Introduction of SVCJ

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P measure

According to Hou et al. (2020) and Asgharian et al. (2006) , the stochastic volatility with correlated jumps (SVCJ) model can be written as :

- $d\log S_t = \mu dt + \sqrt{V_t} dW_t^S + Z_t^y dN_t$
- $dV_t = \kappa(\theta - V_t)dt + \sigma_V \sqrt{V_t} dW_t^V + Z_t^v dN_t$
- $\mathrm{Cov}(dW_t^S, dW_t^V) = \rho dt$
- $P(dN_t = 1) = \lambda dt$
- $Z_t^y \,|\, Z_t^v \sim N(\mu_y + \rho_j Z_t^v, \sigma_y^2);\ \ Z_t^v \sim \mathrm{Exp}(\mu_v)$

Definition

- $d\log S_t = \mu dt + \sqrt{V_t} dW_t^S + \boxed{Z_t^y} \boxed{dN_t}$

Jump size

Jump process with intensity parameter λ

- $dV_t = \boxed{\kappa(\theta - V_t)dt + \sigma_V \sqrt{V_t} dW_t^V} + \boxed{Z_t^v} \boxed{dN_t}$

CIR model :
Assume rates tend to move toward a long-term equilibrium level over time.

Definition

Expected log return

- $d\log S_t = \boxed{\mu} dt + \sqrt{V_t} dW_t^S + Z_t^y dN_t$

Mean reversion rate

Volatility of process

- $dV_t = \boxed{\kappa}(\boxed{\theta} - V_t)dt + \boxed{\sigma_V}\sqrt{V_t} dW_t^V + Z_t^v dN_t$

Mean reversion level

Definition

- $d\log S_t = \mu dt + \sqrt{V_t}\boxed{dW_t^S} + Z_t^y dN_t$

Brownian motion with correlation ρ

$dW \sim N(0,1)$

- $dV_t = \kappa(\theta - V_t)dt + \sigma_V\sqrt{V_t}\boxed{dW_t^V} + Z_t^v dN_t$

Method

- Utilizing Bayesian inference to obtain the posterior distribution of its parameters by Markov Chain Monte Carlo (MCMC) method.
- Discretization interval in one day, then the models become :

$$\begin{pmatrix} Y_{(t+1)\Delta} \\ V_{(t+1)\Delta} \end{pmatrix} = \begin{pmatrix} \mu \\ \alpha + \left(\frac{1}{\Delta} + \beta\right) V_{t\Delta} \end{pmatrix} \Delta + \sqrt{V_{t\Delta}\Delta} \begin{pmatrix} \varepsilon^{Y}_{(t+1)\Delta} \\ \sigma_V \varepsilon^{V}_{(t+1)\Delta} \end{pmatrix} + \begin{pmatrix} Z^{Y}_{(t+1)\Delta} \\ Z^{V}_{(t+1)\Delta} \end{pmatrix} J_{(t+1)\Delta}$$

$$\text{Parameters} : \Theta = \{\mu, \mu_y, \sigma_y, \lambda, \alpha, \beta, \sigma_v, \rho, \rho_j, \mu_v\}$$

$$\text{Latent variables} : X_t = \{V_t, Z_t^y, Z_t^v, J_t\}$$

Parameters

$$p(\mu) = \frac{1}{\sqrt{50\pi}} e^{-\frac{\mu^2}{50}} \rightarrow \mu \sim N(0,25)$$

$$p(\alpha, \beta) = \frac{1}{2\pi} e^{-\frac{1}{2}(\alpha^2+\beta^2)} \rightarrow (\alpha, \beta) \sim N(0_{2\times 1}, I_{2\times 2})$$

$$p(\sigma_v^2) = \frac{0.1^{2.5}}{\Gamma(2.5)} (\frac{1}{\sigma_v^2})^{\alpha+1} e^{-\frac{\beta}{\sigma_v^2}} \rightarrow \sigma_v^2 \sim IG(2.5,0.1)$$

$$p(\mu_y) = \frac{1}{\sqrt{200\pi}} e^{-\frac{\mu_y^2}{200}} \rightarrow \mu_y \sim N(0,100)$$

$$p(\sigma_y^2) = \frac{40^{10}}{\Gamma(10)} (\frac{1}{\sigma_y^2})^{\alpha+1} e^{-\frac{\beta}{\sigma_y^2}} \rightarrow \sigma_y^2 \sim IG(10,40)$$

$$p(\rho) = \frac{1}{2}, \ -1 \leq \rho \leq 1 \rightarrow \rho \sim U(-1,1)$$

$$p(\rho_j) = \frac{1}{\sqrt{8\pi}} e^{-\frac{\rho_j^2}{8}} \rightarrow \rho_j \sim N(0,4)$$

$$p(\mu_v) = \frac{20^{10}}{\Gamma(10)} (\frac{1}{\mu_v})^{\alpha+1} e^{-\frac{\beta}{\mu_v}} \rightarrow \mu_v \sim IG(10,20)$$

$$p(\lambda) = \frac{\Gamma(42)}{\Gamma(2)\Gamma(40)} \lambda(1-\lambda)^{39}, \ 0 \leq \lambda \leq 1 \rightarrow \lambda \sim Beta(2,40)$$

Latent variables

$$p(V_t \mid V_{t-1}, Z_t^v, J_t, \alpha, \beta, \sigma_v) = \frac{1}{\sqrt{2\pi(\sigma_v^2 V_{t-1})}} e^{-\frac{(V_t - (\alpha + \beta V_{t-1} + Z_t^v J_t))^2}{2(\sigma_v^2 V_{t-1})}} \rightarrow V_t \sim N(\alpha + \beta V_{t-1} + Z_t^v J_t, \sigma_v^2 V_{t-1})$$

$$p(Z_t^v \mid \mu_v) = \frac{1}{\mu_v} e^{-\frac{Z_t^v}{\mu_v}} \rightarrow Z_t^v \sim Exp(\mu_v)$$

$$p(Z_t^y \mid Z_t^v, \mu_y, \rho_j, \sigma_y^2) = \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{-\frac{(Z_t^y - (\mu_y + \rho_j Z_t^v))^2}{2\sigma_y^2}} \rightarrow Z_t^y \mid Z_t^v \sim N(\mu_y + \rho_j Z_t^v, \sigma_y^2)$$

$$p(J_t = j \mid \lambda) = \lambda^j (1 - \lambda)^{1-j} \rightarrow J_t \sim Ber(\lambda)$$

Method

$$\begin{pmatrix} Y_{(t+1)\Delta} \\ V_{(t+1)\Delta} \end{pmatrix} = \begin{pmatrix} \mu \\ \underbrace{\boxed{\alpha}}_{\kappa\theta} + \left(\frac{1}{\Delta} + \underbrace{\boxed{\beta}}_{1-\kappa}\right) V_{t\Delta} \end{pmatrix} \Delta + \sqrt{V_{t\Delta}\Delta} \begin{pmatrix} \varepsilon^{Y}_{(t+1)\Delta} \\ \sigma_V \varepsilon^{V}_{(t+1)\Delta} \end{pmatrix} + \begin{pmatrix} Z^{Y}_{(t+1)\Delta} \\ Z^{V}_{(t+1)\Delta} \end{pmatrix} \boxed{J_{(t+1)\Delta}}$$

$J_{(t+1)\Delta} = 1$ indicates a jump arrival which occurs with probability $\Delta\lambda$

Method

- The Bayesian formula gives the posterior distribution.
- For example, finding the posterior distribution of μ.

$$p(\mu \mid Y, V) \propto p(Y, V \mid \mu) p(\mu)$$

$$\mu^{(i+1)} \mid Y, \alpha^{(i)}, \beta^{(i)}, \mu_y^{(i)}, \sigma_y^{2(i)}, \rho_j^{(i)}, \mu_v^{(i)}, \lambda^{(i)}, \rho^{(i)}, \sigma_v^{2(i)}, X^{(i)} \sim N(a, A)$$

$$, where\ a = A\left(\frac{\Delta}{1-\rho^2} \sum_{t=1}^{T} \frac{e_{Y,t}^{\mu} - \frac{\rho}{\sigma_v} e_{V,t}^{\mu}}{V_{t-1}}\right),\ A = \left(\frac{\Delta^2}{1-\rho^2} \sum_{t=1}^{T} \frac{1}{V_{t-1}} + \frac{1}{25}\right)^{-1}$$

Method

- Because our posterior distribution generally has no close-form, using MCMC to approximate probability distribution.
- Since we have many variables, Gibb Sampling can help us to deal with high-dimensional distribution.

MCMC algorithm

$$\mu^{(i+1)} \,|\, Y, \alpha^{(i)}, \beta^{(i)}, \mu_y^{(i)}, \sigma_y^{2(i)}, \rho_j^{(i)}, \mu_v^{(i)}, \lambda^{(i)}, \rho^{(i)}, \sigma_v^{2(i)}, X^{(i)} \sim N(a, A)$$

$$\alpha^{(i+1)} \,|\, Y, \mu^{(i+1)}, \beta^{(i)}, \mu_y^{(i)}, \sigma_y^{2(i)}, \rho_j^{(i)}, \mu_v^{(i)}, \lambda^{(i)}, \rho^{(i)}, \sigma_v^{2(i)}, X^{(i)} \sim N(b, B)$$

$$\vdots$$

Volatility (for t = 1, … ,T), using a random walk Hastings-Metropolis algorithm for V_t

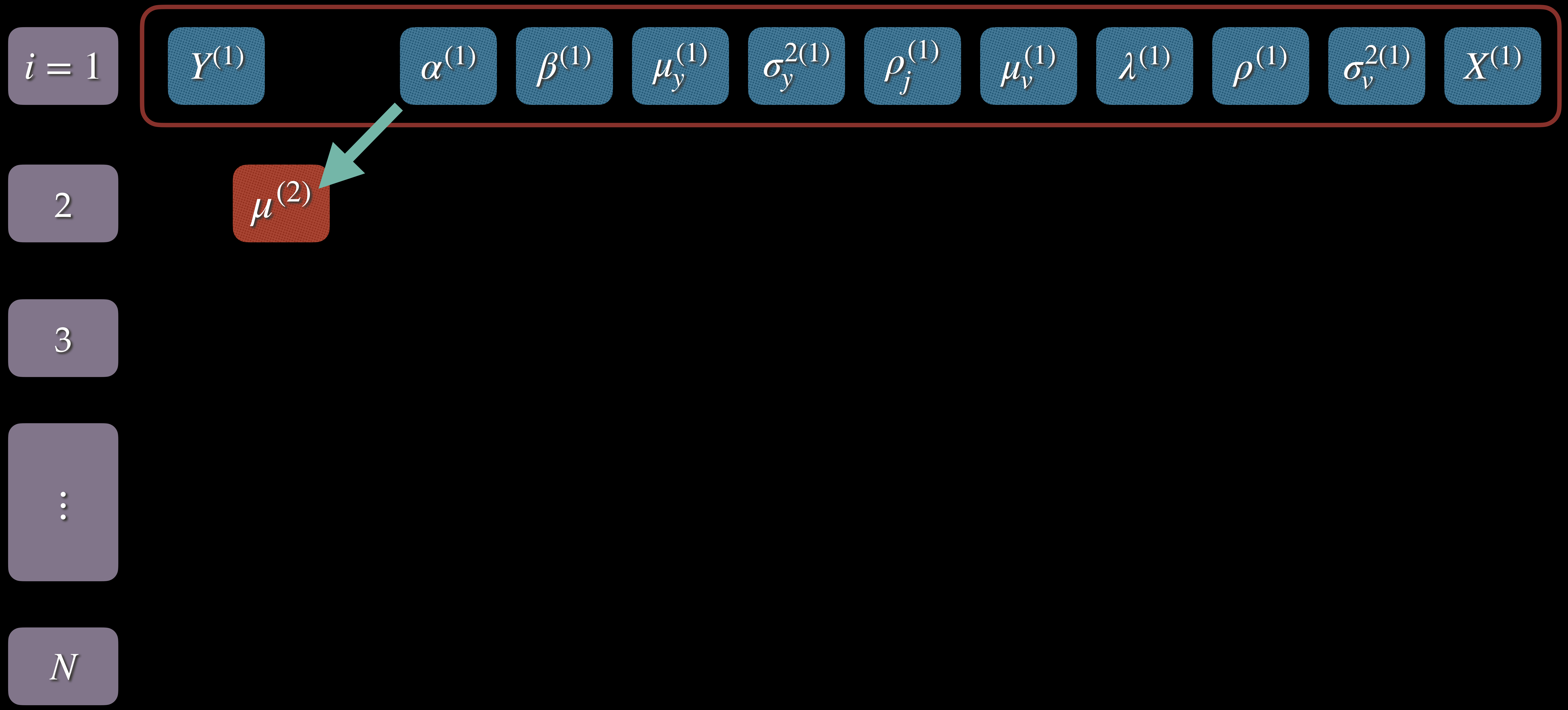
$$f(V_t) \propto V_t^{-1} e^{-\frac{1}{2}\sum_{i=0}^{1}\left(\frac{Y_{t+i} - \mu\Delta - Z_{t+i}^{y} J_{t+i} - \frac{\rho}{\sigma_v} V_{t+i} - V_{t-1+i} - \alpha\Delta - \beta V_{t-1+i}\Delta - Z_{t+i}^{v} J_{t+i}}{(1-\rho^2) V_{t-1+i}\Delta} + \frac{V_{t+i} - V_{t-1+i} - \alpha\Delta - \beta V_{t-1+i}\Delta - Z_{t+i}^{v} J_{t+i}}{\sigma_v^2 V_{t-1+i}\Delta}\right)}$$

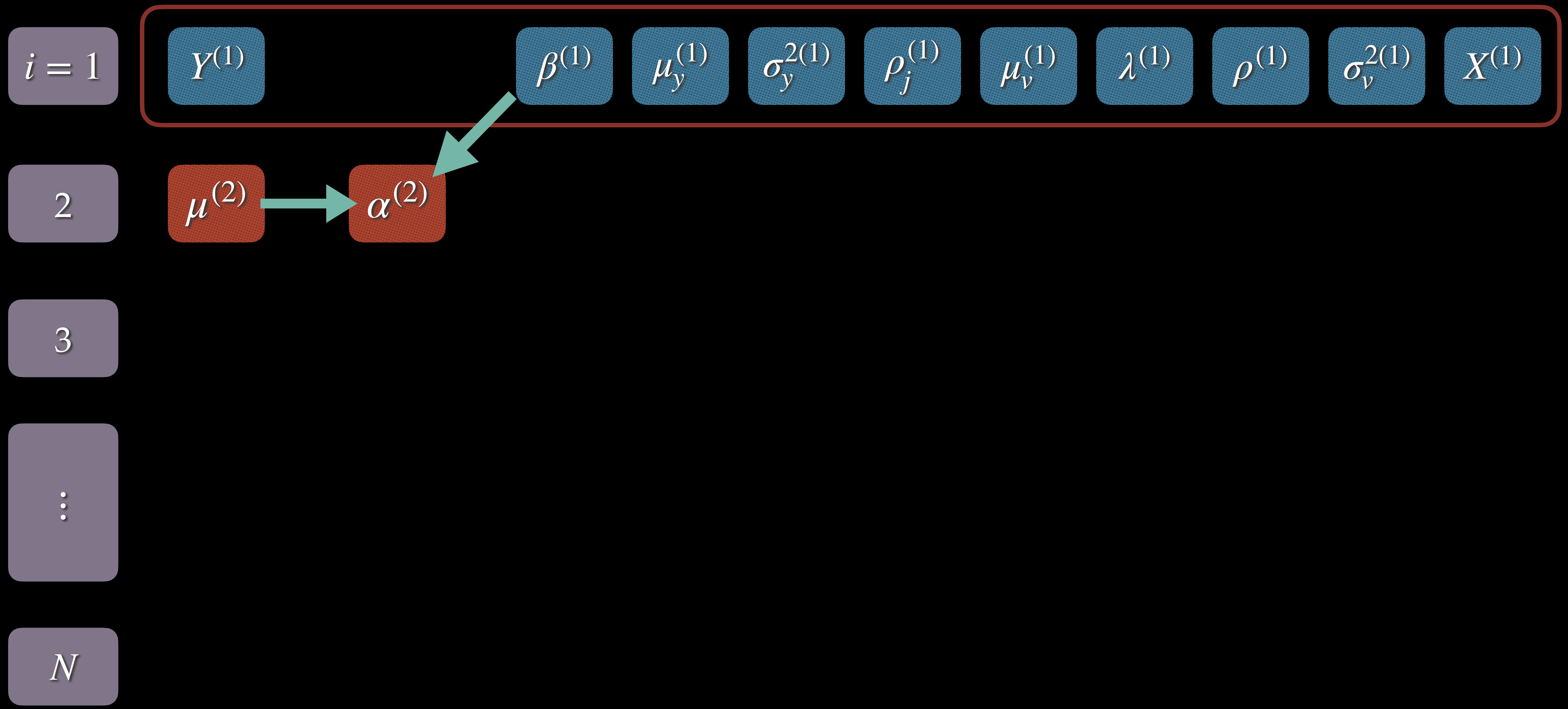
$$\text{Proposal} : V_t^{(i+1)} = V_t^{(i)} + \varepsilon_t \;\; \text{where} \;\; \varepsilon_t \sim t(df = 4.5)$$

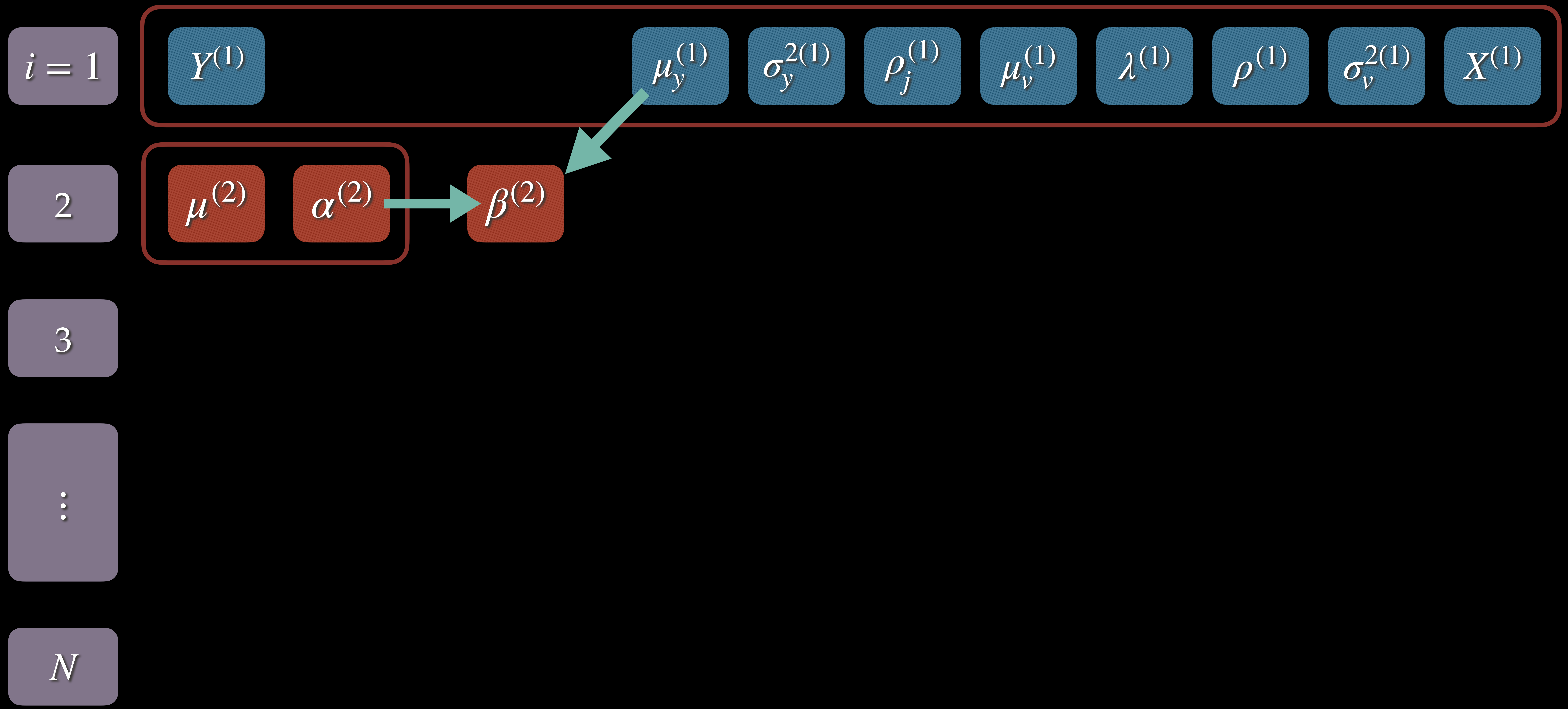
$$\text{Acceptance rate} : A(V_t^{(i)}, V_t^{(i+1)}) = \min\left(\frac{f(V_t^{(i+1)}) q(-\varepsilon_t)}{f(V_t^{(i)}) q(\varepsilon_t)}, 1\right)$$

Thus, we can draw $V_1^{i+1}, V_2^{i}, \cdots V_T^{i}$

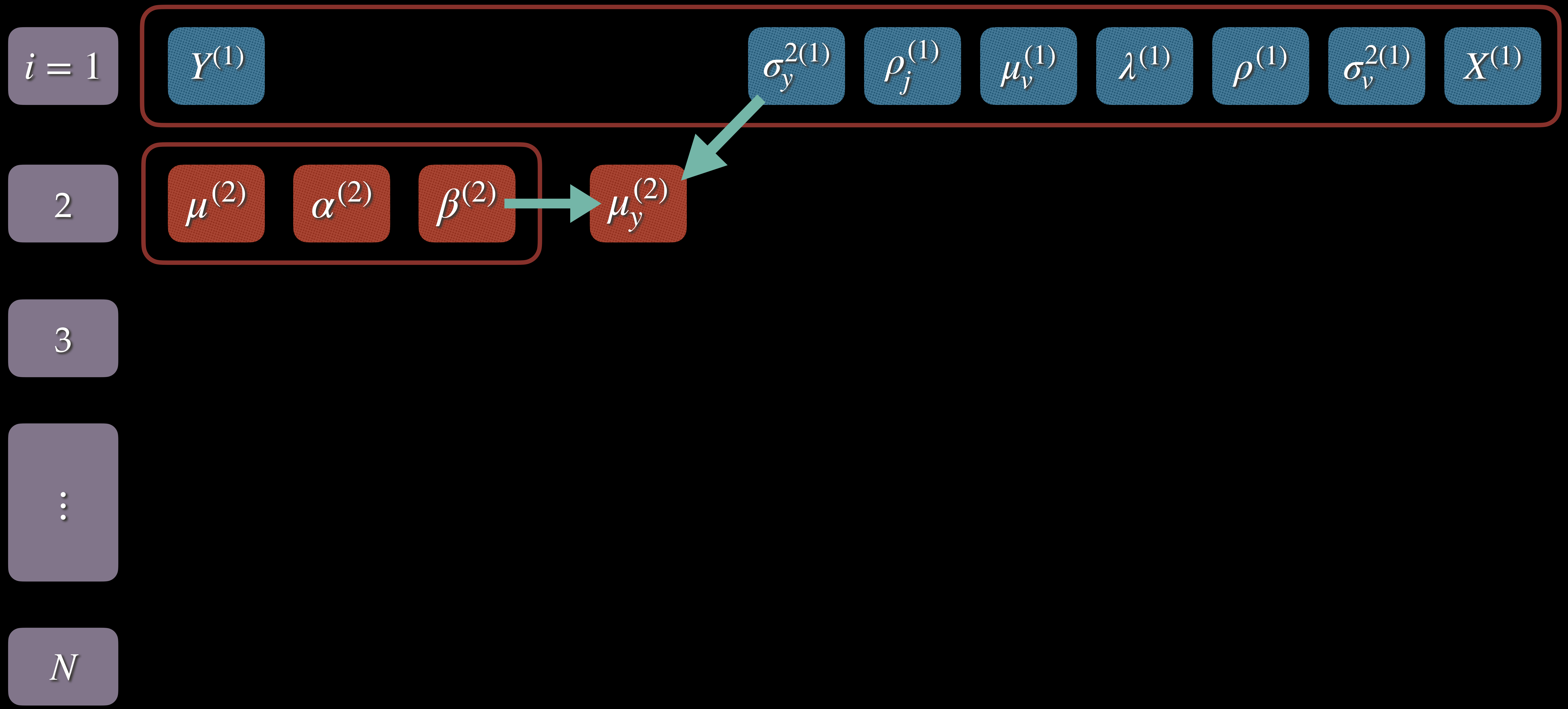
$i = 1$	$Y^{(1)}$	$\mu^{(1)}$	$\alpha^{(1)}$	$\beta^{(1)}$	$\mu_y^{(1)}$	$\sigma_y^{2(1)}$	$\rho_j^{(1)}$	$\mu_v^{(1)}$	$\lambda^{(1)}$	$\rho^{(1)}$	$\sigma_v^{2(1)}$	$X^{(1)}$
2	$\mu^{(2)}$											
3												
$\vdots$												
N												

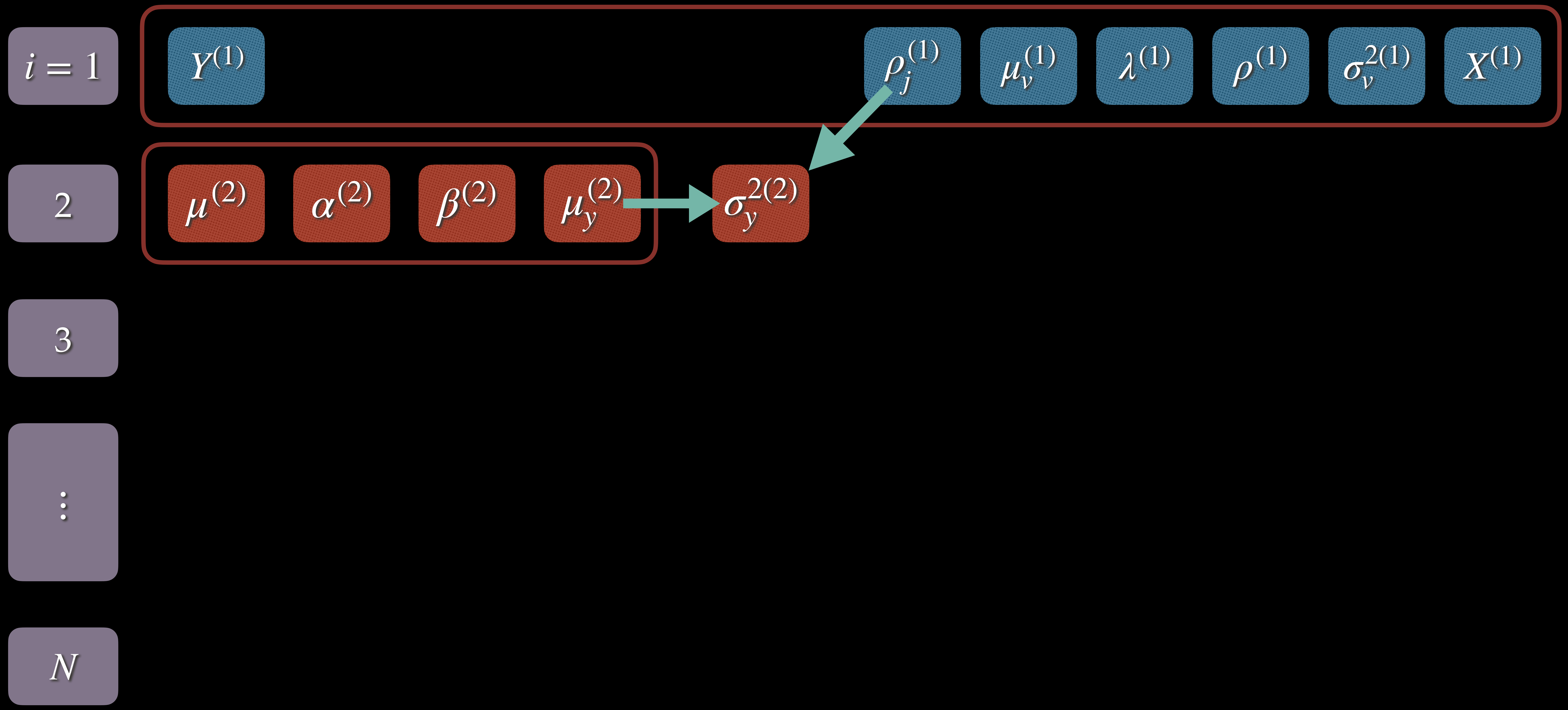

$i = 1$
$Y^{(1)}$
$\alpha^{(1)}$
$\beta^{(1)}$
$\mu_y^{(1)}$
$\sigma_y^{2(1)}$
$\rho_j^{(1)}$
$\mu_v^{(1)}$
$\lambda^{(1)}$
$\rho^{(1)}$
$\sigma_v^{2(1)}$
$X^{(1)}$
2
$\mu^{(2)}$
3
$\vdots$
N

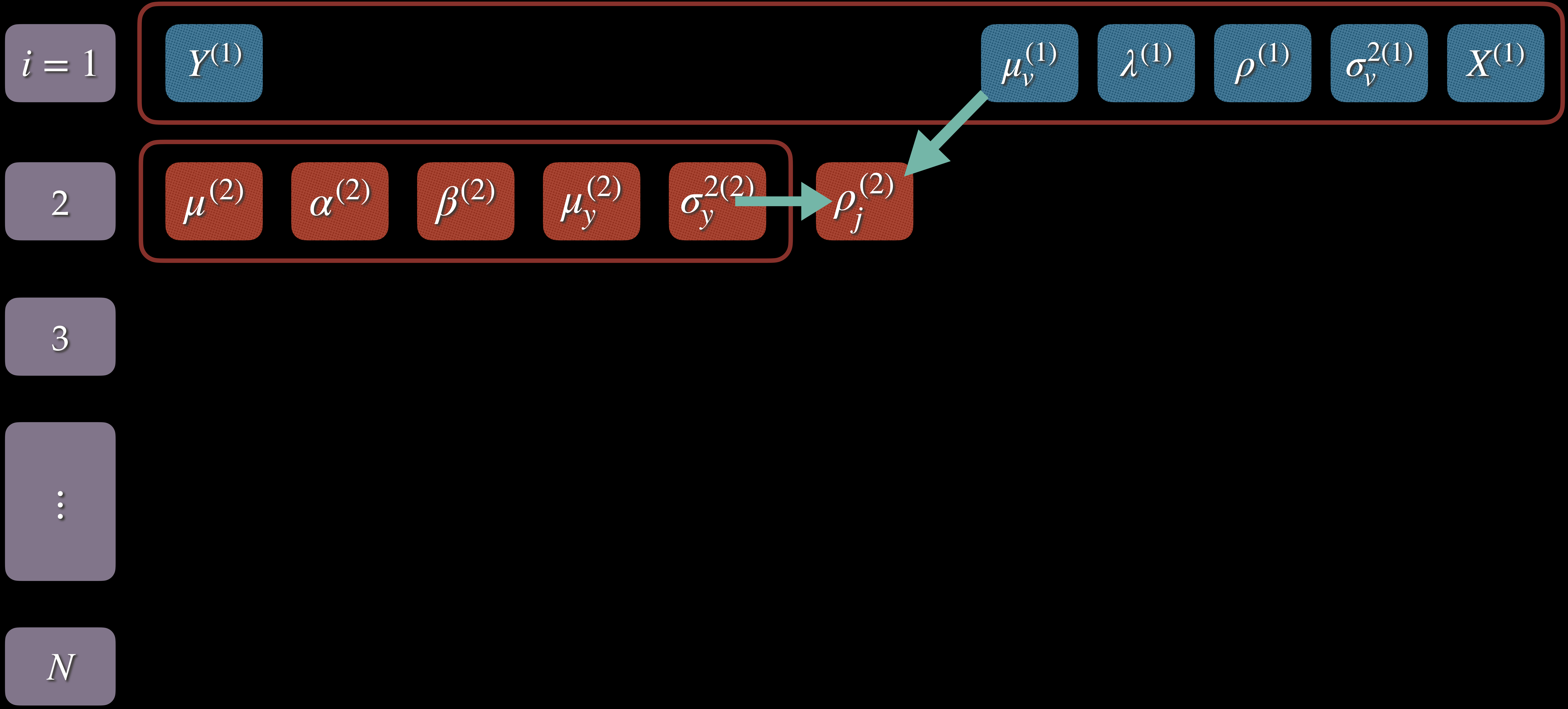

$i = 1$
$Y^{(1)}$
$\beta^{(1)}$
$\mu_y^{(1)}$
$\sigma_y^{2(1)}$
$\rho_j^{(1)}$
$\mu_v^{(1)}$
$\lambda^{(1)}$
$\rho^{(1)}$
$\sigma_v^{2(1)}$
$X^{(1)}$
2
$\mu^{(2)}$
$\alpha^{(2)}$
3
$\vdots$
N

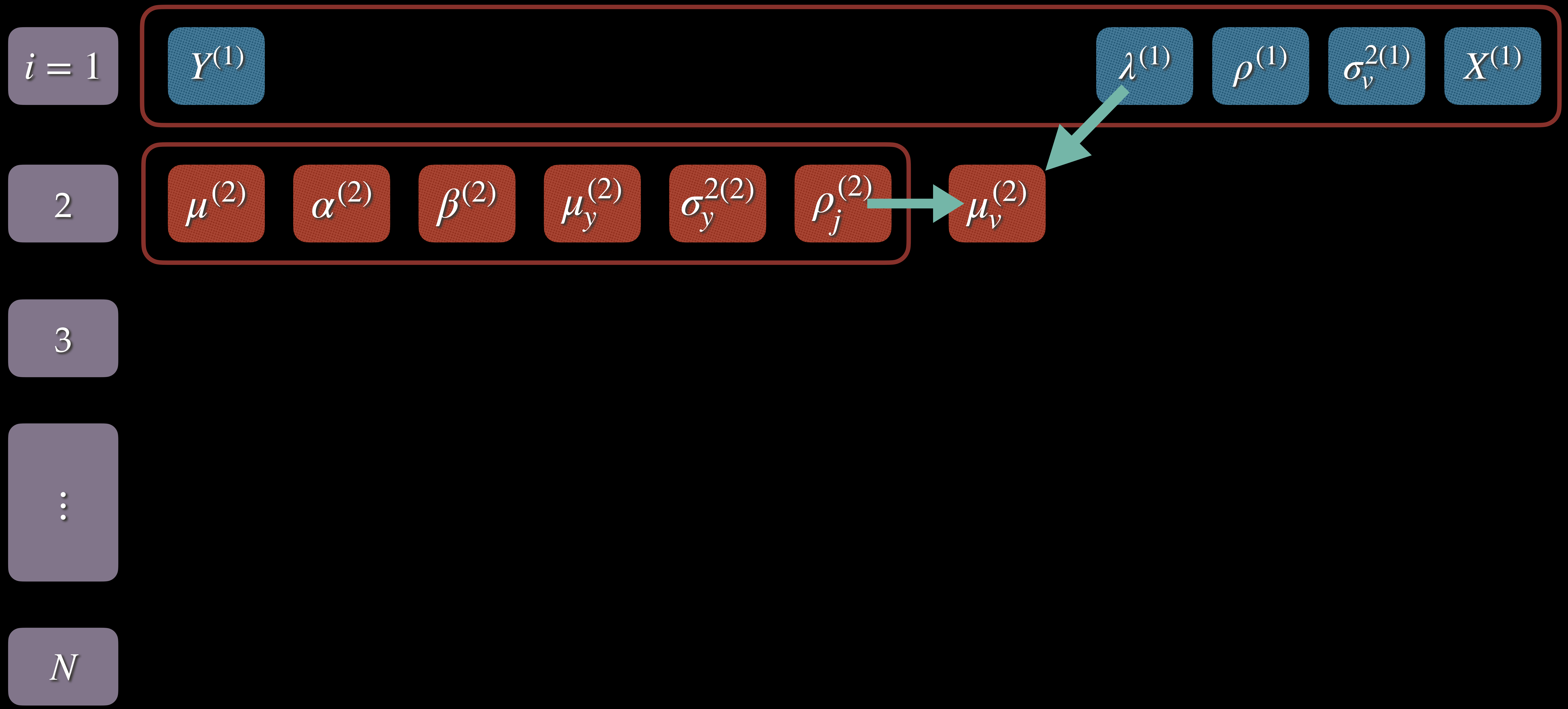


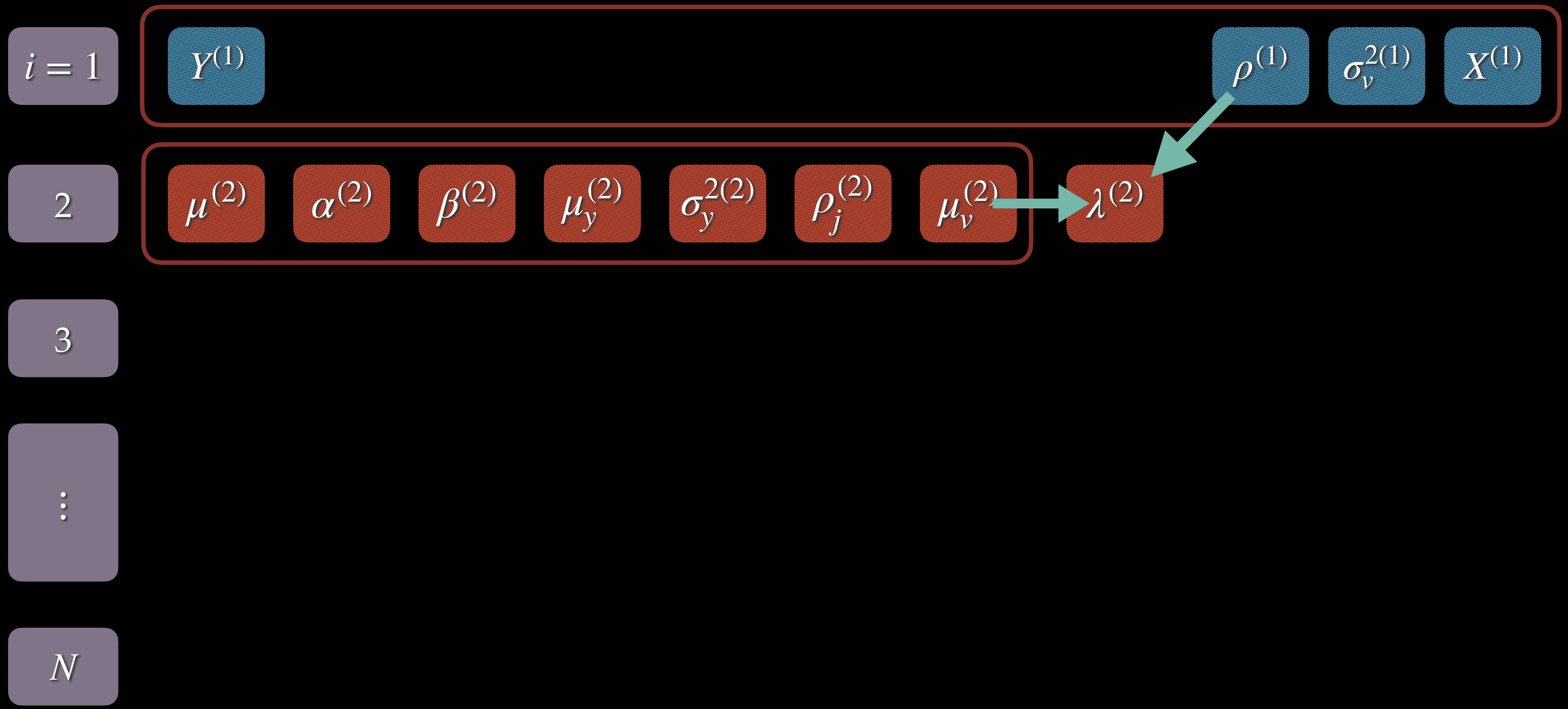
$i = 1$
2
3
⋮
N
$Y^{(1)}$
$\mu_y^{(1)}$
$\sigma_y^{2(1)}$
$\rho_j^{(1)}$
$\mu_v^{(1)}$
$\lambda^{(1)}$
$\rho^{(1)}$
$\sigma_v^{2(1)}$
$X^{(1)}$
$\mu^{(2)}$
$\alpha^{(2)}$
$\beta^{(2)}$

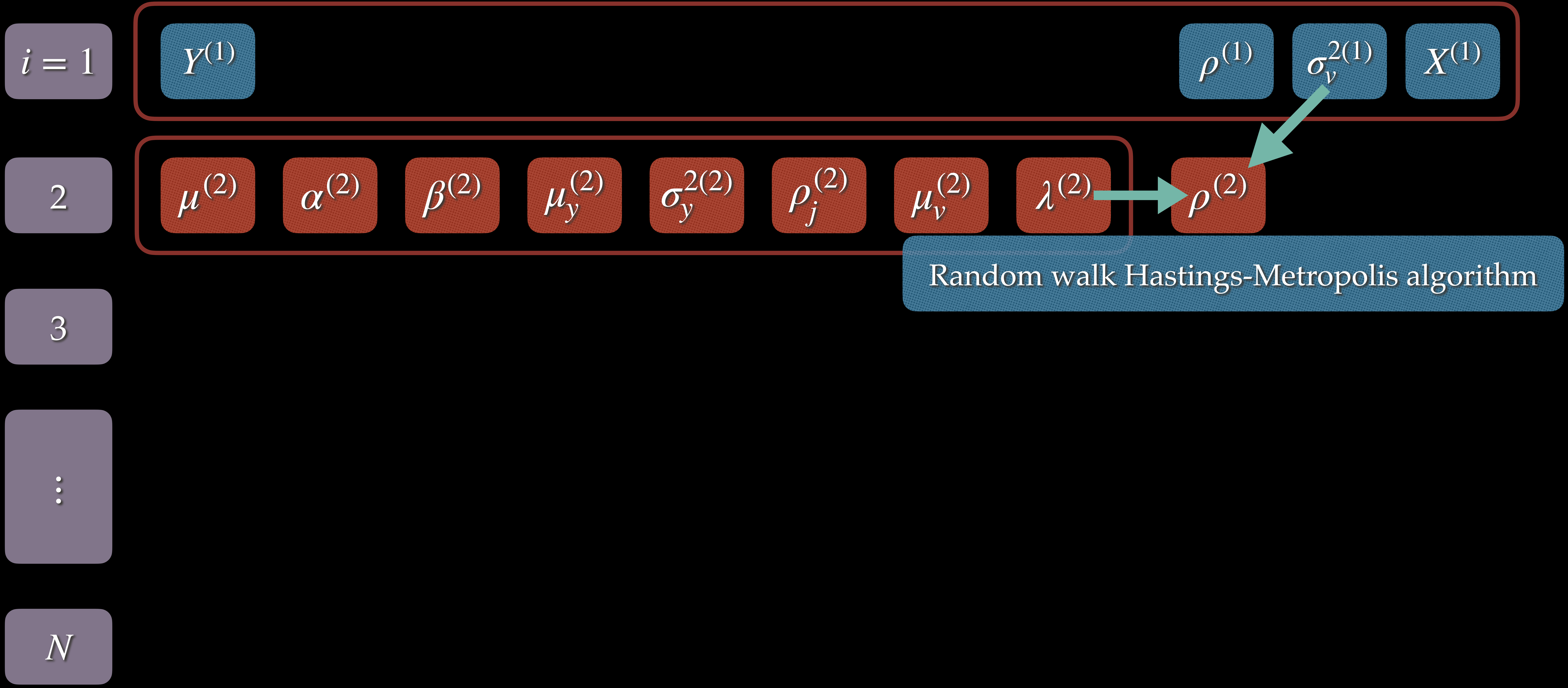

$i = 1$
$Y^{(1)}$
$\sigma_y^{2(1)}$
$\rho_j^{(1)}$
$\mu_v^{(1)}$
$\lambda^{(1)}$
$\rho^{(1)}$
$\sigma_v^{2(1)}$
$X^{(1)}$
2
$\mu^{(2)}$
$\alpha^{(2)}$
$\beta^{(2)}$
$\mu_y^{(2)}$
3
$\vdots$
N

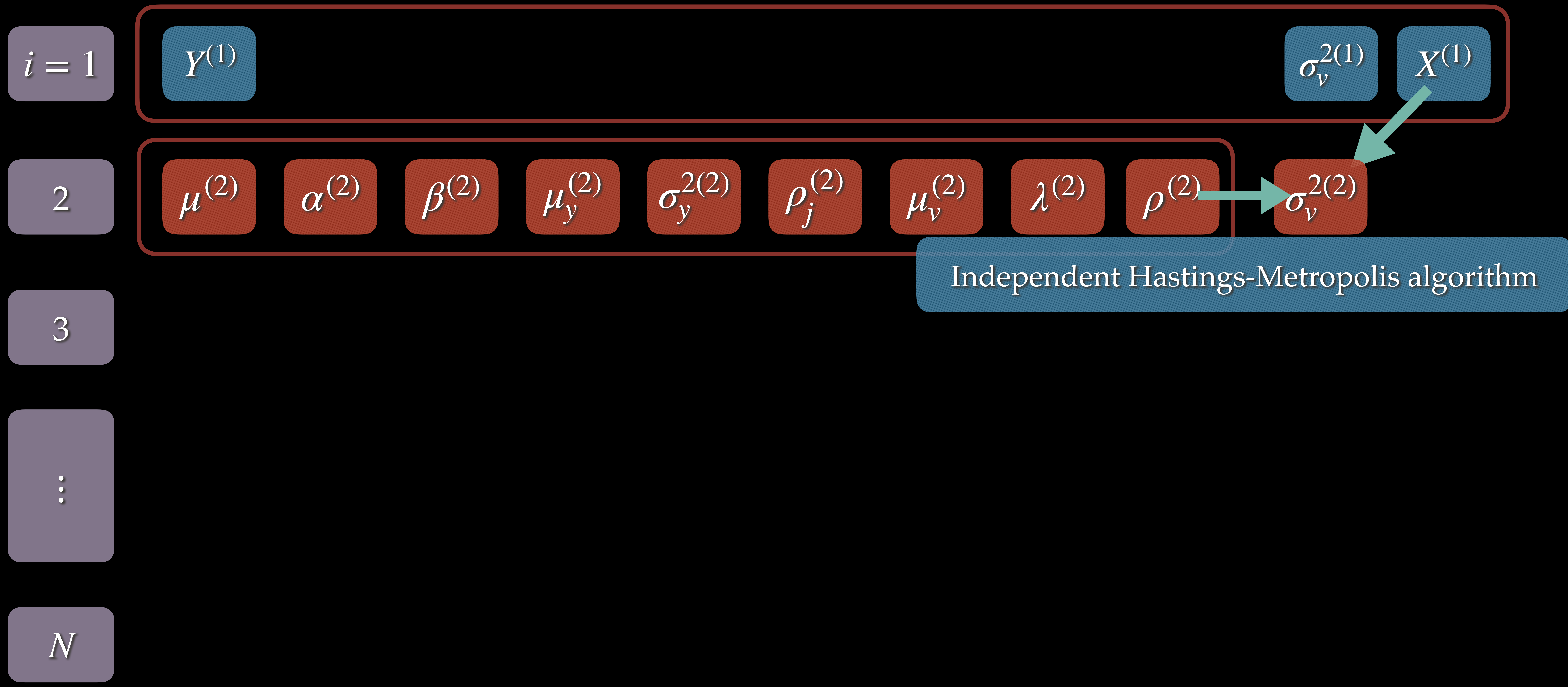

$i = 1$
$Y^{(1)}$
$\rho_j^{(1)}$
$\mu_v^{(1)}$
$\lambda^{(1)}$
$\rho^{(1)}$
$\sigma_v^{2(1)}$
$X^{(1)}$
2
$\mu^{(2)}$
$\alpha^{(2)}$
$\beta^{(2)}$
$\mu_y^{(2)}$
$\sigma_y^{2(2)}$
3
$\vdots$
N


$i = 1$
$Y^{(1)}$
$\mu_v^{(1)}$
$\lambda^{(1)}$
$\rho^{(1)}$
$\sigma_v^{2(1)}$
$X^{(1)}$
2
$\mu^{(2)}$
$\alpha^{(2)}$
$\beta^{(2)}$
$\mu_y^{(2)}$
$\sigma_y^{2(2)}$
$\rho_j^{(2)}$
3
$\vdots$
N

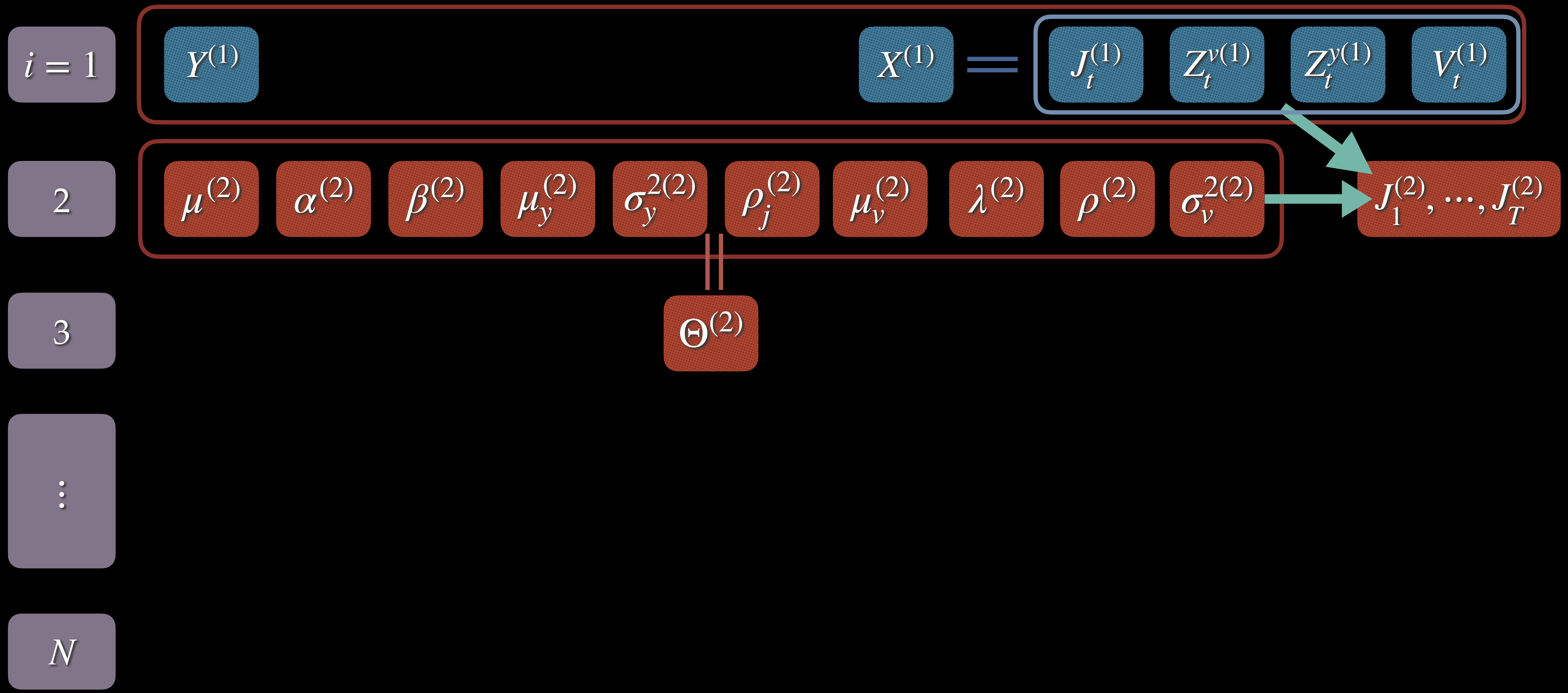

$i = 1$
$Y^{(1)}$
$\lambda^{(1)}$
$\rho^{(1)}$
$\sigma_v^{2(1)}$
$X^{(1)}$
2
$\mu^{(2)}$
$\alpha^{(2)}$
$\beta^{(2)}$
$\mu_y^{(2)}$
$\sigma_y^{2(2)}$
$\rho_j^{(2)}$
$\mu_v^{(2)}$
3
$\vdots$
N


$i = 1$
2
3
⋮
N
$Y^{(1)}$
$\rho^{(1)}$
$\sigma_v^{2(1)}$
$X^{(1)}$
$\mu^{(2)}$
$\alpha^{(2)}$
$\beta^{(2)}$
$\mu_y^{(2)}$
$\sigma_y^{2(2)}$
$\rho_j^{(2)}$
$\mu_v^{(2)}$
$\lambda^{(2)}$

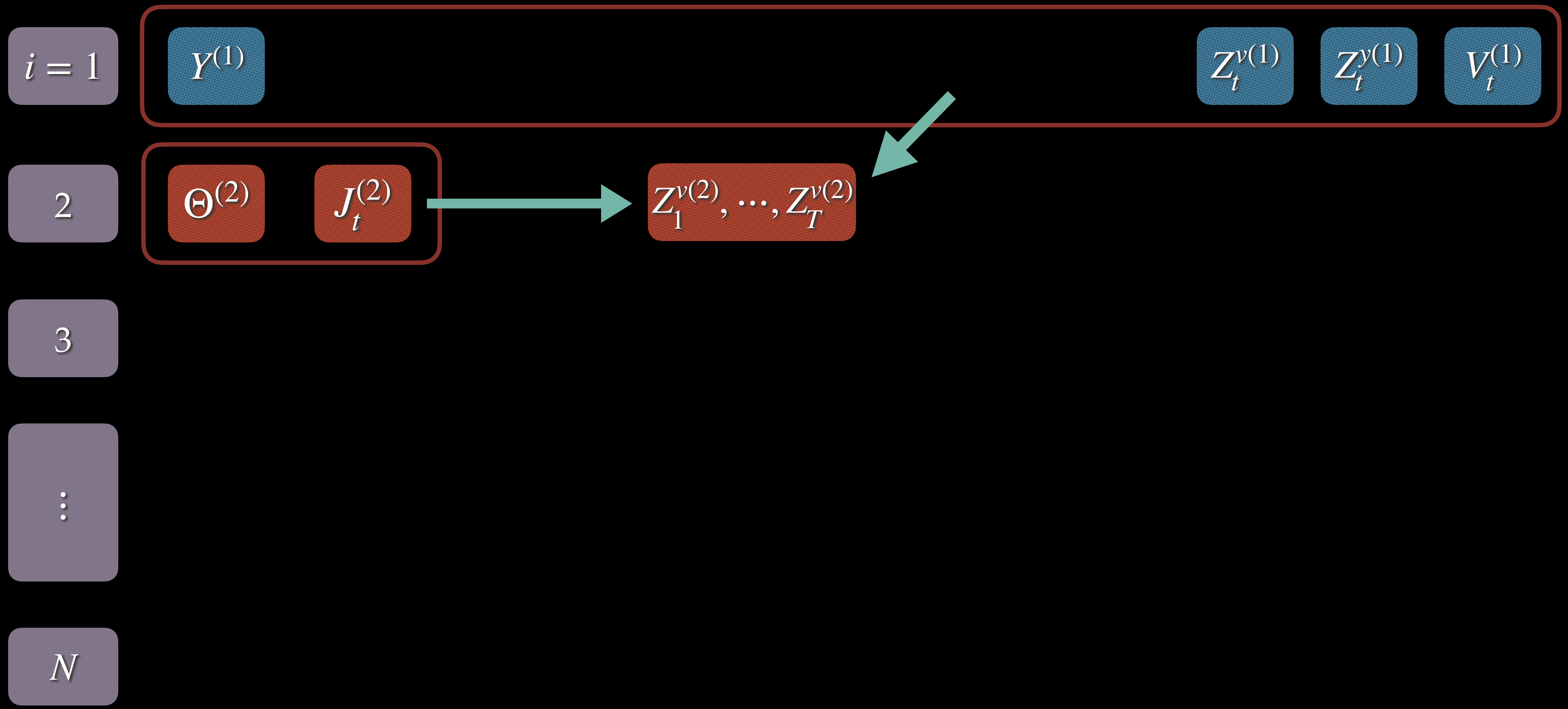

$i = 1$
$Y^{(1)}$
$\rho^{(1)}$
$\sigma_v^{2(1)}$
$X^{(1)}$
2
$\mu^{(2)}$
$\alpha^{(2)}$
$\beta^{(2)}$
$\mu_y^{(2)}$
$\sigma_y^{2(2)}$
$\rho_j^{(2)}$
$\mu_v^{(2)}$
$\lambda^{(2)}$
$\rho^{(2)}$
Random walk Hastings-Metropolis algorithm
3
$\vdots$
N



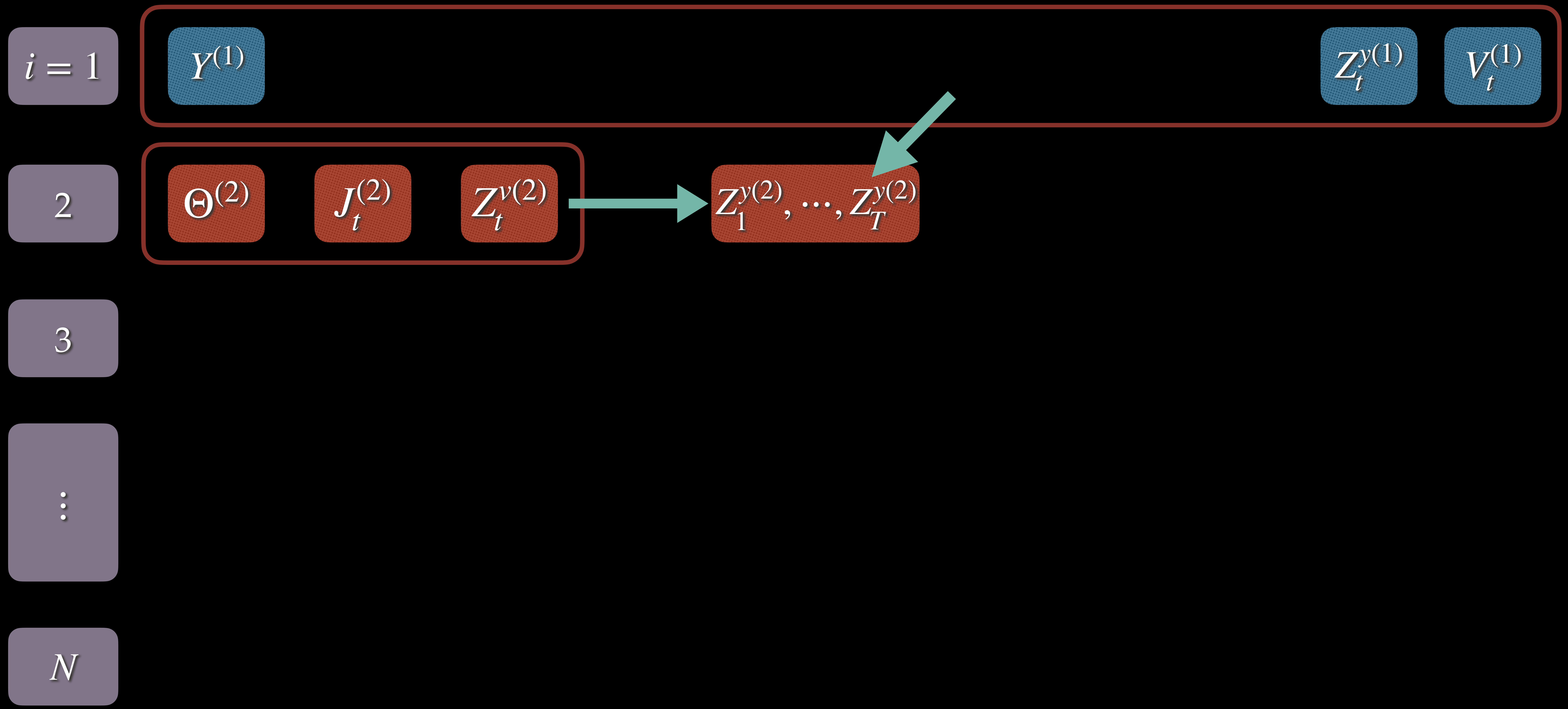
$i = 1$
$Y^{(1)}$
$\sigma_v^{2(1)}$
$X^{(1)}$
2
$\mu^{(2)}$
$\alpha^{(2)}$
$\beta^{(2)}$
$\mu_y^{(2)}$
$\sigma_y^{2(2)}$
$\rho_j^{(2)}$
$\mu_v^{(2)}$
$\lambda^{(2)}$
$\rho^{(2)}$
$\sigma_v^{2(2)}$
Independent Hastings-Metropolis algorithm
3
$\vdots$
N

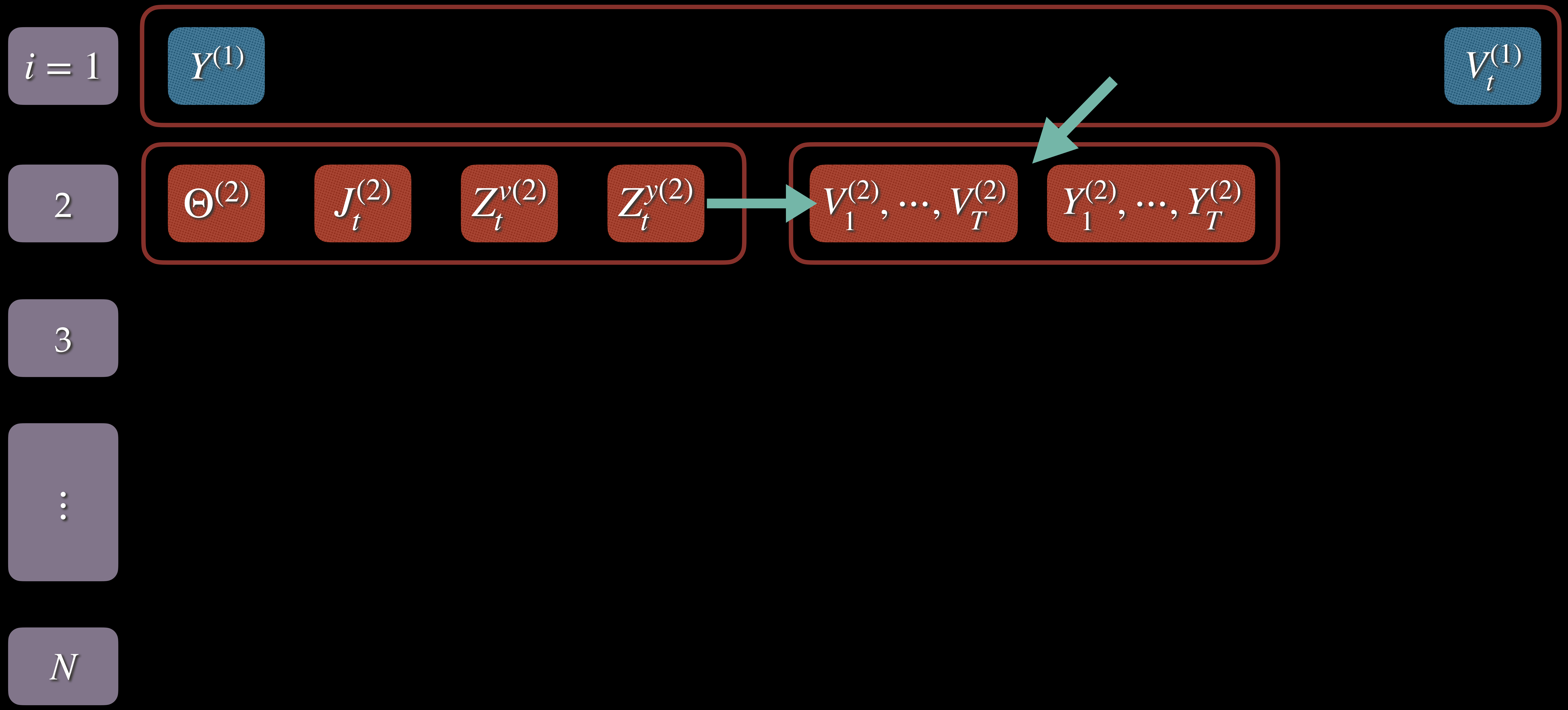


$i = 1$
$Y^{(1)}$
$X^{(1)}$
$J_t^{(1)}$
$Z_t^{v(1)}$
$Z_t^{y(1)}$
$V_t^{(1)}$
2
$\mu^{(2)}$
$\alpha^{(2)}$
$\beta^{(2)}$
$\mu_y^{(2)}$
$\sigma_y^{2(2)}$
$\rho_j^{(2)}$
$\mu_v^{(2)}$
$\lambda^{(2)}$
$\rho^{(2)}$
$\sigma_v^{2(2)}$
$J_1^{(2)}, \dots, J_T^{(2)}$
3
$\Theta^{(2)}$
$\vdots$
N



$i = 1$
2
3
$\vdots$
N
$Y^{(1)}$
$Z_t^{v(1)}$
$Z_t^{y(1)}$
$V_t^{(1)}$
$\Theta^{(2)}$
$J_t^{(2)}$
$Z_1^{v(2)}, \ldots, Z_T^{v(2)}$


$i = 1$
$Y^{(1)}$
$Z_t^{y(1)}$
$V_t^{(1)}$
2
$\Theta^{(2)}$
$J_t^{(2)}$
$Z_t^{v(2)}$
$Z_1^{y(2)}, \dots, Z_T^{y(2)}$
3
$\vdots$
N



$i = 1$
$Y^{(1)}$
$V_t^{(1)}$
2
$\Theta^{(2)}$
$J_t^{(2)}$
$Z_t^{v(2)}$
$Z_t^{y(2)}$
$V_1^{(2)}, \ldots, V_T^{(2)}$
$Y_1^{(2)}, \ldots, Y_T^{(2)}$
3
$\vdots$
N

$i = 1$	$Y^{(1)}$	$\mu^{(1)}$	$\alpha^{(1)}$	$\beta^{(1)}$	$\mu_y^{(1)}$	$\sigma_y^{2(1)}$	$\rho_j^{(1)}$	$\mu_v^{(1)}$	$\lambda^{(1)}$	$\rho^{(1)}$	$\sigma_v^{2(1)}$	$X^{(1)}$
2	$Y^{(2)}$	$\mu^{(2)}$	$\alpha^{(2)}$	$\beta^{(2)}$	$\mu_y^{(2)}$	$\sigma_y^{2(2)}$	$\rho_j^{(2)}$	$\mu_v^{(2)}$	$\lambda^{(2)}$	$\rho^{(2)}$	$\sigma_v^{2(2)}$	$X^{(2)}$
3												
$\vdots$												
N												

$i = 1$	$Y^{(1)}$	$\mu^{(1)}$	$\alpha^{(1)}$	$\beta^{(1)}$	$\mu_y^{(1)}$	$\sigma_y^{2(1)}$	$\rho_j^{(1)}$	$\mu_v^{(1)}$	$\lambda^{(1)}$	$\rho^{(1)}$	$\sigma_v^{2(1)}$	$X^{(1)}$
2	$Y^{(2)}$	$\mu^{(2)}$	$\alpha^{(2)}$	$\beta^{(2)}$	$\mu_y^{(2)}$	$\sigma_y^{2(2)}$	$\rho_j^{(2)}$	$\mu_v^{(2)}$	$\lambda^{(2)}$	$\rho^{(2)}$	$\sigma_v^{2(2)}$	$X^{(2)}$
3	$Y^{(3)}$	$\mu^{(3)}$	$\alpha^{(3)}$	$\beta^{(3)}$	$\mu_y^{(3)}$	$\sigma_y^{2(3)}$	$\rho_j^{(3)}$	$\mu_v^{(3)}$	$\lambda^{(3)}$	$\rho^{(3)}$	$\sigma_v^{2(3)}$	$X^{(3)}$
$\vdots$												
N												

$i = 1$	$Y^{(1)}$	$\mu^{(1)}$	$\alpha^{(1)}$	$\beta^{(1)}$	$\mu_y^{(1)}$	$\sigma_y^{2(1)}$	$\rho_j^{(1)}$	$\mu_v^{(1)}$	$\lambda^{(1)}$	$\rho^{(1)}$	$\sigma_v^{2(1)}$	$X^{(1)}$
2	$Y^{(2)}$	$\mu^{(2)}$	$\alpha^{(2)}$	$\beta^{(2)}$	$\mu_y^{(2)}$	$\sigma_y^{2(2)}$	$\rho_j^{(2)}$	$\mu_v^{(2)}$	$\lambda^{(2)}$	$\rho^{(2)}$	$\sigma_v^{2(2)}$	$X^{(2)}$
3	$Y^{(3)}$	$\mu^{(3)}$	$\alpha^{(3)}$	$\beta^{(3)}$	$\mu_y^{(3)}$	$\sigma_y^{2(3)}$	$\rho_j^{(3)}$	$\mu_v^{(3)}$	$\lambda^{(3)}$	$\rho^{(3)}$	$\sigma_v^{2(3)}$	$X^{(3)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	$Y^{(N)}$	$\mu^{(N)}$	$\alpha^{(N)}$	$\beta^{(N)}$	$\mu_y^{(N)}$	$\sigma_y^{2(N)}$	$\rho_j^{(N)}$	$\mu_v^{(N)}$	$\lambda^{(N)}$	$\rho^{(N)}$	$\sigma_v^{2(N)}$	$X^{(N)}$

$i = 1$
$Y^{(1)}$ $\mu^{(1)}$ $\alpha^{(1)}$ $\beta^{(1)}$ $\mu_y^{(1)}$ $\sigma_y^{2(1)}$ $\rho_j^{(1)}$ $\mu_v^{(1)}$ $\lambda^{(1)}$ $\rho^{(1)}$ $\sigma_v^{2(1)}$ $X^{(1)}$
2
$Y^{(2)}$ $\mu^{(2)}$ $\alpha^{(2)}$ $\beta^{(2)}$ $\mu_y^{(2)}$ $\sigma_y^{2(2)}$ $\rho_j^{(2)}$ $\mu_v^{(2)}$ $\lambda^{(2)}$ $\rho^{(2)}$ $\sigma_v^{2(2)}$ $X^{(2)}$
3
$Y^{(3)}$ $\mu^{(3)}$ $\alpha^{(3)}$ $\beta^{(3)}$ $\mu_y^{(3)}$ $\sigma_y^{2(3)}$ $\rho_j^{(3)}$ $\mu_v^{(3)}$ $\lambda^{(3)}$ $\rho^{(3)}$ $\sigma_v^{2(3)}$ $X^{(3)}$
Burn-in period
$\vdots$
N
$Y^{(N)}$ $\mu^{(N)}$ $\alpha^{(N)}$ $\beta^{(N)}$ $\mu_y^{(N)}$ $\sigma_y^{2(N)}$ $\rho_j^{(N)}$ $\mu_v^{(N)}$ $\lambda^{(N)}$ $\rho^{(N)}$ $\sigma_v^{2(N)}$ $X^{(N)}$
$\hat{\mu}$ $\hat{\alpha}$ $\hat{\beta}$ $\hat{\mu}_y$ $\hat{\sigma}_y^2$ $\hat{\rho}_j$ $\hat{\mu}_v$ $\hat{\lambda}$ $\hat{\rho}$ $\hat{\sigma}_v^2$