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A systematic and efficient simulation scheme for the Greeks of financial derivatives

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Greeks are the price sensitivities of financial derivatives and are essential for pricing, speculation, risk management, and model calibration. Although the pathwise method has been popular for calculating them, its applicability is problematic when the integrand is discontinuous. To tackle this problem, this paper defines and derives the parameter derivative of a discontinuous integrand of certain functional forms with respect to the parameter of interest. The parameter derivative is such that its integration equals the differentiation of the integration of the aforesaid discontinuous integrand with respect to that parameter. As a result, unbiased Greek formulas for a very broad class of payoff functions and models can be systematically derived. This new method is applied to the Greeks of (1) Asian options under two popular Lévy processes, i.e. Merton's jump-diffusion model and the variance-gamma process, and (2) collateralized debt obligations under the Gaussian copula model. Our Greeks outperform the finite-difference and likelihood ratio methods in terms of accuracy, variance, and computation time.

Keywords: Greeks; Dirac delta function; Variance-gamma processes; Jump-diffusion processes; Credit derivatives; Monte Carlo simulation

JEL Classification: C15, C53

1. Introduction

Greeks letters (or simply Greeks) are the price sensitivities of financial derivatives with respect to some parameters of interest such as stock price, volatility, interest rate, default rate, mortality rate, to name just a few. (Derivatives will be called options from now on for brevity.) They are essential for pricing, speculation, risk management, and model calibration (Cui *et al.* 2017, Hull 2014, Jorion 2001). Studies on hedging with Greeks and their calculations can be found in Dumas *et al.* (1998), Pelsser and Vorst (1994) and Bernis *et al.* (2003). Mathematically speaking, the price of a financial derivative equals the expectation of its discounted payoff function under a suitable pricing measure (Duffie 1996). Hence a Greek is the partial derivative of that integral with respect to a parameter. Some Greeks such as Gamma are higher-order partial derivatives.

Numerical methods for calculating the Greeks efficiently and accurately are indispensable when closed-form formulas are not available. Among them, the finite-difference method (FD), with common random numbers if simulation is employed, is straightforward and easy to implement. As its name implies, FD uses finite differences to approximate partial derivatives. As FD recalculates the option price with a perturbed parameter, it is called a bump-and-revalue method or an indirect resimulation method (Broadie and Glasserman 1996). Unfortunately, FD usually produces biased Greeks and, worse still, highly unstable higher-order Greeks. Moreover, the choice of the perturbation size poses a dilemma: If it is too large, the Greeks may be inaccurate, but if it is too small, they can be highly unstable

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(Jäckel 2002). These numerical issues are particularly challenging for higher-order Greeks.

The option price is the integration of its discounted payoff function weighted by the density associated with the pricing measure (Hull 2014). The differentiation operator is applied to that integral, sometimes more than once, to produce the Greeks. This operation will be greatly simplified if the order of differentiation and integration can be switched. Broadie and Glasserman (1996) introduce direct methods and identify conditions under which such interchange produces unbiased estimators. Direct methods are divided into two categories. The pathwise method (also known as the infinitesimal perturbation analysis, or IPA) differentiates the payoff function (Cao 1985). The result is furthermore unbiased when the payoff function is Lipschitz continuous besides satisfying additional regularity conditions. The requirement of Lipschitz continuity, however, limits the use of the pathwise method, particularly for higher-order Greeks.

The likelihood ratio method (LRM) belongs in the second type of direct methods and produces unbiased Greeks when applicable. LRM differentiates the probability density function (pdf) of the asset, which needs to be explicitly given. But LRM estimators tend to have larger variances, especially when the number of discretization steps in the simulation is large (Broadie and Glasserman 1996, Chen and Glasserman 2007). Variants of LRM include the Malliavin calculus (Davis and Johansson 2006, Kawai and Takeuchi 2011) and measure-valued differentiation (Heidergott and Leahu 2010, Thoma 2012).

The standard pathwise method is popular because it is easy to implement and produces unbiased Greek estimators with a small variance (Broadie and Glasserman 1996). But this method requires principally Lipschitz continuity of the integrand to guarantee unbiasedness. To lift the requirement of the Lipschitz continuity of the pathwise method, this paper defines and derives the *parameter derivative* of a discontinuous integrand with respect to the parameter of interest, assuming the integrand has a certain functional form. The parameter derivative is such that its integration equals the differentiation of the integration of the aforesaid discontinuous integrand with respect to that parameter of interest. The parameter derivative differs sharply from the classic distributional derivative: The former is differentiation with respect to the parameter whereas the latter is that with respect to the function argument. This critical distinction may account for some of the technical errors found in the literature.

The Dirac delta function has appeared in a few pathwise method-based Greek formulas. The discontinuous perturbation analysis proposed by Shi (1996) uses this function in sensitivity analysis of discrete-event dynamic systems. But it focuses on the univariate cases, and the Dirac delta function's applicability to Greeks remains unexplored. Other examples include the Gamma for Asian options under the Black-Scholes model (Broadie and Glasserman 1996) and the Delta for the nth to default credit swaps under the Gaussian copula model (Joshi and Kainth 2004). However, the use of the Dirac delta function in deriving the Greek formulas has not been systematized for the pathwise method.

Under the simple Black-Scholes model, Lyuu and Teng (2011) offer a pathwise method that yields unbiased Greeks for a broad class of payoff functions, including many discontinuous ones. Greek formulas for vanilla, maximum, maximum digital, spread, and Asian options are also given by them. However, their formulations are complicated. With the parameter derivative, this paper succeeds in building on a key result of theirs so that unbiased Greeks can be derived systematically and easily. The formulas are also more compact. Method in this paper should help investigations on hedging, risk management, and calibration that require numerically accurate Greeks.

Two applications are studied to demonstrate the general applicability of our method. First, it has been empirically documented that the more general Lévy process fits option prices better than the geometric Brownian motion of the Black-Scholes model (Bates 1996, Madan *et al.* 1998). So our method is first applied to yield unbiased Greeks for Asian options under two popular Lévy processes: Merton's jump-diffusion model and the variance-gamma process. These formulas are the first in the literature. We mention that our Greek formulas for European and Asian options remain valid as long as stock price process satisfies certain separation conditions.

Glasserman and Liu (2010) combine the pathwise method and the LRM to derive the first-order Greeks of Asian options under the normal-inverse-Gaussian model. Because the score function in the LRM is approximated by a saddlepoint method, their resulting Greeks are biased. As for unbiased Greeks, Kawai and Takeuchi (2010) derive the first- and second-order Greeks for Asian options under the inverse-gamma process; Kawai and Takeuchi (2011) adopt Mallliavin calculus to derive the Greek formulas of European options under gamma processes; Joshi and Zhu (2016) propose rejection techniques to compute the first-order Greeks of barrier options under the normal-inverse-Gaussian process and of European options under the variance-gamma process. Our contributions here are the first- and second-order Greeks of Asian options under Merton's jump-diffusion model and the variance-gamma model, the first in the literature.

Greeks are critical for measuring the risk of credit derivatives. In the second application, our method yields unbiased Delta and Gamma formulas for collateralized debt obligations (CDOs) under the Gaussian copula model of Li (2000). Under the same model, Chen and Glasserman (2008) propose a smoothing technique to obtain the Delta formula of the nth to default credit swaps and CDOs, whereas Joshi and Kainth (2004) use the Dirac delta function to derive the Delta formula of the nth to default credit swaps. Joshi and Kainth (2004) briefly mention that they use the Leibniz rule and the Dirac delta function (as the distributional derivative of an indicator function) in deriving their Delta formula. (But a potential error in that formula is identified in Chen and Glasserman (2008).) But it is unclear how to apply such a method to other derivatives. In contrast, our method yields unbiased Greeks of Asian options under Merton's jump-diffusion model and the variance-gamma process.

Extensions to the pathwise method include the smoothed perturbation analysis (Fu and Hu 1995), the support independent unified likelihood ratio and infinitesimal perturbation analysis (SLRIPA) (Wang *et al.* 2012), and the kernel method (Hong and Liu 2011, Hong *et al.* 2014). Besides the aforementioned methods, several alternatives have been proposed to calculate the Greeks. We mention only the weak derivative approach (Heidergott 2001) and the various proxy schemes (Fries and Joshi 2008, Chan and Joshi 2013). These methods may produce biased Greeks, however.

In summary, this paper contributes to the existing literature by defining and deriving the parameter derivative of a discontinuous integrand of certain functional form with respect to the parameter of interest. Our method utilizes the Dirac delta function and yields provably unbiased Greeks for a broad class of models and payoff functions.

The rest of this paper is organized as follows. Section 2 provides fundamental results whereby unbiased Greeks can be easily derived with the Dirac delta function. Section 3 provides the Greek formulas for European and Asian options under two popular Lévy processes: Merton's jump-diffusion model and the variance-gamma process. Section 4 gives the Greek formulas for a collateralized debt obligation. The last section concludes.

2. Fundamental results

2.1. Notations and terminologies

Consider pricing a financial derivative with a risky underlying asset, S, which can be generated as a function with a parameter of interest θ and an m-variate random vector $X = (X_1, \ldots, X_m)'$. Let $X_{-k} = (X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_m)'$, which is an $(m-1)$-variate random vector equal to X with X_k removed. Furthermore, $x = (x_1, \ldots, x_m)'$ is a realization of X, where $x_{-k} = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m)'$.

This financial derivative has a discounted payoff function $\wp$ associated with the underlying asset and is denoted by

$$\wp = \wp(\theta, X).$$

Then the financial derivative has price equal to

$$c(\theta) = E[\wp(\theta, X)] = \int_{\Re^m} \wp(\theta, x) f(x)\,\mathrm{d}x, \tag{1}$$

where $f(x)$ is the probability density function (pdf) of X under the pricing measure. Greeks involve the partial differentiation of $c(\theta)$ with respect to θ, sometimes more than once.

If a closed-form formula for the price $c(\theta)$ in equation (1) does not exist or when the curse of dimensionality is a concern because m is large enough, $c(\theta)$ can be calculated by Monte Carlo simulation (Glasserman 2004). Monte Carlo simulation is a general, flexible, and easy-to-implement approach to approximate an expectation (Ross 2013). Standard Monte Carlo method approximates $c(\theta)$ by

$$\hat{c}(\theta) = \frac{1}{n} \sum_{i=1}^{n} \wp(\theta, X^{(i)}),$$

where $X^{(i)}$ is the ith simulation path for $i = 1, \ldots, n$ with n denoting the sample size. We will soon provide a theorem which establishes the Greek $\partial c(\theta)/\partial\theta$ as an expectation, which will be open to Monte Carlo simulation.

Interchanging the order of integration and differentiation, i.e.

$$\frac{\partial c(\theta)}{\partial\theta} = \int_{\Re^m} \frac{\partial \wp(\theta, x)}{\partial\theta} f(x)\,\mathrm{d}x, \tag{2}$$

if valid, produces a Greek formula that is useful for Monte Carlo simulation. However, the above identity typically requires the payoff function to be Lipschitz continuous so that the dominance convergence theorem applies (Broadie and Glasserman 1996). For this reason, the pathwise method, which relies on the validity of equation (2), is severely limited in its applications despite its simplicity and excellent practical performance. With the new notion of parameter derivative, however, we will be able to go beyond Lipschitz-continuous functions for the applicability of the pathwise method.

Let I_A be the indicator function with the support set A. A function is said to be of class C^k if it has continuous derivatives up to order k. A function is said to be of class C^∞, or simply smooth, if it is infinitely differentiable. Consider discounted payoff functions that assume this form,

$$\wp(\theta, x) = h(\theta, x) \left[\prod_{i\in\mathcal{I}} I_{\{x_k > \chi_{Li}(\theta, x_{-k})\}}\right] \left[\prod_{j\in\mathcal{J}} I_{\{x_k < \chi_{Rj}(\theta, x_{-k})\}}\right], \tag{3}$$

where $h(\theta, x)$ is a differentiable function with respect to θ for every x, and $\chi_{Li}(\theta, x_{-k})$ and $\chi_{Rj}(\theta, x_{-k})$ are of class C^1 with respect to θ for every x_{-k}. Here, k is a positive integer between 1 and m. Above, $\mathcal{I}$ and $\mathcal{J}$ are finite sets of natural numbers. The subscripts L and R in $\chi_{Li}(\theta, x)$ and $\chi_{Rj}(\theta, x)$ indicate the endpoint for a left-bounded and right-bounded interval, respectively. The boundaries of the payoff function's support are required to be $\{x \in \Re^m : x_k = \chi_{Li}(\theta, x_{-k})\}$ and $\{x \in \Re^m : x_k = \chi_{Rj}(\theta, x_{-k})\}$.

Lyuu and Teng (2011) tackle payoff functions of the following form,

$$\wp(\theta, x) = h(\theta, x) \prod_{j\in\mathcal{B}} I_{\{g_j(\theta, x) > 0\}}, \tag{4}$$

where $g_j(\theta, x)$ satisfies certain regularity conditions and $\mathcal{B}$ is a finite set of natural numbers. The form in equation (3) is a special case of equation (4). This paper adopts equation (3) because the functions $\chi_{Ri}(\theta, x_{-k})$ and $\chi_{Lj}(\theta, x_{-k})$ can be easily obtained for many Greeks so that mathematical presentations can be greatly simplified. In contrast, the results in Lyuu and Teng (2011) rely on the implicit function theorem, and the formulations are complicated.

2.2. The dirac delta function and the parameter derivative

In mathematical analysis, generalized functions (or distributions) generalize the classical notion of functions and make possible the differentiation of functions which are not differentiable in the classical sense. Informally, the kth shifted Dirac delta function shifted by a constant θ, denoted by $\delta_\theta^k(x)$, is a function on $\Re^m$ which is zero everywhere except where $x_k = \theta$, is infinite. In addition, $\delta_\theta^k(x)$ satisfies the identity

$$\begin{aligned}&\int_{\Re^m} \delta_\theta^k(x)\psi(x)\,\mathrm{d}x \\ &\quad = \int_{\Re^{m-1}} \psi(x_1, \ldots, x_{k-1}, \theta, x_{k+1}, \ldots, x_m)\,\mathrm{d}x_{-k}\end{aligned}$$

for any sufficiently 'good' function $\psi(x)$. When $m=1$, we denote $\delta_\theta^1(x)$ by $\delta_\theta(x)$ for notional convenience.

Suppose the $c(\theta)$ in equation (1) is differentiable with respect to θ. A function $D_\theta \wp(\theta,x)$ on $\Re^m$ with a parameter θ is said to be the *parameter derivative* of $\wp(\theta,x)$ with respect to θ if

$$\int_{\Re^m} D_\theta \wp(\theta,x) f(x)\,\mathrm{d}x = \frac{\partial}{\partial\theta}\int_{\Re^m} \wp(\theta,x) f(x)\,\mathrm{d}x. \tag{5}$$

In other words, the integration of the parameter derivative equals the differentiation of the integral. The Greek formula is simply this integral

$$\frac{\partial c(\theta)}{\partial\theta} = \int_{\Re^m} D_\theta \wp(\theta,x) f(x)\,\mathrm{d}x.$$

The Greeks hence can be calculated directly, if simple formulas obtain, or integrated numerically with, for example, Monte Carlo simulation. Note that numerical differentiation is completely avoided now. Furthermore, the Greek is automatically unbiased. Note that $D_\theta \wp(\theta,x)$ refers to differentiation with respect to the parameter θ instead of a function argument x_k.

2.3. Main mathematical results

For Greeks calculations, we are interested in $\partial c(\theta)/\partial\theta$, where $c(\theta)$ is given in equation (1). Let X be a random vector with pdf $f(x)$. Let $h_\theta(\theta,x)$, $\chi_{Li,\theta}(\theta,x_{-k})$ and $\chi_{Rj,\theta}(\theta,x_{-k})$ denote the partial differentiations of $h(\theta,x)$, $\chi_{Li}(\theta,x_{-k})$, and $\chi_{Rj}(\theta,x_{-k})$ with respect to θ, respectively. We now summarize the four assumptions for $h(\theta,x)$, $\chi_{Li}(\theta,x_{-k})$, $\chi_{Rj}(\theta,x_{-k})$, and $f(x)$ needed by the theorem below to provide the parameter derivative of the payoff function with respect to θ:

Assumption 1: $h(\theta,x)$ is differentiable with respect to θ for every x.
Assumption 2: $h(\theta,x)$ and $h_\theta(\theta,x)$ are absolutely integrable.
Assumption 3: $h(\theta,x)f(x)$ and $h_\theta(\theta,x)f(x)$ are uniformly continuous with respect to θ and x_k.
Assumption 4: $\chi_{Li}(\theta,x_{-k})$ and $\chi_{Rj}(\theta,x_{-k})$ are of class C^1 with respect to θ for every x_{-k}.

THEOREM 2.1 *Given the four assumptions above, the parameter derivative of $\wp(\theta,x)$ of the form in equation* (3) *with respect to θ equals*

$$\begin{aligned}
&D_\theta h(\theta,x)\left[\prod_{i\in\mathcal{I}} I_{\{x_k>\chi_{Li}(\theta,x_{-k})\}}\right]\left[\prod_{j\in\mathcal{J}} I_{\{x_k<\chi_{Rj}(\theta,x_{-k})\}}\right]\\
&= h_\theta(\theta,x)\left[\prod_{i\in\mathcal{I}} I_{\{x_k>\chi_{Li}(\theta,x_{-k})\}}\right]\left[\prod_{j\in\mathcal{J}} I_{\{x_k<\chi_{Rj}(\theta,x_{-k})\}}\right]\\
&\quad -\sum_{\ell\in\mathcal{I}} h(\theta,x)\left[\prod_{i\in\mathcal{I}\setminus\{\ell\}} I_{\{x_k>\chi_{Li}(\theta,x_{-k})\}}\right]\\
&\quad \times\left[\prod_{j\in\mathcal{J}} I_{\{x_k<\chi_{Rj}(\theta,x_{-k})\}}\right]\delta^k_{\chi_{Li}(\theta,x_{-k})}(x)\,\chi_{Li,\theta}(\theta,x_{-k})\\
&\quad +\sum_{\ell\in\mathcal{J}} h(\theta,x)\left[\prod_{i\in\mathcal{I}} I_{\{x_k>\chi_{Li}(\theta,x_{-k})\}}\right]\\
&\quad \times\left[\prod_{j\in\mathcal{J}\setminus\{\ell\}} I_{\{x_k<\chi_{Rj}(\theta,x_{-k})\}}\right]\delta^k_{\chi_{Rj}(\theta,x_{-k})}(x)\,\chi_{Rj,\theta}(\theta,x_{-k}).
\end{aligned}\tag{6}$$

Proof of Theorem 2.1 See Appendix 2. ■

In most cases, we are interested in a payoff containing just one indicator function. Then the parameter derivative of $h(\theta,x)I_{\{x_k>\chi(\theta,x_{-k})\}}$ with respect to θ equals

$$\begin{aligned}D_\theta h(\theta,x)I_{\{x_k>\chi(\theta,x_{-k})\}} &= h_\theta(\theta,x)I_{\{x_k>\chi(\theta,x_{-k})\}}\\ &\quad - h(\theta,x)\delta^k_{\chi(\theta,x_{-k})}(x)\,\chi_\theta(\theta,x_{-k}).\end{aligned}\tag{7}$$

Likewise, the parameter derivative of $h(\theta,x)I_{\{x_k<\chi(\theta,x_{-k})\}}$ with respect to θ is

$$\begin{aligned}D_\theta h(\theta,x)I_{\{x_k<\chi(\theta,x_{-k})\}} &= h_\theta(\theta,x)I_{\{x_k<\chi(\theta,x_{-k})\}}\\ &\quad + h(\theta,x)\delta^k_{\chi(\theta,x_{-k})}(x)\,\chi_\theta(\theta,x_{-k}).\end{aligned}\tag{8}$$

Theorem 2.1 is applied to obtain Greek formulas as an expectation, which can then be calculated by Monte Carlo simulation. Moreover, Theorem 2.1 guarantees the validity of exchanging the order of differentiation and integration and gives explicit unbiased Greek formulas in terms of expectations only.

A few observations are called for here. When $m=1$, $h(\theta,x)=1$, and $\chi(\theta,x_{-k})=\theta$, equation (8) reduces to

$$D_\theta I_{\{x<\theta\}} = \delta_\theta(x),$$

which gives the parameter derivative of the indicator function $I_{\{x<\theta\}}(x)$ with respect to θ. When $m=1$ and $h(\theta,x)=1$, equation (8) reduces to

$$D_\theta I_{\{x<\chi(\theta)\}} = \delta_{\chi(\theta)}(x)\frac{\partial\chi(\theta)}{\partial\theta},$$

which is the chain rule. Equation (6), incidentally, is the product rule.

We emphasize again that Theorem 2.1 is not merely a trivial application of the fact that the Dirac delta function is the distributional derivative of the indicator function. A brief review on the Dirac delta function and its distributional derivative is covered in Appendix 1.

2.4. Systematic steps to obtain the Greeks

Theorem 2.1 immediately yields the following explicit steps to obtain the Greeks.

(i) Recast, if necessary, the payoff function $\wp(\theta,x)$ in the form of equation (3).
(ii) Identify the random vector x and its distribution.
(iii) Apply Theorem 2.1 to obtain $D_\theta\wp(\theta,x)$, the parameter derivative of $\wp(\theta,x)$ with respect to θ.

(iv) For convenience and where helpful, express the Greek formula as an integral, i.e.

$$\begin{aligned}\frac{\partial c(\theta)}{\partial\theta} &= \int D_\theta \wp(\theta,x) f(x)\,\mathrm{d}x \\ &= \int h_\theta(\theta,x)\left[\prod_{i\in\mathcal{I}} I_{\{x_k>\chi_{Li}(\theta,x_{-k})\}}\right] \\ &\quad\times\left[\prod_{j\in\mathcal{J}} I_{\{x_k<\chi_{Rj}(\theta,x_{-k})\}}\right] f(x)\,\mathrm{d}x \\ &\quad-\sum_{\ell\in\mathcal{I}}\int h(\theta,\tilde{x}_{L\ell}) \\ &\quad\times\left[\prod_{i\in\mathcal{I}\setminus\{\ell\}} I_{\{\chi_{L\ell}(\theta,x_{-k})>\chi_{Li}(\theta,x_{-k})\}}\right] \\ &\quad\times\left[\prod_{j\in\mathcal{J}} I_{\{\chi_{L\ell}(\theta,x_{-k})<\chi_{Rj}(\theta,x_{-k})\}}\right] \\ &\quad\chi_{L\ell,\theta}(\theta,x_{-k}) f(\tilde{x}_{L\ell})\,\mathrm{d}x_{-k} \\ &\quad+\sum_{\ell\in\mathcal{J}}\int h(\theta,\tilde{x}_{R\ell}) \\ &\quad\times\left[\prod_{i\in\mathcal{I}} I_{\{\chi_{R\ell}(\theta,x_{-k})>\chi_{Li}(\theta,x_{-k})\}}\right] \\ &\quad\times\left[\prod_{j\in\mathcal{J}\setminus\{\ell\}} I_{\{\chi_{R\ell}(\theta,x_{-k})<\chi_{Rj}(\theta,x_{-k})\}}\right] \\ &\quad\chi_{R\ell,\theta}(\theta,x_{-k}) f(\tilde{x}_{R\ell})\,\mathrm{d}x_{-k},\end{aligned}$$

where

$$\begin{aligned}\tilde{x}_{L\ell} &= (x_1,\ldots,x_{\ell-1},\chi_{L\ell,\theta}(\theta,x_{-k}),x_{\ell+1},\ldots,x_m),\\ \tilde{x}_{R\ell} &= (x_1,\ldots,x_{\ell-1},\chi_{R\ell,\theta}(\theta,x_{-k}),x_{\ell+1},\ldots,x_m).\end{aligned}$$

(v) The Greek formula, being a sum of integrals, can be integrated by Monte Carlo simulation or other numerical methods.

Second- or higher-order Greeks can be obtained similarly. It is clear that (i) is the only step that is not mechanical.

2.5. *The Delta of digital call options under the Black-Scholes model: an illustration*

Consider the digital call option under the Black-Scholes model, where S_0 denotes the initial stock price, σ the volatility, T the time to maturity, r the interest rate, and K the strike price. The stock price at maturity is generated by

$$S_T = S_0 \mathrm{e}^{(r-\sigma^2/2)T+\sigma\sqrt{T}X},$$

where X is the standard normal distribution. The European digital call option has the discounted payoff function

$$\wp = \mathrm{e}^{-rT} I_{\{S_T>K\}}.$$

The price of the digital call option is (Hull 2014)

$$\begin{aligned}c &= E\left[\mathrm{e}^{-rT} I_{\{S_T>K\}}\right] = \int_{-\infty}^{\infty} \mathrm{e}^{-rT} I_{\{S_0\mathrm{e}^{(r-\sigma^2/2)T+\sigma\sqrt{T}x}>K\}}(x)\phi(x)\,\mathrm{d}x \\ &= \mathrm{e}^{-rT}\Phi(d_2).\end{aligned}$$

Above, $\phi(\cdot)$ and $\Phi(\cdot)$ are the pdf and the cumulative distribution function (cdf) of the standard normal distribution, respectively, and

$$d_2 := d_2(S_0) = \frac{\log\frac{S_0}{K} + (r-\sigma^2/2)T}{\sigma\sqrt{T}}.$$

(Throughout the paper, the notation '$A := B$' means A abbreviates B.) Its Delta is well-known:

$$\frac{\partial c}{\partial S_0} = \frac{\mathrm{e}^{-rT}\phi(d_2)}{S_0\sigma\sqrt{T}}. \tag{9}$$

Before applying Theorem 2.1 to re-establish equation (9), we rewrite c as

$$c = \int_{-\infty}^{\infty} \mathrm{e}^{-rT} I_{\{x>-d_2(x)\}}\phi(x)\,\mathrm{d}x.$$

Because

$$\frac{\partial(-d_2)}{\partial S_0} = -\frac{1}{S_0\sigma\sqrt{T}},$$

we obtain

$$\frac{\partial I_{\{x>-d_2\}}(x)}{\partial S_0} = -\delta_{(-d_2)}(x)\frac{-1}{S_0\sigma\sqrt{T}}.$$

Hence the Delta by equation (6) equals

$$\begin{aligned}&\int_{-\infty}^{\infty} \mathrm{e}^{-rT}\delta_{(-d_2)}(x)\frac{1}{S_0\sigma\sqrt{T}}\phi(x)\,\mathrm{d}x \\ &\quad= \frac{\mathrm{e}^{-rT}\phi(-d_2)}{S_0\sigma\sqrt{T}} = \frac{\mathrm{e}^{-rT}\phi(d_2)}{S_0\sigma\sqrt{T}},\end{aligned}$$

which matches equation (9) exactly. As mentioned in Fu (2007), suppose equation (2) is applied without first checking if the Lipschitz continuity holds for the payoff of the digital option. Then a claim for a zero Delta may be obtained, erroneously, because the derivative of the payoff function is zero almost everywhere.

3. Greek formulas for financial options

This section will present our Greek formulas for two popular Lévy processes: Merton's jump-diffusion model and the variance-gamma processes. Our estimators are unbiased and straightforward to implement. Generalizations of our results will be commented on, too.

3.1. *General results for European and Asian options*

Let c denote the option price, S_0 the initial stock price, and σ the volatility. We provide Greek formulas for the Deltas ($\Delta = \partial c/\partial S_0$), Gammas ($\Gamma = \partial^2 c/\partial S_0^2$), and Vegas ($\nu = \partial c/\partial\sigma$) of European and discretely monitored Asian options under Merton's jump-diffusion model in Section 3.2 and the variance-gamma process in Section 3.3. Throughout this paper, Asian options are discretely monitored.

The European call option has this payoff function

$$\wp = (S_T - K)^+, \tag{10}$$

where S_T is the stock price at maturity T, K is the strike price, and $(\cdot)^+$ is the positive function. Under a suitable pricing measure, assume the stock price at maturity T can be generated by

$$S_T := S_T(S_0, \sigma; X) = S_0 e^{\gamma_T(\sigma;X)}, \tag{11}$$

where $\gamma_T := \gamma_T(\sigma; X)$ is a function that depends on the random vector X and the volatility parameter σ. (The semicolon separates the parameters of interest and the random vector.) Here, X is an m-dimensional random vector each element of which may follow its own distribution. For example, in Merton's jump-diffusion model, some dimensions of X are normal whereas others are compound Poisson; in the variance-gamma process, as another example, some dimensions of X are normal whereas others are gamma. Let Ω denote the range of X. We further assume there exists a random variable X_k for some positive integer k and a smooth function $\chi := \chi(S_0, \sigma; X_{-k})$ so that

$$\begin{aligned}&\{X \in \Omega : S_T(S_0, \sigma; X) > K\}\\ &\quad = \{X \in \Omega : X_k > \chi(S_0, \sigma; X_{-k})\}.\end{aligned} \tag{12}$$

COROLLARY 3.1 *Suppose that the stock price is generated by equations* (11) *and* (12) *holds for some positive integer k and smooth function $\chi(S_0, \sigma; X_{-k})$. Then the Delta, Gamma, and Vega of the European call option equal, respectively,*

$$\Delta = e^{-rT} E\left[\frac{S_T}{S_0} I_{\{S_T > K\}}\right], \tag{13}$$

$$\Gamma = -e^{-rT}\frac{K}{S_0} E\left[f_{X_k}(\chi(S_0, \sigma; X_{-k}))\frac{\partial\chi(S_0, \sigma; X_{-k})}{\partial S_0}\right], \tag{14}$$

$$\nu = e^{-rT} E\left[I_{\{S_T > K\}} S_T \frac{\partial\gamma_T(\sigma; X)}{\partial\sigma}\right], \tag{15}$$

where $f_{X_k}(\cdot)$ is the pdf of X_k.

Proof of Corollary 3.1 See Appendix 4. ■

The payoff function of the Asian option is

$$\wp = (\bar{S} - K)^+,$$

where

$$\bar{S} := \bar{S}(S_0, \sigma; X) = \frac{1}{m}\sum_{i=1}^{m} S_{t_i}$$

is the arithmetic average of the stock prices monitored at times $t_1 < t_2 < \cdots < t_m = T$. Under a suitable pricing measure, suppose that the stock price at time t_i can be generated by

$$S_{t_i} := S_{t_i}(S_0, \sigma; X) = S_0 e^{\gamma_{t_i}(\sigma;X)}, \tag{16}$$

where $\gamma_{t_i} := \gamma_{t_i}(\sigma; X)$ is a function that depends on the random vector X and the volatility parameter σ. Let Ω denote the range of X. Furthermore, assume there exists a random variable X_k for some positive integer k and a smooth function $\chi := \chi(S_0, \sigma; X_{-k})$ so that

$$\{X \in \Omega : \bar{S}(S_0, \sigma; X) > K\} = \{X \in \Omega : X_k > \chi(S_0, \sigma; X_{-k})\}. \tag{17}$$

COROLLARY 3.2 *Suppose that a series of m stock prices are generated by equations* (16) *and* (17) *holds for some positive integer k and smooth function $\chi(\theta; X_{-k})$. Then the Delta, Gamma, and Vega of the Asian call option equal, respectively,*

$$\Delta = e^{-rT} E\left[\frac{\bar{S}}{S_0} I_{\{\bar{S} > K\}}\right], \tag{18}$$

$$\Gamma = -e^{-rT}\frac{K}{S_0} E\left[f_{X_k}(\chi(S_0, \sigma; X_{-k}))\frac{\partial\chi(S_0, \sigma; X_{-k})}{\partial S_0}\right], \tag{19}$$

$$\nu = e^{-rT} E\left[\left(\frac{1}{m}\sum_{i=1}^{m} S_{t_i}\frac{\partial\gamma_{t_i}(\sigma; X)}{\partial\sigma}\right) I_{\{\bar{S} > K\}}\right], \tag{20}$$

where $f_{X_k}(\cdot)$ is the pdf of X_k.

Proof of Corollary 3.2 See Appendix 5. ■

When $m = 1$, Corollary 3.2 reduces to Corollary 3.1, as expected. Note that Corollaries 3.1 and 3.2 require a closed-form formula for f_{X_k}. All examples in this paper have f_{X_k} available in closed form. In general, since X_k is a variate of a random vector X, it is a random variable with an explicit probability density function. If f_{X_k} is not available in closed form, f_{X_k} needs to be estimated before applying the above results.

Corollaries 3.1 and 3.2 are valid to a wide set of models for which stock prices satisfy the following separation conditions:

(1) The stock price at time t can be presented as

$$S_t = S_0\alpha(\sigma; X),$$

for some function α.

(2) For a European call option, the boundary of the support for the payoff function can be written in terms of X_k and $\chi(S_0, \sigma, X_{-k})$, i.e.

$$\{X \in \Omega : S_T = K\} = \{X \in \Omega : X_k = \chi(S_0, \sigma; X_{-k})\},$$

where an explicit closed-form formula for $\chi(S_0, \sigma; X_{-k})$ exists. (For Asian-type and put-type options, the needed changes are straightforward.)

In general, with the above separation conditions, the payoff function of the European or Asian option is in the form ofequation (3). For example, we obtain a non-Lévy process when a jump-diffusion model features conditional heteroscedasticity (Jorion 1988) or has a mean-reverting diffusion component (Cartea and Figueroa 2005). But both models satisfy the separation conditions.

3.2. *Merton's jump-diffusion model*

Let S_t denote the stock price at time t with S_0 being the initial stock price. The dynamics of Merton's jump-diffusion model under a suitable pricing measure is

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t + S_t\left(d\sum_{j=0}^{N_t} Y_j\right). \tag{21}$$

In the diffusion part, r is the risk-free rate, σ is the volatility, and W_t is the standard Brownian motion. In the jump part, the number of jumps at time t, or N_t, is a Poisson process with intensity λ, and the jump sizes Y_i are independently and identically distributed as $\log(1+Y_i) \sim N(\gamma, \delta^2)$. Here, $N(\gamma, \delta^2)$ denotes the normal distribution with mean γ and variance δ^2. Recall that $\phi(\cdot)$ denotes the pdf of the standard normal distribution. Finally, $P(\mu)$ stands for the Poisson distribution with mean μ.

To calculate the Greeks of the Asian option, we assume the monitoring times are evenly spaced at $\Delta t = T/m$; hence $t_i = i\Delta t$, $i = 1, 2, \ldots, m$. For convenience, define

$$\begin{aligned} Z &= (Z_1, \ldots, Z_m), \\ N &= (N_1, \ldots, N_m), \\ Y &= (Y_{1,1}, \ldots, Y_{1,N_1}, Y_{2,1}, \ldots, Y_{2,N_2}, \ldots, Y_{m,N_m}). \end{aligned}$$

We simulate a path of stock prices through times $t_1, t_2, \ldots, t_m$ by

$$S_{t_i} := S_{t_i}(S_0, \sigma; Z, N, Y) = S_0 \mathrm{e}^{\gamma_{t_i}(\sigma; Z, N, Y)} \tag{22}$$

where

$$\begin{aligned} \gamma_{t_i} &:= \gamma_{t_i}(\sigma; Z, N, Y) \\ &= \left((r - \lambda\eta - \sigma^2/2)\, t_i + \sigma\sqrt{\Delta t}\sum_{j=1}^{i} Z_j \right) \qquad (23) \\ &\quad + \sum_{j=1}^{i}\sum_{k_j=0}^{N_j} \log(1 + Y_{j,k_j}), \\ Z_j &\overset{\text{i.i.d.}}{\sim} N(0,1), \\ N_j &\overset{\text{i.i.d.}}{\sim} P(\lambda\Delta t), \\ \log(1 + Y_{j,k_j}) &\overset{\text{i.i.d.}}{\sim} N(\gamma, \delta^2). \end{aligned}$$

Recall that Z, N, and Y are assumed to be mutually independent.

Let Ω denote the range of (Z, N, Y). Note that the support equals

$$\begin{aligned} &\{(Z, N, Y) \in \Omega : \bar{S}(S_0, \sigma; Z, N, Y) > K\} \\ &\quad = \{(Z, N, Y) \in \Omega : Z_m > \chi(S_0, \sigma; Z_{-m}, N, Y)\}, \end{aligned}$$

where

$$\begin{aligned} &\chi(S_0, \sigma; Z_{-m}, N, Y) \\ &\quad = \frac{\log\left(\frac{mK - \sum_{i=1}^{m-1} S_{t_i}}{S_0 \prod_{j=1}^{m}\prod_{k=0}^{N_j}(1+Y_{j,k_j})}\right) - \left(r - \lambda\eta - \frac{\sigma^2}{2}\right) t_m}{\sigma\sqrt{\Delta t}} - \sum_{i=1}^{m-1} Z_i. \end{aligned} \tag{24}$$

Standard calculation yields

$$\begin{aligned} \frac{\partial\gamma_{t_i}(\sigma; Z, N, Y)}{\partial\sigma} &= \sqrt{\Delta t}\left(\sum_{j=1}^{i} Z_j\right) - \sigma t_i, \\ \frac{\partial\chi(S_0, \sigma; Z_{-m}, N, Y)}{\partial S_0} &= -\frac{mK}{S_0\sigma\sqrt{\Delta t}\left(mK - \sum_{i=1}^{m-1} S_{t_i}\right)}. \end{aligned}$$

Applying Corollary 3.2, the Delta, Gamma and Vega of the Asian call option under Merton's jump-diffusion model equal, respectively,

$$\begin{aligned} \Delta &= e^{-rT} E\left[\frac{\bar{S}}{S_0} I_{\{\bar{S}>K\}}\right], \\ \Gamma &= e^{-rT}\frac{mK^2}{S_0^2\sigma\sqrt{\Delta t}} E\left[\frac{\phi(\chi(S_0, \sigma; Z_{-m}, N, Y))}{mK - \sum_{i=1}^{m-1} S_{t_i}}\right], \\ \nu &= e^{-rT} E\left[\left(\frac{1}{m}\sum_{i=1}^{m} S_{t_i}\left(\sqrt{\Delta t}\sum_{j=1}^{i} Z_j - \sigma t_i\right)\right) I_{\{\bar{S}>K\}}\right], \end{aligned}$$

where S_{t_i} is generated by equation (22) and $\chi(S_0, \sigma; Z_{-m}, N, Y)$ is given in equation (24). The formulas reduce to the Greek formulas for the European call option when $m = 1$.

3.3. *The variance-gamma process*

Let W_t be a standard Brownian motion, and Γ_t be a gamma process with mean rate 1 and variance rate v. Then Γ_t has a gamma distribution with shape parameter t/v and scale parameter v written as $\Gamma_t \sim \Gamma(t/v, v)$. Following Madan *et al.* (1998), the risk-neutral process of the stock price is

$$S_t = S_0 \mathrm{e}^{(r+\omega)t + \theta\Gamma_t + \sigma W_{\Gamma_t}}$$

with

$$\omega := \omega(\sigma) = \frac{\log(1 - \theta v - (\sigma^2 v/2))}{v}.$$

To calculate the Greeks of an Asian option, we simulate the stock price through times $t_i = i\Delta t$, where $\Delta t = T/m$ and $i = 1, 2, \ldots, m$, by

$$S_{t_i} := S_{t_i}(S_0, \sigma; Z, G) = S_0 \mathrm{e}^{\gamma_{t_i}(\sigma; Z, G)}, \tag{25}$$

where

$$\gamma_{t_i} := \gamma_{t_i}(\sigma; Z, G) = (r + \omega)t_i + \theta\sum_{j=1}^{i} G_j + \sigma\sum_{j=1}^{i}\sqrt{G_j} Z_j, \tag{26}$$

$$Z_j \overset{\text{i.i.d.}}{\sim} N(0,1), \tag{27}$$

$$G_j \overset{\text{i.i.d.}}{\sim} \Gamma(\Delta t/v, v), \tag{28}$$

For convenience, define $Z = (Z_1, \ldots, Z_m)$ and $G = (G_1, \ldots, G_m)$. Recall that Z and G are independent.

Let Ω denote the range of (Z, G). Note that the support equals

$$\begin{aligned}&\{(Z,G) \in \Omega : \bar{S}(S_0, \sigma; Z, G) > K\}\\ &\quad = \{(Z,G) \in \Omega : Z_m > \chi(S_0, \sigma; Z_{-m}, G)\},\end{aligned}$$

where

$$\chi := \chi(S_0, \sigma; Z_{-m}, G) = \frac{\log\left(\frac{mK - \sum_{i=1}^{m-1} S_{t_i}}{S_0}\right) - (r+\omega)t_m - \left(\sum_{i=1}^{m} \theta G_i\right) - \left(\sum_{i=1}^{m-1} \sigma\sqrt{G_i} Z_i\right)}{\sigma\sqrt{G_m}}. \tag{29}$$

Standard calculation gives

$$\begin{aligned}\frac{\partial \gamma_{t_i}(\sigma; Z, G)}{\partial \sigma} &= \left(\sum_{j=1}^{i} \sqrt{G_j} Z_j\right) - \frac{\sigma t_i}{1 - \theta v - (\sigma^2 v/2)},\\ \frac{\partial \chi(S_0, \sigma; Z_{-m}, G)}{\partial S_0} &= -\frac{mK}{S_0 \sigma \sqrt{G_m}(mK - \sum_{i=1}^{m-1} S_{t_i})}.\end{aligned}$$

Applying Corollary 3.2, the Delta and Gamma of the Asian call option under the variance-gamma process equal, respectively,

$$\begin{aligned}\Delta &= \mathrm{e}^{-rT} E\left[\frac{\bar{S}}{S_0} I_{\{\bar{S}>K\}}\right],\\ \Gamma &= \mathrm{e}^{-rT} \frac{mK^2}{S_0^2 \sigma} E\left[\frac{\phi(\chi(S_0, \sigma; Z_{-m}, G))}{\sqrt{G_m}(mK - \sum_{i=1}^{m-1} S_{t_i})}\right],\end{aligned}$$

where S_{t_i} is generated by equation (25) and $\chi(S_0, \sigma; Z_{-m}, G)$ is given in equation (29). The formulas reduce to the Greek formulas for the European call option when $m = 1$.

3.4. Numerical results

This section compares our method with the central finite-difference method (also abbreviated as FD in this section as there is no ambiguity) for calculating the Deltas, Gammas, and Vegas of European and Asian call options under Merton's jump-diffusion model and the variance-gamma process. Following Glasserman (2004), we estimate the Greeks by FD with common random numbers as follows,

$$\hat{\Delta}_{\mathrm{FD}} = \frac{1}{n} \sum_{i=1}^{n} \frac{\wp(S_0 + h/2, X^{(i)}) - \wp(S_0 - h/2, X^{(i)})}{h},$$

$$\hat{\Gamma}_{\mathrm{FD}} = \frac{1}{n} \sum_{i=1}^{n} \frac{\wp(S_0 + h, X^{(i)}) - 2\wp(S_0, X^{(i)}) + \wp(S_0 - h, X^{(i)})}{h^2},$$

$$\hat{\nu}_{\mathrm{FD}} = \frac{1}{n} \sum_{i=1}^{n} \frac{\wp(\sigma + h/2, X^{(i)}) - c(\sigma - h/2, X^{(i)})}{h},$$

where h is some small perturbation size and $X^{(i)}$ is the ith simulation path for $i = 1, \ldots, n$ with the sample size n set to be 100 000. We use $h = 10^{-1}, 10^{-2}, 10^{-8}$ for the FD-based Greeks. Assessments will be based on the variance of the estimators and computation time.

To obtain accurate benchmarks, we provide the FD-based Greeks based on 1 000 000 repetitions in simulation. For Merton's jump-diffusion model, we set the perturbation size $h = 10^{-2}$ for the Deltas and Vegas and $h = 10^{-1}$ for the Gammas. For the variance-gamma process, we set the perturbation size $h = 10^{-2}$ for the Deltas and $h = 10^{-1}$ for the Gammas.

3.4.1. Accuracy and efficiency. Table 1 compares the Deltas, Gammas, and Vegas of Asian call options together with their variances under Merton's jump-diffusion model for $m = 1$ (yearly), 12 (monthly), 52 (weekly), and 250 (daily). Note that an Asian option degenerates to a European option when $m = 1$. Table 1 also compares Greeks estimates for in-the-money ($K = 36$), at-the-money ($K = 40$), and out-of-the-money options ($K = 44$). In terms of accuracy, the Deltas, Gammas, and Vegas using our method are closest to the benchmarks.

Overall, our method and FD produce similar Deltas and Vagas for all perturbation sizes. But this parallelism breaks down with the Gamma. For $m = 1,2$, our method produces Gammas with the least variances compared with the Gammas using FD. For $m = 52\,250$, our method produces Gammas with moderate scale of variances, but variances of Gammas using FD vary a lot with different purturbation sizes. Indeed, when the perturbation size is very small, FD produces widly unstable Gammas and with large variances. This is a common dilemma FD faces in practice, particularlly for higher-order Greeks: When the perturbation size is large, the Greeks estimates may be biased, whereas when the perturbation size is small, the Greeks estimates are unstable. Lyuu and Teng (2011) reach the same conclusion for spread options on two assets, maximum options on two assets, binary maximum options on two assets, and down-and-out options, all under the much simpler Black-Scholes model.

Table 2 compares the Deltas and Gammas of Asian call options together with their variances under the variance-gamma process for $m = 1,12,52,250$ and $K = 36,40,44$. Again, an Asian option degenerates to a European option when $m = 1$. In terms of accuracy, the Deltas using our method are closest to the benchmarks. The Gammas using our method are also closest to the benchmarks for $m = 1,12$. This incidentally confirms our Gamma formula. For $m = 52\,250$, the Gammas using our method are not close the benchmarks; those using FD with $h = 10^{-1}, 10^{-2}$ are closer. But, FD with $h = 10^{-8}$ produces unstable Gammas. As the Gammas using our method are provably unbiased and have much lower variances in most cases, it is most likely that the Gammas using FD, including the benchmarks, are biased.

Similar conclusion to Merton's jump-diffusion model can be drawn here: Although FD performs comparably with our method on Deltas, it fails for Gammas while perturbations

Table 1. Estimates and variances for the Deltas, Gammas, and Vegas of Asian call options under Merton's jump-diffusion model for $m = 1,12,52,250$ by our method and FD with perturbation size $h = 10^{-1}, 10^{-2}, 10^{-8}$.

			$K=36$		$K=40$		$K=44$	
Greek	m		Estimate	Variance	Estimate	Variance	Estimate	Variance
Delta	1	Benchmark	0.854	0.182	0.652	0.302	0.421	0.320
		Our method	0.853	0.182	0.653	0.302	0.418	0.319
		FD: $h = 10^{-1}$	0.853	0.182	0.653	0.301	0.418	0.318
		FD: $h = 10^{-2}$	0.853	0.182	0.653	0.302	0.418	0.318
		FD: $h = 10^{-8}$	0.853	0.182	0.653	0.302	0.418	0.319
	12	Benchmark	0.898	0.093	0.610	0.265	0.253	0.218
		Our method	0.900	0.092	0.609	0.266	0.253	0.217
		FD: $h = 10^{-1}$	0.900	0.091	0.609	0.264	0.253	0.216
		FD: $h = 10^{-2}$	0.900	0.091	0.609	0.266	0.253	0.217
		FD: $h = 10^{-8}$	0.900	0.092	0.609	0.266	0.253	0.217
	52	Benchmark	0.903	0.085	0.608	0.263	0.236	0.206
		Our method	0.903	0.086	0.609	0.263	0.239	0.208
		FD: $h = 10^{-1}$	0.902	0.085	0.609	0.262	0.239	0.206
		FD: $h = 10^{-2}$	0.903	0.085	0.609	0.263	0.239	0.208
		FD: $h = 10^{-8}$	0.903	0.086	0.609	0.263	0.239	0.208
	250	Benchmark	0.904	0.084	0.606	0.262	0.233	0.203
		Our method	0.905	0.083	0.604	0.263	0.232	0.203
		FD: $h = 10^{-1}$	0.905	0.083	0.604	0.261	0.232	0.202
		FD: $h = 10^{-2}$	0.905	0.083	0.604	0.262	0.232	0.203
		FD: $h = 10^{-8}$	0.905	0.083	0.604	0.263	0.232	0.203
Gamma	1	Benchmark	0.036	0.203	0.058	0.366	0.062	0.430
		Our method	0.036	0.000	0.058	0.000	0.061	0.000
		FD: $h = 10^{-1}$	0.038	0.215	0.060	0.385	0.060	0.415
		FD: $h = 10^{-2}$	0.036	2.041	0.059	3.714	0.064	4.355
		FD: $h = 10^{-8}$	−0.234	1976.440	−0.044	1410.290	−0.045	852.174
	12	Benchmark	0.038	0.214	0.094	0.587	0.080	0.552
		Our method	0.037	0.060	0.096	0.157	0.079	0.138
		FD: $h = 10^{-1}$	0.037	0.215	0.096	0.599	0.078	0.540
		FD: $h = 10^{-2}$	0.041	2.204	0.093	5.835	0.080	5.666
		FD: $h = 10^{-8}$	0.054	10221.300	0.349	7145.550	−0.205	3342.750
	52	Benchmark	0.036	0.204	0.099	0.619	0.081	0.558
		Our method	0.037	0.562	0.096	1.482	0.083	1.386
		FD: $h = 10^{-1}$	0.037	0.207	0.099	0.619	0.081	0.557
		FD: $h = 10^{-2}$	0.035	2.202	0.101	6.415	0.080	5.442
		FD: $h = 10^{-8}$	1.065	34546.500	0.360	25296.500	0.194	10952.600
	250	Benchmark	0.036	0.206	0.099	0.620	0.080	0.551
		Our method	0.033	4.761	0.103	17.365	0.074	12.608
		FD: $h = 10^{-1}$	0.035	0.195	0.102	0.631	0.082	0.566
		FD: $h = 10^{-2}$	0.035	1.991	0.107	6.879	0.081	5.630
		FD: $h = 10^{-8}$	0.271	176037.000	−0.121	121358.000	−1.467	47455.600
Vega	1	Benchmark	9.213	1128.120	14.784	847.750	15.706	808.057
		Our method	9.209	1132.880	14.807	852.877	15.537	796.738
		FD: $h = 10^{-1}$	8.896	1114.240	14.728	853.040	15.469	797.565
		FD: $h = 10^{-2}$	9.209	1129.040	14.807	852.501	15.537	796.684
		FD: $h = 10^{-8}$	9.209	1132.880	14.807	852.877	15.537	796.738
	12	Benchmark	3.557	450.174	9.096	272.304	7.876	271.490
		Our method	3.555	451.016	9.139	268.370	7.883	275.313
		FD: $h = 10^{-1}$	3.454	436.047	9.100	268.578	7.722	267.658
		FD: $h = 10^{-2}$	3.553	449.020	9.139	268.270	7.882	274.782
		FD: $h = 10^{-8}$	3.555	451.016	9.139	268.370	7.883	275.313
	52	Benchmark	3.168	415.456	8.690	245.588	7.245	243.364
		Our method	3.069	413.511	8.816	250.100	7.304	246.397
		FD: $h = 10^{-1}$	2.976	399.221	8.782	250.283	7.122	237.634
		FD: $h = 10^{-2}$	3.066	411.595	8.815	250.048	7.299	245.778
		FD: $h = 10^{-8}$	3.069	413.511	8.816	250.100	7.304	246.397
	250	Benchmark	3.049	407.637	8.611	240.893	7.118	238.898
		Our method	3.076	412.308	8.513	235.772	7.081	235.975
		FD: $h = 10^{-1}$	3.000	397.718	8.478	235.942	6.906	226.995
		FD: $h = 10^{-2}$	3.079	410.300	8.512	235.715	7.080	235.337
		FD: $h = 10^{-8}$	3.076	412.308	8.513	235.772	7.081	235.975

Notes: Parameters: $S_0 = 40$, $r = 0.05$, $\sigma = 0.16$, $T = 1$ (year), $\delta = \sqrt{0.05}$, $\gamma = -0.025$, $\lambda = 1$, and $K = 36,40,44$. Sample size for the Monte Carlo simulation is 100 000. The benchmarks use $h = 10^{-2}$ for the Deltas and $h = 10^{-1}$ for the Gammas with 1 000 000 repetitions.

Table 2. Estimates and variances for the Deltas and Gammas of Asian call options under the variance-gamma process for $m = 1,12,52,250$ by our method and FD with perturbation size $h = 10^{-1}, 10^{-2}, 10^{-8}$.

			$K = 36$		$K = 40$		$K = 44$	
			Estimate	Variance	Estimate	Variance	Estimate	Variance
Delta	1	Benchmark	0.725	0.984	0.663	1.022	0.604	1.038
		Our method	0.723	0.963	0.660	1.011	0.604	1.041
		FD: $h = 10^{-1}$	0.723	0.962	0.660	1.011	0.604	1.041
		FD: $h = 10^{-2}$	0.723	0.962	0.660	1.011	0.604	1.041
		FD: $h = 10^{-8}$	0.723	0.963	0.660	1.011	0.604	1.041
	12	Benchmark	0.673	0.483	0.564	0.526	0.465	0.531
		Our method	0.672	0.483	0.562	0.525	0.464	0.528
		FD: $h = 10^{-1}$	0.672	0.483	0.562	0.525	0.464	0.527
		FD: $h = 10^{-2}$	0.672	0.483	0.562	0.525	0.464	0.527
		FD: $h = 10^{-8}$	0.672	0.483	0.562	0.525	0.464	0.528
	52	Benchmark	0.670	0.456	0.555	0.503	0.454	0.507
		Our method	0.677	0.464	0.555	0.502	0.453	0.508
		FD: $h = 10^{-1}$	0.677	0.463	0.555	0.502	0.453	0.507
		FD: $h = 10^{-2}$	0.677	0.464	0.555	0.502	0.453	0.507
		FD: $h = 10^{-8}$	0.677	0.464	0.555	0.502	0.453	0.508
	250	Benchmark	0.671	0.453	0.553	0.494	0.451	0.502
		Our method	0.670	0.456	0.556	0.562	0.448	0.494
		FD: $h = 10^{-1}$	0.670	0.455	0.556	0.561	0.448	0.493
		FD: $h = 10^{-2}$	0.670	0.456	0.556	0.562	0.448	0.494
		FD: $h = 10^{-8}$	0.670	0.456	0.556	0.562	0.448	0.494
Gamma	1	Benchmark	0.014	0.073	0.015	0.092	0.016	0.103
		Our method	0.014	0.000	0.015	0.000	0.016	0.000
		FD: $h = 10^{-1}$	0.014	0.074	0.015	0.090	0.014	0.095
		FD: $h = 10^{-2}$	0.010	0.506	0.015	0.864	0.014	0.892
		FD: $h = 10^{-8}$	− 0.035	4571.740	− 0.079	4382.300	0.151	3871.210
	12	Benchmark	0.025	0.135	0.026	0.155	0.024	0.158
		Our method	0.024	0.049	0.029	0.530	0.024	0.064
		FD: $h = 10^{-1}$	0.024	0.131	0.027	0.170	0.023	0.153
		FD: $h = 10^{-2}$	0.026	1.337	0.033	1.988	0.023	1.500
		FD: $h = 10^{-8}$	0.269	11209.700	− 0.706	9756.810	0.094	8616.920
	52	Benchmark	0.027	0.144	0.028	0.169	0.025	0.165
		Our method	0.013	0.376	0.013	0.176	0.014	0.449
		FD: $h = 10^{-1}$	0.025	0.132	0.027	0.165	0.026	0.169
		FD: $h = 10^{-2}$	0.023	1.185	0.033	2.024	0.026	1.696
		FD: $h = 10^{-8}$	− 0.651	33710.300	1.192	29536.900	− 0.347	25589.100
	250	Benchmark	0.027	0.144	0.028	0.168	0.026	0.171
		Our method	0.005	0.525	0.004	0.480	0.004	0.792
		FD: $h = 10^{-1}$	0.025	0.131	0.030	0.175	0.026	0.169
		FD: $h = 10^{-2}$	0.023	1.179	0.025	1.515	0.027	1.864
		FD: $h = 10^{-8}$	− 0.467	149486.000	− 2.569	131136.000	− 1.043	107070.000

Notes: Parameters: $S_0 = 40$, $\theta = r = 0.1$, $\sigma = \sqrt{0.4}$, $T = 1$ (year), $\nu = 0.25$, and $K = 36,40,44$. Sample size for the Monte Carlo simulation is 100 000. The benchmarks use $h = 10^{-2}$ for the Deltas and $h = 10^{-1}$ for the Gammas with 1 000 000 repetitions.

size are small. In contrast, our method produces very stable Gammas throughout. A major challenge in calculating the Greeks is about higher-order ones like Gammas, as they have a discontinuous integrand. The intuition behind the better performance of our method lies in its explicit formula for differentiating an integral with a discontinuous integrand.

3.4.2. Computation time. Figures 1 and 2 compare the computation times for calculating the Deltas, Gammas, and Vegas (while applicable) of Asian call options under Merton's jump-diffusion model and the variance-gamma process for $m = 1,12,52,250$. With Merton's jump-diffusion model, our method and FD consume comparable computation time for the Delta, Gamma, and Vega for all perturbation sizes except that our method is reliably faster at $m = 250$. With the variance-gamma process, our method and FD expend comparable computation time.

4. Greeks of credit derivatives: CDOs

Credit derivatives are financial products linked with default events. They have generated tremendous interest in the academia and the financial industry alike (Schönbucher 2003). Risk management for credit derivatives is therefore crucial.

For an illustration of the broad applicability of our method, we calculate the Delta and Gamma of a COD, which is a security backed by a pool of defaultable assets. After reviewing the Gaussian copula model, this section derives the Delta and Gamma of a CDO using our method, FD, and LRM. This section ends with a numerical evaluation of the three methods.

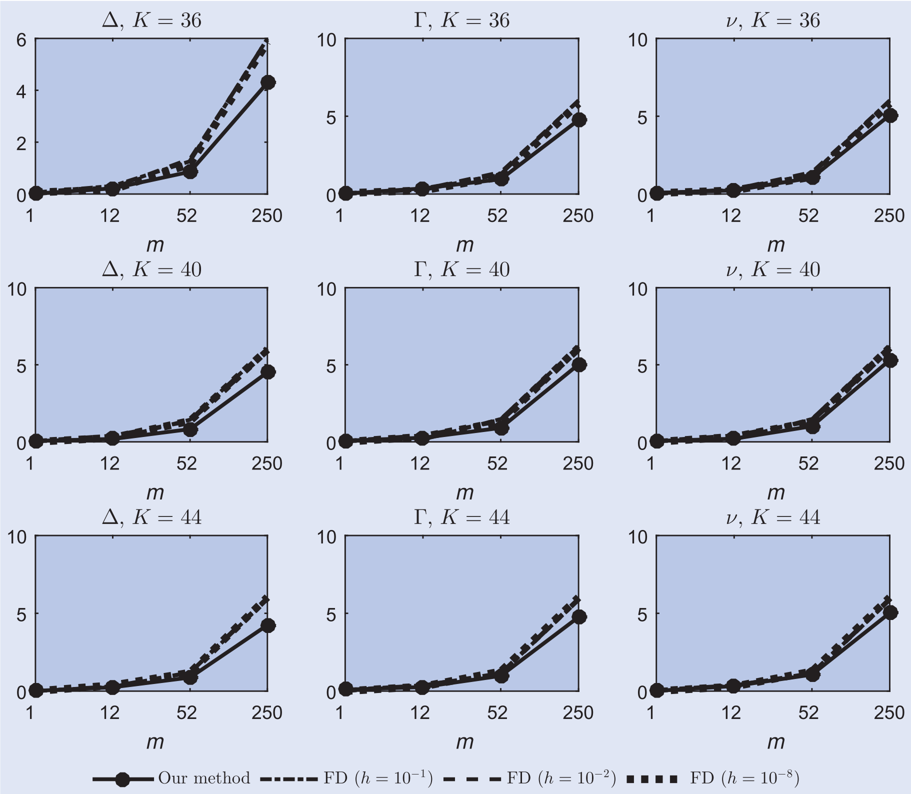

Figure 1. Computation times for Deltas, Gammas, and Vegas of Asian call options under Merton's model for $m = 1,12,52,250$ by our method and FD with perturbation size $h = 10^{-1}, 10^{-2}, 10^{-8}$. Parameters: $S_0 = 40$, $r = 0.05$, $\sigma = 0.16$, $T = 1$ (year), $\delta = \sqrt{0.05}$, $\gamma = -0.025$, $\lambda = 1$, and $K = 36,40,44$. Sample size for the Monte Carlo simulation is 100 000.

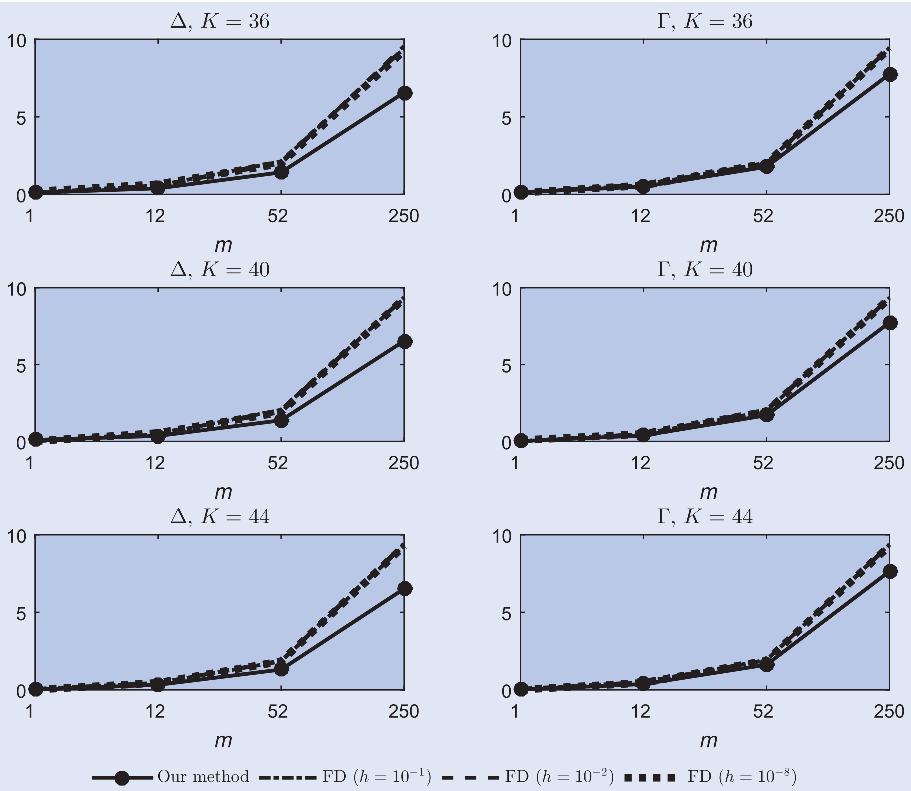

Figure 2. Computation times for Deltas and Gammas of Asian call options under the variance-gamma process for $m = 1,12,52,250$ by our method and FD with with perturbation size $h = 10^{-1}, 10^{-2}, 10^{-8}$. Parameters under the variance-gamma process: $S_0 = 40$, $\theta = r = 0.1$, $\sigma = \sqrt{0.4}$, $T = 1$ (year), $\nu = 0.25$, and $K = 36,40,44$. Sample size for the Monte Carlo simulation is 100 000.

4.1. The Gaussian copula model for pricing credit derivatives

Let $v = (v_1, \ldots, v_N)'$. The Gaussian copula with a positive definite correlation matrix Σ is

$$C_\Sigma(v) = \int_{-\infty}^{\Phi^{-1}(v_1)} \cdots \int_{-\infty}^{\Phi^{-1}(v_N)} \frac{1}{\sqrt{(2\pi)^N |\Sigma|}} \exp \times \left(-\frac{v'\Sigma^{-1}v}{2}\right) \mathrm{d}v_1 \ldots \mathrm{d}v_N, \tag{30}$$

where $\Phi(\cdot)$ is the cdf of the standard Gaussian distribution and $\Phi^{-1}(\cdot)$ its inverse function. Differentiate equation (30) with respect to $v_1, \ldots, v_N$ to obtain the Gaussian copula density function:

$$c_\Sigma(v) = \frac{1}{|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(\Phi^{-1}(v_1), \ldots, \Phi^{-1}(v_N))\Lambda \times (\Phi^{-1}(v_1), \ldots, \Phi^{-1}(v_N))'\right),$$

where

$$\Lambda = \Sigma^{-1} - \mathbf{I}$$

and $\mathbf{I}$ is the identity matrix. The following key fact holds for the Gaussian copula: If a random vector $V = (V_1, \ldots, V_N)'$ has the joint cdf in equation (30) so that

$$P(V_1 \le v_1, \ldots, V_N \le v_N) = C_\Sigma(v),$$

then it has the following stochastic representation (McNeil *et al.* 2005),

$$V \sim (\Phi(X_1), \ldots, \Phi(X_N)), \quad \text{where } X \sim N(0, \Sigma).$$

Following Li (2000), we employ the Gaussian copula to build a joint distribution for the default times with given marginal distributions. Let there be N credit entities with $\tau = (\tau_1, \ldots, \tau_N)'$ denoting their default times. Assume the default time of the ith credit entity is exponentially distributed with default rate h_i; equivalently, τ_i has the exponential distribution with rate h_i, or simply $\tau_i \sim \mathrm{Exp}(h_i)$. This means τ_i has the cdf

$$F_i(s_i, h_i) = 1 - \mathrm{e}^{-h_i s_i}.$$

The inverse function of $F_i(s_i, h_i)$ is

$$F_i^{-1}(\omega, h_i) = -\frac{\log(1-\omega)}{h_i}.$$

The joint cdf of the default times $\tau = (\tau_1, \ldots, \tau_N)'$ via the Gaussian copula is

$$\begin{aligned} &P(\tau_1 \le s_1, \ldots, \tau_N \le s_N) \\ &\quad = C_\Sigma((F_1(s_1, h_1), \ldots, F_N(s_N, h_N))'). \end{aligned} \tag{31}$$

By change of variables, the τ defined in equation (31) has this stochastic representation,

$$\begin{aligned} \tau &\sim \left(F_1^{-1}(\Phi(X_1), h_1), \ldots, F_N^{-1}(\Phi(X_N), h_N)\right) \\ &= \left(-\frac{\log(1-\Phi(X_1))}{h_1}, \ldots, -\frac{\log(1-\Phi(X_N))}{h_N}\right), \end{aligned} \tag{32}$$

where $X \sim N(0, \Sigma)$. The above equation links the default times to the multi-dimensional normal random vector.

4.2. An example CDO

Following Chen and Glasserman (2008), this section considers a CDO such that loss on the pool is tranched and sold to investors. In this CDO, lower seniority tranches act as protection for higher seniority tranches. A tranche of a CDO absorbs losses from the attachment point S_l through the detachment point S_u. The cashflows of this tranche are generated as follows. At coupon dates $0 < T_1 < \cdots < T_m \le T$, the tranche holder receives payment proportional to the notional principal remaining in the tranche. If there are default losses in the portfolio, the tranche covers the cumulative portfolio loss in excess of S_l but only up to S_u. For simplicity, we assume the net default payments occur only at coupon dates and the notional principal is \$1. When the ith credit entity defaults, it causes a loss of $l_i = 1 - r_i$ in the portfolio, where l_i is the loss given default (LGD) of the ith asset and r_i is the recovery rate of that asset.

Define $h = (h_1, \ldots, h_N)$, where we recall that $\tau_i \sim \mathrm{Exp}(h_i)$. Let $L(t, h; \tau)$ be the cumulative loss on the portfolio at time $t \le T$,

$$L(t, h; \tau) = \sum_{i=1}^N l_i I_{\{\tau_i \le t\}},$$

and $M(t, h; \tau)$ be the cumulative loss on the tranche with attachment point S_l and detachment point S_u at time t,

$$M(t, h; \tau) = (L(t, h; \tau) - S_l)^+ - (L(t, h; \tau) - S_u)^+.$$

Let D_j represent the discount factor for time T_j, $j = 1, \ldots, m$. For simplicity, assume $D_j = \exp(-rT_j)$ with r being the interest rate. Note that D_j are independent of our parameters of interest $h_1, h_2, \ldots, h_N$. The discounted payoff of the default payment leg and the premium payment leg are, respectively,

$$V_\mathrm{d}(h; \tau) = \sum_{j=1}^m D_j \left(M(T_j, h; \tau) - M(T_{j-1}, h; \tau)\right), \tag{33}$$

$$V_\mathrm{p}(h; \tau) = \kappa \sum_{j=1}^m D_j (S_u - S_l - M(T_j, h; \tau)), \tag{34}$$

where κ denotes the coupon rate of this tranche. The discounted payoff of a CDO tranche is the difference between the discounted payoffs of the default payment leg and the premium payment leg,

$$\begin{aligned} V(h; \tau) &= V_\mathrm{d}(h; \tau) - V_\mathrm{p}(h; \tau), \\ &= (1+\kappa) \sum_{j=1}^m D_j M(T_j, h; \tau) - \sum_{j=1}^{m-1} D_{j+1} M(T_j, h; \tau) \\ &\quad - \kappa (S_u - S_l) \sum_{j=1}^m D_j. \end{aligned} \tag{35}$$

4.3. Calculating the Greeks of a CDO tranche using our method

Let c denote the value of a CDO tranche. Then

$$c = E[V(h;\tau)],$$

We are interested in calculating its Delta and Gamma, which are the sensitivities of c with respect to h_i, i.e.

$$\Delta_i = \frac{\partial c}{\partial h_i}, \quad \Gamma_{ii} = \frac{\partial^2 c}{\partial h_i^2}.$$

Let B be an $N \times N$ matrix. The $(N-1) \times (N-1)$ matrix $B_{-[i,i]}$ equals B after its ith row and ith column are removed. The following lemma will be useful later.

LEMMA 4.1 *If B is positive definite, then $B_{-[i,i]}$ is positive definite.*

Proof of Lemma 4.1 See Appendix 6. ■

By the above lemma, $\Sigma_{-[i,i]}$ is positive definite when Σ is. The next theorem summarizes the Greek formulas for the CDO tranche.

COROLLARY 4.2 *The Deltas and Gammas of the CDO tranche having the discounted payoff $V(h;\tau)$ in equation* (35) *with default times τ modeled by equation* (32) *equal, respectively,*

$$\Delta_i = (1+\kappa)\sum_{j=1}^{m} D_j a(T_j, h; \tau) - \sum_{j=1}^{m-1} D_{j+1} a(T_j, h; \tau), \quad (36)$$

$$\Gamma_{ii} = (1+\kappa)\sum_{j=1}^{m} D_j b(T_j, h; \tau) - \sum_{j=1}^{m-1} D_{j+1} b(T_j, h; \tau), \quad (37)$$

where

$$\begin{aligned}
a(t,h;\tau) &= t\mathrm{e}^{-h_i t} E\left[A(t, h_{-i}; U_{-i}) w(t, h_i; U_{-i})\right], \\
b(t,h;\tau) &= -\sqrt{2\pi}\, t\mathrm{e}^{-h_i t + U_i^2/2} E \\
&\quad \times \left[\left(\sum_{k=1}^{N} \Phi^{-1}(U_k)\Lambda_{ik}\right) A(t, h_{-i}; U_{-i})\right. \\
&\qquad \left. w(t, h_i; U_{-i})\right] - t a(t,h;\tau),
\end{aligned}$$

$$\begin{aligned}
A(t, h_{-i}; U_{-i}) &= (l_i + \alpha(t) - S_l) I_{\{S_l - l_i < \alpha(t) \le S_l\}} + l_i I_{\{S_l < \alpha(t)\}} \\
&\quad - (l_i + \alpha(t) - S_u) I_{\{S_u - l_i < \alpha(t) \le S_u\}} - l_i I_{\{S_u < \alpha(t)\}}, \\
w(t, h_i; U_{-i}) &= \frac{c(U;\Sigma)}{c(U_{-i}; \Sigma_{-[i,i]})}, \\
\alpha(t) &= \sum_{k \in \{1,\dots,N\} \setminus \{i\}} l_k I_{\{\tau_k \le t\}}.
\end{aligned}$$

Above, $U = (U_1, \dots, U_N)'$, where $U_i = 1 - \mathrm{e}^{-h_i t}$, and U_{-i} is an $(N-1)$-dimensional random vector following a Gaussian copula with the correlation matrix $\Sigma_{-[i,i]}$ for $i = 1, 2, \dots, N$.

Proof of Corollary 4.2 See Appendix 7. ■

4.4. Numerical results

Consider a CDO backed by a pool of $N = 10$ underlying assets and with a maturity of 5 years. We will concentrate on the tranche 3%–10%. Assume the coupon rate of this tranche is 3% paid quarterly and in arrears. The risk-free rate is $r = 5\%$, and the default rates h_i are [0.1, 0.02, 0.015, 0.025, 0.1, 0.3, 0.01, 0.25, 0.15, 0.03]. The recovery rates are all set to be 0.3. Finally, we focus on the one-factor correlation matrix $\Sigma = [\Sigma_{ij}]$ with $\Sigma_{ii} = 1$ and $\Sigma_{ij} = \rho$ for $i \neq j$. Hence ρ is the single parameter that determines the correlation matrix Σ. Specifically, we consider two one-factor correlation matrices: $\rho = 0$ for the independent default times and $\rho = 0.5$ for correlated default times.

All Monte Carlo simulation results are based on 100 000 replications. Assessments are made for the Deltas and Gammas using our method, FD, and LRM with perturbation size $h = 0.01$. The criteria are the variance of the estimators and computation time. To obtain accurate benchmarks, we provide the FD-based Greeks based on 5 000 000 repetitions in simulation and set the perturbation size $h = 0.01$ for the Deltas and Gammas.

The Delta and Gamma using LRM are provided for completeness in the following corollary.

COROLLARY 4.3 *Let $V(h;\tau)$ be the discounted payoff of the CDO tranche with default times τ modeled by equation* (32). *The Deltas and Gammas of this CDO tranche using LRM equal, respectively,*

$$\begin{aligned}
\Delta_i &= E\left[V(h;\tau)\left(B(h;\tau) + \frac{1}{h_i} - \tau_i\right)\right], \\
\Gamma_{ii} &= E\left[V(h;\tau)\left(\frac{\partial B(h;\tau)}{\partial h_i} - \frac{1}{h_i^2}\right.\right. \\
&\qquad \left.\left. + \left(B(h;\tau) + \frac{1}{h_i} - \tau_i\right)^2\right)\right],
\end{aligned}$$

where

$$\begin{aligned}
B(h;\tau) &= -\left(\sum_{k=1}^{N} \Phi^{-1}(F_k(\tau_k, h_k))\Lambda_{ik}\right)\sqrt{2\pi}\,\tau_i \exp \\
&\qquad \times \left(-h_i\tau_i + \frac{F_i^2(\tau_i, h_i)}{2}\right), \\
\frac{\partial B(h;\tau)}{\partial h_i} &= -2\pi\tau_i^2 \mathrm{e}^{-2h_i\tau_i + F_i^2(\tau_i,h_i)}\Lambda_{ii} + B(h;\tau) \\
&\qquad \times \left(-\tau_i + F_i(\tau_i, h_i)\tau_i \mathrm{e}^{-h_i\tau_i}\right),
\end{aligned}$$

for $i = 1, 2, \dots, N$.

Proof of Corollary 4.3 See Appendix 8. ■

4.4.1. Accuracy and efficiency. Table 3 compares Deltas together with their variances using FD (with perturbation size $h = 0.01$), LRM, and our method, against various times to maturity with $\rho = 0$, 0.5. In terms of accuracy, the Deltas using our method are closest to the benchmarks. In addition, when $\rho = 0$, the Deltas using our method, FD, and LRM are

Table 3. Estimates and variances for the Deltas of a CDO using our method, FD with perturbation size $h = 0.01$, and LRM, across various times to maturity $T = 1, 1.5, 2, \dots, 5$, and $\rho = 0, 0.5$.

		$\rho = 0$		$\rho = 0.5$	
T	Method	Estimate	Variance	Estimate	Variance
1	Benchmark	0.134	4.895	0.110	4.036
	Our method	0.134	0.025	0.109	0.022
	FD	0.138	5.027	0.107	3.916
	LRM	0.137	2.538	0.119	1.984
1.5	Benchmark	0.118	3.990	0.119	4.109
	Our method	0.119	0.036	0.119	0.030
	FD	0.119	4.020	0.119	4.116
	LRM	0.114	3.137	0.126	2.580
2	Benchmark	0.096	2.938	0.120	3.882
	Our method	0.096	0.037	0.120	0.036
	FD	0.098	3.020	0.122	3.971
	LRM	0.099	3.466	0.144	3.158
2.5	Benchmark	0.075	2.044	0.116	3.525
	Our method	0.075	0.033	0.117	0.041
	FD	0.071	1.905	0.125	3.817
	LRM	0.070	3.761	0.145	3.598
3	Benchmark	0.058	1.316	0.109	3.054
	Our method	0.058	0.026	0.109	0.042
	FD	0.059	1.371	0.107	3.000
	LRM	0.057	3.964	0.146	4.016
3.5	Benchmark	0.047	0.870	0.102	2.626
	Our method	0.047	0.019	0.102	0.042
	FD	0.047	0.895	0.104	2.719
	LRM	0.054	4.064	0.150	4.306
4	Benchmark	0.039	0.549	0.094	2.216
	Our method	0.038	0.014	0.095	0.040
	FD	0.035	0.429	0.090	2.070
	LRM	0.030	4.268	0.140	4.646
4.5	Benchmark	0.033	0.341	0.087	1.852
	Our method	0.034	0.010	0.088	0.038
	FD	0.032	0.326	0.089	1.909
	LRM	0.042	4.335	0.143	4.851
5	Benchmark	0.030	0.213	0.080	1.525
	Our method	0.030	0.007	0.081	0.036
	FD	0.030	0.233	0.083	1.583
	LRM	0.031	4.465	0.137	5.090

Notes: Sample size for the Monte Carlo simulation is 100 000. The benchmarks use $h = 0.01$ with 5 000 000 repetitions.

Table 4. Estimates and variances for the Gammas of a CDO using our method, FD with perturbation size $h = 0.01$, and LRM, across various times to maturity $T = 1, 1.5, \dots, 5$, and $\rho = 0, 0.5$.

		$\rho = 0$		$\rho = 0.5$	
T	Method	Estimate	Variance	Estimate	Variance
1	Benchmark	−0.011	3462.590	0.042	2325.520
	Our method	−0.025	0.001	0.024	0.001
	FD	0.041	3668.930	0.137	2445.940
	LRM	−0.035	212.443	−0.114	83.416
1.5	Benchmark	−0.036	2997.970	0.115	2526.720
	Our method	−0.034	0.003	0.022	0.001
	FD	−0.107	3132.040	0.214	2487.700
	LRM	−0.031	283.578	−0.185	105.985
2	Benchmark	−0.045	2341.530	0.133	2525.890
	Our method	−0.035	0.006	0.018	0.002
	FD	−0.052	2204.680	0.398	2464.150
	LRM	−0.064	260.655	−0.228	125.063
2.5	Benchmark	−0.018	1677.550	0.085	2420.850
	Our method	−0.032	0.009	0.012	0.002
	FD	−0.229	1659.860	0.095	2376.500
	LRM	−0.020	304.798	−0.355	132.070
3	Benchmark	−0.032	1163.100	0.101	2244.170
	Our method	−0.028	0.011	0.006	0.002
	FD	−0.199	1116.780	0.230	2252.850
	LRM	−0.039	278.659	−0.430	146.849
3.5	Benchmark	−0.034	779.101	0.095	2051.640
	Our method	−0.022	0.011	0.000	0.003
	FD	−0.024	827.904	−0.073	1934.210
	LRM	−0.030	278.669	−0.437	150.458
4	Benchmark	−0.014	522.272	0.132	1874.560
	Our method	−0.017	0.010	−0.005	0.003
	FD	0.106	494.807	0.077	1801.550
	LRM	−0.030	270.334	−0.447	157.892
4.5	Benchmark	−0.016	328.548	0.098	1660.300
	Our method	−0.013	0.010	−0.009	0.004
	FD	−0.098	367.921	−0.017	1693.800
	LRM	−0.109	251.019	−0.525	156.665
5	Benchmark	−0.013	212.070	0.110	1506.430
	Our method	−0.009	0.008	−0.013	0.005
	FD	−0.090	208.092	0.034	1507.150
	LRM	−0.013	261.527	−0.419	169.091

Notes: Sample size for the Monte Carlo simulation is 100 000. The benchmarks use $h = 0.01$ with 5 000 000 repetitions.

quite close. However, FD and LRM suffer from much higher variances than our method. When $\rho = 0.5$, the Deltas using LRM are higher than those using FD and our method for longer time to maturity, possibly because the variance of the Delta using LRM becomes large then. Overall, our method outperforms both FD and LRM in terms of variance.

Table 4 compares the Gammas with their variances using our method, FD, and LRM across various times to maturity and ρ. The Greeks using our method are stable as signified by their small variances compared with those using FD and LRM.

4.4.2. Computation time. Figure 3 compares the computation times for calculating the Delta of the CDO using our method, FD, and LRM. It is clear that FD consumes slightly more time than LRM across all times to maturity because of the need to re-evaluate the payoff function. In contrast, our method takes more time than both FD and LRM, particularly for longer times to maturity. This is because it requires the calculation of $A(t, h_{-1}; U_{-1})$ and $w_t(t, h_1; U_{-1})$, which is computation intensive. Hence, compared with FD and LRM, our method takes more time but has lower variances.

A fair assessment of an estimator, however, should consider both its variance and the computation time needed to achieve it (Fishman 1996). We follow L'Ecuyer (1994) by defining the efficiency number of an estimator to be the reciprocal of the product of its mean squared error (MSE) and computation time:

$$\text{efficiency number} = \frac{1}{\text{MSE} \times \text{computation time}}.$$

This number measures the reduction in variance per unit of computation time. An estimator that produces a large efficiency number is thus preferred. For the MSE of an estimator, our method's estimate will be employed as the mean because it is provably unbiased and has the least variance. Figure 4

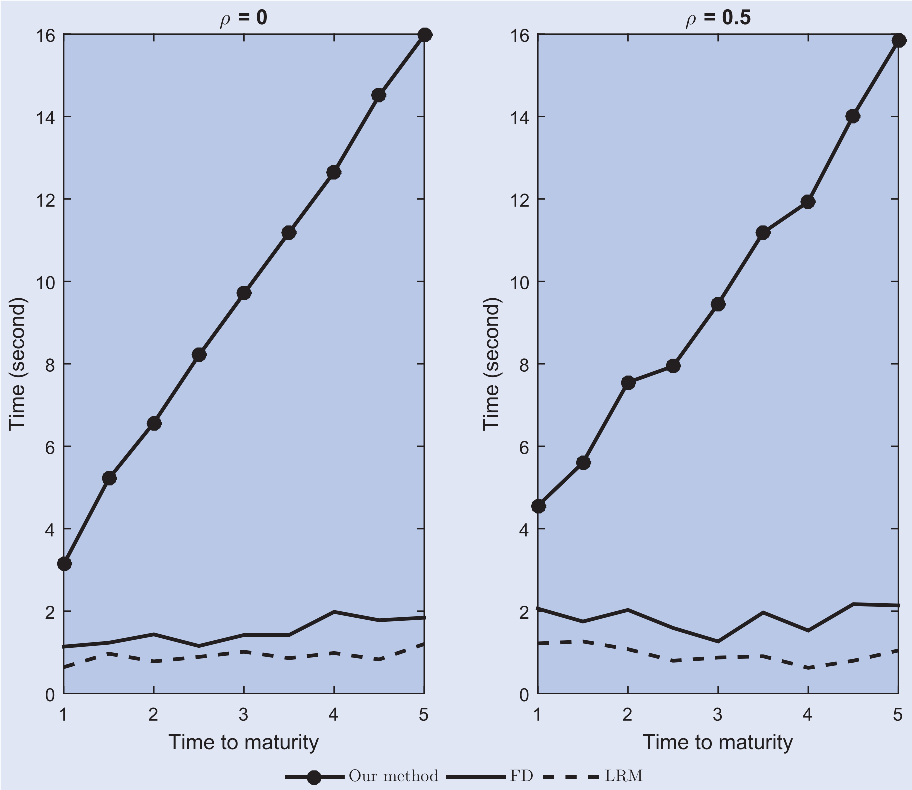

Figure 3. Computation times for Deltas of a CDO using our method, FD, and LRM with perturbation size $h = 0.01$, across various times to maturity and ρ. Sample size for the Monte Carlo simulation is 100 000.

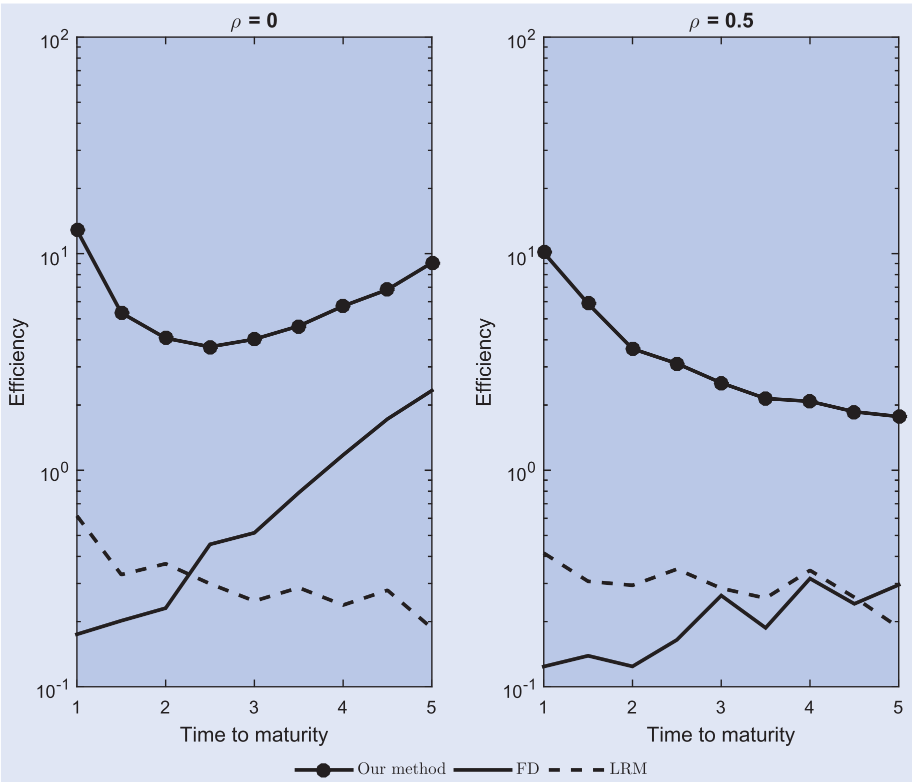

Figure 4. Efficiency numbers for Deltas of a CDO using our method, FD, and LRM with perturbation size $h = 0.01$, across various times to maturity and ρ. Sample size for the Monte Carlo simulation is 100 000.

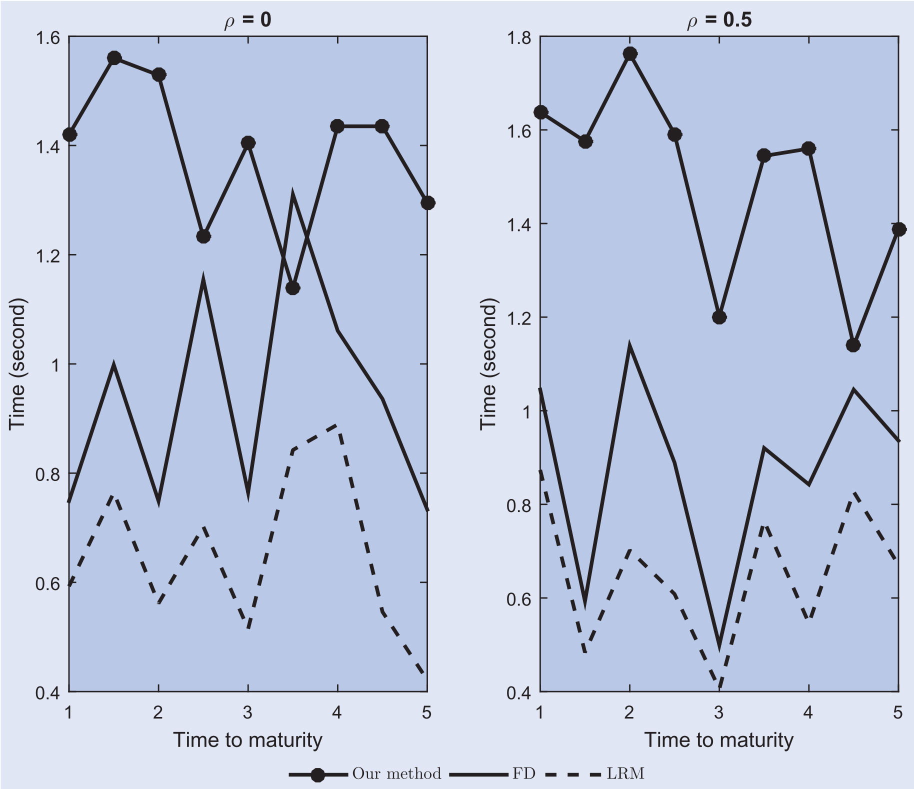

Figure 5. Computation times for Gammas of a CDO using our method, FD, and LRM with perturbation size $h = 0.01$, across various times to maturity and ρ. Sample size for the Monte Carlo simulation is 100 000.

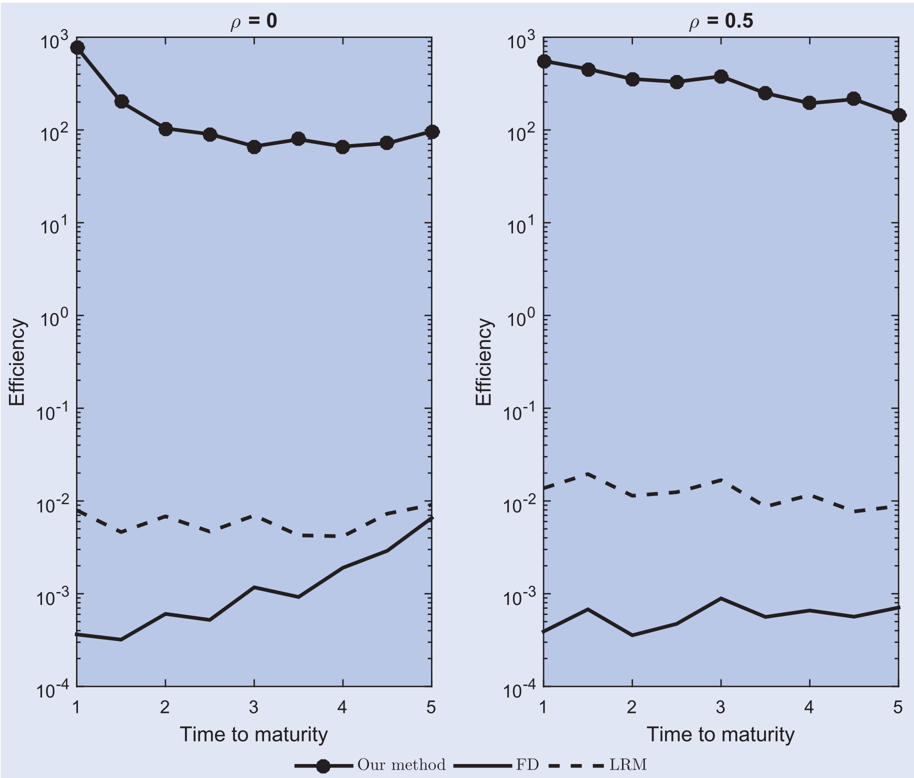

Figure 6. Efficiency numbers for Gammas of a CDO using our method, FD, and LRM with perturbation size $h = 0.01$, across various times to maturity and ρ. Sample size for the Monte Carlo simulation is 100 000.

plots the efficiency numbers for the Deltas of the CDO using our method, FD, and LRM. Based on efficiency numbers, our method emerges as an obvious winner.

Figure 5 compares the computation times for the Gammas of the CDO using our method, FD, and LRM. Our method takes more time than both FD and LRM but only moderately so. Finally, figure 6 compares the efficiency numbers for the Gammas of the CDO using our method, FD, and LRM: Our method now dominates both FD and LRM.

5. Conclusion

The requirement of Lipschitz-continuous payoff functions severely restricts the scope of the standard pathwise method to calculating the Greeks. This paper builds a more general pathwise method on the notion of parameter derivative. And it covers a much broader class of payoff functions and models. As illustrations, the new method yields the unbiased Greeks of (1) European and Asian options under Merton's jump-diffusion model and the variance-gamma process and (2) collateralized debt obligations under the Gaussian copula model. This method is particularly competitive for higher-order Greeks such as Gamma. Our Greeks outperform the finite-difference and likelihood ratio methods in terms of accuracy, variance, and computation time.

Future research along the same line includes the Greeks of path-dependent options for stochastic-volatility models with jumps in return or volatility. Sensitivity analysis of interest rate-sensitive products, Value-at-Risk, and catastrophe options may also benefit from similar treatments (Ahn and Thompson 1988, Cox and Pedersen 2000, Jaimungal and Wang 2006).

Disclosure statement

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Appendices

Appendix 1. Review of the Dirac delta function in the distribution theory

We review the Dirac delta function, the Heaviside function, and the distributional derivative using the theory of distributions (Rudin 1991, Strichartz 2003). Let $C_c^\infty(\Re)$ denote the set of smooth functions with compact support in real numbers. A distribution is a continuous linear functional from $C_c^\infty(\Re)$ to $\Re$. Suppose $q(x):\Re\to\Re$ is a locally integrable function. Its associated distribution, denoted by $\mathcal{T}_q$, is defined as

$$\langle \mathcal{T}_q,\psi\rangle = \int_{\Re} q(x)\psi(x)\,\mathrm{d}x$$

for every test function $\psi(x)\in C_c^\infty(\Re)$. The Heaviside function shifted by a constant $\theta\in\Re$, denoted as H_θ, is the associated distribution of the indicator function $I_{\{x\geq\theta\}}$, in other words,

$$\langle H_\theta,\psi\rangle = \int_{\Re} I_{\{x\geq\theta\}}\psi(x)\,\mathrm{d}x$$

for every test function $\psi(x)\in C_c^\infty(\Re)$.

As a distribution, the Dirac delta function shifted by a constant θ, denoted by δ_θ, is defined as

$$\langle \delta_\theta,\psi\rangle = \psi(\theta) \tag{A1}$$

for every test function $\psi(x)\in C_c^\infty(\Re)$. For a distribution d, its distributional derivative, denoted by d', is a distribution that satisfies

$$\langle d',\psi\rangle = -\langle d,\psi'\rangle$$

for every test function $\psi(x)\in C_c^\infty(\Re)$. The distributional derivative of the Heaviside function is the Dirac delta function.

For Greeks calculation, we are interested in the differentiation of $c(\theta)$ in equation (1) with respect to θ. Since most probability density functions considered in this paper do not have a compact support, we consider the Schwartz space, denoted by $C^\infty(\Re^m)$, which is the set of smooth functions in $\Re^m$ that are rapidly decreasing at infinity along with all partial derivatives.

A tempered distribution is a continuous linear functional from $C^\infty(\Re^m)$ to $\Re$. Suppose that $q(x):\Re^m\to\Re$ is a locally integrable function. Its associated tempered distribution, denoted by T_q, is defined as

$$\langle T_q,\psi\rangle = \int_{\Re^m} q(x)\psi(x)\,\mathrm{d}x$$

for every test function $\psi(x)\in C^\infty(\Re^m)$. As a tempered distribution, the kth shifted Dirac delta function shifted by a constant θ, denoted by δ_θ^k, is defined as

$$\langle \delta_\theta^k,\psi\rangle = \int_{\Re^{m-1}} \psi(x_1,\ldots,x_{k-1},\theta,x_{k+1},\ldots,x_m)\,\mathrm{d}x_{-k}$$

for every test function $\psi\in C^\infty(\Re^m)$.

Note that $\wp(\theta,x):\Re^m\to\Re$ is a one-parameter family of functions where θ is the parameter of interest. We are now ready to define the parameter derivative of $\wp(\theta,x)$ with respect to θ.

Definition A.1 Suppose that $\wp(\theta,x)$ is a locally integrable function and that $\langle\mathcal{T}_\wp,\psi\rangle$ is differentiable with respect to θ. The parameter derivative of $\wp(\theta,x)$ with respect to θ, denoted as $\mathcal{D}_\theta\wp$, is defined as a tempered distribution that satisfies the following identity,

$$\langle\mathcal{D}_\theta\wp,\psi\rangle = \frac{\partial}{\partial\theta}\langle\mathcal{T}_\wp,\psi\rangle.$$

for every test function $\psi\in C^\infty(\Re^m)$.

Appendix 2. Proof of Theorem 2.1

The payoff function considered inequation (3) is a subset of the payoff functions considered in Lyuu and Teng (2011) in that it requires the existence of explicit closed-form formulas for $\chi_{Li}(\theta,x_{-k})$ and $\chi_{Rj}(\theta,x_{-k})$. After imposing this condition, it can be checked that Assumptions 1 to 4 satisfy the conditions of Theorem 3.2 in Lyuu and Teng (2011). See Appendix 3 for a more formal proof.

Let $f(x)$ be the pdf of x. Apply Theorem 3.2 of Lyuu and Teng (2011) in the case that $g_j(\theta,x)=x_k-\chi_{Li}(\theta,x_{-k})$ and $g_j(\theta,x)=\chi_{Rj}(\theta,x_{-k})-x_k$ to obtain

$$\begin{aligned}
&\frac{\partial}{\partial\theta}\int_{\Re^m} h(\theta,x)\left[\prod_{i\in\mathcal{I}} I_{\{x_k>\chi_{Li}(\theta,x_{-k})\}}\right]\left[\prod_{j\in\mathcal{J}} I_{\{x_k<\chi_{Rj}(\theta,x_{-k})\}}\right] f(x)\,\mathrm{d}x\\
&= \int_{\Re^m} h_\theta(\theta,x)\left[\prod_{i\in\mathcal{I}} I_{\{x_k>\chi_{Li}(\theta,x_{-k})\}}\right]\left[\prod_{j\in\mathcal{J}} I_{\{x_k<\chi_{Rj}(\theta,x_{-k})\}}\right] f(x)\,\mathrm{d}x\\
&\quad+\sum_{l\in\mathcal{I}}\int_{\Re^{m-1}}\left[h(\theta,x)J_{Ll}(\theta,x)\left[\prod_{i\in\mathcal{I}\backslash\{l\}} I_{\{x_k>\chi_{Li}(\theta,x_{-k})\}}\right]\right.\\
&\quad\left.\times\left[\prod_{j\in\mathcal{J}} I_{\{x_k<\chi_{Rj}(\theta,x_{-k})\}}\right] f(x)\right]_{x_k=\chi_{Ll}(\theta,x_{-k})}\mathrm{d}x_{-k}
\end{aligned}$$

$$+\sum_{l\in\mathcal{J}}\int_{\Re^{m-1}}\left[h(\theta,x)\left[\prod_{i\in\mathcal{I}}I_{\{x_k>\chi_{Li}(\theta,x_{-k})\}}\right]J_{Rl}(\theta,x)\right.$$
$$\left.\times\left[\prod_{j\in\mathcal{J}\setminus\{l\}}I_{\{x_k<\chi_{Rj}(\theta,x_{-k})\}}\right]f(x)\right]_{x_k=\chi_{Rl}(\theta,x_{-k})}\mathrm{d}x_{-k}$$

where

$$J_{Ll}(\theta,x)=\operatorname{sign}\left(\frac{\partial(x_k-\chi_{Ll}(\theta,x_{-k}))}{\partial x_k}\right)\frac{\partial(x_k-\chi_{Ll}(\theta,x_{-k})/\partial\theta}{\partial(x_k-\chi_{Ll}(\theta,x_{-k})/\partial x_k}$$
$$=(+1)\frac{-\chi_{Ll,\theta}(\theta,x_{-k})}{1}=-\chi_{Ll,\theta}(\theta,x_{-k}),$$
$$J_{Rl}(\theta,x)=\operatorname{sign}\left(\frac{\partial(\chi_{Rl}(\theta,x_{-k}-x_k)}{\partial x_k}\right)\frac{\partial(\chi_{Rl}(\theta,x_{-k}-x_k)/\partial\theta}{\partial(\chi_{Rl}(\theta,x_{-k}-x_k)/\partial x_k}$$
$$=(-1)\frac{\chi_{Rl,\theta}(\theta,x_{-k})}{-1}=\chi_{Rl,\theta}(\theta,x_{-k}).$$

Finally, by the shifting property of the Dirac delta function,

$$\int_{\Re^{m-1}}\left[h(\theta,x)(-\chi_{Ll,\theta}(\theta,x_{-k}))\left[\prod_{i\in\mathcal{I}\setminus\{l\}}I_{\{x_k>\chi_{Li}(\theta,x_{-k})\}}\right]\right.$$
$$\left.\times\left[\prod_{j\in\mathcal{J}}I_{\{x_k<\chi_{Rj}(\theta,x_{-k})\}}\right]f(x)\right]_{x_k=\chi_{Ll}(\theta,x_{-k})}\mathrm{d}x_{-k}$$
$$=-\int_{\Re^m}\left[h(\theta,x)\left[\prod_{i\in\mathcal{I}\setminus\{l\}}I_{\{x_k>\chi_{Li}(\theta,x_{-k})\}}\right]\right.$$
$$\left.\times\left[\prod_{j\in\mathcal{J}}I_{\{x_k<\chi_{Rj}(\theta,x_{-k})\}}\right]f(x)\right]\delta^k_{\chi_{Ll}(\theta,x_{-k})}(x)\chi_{Ll,\theta}(\theta,x_{-k})\,\mathrm{d}x,$$

and

$$\int_{\Re^{m-1}}\left[h(\theta,x)\left[\prod_{i\in\mathcal{I}}I_{\{x_k>\chi_{Li}(\theta,x_{-k})\}}\right]\chi_{Rl,\theta}(\theta,x_{-k})\right.$$
$$\left.\times\left[\prod_{j\in\mathcal{J}\setminus\{l\}}I_{\{x_k<\chi_{Rj}(\theta,x_{-k})\}}\right]f(x)\right]_{x_k=\chi_{Rl}(\theta,x_{-k})}\mathrm{d}x_{-k}$$
$$=\int_{\Re^m}\left[h(\theta,x)\left[\prod_{i\in\mathcal{I}}I_{\{x_k>\chi_{Li}(\theta,x_{-k})\}}\right]\right.$$
$$\left.\times\left[\prod_{j\in\mathcal{J}\setminus\{l\}}I_{\{x_k<\chi_{Rj}(\theta,x_{-k})\}}\right]f(x)\right]$$
$$\delta^k_{\chi_{Rl}(\theta,x_{-k})}(x)\chi_{Rl,\theta}(\theta,x_{-k})\,\mathrm{d}x.$$

we obtain equation (6).

Appendix 3. Checking the regularity conditions of Theorem 3.2 in Lyuu and Teng (2011).

Here, we explain why the three regularity conditions of Theorem 3.2 in Lyuu and Teng (2011) are satisfied by Theorem 2.1. To avoid confusion, whenever there is discrepancy in notations, we follow the notations used in this paper.

First, we recall the needed definitions from Lyuu and Teng (2011). A function $h(\theta,x)\in\mathcal{H}_k$ if the following properties are satisfied:

(i) $h(\theta,x)$ is θ-differentiable;
(ii) $\int|h(\theta,x)|f(x)\,\mathrm{d}x<\infty$ and $\int|h_\theta(\theta,x)|f(x)\,\mathrm{d}x<\infty$;
(iii) $h(\theta,x)f(x)$ and $h_\theta(\theta,x)f(x)$ are uniformly continuous with respect to θ and x_k on a compact set.

A function $g(\theta,x)\in\mathcal{G}_k$ if the following properties are satisfied:

(i) $g(\theta,x)$ is θ-differentiable and $\partial g(\theta,x)/\partial\theta$ is continuous in θ;
(ii) $g(\theta,x)$ is x_k-differentiable and $\partial g(\theta,x)/\partial x_k$ is continuous in x_k;
(iii) $g(\theta,x)$ is strictly monotone in x_k;
(iv) there exists a point for x_k depending on θ and x_k, written as $\chi(\theta,x_{-k})$, such that

$$g(\theta,x)|_{x_k=\chi(\theta,x_{-k})}$$
$$:=g(\theta,x_1,\ldots,x_{k-1},\chi(\theta,x_{-k}),x_{k+1},\ldots,x_n)=0.$$

Finally, recall that Theorem 3.2 of Lyuu and Teng (2011) requires the payoff function be of the following form:

$$\wp(\theta,x)=h(\theta,x)\prod_{j\in\mathcal{B}}I_{\{g_j(\theta,x)>0\}},\tag{A2}$$

where $\mathcal{B}$ is a finite set of natural numbers, $h(\theta,x)\in\mathcal{H}_k$, and $g(\theta,x)\in\mathcal{G}_k$.

This paper considers payoff functions of the following form:

$$\wp(\theta,x)=h(\theta,x)\left[\prod_{i\in\mathcal{I}}I_{\{x_k>\chi_{Li}(\theta,x_{-k})\}}\right]\left[\prod_{j\in\mathcal{J}}I_{\{x_k<\chi_{Rj}(\theta,x_{-k})\}}\right].\tag{A3}$$

First, note that Assumptions 1 to 3 ensure that $h(\theta,x)\in\mathcal{H}_k$. Second, the form of the payoff function in equation (.3) together with Assumption 4 ensure that $g_j(\theta,x)\in\mathcal{G}_k$.

Appendix 4. Proof of Corollary 3.1

Let $f(x)$ be the pdf of x under the pricing measure. The European call option has a price of

$$\int(S_T(S_0,\sigma;x)-K)I_{\{S_T(S_0,\sigma;x)>K\}}f(x)\,\mathrm{d}x$$
$$=\int(S_T(S_0,\sigma;x)-K)I_{\{x_k>\chi(S_0,\sigma;x_{-k}\}}f(x)\,\mathrm{d}x.$$

The following identities are straightforward:

$$\frac{\partial S_T}{\partial S_0}=\mathrm{e}^{\gamma_T(\sigma;X)}=\frac{S_T}{S_0},\quad\frac{\partial^2S_T}{\partial S_0^2}=0,\quad\frac{\partial S_T}{\partial\sigma}=S_T\frac{\partial\gamma_T(\sigma;X)}{\partial\sigma}.$$

Apply Theorem 2.1 to yield the desired results.

Appendix 5. Proof of Corollary 3.2

Let $f(x)$ be the pdf of x under the pricing measure. The Asian call option has a price of

$$\int(\bar S(S_0,\sigma;x)-K)I_{\{\bar S(S_0,\sigma;x)>K\}}f(x)\,\mathrm{d}x$$
$$=\int(\bar S(S_0,\sigma;x)-K)I_{\{x_k>\chi(S_0,\sigma;x_{-k}\}}f(x)\,\mathrm{d}x.$$

The following identities are straightforward:

$$\frac{\partial\bar S}{\partial S_0}=\frac{\bar S}{S_0},\quad\frac{\partial^2\bar S}{\partial S_0^2}=0,\quad\frac{\partial\bar S}{\partial\sigma}=\frac{1}{m}\sum_{i=1}^m S_{t_i}\frac{\partial\gamma_{t_i}(\sigma;X)}{\partial\sigma}.$$

Apply Theorem 2.1 to yield the desired results.

Appendix 6. Proof of Lemma 4.1

Let $\tilde{B}$ be B after moving the ith row to the last row and the ith column to the last column. We only need to prove that $\tilde{B}$ remains positive definite for the following reason. Because the principal minors of $B_{-[i,i]}$ and those of $\tilde{B}_{-[N,N]}$ are identical, all principal minors of $B_{-[i,i]}$ and $\tilde{B}$ will have positive determinants once the claim is proved. Hence, by Sylvester's criterion, $B_{-[i,i]}$ is positive definite (Gilbert 1991).

As $\tilde{B}$ results from a sequence of exchanging adjacent rows and adjacent columns, it suffices to prove the positive definiteness of $\tilde{B}$ resulting from exchanging the ith row with the $(i+1)$th one and the ith column with the $(i+1)$th one. The row-switching transformation and column-switching transformation are the same. Call the transformation R. Then $\tilde{B} = RBR$. Note that R is symmetric and has a full rank. As B is positive definite, $B = L'L$ for some L with a full rank. Now $\tilde{B} = RL'LR = (LR)'(LR)$. Finally, LR has a full rank because both L and R do.

Appendix 7. Proof of Corollary 4.2

To derive the Delta and Gamma of the CDO, it suffices to calculate the first and second partial derivatives of $M(t, h; \tau)$ with respect to h_i because the only terms that involve h_i in equation (35) are the $M(T_j, h; \tau)$s. For ease of notations, define

$$\chi(h_i) = F_i(h_i, t) = 1 - \mathrm{e}^{-h_i t};$$

hence

$$\chi'(h_i) = \frac{\partial \chi(h_i)}{\partial h_i} = t\mathrm{e}^{-h_i t}.$$

As $\alpha(t)$ does not involve h_i, it will be convenient to write

$$L(t, h; \tau) = \sum_{i=1}^{N} l_i I_{\{\tau_i \le t\}} = l_i I_{\{\tau_i \le t\}}(V_i) + \alpha(t).$$

The identity $I_{\{\tau_i \le t\}}(V_i) = I_{\{U_i \le \chi(h_i)\}}(V_i)$ and equation (6) of Theorem 2.1 then imply

$$\frac{\partial L(t, h; \tau)}{\partial h_i} = l_i \delta^i_{\chi(h_i)} \chi'(h_i) = l_i t\mathrm{e}^{-h_i t} \delta^i_{\chi(h_i)}.$$

Define $M_1(t, h; \tau) = (L(t, h; \tau) - S_l) I_{\{L(t,h;\tau) > S_l\}}$ and $M_2(t, h; \tau) = (L(t, h; \tau) - S_u) I_{\{L(t,h;\tau) > S_u\}}$ so that

$$M(t, h; \tau) = M_1(t, h; \tau) - M_2(t, h; \tau).$$

Depending on the value of $\alpha(t)$, $M_1(t, h; \tau)$ equals

$$M_1(t, h; \tau) = \begin{cases} 0, & \text{for } \alpha(t) \le S_l - l_i, \\ (l_i I_{\{\tau \le t\}} + \alpha(t) - S_l) I_{\{\tau_i \le t\}}, & \text{for } S_l - l_i < \alpha(t) \le S_l, \\ l_i I_{\{\tau_i \le t\}} + \alpha(t) - S_l, & \text{for } S_l < \alpha(t). \end{cases}$$

Because $(l_i I_{\{\tau_i \le t\}} + \alpha(t) - S_l) I_{\{\tau_i \le t\}} = (l_i + \alpha(t) - S_l) I_{\{\tau_i \le t\}}$, apply Theorem 2.1 to obtain

$$\frac{\partial M_1(t, h; \tau)}{\partial h_i} = \begin{cases} 0, & \text{for } \alpha(t) \le S_l - l_i, \\ (l_i + \alpha(t) - S_l) \delta^i_{\chi(h_i)} \chi'(h_i), & \text{for } S_l - l_i < \alpha(t) \le S_l, \\ l_i \delta^i_{\chi(h_i)} \chi'(h_i), & \text{for } S_l < \alpha(t). \end{cases}$$

Similarly, depending on the value of $\alpha(t)$, $M_2(t, h; \tau)$ equals

$$M_2(t, h; \tau) = \begin{cases} 0, & \text{for } \alpha(t) \le S_u - l_i, \\ (l_i I_{\{\tau_i \le t\} + \alpha(t) - S_u}) I_{\{\tau_i \le t\}}, & \text{for } S_u - l_i < \alpha(t) \le S_u, \\ (l_i I_{\{\tau_i \le t\}} + \alpha - S_u), & \text{for } S_u < \alpha(t). \end{cases}$$

Because $(l_i I_{\{\tau_i \le t\}} + \alpha(t) - S_u) I_{\{\tau_i \le t\}} = (l_i + \alpha(t) - S_u) I_{\{\tau_i \le t\}}$, apply Theorem 2.1 to obtain

$$\frac{\partial M_2(t, h; \tau)}{\partial h_i} = \begin{cases} 0, & \text{for } \alpha(t) \le S_u - l_1, \\ (l_i + \alpha(t) - S_u) \delta^i_{\chi(h_i)} \chi'(h_i), & \text{for } S_u - l_i < \alpha(t) \le S_u, \\ l_i \delta^i_{\chi(h_i)} \chi'(h_i), & \text{for } S_u < \alpha(t). \end{cases}$$

By the shifting property of the Dirac delta function and the definition of generalized derivatives (Strichartz 2003),

$$\frac{\partial E[M(t, h; \tau)]}{\partial h_i} = t\mathrm{e}^{-h_i t} E[A(t, h_{-i}; U_{-i}) w(t, h_i; U_{-i})] = a(t).$$

Because $A(t, h_{-i}; U_{-i})$ does not involve h_i, easy calculations yield

$$\frac{\partial^2 E[M(t, h_i; \tau)]}{\partial h_i^2} = b(t).$$

Appendix 8. Proof of Corollary 4.3

When the pdf of the underlying assets is explicit and differentiable, LRM calculates the Greeks by differentiating the pdf of the underlying assets. Recall that the default times τ have the joint cdf inequation (31). Let $s = (s_1, \ldots, s_N)'$. The joint pdf of τ is then

$$f(s, h) = \frac{1}{|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(\Phi^{-1}(F(s)))' \Lambda \Phi^{-1}(F(s))\right) \times \prod_{k=1}^{N} h_k \mathrm{e}^{-h_k s_k}, \tag{A4}$$

where $\Phi^{-1}(F(s)) = (\Phi^{-1}(F_1(s_1, h_1)), \Phi^{-1}(F_2(s_2, h_2)), \ldots, \Phi^{-1}(F_N(s_N, h_N)))'$. LRM yields the following Delta and Gamma of the CDO having a discounted payoff $V(h; \tau)$ with τ given in equation (31), respectively,

$$\Delta_i = E\left[V(h; \tau) \frac{\partial \log f(\tau, h)}{\partial h_i}\right],$$

$$\Gamma_{ii} = E\left[V(h; \tau) \frac{\partial^2 f(\tau, h)}{\partial h_i^2} \frac{1}{f(\tau, h)}\right].$$

The term $\partial \log f(\tau, h)/\partial h_i$ is known as the score function (Glasserman 2004).

Define

$$\begin{aligned} B(h; \tau) &:= \frac{\partial \left(-\frac{1}{2}\left(\Phi^{-1}(F(v))' \Lambda \Phi^{-1}(F(v))\right)\right)}{\partial h_i} \\ &= -\left(\sum_{k=1}^{N} \Phi^{-1}(F_k(\tau_k, h_k)) \Lambda_{ik}\right) \sqrt{2\pi} \tau_i \exp \\ &\quad \times \left(-h_i \tau_i + \frac{F_i^2(\tau_i, h_i)}{2}\right). \end{aligned}$$

Because the Gaussian copula is smooth, standard calculations yield

$$\frac{\partial \log f(\tau, h)}{\partial h_i} = B(h;\tau) + \frac{1}{h_i} - \tau_i.$$

We remark that Joshi and Kainth (2004) and Chen and Glasserman (2008) derive the same formula for constant hazard rates.

Before deriving the Gamma, note that

$$\frac{\partial f(\tau, h)}{\partial h_i} = \left(B(h;\tau) + \frac{1}{h_i} - \tau_i\right) f(\tau, h).$$

Hence

$$\frac{\partial^2 f(\tau, h)}{\partial h_i^2} = \left(\frac{\partial B(h;\tau)}{\partial h_i} - \frac{1}{h_i^2} + \left(B(h;\tau) + \frac{1}{h_i} - \tau_i\right)^2\right) f(\tau, h),$$

where

$$\begin{aligned}\frac{\partial B(h;\tau)}{\partial h_i} &= -2\pi \tau_i^2 \mathrm{e}^{-2h_i\tau_i + F_i^2(\tau_i, h_i)} \Lambda_{ii} + B(h;\tau) \\ &\quad \times \left(-\tau_i + F_i(\tau_i, h_i)\tau_i \mathrm{e}^{-h_i\tau_i}\right).\end{aligned}$$