以球狀蒙地卡羅模擬法評價匯率連動選擇權

On Pricing Quanto Options with Spherical Monte Carlo Simulation

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摘　要

匯率連動選擇權 (Quanto Options) 為一同時考量外國資產與匯率之新奇選擇權，其提供投資人於投資外國資產的同時，得以避免匯率風險之保護機制。隨著全球化的快速蔓延以及國外投資機會，該類選擇權可以使投資人有效率地對匯率風險進行避險。Generalized Autoregressive Conditional Heteroskedasticity (GARCH) 模型具有捕捉資產報酬時間序列之波動率群聚與厚尾的實證現象，然而此模型於該選擇權定價上無法得到封閉解。因此，如何準確及有效地評價匯率連動選擇權成為一重要議題。本文介紹球狀蒙地卡羅方法，並應用於匯率連動選擇權於 GARCH 模型下的定價。透過數值分析，我們驗證了所提的球狀蒙地卡羅法可以有效降低變異數並縮短計算時間。

關鍵詞：匯率連動選擇權、GARCH 模型、模擬、球狀分布、變異數降低方法

Abstract

Quanto options are financial options that involve a foreign stock of interest and its exchange rate. Alone with the rapid spread of globalization and foreign investment opportunities, quanto options enable investors to effectively hedge exchange rate risk. The generalized autoregressive conditional heteroskedasticity (GARCH) model allows to capture stylized features such as fat tail and volatility clustering of financial time series data. However, there exists no closed-form formula for the price of quanto options under GARCH model. Therefore, pricing quanto options accurately and efficiently is of considerable importance. To tackle this problem, this paper introduces a spherical Monte Carlo simulation scheme and applies it to price quanto options. Our numerical experiments confirm the superiority of the proposed spherical estimator in terms of variance reduction and computing time.

Keywords: Quanto Options, GARCH Models, Simulation, Spherical Distributions, Variance Reduction Techniques

I. Introduction

A quanto option is a financial derivative where the underlying is denominated in foreign currency, but the instrument itself is settled in domestic currency at a predetermined exchange rate. Quanto options are also called cross-currency options, or foreign equity options. They are popular in the market because they protect investors from exchange rate risk while investing foreign indexes or assets. As a result, accurately and efficiently pricing and hedging quanto options with more realistic models are considerably important. To price quanto options, it is necessary to simultaneously model the foreign underlying asset (index or stock) and the exchange rate.

Romo (2012) indicates that the adoption of Black-Scholes (BS) model for pricing quanto options leads to pricing errors compared with the local volatility model. With the continuing attention to quanto options, more complicated models have been proposed for pricing quanto options. For example, Branger and Muck (2012) and Romo (2014) employ Wishart process to include stochastic volatility and stochastic correlation; Kim, Lee, Mittnik and Park (2015) consider normal tempered stable process to capture heavy tails for Nikkei 225 Index and yen–dollar exchange rate; and Ballotta, Deelstra and Rayee (2017) employ Lévy processes to capture excess kurtosis and asymmetry. The above studies derive approximate pricing formulas for quanto options using Fourier transformation.

On the other hand, Lehar, Scheicher and Schittenkopf (2002) show that generalized autoregressive conditional heteroskedasticity (GARCH) model is substantially superior to the BS and Hull-White models. In addition, Duan and Zhang (2001), and Hsieh and Ritchken (2005), empirically justify that nonlinear GARCH (NGARCH) option pricing model outperforms the BS model for both in-sample and out-of-sample valuation. However, closed-form formulas for pricing quanto options under GARCH models are difficult to obtain.

Although Monte Carlo simulation is indispensable for options pricing, it is notoriously known for its slow convergence. To overcome this problem, variance reduction techniques seek an alternative unbiased estimator which produces less variance. Commonly known variance reduction techniques include antithetic variates, control variables, importance sampling, stratified sampling, and low-discrepancy sequences (Ross, 2013). A good review on Monte Carlo simulation with applications in finance can be found in Boyle, Broadie and Glasserman (1997) and Glasserman (2003).

Some variance reduction methods are problem dependent and require additional sophisticated mathematical analysis. For example, conditioning method needs to determine which variable to condition, whereas importance sampling method needs to select a proper sampling distribution.

The method of antithetic variates is generally applicable and reduces variance by using negatively correlated pairs of replications. It is shown that if the target function is monotone in each variate, antithetic variates method outperforms the standard Monte Carlo method (Ross, 2013). The systematic sampling method generalizes antithetic variables, but the condition to ensure its superiority in terms of variance reduction is difficult to obtained (Glasserman, 2003).

There remains a growing body of studies providing various variance reduction techniques for financial applications. Examples include non-parametric importance sampling (Neddermeyer, 2011), empirical martingale techniques (Duan and Simonato, 1998; Duan, Gauthier and Simonato, 2001), primal-simulation methods (Stentoft, 2005), Markov chain techniques (Duan, Dudley, Gauthier and Simonato, 1999), and extended antithetic variables (Park and Choe, 2016).

This paper aims at introducing the spherical Monte Carlo simulation and applies it in pricing quanto options under GARCH model. The essential idea of the spherical Monte Carlo simulation is to exploit the symmetric geometry of the spherical distribution. Like antithetic variates, a spherical estimator also uses multiple replications to obtain a Monte Carlo sample. It includes various estimators, depending on whether the innermost integral is with respect to the radius or sphere, and the predetermined set of points on the unit sphere.

For example, Teng, Kang and Fuh (2015) provide theoretical analysis on variance reduction for the spherical estimator and suggest to select the predetermined set of unit vectors related to the maximal kissing number in sphere packing problem. It is also difficult to identify conditions under which the spherical estimator is preferred. However, the spherical estimator has been successfully applied in calculating multivariate normal probabilities in econometrics (Hajivassiliou, McFadden and Ruud, 1996) and high-dimensional integrals in Bayesian inference (Monahan and Genz, 1997). As a remark, the spherical Monte Carlo simulation can be applied for non-spherical random variables with importance sampling (Monahan and Genz, 1997). Instead of sampling samples from the original non-spherical distribution, we generate samples from a spherical distribution and adjust these samples by multiplying them with the importance sampling weight.

Although the computing power has been substantially advanced nowadays, many practical problems in finance remain challenging and cannot be solved by simply increasing the sample size with the standard Monte Carlo method. For example, calculating price sensitivities for complicated financial derivatives is particularly difficult because of the curse of differentiation (Glasserman, 2003). In addition, evaluating risk measures (such as value-at-risk and conditional value-at-risk) is closely related to rare-event simulation and additional variance reduction techniques are usually required (Glasserman, Heidelberger and Shahabuddin, 2000, 2002).

The spherical Monte Carlo simulation is rarely applied in finance, except that Teng (2017) employs the spherical estimator in financial risk management on the calculation of value-at-risk and expected shortfall for the quadratic portfolio under the *t*-distributed risk factors. This paper aims at bridging this gap in the literature, showing the efficiency of the spherical estimator in pricing quanto options, and addressing its applicability for a general option pricing problem.

The rest of this paper is organized as follows. Section II overviews quanto options, and Section III reviews how to price a quanto option under GARCH model and Monte Carlo simulation for integration. Section IV introduces the spherical estimator for the spherical Monte Carlo method. Section V summarizes numerical experiments. To address the generality of the spherical estimator, Section VI demonstrates the generality and superiority of the spherical estimator in pricing multi-asset options. The last section concludes.

II. Overview of the Quanto Option

To have an overview on quanto options, the first sub-section of Section II reports over-the-counter (OTC) derivatives from the Bank of International Settlements (BIS), the second sub-section summarizes examples of quanto options, and the third sub-section discusses market potential of quanto options in Taiwan.

1. Global OTC Derivatives from BIS

A foreign exchange derivative is a financial derivative whose payoff is related to foreign exchange rate between two or more currencies. Examples of foreign derivatives include foreign exchange forwards, foreign exchange swaps, currency swaps, and currency options. These instruments are commonly used for speculation, arbitrage, or hedging exchange risk. For example, Geczy, Minton and Schrand (1997) and Allayannis and Weston (2001) document that whether firms use foreign exchange derivatives to hedge exchange rate risk is positively related to their growth opportunities.

Figure 1 compares the outstanding notional amounts and gross market values for global OTC derivatives markets from BIS. It is shown that the outstanding notional amounts of OTC derivatives (particularly the foreign exchange derivatives) have constantly upward trends. In terms of the gross market values, interest rate derivatives start to decline after 2016, whereas foreign exchange derivatives remain at a constant scale of trading volume. These statistics show the popularity of foreign exchange derivatives.

2. Examples of Quanto Options

Let S_t and E_t denote the foreign underlying asset and exchange rate at time t, respectively. Specifically, S_1 and E_1 are the initial price of a foreign underlying asset and exchange rate, respectively. Let T denote the maturity time. The first type of payoff function is with a predetermined fixed exchange rate E_1:

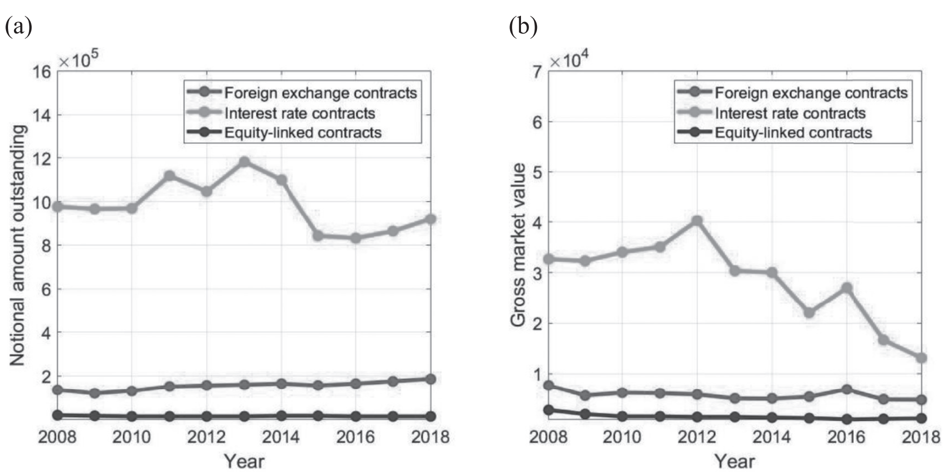

Figure 1 Global OTC Derivatives Market in Billions of USD: (a) National Amounts Outstanding; (b) Gross Market Values

Source: Bank of International Settlements (n.d.).

$$E_1 \max(S_T - K_f, 0), \tag{1}$$

where K_f is the strike price of the foreign underlying denominated in foreign currency. The second type of payoff function is with a random exchange rate at maturity time E_T:

$$E_T \max(S_T - K_f, 0). \tag{2}$$

A quanto option has the second type of payoff function is also called a Compo option.

The third type of payoff function is struck in domestic currency:

$$\max(E_T S_T - K_d, 0), \tag{3}$$

where K_d is the strike price of the foreign underlying denominated in domestic currency. The fourth type of payoff function is

$$S_T \max(E_T - K_E, 0), \tag{4}$$

where K_E is the strike price for the exchange rate. In addition to the above European-typed options, barrier-typed quanto options also exist (Fink and Mittnik, 2016).

Table 1 summarizes the underlying asset, and exchange rate, and types of quanto options. It is shown that indexes, such as Down Jone Industrial Average Index, Standard & Poor's 500 Index, and Nikkei 225 Index, and exchange rates among USD, EUR, GBP, and JPY, are most frequently considered in quanto options market. However, it is difficult to find the trading volume of quanto options, because most of them are traded over the counter (Kim et al., 2015).

Table 1 Summary of the Quanto Options Including Underlying Asset, Exchange Rate, Maturity, and Payoff Function

Reference	Underlying	Exchange rate	Option maturity	Payoff function	Remark
Duan and Wei (1999)	Nikkei 225 Index	USD/JPY	20, 60, 120, 360 days	Equation (1)	n.a.
Romo (2012)	Nikkei 225 Index	EUR/JPY	2, 3 years	Equation (1)	n.a.
Teng et al. (2015)	Deutsche Bank	EUR/USD	30, 90, 180, 240 days	Equation (1)	n.a.
Kim et al. (2015)	Nikkei 225 Index	USD/JPY	n.a.	Equation (1)	n.a.
Fink and Mittnik (2016)	Nikkei 225 Index	EUR/JPY	354 days	Equations (1) and (2), double-barrier	Issued by Société Générale
Ballotta et al. (2017)	Nikkei 225 Index	JPY/USD	n.a.	Equation (1)	n.a.
Ballotta et al. (2017)	Brent Crude Oil	ZAR/USD	n.a.	Equation (1)	n.a.
Li, Zhang and Liu (2018)	Shanghai Stock Exchange (SSE) 50 Exchange Traded Funds (ETF)	HKD/CNY	< 60, 60–120, > 120 days	Equation (2)	n.a.
Fallahgoul, Kim, Fabozzi and Park (2019)	Standard & Poor's 500	EUR/USD	< 60, 20–60, 60–160, > 160 days	Equation (1)	n.a.
Chang, Ho, Liao and Wang (2019)	Dow Jones Industrial Average Index	USD/GBP	5–40, 40–120 days	n.a.	Issued by Société Générale
Chang et al. (2019)	Nasdaq 100 Stock Index	USD/GBP	5–40, 40–120 days	n.a.	Issued by Société Générale
Chang et al. (2019)	Nikkei 225 Index	JPY/USD	5–40, 40–120 days	n.a.	Issued by Société Générale

n.a. = not applicable.

3. Market Potential of Quanto Options in Taiwan

Figure 2 depicts outstanding notional amounts for foreign exchange, interest rate, and equity-linked, derivatives in Taiwan. In contrast to Figure 1, the outstanding notional amount of foreign exchange derivatives substantially exceeds those of interest rate and equity-linked derivatives. This indicates that hedging exchange risk is of considerable importance for both finance and industry sectors in Taiwan.

Table 2 summarizes number of contracts of foreign futures traded in Taiwan from Taiwan Futures Exchange from 2015 to 2019. The underlying indexes include Tokyo Stock Price Index, Standard & Poor’s 500 Index, Dow Jones Industrial Average Index, Nasdaq 100 Stock Index, gold with a purity of 0.995, and Brent Crude Oil. It is shown that Dow Jones Industrial Average Index is

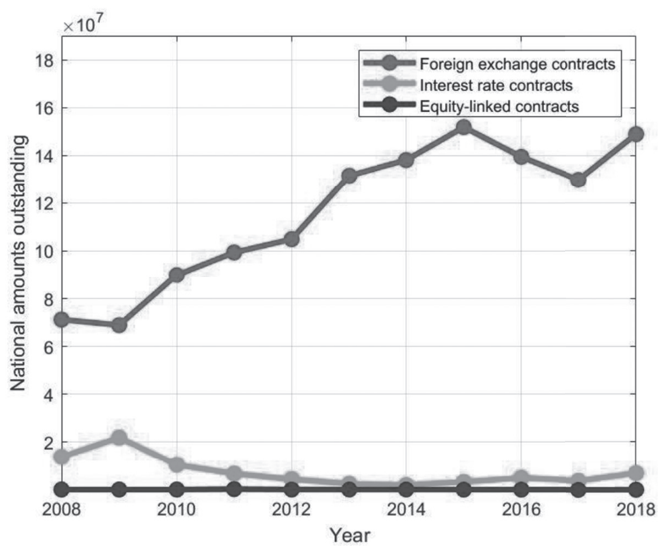

Figure 2 The National Amounts Outstanding of the OTC Derivatives Market in Millions of NTD

Source: Central Bank of the Republic of China (Taiwan) (n.d.).

Table 2 Number of Contracts of the Foreign Futures in Taiwan from Taiwan Futures Exchange from 2015 to 2019 with Different Underlying

Underlying	2015	2016	2017	2018	2019
Tokyo Stock Price Index	17,393	316,578	176,841	88,285	165,075
Standard & Poor's 500 Index	n.a.	n.a.	38,658	125,444	233,038
Dow Jones Industrial Average Index	n.a.	n.a.	403,078	1,466,111	2,209,699
Nasdaq 100 Stock Index	n.a.	n.a.	n.a.	n.a.	75,851
Gold with a purity of 0.995	1	2	12,933	20,396	26,465
Brent Crude Oil	n.a.	n.a.	n.a.	27,118	28,562

the most frequently traded one, followed by Standard & Poor's 500 Index and Tokyo Stock Price Index. Because quanto options are able to provide investors protection from exchange rate risk while investing foreign assets, they could gain great popularity in the financial market of Taiwan. Indeed, the TWD-Denominated Gold Option (ticker symbol: TGO) traded in Taiwan Futures Exchange is an example of quanto options, because the underlying (gold with a purity of 0.9999) is denominated in USD, but the instrument itself is settled in NTD.

III. Preliminaries

The first subsection of Section III reviews the dynamics of the exchange rate and the foreign asset in Duan and Wei (1999), and gives the formula for pricing quanto options. The second subsection reviews the standard Monte Carlo method, and the third subsection introduces two variance reduction techniques: the antithetic variates and systematic sampling methods.

1. Quanto Options under GARCH Models

Let $N_d(\mu, \Sigma)$ denote the d-variate normal random vector with mean vector μ and covariance matrix Σ. Specifically, let $N_d(\mathbf{0}, I)$ denote the d-variate standard normal random vector. Here, $\mathbf{0}$ is the d-variate zero vector and I is the $d \times d$

identity matrix. Without loss of generality, we assume the observations are in daily basis. Let $r_{d,t}$ and $r_{f,t}$ denote the risk-free rate for the domestic currency and foreign currency at time t, respectively. The returns of the exchange rate and foreign stock price are defined as $e_{t+1} = \log\frac{E_{t+1}}{E_t}$ and $s_{t+1} = \log\frac{S_{t+1}}{S_t}$, respectively.

Following Duan and Wei (1999), we assume that e_t and s_t under the physical measure P follow

$$\begin{cases} e_{t+1} = r_{d,t+1} - r_{f,t+1} + \delta_{t+1}\sqrt{q_{t+1}} - \frac{q_{t+1}}{2} + \sqrt{q_{t+1}}\,\varepsilon_{t+1} \\ q_{t+1} = \alpha_0 + \alpha_1 q_t(\varepsilon_t - a)^2 + \alpha_2 q_t \end{cases}$$
$$\begin{cases} s_{t+1} = r_{f,t+1} + \lambda_{t+1}\sqrt{h_{t+1}} - \frac{h_{t+1}}{2} + \sqrt{h_{t+1}}\,\xi_{t+1} \\ h_{t+1} = \beta_0 + \beta_1 h_t(\xi_t - b)^2 + \beta_2 h_t \end{cases} \tag{5}$$

where

$$\begin{pmatrix} \varepsilon_t \\ \xi_t \end{pmatrix} \overset{i.i.d.}{\sim} N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{bmatrix} 1 & \rho_t \\ \rho_t & 1 \end{bmatrix}\right),$$

for $t = 1, \ldots, T$. Here, q_t is the conditional variance of exchange rate. For parameters interpretations, δ_t is the unit risk premium of exchange rate, and a is the leverage effect parameter. The empirical phenomenon that returns become more volatile after a drop in price is known as the leverage effect. Black (1976) first explains the negative relationship between the stock return and volatility using the idea of debt-equity ratio. Duan and Wei (1999) point out that a positive a leads to a negative correlation between the exchange rate return and its volatility. Similarly, h_t is the conditional variance of the stock return, λ_t is the unit risk premium of stock return, and b is the leverage affect parameter.

It is worthy mentioning that these parameters need to satisfy the following conditions: $\alpha_0 > 0$, $\alpha_1, \alpha_2 \geq 0$, $\alpha_1(1 + a_2) + \alpha_2 < 1$, $\beta_0 > 0$, $\beta_1, \beta_2 \geq 0$, and $\beta_1(1 + b_2) + \beta_2 < 1$ to ensure positive variance processes. The unconditional variances are therefore given by $\alpha_0/[1 - \alpha_1(1 + a_2) - \alpha_2]$ and $\beta_0/[1 - \beta_1(1 + b_2) - \beta_2]$.

For the quanto option, we need to convert its payoff into domestic currency and price it under the domestic equilibrium price measure Q_d. Under the domestic equilibrium price measure Q_d the dynamics of e_t and s_t are assumed to follow

$$\begin{cases} e_{t+1} = r_{d,t+1} - r_{f,t+1} - \dfrac{q_{t+1}}{2} + \sqrt{q_{t+1}}\,\varepsilon^*_{t+1} \\ q_{t+1} = \alpha_0 + \alpha_1 q_t(\varepsilon^*_t - \delta_t - a)^2 + \alpha_2 q_t \end{cases}$$
$$\begin{cases} s_{t+1} = r_{f,t+1} + \rho_{t+1}\sqrt{h_{t+1}q_{t+1}} - \dfrac{h_{t+1}}{2} + \sqrt{h_{t+1}}\,\xi^*_{t+1} \\ h_{t+1} = \beta_0 + \beta_1 h_t(\xi^*_t - \lambda_t - \rho_t\sqrt{q_t} - b)^2 + \beta_2 h_t \end{cases} \tag{6}$$

where

$$\begin{pmatrix} \varepsilon^*_t \\ \xi^*_t \end{pmatrix} \overset{i.i.d.}{\sim} N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{bmatrix} 1 & \rho_t \\ \rho_t & 1 \end{bmatrix}\right).$$

Let $\mathbb{E}[\bullet]$ denote the expectation operator and $g(\bullet)$ denote a payoff function of interest. Take the first type of quanto option as an example. The price of the quanto call option is

$$c = \mathbb{E}\left[\exp - \left(\sum_{t=2}^{T} r_{d,t}\right) E_1 \max(S_T - K_f, 0)\right], \tag{7}$$

where the return of stock price S_T is modelled under the domestic equilibrium price measure Q_d in Equation (6). Because the complexity in Equation (6), closed-form formula for pricing this quanto option does not exist. As a result, pricing quanto options using efficient Monte Carlo integration is a timely issue both in theory and practice.

We remark that there exists other heteroskedasticity models to capture the leverage effect. For example, the exponential GARCH (EGARCH) model in Yung and Zhang (2003) assume the variance process to follow

$$\ln(h_t) = w_0 + w_1 \ln(h_{t-1}) + w_2 z_{t-1} + \gamma[|z_{t-1}| - \mathbb{E}(|z_{t-1}|)], \tag{8}$$

where $z_t \overset{i.i.d.}{\sim} N(0, 1)$ with w_0, w_1, w_2 being parameters. The logarithm transformation on the conditional variance allows parameters to be negative.

2. Standard Monte Carlo Method

Let $X = (X_1, \ldots, X_d)'$ be a d-variate random vector with distribution function $F(x) = F(x_1, \ldots, x_d)$, denoted as $X \sim F$. Here, the prime denotes vector or matrix transpose. In addition, $\overset{d}{=}$ enotes equality in distribution. Let $g(\bullet)$ be a real-valued function from $\mathbb{R}^d$ to $\mathbb{R}$ that represents a performance function of interest. We are interested in calculating the expectation of $g(X)$,

$$\mathbb{E}[g(X)], \tag{9}$$

where $\mathbb{E}[\bullet]$ is the expectation operator. We denote the standard estimator (or the standard Monte Carlo estimator) by

$$M_0 = g(X), \tag{10}$$

Algorithm 1 evaluates $\mathbb{E}[g(X)]$ using Monte Carlo simulation with the standard estimator M_0 defined in Equation (10).

Algorithm 1. *Let X be a random vector with distribution function F, $X \sim F$.*

The following steps evaluate $\mathbb{E}[g(X)]$ *with the standard estimator* M_0 *given in Equation (10).*

(1) *Set the Monte Carlo sample size n.*

(2) *Generate independent realizations* $X^{(i)} \sim F$ *for* $i = 1, \ldots, n$.

(3) *Evaluate* $\mathbb{E}[g(X)]$ *by the average of n realizations*: $\hat{M}_0 = \frac{1}{n}\sum_{i=1}^{n} g(x^{(i)})$.

Algorithm 1 shows that estimating a general expectation with simulation using the standard estimator is simple. To apply Algorithm 1 to price quanto options, we first set the initial volatility (q_1, h_1), maturity T, and GARCH parameters $r_{d,t}$, $r_{f,t}$, ρ_t, a, δ_t, α_0, α_1, α_2, b, λ_t, β_0, β_1, β_2, and assume the two dynamics have constant correlation $\rho_t = \rho$.

Then, we set $F \sim N_{2T}(\mathbf{0}, \Sigma)$. Let Σ_{ij} denote the i-th row and j-th column of Σ, then we set $\Sigma_{ii} = 1$ for $i = 1, \ldots, 2T$; $\Sigma_{2i-1,2i} = \rho$ for $i = 1, \ldots, T$; $\Sigma_{2i,2i-1} = \rho$ for $i = 1, \ldots, T$; and $\Sigma_{ij} = 0$ otherwise.

Or equivalently, we write Σ as a matrix:

$$\Sigma = \begin{bmatrix} 1 & \rho & 0 & \cdots & 0 \\ \rho & 1 & 0 & \cdots & 0 \\ 1 & 0 & \ddots & \cdots & 0 \\ 1 & 0 & & 1 & \rho \\ 1 & 0 & \cdots & \rho & 1 \end{bmatrix}.$$

In Step 2, we generate independent realizations $X^{(i)} \sim N_{2T}(0, \Sigma)$ for $i = 1, \ldots, n$. Details for Step 3 are given as follows.

(1) For $i = 1, \ldots, n$, do the following:

a. Set $(\varepsilon_1^{*(i)}, \varepsilon_2^{*(i)}, \ldots, \varepsilon_T^{*(i)}) = (X_1^{*(i)}, X_3^{*(i)}, \ldots, X_{2T-1}^{*(i)})$ and $(\xi_1^{*(i)}, \xi_2^{*(i)}, \ldots, \xi_T^{*(i)}) = (X_2^{*(i)}, X_4^{*(i)}, \ldots, X_{2T}^{*(i)})$.

b. Set $\begin{cases} q_{t+1}^{(i)} = a_0 + a_1 q_t^{(i)}(\varepsilon_t^{(i)} - \delta_t - a)^2 + a_2 q_t^{(i)} \\ e_{t+1}^{(i)} = r_{d,t+1} - r_{f,t+1} - \dfrac{q_{t+1}^{(i)}}{2} + \sqrt{q_{t+1}^{(i)}}\varepsilon_{t+1}^{(i)} \end{cases}$, for $t = 1, \ldots, (T-1)$.

c. Set $\begin{cases} h_{t+1}^{(i)} = \beta_0 + \beta_1 h_t^{(i)}(\xi_t^{*(i)} - \lambda_t - \rho\sqrt{q_t^{(i)}} - b)^2 + \beta_2 h_t^{(i)} \\ s_{t+1}^{(i)} = r_{f,t+1} - \rho\sqrt{h_{t+1}^{(i)} q_{t+1}^{(i)}} - \dfrac{h_{t+1}^{(i)}}{2} + \sqrt{h_{t+1}^{(i)}}\,\xi_{t+1}^{*(i)} \end{cases}$, for $t = 1, \ldots, (T-1)$.

d. Set $S_T^{(i)} = S_1 \exp(s_2^{(i)} + \ldots + s_T^{(i)})$.

e. Set $c^{(i)} = E_1 \exp(-\sum_{i=1}^{n} r_{d,t}) \max(s_T^{(i)} - K, 0)$.

(2) Evaluate the price of the quanto option by the average of n realizations:

$$\hat{M}_0 = \frac{1}{n}\sum_{i=1}^{n} c^{(i)}.$$

3. An Overview on the Antithetic Variates and Systematic Sampling Methods

Suppose V and W be two d-variate random vectors having the same distribution as X. The antithetic variates estimator (or antithetic variates method) is

$$M_A = \frac{g(V) + g(W)}{2}.$$

The antithetic variates estimator is unbiased, because

$$E[M_A] = \frac{g(V) + g(W)}{2} = E[g(X)].$$

The variance of the antithetic variates estimator is

$$\begin{aligned}\text{var}(M_A) &= \text{var}\left[\frac{g(V)+g(W)}{2}\right] \\ &= \frac{\text{var}[g(V)] + \text{var}[g(W)] + 2\text{Cov}(g(V), g(W))}{4} \\ &= \frac{\text{var}[g(V)]}{2} + \frac{\text{Cov}(g(V), g(W))}{2}.\end{aligned}$$

The variance of antithetic variates estimator can be reduced compared with the estimator taking average function values of two independent random vectors, if $\text{Cov}(g(V), g(W)) < 0$. Conditions to guarantee variance reduction for the antithetic variates estimator can be found in Ross (2013). When X is a standard normal random vector, it is common to set $V \overset{d}{=} X$ and $W = -V$. Algorithm 2 evaluates $\mathbb{E}[g(X)]$ using the antithetic variates estimator.

Algorithm 2. *Let X be the d-variate standard normal random vector. The following steps evaluate $\mathbb{E}[g(X)]$ with the antithetic variates estimator M_A.*

(1) *Set the Monte Carlo sample size n.*

(2) *Generate independent realizations $X^{(i)} \sim N_d(\mathbf{0}, I)$ for $i = 1, \ldots, n$.*

(3) *Evaluate $\mathbb{E}[g(X)]$ by the average of n realizations: $\hat{M}_A = \frac{1}{n}\sum_{i=1}^{n}\frac{g(X^{(i)}) + g(-X^{(i)})}{2}$.*

For the systematic sampling method, when X is the d-variate standard normal random vector, the systematic sampling estimator is

$$M_s = \frac{1}{k}\sum_{i=1}^{k} g(T^i X),$$

where T is the $d \times d$ random orthogonal matrix. A random orthogonal matrix T allows us to rotate a set of unit vectors simultaneously and plays a key role in the proposed spherical estimator. To generate a sample of random orthogonal matrix T, we first generate a $d \times d$ matrix where each entry is independently and identically standard normal random variable, and apply the Gram-Schmidt procedure to the above matrix (Heiberger, 1978). More efficient but sophisticated algorithms using only $(d-1)(d+2)/2$ standard normal random variables can be found in Stewart (1980), Diaconis and Shahshahani (1987), and Anderson, Olkin and Underhill (1987), for example. For simplicity, we use the method by Heiberger (1978) to generate a random orthogonal matrix in this paper. Note the systematic sampling estimator is limited to spherical distribution (Glasserman, 2003). Algorithm 3 evaluates $\mathbb{E}[g(X)]$ using the systematic sampling estimator.

Algorithm 3. *Let X be the d-variate standard normal random vector. The following steps evaluate* $\mathbb{E}[g(X)]$ *with the systematic sampling estimator.*

(1) *Set the Monte Carlo sample size n.*

(2) *Generate independent realizations of random orthogonal matrices* $T^{(i)}$ *and* $X^{(i)} \sim N_d(\mathbf{0}, I)$ *for* $i = 1, \ldots, n$.

(3) *Evaluate* $\mathbb{E}[g(X)]$ *by the average of n realizations:*

$$\hat{M}_S = \frac{1}{n}\sum_{i=1}^{n}\frac{g(T^{(i)}Z^{(i)}) + g(T^{(i)2}Z^{(i)}) + \ldots + g(T^{(i)k}Z^{(i)})}{k}.$$

IV. The Spherical Monte Carlo Integration

In the following, the first subsection of Section IV introduces the spherical Monte Carlo simulation, the second subsection contrasts the standard estimator, systematic sampling estimator, and the spherical estimator in the case of the bivariate standard random vector, the third subsection compares the variance between the systematic sampling estimator and the spherical estimator, and the fourth subsection provides a simple example to compare estimators.

1. The Spherical Estimator

A random vector $X \in \mathbb{R}^d$ has a spherical distribution, if it is invariant under rotations. Spherical distributions include the d-variate standard normal, t, and Cauchy random vectors and are of critical importance in finance (McNeil, Frey and Embrechts, 2005).

Let S^{d-1} denote the d-variate unit sphere, $S^{d-1} = \{s \in \mathbb{R}^d\colon s's = 1\}$. A spherical distribution X has the following stochastic representation,

$$X \stackrel{d}{=} RT_v, \tag{12}$$

where v is an arbitrary unit vector on S^{d-1}, R is a non-negative random variable, T is the random orthogonal matrix, and R and T is independent. See the third subsection of Section III for generating a random orthogonal matrix.

Two commonly used spherical distributions are given below to explain how to obtain the distribution of R in Equation (11). Before introducing them, we recall the construction of the chi and chi-square distribution with degrees of freedom d, denoted by χ_d and χ^2, respectively. Note that the chi-square distribution is also known as the chi-squared or χ^2 distribution. Let $Z_1, \ldots, Z_d$ be independent and identical univariate standard normal random variables. Then, it is known that

$$\sqrt{\sum_{i=1}^{d} Z_i^2} \stackrel{d}{=} x_d \text{ and } \sum_{i=1}^{d} Z_i^2 \stackrel{d}{=} x_d^2.$$

Denote a d-variate standard normal random vector by $Z \sim N_d(0, I)$. Expressing Z with Equation (12), we have

$$Z \stackrel{d}{=} RT_v, \tag{13}$$

where $R \sim \chi_d$, T is the random orthogonal matrix, and R and T are independent.

Likewise, let $t_{d,v}$ denote the d-variate t random vector with degree of freedom v. Then, $t_{d,v}$ has the following stochastic representation,

$$t_{d,v} \overset{d}{=} Z\sqrt{\frac{v}{V}}, \tag{14}$$

where $Z \sim N_d(0, I)$, $V \sim \chi_v^2$, and Z and V are independent. Combining the stochastic representations in Equations (13) and (14), we obtain

$$t_{d,v} \overset{d}{=} \left(R\sqrt{\frac{v}{V}}\right)T_v, \tag{15}$$

where $R \sim \chi_d$, $V \sim \chi_v^2$, T is the random orthogonal matrix, and R, V, and T are mutually independent.

We are now ready to propose the spherical estimator. Let $V = \{v_j \in S^{d-1}, j = 1, \ldots, k\}$ be a set of predetermined unit vectors. Let $\overset{i.i.d.}{\sim}$ abbreviate for independently and identically distributed as. A spherical estimator associated with V is defined by

$$M_V = \frac{1}{k}\sum_{i=1}^{k} g(R_j T v_j), \tag{16}$$

where $R_j \overset{i.i.d.}{\sim} F_R$ for some distribution F_R, T is the random orthogonal matrix, and R_j is independent of T, for $j = 1, \ldots, k$. Algorithm 4 evaluates $\mathbb{E}[g(X)]$ with the spherical estimator M_V defined in Equation (16).

Algorithm 4. *Let X be a spherical distribution having the stochastic representation in Equation (12). Suppose $R \sim F_R$ for some distribution F_R. The following steps evaluate $\mathbb{E}[g(X)]$ with the spherical estimator M_V defined in Equation (16).*

(1) *Set the Monte Carlo sample size n.*

(2) *Set the set of unit vectors,* $V = \{v_j \in S^{d-1}, j = 1, \ldots, k\}$.

(3) *Generate independent realizations of random orthogonal matrix* $T^{(i)}$ for $i = 1, \ldots, n$.

(4) *Generate independent realizations,* $R_j^{(i)} \sim F_R$ for $j = 1, \ldots, k$ *and* $i = 1, \ldots, n$.

(5) *Evaluate* $\mathbb{E}[g(X)]$ *by the average of n realizations:*

$$\hat{M}_V = \frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k} g(R_j^{(i)} T^{(i)} v_j).$$

In Step 4 of Algorithm 4, for the d-variate standard normal random vector, F_R is the chi distribution with degree of freedom d. For the d-variate t random vector with degree of freedom ν, we sample independent $R \sim \chi_d$ and $V \sim \chi^2$ to obtain F_R through the stochastic representation in Equation (15):

$$F_R \stackrel{d}{=} R\sqrt{\frac{\nu}{V}}.$$

In pricing quanto options, we have $X \sim N_d(\mathbf{0}, \Sigma)$, where Σ is not identity matrix. Let Γ be the Cholesky decomposition of Σ that satisfies $\Sigma = \Gamma\Gamma'$. Suppose $Z \sim N_{2T}(0, I)$. Then, X has the same distribution as ΓZ, $X \stackrel{\mathrm{d}}{=} \Gamma Z$. In Step 4 of Algorithm 4, we generate independent realizations of radii $R_j^{(i)} \sim X2T$, for $j = i, \ldots, k$ and $i = 1, \ldots, n$. Details of Step 5 are given in the following.

(1) For $i = 1, \ldots, n$, do the following steps:

  a. Set $Z^{(i,j)} = R^{(i,j)} T^{(i)} v^{(j)}$ for $j = 1, \ldots, k$.

  b. Set $X^{(i,j)} = \Gamma Z^{(i,j)}$ for $j = 1, \ldots, k$.

  c. Set $(\varepsilon_1^{*(i,j)}, \varepsilon_2^{*(i,j)}, \ldots, \varepsilon_T^{*(i,j)}) = (X_1^{*(i,j)}, X_3^{*(i,j)}, \ldots, X_{2T-1}^{*(i,j)})$ and $(\xi_1^{*(i,j)}, \xi_2^{*(i,j)}, \ldots, \xi_T^{*(i,j)}) = (X_2^{*(i,j)}, X_4^{*(i,j)}, \ldots, X_{2T}^{*(i,j)})$ and for $j = 1, \ldots, k$.

d. Set $\begin{cases} q_{t+1}^{(i,j)} = \alpha_0 + \alpha_1 q_t^{(i,j)}(\varepsilon_t^{(i,j)} - \delta_t - a)^2 + \alpha_2 q_t^{(i,j)} \\ e_{t+1}^{(i,j)} = r_{d,t+1} - r_{f,t+1} - \dfrac{q_{t+1}^{(i,j)}}{2} + \sqrt{q_{t+1}^{(i,j)}}\,\varepsilon_{t+1}^{(i,j)} \end{cases}$, for $t = 1, (T-1)$ and $j = 1, \ldots, k$.

e. Set $\begin{cases} h_{t+1}^{(i,j)} = \beta_0 + \beta_1 h_t^{(i,j)}(\xi_t^{*(i,j)} - \lambda_t - \rho\sqrt{q_t^{(i,j)}} - b)^2 + \beta_2 h_t^{(i,j)} \\ s_{t+1}^{(i,j)} = r_{f,t+1} - \rho\sqrt{h_{t+1}^{(i,j)} q_{t+1}^{(i,j)}} - \dfrac{h_{t+1}^{(i,j)}}{2} + \sqrt{h_{t+1}^{(i,j)}}\,\xi_{t+1}^{*(i,j)} \end{cases}$, for $t = 1, (T-1)$ and $j = 1, \ldots, k$.

f. Set $S_T^{(i,j)} = S_1 \exp(s_2^{(i,j)} + \ldots + s_T^{(i,j)})$.

g. Set $c^{(i,j)} = E_1 \exp(-\sum_{i=2}^{T} r_{d,t}) \max(s_T^{(i,j)} - K, 0)$.

(2) Evaluate the price of the quanto option by the average of n realizations:

$$\hat{M}_V = \frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k} c^{(i,j)}.$$

2. Demonstration of the Spherical Estimator

For comparison, we focus on the two-dimensional case ($d = 2$) for the ease of exposition. We first depict four independent realizations of the bivariate standard normal random vector in Figure 3. Specifically,

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right).$$

In this case, the correlation between X_1 and X_2 is zero.

The system sampling estimator uses $T^i Z$ for $i = 1, \ldots, k$ and takes average over the k samples. Figure 4 illustrates the four points used by the systematic

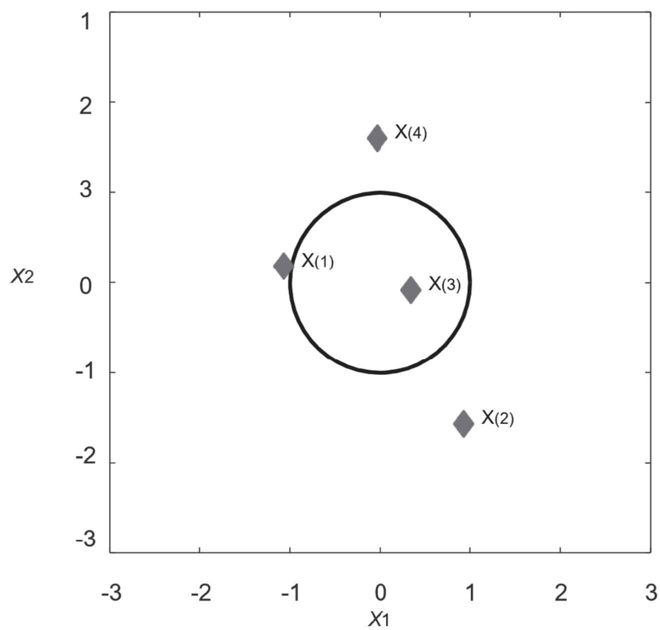

Figure 3 Four Independent Realizations ($X^{(i)}$) of the Two-Dimensional Standard Normal Random Variable Are Depicted in Diamonds for i = 1, …, 4

sampling with a realization of Z = (2.1778, 1.1385). A realization of the random orthogonal matrix is

$$T = \begin{bmatrix} 0.7787 & -0.6274 \\ 0.6274 & 0.7787 \end{bmatrix}, \tag{17}$$

which is used to rotate Z. It is interesting to mention that the above T corresponds to a counterclockwise rotation for about 38.86°: 360° arcsin(-0.6782)/2π ≈ 38.86°, with arcsin(•) being the inverse sin function.

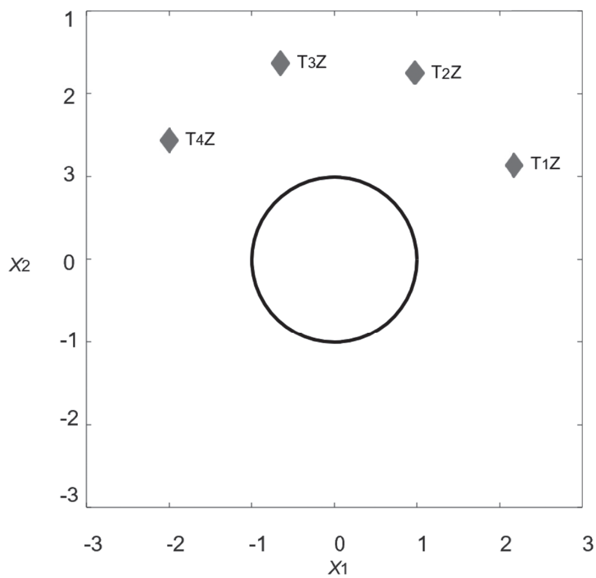

Figure 4 Four Realizations (T^iZ) for Two-Dimensional Standard Normal Random Vector Using the Systematic Sampling Are Depicted in Diamonds for $i = 1, \ldots, 4$

To illustrate the spherical estimator, we set $V = \{v_1 = (1, 0), v_2 = (-1, 0), v_3 = (0, 1), v_4 = (0, -1)\}$, as depicted in the upper panel of Figure 5, and hence we have $k = 4$ for the spherical estimator. Then, the rotated four points Tv_i for $i = 1, \ldots, 4$ using the above realization of T in Equation (17) are depicted in the middle panel of Figure 5. With four independent realizations of radii from the chi distribution with degrees of freedom 2:

$$R = \begin{pmatrix} 1.4487 \\ 1.0319 \\ 2.0529 \\ 0.4187 \end{pmatrix},$$

the final four realizations, R_iTv_i for $i = 1, \ldots, 4$, are depicted in the bottom panel of Figure 5.

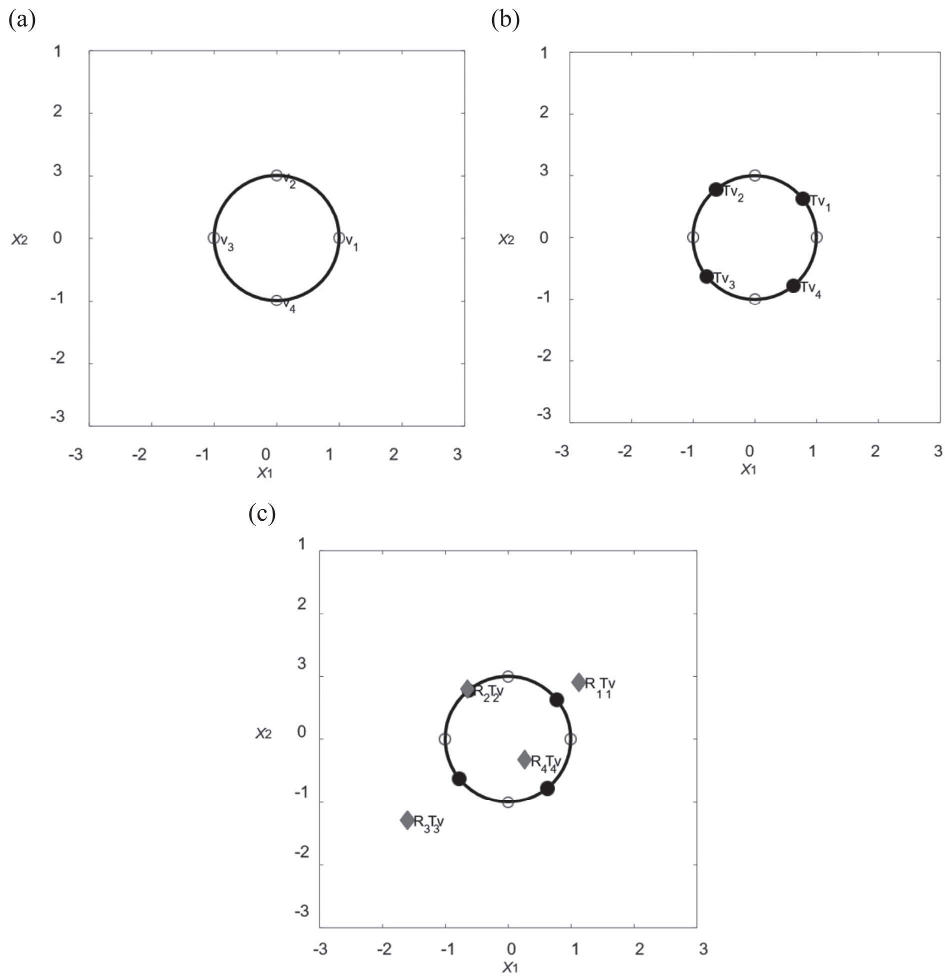

Figure 5 (a) Depicts the Four Unit Vectors in the Set $V(v_i)$ in Hollow Circles, (b) Adds the Rotated Unit Vectors (Tv_i) in Black Circles, and (c) Adds the Final Four Realizations (R_iTv_i) for the Spherical Estimator in Diamonds, for $i = 1, \ldots, 4$

To compare the systematic sampling and spherical estimators, first, we note that implementing the systematic sampling is easier because it only needs to generate the standard normal random vector and random orthogonal matrix. But, implementing the spherical estimator requires to decide a set of unit vectors V, and generate the random orthogonal matrix and k independent radii. Second, the positive integer k in the systematic sampling could be arbitrary, but the k in the spherical estimator is the number of elements of the set V. It is interesting that Figures 4 and 5 show the systematic sampling appears to use four less evenly located points. In contrast, the spherical estimator uses four more evenly located points.

As a remark, the spherical estimator relies heavily on the selection of V. Recall that a lattice L is the set of all integral linear combinations of a given basis of $\mathbb{R}^d$. It is easy to obtain V generically from a lattice L: we simply collect all the shortest vectors in L and normalize them to unit vectors. For simplicity, we set $V = \{\pm e_1, \pm e_2, \ldots, \pm e_d\}$, where e_i for $i = 1, \ldots, d$ is the natural basis of $\mathbb{R}^d$. Hence, $k = 2d$ in our case.

3. Comparisons of the Variances between the Systematic Sampling and Spherical Estimators

To contrast the variance using the four points in Figures 3–5, let Z denote the bivariate standard normal random vector for notational simplicity. Note that the variance of estimator using $X^{(i)}$ for $i = 1, \ldots, 4$ in Figure 3 is

$$
\begin{aligned}
&\operatorname{var}\left(\frac{g(X^{(1)}) + g(X^{(2)}) + g(X^{(3)}) + g(X^{(4)})}{4}\right) \\
&= \frac{\operatorname{var}(g(Z))}{4} + 2\sum_{1 = i < \ldots < j = 4} \operatorname{Cov}\left(g(X^{(i)}), g(X^{(j)})\right) \\
&= \frac{\operatorname{var}(g(Z))}{4}.
\end{aligned}
$$

Because $X^{(i)}$'s are independent two-dimensional standard normal random vectors, all pairwise covariances are zeros and yields the last equality.

For the systematic sampling, the variance of estimator using average function values at points T^iZ for $i = 1, \ldots, 4$ in Figure 4 is

$$\mathrm{var}\left(\frac{g(TZ)+g(T^2Z)+g(T^3Z)+g(T^4Z)}{4}\right)$$

$$=\frac{\mathrm{var}(g(Z))}{4}+2\sum_{1=i<\ldots<j=4}\mathrm{Cov}\big(g(T^iZ),\, g(T^jZ)\big).$$

Recall that the multivariate standard normal random vector is a spherical distribution, its distribution under rotation is invariant, i.e., $TZ \overset{\mathrm{d}}{=}$ (McNeil et al., 2005). However, as mentioned in Glasserman (2003), theoretical analysis for the covariances is difficult. For the spherical estimator, the variance is

$$\mathrm{var}\left(\frac{g(R_1Tv_1)+g(R_2Tv_2)+g(R_3Tv_3)+g(R_4Tv_4)}{4}\right)$$

$$=\frac{\mathrm{var}(g(Z))}{4}+2\sum_{1=i<\ldots<j=4}\mathrm{Cov}\big(g(R_iTv_i),\, g(R_jTv_j)\big).$$

Again, there is no theoretical analysis to calculate these covariances because of the complexity of the estimator itself. Recently, Teng et al. (2015) obtain an upper bound of the variance when $g(\bullet)$ is restricted to the indicator function and radii are fixed at one.

For $i \neq j$, because R_i and R_j are independent, the covariance between R_iTv_i and R_jTv_j mainly comes from the covariance between Tv_i and Tv_j. As v_i and v_j are predetermined in the set of unit vectors V, if they are of opposite direction (or with negative inner product), the covariance between Tv_i and Tv_j tends to be negative. As a result, when more pairs of v_i and v_j are of opposite direction, more negative covariances could be obtained. This may lead to a spherical estimator with smaller variance compared with the estimator using four independent bivariate normal random vectors.

4. Evaluating an Expectation with a Simple Closed-Form Formula

For a positive even number d, consider a problem of calculating

$$\mathbb{E}\left[\sum_{i=1}^{d}(X_i - 1)^2\right],$$

where $X \sim N_d(0, \Sigma)$ and

$$\Sigma = \begin{bmatrix} I_{d/2} & \rho I_{d/2} \\ I_{d/2} & \rho I_{d/2} \end{bmatrix}.$$

Here, $I_{d/2}$ is a $d/2 \times d/2$ identity matrix. Standard calculation gives

$$\mathbb{E}\left[\sum_{i=1}^{d}(X_i - 1)^2\right] = \sum_{i=1}^{d}\mathbb{E}[X_i^2 - 2X_i + 1] = 2d.$$

This theoretical allows us to compare the precision of various estimators.

We consider $d = 2, 10, 50$ and $\rho = -0.5, 0, 0.5$ and set the sample size n = 10,000 for the four estimators: M_0, M_A, M_S, and M_V. Recall that we choose $V = \{\pm e_1, ..., \pm e_d\}$ for M_V, and we set $k = 2d$ for M_S. Therefore, we can fairly compare M_S and M_V. We report the estimated values, errors, variances, and computing times for each estimator in Table 3. Here, the estimated error is calculated as: *estimated error = estimated value − theoretical value*.

Table 3 shows that M_V in general produces the least estimated errors among

Table 3 Calculating $\mathbb{E}\left[\sum_{i=2}^{d}(X_i-1)^2\right]$ Using M_0, M_A, M_S, and M_V

	d = 2			d = 10			d = 50		
Estimator	ρ = -0.5	ρ = 0	ρ = 0.5	ρ = -0.5	ρ = 0	ρ = 0.5	ρ = -0.5	ρ = 0	ρ = 0.5
Estimated value									
M_0	4.02	4.01	4.08	20.04	19.94	20.08	100.13	100.11	100.10
M_A	4.00	3.99	4.01	20.05	19.94	19.97	100.11	100.11	99.93
M_S	3.97	4.02	4.04	19.96	19.97	20.06	99.93	100.05	99.85
M_V	4.00	3.98	4.00	20.01	19.99	19.99	100.02	100.02	100.00
Estimated error									
M_0	0.02	0.01	0.08	0.04	-0.06	0.08	0.13	0.11	0.10
M_A	0.00	0.01	0.01	0.05	0.06	-0.03	0.11	0.11	-0.07
M_S	-0.03	0.02	0.04	-0.04	-0.03	0.06	-0.07	0.05	-0.15
M_V	0.00	-0.02	0.00	0.01	-0.01	-0.01	0.02	0.02	0.00
Variance									
M_0	9.30	11.80	16.90	44.98	60.59	87.42	223.59	306.41	426.45
M_A	4.98	3.90	4.98	25.30	19.81	24.80	126.06	100.48	121.02
M_S	7.25	8.70	10.37	22.52	23.34	26.58	102.88	103.28	104.21
M_V	1.34	1.41	1.79	1.08	1.12	1.19	1.01	1.03	1.04
Computing time									
M_0	0.01	0.01	0.01	0.01	0.00	0.01	0.01	0.01	0.01
M_A	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
M_S	0.21	0.19	0.19	0.51	0.51	0.52	7.68	6.90	7.24
M_V	0.15	0.14	0.14	0.38	0.38	0.38	5.87	5.55	5.57

In this table, the estimated error is estimated value minus theoretical value.

other estimators. This observation is consistent with that M_V produces the least variance. In addition, when the dimension increases, M_V produces estimated errors of similar scale, but this is not the case for M_0, M_A and M_S. Table 3 also shows that M_S and M_V require longer computing times. We will discuss the tradeoff between variance reduction and computing time in Section V in details.

V. Simulation Studies

This section is organized as follows. The first subsection of Section V describes the parameter estimation of the GARCH model and the simulation setting. The second subsection sets up criteria to compare four estimators, the third subsection summarizes the simulation results, and the fourth subsection compares quanto options pricing between GARCH model and BS model. We remark that the simulation results are programmed using Matlab version R2019a in a personal computer (Intel Core i5 CPU with RAM16 GB).

1. Parameter Estimation and Simulation Setting

In our simulation studies, we take TWD/USD exchange rate and Dow Jones Industrial Average Index for example due to its popularity as shown in the second subsection of Section II. We use Treasury Bills from Federal Reserve Economic Data as the risk free rate for the United States, and the fixed rate on one-year deposit from Bank of Taiwan as the risk free rate for Taiwan. The study period is from January 4, 2010 to December 31, 2019. Descriptive statistics of the data are summarized in Table 4. We perform the maximum likelihood method to estimate parameters using Equation (5), and report estimated parameters and standard errors in Table 5.

We see that both Dow Jones Industrial Average Index and TWD/USD exchange rate have GARCH effects clearly, which suggests that it is inadequate to price quanto option under the BS model. In addition, the estimated correlation

Table 4 Descriptive Statistics and Correlations for Dow Jone Index and TWD/USD Historical Returns from January 4, 2010 to December 31, 2019, a Total of 2,473 Daily Observations

Descriptive statistics	Dow Jones Industrial Average Index	TWD/USD
Mean	4.07E − 06	-2.72E − 05
Median	6.52E − 04	0
Max	0.0486	0.0167
Min	-0.0571	-0.0181
SD	0.0089	0.0030
Skewness	-0.4924	-0.3075
Kurtosis	7.0220	7.4269
Correlation	**Dow Jones Industrial Average Index**	**TWD/USD**
Dow Jones Industrial Average Index	1	-0.2129
TWD/USD	-0.2129	1

is significantly different from zero. Hence, ignoring correlation may lead to mispricing of quanto options. Furthermore, our estimation results shows that there is a leverage effect in the equity market.

Here, we adopt the first type of payoff function Equation (1), because of its popularity. We set $E_1 = 29.9075$, $S_1 = 28{,}538.43945/100$,[1] $r_d = 0.01035$, and $r_f = 0.0154$, which are prices and risk-free rates recorded on December 31, 2019. As for the maturity, we set $T = 5$, 20 for short-dated quanto options and $T = 90$, 120 for long-dated quanto options.

We compare the standard estimator M_0, the antithetic variate estimator M_A, the systematic sampling estimator M_S, and the spherical estimator M_V. Again, we

[1] The Dow Jones Industrial Average Index option contract has an underlying value equal to 1/100 of the current index level. For example, when the Dow Jones Industrial Average Index is at 12,000, the underlying value is 120.

Table 5 The Maximum Likelihood Estimation of the GARCH Model Using the Dow Jones Industrial Average Index and TWD/USD Exchange Rate from January 4, 2010 to December 31, 2019, a Total of 2473 Daily Observations

TWD/USD	Estimated parameter	Standard error
α_0	1.26E − 07	2.57E − 08
α_1	0.0683	0.0089
α_2	0.9198	0.0094
a	0.0541	0.0717
δ	-0.0203	0.0194
Dow Jones Industrial Average Index	**Estimated parameter**	**Standard error**
β_0	3.38E − 06	4.25E − 07
β_1	0.1267	0.0106
β_2	0.7299	0.0171
b	0.9176	0.0406
λ	0.0642	0.0177
Correlation	**Estimated parameter**	**Standard error**
ρ	-0.2176	0.0195
Log-likelihood	1,9738.46	

set $V = \{\pm e_1, \ldots, \pm e_{2T}\}$ for the spherical estimator. For fair comparison, we set sample size 10,000 for each estimator. The predetermined $V = \{\pm e_1, \ldots, \pm e_{2T}\}$ is used for M_V. For M_S, we set $k = 4T$ so that it uses the same number of function values as M_V. We obtain the benchmark price using the standard estimator with 10^8 sample size. The pricing error is the estimated price minus benchmark value.

2. Simulation Criteria

The variance and computing time are critical measures for comparing superiority of different estimators. Abbreviate an estimator as a function of a random vector X by $M := M(X)$. To compare the superiority of different estimators, we follow L'Ecuyer (1994) to calculate the efficiency number of an estimator M:

$$\mathrm{eff}(M) = \frac{1}{\mathrm{time}(M)\mathrm{var}[M]}, \tag{18}$$

where time(•) gives average computing time of obtaining a Monte Carlo sample of an estimator and var[•] is the variance operator. The efficiency number defined in Equation (18) is a convenience measure to combine both the variance and computing time. With the efficiency number, for two estimators M_1 and M_2, we say M_1 is more efficient than M_2, if $\mathrm{eff}(M_1) > \mathrm{eff}(M_2)$.

To estimate the efficiency number of an estimator M, we first need to estimate $\mathbb{E}(M)$ by the average of n realizations:

$$\hat{M} = \frac{1}{n}\sum_{i=1}^{n} M(X^{(i)}) \tag{19}$$

where $X^{(i)}$ is the i-th realization. Second, we estimate var(M) by

$$\hat{\mathrm{var}}(M) = \frac{1}{n-1}\sum_{i=1}^{n}\left(M(X^{(i)}) - \hat{M}\right)^2. \tag{20}$$

The estimated computing time time($\hat{M}$) is the total computing time of obtaining $\hat{M}$ divided by the Monte Carlo sample size n. Last, the efficiency number is estimated by

$$\hat{\mathrm{eff}}(M) = \frac{1}{\hat{\mathrm{time}}(M)\hat{\mathrm{var}}(M)}.$$

Furthermore, we calculate the minimum sample size and minimum computing time to achieve a predetermined precision level. Recall that the margin of error for the 95% confidence interval is $1.96\sqrt{\text{var}[M]/n}$. For simplicity, we assume that the margin of error needs to be less than one percent of the option price. Therefore, we require the minimum sample size n^* to satisfy $1.96\sqrt{\text{var}(M)/n^*} \leq 0.01\mathbb{E}[M]$, or equivalently, $n^* \geq \dfrac{38{,}416\text{var}(M)}{(\mathbb{E}[M])^2}$.

Therefore, we set the minimum sample size n^* to be

$$n^* = \text{ceil}\left(\frac{38{,}416\hat{\text{var}}(M)}{\hat{M}^2}\right). \tag{21}$$

where ceil(x) gives the smallest integer that is not less than x.

In practice, we use a large enough sample size n to estimate $\mathbb{E}(M)$ by $\hat{M}$ in Equation (19) and var(M) by $\hat{\text{var}}(M)$ in Equation (20) and plug them in Equation (21) to obtain an estimate of the minimum sample size. Once the minimum sample size is obtained, the minimum computing time is calculated by multiplying $\hat{n}^*$ and time($\hat{M}$).

3. Simulation Results

We compare the evaluated price, variance, computing time, and efficiency number for the standard estimator M_0, the antithetic variates estimator M_A, the systematic sampling estimator M_S, and the spherical estimator M_V. Then, we study the minimum sample size and minimum computing time required to meet predetermined precision level. Finally, we analyze the percentage error versus the Monte Carlo sample size for each estimator.

Tables 6 and 7 compares the evaluated price and estimated variance across various moneyness and maturities among the four estimators, respectively. It is

Table 6 The Evaluated Prices of Quanto Options Using M_0, M_A, M_S, and M_V at Moneyness K/S_1 = 0.9, 0.95, 1.0, 1.05, 1.1 and Maturity T = 5, 20, 90, 120 Days

	Moneyness				
Estimator	0.9	0.95	1.0	1.05	1.1
$T = 5$					
M_0	859.86	432.60	73.31	0.37	0.00
M_A	856.90	432.05	72.86	0.32	0.00
M_S	855.72	432.52	72.93	0.35	0.00
M_V	856.56	432.25	73.10	0.35	0.00
$T = 20$					
M_0	870.46	469.45	148.36	12.42	0.37
M_A	873.54	473.98	147.21	12.64	0.24
M_S	874.69	473.51	148.38	12.54	0.29
M_V	874.66	473.72	148.62	12.65	0.28
$T = 90$					
M_0	995.48	635.85	341.27	143.92	39.75
M_A	992.03	637.31	343.00	144.49	40.61
M_S	991.31	636.65	343.06	141.52	41.28
M_V	992.62	637.22	343.00	141.90	40.94
$T = 120$					
M_0	1,039.54	702.57	406.19	204.83	75.36
M_A	1,037.06	696.65	406.38	198.35	74.87
M_S	1,040.04	694.81	408.36	198.50	76.32
M_V	1,040.20	695.32	407.46	198.46	75.98

shown that the evaluated prices of these four estimators are very close. But, in terms of variance, M_V clearly produces the substantially least variances compared with other estimators. Compared with the variance of M_0, the variance of M_V is about 1,000 times less for T = 5, 20, and about 10,000 times less when T = 90,

Table 7 The Estimated Variances of Pricing Quanto Options Using M_0, M_A, M_S, and M_V at Moneyness K/S_1 = 0.9, 0.95, 1.0, 1.05, 1.1 and Maturity T = 5, 20, 90, 120 Days

	Moneyness				
Estimator	0.9	0.95	1.0	1.05	1.1
$T = 5$					
M_0	33,619.88	30,871.65	9,740.08	57.64	–
M_A	610.34	401.26	2,257.77	18.08	–
M_S	3,517.68	3,432.40	1,332.29	4.19	–
M_V	117.65	97.92	74.46	2.14	–
$T = 20$					
M_0	120,878.81	88,470.55	34,226.50	2,653.72	83.32
M_A	4,075.60	2,186.60	6,618.67	1,320.21	20.64
M_S	3, 023.71	2, 587.98	1, 649.40	191.69	2.20
M_V	70.26	55.05	53.40	17.10	0.78
$T = 90$					
M_0	417,875.58	290,807.13	173,630.62	73,329.73	18,800.68
M_A	22,021.69	25,247.32	34,401.37	26,698.71	8,714.32
M_S	3,992.88	4,011.89	3,692.45	2,054.98	681.04
M_V	58.04	46.95	52.56	41.40	18.81
$T = 120$					
M_0	536,097.67	403,262.09	249,188.34	127,669.60	43,801.77
M_A	36,565.35	40,393.19	49,479.57	41,272.85	19,543.68
M_S	4,397.94	4,641.92	4,399.47	2,873.46	1,228.56
M_V	55.48	49.26	54.02	47.35	24.06

120, whereas the variance of M_A is only 100 times less for in-the-money quanto option. Furthermore, the variances of M_0 and M_A increase significantly as the time to maturity increases, but the variance of M_V remains at similar level.

Table 8 shows that implementing M_V is the most time consuming, because it needs to generate the random orthogonal matrix and calculate function values at each sample. The computation cost increases for options with longer maturities indeed. However, as previously said, M_V enjoys substantial reduction in variance than the other three estimators particularly for long-dated options. To consider both

Table 8 The Total Computing Times in Seconds of Pricing Quanto Options Using M_0, M_A, M_S, and M_V at Moneyness K/S_1 = 0.9, 0.95, 1.0, 1.05, 1.1 and Maturity T = 5, 20, 90, 120 Days

	Moneyness				
Estimator	0.9	0.95	1.0	1.05	1.1
$T = 5$					
M_0	0.30	0.25	0.25	0.25	0.26
M_A	0.23	0.20	0.19	0.18	0.18
M_S	0.84	0.78	0.76	0.77	0.76
M_V	0.46	0.42	0.42	0.42	0.42
$T = 20$					
M_0	0.27	0.26	0.27	0.27	0.27
M_A	0.20	0.20	0.21	0.21	0.21
M_S	5.32	5.28	5.29	5.33	5.34
M_V	3.42	3.36	3.38	3.39	3.38
$T = 90$					
M_0	0.31	0.31	0.31	0.31	0.31
M_A	0.26	0.27	0.26	0.27	0.27
M_S	96.42	93.64	103.36	99.53	97.72
M_V	53.35	50.18	50.28	50.40	50.28
$T = 120$					
M_0	0.33	0.32	0.32	0.33	0.33
M_A	0.29	0.28	0.29	0.29	0.29
M_S	183.84	194.11	204.32	206.88	220.52
M_V	92.84	93.74	91.93	92.63	93.80

the variance reduction and computing time, we investigate the efficiency number for the four estimators in Table 9: M_V produces the highest efficiency number across all maturities and moneyness and dominates the other three estimators.

Now, we calculate the minimum sample sizes of each estimator. Table 10

Table 9 The Efficiency Numbers of Pricing Quanto Options Using M_0, M_A, M_S, and M_V at Moneyness K/S_1 = 0.9, 0.95, 1.0, 1.05, 1.1 and Maturity T = 5, 20, 90, 120 Days

	Moneyness				
Estimator	0.9	0.95	1.0	1.05	1.1
$T = 5$					
M_0	9.78E − 05	1.32E − 04	4.14E − 04	7.00E − 02	–
M_A	7.12E − 03	1.27E − 02	2.34E − 03	3.07E − 01	–
M_S	3.39E − 04	3.75E − 04	9.85E − 04	3.11E − 01	–
M_V	1.84E − 02	2.41E − 02	3.17E − 02	1.11E + 00	–
$T = 20$					
M_0	3.10E − 05	4.31E − 05	1.08E − 04	1.39E − 03	4.38E − 02
M_A	1.21E − 03	2.24E − 03	7.30E − 04	3.66E − 03	2.34E − 01
M_S	6.22E − 05	7.32E − 05	1.15E − 04	9.78E − 04	8.51E − 02
M_V	4.16E − 03	5.40E − 03	5.54E − 03	1.73E − 02	3.77E − 01
$T = 90$					
M_0	7.74E − 06	1.10E − 05	1.86E − 05	4.36E − 05	1.71E − 04
M_A	1.73E − 04	1.48E − 04	1.11E − 04	1.40E − 04	4.30E − 04
M_S	2.60E − 06	2.66E − 06	2.62E − 06	4.89E − 06	1.50E − 05
M_V	3.23E − 04	4.24E − 04	3.78E − 04	4.79E − 04	1.06E − 03
$T = 120$					
M_0	5.70E − 06	7.68E − 06	1.25E − 05	2.40E − 05	7.00E − 05
M_A	9.43E − 05	8.69E − 05	6.90E − 05	8.41E − 05	1.76E − 04
M_S	1.24E − 06	1.11E − 06	1.11E − 06	1.68E − 06	3.69E − 06
M_V	1.94E − 04	2.17E − 04	2.01E − 04	2.28E − 04	4.43E − 04

shows that M_V requires the least minimum sample sizes. Table 11 furthermore justifies that although M_V needs longer average computing time per run, it still takes the least minimum computing times compared with the other three estimators.

Next, we compare estimated variances and computing times versus sample

Table 10 The Estimated Minimum Sample Sizes of Pricing Quanto Options Using M_0, M_A, M_S, and M_V at Moneyness K/S_1 = 0.9, 0.95, 1.0, 1.05, 1.1 and Maturity T = 5, 20, 90, 120 Days

	Moneyness				
Estimator	0.9	0.95	1.0	1.05	1.1
$T = 5$					
M_0	1,747	6,338	69,623	16,070,249	–
M_A	32	83	16,338	6,969,451	–
M_S	185	705	9,624	1,342,494	–
M_V	7	21	536	677,264	–
$T = 20$					
M_0	6,129	15,422	59,737	660,601	23,289,059
M_A	206	374	11,734	317,224	13,465,939
M_S	152	444	2,878	46,814	1,011,316
M_V	4	10	93	4,108	384,406
$T = 90$					
M_0	16,200	27,632	57,274	135,995	457,032
M_A	860	2,388	11,234	49,129	203,042
M_S	157	381	1,206	3,942	15,351
M_V	3	5	18	79	432
$T = 120$					
M_0	19,058	31,385	58,020	116,899	296,297
M_A	1,307	3,198	11,511	40,301	133,932
M_S	157	370	1,014	2,802	8,103
M_V	2	4	13	47	161

Table 11 The Estimated Minimum Computing Times of Pricing Quanto Options Using M_0, M_A, M_S, and M_V for Moneyness K/S_1 = 0.9, 0.95, 1.0, 1.05, 1.1 and Maturity T = 5, 20, 90, 120 Days

	Moneyness				
Estimator	0.9	0.95	1.0	1.05	1.1
$T = 5$					
M_0	0.05	0.16	1.73	398.54	–
M_A	0.00	0.00	0.31	125.45	–
M_S	0.02	0.05	0.73	102.84	–
M_V	0.00	0.00	0.02	28.43	–
$T = 20$					
M_0	0.16	0.40	1.61	17.90	638.12
M_A	0.00	0.01	0.24	6.57	278.74
M_S	0.08	0.23	1.52	24.97	540.35
M_V	0.00	0.00	0.03	1.39	130.07
$T = 90$					
M_0	0.50	0.86	1.78	4.26	14.21
M_A	0.02	0.06	0.30	1.32	5.42
M_S	1.51	3.57	12.47	39.24	150.00
M_V	0.02	0.03	0.09	0.40	2.17
$T = 120$					
M_0	0.62	1.01	1.87	3.81	9.66
M_A	0.04	0.09	0.34	1.16	3.88
M_S	2.89	7.18	20.72	57.97	178.68
M_V	0.02	0.04	0.12	0.44	1.51

sizes for the four estimators. Here, we discuss deep in-the-money, at-the-money, and deep out-of-the-money quanto options with T = 120 in Figures 6–8, respectively. In general, M_S takes longest computing time, followed by M_V. And, M_0 and M_A take about the least computing time.

For in-the-money options (K/S_1 = 0.9), variances of the four estimators

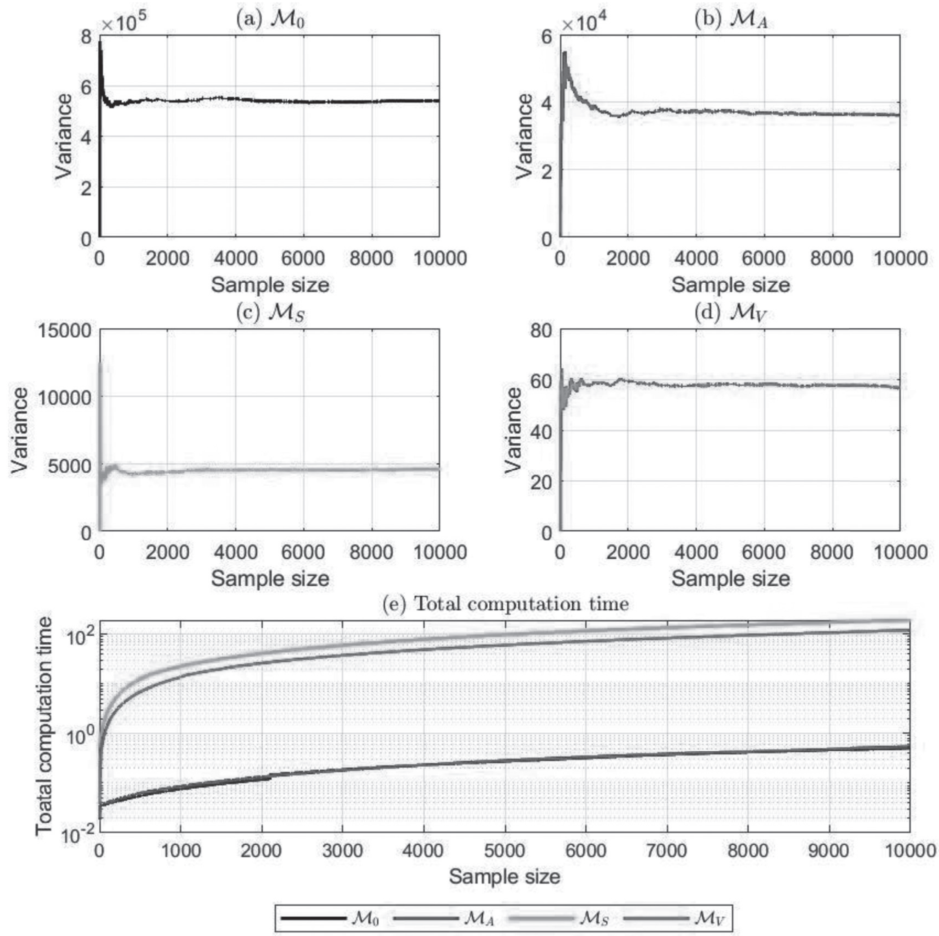

Figure 6 For In-the-Money Options ($K/S_1 = 0.9$), Estimated Variances of M_0, M_A, M_S, and M_V Versus the Sample Size Are Depicted in (a), (b), (c), and (d), Respectively. The Corresponding Total Computation Times in Seconds Are Illustrated in (e)

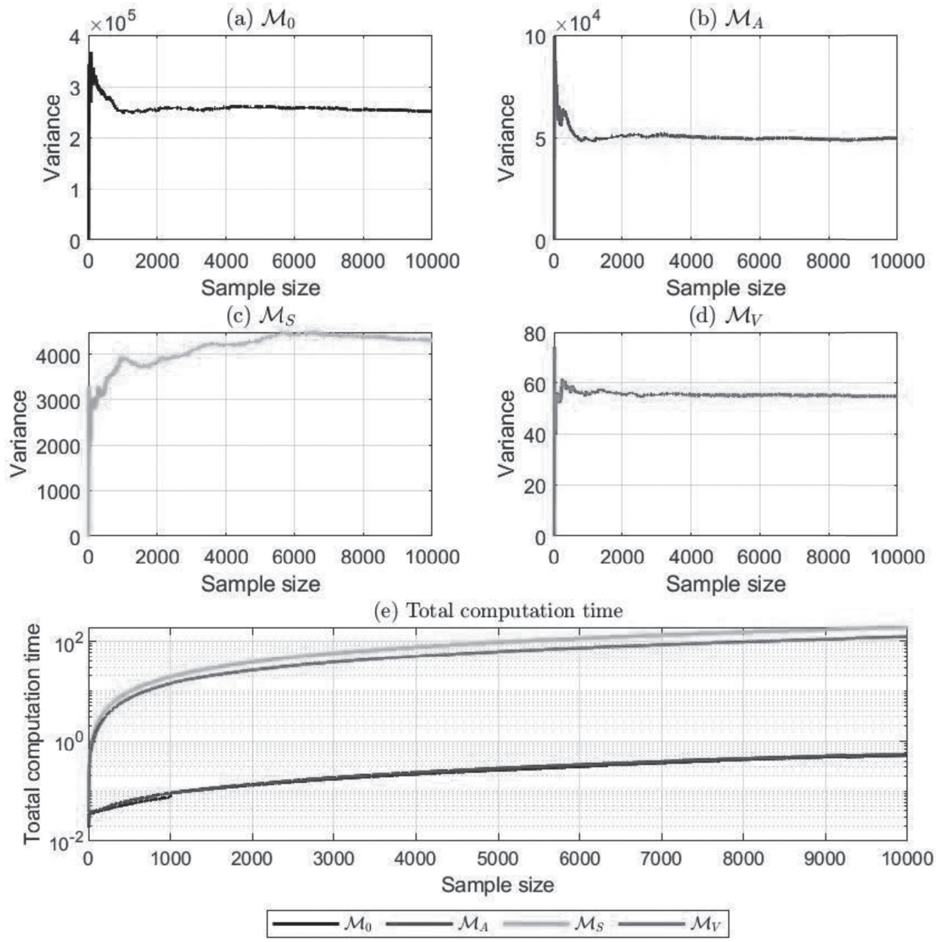

Figure 7 For At-the-Money Options (K/S_1 = 1.0), Estimated Variances of M_0, M_A, M_S, and M_V Versus the Sample Size are Depicted in (a), (b), (c), and (d), Respectively. The Corresponding Total Computation Times in Seconds Are Illustrated in (e)

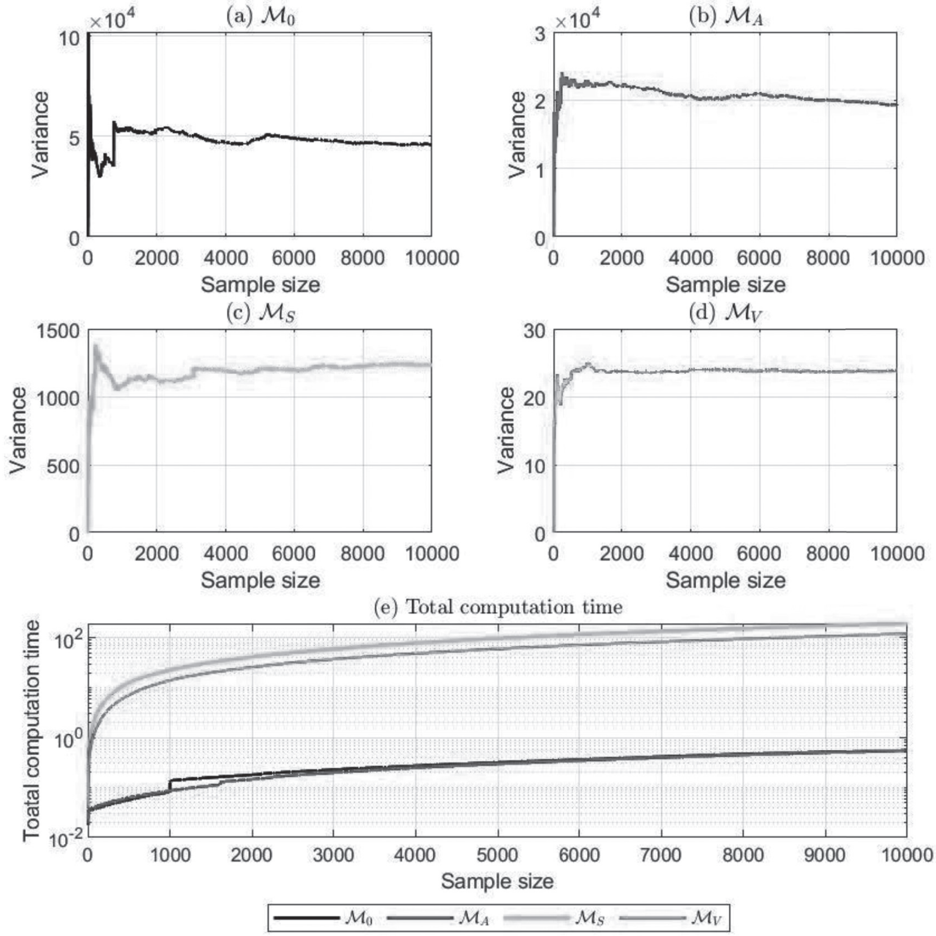

Figure 8 For Out-of-the-Money Options (K/S_1 = 1.1), Estimated Variances of M_0, M_A, M_S, and M_V Versus the Sample Size Are Depicted in (a), (b), (c), and (d), Respectively. The Corresponding Total Computation Times in Seconds Are Illustrated in (e)

converge when the sample size is larger than 1,000. For at-the-money options ($K/S_1 = 1.0$), M_0, M_A and M_V converge when the sample size is larger than 1,000, but M_S starts to converge when the sample size is larger than 5,000. But, for out-of-the-money options ($K/S_1 = 1.1$), the variances of both M_0 and M_S oscillate more drastically than those of M_A and M_V: the variance of M_V starts to stabilize with a smaller sample size than M_A.

In summary, the variance of M_V starts to converge with a substantially small sample size, in contrast to the other three estimators: M_V produces stable variances when the sample size is larger than 2,000. The above experiments numerically confirm the fast convergence and robustness of the proposed spherical estimator.

4. Comparisons of Prices for the Quanto Option between the BS and GARCH Models

To compare the price of quanto options under the BS and GARCH models, the volatility for the BS model is estimated as the square root of the annually unconditional variance in the GARCH model. Doing so allows us to observe the contribution solely from the stochastic nature in volatility while holding the long-run variances fixed. Specifically, we estimate the two volatilities by

$$\sigma_E = \sqrt{\frac{252\alpha_0}{(1 - \alpha_1(1 + a^2) - \alpha_2)}} \text{ and } \sigma_S = \sqrt{\frac{252\beta_0}{(1 - \beta_1(1 + b^2) - \beta_2)}},$$

where σ_S is volatility of the Dow Jones Industrial Average Index and σ_E is volatility of exchange rate between TWD/USD. Correlation is estimated using the maximum likelihood estimate under the GARCH model, which is used for both the BS and GARCH models.

As a note, because M_V is the most efficient estimator, we use it to price quanto options under the GARCH model. The numerical results are summarized in Table 12. For in-the-money quanto options, prices under both models are about the same for $T = 5$, but prices under the BS model are lower than those under the GARCH model as T increases. For out-of-the-money quanto options: prices under both models are about the same for $T = 5$, but prices under the BS model are higher than those under the GARCH model as time to maturity increases. For at-the-money quanto options, prices under the BS model are higher than those under the GARCH model for $T = 5, 20, 90, 120$.

Although it is shown that prices under the BS and GARCH models are about the same of deep in-the-money and deep out-of-the-money for short-dated

Table 12 The Prices of Quanto Options of BS and GARCH Models for Moneyness K/S_1 = 0.9, 0.95, 1.0, 1.05, 1.1 and Maturity T = 5, 20, 90, 120 Days

	Moneyness				
Model	0.9	0.95	1.0	1.05	1.1
$T = 5$					
BS	856.24	430.04	74.41	0.77	0.00
GARCH	856.56	432.25	73.10	0.35	0.00
$T = 20$					
BS	865.12	456.85	151.75	25.36	1.91
GARCH	874.66	473.72	148.62	12.65	0.28
$T = 90$					
BS	941.58	598.34	336.01	164.69	70.17
GARCH	992.62	637.22	343.00	141.90	40.94
$T = 120$					
BS	979.23	649.46	392.79	215.32	106.86
GARCH	1,040.20	695.32	407.46	198.46	75.98

quanto options, this is not the case for long-dated quanto options. As a result, ignoring time-varying volatility may lead to pricing error for pricing long-dated quanto options.

VI. Extensions: Pricing Multi-Asset Options

Multi-assets options (also known as rainbow options) refer to options whose payoff functions depend on multiple underlying assets. To address the generality of the spherical estimator, the spherical estimator can be extended to price more exotic options, such as multi-asset options, in a straightforward manner. Table 13 lists payoff functions of popular multi-asset options.

For simplicity, we assume the underlying assets follow the BS framework. That is, under the pricing measure, the i-th underlying asset S_{it} at time t follows the stochastic differential equation:

$$dS_{it} = rS_{it}dt + \sigma_i S_{it} dW_{it}$$

where W_{it} and W_{jt} are two correlated Wiener processes with correlation $\mathrm{Corr}(dW_{it}, dW_{jt}) = \rho_{ij}dt$. Here, r is the risk-free rate, σ_i is the volatility of the i-th underlying.

Because the payoff function of multi-asset options involves the asset prices at maturity time T, we only need to generate the stock prices at maturity time T for $i = 1, \ldots, d$. Ito's formula allows us to generate the asset prices at maturity time T by

$$S_{iT} = S_{i0}\exp\left(\left(r - \frac{\sigma_i^2}{2}\right)T + \sigma_i\sqrt{T}X\right),$$

where S_{i0} is the initial price of the i-th asset, $X \sim N_d(\mathbf{0}, \Lambda)$. Let d be an even number for simplicity. Suppose that the covariance matrix Λ is

Table 13 List of the Payoff Function for Multi-Asset Options. S_{iT} and K_i is the i-th Asset Price at Maturity and Corresponding Strike Price for $i = 1, \ldots, d$

Type of rainbow options	Pricing formula	Reference
Margrabe option	$\max(S_{2T} - S_{1T}, 0)$	Margrabe (1978)
Spread option	$\max(S_{2T} - S_{1T} - K, 0)$	Pearson (1995); Carmona and Durrleman (2003)
Better-off option	$\max(S_{1T}, \ldots, S_{dT})$	Nenkin (1996); Wystup (2006)
Worst-off option	$\min(S_{1T}, \ldots, S_{dT})$	Nenkin (1996); Wystup (2006)
Maximum option	$\max(\max(S_{1T}, \ldots, S_{dT}) - K, 0)$	Stulz (1982); Johnson (1987)
Minimum option	$\max(\min(S_{1T}, \ldots, S_{dT}) - K, 0)$	Stulz (1982); Johnson (1987)
Multi-strike option	$\max(S_{1T} - K_1, \ldots, S_{dT} - K_d, 0)$	Nenkin (1996); Wystup (2006)
Madonna rainbow option	$\max(\sqrt{(S_{1T} - K_1)^2 + \ldots + (S_{dT} - K_d)^2} - K, 0)$	Nenkin (1996); Wystup (2006)
Pyramid rainbow option	$\max(\lvert S_{1T} - K_1\rvert + \ldots + \lvert S_{dT} - K_d\rvert - K)$	Nenkin (1996); Wystup (2006)
Arithmetic average option	$\max\left(\frac{S_{1T} + \ldots + S_{dT}}{d} - K, 0\right)$	Hull (2018)

$$\Lambda = \begin{bmatrix} I_{d/2} & \rho I_{d/2} \\ \rho I_{d/2} & I_{d/2} \end{bmatrix}.$$

We consider pricing arithmetic average options with payoff function:

$$\max\left(\frac{S_{1T} + \ldots + S_{dT}}{d} - K, 0\right),$$

where K is the strike price. Furthermore, we set sample size to be 10,000 for the four estimators: M_0, M_A, M_S, and M_V. We use the standard estimator with sample size 10^8 to obtain the benchmark value. Furthermore, we set $T = 1$, $r = 0$, $S_{i0} = 90 + 5i$, $\sigma_i = 0.25$ for $i = 1, \ldots, d$, $\rho = 0.5$, and $d = 10, 20, 50$. We define the moneyness for the arithmetic average option by

$$Moneyness = \frac{Strike\ price}{\frac{1}{d}\sum_{i=1}^{d} S_{i0}}.$$

The estimated price, variance, and efficiency number are reported in Tables 14–16, respectively. The same conclusion is drawn: M_V produces the least estimated errors and variances, and the highest efficiency numbers, in most cases. We remark that the same conclusion could be drawn for $\rho = 0, -0.5$ and we omit reporting these numerical results for brevity.

VII. Summary

1. Conclusion

We consider to price quanto options based on TWD/USD exchange rate and Dow Jones Industrial Average Index because of its popularity in the financial market of Taiwan. We introduce the spherical estimator and compare it with the

Table 14 The Estimated Prices of Arithmetic Average Options with Time to Maturity One Year Using M_0, M_A, M_S, and M_V at Moneyness 0.9, 0.95, 1.0, 1.05, 1.1

	Moneyness				
Estimator	0.9	0.95	1.0	1.05	1.1
$n = 10$					
M_0	12.01	7.81	4.39	2.27	1.05
M_A	12.00	7.64	4.39	2.29	1.03
M_S	11.97	7.65	4.43	2.26	1.01
M_V	12.01	7.67	4.41	2.24	1.02
$n = 20$					
M_0	13.99	8.10	3.89	1.37	0.39
M_A	13.99	8.15	3.80	1.47	0.41
M_S	13.98	8.16	3.88	1.41	0.41
M_V	14.01	8.14	3.85	1.42	0.41
$n = 30$					
M_0	16.39	8.84	3.63	1.12	0.19
M_A	16.35	9.06	3.69	1.05	0.20
M_S	16.38	9.06	3.75	1.07	0.20
M_V	16.37	9.05	3.73	1.06	0.20

standard, antithetic variates, and systematic sampling estimators. Our numerical studies show that given a precision level, the spherical estimator requires the least minimum sample sizes and shortest computing time. Furthermore, it is shown that the estimated variance of the spherical estimator starts to converge with the smallest sample size. At last, the spherical estimator can be applied to pricing rainbow options in a straightforward manner. In summary, our simulation results confirm the superiority of the proposed spherical estimator in general in terms of variance reduction and computing time.

Table 15 The Estimated Variances of Pricing Arithmetic Average Options with Time to Maturity One Year Using M_0, M_A, M_S, and M_V at Moneyness 0.9, 0.95, 1.0, 1.05, 1.1

	Moneyness				
Estimator	0.9	0.95	1.0	1.05	1.1
$n = 10$					
M_0	100.66	74.26	46.15	24.85	11.41
M_A	7.59	11.88	13.88	9.89	5.09
M_S	13.33	11.18	7.74	4.39	1.69
M_V	0.42	0.43	0.48	0.35	0.23
$n = 20$					
M_0	83.01	62.83	34.52	12.84	3.17
M_A	3.59	7.08	9.68	5.56	1.67
M_S	6.29	5.47	3.38	1.30	0.32
M_V	0.11	0.11	0.14	0.08	0.04
$n = 30$					
M_0	82.62	63.22	31.26	9.32	1.39
M_A	2.84	5.53	8.94	3.87	0.78
M_S	4.58	4.04	2.40	0.69	0.09
M_V	0.06	0.06	0.08	0.03	0.01

2. Future Works

This paper focuses on pricing quanto options, because foreign exchange market is one of the largest financial markets. On the other hand, another type of quanto options, i.e., the energy quantos, also arouse great popularity in recent studies. Therefore, it is interesting to apply an efficient simulation scheme for pricing energy quantos. In addition, it is worthy of employing the idea of importance sampling for pricing deep out-of-sample quanto options. On the

Table 16 The Efficiency Numbers of Pricing Arithmetic Average Options with Time to Maturity One Year Using M_0, M_A, M_S, and M_V at Moneyness 0.9, 0.95, 1.0, 1.05, 1.1

	Moneyness				
Estimator	0.9	0.95	1.0	1.05	1.1
$n = 10$					
M_0	4.08E − 01	5.92E − 01	9.93E − 01	1.77E + 00	3.66E + 00
M_A	3.06E + 00	2.00E + 00	1.64E + 00	2.21E + 00	4.90E + 00
M_S	7.84E − 02	9.34E − 02	1.32E − 01	2.30E − 01	6.14E − 01
M_V	5.17E + 00	5.15E + 00	4.43E + 00	6.16E + 00	9.85E + 00
$n = 20$					
M_0	5.07E − 01	6.75E − 01	1.21E + 00	3.03E + 00	1.30E + 01
M_A	6.00E + 00	3.23E + 00	2.36E + 00	4.14E + 00	1.34E + 01
M_S	7.50E − 02	8.63E − 02	1.40E − 01	3.67E − 01	1.52E + 00
M_V	8.26E + 00	8.54E + 00	6.79E + 00	1.24E + 01	2.39E + 01
$n = 30$					
M_0	4.78E − 01	5.84E − 01	1.30E + 00	4.19E + 00	2.83E + 01
M_A	7.59E + 00	3.87E + 00	2.42E + 00	5.48E + 00	2.76E + 01
M_S	5.62E − 02	6.47E − 02	1.07E − 01	3.79E − 01	2.77E + 00
M_V	6.69E + 00	7.12E + 00	5.23E + 00	1.18E + 01	2.74E + 01

other hand, how to efficiently calculate price sensitivities of quanto options is also critical for risk management. Last, the question about which models work better for quanto options can only be answered by empirical analysis. Therefore, models calibration and comparisons of GARCH-typed models versus continuous time series models, such as stochastic volatility models (Romo, 2012) and Lévy processes, is worthy of further research.

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