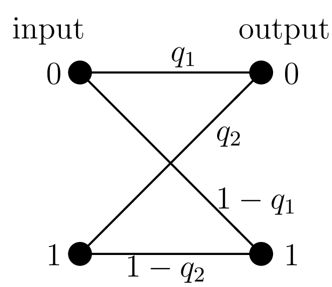
UNIVERSITY OF BRITISH COLUMBIA
ELEC/STAT 321: Stochastic Signals and Systems
Assignment 1

The assignment is due on **Wednesday, September 28** at **9:00pm**.

- Submit your assignment online in the **pdf format** under module "Assignments". You can either typeset your solutions or scan a handwritten copy.
- Assignments are to be completed individually.
- Define notation for events and random variables, and include all steps of your derivations. Writing down the final answer will not be sufficient to receive full marks.
- Please make sure your submission is clear and neat. It is a student's responsibility that the submitted file is in good order (e.g., not corrupted and contains what you intend to submit).
- **Late submission penalty**: 1% per hour or fraction of an hour. (In the event of technical issues with submission, you can email your assignment to the instructor to get a time stamp but submit on canvas as soon as it becomes possible to make it available for grading.)

1. Consider a binary communications channel shown in the figure below.



The input is 0 with probability p and 1 with probability $1 - p$. When input is 0, the output is 0 with probability q_1 and 1 with probability $1 - q_1$. The corresponding probabilities are q_2 and $1 - q_2$, respectively, when the input is 1.

   (a) Find the probability that the output is 0.
   (b) Find the probability that the input was 0 given that the output is 1.
   (c) Find the probability that the input was 1 given that the output is 1.
   (d) If the output is 1, which input is more probable? Explain.

2. A student's alarm clock fails to go off 1 time out of 3. In addition, the student misses the bus with probability 1/2 if his alarm clock fails. If his alarm clock works, he misses the bus with probability 1/3. If the student's alarm clock fails and he misses the bus, he arrives late to class with probability 1/2. This probability is reduced to 1/6 if the alarm clock works and he does not miss the bus. In all other situations (i.e., when the alarm clock fails and he does not miss the bus, and alarm clock works and he misses the bus), the probability of the student being late to class is 1/4.

(a) On a random day, what is the probability that the student's alarm clock works, he does not miss the bus and he is not late to class? Make sure to clearly define all notation used.

(b) What is the probability that this student arrives late to class?

(c) What is the probability that the alarm clock failed, given that he arrived late to class?

3. Three transmitters send messages through bursts of radio signals to an antenna. During each time slot, each transmitter sends a message with probability 1/2. Simultaneous transmissions from more than one transmitter lead to the loss of the messages. Define a random variable X as the number of time slots until the first message gets through.

   (a) Describe the underlying sample space $\mathcal{S}$ of this random experiment and specify the probabilities of its elementary events.

   (b) Show the mapping from $\mathcal{S}$ to $\mathcal{S}_X$, the range of possible values of X.

   (c) Find the probability mass function of random variable X.

   (d) Compute the expected value and standard deviation of X.

4. The random variable X follows the binomial distribution with parameters $n = 4$ and $p = 0.4$. Compute the expected value of $g(X)$ for $g(x) = 2x^2 + 1$.

5. Phone calls arrive at a switchboard at an average rate of $\lambda = 2$ per minute. Assume that the number of calls in a minute follows the Poisson distribution with parameter λ.

   (a) What is the probability that exactly two calls arrive in a minute?

   (b) Find the probability that more than two calls arrive in a minute.

1. I_p := input O_p := output

$P(I_p=0)=p$ $P(I_p=1)=1-p$

$P(O_p=0 \mid I_p=0)=q_1$ $P(O_p=1 \mid I_p=0)=1-q_1$ $P(O_p=0 \mid I_p=1)=q_2$ $P(O_p=1 \mid I_p=1)=1-q_2$

a) $P(O_p=0) = P(I_p=0)P(O_p=0 \mid I_p=0) + P(I_p=1)P(O_p=0 \mid I_p=1)$

$= p(q_1) + (1-p)(q_2)$

$= pq_1 + q_2 - pq_2$ ANS.

b) $P(I_p=0 \mid O_p=1) = \dfrac{P(O_p=1 \mid I_p=0)P(I_p=0)}{P(O_p=1)}$

$P(O_p=1) = P(I_p=0)P(O_p=1 \mid I_p=0) + P(I_p=1)P(O_p=1 \mid I_p=1)$

$= p(1-q_1) + (1-p)(1-q_2)$

$= \cancel{p} - pq_1 + 1 - q_2 - \cancel{p} + pq_2$

$= 1 - pq_1 - q_2 + pq_2$

$P(I_p=0 \mid O_p=1) = \dfrac{(1-q_1)p}{1-pq_1-q_2+pq_2}$ ANS.

c) $P(I_p=1 \mid O_p=1) = \dfrac{P(O_p=1 \mid I_p=1)P(I_p=1)}{P(O_p=1)}$

$= \dfrac{(1-q_2)(1-p)}{1-pq_1-q_2+pq_2}$ ANS.

d. $P(I_p=0 \mid O_p=1) = \dfrac{(1-q_1)p}{1-pq_1-q_2+pq_2}$ $P(I_p=1 \mid O_p=1) = \dfrac{(1-q_2)(1-p)}{1-pq_1-q_2+pq_2}$

ANS. Comparing numerator, $(1-q_1)p$ & $(1-q_2)(1-p)$, the more probable input depends on the value of q_1 & q_2 & p as the denominator is same for both $P(I_p=0 \mid O_p=1)$ & $P(I_p=1 \mid O_p=1)$.

2. $\bar{A}$ - alarm fail $\bar{B}$ - miss bus $\bar{C}$ - late to class

A - alarm work B - get bus C - not late to class

$P(\bar{A}) = 1/3$ $P(\bar{B}|\bar{A}) = 1/2$ $P(\bar{C} | \bar{B} \cap \bar{A}) = 1/2$

$P(\bar{B}|A) = 1/3$ $P(\bar{C} | B \cap A) = 1/6$

$P(\bar{C} | \bar{B} \cap A)$ & $P(\bar{C} | B \cap \bar{A}) = 1/4$

a) $P(A \cap B \cap C) = P(A) \cdot P(B|A) \cdot P(C | A \cap B)$

$= (1-1/3)(1-1/3)(1-1/6)$

$= 2/3 \cdot 2/3 \cdot 5/6$

$= \frac{20}{54} = \frac{10}{27} \approx 0.37037$ ANS.

b) $P(\bar{C}) = P(\bar{B} \cap \bar{A}) P(\bar{C} | \bar{B} \cap \bar{A}) + P(B \cap A) P(\bar{C} | B \cap A) + P(\bar{B} \cap A) P(\bar{C} | \bar{B} \cap A) + P(B \cap \bar{A}) P(\bar{C} | B \cap \bar{A})$

$P(\bar{B} \cap \bar{A}) = P(\bar{A}) P(\bar{B}|\bar{A})$ $P(B \cap A) = P(A) P(B|A)$ $P(\bar{B} \cap A) = P(A) P(\bar{B}|A)$

$= 1/3 \cdot 1/2 = 1/6$ $= (1-1/3)(1-1/3) = \frac{4}{9}$ $= (1-1/3)(1/3) = \frac{2}{9}$

$P(B \cap \bar{A}) = P(\bar{A}) P(B|\bar{A})$

$= 1/3 (1-1/2) = 1/6$

$P(\bar{C}) = 1/6 \cdot 1/2 + \frac{4}{9} \cdot 1/6 + \frac{2}{9} \cdot 1/4 + 1/6 \cdot 1/4$

$= 1/12 + 4/54 + 2/36 + 1/24$

$= \frac{55}{216} \approx 0.25463$ ANS.

c) $P(\bar{A} | \bar{C}) = \frac{P(\bar{A} \cap \bar{C})}{P(\bar{C})}$

Expansion of $P(\bar{A} \cap \bar{C}) = P(\bar{C} \cap B \cap \bar{A}) + P(\bar{C} \cap \bar{B} \cap \bar{A})$

$= P(B \cap \bar{A}) P(\bar{C} | B \cap \bar{A}) + P(\bar{B} \cap \bar{A}) P(\bar{C} | \bar{B} \cap \bar{A})$ $P(B \cap \bar{A})$ & $P(\bar{B} \cap \bar{A})$ from part b.

$= 1/6 \cdot 1/4 + 1/6 \cdot 1/2 = 0.125$

$P(\bar{A} | \bar{C}) = \frac{P(\bar{A} \cap \bar{C})}{P(\bar{C})} = \frac{0.125}{\frac{55}{216}} \approx 0.[illegible]909[illegible]$ ANS.

3. T_1 T_2 T_3 $\rightarrow$ Antenna $X \Rightarrow$ # of time slots untill 1st msg gets through.

$\frac{1}{2}$

a) $S = \{TTT, TFT, TFF, FFF,$

$TTF, FTT, FTF, FFT\}$

$= 2^3$

$= 8$

$P(\text{elementary}) = \frac{1}{8}$, equal prob. for all elem. events. ANS.

b) $S \Rightarrow S_X : X = \{0, 1, 2 \ldots n\}$ as there is prob. of no msg going through untill n trials, no matter how small that probability is.

c) Using geometric r.v. $X \sim Geom(p)$

Getting p $\Rightarrow$ P(only one T) = $\{TFF, FTF, FFT\}$

$= \frac{3}{8}$

$P(X) = (1-p)^{x-1} p$

$= (1 - \frac{3}{8})^{x-1} (\frac{3}{8})$

$= (\frac{5}{8})^{x-1} (\frac{3}{8})$ ANS.

d) $E(X)$ for $Geom(p) = \frac{1}{p}$

$= 1/_{3/8}$

$= 8/3$

$= 2.66667$ ANS.

$Var(X) = \frac{1-p}{p^2}$

$= \frac{1-(3/8)}{(3/8)^2}$

$= 4.\dot{4}$

$SD = \sqrt{Var(X)}$

$= \sqrt{\frac{1-(3/8)}{(3/8)^2}}$

$= 2.10818$ ANS.

4. $X \sim Bin(n, p)$ with $n = 4$ $p = 0.4$, $g(x) = 2x^2 + 1$

$E[g(x)] = E[2x^2+1]$
$= 2E[x^2] + E[1]$

$E(x) = np$ $Var(X) = np(1-p)$
$= E(x^2) - E(x)^2$
$np(1-p) + E(x)^2 = E(x^2)$
$E(x^2) = np(1-p) + (np)^2$

$E[g(x)] = 2[np(1-p) + (np)^2] + 1$
$= 2[1.6(0.6) + (1.6)^2] + 1$
$= 8.04$ ANS.

5. $\underbrace{\text{Call}}_{\lambda = 2} \sim$ Switchboard , X (# of calls) $\sim Pois(\lambda)$

a) P(exactly 2) = $P(X = 2)$

$P(X=2) = \dfrac{e^{-\lambda}\lambda^x}{X!}$

$= \dfrac{e^{-2}2^2}{2!}$

$= 0.27067$ ANS.

b) $P(X > 2) = 1 - P(X \leq 1)$
$= 1 - [P(X=0) + P(X=1)]$
$= 1 - \left[\dfrac{e^{-2}2^0}{0!} + \dfrac{e^{-2}2^1}{1!}\right]$

$= 0.59399$ ANS.