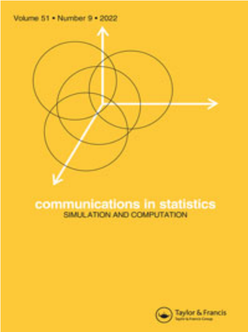

Communications in Statistics - Simulation and Computation

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/lssp20

Simulating false alarm probability in *K*-distributed sea clutter

Huei-Wen Teng & Cheng-Der Fuh

COMMUNICATIONS IN STATISTICS - SIMULATION AND COMPUTATION®
2022, VOL. 51, NO. 9, 5081–5098
https://doi.org/10.1080/03610918.2020.1757707

Taylor & Francis
Taylor & Francis Group

Simulating false alarm probability in *K*-distributed sea clutter

Huei-Wen Teng[a] and Cheng-Der Fuh[b]

[a]Department of Information Management and Finance, National Chiao Tung University, Hsinchu, Taiwan; [b]Fanhai International School of Finance, Fudan University, Shanghai, China

ABSTRACT
Efficient and precise calculation of the probability of false alarms for the *K*-distributed sea clutter and noise is crucial in radar detection system. We propose efficient and optimal importance sampling algorithms, in which two importance sampling estimators are constructed for both the product form and the compound form of the underlying *K* distribution. Moreover, we prove the existence and uniqueness of the optimal tilting parameters. Detection accuracy and efficiency are analyzed in simulation studies.

ARTICLE HISTORY
Received 12 February 2018
Accepted 13 April 2020

KEYWORDS
Constant false alarm rate; Importance sampling; Simulation

1. Introduction

It is known that the constant false alarm rate (CFAR) detection algorithm, based on the analysis of sea clutter statistical properties, is a useful tool for sea clutter detection rule, and has been widely used to search for ship targets surrounded by sea clutters in Synthetic Aperture Radar (SAR) imagery for ocean surveillance, cf. Ward, Watts, and Tough (2006). In CFAR alrogithm, the threshold is calculated depending on the statistics of the surrounding area to ensure that the false alarm rate is constant. Efficient calculation of the probability of false alarm (PFA) that determines an extreme probability of how many pixels belong to target or over the decision threshold is essential in detection theory.

Due to recent improvements in SAR resolution, an interest in the detection of targets in correlated and non-Gaussian radar clutter environments has led to the multivariate spherically invariant random vector (SIRV) clutter model, cf. Conte and Longo (1987) and Rangaswamy, Weiner, and Ozturk (1993). The SIRV model generalizes non-Gaussian distributions and are commonly used as statistical fits to real clutter data. It results in the lognormal, alpha-stable, Weibull distribution, the Student-t distribution, and the K-distribution, which are commonly used to model sea clutter. Various CFAR schemes along these lines have been proposed, cf. di Bisceglie and Galdi (2005) and Gao et al. (2009), and so on. Among all of these models, a more sophisticated K-distributed sea clutter model, can be used to calculate PFA in the CFAR algorithm in detection systems, cf. Ward, Watts, and Tough (2006), Watts (1987), Jiang et al. (1998),

CONTACT Huei-Wen Teng venteng@gmail.com Department of Information Management and Finance, National Chiao Tung University, Hsinchu 30020, Taiwan.

Delignon and Pieczynski (2002), and Paes, Lorenzzetti, and Gherardi (2010). The simple cell-averaging CFAR algorithm in a SAR image searches for pixel values which are unusually right compared to those in the surrounding area. Pixel values which lie above the detection threshold are declared abnormally bright and therefore likely to be samples from a target. The reader is referred to Chapter 9 of Ward, Watts, and Tough (2006) for details.

For the statistical modeling of sea clutter in CFAR, a compound form of the K-distribution has been shown to have excellent agreement with observed data, cf. Chapter 4 of Ward, Watts, and Tough (2006). Moreover, Watts (1987) shows that this model not only give a good fit to clutter amplitude distributions over a wide range of conditions, more importantly it also allows the pulse-to-pulse correlation of the returns to be correctly modeled, which is very important for the accurate prediction of target detection performance following the pulse-to-pulse integration of radar returns.

An important step in detection is to calculate a PFA, which is defined as P_{fa} in the following equation,

$$P_{fa} := P_{fa}(T) = \int_T^{\infty} f(x)dx, \tag{1}$$

where T is a pre-specified threshold of detection and $f(x)$ is a probability density function of sea clutter. Computing a tolerable and small PFA given a fixed threshold is prominent in detection theory.

It is worth mentioning that in practical CFAR detector systems, the parameters of the underlying distribution are unknown and need to be estimated from the sea clutter data. This is the problem usually considered in radar detection when the noise distribution is non-Gaussian with unknown parameters. Note that if the underlying distribution is Gaussian with unknown mean and variance, one typically uses the maximum likelihood estimators, and then the threshold can be expressed as a linear function of the estimated parameters. However, when one uses K-distributed sea clutter for the CFAR detection systems, this becomes a practically important issue. In general, the detection threshold required to maintain a constant false alarm rate depends on the amplitude statistics of non-Gaussian, such as K-distributed clutter. For a CFAR system, the surrounding clutter may be used to estimate the K-distribution parameters and hence the threshold multiplier. The reader is referred to Section 9.3 of Ward, Watts, and Tough (2006) for details. In this paper, we focus on the problem of estimating the PFA via simulation other than studying the practical statistical issue.

Although the K distribution has an excellent agreement with observed data in virtue of a more flexible quality, such as heavy tails, leptokurtosis, or asymmetry of the amplitude of a single pulse of radar sea clutter data, it is known that the computation based on the K distribution is challenging. It is effortlessly perceived that there is no explicit formula of the PFA, and also, typically a numerical method may be needed to resolve the threshold of K based the CFAR because of the appearance of modified Bessel function. Calculating the PFA for K-distributed sea clutter appears to be computationally tedious and analytically difficult, cf. Ward, Watts, and Tough (2006). Based on infinite series expansion given by Shnidman (1989) and Shnidman (2008), Bocquet (2011) develops a numerical method to carry out these calculations. In particular, the Gauss-Laguerre quadrature is used for the integration, with the nodes and weights calculated using matrix methods, so that a general purpose numerical integration routine is not required.

Due to the nonlinear nature of the non-Gaussian receivers and the additional mathematical complexity of the non-Gaussian distributions, it is frequently impossible to obtain closed form analytical results for the threshold, probability of false alarm and probability of detection. An alternative way of approximating the PFA defined in (1) is using numerical method. However as shown in Chapter 8.4 of Ward, Watts, and Tough (2006) that for small shape parameter ν defined in (3), the 'spikier' clutter tends to obscure targets and hence tends to reduce detection performance. Consequently, Monte Carlo simulation is used to obtain estimates of performance. The purpose of applying Monte Carlo simulation is its flexibility and simplicity, and can be applied to complicated cases, such as approximating PFA based on generalized likelihood ratio test under spherically symmetric complex Gaussian models. This motivates us to apply Monte Carlo simulation method to calculate the PFA defined in (1). However, it is known that direct Monte Carlo simulation suffers with large variance, especially for estimating small probabilities such as PFA. To remedy this difficulty, we propose an importance sampling algorithm, to which one uses samples from an alternative distribution Q other than the original distribution P, to have efficient simulation for estimating PFA in K-distributed sea clutter. Note that other than compare our proposed method with the numerical method systematically, in this paper, we only study importance sampling in Monte Carlo simulations to make it more coherent.

Albeit standard Monte Carlo method can be used to estimate a small PFA, it is notoriously known of its low convergence and a large number of trials are required to guaranteed the precision of the estimator. To improve the efficiency of the crude Monte Carlo method, a typical method is importance sampling. The reader is referred to Asmussen and Glynn (2007) and Jeruchim, Balaban, and Shanmugan (2000) for general references. The applications of importance sampling to PFA in SAR images can be found in Orsak (1993), Smith and Orsak (1995), and Srinivasan (1998). However, all of these papers proposed importance sampling techniques to obtain an efficient estimation for PFA under Gaussian clutter.

To remedy the same difficulty arisen from K-distributed sea clutter model, previous study of using importance sampling techniques are given in the literature. For instance Stadelman, Weiner, and Keckler (2002), Gonzalez-Garcia, Sanz-Gonzalez, and Vaquero (2004), and Srinivasan and Rangaswamy (2007) consider the problem in a testing of hypothesis setting. By using the technique in Jeruchim, Balaban, and Shanmugan (2000) and Smith, Shafi, and Gao (1997), they define the tilting probability as the probability under the alternative hypothesis. In this paper, we provide an efficient tilting probability measure to simulate the PFA under the K-distributed sea clutter and noise. The optimality of the proposed tilting probability measure is given by minimizing the variance of the importance sampling estimator within a parametric family. To select the parametric family, we consider two types of simulation probability measure. First, by using to the scale property of the K distribution, we consider the product form of the K distribution. The resulting estimator is basically a product of two independent gamma distributions and needs two tilting parameters. Second, we study the compound form of the K distribution, and the resulting estimator is a function of a gamma distribution and needs only one tilting parameter. Based on the above two approaches, by making use of the exponential tilting measure, we prove the existence and uniqueness of the optimal tilting parameter, which is characterized as the root of a simple system of nonlinear equations.

Note that although the importance sampling algorithm is derived under K-distributed sea clutter, the sea clutter modeled by other distributions such as log-normal, gamma, Weibull, Rayleigh distribution, among others, can be derived as well, provided the moment generating function exists. A direct application of this study is on sea clutter fluctuations and probability of false alarm in radar detection. The reader is referred to Watts (1987) and Chapter 8.4 of Ward, Watts, and Tough (2006), and references there in for details. Further study along this line is its extension to approximate the PFA of an optimal detector in the sense of generalized likelihood ratio test for coherent radar detection in a compound K-distributed clutter environment, cf. Srinivasan and Rangaswamy (2007).

The rest of the paper is organized as follows. 2 formulates the concerned problem. The main methodology of importance sampling for efficiently estimating PFA and algorithm are derived in 3 based on the product form and the compound form under K-distributed sea clutter and noise, respectively. Numerical simulations are given in 4. 5 concludes. All proofs are deferred to the appendix.

2. Problem formulation

For the estimation of detection threshold, it is known that the detection threshold corresponding to a PFA is indeed indicated as a quantile of the clutter distribution, that is, $T_{P_{fa}} = \inf\{x : F_K(x) := P(K \leq x) \geq 1 - P_{fa}\}$. Standard method of computing the sample quantiles can be applied, see Glynn (1996), Liu and Yang (2012), and references therein. In this section, we simulate the probability of false alarm related to the problem considered in Section 8.4 of Ward, Watts, and Tough (2006). More complicated and real cases involving the calculation of the probability of a likelihood ratio statistics exceeding a threshold under the SIRV clutter model will be investigated in a separate paper.

Let $\mathcal{E}(\mu)$ denote the exponential distribution with mean parameter μ, and $\Gamma(\alpha, \beta)$ denote the gamma distribution with shape parameter α and scale parameter β. In the compound form of the K distribution, the clutter power, Z, is described by an exponential distribution of mean power, $X = x$, and has the probability density function (pdf) conditional on $\{X = x\}$,

$$p(z|x) = \frac{1}{x} \exp\left(-\frac{z}{x}\right). \tag{2}$$

Here X itself fluctuates with a Gamma distribution with shape parameter ν and scale parameter $1/b$, and has the pdf,

$$p(x) = \frac{b^{\nu}}{\Gamma(\nu)} x^{\nu-1} \exp(-bx), \tag{3}$$

where ν/b is the sea clutter mean power. We have the following equivalent stochastic representation for Z and X,

$$Z|\{X = x\} \sim \mathcal{E}(x), \quad X \sim \Gamma(\nu, 1/b).$$

In the following, we add noise to the clutter as described in Chapter 4 of Ward, Watts, and Tough (2006). Assume that frequency agility provides independent samples of speckle, noise is added by offsetting x by the noise power p_n in Equation (2),

$$p(z|x) = \frac{1}{x + p_n} \exp\left(-\frac{z}{x + p_n}\right). \tag{4}$$

We assume that the radar has a 'square law detector'. Therefore, writing the sum of N pulses as

$$Y = \sum_{i=1}^{N} Z_i,$$

the pdf of Y conditional on $X = x$ is

$$p(y|x) = \frac{y^{N-1}}{(x + p_n)^N (N-1)!} \exp\left(-\frac{y}{x + p_n}\right). \tag{5}$$

We have the following equivalent stochastic representation for Y and X,

$$Y|X = x \sim \Gamma(N, x + p_n), \quad X \sim \Gamma(\nu, 1/b). \tag{6}$$

Let $\Gamma_{\alpha,\beta}(\cdot)$ denote the survival function for the Gamma distribution with shape parameter α and scale parameter β. The probability of false alarm, given $X = x$, for a threshold T is

$$\begin{aligned} p_{fa}(T|x) &:= P\{Y > T|X = x\} \\ &= \int_T^\infty \frac{y^{N-1}}{(x + p_n)^N (N-1)!} \exp\left(-\frac{y}{x + p_n}\right) dy \\ &= \Gamma_{N, x+p_n}(T). \end{aligned}$$

The false alarm probability is therefore

$$\begin{aligned} p_{fa}(T) &: = P\{Y > T\} \\ &= \int_0^\infty P\{Y > T|X = x\} p(x) dx \\ &= \int_0^\infty \Gamma_{N, x+p_n}(T) p(x) dx, \end{aligned} \tag{7}$$

where $p(x)$ is given in (3). That is, we aims at proposing an efficient and accurate simulation scheme to calculate the PFA defined in (7).

3. Efficient simulation methods

In the beginning of this section, we describe the importance sampling in a general setting. Let $\xi = (\xi_1, ..., \xi_d)'$ be a random vector under the original probability measure P, where the prime denotes the transpose. We are interested in calculating

$$m = E_P[\wp(\xi)], \tag{8}$$

where $\wp(\cdot)$ is a real valued function of interest and $E_P[\cdot]$ denotes the expectation under P.

Assume that the cumulant function of ξ, $\psi(\theta) = \log E[e^{\theta'\xi}]$, exists. Define the exponentially tilting measure Q_θ

$$\frac{dQ_\theta}{dP} = \exp\{\theta'\xi - \psi(\theta)\}.$$

By using change of measure, the following identity holds,

$$\begin{aligned} m = E_P[\wp(\xi)] &= \int \wp(\xi)dP \\ &= \int \wp(\xi)\frac{dP}{dQ_\theta}dQ_\theta = E_{Q_\theta}\left[\wp(\xi)\frac{dP}{dQ_\theta}\right], \end{aligned}$$

and the importance sampling estimator is

$$\hat{m}_\theta = \wp(\xi)\frac{dP}{dQ_\theta} = \wp(\xi)\exp\{-\theta'\xi + \psi(\theta)\}, \text{where } \xi \sim Q_\theta.$$

Here, dP/dQ_θ is called the Radon-Nykodym derivative or the importance sampling weight.

Note that $\hat{m}_\theta$ is an unbiased estimator. To find the optimal tilting parameter θ^*, under the mean square error loss, we minimize the variance of the importance sampling estimators. Define $G(\theta)$ as the second moment of $\hat{m}_\theta$. Standard algebra yields

$$\begin{aligned} G(\theta) &= E_{Q_\theta}\left[\left(\wp(\xi)e^{-\theta'\xi+\psi(\theta)}\right)^2\right] \\ &= E_P\left[\wp^2(\xi)e^{-\theta'\xi+\psi(\theta)}\right]. \end{aligned} \tag{9}$$

The following lemma guarantees the strict convexity of $G(\theta)$ which ensures the uniqueness of the optimal tilting parameter once it exists.

Lemma 1. *$G(\theta)$ defined in (9) is strictly convex.*

The proof of Lemma 1 will be given in the Appendix A. □

To provide a solution to the optimal tiling parameter, for a given $\wp(\cdot)$ and θ, we define the conjugate measure $\bar{Q}_\theta := \bar{Q}_{\wp,\theta}$ with respect to the original measure P by

$$\frac{d\bar{Q}_\theta}{dP} = \frac{\wp^2(\xi)e^{-\theta'\xi}}{E_P\left[\wp^2(\xi)e^{-\theta'\xi}\right]}. \tag{10}$$

The following lemma characterizes the optimal tilting parameter as a solution of a system of d non-linear equations. Let ∇ denote the gradient.

Lemma 2. *The optimal tiling parameter θ that minimizes $G(\theta)$ in (9) satisfies the following system of d non-linear equations,*

$$\nabla\psi(\theta) = E_{\bar{Q}_\theta}[\xi], \tag{11}$$

where the conjugate measure $\bar{Q}_\theta$ is defined in (10).

The proof of Lemma 2 will be given in the Appendix A. □

Specific applications employing the above importance sampling framework for various distributions can be found for financial options pricing in Teng, Fuh, and Chen (2016). We include it here for the sake of completeness. Next, we employ the above importance sampling framework to calculate the PFA in (7) based on the product form and the structure form of the K-based distribution in Secs. 3.1 and 3.2, respectively.

3.1. The product form of the K distribution

Recall $P_{fa}(T) = P\{Y > T\} = E_P[I_{\{Y>T\}}]$, where $Y|X = x \sim \Gamma(N, x + p_n), X \sim G(\nu, 1/b)$ and $I_{\{A\}}$ denotes the indicator function with a support set A. By the scaling property of the gamma distribution, i.e., $\Gamma(\nu, c/b) \stackrel{d}{=} c\Gamma(\nu, 1/b)$ for a constant $c > 0$, we have

$$Y|X = x \sim \Gamma(N, x + p_n) \stackrel{d}{=} (x + p_n)\Gamma(N, 1).$$

Therefore, the stochastic representation of Y under the product form is

$$Y := Y(W, X) = W(X + p_n), \tag{12}$$

where $W \sim \Gamma(N, 1), X \sim \Gamma(\nu, 1/b)$, and W is independent of X.

Let $\xi = (W, X)$, then the crude Monte Carlo estimator of (8) is

$$\hat{p} = \wp(Y) = \wp(W, X) = I_{\{W(X+p_n)>T\}},$$

where $W \sim \Gamma(N, 1), X \sim \Gamma(\nu, 1/b)$, and W is independent of X. Let $\theta = (\theta_1, \theta_2)'$ be the bivariate tilting parameter for $\theta_1 < 1$ and $\theta_2 < b$. It is straightforward to see that the importance sampling estimator is

$$\begin{aligned} \hat{p}_\theta &= \wp(W, X)\mathrm{e}^{-\theta_1 W - \theta_2 X - N\log(1-\theta_1) - \nu\log(1-\theta_2/b)} \\ &= I_{\{W(X+p_n)>T\}}\mathrm{e}^{-\theta_1 W - \theta_2 X - N\log(1-\theta_1) - \nu\log(1-\theta_2/b)}, \end{aligned} \tag{13}$$

where $W \sim \Gamma(N, 1/(1-\theta_1)), X \sim \Gamma(\nu, 1/(b-\theta_2))$, and W is independent of X.

To find the optimal tilting parameter, note that the second moment of the $\hat{p}_\theta$ is

$$\begin{aligned} &G(\theta) \\ &= E_P\left[I_{\{W(X+p_n)>T\}}\mathrm{e}^{-\theta_1 W - \theta_2 X - N\log(1-\theta_1) - \nu\log(1-\theta_2/b)}\right]. \end{aligned} \tag{14}$$

Define $d\bar{Q}_\theta := d\bar{Q}_{\wp,(\theta_1,\theta_2)}$ as the conjugate measure with respect $\wp(W, X) = I_{\{W(X+p_n)>T\}}$ and the tilting parameter $\theta = (\theta_1, \theta_2)$ by

$$\frac{d\bar{Q}_\theta}{dP} = \frac{I_{\{W(X+p_n)>T\}}\exp^{-\theta_1 W - \theta_2 X}}{E_P\left[I_{\{W(X+p_n)>T\}}\exp^{-\theta_1 W - \theta_2 X}\right]}. \tag{15}$$

By (11), the optimal tilting parameter θ_1^* and θ_2^* are solutions of the following system of non-linear equations,

$$\begin{cases} \dfrac{N}{1-\theta_1} &= E_{\bar{Q}_\theta}[W], \\ \dfrac{\nu}{b-\theta_2} &= E_{\bar{Q}_\theta}[X]. \end{cases} \tag{16}$$

The following theorem summarizes the existence, uniqueness, and characterization of the optimal tilting parameter based on the importance sampling estimator $\hat{p}_\theta$ defined in (13).

Theorem 1. *When Y is a K distribution defined as the product form in (12), the minimizer of $G(\theta)$ defined in (14) exists. Moreover, $G(\theta)$ is a convex function in θ, and then the minimizer of (14) is unique, which satisfies (16).*

The proof of Theorem 1 will be given in the Appendix B. □

To find θ_1^* and θ_2^* in (16), we apply a stochastic fixed-point algorithm. In the following, we only explain how to calculate $E_{\bar{Q}_\theta}[W]$ by simulation, because the calculation of $E_{\bar{Q}_\theta}[X]$ can be done in a similar fashion. By the definition of $\bar{Q}_\theta$ in (15) into $E_{\bar{Q}_\theta}[W]$, we calculate $E_{\bar{Q}_\theta}[W]$ via standard Monte Carlo simulation,

$$
\begin{aligned}
E_{\bar{Q}_\theta}[W] &= \frac{E_P\left[I_{\{W(X+p_n)>T\}} W e^{-\theta_1 W-\theta_2 X}\right]}{E_P\left[I_{\{W(X+p_n)>T\}} e^{-\theta_1 W-\theta_2 X}\right]} \\
&\approx \frac{\sum_{j=1}^m I_{\{W^{(j)}(X^{(j)}+p_n)>T\}} e^{-\theta_1 W^{(j)}-\theta_2 X^{(j)}}}{\sum_{j=1}^m I_{\{W^{(j)}(X^{(j)}+p_n)>T\}} e^{-\theta_1 W^{(j)}-\theta_2 X^{(j)}}},
\end{aligned}
$$

where $W^{(j)} \overset{i.i.d.}{\sim} \Gamma(N,1), X^{(j)} \overset{i.i.d.}{\sim} \Gamma(\nu, 1/b)$, and $W^{(j)}$ and $X^{(j)}$ are mutually independent, for $j = 1, \ldots, m$.

In the case of small false alarm probability, to produce a more accurate estimator, we adaptively adjust the distribution of $W^{(j)}$ and $X^{(j)}$ with proper $\tilde{\theta}_1$ and $\tilde{\theta}_2$:

$$
\begin{aligned}
&E_{\bar{Q}_\theta}[W] \\
&\approx \frac{\sum_{j=1}^m I_{\{W^{(j)}(X^{(j)}+p_n)>T\}} W^{(j)} e^{-W^{(j)}(\theta_1+\tilde{\theta}_1)-X^{(j)}(\theta_2+\tilde{\theta}_2)}}{\sum_{j=1}^m I_{\{W^{(j)}(X^{(j)}+p_n)>T\}} e^{-W^{(j)}(\theta_1+\tilde{\theta}_1)-X^{(j)}(\theta_2+\tilde{\theta}_2)}},
\end{aligned}
$$

where $W^{(j)} \overset{i.i.d.}{\sim} \Gamma(N, 1/(1-\tilde{\theta}_1)), X^{(j)} \overset{i.i.d.}{\sim} \Gamma(\nu, 1/(b-\tilde{\theta}_2))$, and $W^{(j)}$ and $X^{(j)}$ are mutually independent, for $j = 1, \ldots, m$. In iterating the tilting parameter, we simply set the latest tilting parameter to be $\tilde{\theta}_1$ and $\tilde{\theta}_2$. Furthermore, to reduce the computational cost in drawing independent samples for the gamma distribution in the searching phase, we use common random numbers by the scaling property of the gamma distribution.

Algorithm 1 summarizes steps in estimating the false alarm probability based on $\hat{p}_\theta$ in (13) with optimal tiling parameter being the solution of (16) into two phases, the searching phase and the calculation phase. The searching phase is essentially a stochastic fixed-point algorithm.

To determine when to stop the iterations, let ε denote a small precision value, $|| \cdot, \cdot ||$ denote the Euclidean distance, and D denote the squared distance between $\nabla\psi(\theta)$ and $E_{\bar{Q}_\theta}[\xi]$, i.e.,

$$
\begin{aligned}
D &= ||\nabla\psi(\theta), E_{\bar{Q}_\theta}[\xi]||^2 \\
&= (N/(1-\theta_1) - E_{\bar{Q}_\theta}[W])^2 + (\nu/(b-\theta_2) - E_{\bar{Q}_\theta}[X])^2.
\end{aligned}
$$

We stop the iterations in the searching phase when $D < \varepsilon$.

Algorithm 1. *The following steps calculate the false alarm probability defined in (7) based on the importance sampling estimator $\hat{p}_\theta$ in (13), where the optimal tiling parameters $\theta^* = (\theta_1^*, \theta_2^*)'$ is the solution of (16).*

- *The searching phase.*
  1. *Start $\theta^{(0)} = (\theta_1^{(0)}, \theta_2^{(0)})'$ and $\tilde{\theta} = (\tilde{\theta}_1, \tilde{\theta}_2)'$ properly, and set $i = 1$.*
  2. *Generate independent samples, $W^{(j)} \sim \Gamma(N, 1/(1-\tilde{\theta}_1))$ and $X^{(j)} \sim \Gamma(\nu, 1/(b-\tilde{\theta}_2))$ for $j = 1, ..., m$.*
  3. *Calculate $\bar{Q}_W$ as*
$$\frac{\sum_{j=1}^m I_{\{W^{(j)}(X^{(j)}+p_n)>T\}} W^{(j)} e^{-W^{(j)}(\theta_1^{(i-1)}+\tilde{\theta}_1)-X^{(j)}(\theta_2^{(i-1)}+\tilde{\theta}_2)}}{\sum_{j=1}^m I_{\{W^{(j)}(X^{(j)}+p_n)>T\}} e^{-W^{(j)}(\theta_1^{(i-1)}+\tilde{\theta}_1)-X^{(j)}(\theta_2^{(i-1)}+\tilde{\theta}_2)}}.$$
  4. *Update $\theta_1^{(i)} = 1 - N/\bar{Q}_W$.*
  5. *Calculate*
$$\bar{Q}_X = \frac{\sum_{j=1}^m I_{\{W^{(j)}(X^{(j)}+p_n)>T\}} X^{(j)} e^{-W^{(j)}(\theta_1^{(i)}+\tilde{\theta}_1)-X^{(j)}(\theta_2^{(i-1)}+\tilde{\theta}_2)}}{\sum_{j=1}^m I_{\{W^{(j)}(X^{(j)}+p_n)>T\}} e^{-W^{(j)}(\theta_1^{(i)}+\tilde{\theta}_1)-X^{(j)}(\theta_2^{(i-1)}+\tilde{\theta}_2)}}.$$
  6. *Update $\theta_2^{(i)} = b - \nu/\bar{Q}_X$.*
  7. *Calculate*
$$\bar{Q}_W = \frac{\sum_{j=1}^m I_{\{W^{(j)}(X^{(j)}+p_n)>T\}} W^{(j)} e^{-W^{(j)}(\theta_1^{(i)}+\tilde{\theta}_1)-X^{(j)}(\theta_2^{(i)}+\tilde{\theta}_2)}}{\sum_{j=1}^m I_{\{W^{(j)}(X^{(j)}+p_n)>T\}} e^{-W^{(j)}(\theta_1^{(i)}+\tilde{\theta}_1)-X^{(j)}(\theta_2^{(i)}+\tilde{\theta}_2)}},$$
     and
$$\bar{Q}_X = \frac{\sum_{j=1}^m I_{\{W^{(j)}(X^{(j)}+p_n)>T\}} X^{(j)} e^{-W^{(j)}(\theta_1^{(i)}+\tilde{\theta}_1)-X^{(j)}(\theta_2^{(i)}+\tilde{\theta}_2)}}{\sum_{j=1}^m I_{\{W^{(j)}(X^{(j)}+p_n)>T\}} e^{-W^{(j)}(\theta_1^{(i)}+\tilde{\theta}_1)-X^{(j)}(\theta_2^{(i)}+\tilde{\theta}_2)}}.$$
  8. *Calculate $D = \left(\frac{N}{1-\theta_1^{(i)}} - \bar{Q}_W\right)^2 + \left(\frac{\nu}{b-\theta_2^{(i)}} - \bar{Q}_X\right)^2$.*
  9. *If $D < \varepsilon$, stop; Otherwise, set $i = i + 1$ and return to step 3).*
  10. *Denote the converged tilting parameter by $\theta^* = (\theta_1^*, \theta_2^*)'$.*

- *The calculation phase.*

1. *Generate independent samples, $W^{(j)} \sim \Gamma(N, 1/(1-\theta_1^*))$ and $X^{(j)} \sim \Gamma(\nu, 1/(b-\theta_2^*))$ for $j = 1, ..., m$.*
2. *Estimate $p_{fa}(T)$ by*
$$\frac{1}{m}\sum_{j=1}^m I_{\{W^{(j)}\left(X^{(j)}+p_n\right)>T\}} e^{-\theta_1^* W^{(j)} - \theta_2^* X^{(j)} - N\log(1-\theta_1^*) - \nu\log(1-\theta_2^*/b)}.$$

3.2. The compound form of the K distribution

Recalling that the probability of false alarm in (7), we define the conditioning estimator for $p_{fa}(T)$ as

$$\hat{p}_c = \Gamma_{N,X+p_n}(T), \tag{17}$$

where $X \sim \Gamma(\nu, 1/b)$. Note that the conditioning estimator applies Rao-Blackwellization, because it can be seen clearly that

$$\Gamma_{N,X+p_n}(T) = E_P\left[I_{\{Y>T\}}|X\right].$$

As a result, $\hat{p}_c$ enjoys lower variance than the crude Monte Carlo estimator.

Let $\theta < b$, we apply the importance sampling to the conditioning estimator $\hat{p}_c$ to have an alternative importance sampling estimator, denoted by $\hat{p}_{c,\theta}$,

$$\begin{aligned}\hat{p}_{c,\theta} &= \wp(X)\mathrm{e}^{-\theta X-\nu\log(1-\theta/b)} \\ &= \Gamma_{N,X+p_n}(T)\mathrm{e}^{-\theta X-\nu\log(1-\theta/b)},\end{aligned} \tag{18}$$

where θ is the univariate tiling parameter, and $X \sim \Gamma(\nu, 1/(b-\theta))$.

To find the optimal tilting parameter, note that the second moment of the $\hat{p}_{c,\theta}$ is

$$G(\theta) = E_P\left[\Gamma^2_{N,X+p_n}(T)\mathrm{e}^{-\theta X-\nu\log(1-\theta/b)}\right]. \tag{19}$$

Define $\bar{Q}_\theta := \bar{Q}_{\wp,\theta}$ as the conjugate measure with respect to $\wp(X) = \Gamma_{N,X+p_n}(T)$ and the tilting parameter θ by

$$\frac{d\bar{Q}_\theta}{dP} = \frac{\Gamma^2_{N,X+p_n}(T)\mathrm{e}^{-\theta X}}{E_P\left[\Gamma^2_{N,X+p_n}(T)\mathrm{e}^{-\theta X}\right]}.$$

Applying (11), the optimal tilting parameter θ^* is the solution of

$$\frac{\nu}{b-\theta} = E_{\bar{Q}_\theta}[X]. \tag{20}$$

The following theorem summarizes the existence, uniqueness, and characterization of the optimal tilting parameter based on the importance sampling estimator $\hat{p}_{c,\theta}$ defined in (18).

Theorem 2. *When Y is a K distribution defined as the compound form in (7). Then, the minimizer of $G(\theta)$ defined in (19) exists. Moreover, $G(\theta)$ is a convex function in θ, and then the minimizer of (9) is unique, which satisfies (20).*

The proof of Theorem 2 will be given in the Appendix B. □

Note that the condition $T > \nu/b$ required in Theorem 2 does not limit its practical usage, because there is no need to use importance sampling when T is small.

Algorithm 2 summarizes steps in estimating the false alarm probability based on $\hat{p}_{c,\theta}$ proposed in (18) with optimal tiling parameter being the solution of (20) into two phases, the searching phase and the calculation phase. The searching phase is essentially a stochastic fixed-point algorithm. Similarly, we stop the iterations in the searching phase when

$$D = \left(\frac{\nu}{b-\theta} - E_{\bar{Q}_\theta}[X]\right)^2 < \varepsilon.$$

Algorithm 2. *The following steps calculate the false alarm probability defined in (7) based on the importance sampling estimator $\hat{p}_{c,\theta}$ in (18), where the optimal tilting parameter θ^* is the solution of (20).*

- *The searching phase.*
  1) Start $\theta^{(0)}$ and $\tilde{\theta}$ properly, and set $i = 1$.
  2) *Generate independent samples,* $X^{(j)} \sim \Gamma(\nu, 1/(b-\tilde{\theta}))$ *for* $j = 1, ..., m$.
  3) *Calculate* $\bar{Q}_X = \frac{\sum_{j=1}^m \Gamma^2_{N, X^{(j)}+pn}(T) X^{(j)} e^{-X^{(j)}(\theta^{(i-1)}+\tilde{\theta})}}{\sum_{j=1}^m \Gamma^2_{N, X^{(j)}+pn}(T) e^{-X^{(j)}(\theta^{(i-1)}+\tilde{\theta})}}$.
  4) *Update* $\theta^{(i)} = b - \nu/\bar{Q}_X$.
  5) *Calculate* $\bar{Q}_X = \frac{\sum_{j=1}^m \Gamma^2_{N, X^{(j)}+pn}(T) X^{(j)} e^{-X^{(j)}(\theta^{(i)}+\tilde{\theta})}}{\sum_{j=1}^m \Gamma^2_{N, X^{(j)}+pn}(T) e^{-X^{(j)}(\theta^{(i)}+\tilde{\theta})}}$.
  6) *Calculate* $D = \left(\frac{\nu}{b-\theta^{(i)}} - \bar{Q}_X\right)^2$.
  7) *If* $D < \varepsilon$, *stop; Otherwise, set* $i = i + 1$ *and return to step 3).*
  8) *Denote the converged tilting parameter by* θ^*.
- *The calculation phase.*
  1) *Generate independent samples,* $X^{(j)} \sim \Gamma(\nu, 1/(b-\theta^*))$ *for* $j = 1, ..., m$.
  2) *Estimate* $P_{fa}(T)$ *by* $\frac{1}{m}\sum_{j=1}^m \Gamma_{N, X^{(j)}+p_n}(T) e^{-\theta^* X^{(j)} - \nu \log(1-\theta^*/b)}$.

Remark. We compare the product and compound forms of the K distribution. Recall the associated probability model in Equation (6). The sea clutter is originally presented in the compound form, where the amplitude is random. However, with the scaling property of the gamma distribution, the K distribution has an equivalent stochastic representation as a product of two independent gamma distributions, or the so-called product form, as shown in Equation (12).

With the product form in Equation (12), the importance sampling estimator $\hat{p}_\theta$ in Equation (13) has two tilting parameters. Here, the optimal tilting parameters is characterized in the system (16), and the optimization has to be implemented in two dimensions.

In contrast, with the compound form of the K distribution, it is straightforward to obtain the conditioning estimator in Equation (17). The resulting importance sampling estimator in Equation (18) needs just one tilting parameter. Here, the optimal tilting parameter is characterized in Equation (20), and the optimization is implemented in one dimension. However, calculating the survival function of the Gamma distribution is possibly computational demanding.

4. Simulation studies

To measure the efficiency of the proposed estimators, $\hat{p}_\theta$, $\hat{p}_c$, and $\hat{p}_{c,\theta}$, we report the variance ratio and the penalized variance ratio of two estimators as follows. The variance ratio of two estimators is defined as

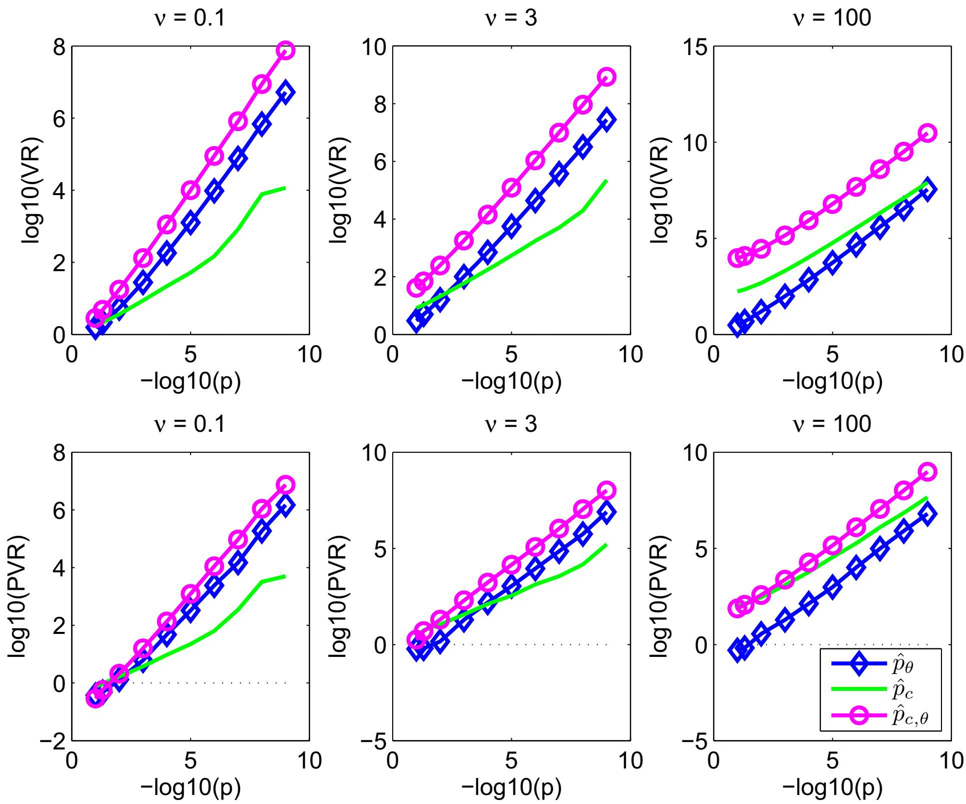

Figure 1. Estimated variance ratios and penalized variance ratios against the false alarm probability at various ν. Parameter settings: $N=1$, $p_n=0$, $b=1$.

$$\mathrm{VR}(\hat{p}_1,\hat{p}_2)=\frac{\mathrm{Var}(\hat{p}_1)}{\mathrm{Var}(\hat{p}_2)}.$$

To define the penalized variance ratio, let δ be a small predetermined precision level. Then the required sample size is $\sqrt{\mathrm{Var}(\hat{p}_i)/m_i}<\delta$ or $m_i\geq \mathrm{Var}(\hat{p}_i)/\delta^2$. Define the total computational time $\mathcal{T}_i=m_iT_i$, where T_i is the running time of implementing one sample of the associated estimator. The penalized variance ratio by computational time (PVR) is defined as the ratio between two total computational times,

$$\mathrm{PVR}(\hat{p}_1,\hat{p}_2)=\frac{\mathcal{T}_1}{\mathcal{T}_2}=\frac{\mathrm{Var}(\widehat{p_1})T_1}{\mathrm{Var}(\widehat{p_2})T_2}.$$

The study plan of our simulation is summarized as follows. We first set the crude Monte Carlo estimator as the benchmark, i.e., we set $\hat{p}_1$ to be the crude Monte Carlo estimator. The parameter combinations are $N\in\{1,10\}, p_n\in\{0,1\}$, $b=1$, $\nu\in\{0.1,3,100\}$, $p_{fa}(T)\in\{0.1,0.05,0.01,0.001,\ldots,10^{-9}\}$. For each case, we set the numbers of simulation to be $m=1,000,000$ and $\varepsilon=0.0001$ to produce precise estimated variances.

The simulation results for various estimators, $\hat{p}_\theta,\hat{p}_c$, and $\hat{p}_{c,\theta}$, based on different parameter combinations, are presented in Figures 1–4. Here, the horizontal axis is the PFA

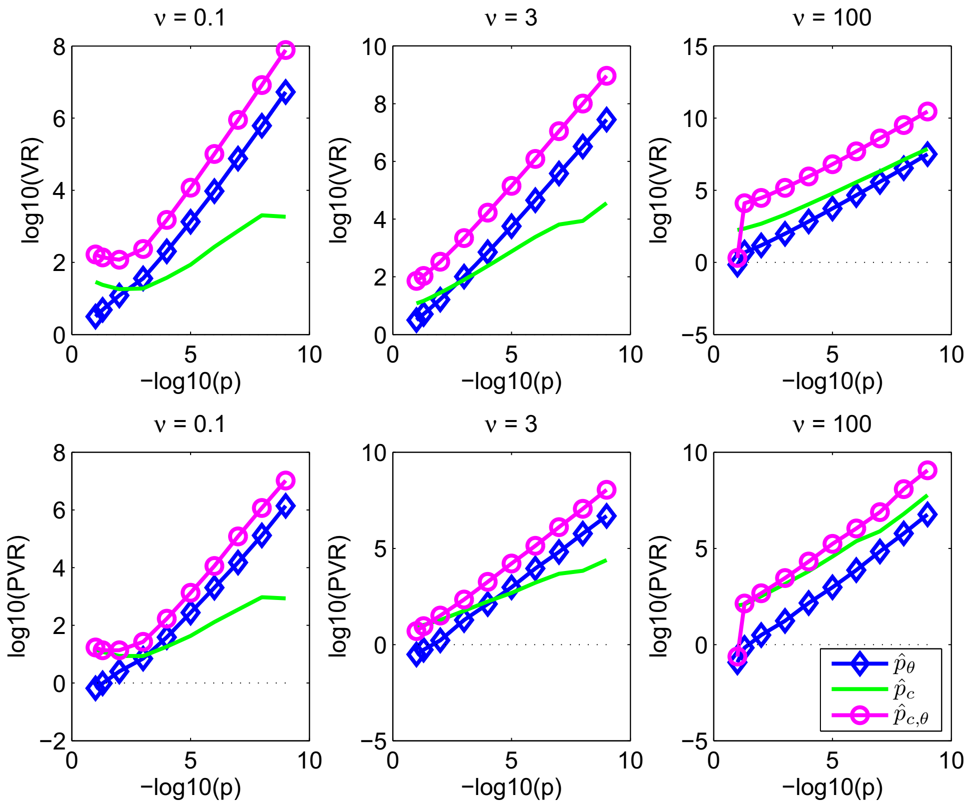

Figure 2. Estimated variance ratios and penalized variance ratios against the false alarm probability at various ν. Parameter settings: $N = 1$, $p_n = 1$, $b = 1$.

in negative base ten logarithm, and the vertical axis is the estimated variance ratio or penalized variance ratio in base ten logarithm.

In summary, we first note that $\hat{p}_{c,\theta}$ performs the best in almost all cases. In terms of variance ratios, three estimators, $\hat{p}_\theta, \theta p_c$ and $\hat{p}_{c,\theta}$ dominate the crude Monte Carlo estimator because all variance ratios are larger than one; the better performance of $\hat{p}_{c,\theta}$ compared to $\hat{p}_c$ is due to importance sampling as expected; the better performance of $\hat{p}_c$ compared to the crude Monte Carlo estimator is also coherent with the Rao-Blackwellization. However, neither $\hat{p}_c$ nor $\hat{p}_\theta$ dominates the others.

In terms of penalized variance ratios, the same conclusions hold, $\hat{p}_{c,\theta}$ outperforms the other estimators; while $\hat{p}_c$ and $\hat{p}_\theta$ are competitive.

To interpret the practical value of the proposed importance sampling method, assume it is required that the estimator of the probability of false alarm has to be precise in two decimals of its theoretical value, p, the least Monte Carlo sample size, m, has to satisfy

$$\sqrt{\frac{\mathrm{Var}(\hat{p})}{m}} \leq 0.01p.$$

In the case of $p = 10^{-9}$. A crude Monte Carlo estimator has variance about $\mathrm{Var}(\widehat{p_0}) = p(1-p) \approx p$, and needs about the least Monte Carlo sample size $m \geq 10000/p = 10^{13}$,

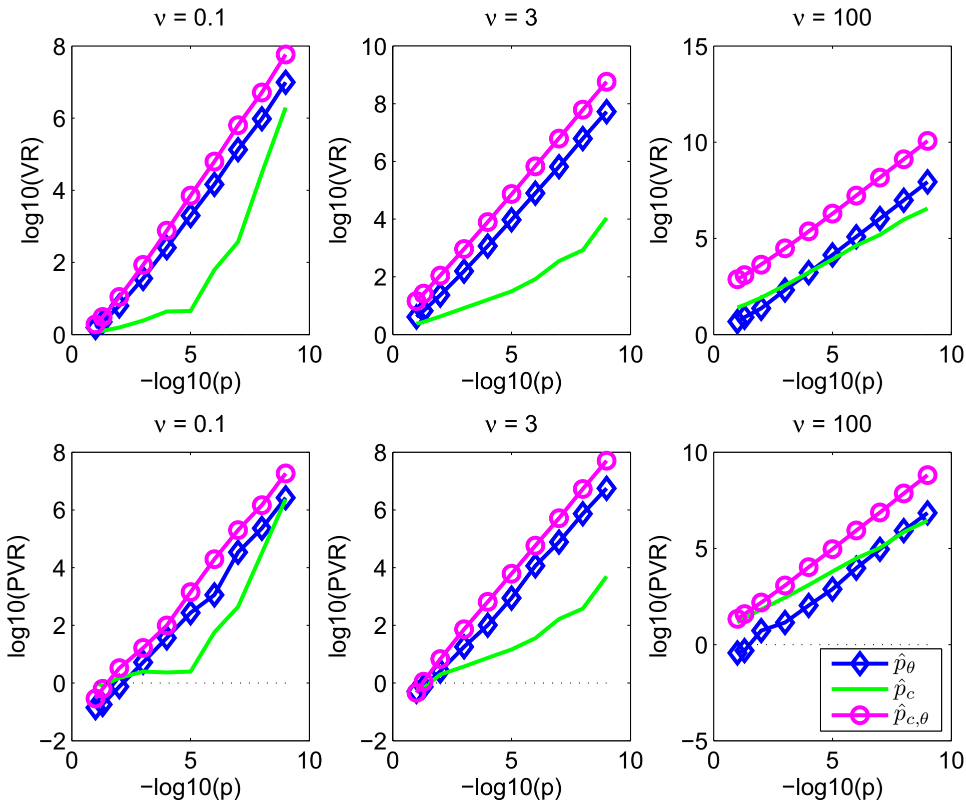

Figure 3. Estimated variance ratios and penalized variance ratios against the false alarm probability at various ν. Parameter settings: $N = 10$, $p_n = 0$, $b = 1$.

which is formidable in practice. On the contrary, because the variance ratio of $\hat{p}_{\theta,c}$ is about 10^8, it needs the least Monte Carlo sample size $m \geq 10^5$. The reduction in the least Monte Carlo sample size is substantial, and allows a feasible and accurate simulation scheme.

5. Conclusion

In radar detection system, computing an excess probability of some statistics is a prominent issue. In this paper, we first present an importance sampling framework, and apply it to obtain two importance sampling estimators, $\hat{p}_\theta$ and $\hat{p}_{c,\theta}$, based on the product form and compound form for the K-distributed sea clutter and noise, respectively, for calculating the PFA defined in (7). For these two importance sampling estimators, we next provide theorems to ensure the existence, uniqueness and characterization of the optimal tilting parameters. Third, we outline algorithms into the searching phase and the calculation phase to calculate the PFA, where the searching phase is essentially a stochastic fixed-point algorithm. Last, we compare the efficiency among three proposed estimators, $\hat{p}_\theta, \hat{p}_c$, and $\hat{p}_{c,\theta}$. The efficiency gain of $\hat{p}_{c,\theta}$ is substantial and the highest in terms of variance ratios and penalized variance ratios.

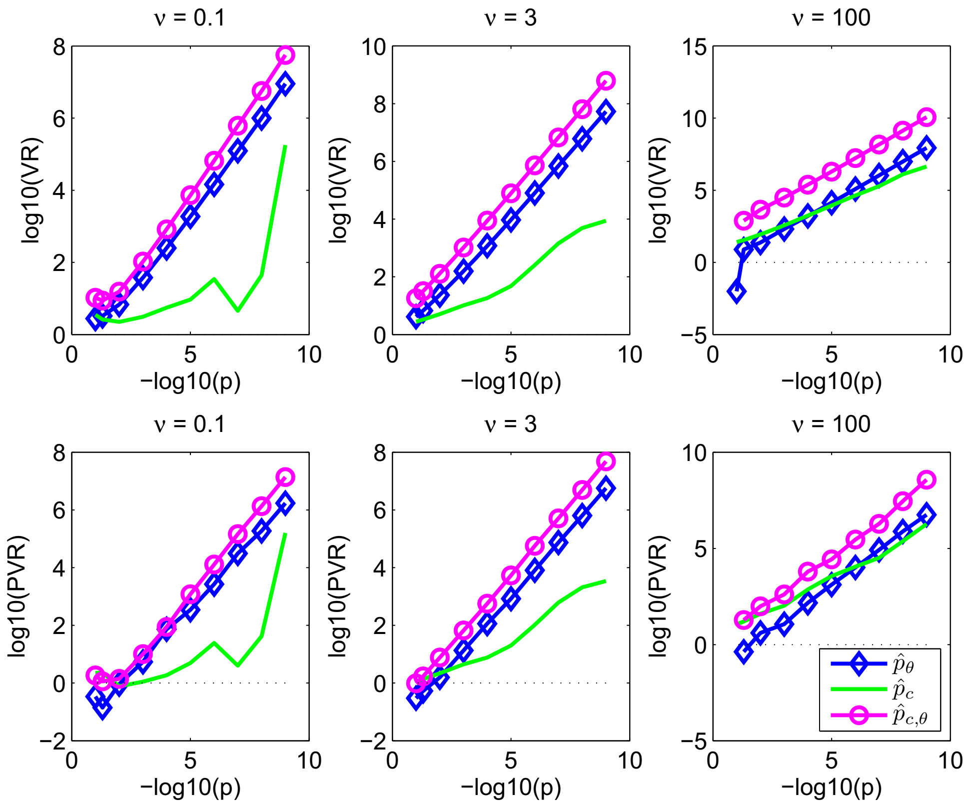

Figure 4. Estimated variance ratios and penalized variance ratios against the false alarm probability at various ν. Parameter settings: $N = 10$, $p_n = 1$, $b = 1$.

Note that the importance sampling propsed in this paper is based on exponential tilting under variance minimization criterion, which is particular useful for the PFA calculation. Based on the product form and compound form for the K-distributed sea clutter, the proposed algorithm is easy to implement. Other than the calculation of the PFA in K-distributed sea clutter, the proposed importance sampling algorithm can be applied to parametric bootstrapping confidence interval under K-distributed statistics, other than the non-parametric bootstrapping confidence interval used in Fuh and Hu (2004).

Further applications involve the course of calculating the probability of a likelihood ratio testing statistics exceeding a threshold, where the sea clutter is modeled via a complex d-dimensional SIRV model. By using the idea of spherical Monte Carlo simulation and importance sampling, we will study this problem as well as the spatial dependence in SAR images in a separate paper. Moreover, in response to flourishing advancements in radar technology, how to effectively deal with the high-resolution data and efficiently calculate PFA in the CFAR algorithm based on a more concrete and realistic model is also a problem of interest.

Acknowledgment

The authors are grateful to the anonymous referee for constructive comments and suggestions.

Appendix A
Proof of Lemmas 1 and 2

Proof of Lemma 1. For any given $\lambda \in (0,1)$, and $\theta, \tilde{\theta} \in \Theta$,

$$\begin{aligned} & G(\lambda\theta + (1-\lambda)\tilde{\theta}) \\ &= E_P\left[\wp^2(\xi)e^{-(\lambda\theta+(1-\lambda)\tilde{\theta})'\xi+\psi(\lambda\theta+(1-\lambda)\tilde{\theta})}\right] \\ &< E_P\left[\wp^2(\xi)(\lambda e^{-\theta'\xi} + (1-\lambda)e^{-\tilde{\theta}'\xi}) + \wp^2(\xi)(\lambda e^{\psi(\theta)} + (1-\lambda)e^{\psi(\tilde{\theta})})\right] \\ &= \lambda E_P\left[\wp^2(\xi)e^{-\theta'\xi+\psi(\theta)}\right] + (1-\lambda)E_P\left[\wp^2(\xi)e^{-\tilde{\theta}'\xi+\psi(\tilde{\theta})}\right] \\ &= \lambda G(\theta) + (1-\lambda)G(\tilde{\theta}). \end{aligned} \tag{A1}$$

Therefore, we conclude that $G(\theta)$ is strictly convex. For completeness, we include the proof here, although similar statements can be found in Teng, Fuh, and Chen (2016). □

Proof of Lemma 2. The first-order-condition to minimize $G(\theta)$ in (9) requires θ^* to satisfy

$$\nabla G(\theta) = E_P\left[\wp^2(\xi)(-\xi + \nabla\psi(\theta))\mathrm{e}^{-\theta'\xi+\psi(\theta)}\right] = 0_d, \tag{A2}$$

with 0_d being a d-variate zero vector. Equation (A2) indeed equals

$$\nabla\psi(\theta) = \frac{E_P\left[\wp^2(\xi)\xi\mathrm{e}^{-\theta'\xi}\right]}{E_P\left[\wp^2(\xi)\mathrm{e}^{-\theta'\xi}\right]}. \tag{A3}$$

Plugging definition of the conjugate measure $\bar{Q}_\theta$ defined in (10), Equation (A3) equals

$$\nabla\psi(\theta) = E_{\bar{Q}_\theta}[\xi].$$

The proof is completed. □

Appendix B
Proof of Theorems 1 and 2

To prove the existence of the minimizer of $G(\theta)$ in Theorems 1 and 2, we first generalize the Weierstrass extreme value theorem in the following lemma. Denote $\Theta = \{\theta := (\theta_1, \dots, \theta_d) \in \Re^d : \theta_i < c_i, \text{for some constants} c_i, i = 1, \dots, d\}$.

Definition 1. (weakly coercive function). A function f is said to be weakly coercive with respect to the set Θ in $\Re^d$ *if* $\lim_{||\theta||\to\infty, \theta\in\Theta} f(\theta) = \infty$ and $\lim_{\theta_i\to c_i} f(\theta) = \infty$, for $i = 1, \dots, d$.

Lemma 3. *Let $f : \Theta \to \Re$ be continuous. If f is weakly coercive with respect to Θ, then f has at least one global minimizer.*

Proof of Lemma 3. Let $\alpha \in \Re$ be chosen so that the set $D = \{\theta \in \Theta : f(\theta) \le \alpha\}$ is non-empty. We will first show that the property of weakly coercive of f implies the compactness of the sets D. We begin by noting that the continuity of f implies the closedness of the sets D. Thus, it remains only to show that any set D is bounded. We show this by contradiction. Suppose to the contrary that there is an $\alpha \in \mathbb{R}$ such that the set D is unbounded. Then there must exist a sequence $\{\theta_\nu\} \in D$ for $\nu = 1, \dots, \infty$ with $||\theta_\nu|| \to \infty$ as $\nu \to \infty$. But then, by the definition of weakly coercive of f, we must also have $f(\theta_\nu) \to \infty$. This contradicts the assumption that $f(\theta_\nu) \le \alpha$ for all $\nu = 1, 2, \dots$. Therefore the set D must be bounded. Now, since D has to be closed and bounded, by Weierstrass extreme value theorem, $f(\theta)$ attains a minimum in D, and consequently $f(\theta)$ also attains a minimum in Θ. □

Proof of Theorem 1. To prove the existence of global minimum of $G(\theta)$ defined in (14) for product form of the K distribution, we recall that

$$G(\theta) = E_P\left[I_{\{W(X+p_n)>T\}} e^{-\theta_1 W - \theta_2 X - N\log(1-\theta_1) - \nu \log(1-\theta_2/b)}\right].$$

It is easy to see that $G(\theta)$ is continuous in $\theta_1 < 1$ and $\theta_2 < b$, and this implies that $G(\theta)$ is continuous for $\theta \in \Theta$. Next, it is easy to see that $G(\theta) \to \infty$ either as $\theta_1 \to 1$, $\theta_2 \to b$, or $\theta_1 \to -\infty$, $\theta_2 \to -\infty$. This implies that $G(\theta)$ is weakly coercive. Hence, by Lemma 3, we prove the existence of a minimizer for $G(\theta)$.

The uniqueness comes from the fact that $G(\theta)$ is strictly convex as shown in Lemma 1. And the characterization of the optimal tiling parameter is a straightforward application of Lemma 2. The proof is completed. □

Proof of Theorem 2. To prove the existence of global minimum of $G(\theta)$ defined in (19) for compound form of the K distribution, we recall that

$$G(\theta) = E_P\left[\Gamma^2_{N, X+p_n}(T) e^{-\theta X - \nu \log(1-\theta/b)}\right].$$

It is straightforward to see that $G(\theta)$ is continuous in $\theta < 1$. Next, it is easy to see that $G(\theta) \to \infty$ either as $\theta \to 1$, or $\theta \to -\infty$. This implies that $G(\theta)$ is weakly coercive. Hence, by Lemma 3, we prove the existence of a minimizer for $G(\theta)$.

The uniqueness comes from the fact that $G(\theta)$ is strictly convex as shown in Lemma 1. And the characterization of the optimal tiling parameter is a straightforward application of Lemma 2. The proof is completed. □

Funding

The author, Huei-Wen Teng, gratefully acknowledges the support by the the Ministry of Education of Taiwan, R.O.C., under the Higher Education Sprout Project, and the Ministry of Science and Technology of Taiwan, R.O.C., under grant MOST 108-2118-M-009-001-MY2, and, grant MOST 108-2634-F-163-001 through Pervasive Artificial Intelligence Research Labs.

References

Asmussen, S., and P. W. Glynn. 2007. *Stochastic simulation: algorithms and analysis*. New York: Springer-Verlag.

Bocquet, S. 2011. Calculation of radar probability of detection in k-distributed sea clutter and noise. Technical report, DSTO-TN-1000.

Conte, E., and M. Longo. 1987. Characterisation of radar clutter as a spherically invariant random process. *IEE Proceedings F Communications, Radar and Signal Processing* 134 (2):191–7. doi:10.1049/ip-f-1.1987.0035.

Delignon, Y., and W. Pieczynski. 2002. Modeling non-rayleigh speckle distribution in SAR images. *IEEE Transactions on Geoscience and Remote Sensing* 40 (6):1430–5. doi:10.1109/TGRS.2002.800234.

di Bisceglie, M., and C. Galdi. 2005. CFAR detection of extended objects in high-resolution sar images. *IEEE Transactions on Geoscience and Remote Sensing* 43 (4):833–43. doi:10.1109/TGRS.2004.843190.

Fuh, C. D., and I. Hu. 2004. Efficient importance sampling for events of moderate deviations with applications. *Biometrika* 91 (2):471–90. doi:10.1093/biomet/91.2.471.

Gao, G.,. L. Liu, L. Zhao, G. Shi, and G. Kuang. 2009. An adaptive and fast CFAR algorithm based on automatic censoring for target detection in high-resolution SAR images. *IEEE Transactions on Geoscience and Remote Sensing* 47 (6):1685–97.

Glynn, P. W. 1996. Importance sampling for Monte Carlo estimation of quantiles. In *Mathematical Methods in Stochastic Simulation and Experimental Design: Proc. 2nd St. Petersburg Workshop on Simulation*, 180–5. St Petersburg, Russia: Publishing House of Saint Petersburg University.

Gonzalez-Garcia, J. E., J. Sanz-Gonzalez, and F. Á. Vaquero. 2004. Optimal detectors for nonfluctuating targets under spiky *k*-distributed clutter in radar applications. In *IEEE Sensor Array and Multichannel Signal Processing Workshop Proceedings*, 191–5.

Jeruchim, M. C., P. B. Balaban, and K. S. Shanmugan. 2000. *Simulation of Communication Systems* 2nd ed. New York, NY: Kluwer Academic/Plenum Publisher.

Jiang, Q., S. Wang, D. Ziou, A. E. Zaart, M. T. Rey, G. B. Bénié, and M. Henschel. 1998. Ship detection in RADARSAT SAR imagery. *IEEE International Conference on Systems, Man, and Cybernetics* 5:4562–6.

Liu, J., and X. Yang. 2012. The convergence rate and asymptotic distribution of bootstrap quantile variance estimator for importance sampling. *Advances in Applied Probability* 44 (3):815–41. doi:10.1239/aap/1346955266.

Orsak, G. C. 1993. A note on estimating false alarm rates via importance sampling. *IEEE Transactions on Communications* 41 (9):1275–7. doi:10.1109/26.237841.

Paes, R. L., J. A. Lorenzzetti, and D. F. M. Gherardi. 2010. Ship detection using TerraSAR-X images in the Campos Basin (Brazil). *IEEE Geoscience and Remote Sensing Letters* 7 (3):545–8. doi:10.1109/LGRS.2010.2041322.

Rangaswamy, M., D. D. Weiner, and A. Ozturk. 1993. Non-gaussian random vector identification using spherically invariant random processes. *IEEE Transactions on Aerospace and Electronic Systems* 29 (1):111–24. doi:10.1109/7.249117.

Shnidman, D. A. 1989. The calculation of the probability of detection and the generalized Marcum Q-function. *IEEE Transactions on Information Theory* 35 (2):389–400. doi:10.1109/18.32133.

Shnidman, D. A. 2008. Update on radar detection probabilities and their calculation. *IEEE Transactions on Aerospace and Electronic Systems* 44 (1):380–3. doi:10.1109/TAES.2008.4517013.

Smith, P. J., M. Shafi, and H. Gao. 1997. Quick simulation: A review of importance sampling techniques in communications systems. *IEEE Journal on Selected Areas in Communications* 15 (4): 597–613. doi:10.1109/49.585771.

Smith, S. L., and G. C. Orsak. 1995. A modified importance sampling scheme for the estimation of detection system performance. *IEEE Transactions on Communications* 43 (2/3/4):1341–6. doi:10.1109/26.380183.

Srinivasan, R. 1998. Some results in importance sampling and an application to detection. *Signal Processing* 65 (1):73–88. doi:10.1016/S0165-1684(97)00208-9.

Srinivasan, R., and M. Rangaswamy. 2007. Importance sampling for characterizing STAP detectors. *IEEE Transactions on Aerospace and Electronic Systems* 43 (1):273–85. doi:10.1109/TAES.2007.357133.

Stadelman, D. L., D. D. Weiner, and A. D. Keckler. 2002. Efficient determination of thresholds via importance sampling for Monte Carlo evaluation of radar performance in non-Gaussian clutter. *Proceedings of the IEEE Radar Conference*, 272–7.

Teng, H. W., C. D. Fuh, and C. C. Chen. 2016. On an automatic and optimal importance sampling approach with applications in finance. *Quantitative Finance* 16 (8):1259–71. doi:10.1080/14697688.2015.1136077.

Ward, K. D., S. R. Watts, and, and J. A. Tough. 2006. Sea clutter: scattering, the k-distribution and radar performance. Technical report, Institution of Engineering and Technology, London, UK.

Watts, S. 1987. Radar detection prediction of k-distributed sea clutter and thermal noise. *IEEE Transactions on Aerospace and Electronic Systems* AES-23 (1):40–5. doi:10.1109/TAES.1987.313334.