THE UNIVERSITY OF BRITISH COLUMBIA
STAT 321 - Stochastic Signals and Systems
Assignment 4

Due: **Wednesday, December 7 at 9:00pm**.

- Submit your assignment online in the ***pdf format*** under module "Assignments". You can either typeset your solutions or scan a handwritten copy.
- Assignments are to be completed individually.
- Define notation for events and random variables, and include all steps of your derivations. Writing down the final answer will not be sufficient to receive full marks.
- Please make sure your submission is clear and neat. It is a student's responsibility to ensure that the submitted file is in good order (e.g., not corrupted and contains what you intend to submit).
- **Late submission penalty**: 1% per hour or fraction of an hour. (In the event of technical issues with submission, you can email your assignment to the instructor to get a time stamp but submit on Canvas as soon as it becomes possible to make it available for grading.)

1. We have 3 coins which we will refer to as A, B and C. They look identical, but they have different biases. For Coin A, $P(H) = 0.3, P(T) = 0.7$; for Coin B, $P(H) = 0.7, P(T) = 0.3$ and for Coin C, $P(H) = 0.5, P(T) = 0.5$.

   (a) Alice chooses one of the 3 coins completely at random. She tosses the coin 2 times; the result of the first toss is a 'H' whereas the result of the second toss is a 'T'. Which coin, A, B or C, should Alice guess she chose if she wishes to minimize the conditional probability of error, $P_{e|HT}$?

   (b) Under a different scenario, Bob gives Alice a coin which he selected with the following probabilities: $P(A) = 0.6, P(B) = 0.4$ and $P(C) = 0$. Alice knows these a priori probabilities. She tosses the coin once. The possible outcome is a 'H' or a 'T'. For each possible outcome, determine the MAP decision rule for Alice to decide which coin, A, B or C, she was given. What is the resulting average probability of error, P_e?

2. A signal S is equally likely to take on one of 3 possible values: $-1, 0, +1$. It is sent over an additive Laplacian noise channel where the pdf of the noise W is

$$f_W(w) \quad = \quad \frac{1}{2c}\, e^{\frac{-|w|}{c}}$$

   The output of the channel is $Y = S + W$, where S and W are independent.

   (a) Determine $f_{Y|S}(y|s)$ for $s = -1, 0, +1$. Sketch your answers (the 3 conditional pdfs) for $c = 1$.

   (b) Determine the minimum P_e decision rule $\widehat{s}_{MAP}(y)$. Your answer should take the following form: choose $\widehat{s} = -1$ if y is a value in this (or these) interval(s); similarly for $\widehat{s} = 0$ and $\widehat{s} = +1$.

   (c) Determine the resulting average probability of error, P_e, in terms of c.

3. Let S be a positive rv with mean μ and variance σ^2. Our goal is to estimate S based on an observation X of the form $X = \sqrt{S}\, W$. We can assume that S and W are independent and W has mean 0 and variance 1.

   (a) Determine the linear LMS (LLMS) estimator of S given $X = x$.

   (b) Suppose that $Y = X^2$. Determine the LLMS estimator of S given $Y = y$. Your answer may include the fourth moment of W, i.e. $E(W^4)$.

4. A signal S is equally likely to take on one of 2 possible values: $-1, +1$. It is sent over an additive noise channel where noise $W \sim U(-2, +2)$. The output of the channel is $Y = S + W$, where S and W are independent.

   (a) Determine the LMS estimate of S given Y.

   (b) Determine the overall average MSE for the estimator in Part (a).

   (c) Suppose a MAP decision rule is used to decide whether $S = -1$ or $S = +1$. Determine the MAP decision rule and the resulting overall average probability of error.

   (d) Compare the MSEs for the LMS estimator and MAP decoder.

5. (a) Derive the distribution of the number of Bernoulli trials required to get r successes.

   (b) Suppose that the probability of success is $p = 0.1$. Sketch the distributions of the number of Bernoulli trials required to get $r = 1$ and $r = 2$ successes.

6. Suppose that earthquakes occur according to a Poisson process with rate $\lambda = 2$ and the time unit is a month. Thus, the rate of occurrence is 2 per month.

   (a) Determine the probability that at least 2 earthquakes occur in the next 3 months.

   (b) Determine the probability distribution of the time from now until the next earthquake.

7. The autocorrelation function of a random sequence $\{A_n\}$ is defined as

$$R_a(m) \triangleq E(A_n^* A_{n+m}).$$

   Suppose that the elements of the sequence $\{A_n\}_{n=-\infty}^{\infty}$ are independent, identically distributed random variables taking on one of three possible values $\{-1, 0, +2\}$ with probabilities $P(-1) = \frac{1}{4}$, $P(0) = \frac{1}{4}$ and $P(+2) = \frac{1}{2}$. Determine $R_a(m)$.

Assignment 4

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1. A: $P(H) = 0.3$ B: $P(H) = 0.7$ C: $P(H) = 0.5$, $P(A) = P(B) = P(C) = \frac{1}{3}$

$P(T) = 0.7$ $P(T) = 0.3$ $P(T) = 0.5$

a) Given 1^{st} H & 2^{nd} T.

$P(HT|A) = 0.3 \cdot 0.7 = 0.21$

$P(HT|B) = 0.7 \cdot 0.3 = 0.21$ →

$P(HT|C) = 0.5 \cdot 0.5 = 0.25$

$P(HT) = \sum P(HT|k)P(k)$

$= P(HT|A)P(A) + P(HT|B)P(B) + P(HT|C)P(C)$

$= 0.2233$

$P(A|HT) = \frac{P(A)P(HT|A)}{P(HT)} = \frac{\frac{1}{3} \cdot 0.21}{0.2233} = 0.3134$

$P(B|HT) = \frac{P(B)P(HT|B)}{P(HT)} = \frac{\frac{1}{3} \cdot 0.21}{0.2233} = 0.3134$

$P(C|HT) = \frac{P(C)P(HT|C)}{P(HT)} = \frac{\frac{1}{3} \cdot 0.25}{0.2233} \cdot 0.3731$

$P_{e|A} = P_{e|B} = 1 - 0.3134 = \underline{0.6865}$ $P_{e|C} = 1 - 0.3731 = \underline{0.6268}$, $P_{e|C}$ is the smallest $\therefore \underline{C}$

ANS.

b) $P(A) = 0.6$ $P(B) = 0.4$ $P(C) = 0$

$P(H) = \sum P(H|k)P(k) = P(H|A)P(A) + P(H|B)P(B) + P(H|C)P(C) = 0.3 \cdot 0.6 + 0.4 \cdot 0.7 = 0.46$

$P(T) =$ " $= P(T|A)P(A) + P(T|B)P(B) + P(T|C)P(C) = 0.7 \cdot 0.6 + 0.3 \cdot 0.4 = 0.54$

Given H,

$P(A|H) = \frac{P(A)P(H|A)}{P(H)} = \frac{0.6 \cdot 0.3}{0.46} = 0.391$ $P(B|H) = \frac{P(B)P(H|B)}{P(H)} = \frac{0.4 \cdot 0.7}{0.46} = \underline{0.609}$

Given T,

$P(A|T) = \frac{P(A)P(T|A)}{P(T)} = \frac{0.6 \cdot 0.7}{0.54} = \underline{0.778}$ $P(B|T) = \frac{P(B)P(T|B)}{P(T)} = \frac{0.4 \cdot 0.3}{0.54} = 0.222$

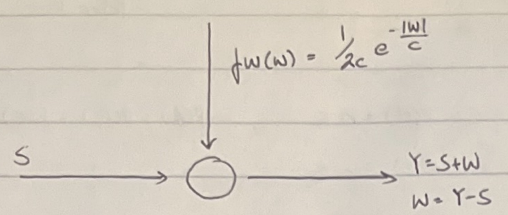
H → choose B T → choose A

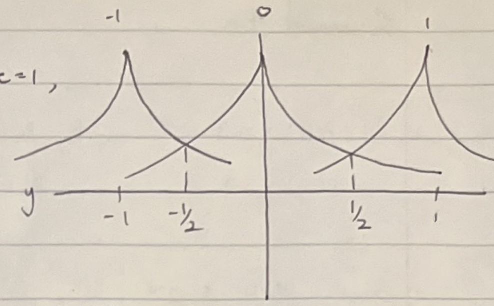
$\underline{\hat{S}(H)_{map} = B}$ $\underline{\hat{S}(T)_{map} = A}$

$P_{e|MAP} = P(A|H)P(H) + P(B|T)P(T) = 0.391 \cdot 0.46 + 0.222 \cdot 0.54 = \underline{0.3}$

ANS.

2.





a) $f_{Y|S}(y|s) = f_{Y|S}(y-s|s)$

$= \frac{1}{2c} e^{-\frac{|y-s|}{c}} \rightarrow$

$s = -1, \quad f_{Y|S}(y|s) = \frac{1}{2c} e^{-\frac{|y+1|}{c}}$

$s = 0, \quad f_{Y|S}(y|s) = \frac{1}{2c} e^{-\frac{|y|}{c}}$

$s = 1, \quad f_{Y|S}(y|s) = \frac{1}{2c} e^{-\frac{|y-1|}{c}}$

b) From the graph choose...

$$s = \begin{cases} -1 : & (-\infty, -\tfrac{1}{2}) \\ 0 : & [-\tfrac{1}{2}, \tfrac{1}{2}] \quad \text{for } y \\ 1 : & (\tfrac{1}{2}, \infty) \end{cases}$$

c) $P(\hat{s} = s) = P(\hat{s} = s, y < -\tfrac{1}{2}) + P(\hat{s} = s, -\tfrac{1}{2} \le y \le \tfrac{1}{2}) + P(\hat{s} = s, y > \tfrac{1}{2})$

$= P(s = -1, y < -\tfrac{1}{2}) + P(s = 0, -\tfrac{1}{2} \le y \le \tfrac{1}{2}) + P(s = 1, y > \tfrac{1}{2})$

$= P(y < -\tfrac{1}{2} | s = -1) P(s = -1) + P(-\tfrac{1}{2} \le y \le \tfrac{1}{2} | s = 0) P(s = 0) + P(y > \tfrac{1}{2} | s = 1) P(s = 1)$

$= \int_{-\infty}^{-1/2} f_{Y|S}(y|s=-1)\,dy \times P(S=-1) + \int_{-1/2}^{1/2} f_{Y|S}(y|s=0)\,dy \times P(s=0) + \int_{1/2}^{\infty} f_{Y|S}(y|s=1)\,dy \times P(s=1) \rightarrow \tfrac{1}{3}$ for all $P(s = -1, 0, 1)$

$= \tfrac{1}{3} \left[(-\tfrac{1}{2} e^{-1/2c} + e^{(1/2c)\frac{1}{2}}) + (e^{1/2c} - e^{-1/2c}) + (-\tfrac{1}{2} e^{-1/2c} + \tfrac{1}{2} e^{1/2c}) \right.$

$= \tfrac{2}{3} e^{-1/2c} (e^{1/c} - 1)$

ANS.

$P(\hat{s} \ne s) = 1 - P(\hat{s} = s)$

$= 1 - \tfrac{2}{3} e^{-1/2c} (e^{1/c} - 1)$

ANS.

3. $S \sim N(\mu, \sigma^2)$ $\quad X \sim$ obsv. with $X = \sqrt{S}W$, $W \sim N(0,1)$

$E(S) = \mu \quad Var(S) = \sigma^2 \quad E(W) = 0 \quad Var(W) = 1$

$Var(S) = E(S^2) - E(S)^2 = E(S^2) - \mu^2$

$E(S^2) = \sigma^2 + \mu^2$

$Var(W) = E(W^2) - E(W)^2$

$E(W^2) = 1$

a) $\hat{S}_{LLMS} = E(S) + \dfrac{Cov(S,X)}{Var(X)}(X - E(X))$

$E(X) = E(\sqrt{S} \cdot W)$
$= E(\sqrt{S})E(W)$ as $S \perp W$
$= 0$

$Var(X) = E(X^2) - E(X)^2$
$= E(SW^2) - 0$
$= E(S)E(W^2)$
$= \mu$

$Cov(S,X) = E(\overbrace{S \cdot \sqrt{S} \cdot W}^{S \cdot X}) - E(S)E(X)$
$= E(S^{3/2}W) - 0$
$= E(S^{3/2})E(W)$
$= 0$

$\hat{S}_{LLMS} = E(S) + 0 = \underline{\mu}$ ANS.

b) $Y = X^2 = SW^2$

$E(Y) = E(SW^2)$
$= E(S)E(W^2)$
$= \mu$

$Var(Y) = E(Y^2) - E(Y)^2$
$= E(S^2W^4) - E(Y)^2$
$= E(S^2)E(W^4) - E(Y)^2$
$= (\sigma^2 + \mu^2)E(W^4) - \mu^2$

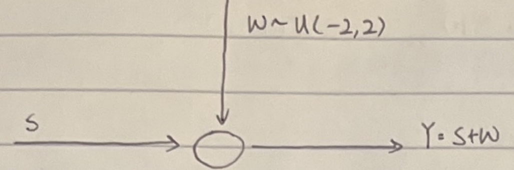
$Cov(S,Y) = E(S \cdot SW^2) - E(S)E(SW^2)$
$= E(S^2)E(W^2) - \mu \cdot \mu$
$= (\sigma^2 + \mu^2)(1) - \mu^2$
$= \sigma^2$

$\hat{S}_{LLMS}(S|y) = E(S) + \dfrac{Cov(S,Y)}{Var(Y)}(Y - E(Y))$

$= \underline{\mu + \dfrac{\sigma^2}{(\sigma^2+\mu^2)E(W^4) - \mu^2}(Y - \mu)}$

ANS.

4.



, $P(S=1) = P(S=-1) = 1/2$

$$f(w) = \frac{1}{b-a} = \frac{1}{2-(-2)} = \frac{1}{4}$$

$$f(s|y) = \frac{f(s)f(y|s)}{f(y)} = \frac{\frac{1}{2}\cdot\frac{1}{4}}{f(y)}$$ where $f(y) = \sum f(y|s_k)f(s_k) = f(y|s=-1)f(s=-1) + f(y|s=1)f(s=1)$ $= \frac{1}{2}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{4}$

$$= \frac{\frac{1}{8}}{\frac{1}{2}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{4}} = \frac{1}{2} \text{ for } s = \pm 1$$ ANS.

a) $\hat{S}_{LMS}(y) = E[S|y] = \sum_s s f(s|y) = (1)(\frac{1}{2}) + (-1)(\frac{1}{2}) = 0$ ANS.

b) $MSE_{LMS} = E[(S-\hat{S}_{LMS})^2] = E[Var(S|X)] = E[\overset{\nearrow 1}{S^2|Y}] - \overset{\nearrow 0}{E[S|Y]^2} = \sum_s s^2 f(s|y) - [\sum_s s f(s|y)]^{2}$

$= (1^2(\frac{1}{2}) + (-1)^2(\frac{1}{2})) - (1(\frac{1}{2}) + -1(\frac{1}{2})) = 1 - 0 = 1$ ANS

c) $\hat{S}_{MAP}(y) = \arg\max_{\hat{s}} f(s|y)$ $\rightarrow$ $f_W(y-s) = \begin{cases} 1/4 & -2 \le y-s \le 2 \\ 0 & \text{o/w} \end{cases}$ $\rightarrow$ choose $s=-1,\ y<0$; $s=1,\ y>0$

$= \arg\max_{\hat{s}} f_W(y-s)$

$$\arg\max_{\hat{s}} f_W(y-s) = \begin{cases} -1 & y<0 \\ 1 & y>0 \end{cases}$$

$P_e = P(y>0|s=-1)P(s=-1) + P(y<0|s=1)P(s=1)$

$= P(-1+W>0) + P(1+W<0)$

$= P(W>1) + P(W<-1)$

$= 1/4 + 1/4 = 1/2$ ANS

d) $MSE_{MAP} = E((s-\hat{s}_{MAP})^2|Y) = \underset{\text{values from b}}{E(s^2|Y)} - 2\hat{s}_{MAP}\underset{\text{from a}}{E(s|Y)} + \hat{s}_{MAP}^2$

$\rightarrow 1 - 2(1)(0) + 1 = 2$ given $s=1$

$1 - 2(1)(0) + (-1)^2 = 2$ given $s=-1$ $\Big\}$ $MSE_{MAP} = 2$

$2MSE_{LMS} = MSE_{MAP}$, MSE MAP is 2x the MSE_{LMS}.

ANS.

5.

a) $X \sim \text{Bern}(p) \rightarrow p(x) = p^x(1-p)^{1-x}$ for r success.

1	1	0	0	0
p	p	$1-p$	$1-p$	$1-p$

p^2 $(1-p)^3$

$\rightarrow$ each trial $\perp\!\!\!\perp$

$\therefore p^2 \cdot (1-p)^3$

Generalising $\rightarrow$ 1 1 1 .... 0 0 0 ....

r times $\qquad$ $n-r$ times

p^r $\qquad$ $(1-p)^{n-r}$, n is the total trial.

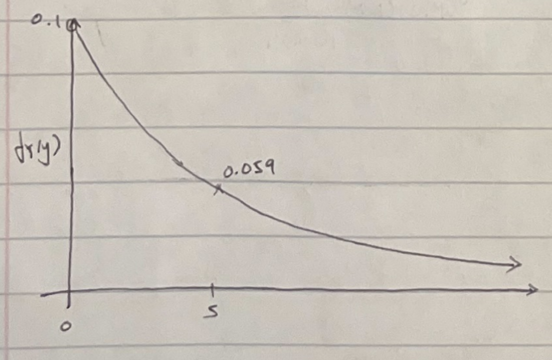
$\rightarrow \therefore p^r(1-p)^{n-r}$ but this dosen't cover observations of different conditions such as having 1 success out of 3 trials where : 100, 010, 001 are all considered successful.

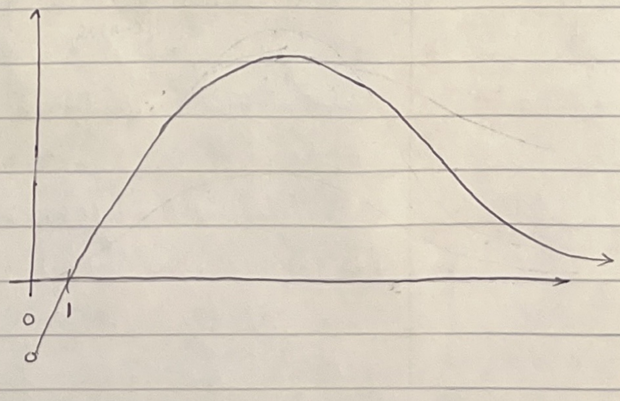
$\rightarrow$ to have this vere use $C(n,x) = \binom{n}{x} = \frac{n!}{x!(n-x)!}$ where n - # of trials, x - # of success we want

$\rightarrow$ resulting prob. function is $\begin{cases} \binom{n-1}{r-1} p^r (1-p)^{n-r} & \text{for } n = r, r+1 \ldots \\ 0 & \text{otw.} \end{cases}$

where $\binom{n-1}{r-1}$ diminates $0-1=-1$ case.

b) $p = 0.1$, $r = 1$: $\binom{n-1}{1-1}(0.1)(0.9)^{n-1}$ & $r=2$: $\binom{n-1}{2-1}(0.1)^2(0.9)^{n-2}$





6. $E \sim Pois(\lambda = 2)$, 2 per month = 6 per 3 months.

a) P(at least 2 E in 3 month) = $P(E \geq 2)$

$= 1 - P(E < 2) = 1 - (P(1) + P(0))$

$P(E = k) = e^{-\lambda} \frac{\lambda^k}{k!}$ where $\lambda = 6$ per 3 month. $\rightarrow$ $P(0) = e^{-6} \frac{6^0}{0!} = 0.002478$

$P(1) = e^{-6} \frac{6^1}{1!} = 0.01487$

$P(E \geq 2) = 1 - (0.01487 + 0.002478)$

$= \underline{0.9826}$ ANS.

b) Let T be the amount of time btw earthquakes.

$T \sim Exp(\lambda = 2 \text{ per month})$

$$f_T(t) = \begin{cases} \lambda \cdot e^{-\lambda t} & t \geq 0 \\ 0 & o/w. \end{cases}$$

$= \underline{2e^{-2t} \text{ for } t \geq 0}$ ANS.

can ignore as all real #.

7. $R_a(m) \triangleq E(A_n^* A_{n+m})$ with $\{A_n\}_{n=-\infty}^{\infty} \rightarrow \{-1, 0, 2\}$ with $P(-1) = 1/4$ $P(0) = 1/4$ $P(2) = 1/2$

$m = 0 \rightarrow R_a(m) = E(A_n^2)$

$m \neq 0 \rightarrow R_a(m) = E(A_n A_{n+m}) = E(A_n)E(A_{n+m})$

identically distrib.

$E(A_n) = \sum A_n \cdot P(A_n) = -1(1/4) + 0(1/4) + 2(1/2) = 3/4 \rightarrow E(A_n) = E(A_{n+m}) = 3/4 = 0.75$

$Var(A_n) = \sum (A_n - E(A_n))^2 \cdot P(A_n) = (-1 - (3/4))^2(1/4) + (0 - (3/4))^2(1/4) + (2 - (3/4))^2(1/2) = 1.6875$

$= E(A_n^2) - E(A_n)^2$

$E(A_n^2) = 1.6875 + (3/4)^2 = 2.25$

$R_a(m) = E(A_n^2) + E(A_n)E(A_{n+m})$

$= 2.25 + (3/4)^2$

$= \underline{2.8125}$ ANS.