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ELEC/STAT 321: Stochastic Signals and Systems

Assignment 2

The assignment is due on **Wednesday, October 12** at **9:00pm**.

- Submit your assignment online in the **pdf format** under module “Assignments”. You can either typeset your solutions or scan a handwritten copy.
- Assignments are to be completed individually.
- Define notation for events and random variables, and include all steps of your derivations. Writing down the final answer will not be sufficient to receive full marks.
- Please make sure your submission is clear and neat. It is a student’s responsibility that the submitted file is in good order (e.g., not corrupted and contains what you intend to submit).
- **Late submission penalty**: 1% per hour or fraction of an hour. (In the event of technical issues with submission, you can email your assignment to the instructor to get a time stamp but submit on canvas as soon as it becomes possible to make it available for grading.)

1. Let X be a point randomly selected from the unit interval $[0,1]$. Consider the random variable

$$Y = (1-X)^{-1/2}.$$

   (a) Sketch Y as a function of X.
   (b) Find and plot the cdf of Y.
   (c) Derive the pdf of Y.
   (d) Compute the following probabilities: $P(Y > 1)$, $P(3 < Y < 6)$, $P(Y \leq 10)$.

2. The pdf of random variable X has the form:

$$f_X(x) = \begin{cases} cx(1-x^2), & 0 \leq x \leq 1; \\ 0, & \text{otherwise} \end{cases}$$

   for some constant $c > 0$.

   (a) Find c and plot the pdf.
   (b) Find and plot the cdf of X.
   (c) Compute $P(0 < X < 0.6)$, $P(X = 1)$ and $P(0.2 < X < 0.5)$.
   (d) Compute the mean and variance of X.

3. Let random variable X denote the length of a wire, assumed to be exponentially distributed with mean 10π cm. The wire is cut to make rings of radius 1 cm. Derive the probability mass function for the number of complete rings that can be produced by each length of wire.

4. A voltage X follows a normal distribution with mean 1 and variance 4. Find the pdf of the power dissipated by an R-ohm resistor $P = RX^2$.

5. A modem transmits a two-dimensional signal (X, Y) given by

$$X = r\cos(2\pi\Theta/8) \quad \text{and} \quad Y = r\sin(2\pi\Theta/8),$$

where Θ is a discrete uniform random variable on the set $\{0, 1, 2, \ldots, 7\}$.

   (a) Show the mapping from $\mathcal{S}$ to $\mathcal{S}_{XY}$, the range of the pair (X, Y).

   (b) Find the joint pmf of X and Y.

   (c) Find the marginal pmf's of X and Y.

   (d) Compute the probability of the following events: $A = \{X = 0\}$, $B = \{Y \leq r/\sqrt{2}\}$, $C = \{X \geq r/\sqrt{2},\ Y \geq r/\sqrt{2}\}$ and $D = \{X < -r/\sqrt{2}\}$.

1. Let X be a point randomly selected from the unit interval $[0,1]$. Consider the random variable

$$Y = (1 - X)^{-1/2}.$$

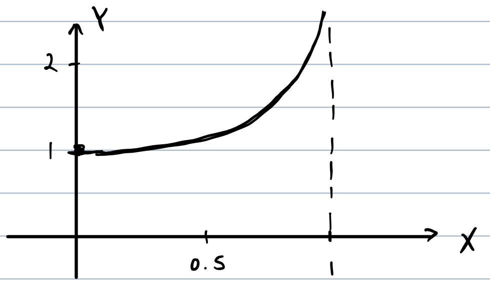
(a) Sketch Y as a function of X.

(b) Find and plot the cdf of Y.

(c) Derive the pdf of Y.

(d) Compute the following probabilities: $P(Y > 1)$, $P(3 < Y < 6)$, $P(Y \leq 10)$.

a) Sketch of $Y = (1-X)^{-1/2}$



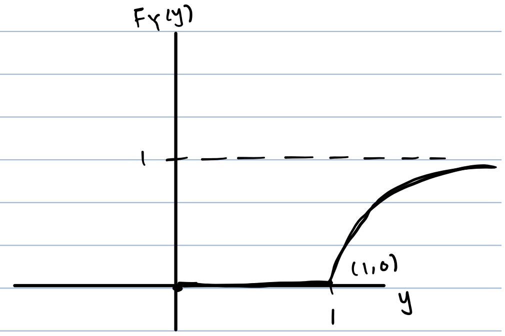
b) cdf of Y

$$F_Y(y) = P\{Y \leq y\}$$
$$= P\{(1-X)^{-1/2} \leq y\}$$
$$= P\{(1-X) \leq \frac{1}{y^2}\}$$
$$= P\{-X \leq \frac{1}{y^2} - 1\}$$
$$= P\{X \leq 1 - \frac{1}{y^2}\}$$

$F_Y(y) = 1 - \frac{1}{y^2}$ with condition. $0 \leq 1 - \frac{1}{y^2} \leq 1$

$$-1 < -\frac{1}{y^2} < 0$$
$$1 > \frac{1}{y^2} > 0$$
$$y^2 > 1 > 0$$
$$y^2 > 1$$
$$y > 1$$

$$\therefore F_Y(y) = \begin{cases} 1 - \frac{1}{y^2}, & y > 1 \\ 0, & \text{other} \end{cases}$$



c) pdf of Y

$$f_Y(y) = \frac{dF_Y(y)}{dy}$$
$$= \frac{d(1 - y^{-2})}{dy}$$
$$= 0 - (-2)y^{-3}$$
$$= \frac{2}{y^3}$$
$$= \begin{cases} \frac{2}{y^3}, & y > 1 \\ 0, & \text{other} \end{cases}$$

d) $P(Y > 1) = 1 - P(Y \leq 1)$
$$= 1 - F_Y(1)$$
$$= 1 - (1 - \frac{1}{1})$$
$$= 1.0000$$

$P(3 < Y < 6) = P(6) - P(3)$
$$= F_Y(6) - F_Y(3)$$
$$= (1 - \frac{1}{36}) - (1 - \frac{1}{9})$$
$$= 0.0833$$

$P(Y \leq 10) = P(10)$
$$= F_Y(10)$$
$$= 1 - \frac{1}{100}$$
$$= 0.9900$$

2. The pdf of random variable X has the form:

$$f_X(x) = \begin{cases} cx(1-x^2), & 0 \le x \le 1; \\ 0, & \text{otherwise} \end{cases}$$

for some constant $c > 0$.

(a) Find c and plot the pdf.
(b) Find and plot the cdf of X.
(c) Compute $P(0 < X < 0.6)$, $P(X = 1)$ and $P(0.2 < X < 0.5)$.
(d) Compute the mean and variance of X.

a) find c

$$f_X(x) = \begin{cases} cx(1-x^2) & 0 \le x \le 1 \\ 0 & \text{other} \end{cases}$$

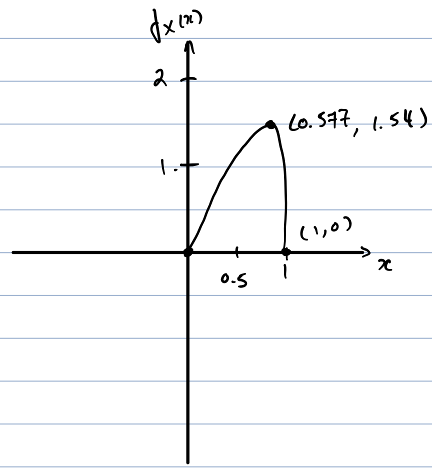
$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^1 cx(1-x^2)\,dx$$

$$= \int_0^1 cx(1-x^2)\,dx$$

$$= c\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = c\left[\frac{1}{2} - \frac{1}{4}\right]$$

$$1 = \frac{c}{4}$$

$$c = 4$$



$$\therefore \text{pdf} \Rightarrow f_X(x) = \begin{cases} 4x(1-x^2) & , 0 \le x \le 1 \\ 0 & , \text{other} \end{cases}$$

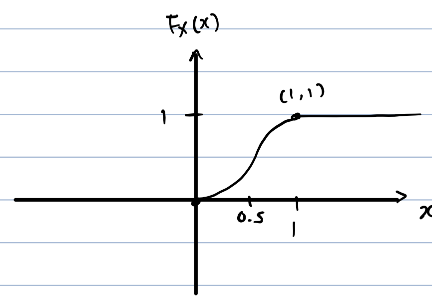
b) cdf of X

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx = \int_0^x 4x(1-x^2)\,dx$$

$$= 4\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^x$$

$$= 2x^2 - x^4$$

$$= \begin{cases} 0 & , x < 0 \\ 2x^2 - x^4 & , 0 \le x \le 1 \\ 1 & , x > 1 \end{cases}$$



c) $P(0 < X < 0.6) = P(0.6) - P(0)$

$= F_X(0.6) - F_X(0)$

$= [2(0.6)^2 - (0.6)^4] - [2(0)^2 - (0)^4]$

$= 0.5904$

$P(0.2 < X < 0.5) = P(0.5) - P(0.2)$

$= F_X(0.5) - F_X(0.2)$

$= [2(0.5)^2 - (0.5)^4] - [2(0.2)^2 - (0.2)^4]$

$= 0.3591$

$P(X=1) = F_X(1) - F_X(1) = 0$

d) Mean $\Rightarrow E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx$

$= \int_0^1 4x^2(1-x^2)$

$= \cancel{4}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$

$= 4\left[\frac{1}{3} - \frac{1}{5}\right]$

$= 4\left[\frac{2}{15}\right]$

$= \frac{8}{15}$

$= 0.5333$

Var $\Rightarrow E(X^2) - E(X)^2$

$\int_0^1 4x^3(1-x^2)\,dx$

$= 4\left[\frac{x^4}{4} - \frac{x^6}{6}\right]_0^1$

$= 4\left[\frac{1}{4} - \frac{1}{6}\right]$

$= 4\left(\frac{2}{24}\right)$

$= \frac{8}{24}$

$= \frac{1}{3}.$

$Var(X) = \frac{1}{3} - \left(\frac{8}{15}\right)^2$

$= 0.0489$

3. Let random variable X denote the length of a wire, assumed to be exponentially distributed with mean 10π cm. The wire is cut to make rings of radius 1 cm. Derive the probability mass function for the number of complete rings that can be produced by each length of wire.

$X \sim Exp(\lambda = \frac{1}{10\pi})$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{other} \end{cases} \rightarrow F_X(x) = 1 - e^{-\lambda x} = 1 - e^{\frac{-x}{10\pi}}$$

$$= \frac{1}{10\pi} e^{-\frac{x}{10\pi}}$$

with $r = 1\,cm$, $P_r = 2\pi r = 2\pi(1)\,cm$ $\rightarrow$ $P_r\ Y=1 = 2\pi$, $P_r\ Y=2 = 4\pi$, $P_r\ Y=3 = 6\pi$, $\vdots$

cdf:

$P\{Y=0\} = P\{X < 2\pi\} = P(2\pi) = F_X(2\pi) = 1 - e^{-\frac{2\pi}{10\pi}} = e^0 - e^{-1/5}$

$P\{Y=1\} = P\{2\pi < X < 4\pi\} = P(4\pi) - P(2\pi) = F_X(4\pi) - F_X(2\pi) = [1 - e^{-\frac{4\pi}{10\pi}}] - [1 - e^{-\frac{2\pi}{10\pi}}] = e^{-1/5} - e^{-2/5}$

$P\{Y=1\} = P\{4\pi < X < 8\pi\} = P(6\pi) - P(4\pi) = F_X(6\pi) - F_X(4\pi) = [1 - e^{-\frac{6\pi}{10\pi}}] - [1 - e^{-\frac{4\pi}{10\pi}}] = e^{-2/5} - e^{-3/5}$

$\vdots$

$$\sum_{x=0}^{n} e^{-x/5} - e^{-\frac{x+1}{5}}$$

$\therefore$ pmf:

$P_Y(Y=y) = e^{-\frac{y}{5}} - e^{-\frac{y+1}{5}}$, $y \in \mathbb{Z}^+$ positive int. $\{0, 1, 2, 3, 4 \ldots\}$

4. A voltage X follows a normal distribution with mean 1 and variance 4. Find the pdf of the power dissipated by an R-ohm resistor $P = RX^2$.

$X \sim N\{\mu=1,\ \sigma^2=4\}$

$$F_P(p) = P\{P \le p\}$$
$$= P\{RX^2 \le p\}$$
$$= P\{X \le \pm\sqrt{\tfrac{p}{R}}\}$$

both increasing & decreasing

$$= P\{-\sqrt{\tfrac{p}{R}} \le X \le \sqrt{\tfrac{p}{R}}\}$$

$\therefore$ $f_Y(y) = f_X(g^{-1}(y)) \cdot \left|\frac{d}{dy}(g^{-1}(y))\right|$ ~~X~~

$$F_P(p) = F_X(\sqrt{\tfrac{p}{R}}) - F_X(-\sqrt{\tfrac{p}{R}}) \rightarrow f_P(p) = \frac{dF_P(p)}{dp}$$

$$= \frac{f_X(\sqrt{\frac{p}{R}})}{2R\sqrt{\frac{p}{R}}} - \frac{f_X(-\sqrt{\frac{p}{R}})}{2R\sqrt{\frac{p}{R}}}$$

$$= \frac{1}{2R\sqrt{\frac{p}{R}}}\left(f_X(\sqrt{\tfrac{p}{R}}) - f_X(-\sqrt{\tfrac{p}{R}})\right)$$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma}}$$
$$= \frac{1}{2\sqrt{2\pi}} e^{-\frac{(x-1)^2}{4}}$$

$$f_P(p) = \frac{1}{2R\sqrt{\frac{p}{R}}}\left(\frac{1}{2\sqrt{2\pi}} e^{-\frac{(\sqrt{\frac{p}{R}}-1)^2}{4}} - \frac{1}{2\sqrt{2\pi}} e^{-\frac{(-\sqrt{\frac{p}{R}}-1)^2}{4}}\right)$$

$$= \frac{1}{4R\sqrt{2\pi}\sqrt{\frac{p}{R}}}\left(e^{-\frac{(\sqrt{\frac{p}{R}}-1)^2}{4}} - e^{-\frac{(-\sqrt{\frac{p}{R}}-1)^2}{4}}\right)$$

5. A modem transmits a two-dimensional signal (X, Y) given by

$$X = r\cos(2\pi\Theta/8) \quad \text{and} \quad Y = r\sin(2\pi\Theta/8),$$

where Θ is a discrete uniform random variable on the set $\{0, 1, 2, \ldots, 7\}$.

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(d) Compute the probability of the following events: $A = \{X = 0\}$, $B = \{Y \le r/\sqrt{2}\}$, $C = \{X \ge r/\sqrt{2},\ Y \ge r/\sqrt{2}\}$ and $D = \{X < -r/\sqrt{2}\}$.

a)

S	S_{XY}
0	$r\cos(2\pi(0)/8)$, $r\sin(2\pi(0)/8) \to \{r, 0\}$
1	$r\cos(2\pi(1)/8)$, $r\sin(2\pi(1)/8) \to \{r/\sqrt{2}, r/\sqrt{2}\}$
2	$r\cos(2\pi(2)/8)$, $r\sin(2\pi(2)/8) \to \{0, r\}$
3	$r\cos(2\pi(3)/8)$, $r\sin(2\pi(3)/8) \to \{-r/\sqrt{2}, r/\sqrt{2}\}$
4	$r\cos(2\pi(4)/8)$, $r\sin(2\pi(4)/8) \to \{-r, 0\}$
5	$r\cos(2\pi(5)/8)$, $r\sin(2\pi(5)/8) \to \{-r/\sqrt{2}, -r/\sqrt{2}\}$
6	$r\cos(2\pi(6)/8)$, $r\sin(2\pi(6)/8) \to \{0, -r\}$
7	$r\cos(2\pi(7)/8)$, $r\sin(2\pi(7)/8) \to \{r/\sqrt{2}, -r/\sqrt{2}\}$

, Range : $\{X \mid -r \le X \le r,\ Y \mid -r \le Y \le r\}$

b) $P(X,Y) = P(X=x, Y=y) \to$

$P(r, 0) = \frac{1}{8}$ $\quad P(-r, 0) = \frac{1}{8}$

$P(r/\sqrt{2}, r/\sqrt{2}) = \frac{1}{8}$ $\quad P(-r/\sqrt{2}, -r/\sqrt{2}) = \frac{1}{8}$

$P(0, r) = \frac{1}{8}$ $\quad P(0, -r) = \frac{1}{8}$

$P(-r/\sqrt{2}, r/\sqrt{2}) = \frac{1}{8}$ $\quad P(r/\sqrt{2}, -r/\sqrt{2}) = \frac{1}{8}$

c) $P_X(-r) = P(r, 0) = 1/8$

$P_X(-r/\sqrt{2}) = P(-r/\sqrt{2}, r/\sqrt{2}) + P(-r/\sqrt{2}, -r/\sqrt{2}) = \frac{2}{8} = \frac{1}{4}$

$P_X(0) = P(0, r) + P(0, -r) = \frac{2}{8} = \frac{1}{4}$

$P_X(r/\sqrt{2}) = P(r/\sqrt{2}, r/\sqrt{2}) + P(r/\sqrt{2}, -r/\sqrt{2}) = \frac{2}{8} = \frac{1}{4}$

$P_X(r) = P(r, 0) = 1/8$

$P_Y(-r) = P(0, -r) = \frac{1}{8}$

$P_Y(-r/\sqrt{2}) = P(-r/\sqrt{2}, -r/\sqrt{2}) + P(r/\sqrt{2}, -r/\sqrt{2}) = 1/4$

$P_Y(0) = P(r, 0) + P(-r, 0) = 1/4$

$P_Y(r/\sqrt{2}) = P(r/\sqrt{2}, r/\sqrt{2}) + P(-r/\sqrt{2}, r/\sqrt{2}) = 1/4$

$P_Y(r) = P(0, r) = 1/8$

d) $P_A\{X=0\} = P_X(0)$

$= 1/4$

$P_B\{Y \le r/\sqrt{2}\} = P_Y(-r) + P_Y(-r/\sqrt{2}) + P_Y(0) + P_Y(r/\sqrt{2}) = \frac{1}{8} + \frac{3}{4} = \frac{7}{8}$

$P_C\{X \geq r/\sqrt{2},\ Y \geq r/\sqrt{2}\} = P(r/\sqrt{2}, r/\sqrt{2}) = \frac{1}{8}$

$P_D\{X < -r/\sqrt{2}\} = P_X(-r) = \frac{1}{8}$