector having 1 in the ith component and 0 elsewhere, and let $\mathbf{x}_{ij}$ be the vector $\mathbf{x}$ with x_i and x_j removed.

Theorem 4.3 *The delta, gamma and cross-gamma of the binary maximum option on n assets are*

$$\Delta_i = \mathrm{e}^{-rT} E_q\Bigg[\prod_{j\in\mathcal{B}\backslash\{i\}} \mathbf{1}_{\{S_j \mathrm{e}^{(r-\sigma_j^2/2)T+\sigma_j\sqrt{T}x_j}<K\}}(\mathbf{x})\tilde{\eta}_{ii}(\theta,\mathbf{x}_i)\Bigg]\Big/\big(S_i\sigma_i\sqrt{T}\big),$$

$$\Gamma_{ii} = \mathrm{e}^{-rT} E_q\Bigg[\bigg(\frac{-1}{S_i} + \frac{\mathbf{v_i}^t\Sigma^{-1}\tilde{\mathbf{x}}_i + \tilde{\mathbf{x}}_i^t\Sigma^{-1}\mathbf{v_i}}{2S_i\sigma_i\sqrt{T}}\bigg)$$
$$\times \prod_{j\in\mathcal{B}\backslash\{i\}} \mathbf{1}_{\{S_j \mathrm{e}^{(r-\sigma_j^2/2)T+\sigma_j\sqrt{T}x_j}<K\}}(\mathbf{x}_i)\tilde{\eta}_{ii}(\mathbf{x}_i)\Bigg]\Big/\big(S_i\sigma_i\sqrt{T}\big),$$

$$\Gamma_{ij} = -\mathrm{e}^{-rT} E_{q,j}\Bigg[\prod_{\ell\in\mathcal{B}\backslash\{i,j\}} \mathbf{1}_{\{S_\ell \mathrm{e}^{(r-\sigma_\ell^2/2)T+\sigma_\ell\sqrt{T}x_\ell}<K\}}(\mathbf{x}_{ij})$$
$$\times\tilde{\eta}_{ii,j}(\theta,\mathbf{x}_{ij})f(\chi_{jj}(\theta);0,1)\Bigg]\Big/\big(S_iS_j\sigma_i\sigma_jT\big),$$

where $\tilde{\mathbf{x}}_i$ is defined in (11.2), $\tilde{\eta}_{ii,j}(\theta,\mathbf{x}_{ij})$ is defined in (11.3), $\chi_{jj}(\theta)$ abbreviates $\chi_{ii}(\theta,\mathbf{x}_i)$ in (4.6), $\mathbf{x}_i$ has pdf $f(\mathbf{x}_i;\mathbf{0},\mathbf{I}_{n-1})$ under E_q for $i \in \mathcal{B}$, and $\mathbf{x}_{ij}$ has pdf $f(\mathbf{x}_{ij};\mathbf{0},\mathbf{I}_{n-2})$ under $E_{q,j}$ for $j \in \mathcal{B}\backslash\{i\}$.

Proof See Appendix E. □

4.4 Greeks of barrier options

There are two broad types of barrier options: the knock-out type, which knocks out when the barrier is touched, and the knock-in type, which comes to exist only after the barrier is touched. A barrier option is clearly cheaper than an otherwise identical vanilla option. Partly because of this property, the barrier feature is frequently incorporated into derivative contracts.

In simple cases, closed-form or semi-closed-form formulas for the prices of barrier options exist [17, 22, 26, 29]. However, a formula is often not useful or accurate enough in practice if the resulting integral is of high dimension, which is often the case when the barrier is monitored discretely. In such cases, it may be necessary to price barrier options by numerical methods [20, 28].

Consider the down-and-out call option with a constant barrier H, where $S > H$ and $H \neq K$. The payoff function depends on the stock prices at each monitored time. Let $\mathcal{B} = \{1, \dots, m+1\}$. Using (2.4) to generate $\mathbf{S}$, the payoff function is

$$\begin{aligned}\wp &= \big(S(T) - K\big)^+ \mathbf{1}_{\{\min(S(t_1),\dots,S(t_m)) > H\}}(\mathbf{S})\\ &= \big(S(T) - K\big)\mathbf{1}_{\{S(T)>K\}}(\mathbf{S}) \prod_{i=1,\dots,m} \mathbf{1}_{\{S(t_i)>H\}}(\mathbf{S}).\end{aligned}$$

Defining

$$\begin{aligned}g_{m+1}(\theta, \mathbf{x}) &= h(\theta, \mathbf{x}) = S\mathrm{e}^{(r-\sigma^2/2)m\Delta t + \sigma\sqrt{\Delta t}(x_1+\cdots+x_m)} - K,\\ g_i(\theta, \mathbf{x}) &= S\mathrm{e}^{(r-\sigma^2/2)i\Delta t + \sigma\sqrt{\Delta t}(x_1+\cdots+x_i)} - H \quad \text{for } i = 1, \dots, m,\end{aligned}$$

we can rewrite the payoff function as

$$\wp(\theta, \mathbf{x}) = h(\theta, \mathbf{x}) \prod_{i\in\mathcal{B}} \mathbf{1}_{\{g_i(\theta,\mathbf{x})>0\}}(\mathbf{x}).$$

We proceed to prove that the payoff function belongs to $\mathcal{C}$. Note that $h(\theta, \mathbf{x}) \in \mathcal{H}_1$ with pdf $f(\mathbf{x}; \mathbf{0}, \mathbf{\Sigma})$ for $\mathbf{x}$ and $g_i(\theta, \mathbf{x})$ is strictly monotone in θ and in x_1 for $i \in \mathcal{B}$. Let

$$\chi_1(\theta, \mathbf{x}_1) = \frac{\log H/S - (r - \sigma^2/2)\Delta t}{\sigma\sqrt{\Delta t}}, \tag{4.8}$$

$$\chi_i(\theta, \mathbf{x}_1) = \frac{\log H/S - (r - \sigma^2/2)i\Delta t}{\sigma\sqrt{\Delta t}} - x_2 - \cdots - x_i \quad \text{for } i = 2, \dots, m, \tag{4.9}$$

$$\chi_{m+1}(\theta, \mathbf{x}_1) = \frac{\log K/S - (r - \sigma^2/2)m\Delta t}{\sigma\sqrt{\Delta t}} - x_2 - \cdots - x_m. \tag{4.10}$$

It is easy to verify that $g_i(\theta, \mathbf{x})|_{x_1=\chi_j(\theta,\mathbf{x}_1)} = 0$ for $i \in \mathcal{B}$; hence $g_i(\theta, \mathbf{x}) \in \mathcal{G}_1$. We conclude that $\wp(\theta, \mathbf{x}) \in \mathcal{C}$.

Theorem 4.4 *The delta and gamma of the down-and-out call option equal*

$$\begin{aligned}\Delta = \mathrm{e}^{-rT}\Bigg\{ &E\Bigg[S(t_m)\mathbf{1}_{\{S(t_m)-K>0\}}(\mathbf{S}) \prod_{i=1,\dots,m} \mathbf{1}_{\{S(t_i)-H>0\}}(\mathbf{S})\Bigg]\Big/ S\\ &+ \sum_{l=1}^{m-1} E_q\Big[\Big(H\mathrm{e}^{(r-\sigma^2/2)(m-\ell)\Delta t + \sigma\sqrt{\Delta t}(x_{\ell+1}+\cdots+x_m)} - K\Big)\end{aligned}$$

$$\times \prod_{i\in\mathcal{B}\setminus\{\ell\}} \mathbf{1}_{\{\chi_\ell(\theta,\mathbf{x}_1)>\chi_i(\theta,\mathbf{x}_1)\}}(\mathbf{x}_1)\ \tilde{\eta}_\ell(\theta,\mathbf{x}_1)\Bigg] \Big/ \big(S\sigma\sqrt{\Delta t}\big)$$

$$+ (H-K)E_q\Bigg[\prod_{i\in\mathcal{B}\setminus\{m\}} \mathbf{1}_{\{\chi_m(\theta,\mathbf{x}_1)>\chi_i(\theta,\mathbf{x}_1)\}}(\mathbf{x}_1)\ \tilde{\eta}_m(\theta,\mathbf{x}_1)\Bigg] \Big/ \big(S\sigma\sqrt{\Delta t}\big)\Bigg\},$$

$$\Gamma = \mathrm{e}^{-rT}\Bigg\{\sum_{\ell=1}^{m-1} E_q\Bigg[\Bigg(\frac{\chi_\ell(\theta,\mathbf{x}_1)(H\mathrm{e}^{(r-\sigma^2/2)(m-\ell)\Delta t+\sigma\sqrt{\Delta t}(x_{\ell+1}+\cdots+x_m)}-K)}{\sigma\sqrt{\Delta t}} + K\Bigg)$$

$$\times \prod_{i\in\mathcal{B}\setminus\{\ell\}} \mathbf{1}_{\{\chi_\ell(\theta,\mathbf{x}_1)>\chi_i(\theta,\mathbf{x}_1)\}}(\mathbf{x})\tilde{\eta}_\ell(\theta,\mathbf{x}_1)\Bigg]$$

$$+ E_q\Bigg[\Bigg(\frac{\chi_m(\theta,\mathbf{x}_1)(H-K)}{\sigma\sqrt{\Delta t}} + K\Bigg)\prod_{i\in\mathcal{B}\setminus\{m\}} \mathbf{1}_{\{\chi_m(\theta,\mathbf{x}_1)>\chi_i(\theta,\mathbf{x}_1)\}}(\mathbf{x})\ \tilde{\eta}_m(\theta,\mathbf{x}_1)\Bigg]$$

$$+ KE_q\Bigg[\prod_{i\in\mathcal{B}\setminus\{m+1\}} \mathbf{1}_{\{\chi_{m+1}(\theta,\mathbf{x}_1)>\chi_i(\theta,\mathbf{x}_1)\}}(\mathbf{x})\ \tilde{\eta}_{m+1}(\theta,\mathbf{x}_1)\Bigg]\Bigg\} \Big/ \big(S^2\sigma\sqrt{\Delta t}\big),$$

where $\chi_\ell(\theta,\mathbf{x}_1)$ *is defined in* (4.8)–(4.10), $\tilde{\eta}_\ell(\theta,\mathbf{x}_1)$ *is defined in* (12.1), *and* $\mathbf{x}_1$ *has pdf* $f(\mathbf{x}_1;\mathbf{0},\mathbf{I}_{m-1})$ *under* E_q.

Proof See Appendix F. □

4.5 Greeks of Asian options

The payoff function of the discretely monitored Asian option is

$$\wp = \max(\bar{S}-K,0) = (\bar{S}-K)\mathbf{1}_{\{\bar{S}-K>0\}}(\mathbf{S}),$$

where $\bar{S} = (S(t_1)+\cdots+S(t_m))/m$ is the average stock price. Let y be a univariate random variable which has the same distribution as

$$\frac{\mathrm{e}^{(r-\sigma^2/2)\Delta t+\sigma\sqrt{\Delta t}x_1} + \cdots + \mathrm{e}^{m(r-\sigma^2/2)\Delta t+\sigma\sqrt{\Delta t}(x_1+\cdots+x_m)}}{m}. \tag{4.11}$$

Note that the average price $\bar{S}$ has the same distribution as Sy by (2.4) and the payoff function of an Asian option can be expressed as

$$\wp(\theta,y) = (Sy-K)\mathbf{1}_{\{Sy-K>0\}}(y).$$

Defining $h(\theta,y) = g(\theta,y) = Sy-K$, the payoff function can be rewritten as

$$\wp(\theta,y) = h(\theta,y)\mathbf{1}_{\{g(\theta,y)>0\}}(y).$$

We now proceed to prove that the payoff function belongs to $\mathcal{C}$. Note that $h(\theta,y)\in\mathcal{H}$ with pdf $f(y)$ for y and $g(y)$ is increasing in y. Let $\chi(\theta) = K/S$. It is easy to verify that $g(\theta,y)|_{y=\chi(\theta)} = 0$; hence $g(\theta,y)\in\mathcal{G}$. We conclude that $\wp(\theta,y)\in\mathcal{C}$.

Theorem 4.5 *The delta and gamma of the discretely monitored Asian call option are*

$$\Delta = \mathrm{e}^{-rT} E\big[y\mathbf{1}_{\{Sy-K>0\}}(y)\big],$$
$$\Gamma = \mathrm{e}^{-rT} K^2 f(K/S)/S^3,$$

where y is a univariate random variable having the same distribution as (4.11), *and $f(\cdot)$ is the pdf of y.*

Proof Recall that $\chi(\theta) = K/S$. Because y is univariate, $\tilde{\eta}(\theta, y)$ reduces to

$$\tilde{\eta}(\theta) = f(y)|_{y=\chi(\theta)} = f(K/S), \tag{4.12}$$

and the expectation operator E_q on the right-hand side of (3.8) can be removed. Now, apply Theorem 3.3 to obtain

$$\begin{aligned}\Delta &= \mathrm{e}^{-rT}\big\{E\big[y\mathbf{1}_{\{Sy-K>0\}}(y)\big] + \big[(Sy-K)J(\theta, y)\big|_{y=\chi(\theta)}\tilde{\eta}(\theta)\big]\big\}\\ &= \mathrm{e}^{-rT} E\big[y\mathbf{1}_{\{Sy-K>0\}}(y)\big],\end{aligned}$$

where $\tilde{\eta}(\theta, y)$ is in (4.12), and $J(\theta, y) = y/S$ by (3.4). Apply Theorem 3.3 again to obtain $\Gamma = \mathrm{e}^{-rT}[yJ(\theta, y)|_{y=\chi(\theta)}\tilde{\eta}(\theta)]$. □

The delta and gamma formulas of Asian options in Theorem 4.5 are identical to those based on the pathwise method after a suitable change of variables [4]. We remark that the gamma in [4] employs techniques in addition to the pathwise method because the integrand of the delta is not Lipschitz-continuous. If the pdf of y were known, the gamma of Asian options could be calculated by $\mathrm{e}^{-rT}K^2 f(K/S)/S^3$. However, being a sum of log-normal distributions, y's pdf is not yet known to have a closed-form formula. As a result, numerical methods are required for $f(K/S)$, which is beyond the scope of this paper.

5 Numerical results

We first compare our method with other methods for the Greeks of three rainbow options and one barrier option. Then we compare the numerical results of the Greeks of maximum options using two different sampling distributions: the standard normal and the shifted normal.

5.1 Greek estimates of rainbow options and barrier options

Although the FD approach is intuitive and easy to understand, those advantages are weakened by its shortcomings. A fundamental weakness of the FD approach is that so far there are no known hard rules for choosing the right h in calculating the Greeks [14]. This delicate issue is especially damaging for higher-order Greeks like gammas and cross-gammas. It also highlights the advantage of our method, which does not have to pick h; h simply does not enter our formulas at all. This section compares our method with various FD methods and the likelihood ratio method. Recall that Table 2 lists standard formulas for the forward and central FDs. The formulas of Greeks for rainbow options using the likelihood ratio method are summarized in Appendix G.

Tables 3, 4, 5 and 6 show the results of applying our method to spread options, maximum options, binary maximum options, and barrier options, respectively. For the benchmark values, we first compute their option prices, then calculate the Greeks by central FD with $h = 0.1$. We use numerical integration to calculate the prices of spread and barrier options and follow Johnson [15] in calculating the prices of maximum and binary maximum options. These tables contain simulation results with 100,000 paths for deltas, gammas, and cross-gammas so that our method can be com-

Table 3 Greeks of spread options on two assets

h	Δ_1		Γ_{11}		Γ_{12}	
Benchmark value						
	−0.4340		0.0414		−0.0419	
Our method						
	−0.4346	(0.0017)	0.0414	(0.0001)	−0.0419	(0.0001)
Likelihood ratio method						
	−0.4308	(0.0033)	0.0407	(0.0007)	−0.0410	(0.0007)
Forward FD						
10^{-1}	−0.4339	(0.0016)	0.0398	(0.0017)	−0.0422	(0.0017)
10^{-2}	−0.4343	(0.0017)	0.0408	(0.0054)	−0.0306	(0.0044)
10^{-3}	−0.4349	(0.0017)	0.0476	(0.0185)	−0.0215	(0.0107)
10^{-4}	−0.4314	(0.0016)	0.0000	(0.0000)	0.0000	(0.0000)
10^{-5}	−0.4304	(0.0016)	0.0000	(0.0000)	0.0000	(0.0000)
10^{-6}	−0.4355	(0.0017)	−0.0031	(0.0000)	0.0000	(0.0000)
10^{-7}	−0.4335	(0.0016)	−0.0011	(0.0012)	0.0000	(0.0000)
10^{-8}	−0.4312	(0.0016)	−0.0636	(0.1196)	0.0013	(0.0018)
10^{-9}	−0.4338	(0.0016)	−3092.6010	(16.8991)	0.2572	(0.2033)
Central FD						
10^{-1}	−0.4338	(0.0016)	0.0420	(0.0017)	−0.0448	(0.0018)
10^{-2}	−0.4351	(0.0017)	0.0415	(0.0055)	−0.0541	(0.0063)
10^{-3}	−0.4343	(0.0016)	0.0405	(0.0182)	−0.0411	(0.0164)
10^{-4}	−0.4330	(0.0016)	0.0804	(0.0804)	−0.0432	(0.0432)
10^{-5}	−0.4335	(0.0016)	0.0000	(0.0000)	0.0000	(0.0000)
10^{-6}	−0.4344	(0.0016)	0.0000	(0.0000)	0.0000	(0.0000)
10^{-7}	−0.4345	(0.0017)	0.0003	(0.0006)	0.0000	(0.0000)
10^{-8}	−0.4321	(0.0016)	−0.0085	(0.0634)	0.0006	(0.0006)
10^{-9}	−0.4342	(0.0016)	3.4397	(6.3518)	0.0643	(0.1114)

Parameters: $S_1 = S_2 = 40$, $K = 0.5$, $r = 10\%$, $T = 1$ year, $\sigma_1 = \sigma_2 = 0.3$, and $\rho_{12} = \rho_{21} = 0.69$. For benchmark values, we obtain spread options' prices using numerical integrations and then calculate the Greeks by central FD with $h = 0.1$. The standard errors are in parentheses. All simulation results are based on 100,000 trials

Table 4 Greeks of maximum options on two assets

h	Δ_1		Γ_{11}		Γ_{12}	
Benchmark value						
	0.4322		0.0425		−0.0286	
Our method						
	0.4354	(0.0020)	0.0424	(0.0001)	−0.0286	(0.0001)
Likelihood ratio method						
	0.4393	(0.0069)	0.0437	(0.0013)	−0.0292	(0.0011)
Forward FD						
10^{-1}	0.4338	(0.0019)	0.0411	(0.0017)	−0.0288	(0.0015)
10^{-2}	0.4342	(0.0019)	0.0397	(0.0057)	−0.0289	(0.0047)
10^{-3}	0.4316	(0.0019)	0.0448	(0.0172)	−0.0113	(0.0113)
10^{-4}	0.4324	(0.0019)	0.0233	(0.0166)	0.0000	(0.0000)
10^{-5}	0.4343	(0.0019)	0.0000	(0.0000)	0.0000	(0.0000)
10^{-6}	0.4306	(0.0019)	0.0030	(0.0000)	0.0000	(0.0000)
10^{-7}	0.4325	(0.0019)	0.0006	(0.0014)	0.0000	(0.0000)
10^{-8}	0.4325	(0.0019)	0.1537	(0.1367)	0.0000	(0.0000)
10^{-9}	0.4334	(0.0019)	3083.1500	(19.9935)	0.0000	(0.0000)
Central FD						
10^{-1}	0.4323	(0.0019)	0.0442	(0.0018)	−0.0267	(0.0015)
10^{-2}	0.4336	(0.0019)	0.0440	(0.0056)	−0.0353	(0.0053)
10^{-3}	0.4305	(0.0019)	0.0430	(0.0168)	−0.0358	(0.0217)
10^{-4}	0.4325	(0.0019)	0.0000	(0.0000)	−0.0154	(0.0154)
10^{-5}	0.4298	(0.0019)	0.0000	(0.0000)	0.0000	(0.0000)
10^{-6}	0.4303	(0.0019)	0.0000	(0.0000)	0.0000	(0.0000)
10^{-7}	0.4311	(0.0019)	−0.0011	(0.0007)	0.0000	(0.0000)
10^{-8}	0.4308	(0.0019)	0.0129	(0.0724)	0.0000	(0.0000)
10^{-9}	0.4288	(0.0019)	7.2651	(7.2940)	0.0000	(0.0000)

Parameters: $S_1 = S_2 = 40$, $K = 40$, $r = 10\%$, $T = 1$ year, $\sigma_1 = \sigma_2 = 0.3$, and $\rho_{12} = \rho_{21} = 0.69$. For benchmark values, we follow Johnson's formula to obtain maximum options' prices [15]. We then calculate the Greeks by central FD with $h = 0.1$. The standard errors are in parentheses. Simulation results are based on 100,000 trials

pared with forward and central FD methods under various h. We also use common random numbers for the FD estimates.

Several conclusions can be drawn. The deltas given by our method and the FD methods are of similar quality. In particular, our method always gives unbiased deltas, whereas under the FD schemes, deltas may be biased when h is large but quickly converge to the correct value as h approaches zero. But the similarity ends here. Although the desired convergence property continues to be shared by our method in the calculations of gammas and cross-gammas, it is no longer shared by any of the FD methods. For large h, the gammas and cross-gammas calculated by the FD methods are clearly biased. On the other hand, for small h, their gammas and cross-gammas are unstable. Our method yields Greek formulas and hence avoids these drawbacks arising from

Table 5 Greeks of binary maximum options on two assets

h	Δ_1		Γ_{11}		Γ_{12}	
Benchmark value						
	0.0139		0.0004		−0.0014	
Our method						
	0.0138	(0.0001)	0.0004	(0.0000)	−0.0014	
Likelihood ratio method						
	0.0138	(0.0003)	0.0004	(0.0000)	−0.0013	(0.0000)
Forward FD						
10^{-1}	0.0147	(0.0012)	−0.0081	(0.0164)	0.0000	(0.0000)
10^{-2}	0.0190	(0.0041)	0.4524	(0.5198)	0.0000	(0.0000)
10^{-3}	0.0090	(0.0090)	−9.0484	(9.0484)	0.0000	(0.0000)
10^{-4}	0.0000	(0.0000)	0.0000	(0.0000)	0.0000	(0.0000)
10^{-5}	0.0000	(0.0000)	0.0000	(0.0000)	0.0000	(0.0000)
10^{-6}	0.0000	(0.0000)	0.0000	(0.0000)	0.0000	(0.0000)
10^{-7}	0.0000	(0.0000)	0.0000	(0.0000)	0.0000	(0.0000)
10^{-8}	0.0000	(0.0000)	0.0000	(0.0000)	0.0000	(0.0000)
10^{-9}	0.0000	(0.0000)	0.0000	(0.0000)	0.0000	(0.0000)
Central FD						
10^{-1}	0.0116	(0.0010)	−0.0009	(0.0158)	−0.0018	(0.0013)
10^{-2}	0.0136	(0.0035)	0.2715	(0.5198)	0.0000	(0.0000)
10^{-3}	0.0362	(0.0181)	−18.0967	(18.0967)	0.0000	(0.0000)
10^{-4}	0.0000	(0.0000)	−904.8374	(904.8374)	0.0000	(0.0000)
10^{-5}	0.0000	(0.0000)	0.0000	(0.0000)	0.0000	(0.0000)
10^{-6}	0.0000	(0.0000)	0.0000	(0.0000)	0.0000	(0.0000)
10^{-7}	0.0000	(0.0000)	0.0000	(0.0000)	0.0000	(0.0000)
10^{-8}	0.0000	(0.0000)	0.0000	(0.0000)	0.0000	(0.0000)
10^{-9}	0.0000	(0.0000)	0.0000	(0.0000)	0.0000	(0.0000)

Parameters: $S_1 = S_2 = 40$, $K = 40$, $r = 10\%$, $T = 1$ year, $\sigma_1 = \sigma_2 = 0.3$, and $\rho_{12} = \rho_{21} = 0.69$. For benchmark values, we follow Johnson's formula to obtain binary maximum options' prices [15]. We then calculate the Greeks by central FD with $h = 0.1$. The standard errors are in parentheses. Simulation results are based on 100,000 trials. Since the formula for Γ_{12} reduces to the closed-form formula for the binary maximum option on two assets, Monte Carlo simulation is not required and no standard error is given

FDs being approximate only. Its effectiveness is clearly demonstrated by the numerical superiority to the FD methods in treating gammas and cross-gammas. Both our method and the likelihood ratio method produce unbiased Greeks for rainbow options. However, our method produces estimates with much lower standard errors. It is also not obvious how to apply the likelihood ratio method for path-dependent options.

5.2 A comparison of sampling distributions

In Sect. 4.2, we derive the Greeks of maximum options using the standard normal distribution as the sampling distribution. Recall the payoff function of the maximum

Table 6 Greeks of down-and-out options with two monitored dates

	Δ		Γ	
Benchmark value	0.8331		0.0123	
Our method	0.8328	(0.0023)	0.0121	(0.0000)
Forward FD				
$h = 10^{-1}$	0.8396	(0.0201)	0.0648	(0.2989)
$h = 10^{-2}$	0.8074	(0.0640)	−4.2400	(9.1968)
$h = 10^{-3}$	0.8987	(0.2391)	−165.9436	(117.5001)
$h = 10^{-4}$	1.3551	(0.6947)	13441.3315	(13441.2765)
$h = 10^{-5}$	0.6582	(0.0021)	0.0000	(0.0000)
$h = 10^{-6}$	0.6605	(0.0021)	0.0000	(0.0000)
$h = 10^{-7}$	0.6548	(0.0021)	0.9419	(0.0050)
$h = 10^{-8}$	0.6591	(0.0021)	94.6567	(0.5063)
$h = 10^{-9}$	0.6598	(0.0021)	−9414.8747	(50.4979)
Central FD				
$h = 10^{-1}$	0.8565	(0.0220)	−0.4548	(0.2882)
$h = 10^{-2}$	0.8076	(0.0534)	−1.4426	(8.3634)
$h = 10^{-3}$	0.9593	(0.2453)	−464.5957	(521.2688)
$h = 10^{-4}$	0.6571	(0.0021)	0.0000	(0.0000)
$h = 10^{-5}$	0.6545	(0.0021)	0.0000	(0.0000)
$h = 10^{-6}$	0.6589	(0.0021)	0.0000	(0.0000)
$h = 10^{-7}$	0.6625	(0.0021)	0.0041	(0.0025)
$h = 10^{-8}$	0.6586	(0.0021)	0.1594	(0.2521)
$h = 10^{-9}$	0.6584	(0.0021)	−14.4015	(25.0361)

Parameters: $S = 100$, $K = 95$, $H = 95$, $r = 10\%$, $T = 1$ year, $\sigma = 0.25$ and $m = 2$. The monitored dates are at years 0.5 and 1. For benchmark values, we obtain down-and-out options' prices using numerical integrations and then calculate the Greeks by central FD with $h = 0.1$. The standard errors are in parentheses. All simulation results are based on 100,000 trials

option in (4.5). The price of the maximum option is

$$C = \mathrm{e}^{-rT} \sum_{i \in \mathcal{B}} E\left[h_i(\theta, \mathbf{x}) \prod_{j \in \mathcal{B}} \mathbf{1}_{\{g_{ij}(\theta,\mathbf{x})>0\}}(\mathbf{x}) \right], \tag{5.1}$$

where $\mathbf{x}$ has pdf $f(\mathbf{x}; \mathbf{0}, \boldsymbol{\Sigma})$ under E, $h_i(\theta, \mathbf{x})$ is defined in (4.2), $g_{ii}(\theta, \mathbf{x})$ is defined in (4.3), and $g_{ij}(\theta, \mathbf{x})$ is defined in (4.4). The price of the maximum option can be estimated by the Monte Carlo method using

$$\frac{1}{N} \mathrm{e}^{-rT} \sum_{i \in \mathcal{B}} \sum_{n=1}^{N} p_i^{(n)},$$

Table 7 Comparisons of two sampling distributions

K	C		Δ_1		Γ_{11}		Γ_{12}	
Benchmark value								
20	25.6660		0.5467		0.0420		−0.0419	
30	16.8012		0.5283		0.0429		−0.0392	
40	9.2682		0.4322		0.0425		−0.0286	
50	4.3395		0.2786		0.0346		−0.0155	
60	1.8012		0.1474		0.0225		−0.0067	
70	0.6924		0.0682		0.0123		−0.0025	
80	0.2547		0.0290		0.0060		−0.0009	
90	0.0917		0.0117		0.0027		−0.0003	
100	0.0328		0.0046		0.0012		−0.0001	
110	0.0118		0.0018		0.0005		0.0000	
120	0.0043		0.0007		0.0002		0.0000	
Standard normal distribution								
20	25.6886	(0.0400)	0.5481	(0.0019)	0.0420	(0.0000)	−0.0419	(0.0000)
30	16.7924	(0.0396)	0.5295	(0.0019)	0.0429	(0.0000)	−0.0391	(0.0001)
40	9.3174	(0.0347)	0.4354	(0.0020)	0.0424	(0.0001)	−0.0286	(0.0001)
50	4.3002	(0.0259)	0.2807	(0.0018)	0.0346	(0.0001)	−0.0154	(0.0001)
60	1.8154	(0.0175)	0.1475	(0.0015)	0.0224	(0.0001)	−0.0067	(0.0001)
70	0.6874	(0.0109)	0.0668	(0.0011)	0.0123	(0.0001)	−0.0025	(0.0000)
80	0.2600	(0.0067)	0.0290	(0.0008)	0.0060	(0.0000)	−0.0009	(0.0000)
90	0.0840	(0.0038)	0.0120	(0.0005)	0.0027	(0.0000)	−0.0003	(0.0000)
100	0.0340	(0.0025)	0.0053	(0.0004)	0.0012	(0.0000)	−0.0001	(0.0000)
110	0.0131	(0.0016)	0.0015	(0.0002)	0.0005	(0.0000)	0.0000	(0.0000)
120	0.0050	(0.0010)	0.0004	(0.0001)	0.0002	(0.0000)	0.0000	(0.0000)
Normal distribution with shifted mean								
20	24.6185	(2.2474)	0.5386	(0.0568)	0.0398	(0.0028)	−0.0397	(0.0028)
30	16.9916	(0.2758)	0.5323	(0.0078)	0.0428	(0.0003)	−0.0391	(0.0003)
40	9.2909	(0.0443)	0.4327	(0.0022)	0.0425	(0.0001)	−0.0285	(0.0001)
50	4.3574	(0.0138)	0.2821	(0.0014)	0.0346	(0.0000)	−0.0154	(0.0001)
60	1.7984	(0.0058)	0.1469	(0.0009)	0.0225	(0.0000)	−0.0067	(0.0000)
70	0.6862	(0.0027)	0.0676	(0.0005)	0.0123	(0.0000)	−0.0025	(0.0000)
80	0.2548	(0.0012)	0.0290	(0.0003)	0.0060	(0.0000)	−0.0009	(0.0000)
90	0.0913	(0.0005)	0.0115	(0.0001)	0.0027	(0.0000)	−0.0003	(0.0000)
100	0.0328	(0.0003)	0.0046	(0.0001)	0.0012	(0.0000)	−0.0001	(0.0000)
110	0.0117	(0.0001)	0.0018	(0.0000)	0.0005	(0.0000)	0.0000	(0.0000)
120	0.0043	(0.0001)	0.0007	(0.0000)	0.0002	(0.0000)	0.0000	(0.0000)

Parameters: $S_1 = S_2 = 40$, $r = 10\%$, $T = 1$ year, $\sigma_1 = \sigma_2 = 0.3$, and $\rho_{12} = \rho_{21} = 0.69$. For benchmark values, we follow Johnson's formula to obtain maximum options' prices [15]. We then calculate the Greeks by central FD with $h = 0.1$. The standard errors are in parentheses. Simulation results are based on 100,000 trials where sample paths $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(N)}$ are drawn from the sampling distribution $f(\mathbf{x}; \mathbf{0}, \boldsymbol{\Sigma})$ and

$$p_i^{(n)} = h_i\big(\theta, \mathbf{x}^{(n)}\big) \prod_{j \in \mathcal{B}} \mathbf{1}_{\{g_{ij}(\theta,\mathbf{x})>0\}}\big(\mathbf{x}^{(n)}\big)$$

for $i \in \mathcal{B}$ and $n = 1, \ldots, N$. Let $\chi_{ii}(\theta)$ abbreviate $\chi_{ii}(\theta, \mathbf{x}_i)$ in (4.6), because $\chi_{ii}(\theta, \mathbf{x})$ does not depend on $\mathbf{x}_i$. Note that the set $\{\mathbf{x} : g_{ii}(\theta, \mathbf{x}) > 0\} = \{\mathbf{x} : x_i > \chi_{ii}(\theta)\}$ is a half-space, and the set $\{\mathbf{x} : g_{ij}(\theta, \mathbf{x}) > 0\} = \{\mathbf{x} : x_i > \chi_{ij}(\theta, \mathbf{x}_j)\}$ for $j \in \mathcal{B}\backslash\{i\}$ is also a half-space. Hence, $p_i^{(n)}$ is nonzero when the sample path $\mathbf{x}^{(n)}$ is in the intersection $\bigcap_{j\in\mathcal{B}}\{\mathbf{x} : g_{ij}(\theta, \mathbf{x}) > 0\}$ of these half-spaces, and zero otherwise. When the maximum option is deep out-of-the money, $\chi_{ii}(\theta)$ is far away from zero. In this case, when the sample path $\mathbf{x}^{(n)}$ is drawn from $f(\mathbf{x}; \mathbf{0}, \boldsymbol{\Sigma})$, many $p_i^{(n)}$ would be zero and, as a result, a large number of sample paths are required to obtain an accurate estimate. To overcome this problem, a simple solution is to use the shifted normal with mean $(\chi_{11}(\theta), \ldots, \chi_{nn}(\theta))$ as the sampling distribution.

Similarly, we use the shifted normal as the sampling distribution for the Greeks of deep out-of-the money maximum options, because the integrands of the Greek formulas consist of several indicator functions with the same support sets. The price and Greek formulas for maximum options using the shifted normal distribution as the sampling distribution are given in the next theorem. The proof is similar to the proof of Theorem 4.2 and is thus omitted.

Theorem 5.1 *The price, delta, gamma and cross-gamma of the maximum option on n assets are*

$$\begin{aligned}
C &= \mathrm{e}^{-rT} E^s\big[\max\big(\max\big(S_1(T), \ldots, S_n(T)\big) - K, 0\big)\eta^s(\mathbf{x})\big],\\
\Delta_i &= \mathrm{e}^{-rT} E^s\Bigg[\mathrm{e}^{(r-\sigma_i^2/2)T+\sigma_i\sqrt{T}x_i}\mathbf{1}_{\{S_i\mathrm{e}^{(r-\sigma_i^2/2)T+\sigma_i\sqrt{T}x_i}>K\}}(\mathbf{x})\\
&\qquad \times \prod_{j\in\mathcal{B}\backslash\{i\}} \mathbf{1}_{\{S_i\mathrm{e}^{(r-\sigma_i^2/2)T+\sigma_i\sqrt{T}x_i}>S_j\mathrm{e}^{(r-\sigma_j^2/2)T+\sigma_j\sqrt{T}x_j}\}}(\mathbf{x})\eta^s(\mathbf{x})\Bigg],\\
\Gamma_{ii} &= \mathrm{e}^{-rT}\Bigg\{E_{q^s}\Bigg[K\prod_{j\in\mathcal{B}\backslash\{i\}}\mathbf{1}_{\{S_j\mathrm{e}^{(r-\sigma_j^2/2)T+\sigma_j\sqrt{T}x_j}<K\}}(\mathbf{x}_i)\tilde{\eta}_{ii}^s(\theta, \mathbf{x}_i)\Bigg]\\
&\qquad + \sum_{\ell\in\mathcal{B}\backslash\{i\}} E_{q^s}\Bigg[S_\ell\mathrm{e}^{(r-\sigma_\ell^2/2)T+\sigma_\ell\sqrt{T}x_\ell}\mathbf{1}_{\{S_\ell\mathrm{e}^{(r-\sigma_\ell^2/2)T+\sigma_\ell\sqrt{T}x_\ell}>K\}}(\mathbf{x}_i)\\
&\qquad \times \prod_{j\in\mathcal{B}\backslash\{\ell,i\}}\mathbf{1}_{\{S_\ell\mathrm{e}^{(r-\sigma_\ell^2/2)T+\sigma_\ell\sqrt{T}x_\ell}>S_j e^{(r-\sigma_j^2/2)T+\sigma_j\sqrt{T}x_j}\}}(\mathbf{x}_i)\tilde{\eta}_{i\ell}^s(\theta, \mathbf{x}_i)\Bigg]\\
&\qquad \Big/\big(S_i^2\sigma_i\sqrt{T}\big)\Bigg\},
\end{aligned}$$

$$\Gamma_{ij} = -\mathrm{e}^{-rT} E_{q^s}\Bigg[\mathrm{e}^{(r-\sigma_j^2/2)T+\sigma_j\sqrt{T}x_j}\mathbf{1}_{\{S_j\mathrm{e}^{(r-\sigma_j^2/2)T+\sigma_j\sqrt{T}x_j}>K\}}(\mathbf{x}_i)$$
$$\times \prod_{\ell\in\mathcal{B}\backslash\{j,i\}} \mathbf{1}_{\{S_j\mathrm{e}^{(r-\sigma_j^2/2)T+\sigma_j\sqrt{T}x_j}>S_\ell e^{(r-\sigma_\ell^2/2)T+\sigma_\ell\sqrt{T}x_\ell}\}}(\mathbf{x}_i)\tilde{\eta}^s_{ij}(\theta,\mathbf{x}_i)\Bigg]\Big/\big(S_i\sigma_i\sqrt{T}\big),$$

where $\chi_{ij}(\theta,\mathbf{x}_i)$ *is defined in* (4.6) *and* (4.7), $\chi_{ii}(\theta)$ *abbreviates* $\chi_{ii}(\theta,\mathbf{x}_i)$,

$$\eta^s(\mathbf{x}) = \frac{f(\mathbf{x};\mathbf{0},\boldsymbol{\Sigma})}{f(\mathbf{x};\boldsymbol{\mu},\boldsymbol{\Sigma})},$$
$$\tilde{\eta}^s_{ij}(\mathbf{x}) = \frac{f(\mathbf{x};\mathbf{0},\boldsymbol{\Sigma})}{f(\mathbf{x}_i;\boldsymbol{\mu}_i,\mathbf{I}_{n-1})}\bigg|_{x_i=\chi_{ij}(\theta,\mathbf{x}_i)}.$$

Moreover, E^s *is the expectation operator of* $\mathbf{x}$, *under which* $\mathbf{x}$ *has pdf* $f(\mathbf{x},\boldsymbol{\mu},\boldsymbol{\Sigma})$ *with* $\boldsymbol{\mu}=[\chi_{11}(\theta),\ldots,\chi_{nn}(\theta)]^t$. E_{q^s} *is the expectation operator of* $\mathbf{x}_i$, *under which* $\mathbf{x}_i$ *has pdf* $f(\mathbf{x}_i;\boldsymbol{\mu}_i,\mathbf{I}_{n-1})$ *with* $\boldsymbol{\mu}_i=[\chi_{11}(\theta),\ldots,\chi_{i-1,i-1}(\theta),\chi_{i+1,i+1}(\theta),\ldots,\chi_{nn}(\theta)]^t$ *for* $i\in\mathcal{B}$.

Table 7 compares the numerical results of the Greeks of the maximum option using the standard normal and the shifted normal as the sampling distributions for strike prices ranging from 20 to 120. The range of strike prices covers deep in-the-money and deep out-of-the-money options. The Greek formulas from Theorem 4.2 and 5.1 both give unbiased Greek estimates. Numerical results show that using the shifted normal as the sampling distribution can improve the Greek estimates for deep out-of-the-money options in terms of accuracy and standard error, whereas the standard normal works well for at-the-money and in-the-money options.

6 Conclusions

Mathematically, Greeks are partial differentiations of an option's price with respect to a parameter of interest. For complex options such as rainbow and path-dependent options, there are usually no analytic formulas for their prices; hence Monte Carlo simulation is often the only available method to estimate them. The FD method is subsequently used to approximate the Greeks. But FD as an approximate differentiation operator can be rather unstable and biased. Two direct methods, the pathwise method and the likelihood ratio method, have been proposed to overcome the drawbacks of FD. However, the Greeks of options whose payoff functions are not Lipschitz-continuous cannot be obtained using the pathwise method without modifications if convergence is to be guaranteed.

The major theoretical advantage of our method over the pathwise method is a new mathematical formulation so that the Lipschitz-continuity restriction on the payoff function is lifted. We present a rule to interchange the order of integration and differentiation when the integrand can be decomposed into a sum of products of differentiable functions and certain indicator functions. As a result, the Greek formulas of a wide variety of options can be derived systematically. For practical purposes, we propose a useful importance sampling method to estimate these Greeks using the formulas mentioned above. For illustration purposes, formulas and numerical results for the Greeks are given for popular rainbow and path-dependent options. Another key feature of our method is that it is easier to implement and its application is almost mechanical compared with such methods as the likelihood ratio method and the conditional Monte Carlo method. Although the focus of this paper is on the Greeks of rainbow and path-dependent options under the Black–Scholes model, our method is applicable to alternative models for the underlying assets as long as the payoff function belongs to $\mathcal{C}$.

Numerically, our estimators for Greeks are unbiased, whereas estimators obtained from the FD methods are not. Furthermore, the FD methods have to solve the difficult problem of determining the right perturbed size, which is completely eliminated by our method. Also eliminated by our method is resimulation. When the payoff function is not smooth, the FD methods tend to fail, particularly for higher-order Greeks like gammas. In contrast, our method can handle a rich class of payoff functions. These results make our method more generally applicable and useful than FD schemes. Finally, our method enjoys lower variances for rainbow options than the likelihood ratio method.

Acknowledgements We thank William Wei-Yuan Hsu and Chun-Ying Chen of National Taiwan University for programming assistance. We thank Ming-Hsuan Kang of Pennsylvania State University for useful discussions. We also thank the associate editor and three anonymous reviewers, whose comments on the paper greatly improved its quality.

Appendix A: Proof of Theorem 3.1

Proof Let $f(\theta,x)=h(\theta,x)f(x)$. Define $I_j=\{x\in\Re: g_j(\theta,x)>0\}$ for $j\in\mathcal{B}$ and $I=\bigcap_{j\in\mathcal{B}} I_j$. The third assumption for $g_j(\theta,x)\in\mathcal{G}_k$ requires $g_j(\theta,x)$ to be strictly monotone in x. Since $g_j(\theta,x)$ is strictly increasing in x if $\operatorname{sign}(\partial g_j(\theta,x)/\partial x)=1$ and $g_j(\theta,x)$ is strictly decreasing in x if $\operatorname{sign}(\partial g_j(\theta,x)/\partial x)=-1$, we split the set $\mathcal{B}$ into two disjoint sets $\mathcal{B}_R$ and $\mathcal{B}_L$ defined by

$$\mathcal{B}_{R,L}=\bigl\{j\in\mathcal{B}:\ \operatorname{sign}\bigl(\partial g_j(\theta,x)/\partial x\bigr)=\pm 1\bigr\}.$$

As a result, we have $I_j=(\chi_j(\theta),\infty)$ for $j\in\mathcal{B}_R$ and $I_j=(-\infty,\chi_j(\theta))$ for $j\in\mathcal{B}_L$. When $\mathcal{B}_R(\mathcal{B}_L)$ is nonempty, let $R(L)$ denote the index such that $\chi_R(\theta)(\chi_L(\theta))$ is the maximum (minimum) of $\chi_j(\theta)$ for $j\in\mathcal{B}_R(\mathcal{B}_L)$. As a result, I equals

$$\begin{cases}\text{case 1: } I_R\cap I_L, \text{ when } \mathcal{B}_L \text{ and } \mathcal{B}_R \text{ are both nonempty and } \chi_L(\theta)>\chi_R(\theta);\\ \text{case 2: } \emptyset, \text{ when } \mathcal{B}_L \text{ and } \mathcal{B}_R \text{ are both nonempty and } \chi_L(\theta)\le\chi_R(\theta);\\ \text{case 3: } I_R, \text{ when } \mathcal{B}_L \text{ is empty;}\\ \text{case 4: } I_L, \text{ when } \mathcal{B}_R \text{ is empty.}\end{cases}$$

In case 1, $I=I_R\cap I_L=(\chi_R(\theta),\chi_L(\theta))$ is a finite open interval. When ℓ is neither R nor L, it is clear that $\prod_{j\in\mathcal{B}\setminus\{j\}}\{x: g_j(\theta,x)>0\}$ equals $I_R\cap I_L$, and hence

$$\Biggl[\prod_{j\in\mathcal{B}\setminus\{j\}}\mathbf{1}_{\{g_j(\theta,x)>0\}}\Biggr]_{x=\chi_\ell(\theta)}=0.$$

On the other hand, when ℓ is R or L, it is clear that $\prod_{j\in\mathcal{B}\setminus\{j\}}\{x : g_j(\theta,x) > 0\}$ dominates $I_R \cap I_L$, and hence

$$\left[\prod_{j\in\mathcal{B}\setminus\{j\}} \mathbf{1}_{\{g_j(\theta,x)>0\}}\right]_{x=\chi_\ell(\theta)} = 1.$$

As a result, we only need to prove the reduced formula for (3.1),

$$\begin{aligned}
&\frac{\partial}{\partial\theta}\int_{\Re} f(\theta,x)\mathbf{1}_{\{g_R(\theta,x)>0\}}(x)\mathbf{1}_{\{g_L(\theta,x)>0\}}(x)\,dx \\
&\quad= \int_{\Re} f_\theta(\theta,x)\mathbf{1}_{\{g_R(\theta,x)>0\}}(x)\mathbf{1}_{\{g_L(\theta,x)>0\}}(x)\,dx \\
&\qquad + \big[f(\theta,x)J_L(\theta,x)\big]_{x=\chi_L(\theta)} + \big[f(\theta,x)J_R(\theta,x)\big]_{x=\chi_R(\theta)}.
\end{aligned} \tag{7.1}$$

Note that the left-hand side of (7.1) equals

$$\begin{aligned}
\frac{\partial}{\partial\theta}\int_{\chi_R(\theta)}^{\chi_L(\theta)} f(\theta,x)\,dx &= \int_{\chi_R(\theta)}^{\chi_L(\theta)} f_\theta(\theta,x)dx + f\big(\theta,\chi_L(\theta)\big)\frac{\partial\chi_L(\theta)}{\partial\theta} \\
&\quad - f\big(\theta,\chi_R(\theta)\big)\frac{\partial\chi_R(\theta)}{\partial\theta}
\end{aligned} \tag{7.2}$$

by the Leibniz rule, where the partial differentiation of $\chi_\ell(\theta)$ with respect to θ can be calculated alternatively as

$$\frac{\partial\chi_\ell(\theta)}{\partial\theta} = -\left[\frac{\partial g_j(\theta,x)/\partial\theta}{\partial g_j(\theta,x)/\partial x}\right]_{x=\chi_\ell(\theta)}, \quad \ell\in\mathcal{B},$$

by the implicit function theorem. Recall that (3.2) defines

$$J_\ell(\theta,x) = \operatorname{sign}\left(\frac{\partial g_\ell(\theta,x)}{\partial x}\right)\frac{\partial g_\ell(\theta,x)/\partial\theta}{\partial g_\ell(\theta,x)/\partial x}, \quad \ell\in\mathcal{B}.$$

Now, it is clear that (7.2) equals

$$\begin{aligned}
&\int_{\chi_R(\theta)}^{\chi_L(\theta)} f_\theta(\theta,x)\,dx - f\big(\theta,\chi_L(\theta)\big)\frac{\partial g_L(\theta,x)/\partial\theta}{\partial g_L(\theta,x)/\partial x} + f\big(\theta,\chi_R(\theta)\big)\frac{\partial g_R(\theta,x)/\partial\theta}{\partial g_R(\theta,x)/\partial x} \\
&= \int_{\Re} f_\theta(\theta,x)\mathbf{1}\big\{g_R(\theta,x)>0\big\}(x)\mathbf{1}\big\{g_L(\theta,x)>0\big\}(x)\,dx \\
&\quad - f\big(\theta,\chi_L(\theta)\big)\frac{\partial g_L(\theta,x)/\partial\theta}{\partial g_L(\theta,x)/\partial x} + f\big(\theta,\chi_R(\theta)\big)\frac{\partial g_R(\theta,x)/\partial\theta}{\partial g_R(\theta,x)/\partial x} \\
&= \int_{\Re} f_\theta(\theta,x)\mathbf{1}_{\{g_R(\theta,x)>0\}}(x)\mathbf{1}_{\{g_L(\theta,x)>0\}}(x)\,dx \\
&\quad + \big[f(\theta,x)J_L(\theta,x)\big]_{x=\chi_L(\theta)} + \big[f(\theta,x)J_R(\theta,x)\big]_{x=\chi_R(\theta)}
\end{aligned}$$

which is exactly the right-hand side of (7.1). Hence, the proof of (3.1) for case 1 is done. The proofs for cases 2, 3, and 4 are similar and therefore omitted. ☐

Appendix B: Proof of Theorem 3.2

Proof Let $f(\theta, \mathbf{x}) = h(\theta, \mathbf{x}) f(\mathbf{x})$. With $\mathbf{x}_k$ fixed, Theorem 3.1 implies

$$\begin{aligned}
&\frac{\partial}{\partial \theta} \int_{\Re} f(\theta, \mathbf{x}) \prod_{j \in \mathcal{B}} \mathbf{1}_{\{g_j(\theta, \mathbf{x}) > 0\}}(\mathbf{x})\, dx_k \\
&= \int_{\Re} f_\theta(\theta, \mathbf{x}) \prod_{j \in \mathcal{B}} \mathbf{1}_{\{g_j(\theta, \mathbf{x}) > 0\}}(\mathbf{x})\, dx_k \\
&\quad + \sum_{\ell \in \mathcal{B}} \left[f(\theta, \mathbf{x}) J_\ell(\theta, \mathbf{x}) \prod_{j \in \mathcal{B} \setminus \{\ell\}} \mathbf{1}_{\{g_j(\theta, \mathbf{x}) > 0\}}(\mathbf{x}) \right]_{x_k = \chi_\ell(\theta, \mathbf{x}_k)}.
\end{aligned} \tag{8.1}$$

Because $h(\theta, \mathbf{x}) \in \mathcal{H}_k$, $\int_{\Re^n} \left| f(\theta, \mathbf{x}) \prod_{j \in \mathcal{B}} \mathbf{1}_{\{g_j(\theta, \mathbf{x}) > 0\}}(\mathbf{x}) \right| d\mathbf{x}$ is finite, and we can exchange the order of integrals by the Fubini theorem to get

$$\begin{aligned}
&\frac{\partial}{\partial \theta} \int_{\Re^n} f(\theta, \mathbf{x}) \prod_{j \in \mathcal{B}} \mathbf{1}_{\{g_j(\theta, \mathbf{x}) > 0\}}(\mathbf{x})\, d\mathbf{x} \\
&= \frac{\partial}{\partial \theta} \int_{\Re^{n-1}} \left[\int_{\Re} f(\theta, \mathbf{x}) \prod_{j \in \mathcal{B}} \mathbf{1}_{\{g_j(\theta, \mathbf{x}) > 0\}}(\mathbf{x})\, dx_k \right] d\mathbf{x}_k \\
&= \lim_{h \to 0} \int_{\Re^{n-1}} \left[\int_{\Re} \frac{f(\theta+h, \mathbf{x}) \prod_{j \in \mathcal{B}} \mathbf{1}_{\{g_j(\theta+h, \mathbf{x}) > 0\}}(\mathbf{x}) - f(\theta, \mathbf{x}) \prod_{j \in \mathcal{B}} \mathbf{1}_{\{g_j(\theta, \mathbf{x}) > 0\}}(\mathbf{x})}{h}\, dx_k \right] d\mathbf{x}_k.
\end{aligned}$$

Let $\{h_m\}$ be a sequence of numbers with $h_m \to 0$ as $m \to \infty$. Define a sequence of functions $\{q_m(\mathbf{x}_k)\}$ by

$$\begin{aligned}
&q_m(\mathbf{x}_k) \\
&= \int_{\Re} \frac{f(\theta + h_m, \mathbf{x}) \prod_{j \in \mathcal{B}} \mathbf{1}_{\{g_j(\theta + h_m, \mathbf{x}) > 0\}}(\mathbf{x}) - f(\theta, \mathbf{x}) \prod_{j \in \mathcal{B}} \mathbf{1}_{\{g_j(\theta, \mathbf{x}) > 0\}}(\mathbf{x})}{h_m}\, dx_k,
\end{aligned}$$

and define $q(\mathbf{x}_k)$ as the right-hand side of (8.1). Since (8.1) holds, $q_m(\mathbf{x}_k) \to q(\mathbf{x}_k)$ almost everywhere. We only need to show there exists an integrable function $p(\mathbf{x}_k)$ such that $|q_m(\mathbf{x}_k)| \le p(\mathbf{x}_k)$ for all m; then by Lebesgue's dominated convergence theorem,

$$\lim_{m \to \infty} \int_{\Re^{n-1}} q_m(\mathbf{x}_k)\, d\mathbf{x}_k = \int_{\Re^{n-1}} q(\mathbf{x}_k)\, d\mathbf{x}.$$

Since $\{h_m\}$ is an arbitrary sequence with $h_m \to 0$ as $m \to \infty$,

$$\begin{aligned}
&\text{LHS of (3.3)} \\
&= \lim_{h \to 0} \int_{\Re^{n-1}} \left[\int_{\Re} \frac{f(\theta+h, \mathbf{x}) \prod_{j \in \mathcal{B}} \mathbf{1}_{\{g_j(\theta+h, \mathbf{x}) > 0\}}(\mathbf{x}) - f(\theta, \mathbf{x}) \prod_{j \in \mathcal{B}} \mathbf{1}_{\{g_j(\theta, \mathbf{x}) > 0\}}(\mathbf{x})}{h}\, dx_k \right] d\mathbf{x}_k
\end{aligned}$$

$$= \lim_{m\to\infty} \int_{\Re^{n-1}} q_m(\mathbf{x}_k)\, d\mathbf{x}_k = \int_{\Re^{n-1}} q(\mathbf{x}_k)\, d\mathbf{x} = \text{RHS of (3.3)}.$$

We now prove that there exists an integrable function $p(\mathbf{x}_k)$ such that $|q_m(\mathbf{x}_k)| \le p(\mathbf{x}_k)$ for all m. We only need to prove the case when $\mathcal{B}$ contains only one member; the proofs for other cases are similar. Without loss of generality, assume $\partial g(\theta,\mathbf{x})/\partial\theta > 0$ and $\partial g(\theta,\mathbf{x})/\partial x_k > 0$. Also assume that $h(\theta,\mathbf{x})$ and $h_\theta(\theta,\mathbf{x})$ are increasing in θ and x_k. Let $a = \chi(\theta,\mathbf{x}_k)$ and $b_m = \chi(\theta+h_m,\mathbf{x}_k)$. Since $h_m \to 0$, $|h_m| < 1$ for m large enough. Note that $f(\theta,\mathbf{x})$ and $f_\theta(\theta,\mathbf{x})$ are nondecreasing in θ and we may assume that $f(\theta,\mathbf{x})$ is nondecreasing in x_k for $x_k \in [b_m, a]$ without loss of generality. For sufficiently large m,

$$\begin{aligned}
\left|q_m(\mathbf{x}_k)\right| &= \left| \int_{b_m}^{a} \frac{f(\theta+h_m,\mathbf{x})}{h_m}\, dx_k + \int_a^\infty \frac{f(\theta+h_m,\mathbf{x}) - f(\theta,\mathbf{x})}{h_m}\, dx_k \right| \\
&\le \left| \int_{b_m}^{a} \frac{f(\theta+h_m,\mathbf{x})}{h_m}\, dx_k \right| + \left| \int_a^\infty \frac{f(\theta+h_m,\mathbf{x}) - f(\theta,\mathbf{x})}{h_m}\, dx_k \right| \\
&\le \left| f(\theta+1, x_1,\ldots,x_{k-1},a,x_{k+1},\ldots,x_n)\left(\frac{a-b_m}{h_m}\right)\right| \\
&\quad + \left| \int_a^\infty f_\theta(\theta+1,\mathbf{x})\, dx_k \right|.
\end{aligned}$$

Taylor expansion gives

$$a - b_m = \frac{g_\theta(\theta, x_1,\ldots,x_{k-1},a,x_{k+1},\ldots,x_n)}{g_{x_k}(\theta, x_1,\ldots,x_{k-1},a,x_{k+1},\ldots,x_n)} h_m + O\left(h_m^2\right).$$

As a result, if m is large enough,

$$\left|\frac{a-b_m}{h_m}\right| \le \left| \frac{g_\theta(\theta, x_1,\ldots,x_{k-1},a,x_{k+1},\ldots,x_n)}{g_{x_k}(\theta, x_1,\ldots,x_{k-1},a,x_{k+1},\ldots,x_n)} \right| + 1.$$

To bound $q_m(\mathbf{x}_k)$, we choose $p(\mathbf{x}_k)$ as

$$\begin{aligned}
p(\mathbf{x}_k) &= \left| f(\theta+1, x_1,\ldots,x_{k-1},a,x_{k+1},\ldots,x_n)\right| \\
&\quad \times \left(\left| \frac{g_\theta(\theta, x_1,\ldots,x_{k-1},a,x_{k+1},\ldots,x_n)}{g_{x_k}(\theta, x_1,\ldots,x_{k-1},a,x_{k+1},\ldots,x_n)} \right| + 1 \right) \\
&\quad + \left| \int_{\chi(\theta,\mathbf{x}_k)}^\infty f_\theta(\theta+1,\mathbf{x})\, dx_k \right|,
\end{aligned}$$

which is integrable by the assumptions on $f(\theta,\mathbf{x})$ and $g(\theta,\mathbf{x})$. □

Appendix C: Proof of Theorem 4.1

Proof Applying Theorem 3.3 and noticing that $h(\theta,\mathbf{x})|_{x_1=\chi_1(\theta,\mathbf{x}_1)}=0$, we obtain

$$\Delta_1 = \mathrm{e}^{-rT} E\big[-\mathrm{e}^{(r-\sigma_1^2/2)T+\sigma_1\sqrt{T}x_1}\mathbf{1}_{\{g(\theta,\mathbf{x})>0\}}(\mathbf{x})\big],$$

or equivalently

$$\Delta_1 = \mathrm{e}^{-rT} E\big[-S_1(T)\mathbf{1}_{\{S_2(T)-S_1(T)-K>0\}}(\mathbf{S}_T)\big]/S_1.$$

To derive Γ_{11}, note that $-\mathrm{e}^{(r-\sigma_1^2/2)T+\sigma_1\sqrt{T}x_1}\in\mathcal{H}_2$ with pdf $f(\mathbf{x};\mathbf{0},\boldsymbol{\Sigma})$ and $g(\theta,\mathbf{x})\in\mathcal{G}_2$ with $\chi(\theta,\mathbf{x}_2)$ defined in (4.1). As a result, we conclude that $-\mathrm{e}^{(r-\sigma_1^2/2)T+\sigma_1\sqrt{T}x_1}\mathbf{1}_{\{g(\theta,\mathbf{x})>0\}}\in\mathcal{C}$. Note that

$$\operatorname{sign}\left(\frac{\partial g(\theta,\mathbf{x})}{\partial x_2}\right)\frac{\partial g(\theta,\mathbf{x})/\partial S_1}{\partial g(\theta,\mathbf{x})/\partial x_2} = -\frac{\mathrm{e}^{(r-\sigma_1^2/2)T+\sigma_1\sqrt{T}x_1}}{S_2\sigma_2\sqrt{T}\mathrm{e}^{(r-\sigma_2^2/2)T+\sigma_2\sqrt{T}x_2}},$$

and let

$$\tilde{\eta}(\theta,\mathbf{x}_2) =. \frac{f(\mathbf{x};\mathbf{0},\Sigma)}{f(\mathbf{x}_2;\mathbf{0},\mathbf{I}_{n-1})}\bigg|_{x_2=\chi(\theta,\mathbf{x}_2)}, \tag{9.1}$$

where $\chi(\theta,\mathbf{x}_2)$ is defined in (4.1). Now

$$\frac{\mathrm{e}^{(r-\sigma_1^2/2)T+\sigma_1\sqrt{T}x_1}}{S_2\sigma_2\sqrt{T}\mathrm{e}^{(r-\sigma_2^2/2)T+\sigma_2\sqrt{T}x_2}}\bigg|_{x_2=\chi(\theta,\mathbf{x}_2)} = \frac{\mathrm{e}^{(r-\sigma_1^2/2)T+\sigma_1\sqrt{T}x_1}}{S_2\sigma_2\sqrt{T}\mathrm{e}^{(r-\sigma_2^2/2)T+\sigma_2\sqrt{T}\chi(\theta,\mathbf{x}_2)}}.$$

Apply Theorem 3.3 to Δ_1 to obtain

$$\Gamma_{11} = \mathrm{e}^{-rT}E_q\left[\mathrm{e}^{(r-\sigma_1^2/2)T+\sigma_1\sqrt{T}x_1}\frac{\mathrm{e}^{(r-\sigma_1^2/2)T+\sigma_1\sqrt{T}x_1}}{S_2\sigma_2\sqrt{T}\mathrm{e}^{(r-\sigma_2^2/2)T+\sigma_2\sqrt{T}\chi(\theta,\mathbf{x}_2)}}\tilde{\eta}(\theta,\mathbf{x}_2)\right],$$

where $\mathbf{x}_2\sim N(0,1)$ under E_q. To derive Γ_{12}, note that

$$\operatorname{sign}\left(\frac{\partial g(\theta,\mathbf{x})}{\partial x_2}\right)\frac{\partial g(\theta,\mathbf{x})/\partial S_2}{\partial g(\theta,\mathbf{x})/\partial x_2} = \frac{1}{S_2\sigma_2\sqrt{T}}.$$

Hence

$$\Gamma_{12} = \mathrm{e}^{-rT}E_q\left[-\mathrm{e}^{(r-\sigma_1^2/2)T+\sigma_1\sqrt{T}x_1}\frac{1}{S_2\sigma_2\sqrt{T}}\tilde{\eta}(\theta,\mathbf{x}_2)\right]. \qquad \square$$

Appendix D: Proof of Theorem 4.2

Proof To derive Δ_i, rewrite $\wp(\theta,\mathbf{x})$ as

$$\wp(\theta,\mathbf{x}) = h_i(\theta,\mathbf{x})\prod_{j\in\mathcal{B}}\mathbf{1}_{\{g_{ij}(\theta,\mathbf{x})>0\}}(\mathbf{x}) + \sum_{\ell\in\mathcal{B}\setminus\{i\}}h_\ell(\theta,\mathbf{x})\prod_{j\in\mathcal{B}}\mathbf{1}_{\{g_{\ell j}(\theta,\mathbf{x})>0\}}(\mathbf{x}). \tag{10.1}$$

Note that $g_{ij}(\theta,\mathbf{x}) \in \mathcal{G}_i$, $g_{li}(\theta,\mathbf{x}) \in \mathcal{G}_i$,

$$\operatorname{sign}\left(\frac{\partial g_{ij}(\theta,\mathbf{x})}{\partial x_i}\right)\frac{\partial g_{ij}(\theta,\mathbf{x})/\partial S_i}{\partial g_{ij}(\theta,\mathbf{x})/\partial x_i} = \frac{1}{S_i\sigma_i\sqrt{T}} \quad \text{for } i, j \in \mathcal{B}, \tag{10.2}$$

$$\operatorname{sign}\left(\frac{\partial g_{\ell i}(\theta,\mathbf{x})}{\partial x_i}\right)\frac{\partial g_{\ell i}(\theta,\mathbf{x})/\partial S_i}{\partial g_{\ell i}(\theta,\mathbf{x})/\partial x_i} = -\frac{1}{S_i\sigma_i\sqrt{T}} \quad \text{for } \ell \in \mathcal{B}\backslash\{i\}. \tag{10.3}$$

Let

$$\eta_i(\theta,\mathbf{x}) = f(\mathbf{x};\mathbf{0},\boldsymbol{\Sigma})/f(\mathbf{x}_i;\mathbf{0},\mathbf{I}_{n-1}) \quad \text{for } i \in \mathcal{B}. \tag{10.4}$$

Apply Theorem 3.3 to obtain

$$\begin{aligned}
\Delta_i &= \mathrm{e}^{-rT}\Bigg\{E\Bigg[\mathrm{e}^{(r-\sigma_i^2/2)T+\sigma_i\sqrt{T}x_i}\prod_{j\in\mathcal{B}}\mathbf{1}_{\{g_{ij}(\theta,\mathbf{x})>0\}}(\mathbf{x})\Bigg] \\
&\quad + E_q\Bigg[h_i(\theta,\mathbf{x})\frac{1}{S_i\sigma_i\sqrt{T}}\prod_{j\in\mathcal{B}\backslash\{i\}}\mathbf{1}_{\{g_{ij}(\theta,\mathbf{x})>0\}}(\mathbf{x})\eta_i(\theta,\mathbf{x})\Bigg]_{x_i=\chi_{ii}(\theta,\mathbf{x}_i)} \\
&\quad + \sum_{\ell\in\mathcal{B}\backslash\{i\}} E_q\Bigg[h_i(\theta,\mathbf{x})\frac{1}{S_i\sigma_i\sqrt{T}}\prod_{j\in\mathcal{B}\backslash\{\ell\}}\mathbf{1}_{\{g_{ij}(\theta,\mathbf{x})>0\}}(\mathbf{x})\eta_i(\theta,\mathbf{x})\Bigg]_{x_i=\chi_{i\ell}(\theta,\mathbf{x}_i)} \\
&\quad + \sum_{\ell\in\mathcal{B}\backslash\{i\}} E_q\Bigg[h_\ell(\theta,\mathbf{x})\frac{-1}{S_i\sigma_i\sqrt{T}}\prod_{j\in\mathcal{B}\backslash\{i\}}\mathbf{1}_{\{g_{\ell j}(\theta,\mathbf{x})>0\}}(\mathbf{x})\eta_i(\theta,\mathbf{x})\Bigg]_{x_i=\chi_{i\ell}(\theta,\mathbf{x}_i)}\Bigg\}, \\
&= \mathrm{e}^{-rT}\big\{(\mathbf{I})+(\mathbf{II})+(\mathbf{III})+(\mathbf{IV})\big\},
\end{aligned}$$

where $(\mathbf{I})$, $(\mathbf{II})$ and $(\mathbf{III})$ result from differentiating the first term in (10.1) and $(\mathbf{IV})$ results from differentiating the second term in (10.1). Notice that $h_i(\theta,\mathbf{x})|_{x_1=\chi_{ii}(\theta,\mathbf{x}_1)} = 0$, so $(\mathbf{II}) = 0$. For $(\mathbf{III})$, $h_i(\theta,\mathbf{x})|_{x_i=\chi_{i\ell}(\theta,\mathbf{x}_i)} = S\mathrm{e}^{(r-\sigma_\ell^2/2)T+\sigma_\ell\sqrt{T}x_\ell} - K$ and

$$\begin{aligned}
&\prod_{j\in\mathcal{B}\backslash\{\ell\}}\mathbf{1}_{\{g_{ij}(\theta,\mathbf{x})>0\}}(\mathbf{x})\Big|_{x_i=\chi_{i\ell}(\theta,\mathbf{x}_i)} \\
&\quad = \Bigg(\prod_{j\in\mathcal{B}\backslash\{\ell,i\}}\mathbf{1}_{\{g_{ij}(\theta,\mathbf{x})>0\}}(\mathbf{x})|_{x_i=\chi_{i\ell}(\theta,\mathbf{x}_i)}\Bigg)\big(\mathbf{1}_{\{g_{ii}(\theta,\mathbf{x})>0\}}(\mathbf{x})|_{x_i=\chi_{i\ell}(\theta,\mathbf{x}_i)}\big) \\
&\quad = \Bigg(\prod_{j\in\mathcal{B}\backslash\{\ell,i\}}\mathbf{1}_{\{g_{\ell j}(\theta,\mathbf{x})>0\}}(\mathbf{x})\Bigg)\mathbf{1}_{\{g_{\ell\ell}(\theta,\mathbf{x})>0\}}(\mathbf{x}) \\
&\quad = \prod_{j\in\mathcal{B}\backslash\{i\}}\mathbf{1}_{\{g_{\ell j}(\theta,\mathbf{x})>0\}}(\mathbf{x}).
\end{aligned} \tag{10.5}$$

Therefore, for each term in $(\mathbf{III})$,

$$E_q\Bigg[h_i(\theta,\mathbf{x})\frac{1}{S_i\sigma_i\sqrt{T}}\prod_{j\in\mathcal{B}\backslash\{\ell\}}\mathbf{1}_{\{g_{ij}(\theta,\mathbf{x})>0\}}(\mathbf{x})\eta_i(\theta,\mathbf{x})\Bigg]_{x_i=\chi_{i\ell}(\theta,\mathbf{x}_i)}$$

$$= E_q\left[\left(S_\ell \mathrm{e}^{(r-\sigma_\ell^2/2)T+\sigma_\ell\sqrt{T}x_\ell} - K\right)\prod_{j\in\mathcal{B}\setminus\{i\}} \mathbf{1}_{\{g_{\ell j}(\theta,\mathbf{x})>0\}}(\mathbf{x})\tilde{\eta}_{i\ell}(\theta,\mathbf{x}_i)\right]\Big/\left(S_i\sigma_i\sqrt{T}\right),$$

where

$$\tilde{\eta}_{i\ell}(\theta,\mathbf{x}_i) = \eta_i(\theta,\mathbf{x})|_{x_i=\chi_{i\ell}(\theta,\mathbf{x}_i)} \quad \text{for } i,\ell\in\mathcal{B}. \tag{10.6}$$

Similarly, for each term in (**IV**),

$$E_q\left[h_\ell(\theta,\mathbf{x})\frac{-1}{S_i\sigma_i\sqrt{T}}\prod_{j\in\mathcal{B}\setminus\{i\}} \mathbf{1}_{\{g_{\ell j}(\theta,\mathbf{x})>0\}}(\mathbf{x})\eta_i(\theta,\mathbf{x})\right]_{x_i=\chi_{i\ell}(\theta,\mathbf{x}_i)}$$
$$= -E_q\left[\left(S_\ell \mathrm{e}^{(r-\sigma_\ell^2/2)T+\sigma_\ell\sqrt{T}x_\ell} - K\right)\prod_{j\in\mathcal{B}\setminus\{i\}} \mathbf{1}_{\{g_{\ell j}(\theta,\mathbf{x})>0\}}(\mathbf{x})\tilde{\eta}_{i\ell}(\theta,\mathbf{x}_i)\right]$$
$$\Big/\left(S_i\sigma_i\sqrt{T}\right).$$

As a result, (**III**) and (**IV**) sum to zero. Apply Theorem 3.3 to Δ_i to obtain

$$\begin{aligned}\Gamma_{ii} = \mathrm{e}^{-rT}\Bigg\{ & E_q\Bigg[\mathrm{e}^{(r-\sigma_i^2/2)T+\sigma_i\sqrt{T}x_i}\frac{1}{S_i\sigma_i\sqrt{T}} \\ & \times \prod_{j\in\mathcal{B}\setminus\{i\}} \mathbf{1}_{\{g_{ij}(\theta,\mathbf{x})>0\}}(\mathbf{x})\eta_i(\theta,\mathbf{x})\Bigg]_{x_i=\chi_{ii}(\theta,\mathbf{x}_i)} \\ & + \sum_{\ell\in\mathcal{B}\setminus\{i\}} E_q\Bigg[\mathrm{e}^{(r-\sigma_i^2/2)T+\sigma_i\sqrt{T}x_i}\frac{1}{S_i\sigma_i\sqrt{T}} \\ & \times \prod_{j\in\mathcal{B}\setminus\{\ell\}} \mathbf{1}_{\{g_{ij}(\theta,\mathbf{x})>0\}}(\mathbf{x})\eta_i(\theta,\mathbf{x})\Bigg]_{x_i=\chi_{i\ell}(\theta,\mathbf{x}_i)}\Bigg\}.\end{aligned}$$

Note that $g_{ij}(\theta,\mathbf{x})|_{x_i=\chi_{ii}(\theta,\mathbf{x}_i)} = -g_{jj}(\theta,\mathbf{x})$, $g_{ij}(\theta,\mathbf{x})|_{x_i=\chi_{i\ell}(\theta,\mathbf{x}_i)} = g_{\ell j}(\theta,\mathbf{x})$, and

$$\begin{aligned}\prod_{j\in\mathcal{B}\setminus\{i\}} \mathbf{1}_{\{g_{ij}(\theta,\mathbf{x})>0\}}(\mathbf{x})|_{x_i=\chi_{ii}(\theta,\mathbf{x}_i)} &= \prod_{j\in\mathcal{B}\setminus\{i\}} \mathbf{1}_{\{-g_{jj}(\theta,\mathbf{x})>0\}}(\mathbf{x}), \\ \prod_{j\in\mathcal{B}\setminus\{\ell\}} \mathbf{1}_{\{g_{ij}(\theta,\mathbf{x})>0\}}(\mathbf{x})|_{x_i=\chi_{i\ell}(\theta,\mathbf{x}_i)} &= \prod_{j\in\mathcal{B}\setminus\{\ell\}} \mathbf{1}_{\{g_{\ell j}(\theta,\mathbf{x})>0\}}(\mathbf{x}).\end{aligned} \tag{10.7}$$

As a result,

$$\Gamma_{ii} = \mathrm{e}^{-rT}\Bigg\{E_q\left[\frac{K}{S_i}\frac{1}{S_i\sigma_i\sqrt{T}}\prod_{j\in\mathcal{B}\setminus\{i\}} \mathbf{1}_{\{-g_{jj}(\theta,\mathbf{x})>0\}}(\mathbf{x})\tilde{\eta}_{ii}(\theta,\mathbf{x}_i)\right]$$
$$+ \sum_{\ell\in\mathcal{B}\setminus\{i\}} E_q\Bigg[\frac{S_\ell \mathrm{e}^{(r-\sigma_\ell^2/2)T+\sigma_\ell\sqrt{T}x_\ell}}{S_i}\frac{1}{S_i\sigma_i\sqrt{T}}$$

$$\times \prod_{j\in\mathcal{B}\setminus\{\ell\}} \mathbf{1}_{\{g_{\ell j}(\theta,\mathbf{x})>0\}}(\mathbf{x})\tilde{\eta}_{i\ell}(\theta,\mathbf{x}_i)\Bigg]\Bigg\}.$$

To differentiate Δ_i with respect to S_j, note that only $\mathbf{1}_{\{g_{ij}(\theta,\mathbf{x})>0\}}(\theta,\mathbf{x})$ depends on S_j. Since $g_{ij}(\theta,\mathbf{x})\in\mathcal{G}_i$ and

$$\operatorname{sign}\left(\frac{\partial g_{ij}(\theta,\mathbf{x})}{\partial x_i}\right)\frac{\partial g_{ij}(\theta,\mathbf{x})/\partial S_j}{\partial g_{ij}(\theta,\mathbf{x})/\partial x_i}=\frac{-\mathrm{e}^{(r-\sigma_j^2/2)T+\sigma_j\sqrt{T}x_j}}{S_i\sigma_i\sqrt{T}\mathrm{e}^{(r-\sigma_i^2/2)T+\sigma_i\sqrt{T}x_i}},$$

apply Theorem 3.3 to obtain

$$\begin{aligned}\Gamma_{ij}&=\mathrm{e}^{-rT}E_q\Bigg[\mathrm{e}^{(r-\sigma_i^2/2)T+\sigma_i\sqrt{T}x_i}\frac{-\mathrm{e}^{(r-\sigma_j^2/2)T+\sigma_j\sqrt{T}x_j}}{S_i\sigma_i\sqrt{T}\mathrm{e}^{(r-\sigma_i^2/2)T+\sigma_i\sqrt{T}x_i}}\\&\quad\times\prod_{\ell\in\mathcal{B}\setminus\{j\}}\mathbf{1}_{\{g_{i\ell}(\theta,\mathbf{x})>0\}}(\mathbf{x})\eta_i(\theta,\mathbf{x})\Bigg]_{x_i=\chi_{ij}(\theta,\mathbf{x}_j)}\\&=\mathrm{e}^{-rT}E_q\Bigg[\frac{-\mathrm{e}^{(r-\sigma_j^2/2)T+\sigma_j\sqrt{T}x_j}}{S_i\sigma_i\sqrt{T}}\prod_{\ell\in\mathcal{B}\setminus\{j\}}\mathbf{1}_{\{g_{j\ell}(\theta,\mathbf{x})>0\}}(\mathbf{x})\tilde{\eta}_{ij}(\theta,\mathbf{x}_i)\Bigg].\qquad\square\end{aligned}$$

Appendix E: Proof of Theorem 4.3

Proof Similarly to the proof of Theorem 4.2, write $\wp(\theta,\mathbf{x})$ as

$$\wp(\theta,\mathbf{x})=\prod_{j\in\mathcal{B}}\mathbf{1}_{\{g_{ij}(\theta,\mathbf{x})>0\}}(\mathbf{x})+\sum_{\ell\in\mathcal{B}\setminus\{i\}}\prod_{j\in\mathcal{B}}\mathbf{1}_{\{g_{\ell j}(\theta,\mathbf{x})>0\}}(\mathbf{x}).\tag{11.1}$$

Recall (10.2)–(10.4). Apply Theorem 3.3,

$$\begin{aligned}\Delta_i&=\mathrm{e}^{-rT}\Bigg\{E_q\Bigg[\frac{1}{S_i\sigma_i\sqrt{T}}\prod_{j\in\mathcal{B}\setminus\{i\}}\mathbf{1}_{\{g_{ij}(\theta,\mathbf{x})>0\}}(\mathbf{x})\eta_i(\theta,\mathbf{x})\Bigg]_{x_i=\chi_{ii}(\theta,\mathbf{x}_i)}\\&\quad+\sum_{\ell\in\mathcal{B}\setminus\{i\}}E_q\Bigg[\frac{1}{S_i\sigma_i\sqrt{T}}\prod_{j\in\mathcal{B}\setminus\{\ell\}}\mathbf{1}_{\{g_{ij}(\theta,\mathbf{x})>0\}}(\mathbf{x})\eta_i(\theta,\mathbf{x})\Bigg]_{x_i=\chi_{i\ell}(\theta,\mathbf{x}_i)}\\&\quad+\sum_{\ell\in\mathcal{B}\setminus\{i\}}E_q\Bigg[\frac{-1}{S_i\sigma_i\sqrt{T}}\prod_{j\in\mathcal{B}\setminus\{i\}}\mathbf{1}_{\{g_{\ell j}(\theta,\mathbf{x})>0\}}(\mathbf{x})\eta_i(\theta,\mathbf{x})\Bigg]_{x_i=\chi_{i\ell}(\theta,\mathbf{x}_i)}\Bigg\},\\&=\mathrm{e}^{-rT}\big\{(\mathbf{I})+(\mathbf{II})+(\mathbf{III})\big\},\end{aligned}$$

where (**I**) and (**II**) result from differentiating the first term in (11.1) and (**III**) results from differentiating the second term in (11.1). The sum of (**II**) and (**III**) is zero by (10.5). Following (10.7), we have

$$\Delta_i=\mathrm{e}^{-rT}E_q\Bigg[\frac{1}{S_i\sigma_i\sqrt{T}}\prod_{j\in\mathcal{B}\setminus\{i\}}\mathbf{1}_{\{-g_{jj}(\theta,\mathbf{x})>0\}}(\theta,x_j)\tilde{\eta}_{ii}(\theta,\mathbf{x}_i)\Bigg].$$

To derive Γ_{ii}, note that $g_{jj}(\theta, \mathbf{x})$ does not involve S_i. Using the definition of $\tilde{\eta}_{ii}(\theta, \mathbf{x})$ in (10.6), the chain rule gives

$$\frac{\partial}{\partial S_i}\tilde{\eta}_{ii}(\theta, \mathbf{x}_i) = \frac{\mathbf{v_i}^t \boldsymbol{\Sigma}^{-1}\tilde{\mathbf{x}}_i + \tilde{\mathbf{x}}_i^t \boldsymbol{\Sigma}^{-1}\mathbf{v_i}}{2S_i\sigma_i\sqrt{T}}\tilde{\eta}_{ii}(\theta, \mathbf{x}),$$

where

$$\tilde{\mathbf{x}}_i = \big(x_1, \ldots, x_{i-1}, \chi_{ii}(\theta, \mathbf{x}), x_{i+1}, \ldots, x_n\big)^t. \tag{11.2}$$

As a result,

$$\Gamma_{ii} = \mathrm{e}^{-rT} E_q\left[\frac{1}{S_i\sigma_i\sqrt{T}}\left(\frac{-1}{S_i} + \frac{\mathbf{v_i}^t \boldsymbol{\Sigma}\tilde{\mathbf{x}}_i + \tilde{\mathbf{x}}_i \boldsymbol{\Sigma}\mathbf{v_i}}{2S_i\sigma_i\sqrt{T}}\right)\prod_{j\in\mathcal{B}\setminus\{i\}}\mathbf{1}_{\{-g_{jj}(\theta,\mathbf{x})>0\}}(x_j)\tilde{\eta}_{ii}(\mathbf{x}_i)\right].$$

Apply Theorem 3.3 to obtain

$$\begin{aligned}\Gamma_{ij} = \mathrm{e}^{-rT} E_{q,j}\Bigg[&\frac{1}{S_i\sigma_i\sqrt{T}}\frac{-1}{S_j\sigma_j\sqrt{T}}\\ &\times\prod_{\ell\in\mathcal{B}\setminus\{i,j\}}\mathbf{1}_{\{-g_{\ell\ell}(\theta,x_\ell)>0\}}(\theta, x_\ell)\tilde{\eta}_{ii,j}(\theta, \mathbf{x}_{ij})f(\chi_{jj}(\theta); 0, 1)\Bigg],\end{aligned}$$

where $\mathbf{x}_{ij}$ has pdf $f(\mathbf{x}_{ij}; \mathbf{0}, \mathbf{I}_{n-2})$ under $E_{q,j}$, and

$$\tilde{\eta}_{ii,j}(\mathbf{x}_{ij}) = \tilde{\eta}_{ii}(\theta, \mathbf{x}_i)|_{x_j=\chi_{jj}(\theta,\mathbf{x})}. \tag{11.3}$$

□

Appendix F: Proof of Theorem 4.4

Proof Apply Theorem 3.3 to obtain

$$\begin{aligned}\Delta = \mathrm{e}^{-rT}\Bigg\{&E\left[\mathrm{e}^{(r-\sigma^2/2)m\Delta t+\sigma\sqrt{\Delta t}(x_1+\cdots+x_m)}\prod_{i\in\mathcal{B}}\mathbf{1}_{\{g_i(\theta,\mathbf{x})>0\}}(\mathbf{x})\right]\\ &+\sum_{\ell\in\mathcal{B}}E_q\left[h(\theta, \mathbf{x})J_\ell(\theta, \mathbf{x})\prod_{i\in\mathcal{B}\setminus\{\ell\}}\mathbf{1}_{\{g_i(\theta,\mathbf{x})>0\}}(\mathbf{x})\eta(\theta, \mathbf{x}_1)\right]_{x_1=\chi_\ell(\theta,\mathbf{x}_1)}\Bigg\},\end{aligned}$$

where for $\ell \in \mathcal{B}$

$$\begin{aligned}J_\ell(\theta, \mathbf{x}) &= \frac{1}{S\sigma\sqrt{\Delta t}},\\ \eta_\ell(\theta, \mathbf{x}) &= \frac{f(\mathbf{x}; \mathbf{0}, \mathbf{I}_m)}{f(\mathbf{x}_1; \mathbf{0}, \mathbf{I}_{m-1})} = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x_1^2/2}.\end{aligned}$$

Note that

$$
\begin{aligned}
&h(\theta,\mathbf{x})|_{x_1=\chi_\ell(\theta,\mathbf{x}_1)}\\
&\quad=\begin{cases} H\mathrm{e}^{(r-\sigma^2/2)(m-\ell)\Delta t+\sigma\sqrt{\Delta t}(x_{\ell+1}+\cdots+x_m)}-K & \text{for } \ell=1,\ldots,m-1,\\ H-K & \text{for } \ell=m,\\ 0 & \text{for } \ell=m+1.\end{cases}
\end{aligned}
$$

For $\ell\in\mathcal{B}$, define

$$
\begin{aligned}
\tilde{J}_\ell(\theta,\mathbf{x})&=J_\ell(\theta,\mathbf{x})|_{x_1=\chi_\ell(\theta,\mathbf{x}_1)}=\frac{1}{S\sigma\sqrt{\Delta t}},\\
\tilde{\eta}_\ell(\theta,\mathbf{x}_1)&=\eta_\ell(\theta,\mathbf{x})|_{x_1=\chi_\ell(\theta,\mathbf{x}_1)}=\tfrac{1}{\sqrt{2\pi}}\mathrm{e}^{-\chi_\ell(\theta,\mathbf{x}_1)^2/2},
\end{aligned}
\tag{12.1}
$$

where $\chi_\ell(\theta,\mathbf{x}_1)$ is defined in (4.8)–(4.10). Notice that the sets $\{\mathbf{x}:g_i(\theta,\mathbf{x})>0\}$ and $\{\mathbf{x}:x_1>\chi_i(\theta,\mathbf{x}_1)\}$ are equal for $i\in\mathcal{B}$; therefore,

$$
\mathbf{1}_{\{g_i(\theta,\mathbf{x})>0\}}(\mathbf{x})|_{x_1=\chi_\ell(\theta,\mathbf{x}_1)}=\mathbf{1}_{\{x_1>\chi_i(\theta,\mathbf{x}_1)\}}(\mathbf{x})|_{x_1=\chi_\ell(\theta,\mathbf{x}_1)}=\mathbf{1}_{\{\chi_\ell(\theta,\mathbf{x}_1)>\chi_i(\theta,\mathbf{x}_1)\}}(\mathbf{x}).
$$

Because the set $\{\mathbf{x}:\chi_\ell(\theta,\mathbf{x}_1)>\chi_i(\theta,\mathbf{x}_1)\}$ only depends on $\mathbf{x}_1$ but not $\mathbf{x}$, $\mathbf{1}_{\{\chi_\ell(\theta,\mathbf{x}_1)>\chi_i(\theta,\mathbf{x}_1)\}}(\mathbf{x})$ equals $\mathbf{1}_{\{\chi_\ell(\theta,\mathbf{x}_1)>\chi_i(\theta,\mathbf{x}_1)\}}(\mathbf{x}_1)$. Now,

$$
\begin{aligned}
\Delta=\mathrm{e}^{-rT}\Bigg\{&E\Bigg[\mathrm{e}^{(r-\sigma^2/2)m\Delta t+\sigma\sqrt{\Delta t}(x_1+\cdots+x_m)}\prod_{i\in\mathcal{B}}\mathbf{1}_{\{g_i(\theta,\mathbf{x})>0\}}(\mathbf{x})\Bigg]\\
&+\sum_{\ell=1}^{m-1}E_q\Bigg[\Big(H\mathrm{e}^{(r-\sigma^2/2)(m-\ell)\Delta t+\sigma\sqrt{\Delta t}(x_{\ell+1}+\cdots+x_m)}-K\Big)\tilde{J}_\ell(\theta,\mathbf{x})\\
&\times\prod_{i\in\mathcal{B}\setminus\{\ell\}}\mathbf{1}_{\{\chi_\ell(\theta,\mathbf{x}_1)>\chi_i(\theta,\mathbf{x}_1)\}}(\mathbf{x}_1)\tilde{\eta}_\ell(\theta,\mathbf{x}_1)\Bigg]\\
&+E_q\Bigg[(H-K)\tilde{J}_m(\theta,\mathbf{x})\prod_{i\in\mathcal{B}\setminus\{m\}}\mathbf{1}_{\{\chi_m(\theta,\mathbf{x}_1)>\chi_i(\theta,\mathbf{x}_1)\}}(\mathbf{x}_1)\tilde{\eta}_m(\theta,\mathbf{x}_1)\Bigg]\Bigg\}
\end{aligned}
\tag{12.2}
$$

Apply Theorem 3.3 to Δ to derive Γ. Note that

$$
\begin{aligned}
&\mathrm{e}^{(r-\sigma^2/2)m\Delta t+\sigma\sqrt{\Delta t}(x_1+\cdots+x_m)}|_{x_1=\chi_\ell(\theta,\mathbf{x}_1)}\\
&\quad=\begin{cases} H\mathrm{e}^{(r-\sigma^2/2)(m-\ell)\Delta t+\sigma\sqrt{\Delta t}(x_{\ell+1}+\cdots+x_m)}/S & \text{for } \ell=1,\ldots,m,\\ H/S & \text{for } \ell=m,\\ K/S & \text{for } \ell=m+1.\end{cases}
\end{aligned}
$$

From the definition of $\chi_i(\theta,\mathbf{x}_1)$, we have the set identities

$$
\big\{\chi_i(\theta,\mathbf{x}_1)>\chi_j(\theta,\mathbf{x}_1)\big\}=\left\{x_{i+1}+\cdots+x_j>\frac{-(r-\sigma^2/2)(j-i)\Delta t}{\sigma\sqrt{\Delta t}}\right\}
$$

for $i = 1, \ldots, m,\ j = i+1, \ldots, m$,

$$\left\{\chi_i(\theta, \mathbf{x}_1) > \chi_{m+1}(\theta, \mathbf{x}_1)\right\} = \left\{x_{i+1} + \cdots + x_m > \frac{\log K/H - (r - \sigma^2/2)(m-i)\Delta t}{\sigma\sqrt{\Delta t}}\right\}$$

for $i = 1, \ldots, m-1$,

$$\left\{\chi_m(\theta, \mathbf{x}_1) > \chi_{m+1}(\theta, \mathbf{x}_1)\right\} = \{H > K\}$$

hence the only term that depends on S in the integrand under the E_q in (12.2) is $\tilde{J}_\ell(\theta, \mathbf{x}_1)\tilde{\eta}_\ell(\theta, \mathbf{x}_1)$, and

$$\frac{\partial}{\partial S}\left(\tilde{J}_\ell(\theta, \mathbf{x}_1)\tilde{\eta}_\ell(\theta, \mathbf{x}_1)\right) = \frac{\tilde{J}_\ell(\theta, \mathbf{x}_1)\tilde{\eta}_\ell(\theta, \mathbf{x}_1)}{S}\left(-1 + \frac{\chi_\ell(\theta, \mathbf{x}_1)}{\sigma\sqrt{\Delta t}}\right).$$

Therefore, we have

$$\begin{aligned}
\Gamma = \mathrm{e}^{-rT}\Bigg\{ & \sum_{\ell=1}^{m-1} E_q\Bigg[\frac{H}{S}\mathrm{e}^{(r-\sigma^2/2)(m-\ell)\Delta t + \sigma\sqrt{\Delta t}(x_{\ell+1}+\cdots+x_m)}\tilde{J}_\ell(\theta, \mathbf{x}_1) \\
& \times \prod_{i\in\mathcal{B}\setminus\{\ell\}} \mathbf{1}_{\{\chi_\ell(\theta,\mathbf{x}_1) > \chi_i(\theta,\mathbf{x}_1)\}}(\mathbf{x}_1)\tilde{\eta}_\ell(\theta, \mathbf{x}_1)\Bigg] \\
& + E_q\Bigg[\frac{H}{S}\tilde{J}_m(\theta, \mathbf{x}_1) \prod_{i\in\mathcal{B}\setminus\{m\}} \mathbf{1}_{\{\chi_m(\theta,\mathbf{x}_1) > \chi_i(\theta,\mathbf{x}_1)\}}(\mathbf{x}_1)\tilde{\eta}_m(\theta, \mathbf{x}_1)\Bigg] \\
& + E_q\Bigg[\frac{K}{S}\tilde{J}_{m+1}(\theta, \mathbf{x}_1) \prod_{i\in\mathcal{B}\setminus\{m+1\}} \mathbf{1}_{\{\chi_{m+1}(\theta,\mathbf{x}_1) > \chi_i(\theta,\mathbf{x}_1)\}}(\mathbf{x}_1)\tilde{\eta}_{m+1}(\theta, \mathbf{x}_1)\Bigg] \\
& + \sum_{\ell=1}^{m-1} E_q\Bigg[\left(H\mathrm{e}^{(r-\sigma^2/2)(m-\ell)\Delta t + \sigma\sqrt{\Delta t}(x_{\ell+1}+\cdots+x_m)} - K\right) \\
& \times \frac{\tilde{J}_\ell(\theta, \mathbf{x}_1)\tilde{\eta}_\ell(\theta, \mathbf{x}_1)}{S}\left(-1 + \frac{\chi_\ell(\theta, \mathbf{x}_1)}{\sigma\sqrt{\Delta t}}\right) \prod_{i\in\mathcal{B}\setminus\{\ell\}} \mathbf{1}_{\{\chi_\ell(\theta,\mathbf{x}_1) > \chi_i(\theta,\mathbf{x}_1)\}}(\mathbf{x}_1)\Bigg] \\
& + E_q\Bigg[(H - K)\frac{\tilde{J}_m(\theta, \mathbf{x}_1)\tilde{\eta}_m(\theta, \mathbf{x}_1)}{S}\left(-1 + \frac{\chi_m(\theta, \mathbf{x}_1)}{\sigma\sqrt{\Delta t}}\right) \\
& \times \prod_{i\in\mathcal{B}\setminus\{m\}} \mathbf{1}_{\{\chi_m(\theta,\mathbf{x}_1) > \chi_i(\theta,\mathbf{x}_1)\}}(\mathbf{x}_1)\Bigg]\Bigg\},
\end{aligned}$$

where the first three terms result from differentiating the first term on the right-hand side of (12.2), and the last two terms result from differentiating the second and the third terms on the right-hand side of (12.2). □

Appendix G: Greeks using the likelihood ratio method

The relationship between $\mathbf{S_T}$ and $\mathbf{x}$ is in (2.1). Let $\mathbf{s} = (s_1, \ldots, s_n)^t$ and $f_{\mathbf{S_T}}(\mathbf{s})$ be the probability density function of $\mathbf{S_T}$. Define

$$d_i(s_i) = \frac{\log(s_i/S_i) - (r - \sigma_i^2/2)T}{\sigma_i\sqrt{T}} \quad \text{for } i = 1, \ldots, n,$$

and $\mathbf{d}(\mathbf{s}) = (d_1(s_1), \ldots, d_n(s_n))^t$. Then $\frac{\partial d_i(s_i)}{\partial S_i} = -1/(S_i\sigma_i\sqrt{T})$, and

$$f_{\mathbf{S_T}}(\mathbf{s}) = f\big(\mathbf{d}(\mathbf{s}); \mathbf{0}, \mathbf{\Sigma}\big)\left(\prod_{i=1}^{n} \frac{1}{s_i\sigma_i\sqrt{T}}\right). \tag{13.1}$$

Recall that $\mathbf{v_i}$ is an $n \times 1$ vector having 1 in the ith component and 0 elsewhere. By straightforward calculations,

$$\frac{\partial}{\partial S_i} f_{\mathbf{S_T}}(\mathbf{s}) = f_{\mathbf{S_T}}(\mathbf{s})\left[\frac{\mathbf{v_i}^t\Sigma^{-1}\mathbf{d}(\mathbf{s})}{S_i\sigma_i\sqrt{T}}\right],$$

$$\frac{\partial^2}{\partial S_i^2} f_{\mathbf{S_T}}(\mathbf{s}) = f_{\mathbf{S_T}}(\mathbf{s})\left[\frac{(\mathbf{v_i}^t\mathbf{\Sigma}^{-1}\mathbf{d}(\mathbf{s}))^2 - \mathbf{v_i}^t\mathbf{\Sigma}^{-1}\mathbf{v_i} - \mathbf{v_i}^t\Sigma^{-1}\mathbf{d}(\mathbf{s})\sigma_i\sqrt{T}}{S_i^2\sigma_i^2 T}\right],$$

$$\frac{\partial^2}{\partial S_i\partial S_j} f_{\mathbf{S_T}}(\mathbf{s}) = f_{\mathbf{S_T}}(\mathbf{s})\left[\frac{(\mathbf{v_i}^t\mathbf{\Sigma}^{-1}\mathbf{d}(\mathbf{s}))(\mathbf{v_j}^t\mathbf{\Sigma}^{-1}\mathbf{d}(\mathbf{s})) - \mathbf{v_i}^t\Sigma^{-1}\mathbf{v_j}}{S_iS_j\sigma_i\sigma_j T}\right].$$

The Greeks using the likelihood ratio method are [4]:

$$\Delta_i = \mathrm{e}^{-rT}E\left[\wp(\mathbf{S_T})\frac{\mathbf{v_i}^t\mathbf{\Sigma}^{-1}\mathbf{d}(\mathbf{S_T})}{S_i\sigma_i\sqrt{T}}\right],$$

$$\Gamma_{ii} = \mathrm{e}^{-rT}E\left[\wp(\mathbf{S_T})\frac{(\mathbf{v_i}^t\mathbf{\Sigma}^{-1}\mathbf{d}(\mathbf{S_T}))^2 - \mathbf{v_i}^t\Sigma^{-1}\mathbf{v_i} - \mathbf{v_i}^t\mathbf{\Sigma}^{-1}\mathbf{d}(\mathbf{S_T})\sigma_i\sqrt{T}}{S_i^2\sigma_i^2 T}\right],$$

$$\Gamma_{ij} = \mathrm{e}^{-rT}E\left[\wp(\mathbf{S_T})\frac{(\mathbf{v_i}^t\mathbf{\Sigma}^{-1}\mathbf{d}(\mathbf{S_T}))(\mathbf{v_j}^t\mathbf{\Sigma}^{-1}\mathbf{d}(\mathbf{S_T})) - \mathbf{v_i}^t\mathbf{\Sigma}^{-1}\mathbf{v_j}}{S_iS_j\sigma_i\sigma_j T}\right], \quad i \neq j,$$

where the expectation is taken over the random variable $\mathbf{S_T}$ whose probability density function is in (13.1). Or equivalently after change of variables,

$$\Delta_i = \mathrm{e}^{-rT}E\left[\wp(\mathbf{S_T})\frac{\mathbf{v_i}^t\mathbf{\Sigma}^{-1}\mathbf{x}}{S_i\sigma_i\sqrt{T}}\right],$$

$$\Gamma_{ii} = \mathrm{e}^{-rT}E\left[\wp(\mathbf{S_T})\frac{(\mathbf{v_i}^t\mathbf{\Sigma}^{-1}\mathbf{x})^2 - \mathbf{v_i}^t\Sigma^{-1}\mathbf{v_i} - \mathbf{v_i}^t\mathbf{\Sigma}^{-1}\mathbf{x}\sigma_i\sqrt{T}}{S_i^2\sigma_i^2 T}\right],$$

$$\Gamma_{ij} = \mathrm{e}^{-rT}E\left[\wp(\mathbf{S_T})\frac{(\mathbf{v_i}^t\mathbf{\Sigma}^{-1}\mathbf{x})(\mathbf{v_j}^t\mathbf{\Sigma}^{-1}\mathbf{x}) - \mathbf{v_i}^t\Sigma^{-1}\mathbf{v_j}}{S_iS_j\sigma_i\sigma_j T}\right], \quad i \neq j,$$

where the expectation is taken over the random variable $\mathbf{x}$.

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