Math 256 Week 02 Assignment - written

This problem set is to be worked on during this week's Friday (Sept 17) lecture. You can continue to work on it and write your answers up carefully until Wednesday evening (Sept 22) at which time you should upload your work on GradeScope, following the guidelines on Canvas. Points may be deducted for failing to follow the guidelines.

1. (2 pts) The equation $y'' - 3y' + 2y = 0$ has solutions of the form $y(t) = e^{rt}$. Plug this ansatz[1] into the equation and find all values of r that provide solutions to the resulting equation. This equation for r is called the *characteristic equation* for the ODE.

2. (4 pts) Because $L[y] = y'' - 3y' + 2y$ is a linear operator, solutions to the equation $L[y] = 0$ can be added to other solutions. That is, if $L[y_1] = 0$ and $L[y_2] = 0$ then $L[y_1 + y_2] = L[y_1] + L[y_2] = 0 + 0 = 0$. Furthermore, constant multiples of solutions are also solutions because $L[cy_i] = cL[y_i] = 0$ for any constant c. Thus,
$$y_g(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$
is also a solution. Consider any initial condition (IC) $y(0) = y_0,\ y'(0) = v_0$ where y_0 and v_0 are constants that characterize the initial state of the system (e.g. if y is position, y_0 is initial position and v_0 is initial velocity). Show that you can always find some c_1 and c_2 so that the resulting $y_g(t)$ solves the initial value problem (IVP). Give expressions for the c values in terms of y_0 and v_0. When this is true we call $y_g(t)$ a general solution.

3. (a) (6 pts) Define the operator L for the equation
$$t^2 y'' + ty' - 4y = 0$$
and show that it is linear. Note: skipping steps that one might normally do in one's head makes this calculation look trivial so include ALL steps (yes, even reordering the terms in a sum counts as a step and each step is worth 1/2 pt).

   (b) (5 pts) Notice that the coefficients involve t^2, t and a constant respectively. In no more than 2-3 sentences (describing, not calculating), explain why an ansatz of $y(t) = t^n$ might work? Plug this ansatz into the equation and find values of n that provide solutions to the equation. Write down the solutions.

   (c) (4 pts) Show that
$$y_g(t) = c_1 t^{n_1} + c_2 t^{n_2}$$
is a general solution by showing that a c_1 and c_2 can be found for any IC $y(t_0) = y_0,\ y'(t_0) = v_0$ when $t_0 \neq 0$. You just need to show a c_1 and c_2 exists; you don't have to calculate them.
A general solution must provide (by specifying c_1 and c_2) a solution to ALL initial value problems for which a solution exist. What happens with initial conditions at $t_0 = 0$?

4. (4 pt) For what values of a does every solution to
$$y'' - ay' - (1 + a)y = 0$$
eventually approach zero? Be sure to write down the general solution as part of your answer and be careful not to miss a special case of a that has a different form of general solution.

[1] An *ansatz* is a fancy way of saying an educated guess that reduces the complexity of the problem you have to solve.

1) $y(t) = e^{rt} \quad y'(t) = re^{rt} \quad y''(t) = r^2e^{rt}$

$$y'' - 3y' + 2y = 0$$

$r^2e^{rt} - 3re^{rt} + 2e^{rt} = 0$

$\cancel{e^{rt}}(r^2 - 3r + 2)$

$r^2 - 3r + 2 \quad = 0$

$r = 2, 1$

2) $y_g(t) = C_1e^{r_1t} + C_2e^{r_2t} \quad , \quad r_1 = 2 \quad r_2 = 1$

$= C_1e^{2t} + C_2e^t \quad , \quad y(0) = y_0$

$y'(t) = 2C_1e^{2t} + C_2e^t \qquad y'(0) = V_0$

$y(0) = C_1 + C_2 \quad = y_0$

$y'(0) = 2C_1 + C_2 \quad = V_0$

$C_1 = y_0 - C_2 \qquad 2(y_0 - C_2) + C_2 = V_0$

$= y_0 - (2y_0 - V_0) \qquad 2y_0 - 2C_2 + C_2 =$

$= y_0 - 2y_0 + V_0 \qquad 2y_0 - C_2 =$

$C_1 = -y_0 + V_0 \qquad C_2 = 2y_0 - V_0$

$y_g(t) = (-y_0 + V_0)e^{2t} + (2y_0 - V_0)e^t$

3)

a. $L[y(t)] = t^2y'' + ty' - 4y$

$L[cy(t)] = t^2(cy)'' + t(cy)' + 4(cy)$

$= ct^2y'' + cty' + 4cy$

$= c(t^2y'' + ty' + 4y) \quad = cL[y(t)]$

$\therefore L[cy(t)] = cL[y(t)]$

$L[y(t) + z(t)] = t^2(y+z)'' + t(y+z)' + 4(y+z)$

$= t^2y'' + t^2z'' + ty' + tz' + 4y + 4z$

$= t^2y'' + ty' + 4y + tz'' + tz' + 4z \quad = L[y(t)] + 2[z(t)]$

$\therefore L[y(t) + z(t)] = L[y(t)] + L[z(t)]$

$\therefore$ Eq. is linear

b. The order of t in the equation increases linearly. Ansatz $y(t) = t^n$ will compensate the linear increase, allowing to isolate variable t and solve for n

$y(t) = t^n \qquad y'(t) = nt^{n-1} \qquad y''(t) = n(n-1)t^{n-2}$

$t^2y'' + ty' - 4y = 0$

$$\begin{aligned}
t^2(n^2-n)t^{n-2} + tnt^{n-1} - 4t^n &= 0 \\
(n^2-n)t^n + nt^n - 4t^n &= \\
t^n(n^2-n+n-4) &= \\
n^2-4 &= 0
\end{aligned}$$

$n = \pm 2$

c. $y(t) = C_1t^{n_1} + C_2t^{n_2}$

$\quad = C_1t^2 + C_2t^{-2} \qquad , \quad y(t_0) = y_0$

$y'(t) = 2C_1t - 2C_2t^{-3} \qquad , \quad y'(t_0) = V_0$

$y(t_0) = C_1t_0^2 + \frac{C_2}{t_0^2} = y_0 \qquad\qquad y'(t_0) = 2C_1t_0 - \frac{2C_2}{t_0^3} = V_0$

$$\det\begin{pmatrix} t_0^2 & t_0^{-2} \\ 2t_0 & -2t^{-3} \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ V_0 \end{pmatrix}$$

$-2t^{-1} - 2t^{-1} =$

$-4t^{-1} \neq 0 \quad$ when $t \neq 0$

Solution exist $\therefore C_1 \,\&\, C_2$ exist

$t_0 = 0$, sol'n DNE as denominator $\Rightarrow 0$

$$\begin{aligned}
C_1 &= \left(y_0 - \frac{C_2}{t_0^2}\right)\frac{1}{t_0^2} \\
&= \frac{y_0}{t_0^2} - \frac{C_2}{t_0^4} \\
&= \frac{y_0}{t_0^2} - \frac{2y_0t_0^{\cancel{4}} - V_0t_0^{\cancel{3}}}{4t_0^{\cancel{4}\,2}} \\
&= y_0\left(\frac{1}{t_0} - \frac{2 - V_0t_0}{4t_0}\right) \\
&= y_0\left(\frac{\cancel{4-2}\;2 - V_0t_0}{4t_0}\right)
\end{aligned}$$

$C_1 = y_0\left(\frac{2 - V_0t_0}{4t_0}\right)$

$$\begin{aligned}
&= \frac{2y_0}{t_0} - \frac{2C_2}{t_0^3} - \frac{2C_2}{t_0^3} = V_0 \\
&\frac{2y_0}{t_0} - \frac{4C_2}{t_0^3} = V_0 \\
&\frac{2y_0t_0^2}{t_0^3} - \frac{4C_2}{t_0^3} = \\
&2y_0t_0^2 - 4C_2 = V_0t_0^3 \\
&2y_0t_0^2 - V_0t_0^3 = 4C_2 \\
&C_2 = \frac{2y_0t_0^2 - V_0t_0^3}{4}
\end{aligned}$$

* attempt to solve for C.

4) $y'' - ay' - (1+a)y = 0$

$y(t) = e^{rt} \qquad y'(t) = re^{rt} \qquad y''(t) = r^2e^{rt}$

$e^{rt}(r^2 - ar - (1+a)) = 0$

$r^2 - ar - (1+a) = 0$

For sol'n to $\to 0$ as $t \to \infty$, one of two must be negative.

$r = \frac{a \pm \sqrt{a^2 - 4(1)(-1-a)}}{2} \qquad\qquad r = \frac{a \pm (a+2)}{2}$

$a^2 + 4 + 4a \geq 0 \qquad r = \frac{a-a-2}{2} \qquad r = a + a + 2$

$a^2 + 4 + 4a \geq 0 \qquad\qquad\qquad = 2a + 2 \quad$ no restriction as condition is satisfied.

$(a+2)^2 \geq 0 \qquad = -1$

$a \geq -2$

$\therefore a \geq -2$

Math 256 Week 03 Assignment - written

This problem set is to be worked on during this week's Friday (Sept 24) lecture. You can continue to work on it and write your answers up carefully until the due date (Thursday Sept 30) at which time you should upload your work on GradeScope, following the guidelines on Canvas. Points may be deducted for failing to follow the guidelines.

The method for solving inhomogeneous second order ODEs outlined in these problems is called the Method of Undetermined Coefficients.

1. (7 pts) **Families of functions.** Find the general solution to the equation

$$y'' - y' - 2y = 5\cos x$$

y_p & y_h

using the steps outlined below. member of family $y_p = A\cos x + B\sin x$

   (a) Find the general solution of the associated homogeneous equation

   $$y'' - y' - 2y = 0$$

   which is often referred to as the homogeneous solution.

   (b) Try the ansatz $y_p(x) = A\cos x + B\sin x$. Why would the arguably more obvious ansatz $y_p(x) = A\cos x$ not work? plug in cos → sin ∴ ~~to~~ can't compensate

   (c) Add the homogeneous solution to your particular solution y_p to get the general solution to the inhomogeneous equation.

2. (7 pts) **Families of functions.** Find the general solution to

$$y'' - 6y' + 9y = 5xe^{2x}.$$

   (a) Find the homogeneous solution.

   (b) Try the ansatz $y_p(x) = (Ax+B)e^{2x}$. Why would the arguably more obvious ansatz $y_p(x) = Axe^{2x}$ not work?

   (c) State the general solution to the inhomogeneous equation.

3. (7 pts) **When the RHS is in the nullspace.** Find the general solution to

$$y'' + 3y' + 2y = 7e^{-2x}.$$

$(r+2)(r+1)$

   (a) Find the homogeneous solution.

   $L[Ae^{-2x}] \neq 7e^{-2x}$ usual way fail b.c e^{-2x} is in Null space of L. $(A\vec{x}=0)$

   (b) Try the ansatz $y_p(x) = Ae^{-2x}$. Why does this not work?

   (c) Try the ansatz $y_p(x) = Axe^{-2x}$.

   $L[Axe^{-2x}]$ ↓ $0xe^{-2x} + e^{-2x}$

   (d) State the general solution to the inhomogeneous equation.

4. (6 pts) **When the RHS is in the nullspace and there is a repeated root.** Solve the IVP,

$$y'' - 6y' + 9y = 2e^{3x},\ y(0) = y'(0) = 0$$

   (a) Find the homogeneous solution.

   (b) Try the ansatz $y_p(x) = Ax^2e^{3x}$.

   (c) Explain (i) why only one term is sufficient and (ii) why neither Axe^{3x} nor Ae^{3x} would have worked.

   (d) State the general solution to the inhomogeneous equation. Use it to find the particular solution that solves the IC given above.

1. (7 pts) **Families of functions.** Find the general solution to the equation

$$y'' - y' - 2y = 5\cos x \qquad y_p \ \& \ y_h$$

using the steps outlined below. member of family $\quad y_p = A\cos x + B\sin x$

(a) Find the general solution of the associated homogeneous equation

$$y'' - y' - 2y = 0$$

which is often referred to as the homogeneous solution.

(b) Try the ansatz $y_p(x) = A\cos x + B\sin x$. Why would the arguably more obvious ansatz $y_p(x) = A\cos x$ not work? plug in cos → sin ∴ ~~to~~ can't compensate

(c) Add the homogeneous solution to your particular solution y_p to get the general solution to the inhomogeneous equation.

1.

a) $y''-y'-2y = 0 \qquad y = e^{rt} \quad y' = re^{rt} \quad y'' = r^2e^{rt}$

$r^2\cancel{e^{rt}} - r\cancel{e^{rt}} - 2\cancel{e^{rt}} = 0$

$r^2 - r - 2 =$

$(r-2)(r+1) =$

$r = 2, -1$

$y_H(x) = C_1e^{2x} + C_2e^{-x}$

b) $L[y] := y'' - y' - 2y \qquad L[y] = 5\cos x$

Ansatz:

$y_p(x) = A\cos x + B\sin x$

$y_p'(x) = -A\sin x + B\cos x$

$y_p''(x) = -A\cos x - B\sin x$

$L[A\cos x + B\sin x] = (-A\cos x - B\sin x) - (-A\sin x + B\cos x) - 2(A\cos x + B\sin x)$

$= \quad + A\sin x - B\cos x - 2A\cos x - 2B\sin x$

$= -A\cos x + A\sin x - 2A\cos x - B\sin x - B\cos x - 2B\sin x$

$= -3A\cos x - B\cos x + A\sin x - 3B\sin x$

$-3A - B = 5$

$A - 3B = 0$

$A = 3B$

$-9B - B = 5$

$B = -\frac{1}{2}, \quad A = -\frac{3}{2}$

$y_p(x) = -\frac{3}{2}\cos(x) - \frac{1}{2}\sin(x)$

Acos x would not work as the family of the function cos x involves sin x (when taken derivative), while the ansatz does not. Thus, not being able to compensate the entire family of function 5cos x. In this case, the obtained particular solution will be incorrect.

c) $y_g(x) = y_h(x) + y_p(x)$

$= C_1e^{2x} + C_2e^{-x} - \frac{1}{2}\sin x - \frac{3}{2}\cos x$

(7 pts) **Families of functions.** Find the general solution to

$$y'' - 6y' + 9y = 5xe^{2x}.$$

(a) Find the homogeneous solution.

(b) Try the ansatz $y_p(x) = (Ax+B)e^{2x}$. Why would the arguably more obvious ansatz $y_p(x) = Axe^{2x}$ not work?

(c) State the general solution to the inhomogeneous equation.

2.

a) $y'' - 6y' + 9y = 0$ $y = e^{rt}$ $y' = re^{rt}$ $y'' = r^2e^{rt}$

$r^2\cancel{e^{rt}} - 6r\cancel{e^{rt}} + 9\cancel{e^{rt}} = 0$

$r^2 - 6 + 9 =$

$(r-3)^2 =$

$r = 3$

$y_h(x) = C_1e^{3x} + xC_1e^{3x}$

b) $L[y] := y'' - 6y' + 9y$ $L[y] = 5xe^{2x}$

Ansatz :

$y_p(x) = (Ax+B)e^{2x}$

$y_p'(x) = Ae^{2x} + (Ax+B)2e^{2x}$

$y_p''(x) = 2Ae^{2x} + A2e^{2x} + (Ax+B)4e^{2x}$

$= 4(Ax+B)e^{2x} + 4Ae^{2x}$

$L[(Ax+B)e^{2x}] = 4(Ax+B)e^{2x} + 4Ae^{2x} - 6[Ae^{2x} + (Ax+B)2e^{2x}] + 9[(Ax+B)e^{2x}]$

$= 4Axe^{2x} + 4Be^{2x} + 4Ae^{2x} - 6Ae^{2x} - 12Axe^{2x} - 12Be^{2x} + 9Axe^{2x} + 9Be^{2x}$

$= 4Axe^{2x} - 12Axe^{2x} + 9Axe^{2x} + 4Ae^{2x} - 6Ae^{2x} + 4Be^{2x} - 12Be^{2x} + 9Be^{2x}$

$= Axe^{2x} - 2Ae^{2x} + Be^{2x}$

$A = 5$

$-2A + B = 0$

$B = 10$

$y_p(x) = 5xe^{2x} + 10e^{2x}$

Axe^{x} would not work as it does not consider the entire family of the function $5xe^{2x}$ were the derivative involves a constant with e^{2x}. In such case the particular solution obtained will be incorrect.

c) $y_g(x) = C_1e^{3x} + xC_2e^{3x} + 5xe^{2x} + 10e^{2x}$

3. (7 pts) **When the RHS is in the nullspace.** Find the general solution to

$$y'' + 3y' + 2y = 7e^{-2x}.$$

(a) Find the homogeneous solution.
(b) Try the ansatz $y_p(x) = Ae^{-2x}$. Why does this not work?
(c) Try the ansatz $y_p(x) = Axe^{-2x}$.
(d) State the general solution to the inhomogeneous equation.

$(r+2)(r+1)$

$L[Ae^{-2x}] \neq 7e^{-2x}$ usual way fail b.c e^{-2x} is in Null space of L. ($A\vec{x}=0$)

$L[Axe^{-2x}]$

$0xe^{-2x} + e^{-2x}$

3.

a) $y'' + 3y' + 2y = 0 \qquad y = e^{rt} \quad y' = re^{rt} \quad y'' = r^2e^{rt}$

$r^2\cancel{e^{rt}} + 3r\cancel{e^{rt}} + 2\cancel{e^{rt}} = 0$

$r^2 + 3r + 2 =$

$(r+2)(r+1) =$

$r = -2, -1$

$y_h(x) = C_1e^{-2x} + C_2e^{-x}$

b) $L[y] := y'' + 3y' + 2y = 7e^{-2x} \qquad L[y] = 7e^{-2x}$

Ansatz :

$y_p(x) = Ae^{-2x}$

$y'_p(x) = -2Ae^{-2x}$

$y''_p(x) = 4Ae^{-2x}$

$L[Ae^{-2x}] = 4e^{-2x} - 6Ae^{-2x} + 2Ae^{2x}$

$= 0$

Ae^{-2x} would not work as with the homogenous solution with e^{-2x} in its solution, when ansatz is used, the particular solution equates to 0. Thus, unable to obtain a general solution.

c) Ansatz :

$y_p(x) = Axe^{-2x}$

$y'_p(x) = Ae^{-2x} + Ax(-2e^{-2x})$

$y''_p(x) = 0 + A(-2e^{-2x}) + A(-2e^{-2x}) + Ax(4e^{-2x})$

$L[Axe^{-2x}] = A(-2e^{-2x}) + A(-2e^{-2x}) + Ax(4e^{-2x}) + 3[Ae^{-2x} + Ax(-2e^{-2x})] + 2[Axe^{-2x}]$

$= -2Ae^{-2x} - 2Ae^{-2x} + 4Axe^{-2x} + 3Ae^{-2x} - 6Axe^{-2x} + 2Axe^{-2x}$

$= -Ae^{-2x}$

$A = -7$

$y_p(x) = -7xe^{-2x}$

d) $y_g(x) = C_1e^{-2x} + C_2e^{-x} - 7xe^{-2x}$

4. (6 pts) **When the RHS is in the nullspace and there is a repeated root.** Solve the IVP,

$$y'' - 6y' + 9y = 2e^{3x}, \ y(0) = y'(0) = 0$$

(a) Find the homogeneous solution.

(b) Try the ansatz $y_p(x) = Ax^2e^{3x}$.

(c) Explain (i) why only one term is sufficient and (ii) why neither Axe^{3x} nor Ae^{3x} would have worked.

(d) State the general solution to the inhomogeneous equation. Use it to find the particular solution that solves the IC given above.

4. a) $y'' - 6y' + 9 = 0$ $\quad y = e^{rt} \quad y' = re^{rt} \quad y'' = r^2e^{rt}$

$$r^2e^{rt} - 6re^{rt} + 9e^{rt} = 0$$
$$r^2 - 6 + 9 =$$
$$(r-3)^2 =$$
$$r = 3$$

$y_h(x) = C_1e^{3x} + xC_2e^{3x}$

b) $L[y] := y'' - 6y' + 9y \qquad L[y] = 2e^{3x}$

Ansatz:

$$y_p(x) = Ax^2e^{3x}$$
$$y'_p(x) = 2Axe^{3x} + 3Ax^2e^{3x}$$
$$y''_p(x) = 2Ae^{3x} + 6Axe^{3x} + 6Axe^{3x} + 9Ax^2e^{3x}$$
$$= 9Ax^2e^{3x} + 12Axe^{3x} + 2Ae^{3x}$$

$$L[Ax^2e^{3x}] = 9Ax^2e^{3x} + 12Axe^{3x} + 2Ae^{3x} - 6[2Axe^{3x} + 3Ax^2e^{3x}] + 9[Ax^2e^{3x}]$$
$$= \cancel{9Ax^2}e^{3x} + \cancel{12Ax}e^{3x} + 2Ae^{3x} - \cancel{12Ax}e^{3x} - \cancel{18Ax^2}e^{3x} + \cancel{9Ax^2}e^{3x}$$
$$= 0 + 0 + 2Ae^{3x}$$
$$= 2Ae^{3x}$$
$$2A = 2$$
$$A = 1$$

$y_p(x) = x^2e^{3x}$

c)

i) One term is sufficient as the family of the function $2e^{3x}$ only involve itself, the constant with e^{3x}.

ii) Axe^{3x} & Ae^{3x} would not have worked as homogenous solution contained two e^{3x} which if not compensated with x^2, would not have been sufficient to obtain a particular solution as $y_p(x)$ will equate to 0 in both cases of ansatz.

d) $y_g(x) = C_1e^{3x} + xC_2e^{3x} + x^2e^{3x}$

$y(0) = C_1 + 0 + 0$

$y'(x) = 3C_1e^{3x} + C_2e^{3x} + x3C_2e^{3x} + 2xe^{3x} + x^2 3e^{3x}$

$= 3C_1e^{3x} + C_2e^{3x} + 3xC_2e^{3x} + 2xe^{3x} + 3x^2e^{3x}$

$y'(0) = 3C_1 + C_2$

$C_1 = 0$

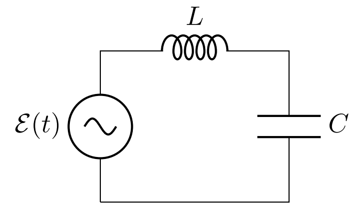
$3C_1 + C_2 = 0$

$C_2 = 0$

$y(x) = x^2e^{3x}$

Math 256 Week 04 Assignment - written

An inductor, a capacitor, and an alternating voltage source are connected to each other as shown in the diagram. The voltages across the elements are given by constitutive laws: $v_L = Li'(t)$, $v_C = q(t)/C$, and the EMF is given by $\mathcal{E}(t) = \mathcal{E}_0 \cos(\omega t)$ where $i(t)$ is the current in the circuit, $q(t)$ is the charge across the capacitor, L is the inductance, C is the capacitance, $\mathcal{E}_0$ is the amplitude of the alternating voltage source, and ω is its angular frequency. As the current, $i(t)$ is the same in all elements and $i = dq/dt$, Kirchhoff's Voltage Law tells us that



$$Lq'' + \frac{q}{C} = \mathcal{E}(t).$$

1. (2 pts) What is the natural frequency of the circuit, ω_0? That is, at what angular frequency does the charge on the capacitor (or equivalently, the current in the circuit) oscillate when $\mathcal{E}_0 = 0$? Write down the solution to the homogeneous equation using ω_0.

2. Consider the case of $\omega \neq \omega_0$.

   (a) (3 pts) Use the Method of Undetermined Coefficients to find the general solution, $q(t)$, to the equation when $\omega \neq \omega_0$.

   (b) (1 pt) Find the particular solution that satisfies $q(0) = q'(0) = 0$.

   (c) (1 pt) Use the identity

   $$\cos(\theta) - cos(\varphi) = -2\sin\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right)$$

   to rewrite the solution from the previous part in terms of sine functions.

   (d) (1 pt) The sine functions are bounded between -1 and 1 so the absolute value of the coefficient out front can be thought of as an amplitude. Sketch the amplitude as a function of ω.

3. (3 pts) Use the Method of Undetermined Coefficients to find the particular solution, $q_{\omega_0}(t)$, to the equation when $\omega = \omega_0$, that satisfies ICs $q_{\omega_0}(0) = q'_{\omega_0}(0) = 0$. What happens to the solution as t gets large?

4. (1 pt) Show that, in the limit of $\omega \to \omega_0$, your answer in 2b converges to your answer in 3. Recall that when the θ is small, linear approximation gives $\sin(\theta) \approx \theta$; this will be a useful approximation.

1. (2 pts) What is the natural frequency of the circuit, ω_0? That is, at what angular frequency does the charge on the capacitor (or equivalently, the current in the circuit) oscillate when $\mathcal{E}_0 = 0$? Write down the solution to the homogeneous equation using ω_0.

1. $Lq'' + \frac{q}{C} = \mathcal{E}_0 \cos(\omega t)$

Ansatz: $q(t) = e^{rt}$ $q'(t) = re^{rt}$ $q''(t) = r^2 e^{rt}$

homog. $Lq'' + \frac{q}{C} = 0$

$q'' + \frac{q}{LC} = 0$

$Lr^2 + \frac{1}{C} = 0$

$Lr^2 = -\frac{1}{C}$

$r^2 = \frac{-1}{LC}$

$r = i\sqrt{1/LC}$

$q_h(t) = C_1 \sin(\sqrt{1/LC}\, t) + C_2 \cos(\sqrt{1/LC}\, t)$, $\omega_0 = \sqrt{1/LC}$

$q_h(t) = C_1 \sin(\omega_0 t) + C_2 \cos(\omega_0 t)$

2. Consider the case of $\omega \neq \omega_0$.

(a) (3 pts) Use the Method of Undetermined Coefficients to find the general solution, $q(t)$, to the equation when $\omega \neq \omega_0$.

(b) (1 pt) Find the particular solution that satisfies $q(0) = q'(0) = 0$.

(c) (1 pt) Use the identity

$$\cos(\theta) - cos(\varphi) = -2\sin\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right)$$

to rewrite the solution from the previous part in terms of sine functions.

(d) (1 pt) The sine functions are bounded between -1 and 1 so the absolute value of the coefficient out front can be thought of as an amplitude. Sketch the amplitude as a function of ω.

2.

a) $Lq'' + q/c = 0$ $\quad\therefore$ Homog. can be written as: $Lq'' + q/c = 0$

$W_0 = \sqrt{1/LC}$ $\qquad Lq'' + w^2 Lq =$

$\qquad L(q'' + w^2 q) =$

$C = \frac{1}{W_0^2 L}$ $\qquad q'' + w_0^2 q = 0$

$q'' + W_0^2 q = \mathcal{E}_0 \cos(wt)$

$L[y] := q'' + W_0^2 q \qquad L[y] = \mathcal{E}_0 \cos(wt)$

Ansatz: $q_p(t) = A\cos wt + B\sin wt$

$q_p'(t) = -A\sin wt + B\cos wt$

$q_p''(t) = -Aw^2\cos wt - Bw^2 \sin wt$

$L[A\cos wt + B\sin wt] = -Aw^2\cos wt - Bw^2\sin wt + w_0^2(A\cos wt + B\sin wt)$

$= (-Aw^2 + Aw_0^2)\cos wt + (-Bw^2 + Bw_0)\sin wt$

$-Aw^2 + Aw_0^2 = \mathcal{E}_0 \qquad -Bw^2 + Bw_0^2 = 0$

$A(-w^2 + w_0^2) = \mathcal{E}_0 \qquad B = 0$

$A = \frac{\mathcal{E}_0}{-W^2 + W_0^2}$

$q_p(t) = \frac{\mathcal{E}_0}{-w^2 + W_0^2}\cos(wt)$

$q_g(t) = C_1\sin(W_0 t) + C_2\cos(W_0 t) + \frac{\mathcal{E}_0}{-W^2 + W_0^2}\cos(wt)$ when $W_0 \neq W$

b) $q_g(t) = C_1\sin(W_0 t) + C_2\cos(W_0 t) + \frac{\mathcal{E}_0}{-W^2 + W_0^2}\cos(wt)$ $\qquad q_g(0) = 0 + C_2 + \frac{\mathcal{E}_0}{-W^2 + W_0^2} = 0$

$C_2 = -\frac{\mathcal{E}_0}{-W^2 + W_0^2}$

$q_g'(t) = C_1 W_0\cos(W_0 t) - C_2 W_0\sin(W_0 t) - \frac{\mathcal{E}_0 W}{-W^2 + W_0^2}\sin(wt)$ $\qquad q_g'(0) = C_1 W_0 = 0$

$C_1 = 0$

$q_g(t) = \frac{-\mathcal{E}_0}{-W^2 + W_0^2}\cos(W_0 t) + \frac{\mathcal{E}_0}{-W + W_0^2}\cos(wt) = \frac{\mathcal{E}_0}{-W^2 + W_0^2}\left[\cos(wt) - \cos(W_0 t)\right]$

c)

$$\cos(\theta) - cos(\varphi) = -2\sin\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right)$$

$$q_g(t) = \frac{\mathcal{E}_0}{-W^2+W_0^2}\left(-\cos(w_0 t) + \cos(wt)\right)$$

$$= \left(\cos(wt) - \cos(w_0 t)\right)$$

$$= \frac{\mathcal{E}_0}{-W^2+W_0^2}\left[-2\sin\left(\frac{wt+w_0t}{2}\right)\sin\left(\frac{wt-w_0t}{2}\right)\right]$$

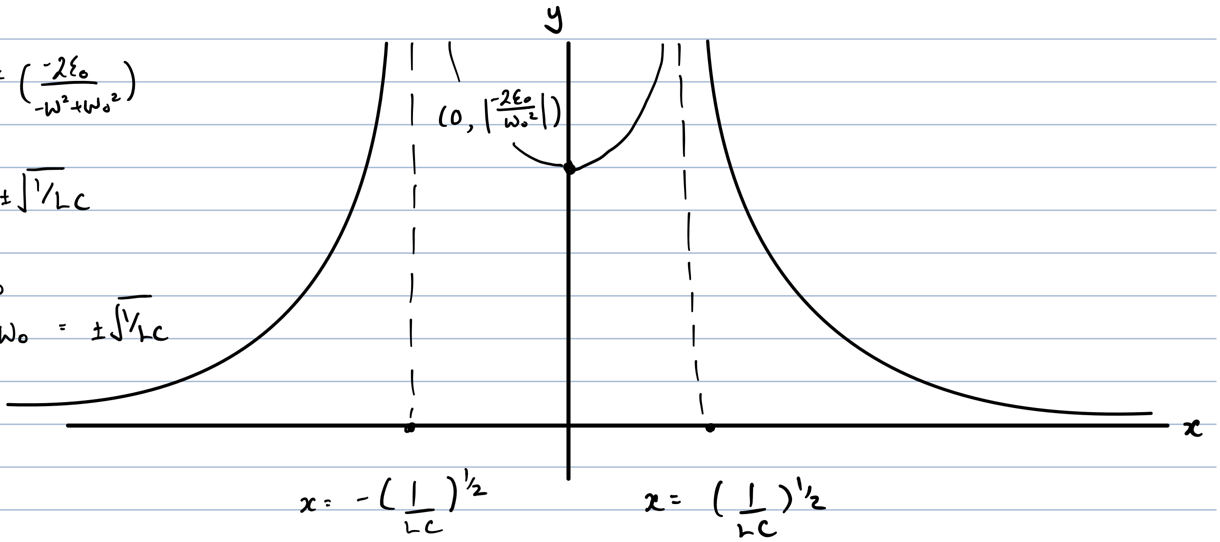
d) $$q_g(t) = \frac{-2\mathcal{E}_0}{-W^2+W_0^2}\left[\sin\left(\frac{wt+w_0t}{2}\right)\sin\left(\frac{wt-w_0t}{2}\right)\right]$$

$$A = \left|\frac{-2\mathcal{E}_0}{-W^2+W_0^2}\right| = \pm\left(\frac{-2\mathcal{E}_0}{-W^2+W_0^2}\right)$$

$$|W_0| = \pm\sqrt{1/LC}$$

asymptotes : $y = 0$

$x = W_0 = \pm\sqrt{1/LC}$



3. (3 pts) Use the Method of Undetermined Coefficients to find the particular solution, $q_{\omega_0}(t)$, to the equation when $\omega = \omega_0$, that satisfies ICs $q_{\omega_0}(0) = q'_{\omega_0}(0) = 0$. What happens to the solution as t gets large?

3. $Lq'' + q/C = \mathcal{E}_0 \cos(w_0 t)$

$q'' + q/LC = 0$

$r^2 + 1/LC = 0$

$r = \pm i\sqrt{1/LC}$

from q |

$q_h(t) = C_1 \sin(\sqrt{1/LC}\, t) + C_2\cos(\sqrt{1/LC}\, t)$, $W_0 = \sqrt{1/LC}$

$= C_1 \sin(W_0 t) + C_2 \cos(W_0 t)$

$W_0^2 L = 1/C$

$C = \frac{1}{W_0^2 L}$ $\therefore$ $Lq'' + W_0^2 L q = 0$

$q'' + W_0^2 q = 0$ has soln $q_h(t)$ too.

$L[y] := q'' + w_0^2 q \qquad L[y] = \mathcal{E}_0 \cos(w_0 t)$

Ansatz : $q_p(t) = (A\cos w_0 t + B\sin w_0 t)t$
$= At\cos w_0 t + Bt\sin w_0 t$

$q'_p(t) = A\cos W_0 t + At(-\sin W_0 t \cdot w_0) + B\sin W_0 t + Bt(\cos W_0 t \cdot w_0)$

$= A\cos W_0 t - Atw_0 \sin(w_0 t) + B\sin W_0 t + Btw_0 \cos W_0 t$

$q''_p(t) = -A\sin W_0 t \cdot W_0 - AW_0 \sin W_0 t + AW_0^2 t \cos W_0 t + B\cos W_0 t \cdot W_0 + Bw_0 \cos W_0 t - Btw_0^2 \sin W_0 t$
$= -AW_0 \sin W_0 t - AW_0 \sin W_0 t + Atw_0^2 \cos W_0 t + Bw_0 \cos W_0 t + Bw_0 \cos W_0 t - Btw_0^2 \sin W_0 t$
$= -AW_0 \sin W_0 t - AW_0 \sin W_0 t - Btw_0^2 \sin W_0 t + Atw_0^2 \cos W_0 t + Bw_0 \cos W_0 t + Bw_0 \cos W_0 t$
$= (-AW_0 - AW_0 - Btw_0^2)\sin W_0 t + (Bw_0 + Bw_0 - Atw_0^2)\cos W_0 t$

$L[A\cos(w_0 t) + B\sin(w_0 t)] = (-AW_0 - AW_0 - \cancel{Btw_0^2})\sin W_0 t + (Bw_0 + Bw_0 - \cancel{Atw_0^2})\cos W_0 t + At\cancel{w_0^2}\cos W_0 t + B\cancel{tw_0^2}\sin W_0 t$
$= -2AW_0 \sin(W_0 t) + 2BW_0 \cos(W_0 t)$

$-2AW_0 = 0 \quad -W_0 \qquad 2BW_0 = \mathcal{E}_0$
$A = 0 \qquad B = \frac{\mathcal{E}_0}{2w_0}$

$q_p(t) = \frac{\mathcal{E}_0}{2W_0} \sin(w_0 t)$

$q_g(t) = C_1 \sin(W_0 t) + C_2 \cos(W_0 t) + \frac{\mathcal{E}_0 t}{2W_0} \sin(W_0 t)$

$q_g(0) = 0 + C_2 + 0 = 0$
$C_2 = 0$

$q'_g(t) = C_1 w_0 \cos(w_0 t) - C_2 W_0 \sin(w_0 t) + \frac{\mathcal{E}_0}{2w_0}\sin(w_0 t) + \frac{\mathcal{E}_0 t}{2}\cos(w_0 t)$

$q_g(0) = C_1 W_0 + 0 = 0$
$C_1 = 0$

$q_g(t) = \frac{\mathcal{E}_0 t}{2w_0} \sin(w_0 t)$

4. (1 pt) Show that, in the limit of $\omega \to \omega_0$, your answer in 2b converges to your answer in 3. Recall that when the θ is small, linear approximation gives $\sin(\theta) \approx \theta$; this will be a useful approximation.

4. $$\lim_{w \to w_0} q_g(t) = \lim_{w \to w_0} \frac{\varepsilon_0}{-w^2 + w_0^2} \left[\cos(wt) - \cos(w_0 t) \right]$$

$$= \frac{\varepsilon_0}{-w^2 + w_0^2} \left[-2 \sin\left(\frac{wt + w_0 t}{2} \right) \sin\left(\frac{wt - w_0 t}{2} \right) \right]$$

$$= \frac{2\varepsilon_0}{w_0^2 - w^2} \left[\sin\left(\frac{wt + w_0 t}{2} \right) \sin\left(\frac{wt - w_0 t}{2} \right) \right]$$

$$= \lim_{w \to w_0} \frac{2\varepsilon_0}{(w_0 + w)(w_0 - w)} \left[\sin\left(\frac{wt + w_0 t}{2} \right) \sin\left(\frac{wt - w_0 t}{2} \right) \right]$$

$$= \frac{2\varepsilon_0}{w_0 + w_0} \left[\sin(w_0 t) \cdot t/2 \right]$$

$$= \frac{2\varepsilon_0}{2w_0} \cdot \frac{t}{2} \cdot \sin(w_0 t)$$

$$= \frac{\varepsilon_0 t}{2w_0} \sin(w_0 t)$$ $\therefore$ 2b converges to answer in 3.

Math 256 Week 05 Assignment - written

1. (9 pts) For each value of a, the system of equations defined by the vector equation $\underline{y}' = A\underline{y}$ with

$$A = \begin{pmatrix} -2 & a \\ 3 & -2 \end{pmatrix}$$

has two independent solutions. For each scenario given below, provide conditions on a for which the scenario occurs or explain why it does not occur for any value of a. Assume a is real.

   (a) When do these two solutions oscillate? complex roots

   (b) When does the system have a repeated eigenvalue? $= 0$ the edge case.

   (c) When do both solutions decay exponentially without oscillating?

   (d) When does one solution decay exponentially and the other grow exponentially?

   (e) When do both solutions grow exponentially (either oscillating or not)?

2. (6 pts) Consider the following equations:

$$\frac{dx}{dt} = 2x - y,$$

$$\frac{dy}{dt} = 2x + 4y.$$

   (a) Write the equations in matrix form.

   (b) Find the real-valued general solution using eigenvalues and eigenvectors.

3. (6 pts) Consider the system

$$y' = \begin{bmatrix} 0 & 3 \\ -3 & 6 \end{bmatrix} y$$

   (a) Find the eigenvalues of the matrix.

   (b) Find the corresponding eigenvectors.

   (c) Find the general solution to the system.

1. $\bar{y}' = A\bar{y}$

$$A = \begin{pmatrix} -2 & a \\ 3 & -2 \end{pmatrix}$$

$$(A - \lambda I)\bar{v} = 0$$

$$\begin{pmatrix} -2-\lambda & a \\ 3 & -2-\lambda \end{pmatrix} = 0$$

$$(-2-\lambda)^2 - 3a = 0$$
$$4 + 4\lambda + \lambda^2 - 3a = 0$$
$$\lambda^2 + 4\lambda + 4 - 3a = 0$$

$$\lambda = \frac{-4 \pm \sqrt{16 - 4(1)(4-3a)}}{2}$$

$$= \frac{-4 \pm (16 - 16 + 12a)^{1/2}}{2}$$

$$= \frac{-4 \pm (12a)^{1/2}}{2}$$

$$= \frac{-\cancel{4}^{2} \pm \cancel{2}\sqrt{3a}}{\cancel{2}}$$

$$= -2 \pm \sqrt{3a}$$

a) $a < 0$
$\lambda = -2 \pm \sqrt{c}i$ complex ev, sin & cos appear in complex ev's.

b) $a = 0$
$\lambda = -2$ Double ev, other values of a will cause $\pm$ to have diff roots.

c) $a = 0$
$\lambda = -2$ Equal & neg ev : e^{-2t} & e^{-2t} $\therefore$ Decay exponencially.

d) $a = 3$
$\lambda = -2 \pm 3$ Oppo sign ev : e^{t} & e^{-5t} $\therefore$ One decay
$\lambda_1 = 1 \quad \lambda_2 = -5$ One grow exponencially

e) $a < 0$
$\lambda = -2 \pm \sqrt{c}i$ Grows exponencially with oscillation as e^{t} always grow regardless of λ.

2. $\frac{dx}{dt} = 2x - y \qquad \frac{dy}{dt} = 2x + 4y$

a) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

b) $\bar{y} = A\bar{y}$

$(A - I\lambda)\bar{v} = 0$

$\det(A - I\lambda) = 0$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{6 \pm (36 - 4(1)(10))^{1/2}}{2}$$
$$= \frac{6 \pm (-4)^{1/2}}{2} = \frac{6 \pm 2i}{2}$$
$$= 3 \pm i$$

$$\det\begin{pmatrix} 2-\lambda & -1 \\ 2 & 4-\lambda \end{pmatrix} = 0$$
$$= (2-\lambda)(4-\lambda) - (-1)(2)$$
$$= \lambda^2 - 6\lambda + 10$$

$\lambda = 3 + i$,

$$\left(\begin{array}{cc|c} 2-3-i & -1 & 0 \\ 2 & 4-3-i & 0 \end{array}\right) \quad \left(\begin{array}{cc|c} -1-i & -1 & 0 \\ 2 & 1-i & 0 \end{array}\right) \div \frac{(-1-i)}{2}$$

$$\left(\begin{array}{cc|c} 1 & \frac{-1+i}{2} & 0 \\ 2 & 1-i & 0 \end{array}\right) - 2R_1 \quad \left(\begin{array}{cc|c} 1 & \frac{-1+i}{2} & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$V_1 + \frac{1-i}{2} V_2 = 0 \qquad V_2 = c$

$V_1 = -c\left(\frac{1-i}{2}\right)$

$V = c\begin{bmatrix} \frac{-1+i}{2} \\ 1 \end{bmatrix}$

$$y(t) = e^{(3+i)t}\begin{bmatrix} \frac{-1+i}{2} \\ 1 \end{bmatrix}$$
$$= e^{3t} e^{it} [\; " \;]$$
$$= e^{3t}(\cos t + i\sin t)\begin{bmatrix} \frac{-1+i}{2} \\ 1 \end{bmatrix}$$

$$= e^{3t}\begin{pmatrix} \frac{-\cos t + i\cos t}{2} + \frac{-i\sin t - \sin t}{2} \\ \cos t + i\sin t \end{pmatrix}$$

$$= e^{3t}\begin{pmatrix} -\frac{\cos t}{2} - \frac{\sin t}{2} & \frac{i\cos t}{2} - \frac{i\sin t}{2} \\ \cos t & i\sin t \end{pmatrix}$$

$$\bar{y}(t) = C_1 e^{3t}\begin{bmatrix} \frac{-\cos t - \sin t}{2} \\ \cos t \end{bmatrix} + C_2 e^{3t}\begin{bmatrix} \frac{\cos t - \sin t}{2} \\ \sin t \end{bmatrix}$$

3. $\bar{y}' = A\bar{y}$

$$A = \begin{pmatrix} 0 & 3 \\ -3 & 6 \end{pmatrix}$$

a) $\det(A - \lambda \mathbb{1})\bar{v} = 0$

$$\begin{pmatrix} 0-\lambda & 3 \\ -3 & 6-\lambda \end{pmatrix} = 0$$

$$-\lambda(6-\lambda) + 9 = 0$$
$$\lambda^2 - 6\lambda + 9 = 0$$
$$(\lambda - 3)^2 = 0$$
$\lambda = 3$ double root.

b) $\lambda = 3$,

$$\left(\begin{array}{cc|c} -3 & 3 & 0 \\ -3 & 3 & 0 \end{array}\right) \div -3 \quad \left(\begin{array}{cc|c} 1 & -1 & 0 \\ -3 & 3 & 0 \end{array}\right) + 3R_1$$

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$V_1 - V_2 = 0$ $\quad V = c\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$V_2 = c$
$V_1 = c$

c) $y(t) = C_1 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$y_2(t) = (\bar{w} + t\bar{u})e^{3t}$$
$$= \bar{w}e^{3t} + t\bar{u}e^{3t}$$
$$y_2'(t) = 3\bar{w}e^{3t} + \bar{u}e^{3t} + 3t\bar{u}e^{3t}$$

$$A\bar{y}_2 = A\bar{w}e^{3t} + tA\bar{u}e^{3t}$$
$$3\bar{w} + \bar{u} + 3t\bar{u} = A\bar{w} + tA\bar{u}$$
$A\bar{u} = 3\bar{u}$ where $\bar{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$$A\bar{w} = 3\bar{w} + \bar{u}$$

$$(A - 3\mathbb{1})\bar{w} = \bar{u}$$
$$\begin{pmatrix} -3 & 3 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -3 & 3 & 1 \\ -3 & 3 & 1 \end{array}\right) \div -3 \;\; +3R_1 \quad \begin{pmatrix} 1 & -1 & -1/3 \\ 0 & 0 & 0 \end{pmatrix}$$

$w_1 - w_2 = -1/3$
$w_2 = d$
$w_1 = -1/3\, d$ $\quad \bar{w} = \begin{pmatrix} -1/3 + d \\ d \end{pmatrix}$

$$= c\begin{pmatrix} -1/3 \\ 0 \end{pmatrix} + d\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\bar{y}(t) = C_1 e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{3t}\left[c\begin{pmatrix} -1/3 \\ 0 \end{pmatrix} + d\begin{pmatrix} 1 \\ 1 \end{pmatrix} + tc\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]$$

$$= C_3 e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_4 e^{3t}\left[\begin{pmatrix} -1/3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]$$

Math 256 Week 07 Assignment - written

For a system of two ODEs in two unknowns with a constant-vector inhomogeneous term,

$$\bar{y}' = A\bar{y} + \bar{b}, \tag{1}$$

there are four possible scenarios (marked at the end by a $\bullet$) that determine the best choice of ansatz for $\bar{y}_p(t)$:

1. A is invertible $\implies$ use $\bar{y}_p(t) = \bar{u}$. $\bullet$
2. A is not invertible and...
   (a) ...$\bar{b}$ is in the range of A (in 2D this is equivalent to $\bar{b}$ being parallel to the columns of A which themselves are parallel to each other when A is not invertible) $\implies$ use $\bar{y}_p(t) = \bar{u}$. $\bullet$
   (b) ...$\bar{b}$ is not in the range of A and...
      i. ...zero is not a repeated eigenvalue or A is the zero matrix $\implies$ use $\bar{y}_p(t) = \bar{u}t + \bar{v}$. $\bullet$
      ii. ...zero is a repeated eigenvalue with only one independent eigenvector $\implies$ use $\bar{y}_p(t) = \bar{u}t^2 + \bar{v}t + \bar{w}$. $\bullet$

1. (12 pts) Consider equation (1) with

$$A = \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}, \qquad \bar{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

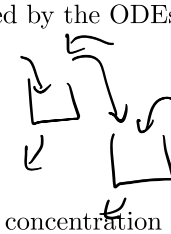
case b

Carry out the following steps to find the general solution.

   (a) Find the general solution to the associated homogeneous equation.
   (b) For the particular solution, try $\bar{y}_p(t) = \bar{u}$. With the four scenarios above in mind, why does this not work? (This one is included for pedagogical purposes - in general, this step can be skipped if you recognize the structure of A and $\bar{b}$.)
   (c) Try $\bar{y}_p(t) = \bar{u}t + \bar{v}$ which should work. Your answer should consist of a sum of two vectors, one of which has an arbitrary constant in front of it. Why do you not need to include the term with the arbitrary constant when assembling the general solution?

2. (17 pts) The general case of two tanks with inflow and outflow of salt water solution into/out of both tanks and pipes connecting the two tanks with flow in both directions is described by the ODEs

$$m_1' = a_1c_1 - \frac{b_{12} + r_1}{v_1} m_1 + \frac{b_{21}}{v_2} m_2,$$
$$m_2' = a_2c_2 + \frac{b_{12}}{v_1} m_1 - \frac{b_{21} + r_2}{v_2} m_2,$$



where a_i is the rate at which the solution is added to Tank i (L/min), c_i is the concentration of the incoming solution into Tank i, r_i is the rate at which solution is removed from Tank i (L/min), b_{ij} is the rate at which solution moves from Tank i to Tank j (L/min), and v_i is the volume of solution in Tank i. For obvious physical reasons, the parameters $(a_i, b_{ij}, c_i, v_i, r_i)$ cannot be negative. The flow rates are set so that the volume in each tank does not change with time:

$$a_1 + b_{21} = b_{12} + r_1 \tag{2}$$
$$a_2 + b_{12} = b_{21} + r_2. \tag{3}$$

As described above, when A is not invertible, the particular solution can grow linearly in time or even quadratically. This question is focused on ruling out that possibility for a two-tank system.

List all (three) sets of conditions (two conditions in each set) that make $\det(A) = 0$. For example, one set of conditions is "$b_{12} = 0$ and $r_1 = 0$". For each set of conditions (a.k.a. scenarios):

1. (12 pts) Consider equation (1) with

$$A = \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}, \qquad \bar{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

case b

Carry out the following steps to find the general solution.

(a) Find the general solution to the associated homogeneous equation.

(b) For the particular solution, try $\bar{y}_p(t) = \bar{u}$. With the four scenarios above in mind, why does this not work? (This one is included for pedagogical purposes - in general, this step can be skipped if you recognize the structure of A and $\bar{b}$.)

(c) Try $\bar{y}_p(t) = \bar{u}t + \bar{v}$ which should work. Your answer should consist of a sum of two vectors, one of which has an arbitrary constant in front of it. Why do you not need to include the term with the arbitrary constant when assembling the general solution?

1.

a) $\bar{y}' = A\bar{y}$ $\quad A = \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}$

$\det(A - \lambda I) = 0$

$\det\begin{pmatrix} 1-\lambda & 1 \\ -2 & -2-\lambda \end{pmatrix} = 0$

$(1-\lambda)(-2-\lambda) + 2 = 0$

$-2 - \lambda + 2\lambda + \lambda^2 + 2 =$

$\lambda^2 + \lambda = 0$

$\lambda(\lambda+1) = 0$

$\lambda = 0, -1$

$\lambda = 0$

$(A - \lambda I)\bar{v} = 0$

$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ -2 & -2 & 0 \end{array}\right)$ $R_1 = R_1 + -\frac{1}{2}R_2$

$\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 1 & 0 \end{array}\right)$

$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$V_1 + V_2 = 0 \qquad V_1 = -V_2$

$V_2 = C$

$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} C$

$\lambda = -1$

$(A - \lambda I)\bar{v} = 0$

$\left(\begin{array}{cc|c} 2 & 1 & 0 \\ -2 & -1 & 0 \end{array}\right)$ $\frac{R_1}{2}$

$\left(\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ -2 & -1 & 0 \end{array}\right)$ $R_2 + 2R_1$

$\left(\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right)$

$V_1 + \frac{1}{2}V_2 = 0 \qquad V_1 = -\frac{1}{2}V_2$

$V_2 = C$

$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} C$

$\bar{y}(t) = C_1 e^{0t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$

b)

$\bar{A} = \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} \quad \bar{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\bar{y}_p = \bar{u}$

$\bar{y}'_p = 0$

$\bar{y}' = \bar{A}\bar{y} + \bar{b} \quad = \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}\bar{u} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$0 =$

$\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}\bar{u} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$

$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ -2 & -2 & -1 \end{array}\right) {}_{+2R_1} \rightarrow \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 0 & -1 \end{array}\right) {}_{-1} \rightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$

u_1 & u_2 can't be solved. $\therefore$ $\bar{y}_p = \bar{u}$ does not work.

c) $A = \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} \quad \bar{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\bar{y} = \bar{u}t + \bar{v}$

$\bar{y}' = \bar{u}$

$\bar{y}' = A\bar{y} + \bar{b} \quad = \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}\bar{y} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\bar{u} =$

$\bar{u} = \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}(\bar{u}t + \bar{v}) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}\bar{u}t + \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}\bar{v} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}\bar{u} = 0$

$\bar{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}c$

$\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}\bar{v} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}c$

$\underbrace{\begin{pmatrix} 1 \\ -2 \end{pmatrix}\bar{v}_1 + \begin{pmatrix} 1 \\ -2 \end{pmatrix}\bar{v}_2}_{} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} =$

redundant. (two of same)

$\begin{pmatrix} 1 \\ -2 \end{pmatrix}\bar{v}_2 + \begin{pmatrix} 1 \\ -1 \end{pmatrix}c = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} v_2 \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ -2 & -1 & -1 \end{array}\right) {}_{+2R_1} \rightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & -1 \end{array}\right)$

$C = -1$

$V_2 + C = 0$

$V_2 = 1$

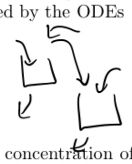
$\bar{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \bar{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\bar{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}c = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$y(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

2. (17 pts) The general case of two tanks with inflow and outflow of salt water solution into/out of both tanks and pipes connecting the two tanks with flow in both directions is described by the ODEs

$$m_1' = a_1c_1 - \frac{b_{12}+r_1}{v_1}m_1 + \frac{b_{21}}{v_2}m_2,$$
$$m_2' = a_2c_2 + \frac{b_{12}}{v_1}m_1 - \frac{b_{21}+r_2}{v_2}m_2,$$



where a_i is the rate at which the solution is added to Tank i (L/min), c_i is the concentration of the incoming solution into Tank i, r_i is the rate at which solution is removed from Tank i (L/min), b_{ij} is the rate at which solution moves from Tank i to Tank j (L/min), and v_i is the volume of solution in Tank i. For obvious physical reasons, the parameters $(a_i, b_{ij}, c_i, v_i, r_i)$ cannot be negative. The flow rates are set so that the volume in each tank does not change with time:

$$a_1 + b_{21} = b_{12} + r_1 \tag{2}$$
$$a_2 + b_{12} = b_{21} + r_2. \tag{3}$$

As described above, when A is not invertible, the particular solution can grow linearly in time or even quadratically. This question is focused on ruling out that possibility for a two-tank system.

List all (three) sets of conditions (two conditions in each set) that make $\det(A) = 0$. For example, one set of conditions is "$b_{12} = 0$ and $r_1 = 0$". For each set of conditions (a.k.a. scenarios):

2. $\frac{d}{dt}\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} -\frac{b_{12}+r_1}{v_1} & \frac{b_{21}}{v_2} \\ \frac{b_{12}}{v_1} & -\frac{b_{21}+r_2}{v_2} \end{pmatrix} + \begin{pmatrix} a_1c_1 \\ a_2c_2 \end{pmatrix}$

$\det\begin{pmatrix} -\frac{b_{12}+r_1}{v_1} & \frac{b_{21}}{v_2} \\ \frac{b_{12}}{v_1} & -\frac{b_{21}+r_2}{v_2} \end{pmatrix}$

$\left(-\frac{b_{12}+r_1}{v_1} \cdot -\frac{b_{21}+r_2}{v_2}\right) - \left(\frac{b_{21}}{v_1} \cdot \frac{b_{12}}{v_2}\right) = 0$

① $b_{12} = 0$ $r_1 = 0$

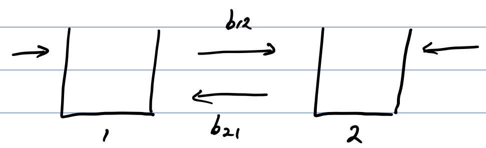
② $b_{21} = 0$ $r_2 = 0$

③ $r_1 = 0$ $r_2 = 0$

$(-b_{12}+r_1)(-b_{21}+r_2) - (b_{21}b_{12})$

$\cancel{b_{12}b_{21}} - b_{12}r_2 - b_{21}r_1 + r_1r_2 - \cancel{b_{21}b_{12}} = 0$

$\underbrace{-b_{12}r_2}_{0} \underbrace{- b_{21}r_1}_{0} + \underbrace{r_1r_2}_{0} = 0$



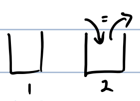
1: $a_1 + b_{21} = b_{12} + r_1$

$a_1 + b_{21} = 0$

a_1 & $b_{21} = 0$

$a_2 + b_{12} = b_{21} + r_2$

$a_2 = r_2$ No in & out flow in tank 1



$\therefore m_1' = 0$ $m(t) = m(0)$

$m_2' = r_2c_2 + 0 - \frac{0 + r_2}{v_2}m_2 = r_2c_2 - \frac{r_2}{v_2}m_2$

$m_2' + \frac{r_2}{v_2}m_2 = r_2c_2$ $N = e^{r_2t/v_2}m_2$

$[e^{r_2t/v_2}m_2]' = r_2c_2e^{r_2t/v_2}$

$[\quad " \quad] = \int "$

$" = c_2v_2e^{r_2t/v_2} + k_1$

$m(t) = c_2v_2 + \frac{k_1}{e^{r_2t/v_2}}$

$m(0) = c_2v_2 + k_1$

$k_1 = m(0) - c_2v_2$

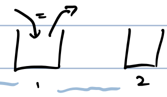
$m(t) = c_2v_2 + \frac{m(0) - c_2v_2}{e^{r_2t/v_2}}$ $(0, 1]$ $\therefore m_1 + m_2$ bounded.

2: $b_{21} = 0 \quad r_2 = 0$

$$a_2 + b_{12} = b_{21} + r_2$$
$$= 0$$
$$a_2 \,\&\, b_{12} = 0$$

$$a_1 + b_{21} = b_{12} + r_1$$
$$a_1 = b_{12} + r_1$$
$$a_1 = r_1$$

$\therefore$ No in & out flow at tank 2.



$$m_1' = r_1 c_1 - \frac{r_1 m_1}{v_1}$$
$$m_1'(t) + \frac{r_1 m_1}{v_1} = r_1 c_1$$

$$[e^{r_1 t/v_1} m_1]' = r_1 c_1 e^{r_1 t/v_1}$$
$$[\quad " \quad] = \int "$$
$$= r_1 c_1 e^{r_1 t/v_1} + k_2$$
$$m_1 = r_1 c_1 + \frac{k_2}{e^{r_1 t/v_1}}$$

$$m_1(0) = r_1 c_1 + k_2$$
$$k_2 = m(0) - r_1 c_1$$
$$m_1(t) = r_1 c_1 + \frac{m(0) - r_1 c_1}{e^{r_1 t/v_1}}$$

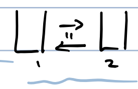
$(0, 1]$ $\quad \therefore m_1 + m_2$ bounded.

$$m_2' = 0$$
$$m_2(t) = m_2(0)$$

3: $r_1 = 0 \quad r_2 = 0$

$$a_1 + b_{21} = b_{12} \qquad a_2 + b_{12} = b_{21}$$
$$\rightarrow \quad a_1 = b_{12} - b_{21} \qquad -a_1 = -b_{12} + b_{21}$$
$$a_2 = b_{21} - b_{12} \qquad \therefore a_1 = -a_2$$
$$a_1 \,\&\, a_2 = 0 \;:\; b_{12} = b_{21}$$

in & out flow from outside of both tanks.



$$m_1' = -\frac{b_{12} m_1}{v_1} + \frac{b_{21} m_2}{v_2}$$
$$m_2' = \frac{b_{12} m_1}{v_1} - \frac{b_{21} m_2}{v_2}$$

$$m_1' + m_2' = 0$$
$$\therefore m_1 + m_2 = k \leftarrow \text{constant.}$$

$$\begin{bmatrix} m_1' \\ m_2' \end{bmatrix} = \begin{bmatrix} -\frac{b_{12}}{v_1} & \frac{b_{21}}{v_2} \\ \frac{b_{12}}{v_1} & -\frac{b_{21}}{v_2} \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$$

using $\bar{u}t + \bar{v} = \bar{y}_p'$
$y_p = \bar{v}$

$$\bar{y}_p' = (\;"\;)(\bar{v}t + \bar{u}) + \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$$

$$\begin{pmatrix} -\frac{b_{12}}{v_1} & \frac{b_{21}}{v_2} \\ \frac{b_{12}}{v_1} & -\frac{b_{21}}{v_2} \end{pmatrix} \bar{v} = 0 \qquad \bar{u} = \begin{pmatrix} -\frac{b_{12}}{v_1} & \frac{b_{21}}{v_2} \\ \frac{b_{12}}{v_1} & -\frac{b_{21}}{v_2} \end{pmatrix} \bar{v}$$

$$\bar{v} = \begin{bmatrix} b_{21}/v_2 \\ b_{12}/v_1 \end{bmatrix} c$$

$$\bar{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} b_{21}/v_2 \\ b_{12}/v_1 \end{bmatrix}$$ constant mass.

Math 256 Week 08 Assignment - written

1. (5 pts) Use Laplace Transforms to solve $y'' + 9y = 0$ with ICs $y(0) = y_0,\ y'(0) = v_0$. Because y_0 and v_0 are not specified, your solution is a general solution.

2. (7 pts) Use Laplace Transforms to solve $y'' - 8y' + 20y = 0$ with ICs $y(0) = 3,\ y'(0) = 0$.

3. (9 pts)

   (a) Use Laplace Transforms to solve $y'' + 9y = u(t-2) - u(t-a)$ with ICs $y(0) = 0,\ y'(0) = 0$.

   (b) Plot the solution when $a = 4.5$ and $a = 6$. This should be done by hand but you can use Desmos or similar to get an idea of what it looks like.

   (c) For what values of $a > 2$ is $y(t) = 0$ for all $t \geq a$? You must give an exact symbolic answer rather than an approximate decimal (e.g. $\pi/2$ rather than 1.57).

   For context, a physical interpretation of this question is as follows. A battery is connected to an LC circuit at time $t = 2$ and is removed at time $t = a$. If the current in the circuit was initially zero, at what time a must you remove it to ensure the current remains zero thereafter?

Summary

$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$

$\mathcal{L}\{u_c(t)\} = e^{-sc} \cdot \frac{1}{s}$

$\mathcal{L}\{f(t-c)\, u(t-c)\} = e^{-sc}\, \mathcal{L}\{f(t)\}$

$\mathcal{L}\{y(t)\} = Y(s)$

$y' = sY(s) - y(0)$

$y'' = s^2 Y(s) - s y(0) - y'(0)$

$\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{\omega^2 + s^2}$

$\mathcal{L}\{\cos(\omega t)\} = \frac{s}{\omega^2 + s^2}$

$\mathcal{L}\{3\} = 3/s$ $\quad$ $\mathcal{L}\{3t\} = 3/s^2$

$\mathcal{L}\{e^{6t}\} = \frac{1}{s-6}$

$\mathcal{L}\{\cos t\} = \frac{s}{1+s^2}$ $\quad$ $\mathcal{L}\{\sin 5t\} = \frac{5}{25+s^2}$

1. (5 pts) Use Laplace Transforms to solve $y'' + 9y = 0$ with ICs $y(0) = y_0,\ y'(0) = v_0$. Because y_0 and v_0 are not specified, your solution is a general solution.

1. $y'' + 9y = 0$

$\mathcal{L}\{y''\} + 9\mathcal{L}\{y\} = 0$

$s^2 Y(s) - sy(0) - y'(0) + 9Y(s) = 0$

$(s^2+9)Y(s) = sy(0) - y'(0)$

$Y(s) = \dfrac{''}{(s^2+9)}$

$Y(s) = \dfrac{sy_0 - v_0}{(s^2+9)}$

$Y(s)^{-1}$: $\sin 5t \mapsto \dfrac{5}{25+s^2}$ $\quad \cos 5t \mapsto \dfrac{s}{25+s^2}$

$y_0 \cos(3t) - \sin(3t)$

$\dfrac{y_0 s}{s^2+9} - \dfrac{v_0}{3}\dfrac{3}{s^2+9}$

$\therefore\ y(t) = y_0 \cos(3t) - \dfrac{v_0}{3}\sin(3t)$

2. (7 pts) Use Laplace Transforms to solve $y'' - 8y' + 20y = 0$ with ICs $y(0) = 3,\ y'(0) = 0$.

2. $y'' - 8y' + 20y = 0$

$\mathcal{L}\{y''\} - 8\mathcal{L}\{y'\} + 20\mathcal{L}\{y\} = 0$

$s^2Y(s) - sy(0) - y'(0) - 8(sY(s) - y(0)) + 20Y(s) = 0$

$s^2Y(s) - 8sY(s) + 20Y(s) = sy(0) + y'(0) - 8y(0) = \dfrac{3s-24}{(s-4)^2+4}$

$Y(s)(s^2 - 8s + 20) =$

$Y(s) = \dfrac{3s-24}{(s^2-8s+20)} = \dfrac{3s-24}{(s-4)^2+4} = \dfrac{3(s-4)-12}{(s-4)^2+4}$

can't ()()

$= \dfrac{3(s-4)}{(s-4)^2+4} - \dfrac{12}{(s-4)^2+4}$

$Y(s)^{-1}$: $F(s-a) = \dfrac{(s-4)}{(s-4)^2+4}$ $\quad F(s-a) = \dfrac{1}{(s-4)^2+4}$

$F(s) = \dfrac{s}{s^2+4},\ a=4$ $\quad F(s) = \dfrac{1}{s^2+4},\ a=4$

$F^{-1}(s) = \cos(2t),\ e^{4t}$ $\quad F^{-1}(s) = \dfrac{\sin(2t)}{2},\ e^{4t}$

* $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$

$\mathcal{L}\{\sin(at)\} = \dfrac{a}{s^2+a^2}$

$\mathcal{L}\{\cos(at)\} = \dfrac{s}{s^2+a^2}$

$\therefore\ y(t) = 3\cos(2t)e^{4t} - 12\dfrac{\sin(2t)}{2}e^{4t} = 3e^{4t}(\cos(2t) - 2\sin(2t))$

3. (9 pts)

(a) Use Laplace Transforms to solve $y'' + 9y = u(t-2) - u(t-a)$ with ICs $y(0) = 0,\ y'(0) = 0$.

(b) Plot the solution when $a = 4.5$ and $a = 6$. This should be done by hand but you can use Desmos or similar to get an idea of what it looks like.

(c) For what values of $a > 2$ is $y(t) = 0$ for all $t \geq a$? You must give an exact symbolic answer rather than an approximate decimal (e.g. $\pi/2$ rather than 1.57).

For context, a physical interpretation of this question is as follows. A battery is connected to an LC circuit at time $t = 2$ and is removed at time $t = a$. If the current in the circuit was initially zero, at what time a must you remove it to ensure the current remains zero thereafter?

a) $s^2Y(s) - sy(0) - y'(0) + 9Y(s) = \mathcal{L}\{u(t-2)\} - \mathcal{L}\{u(t-a)\}$

$$Y(s)(s^2+9) = \frac{e^{-2s}}{s} - \frac{e^{-as}}{s}$$

$$Y(s) = \frac{e^{-2s} - e^{-as}}{s(s^2+9)} = \frac{e^{-2s}}{s(s^2+9)} - \frac{e^{-as}}{s(s^2+9)} = e^{-2s}\cdot\frac{1}{s^3+9s} - e^{-as}\cdot\frac{1}{s^3+9s}$$

$$Y^{-1}(s) = u(t-2)\,\mathcal{L}^{-1}\left\{\frac{1}{s^3+9s}\right\} - u(t-a)\,\mathcal{L}^{-1}\left\{\frac{1}{s^3+9s}\right\}$$

$$= u(t-2)$$

$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}$

$\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}$

$\mathcal{L}\{f(t-c)u(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}$

$$\frac{1}{s^3+9s} = \frac{A}{s} + \frac{Bs+C}{s^2+9}$$

$$1 = As^2 + 9A + Bs^2 + Cs$$

$$= As^2 + Bs^2 + Cs + 9A$$

$$A = \frac{1}{9} \quad B = -\frac{1}{9} \quad C = 0$$

$$= \frac{1}{9s} - \frac{1}{9}\frac{s}{s^2+9}$$

$$\mathcal{L}^{-1}\{\,''\,\} = \mathcal{L}^{-1}\left\{\frac{1}{9s}\right\} - \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}$$

$$= \frac{1}{9} - \frac{1}{9}\cos(3t)$$

$$y(t) = u(t-2)\left[\frac{1}{9} - \frac{1}{9}\cos(3\underbrace{(t-2)}_{f(t-c)})\right] - u(t-a)\left[\frac{1}{9} - \frac{1}{9}\cos(3(t-a))\right]$$

b) $u(t-2)\left[\frac{1}{9} - \frac{1}{9}\cos(3\underbrace{(t-2)}_{f(t-c)})\right] - u(t-a)\left[\frac{1}{9} - \frac{1}{9}\cos(3(t-a))\right]$

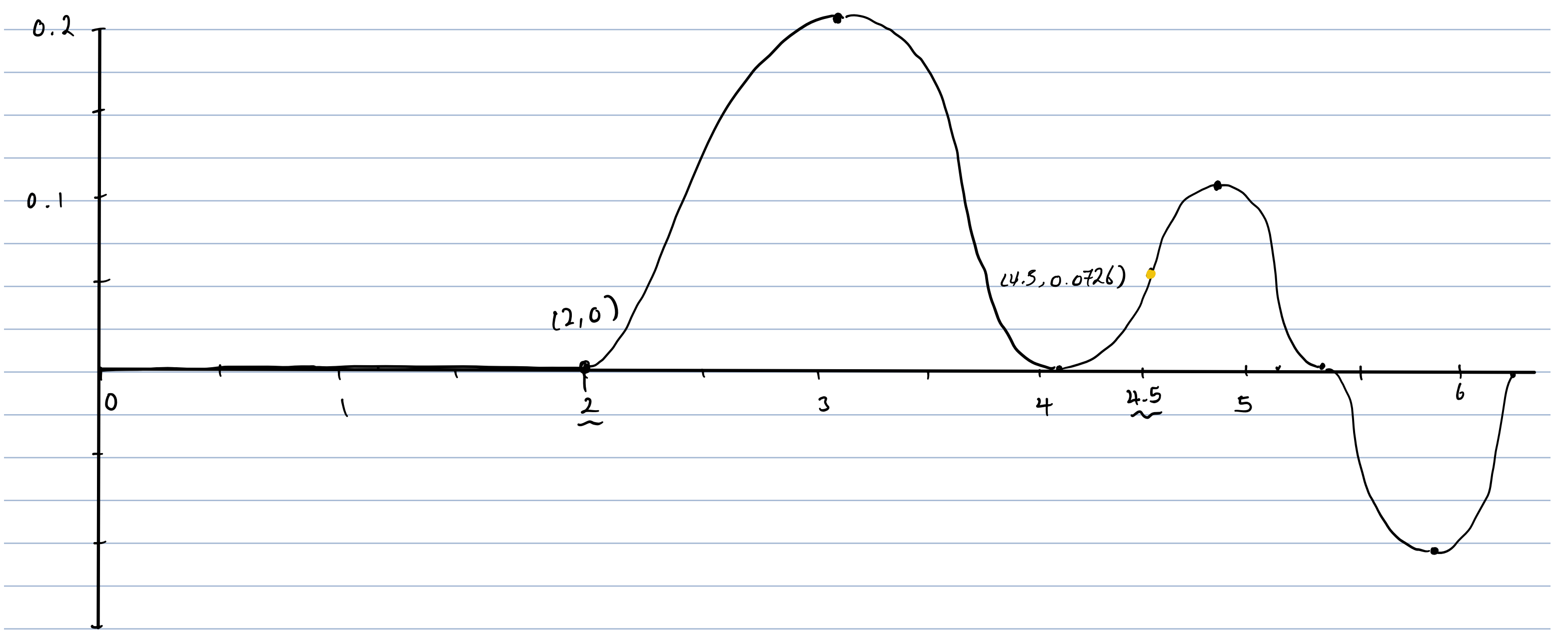
$$u(t-2) = \begin{cases} 0 & t < 2 \\ 1 & t \geq 2 \end{cases} \qquad u(t-4.5) = \begin{cases} 0 & t < 4.5 \\ 1 & t \geq 4.5 \end{cases}$$

$$u(t-6) = \begin{cases} 0 & t < 6 \\ 1 & t \geq 6 \end{cases}$$

1: $0 \sim 2$ $\quad 0$

$2 \sim 4.5$ $\quad \frac{1}{9} - \frac{1}{9}\cos(3(t-2))$

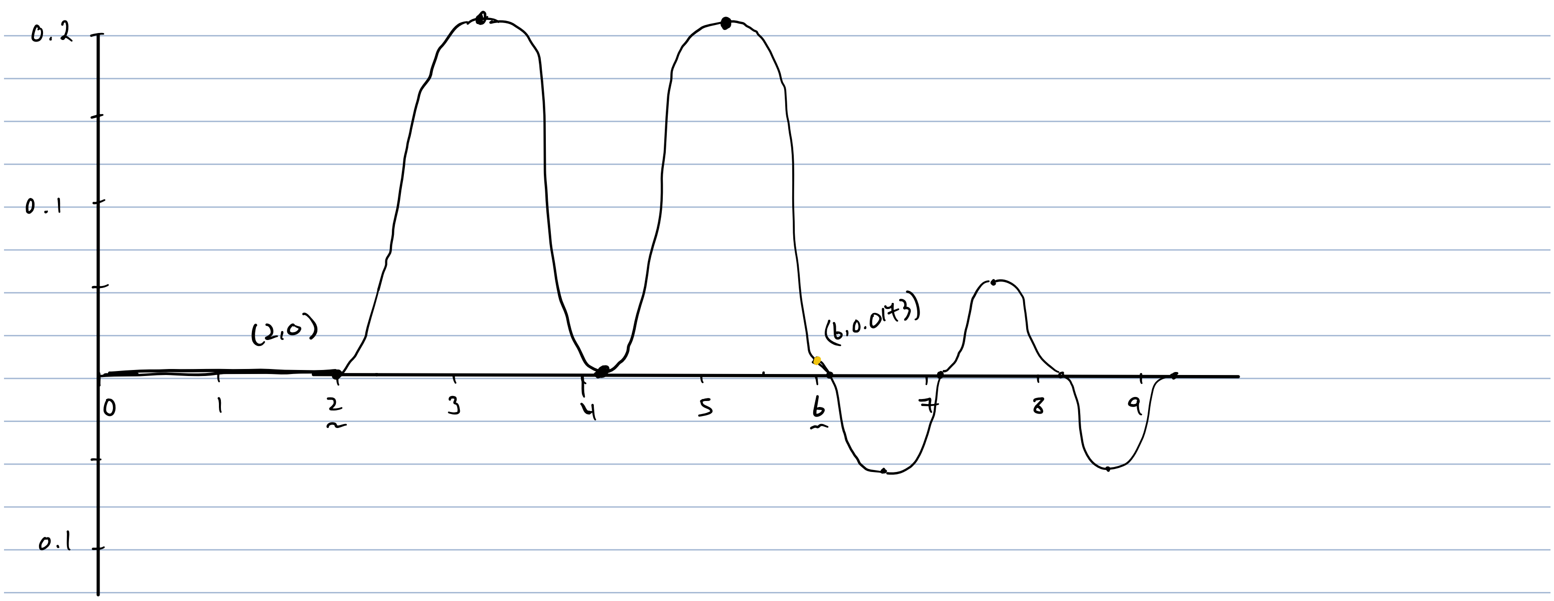
$4.5 \sim \infty$ $\quad Y^{-1}(s) = \frac{1}{9} - \frac{1}{9}\cos(3(t-2)) - \left(\frac{1}{9} - \frac{1}{9}\cos(3(t-4.5))\right)$



2: $0 \sim 2$ $\quad 0$

$2 \sim 6$ $\quad \frac{1}{9} - \frac{1}{9}\cos(3(t-2))$

$6 \sim \infty$ $\quad Y^{-1}(s) = \frac{1}{9} - \frac{1}{9}\cos(3(t-2)) - \left(\frac{1}{9} - \frac{1}{9}\cos(3(t-6))\right)$



c. $y(t) = 0$

$$y(t) = u(t-2)\left[\tfrac{1}{9} - \tfrac{1}{9}\cos(3(t-2))\right] - u(t-a)\left[\tfrac{1}{9} - \tfrac{1}{9}\cos(3(t-a))\right]$$

$$u(t-a)\left[\tfrac{1}{9} - \tfrac{1}{9}\cos(3(t-a))\right] = u(t-2)\left[\tfrac{1}{9} - \tfrac{1}{9}\cos(3(t-2))\right]$$

given $a > 2$
$t \geq a$

$$\tfrac{1}{9} - \tfrac{1}{9}\cos(3(t-a)) = \tfrac{1}{9} - \tfrac{1}{9}\cos(3(t-2))$$

$$\tfrac{1}{9}(1 - \cos(3t-3a)) = \tfrac{1}{9}(1 - \cos(3t-6))$$

$$1 - \cos(\text{''}) = 1 - \cos(\text{''})$$

$$\cos(3t-3a) = \cos(3t-6)$$

assume $t=2$: $\cos(6-3a) = \cos(6-6)$

$\underbrace{\cos(6-3a)}$ $= 1$

$\downarrow$

$\cos(0, 2\pi, 4\pi \ldots) = 1$

$0 = 6 - 3a$

$a = 2$

$2\pi = 6 - 3a$

$a = \frac{2\pi - 6}{-3}$

$4\pi = 6 - 3a$

$a = \frac{4\pi - 6}{-3}$

$6 - 3a = 0, 2\pi, 4\pi \ldots$

$6 - 3a = 2\pi$

$\cancel{-3a} = \frac{2\pi - 6}{-3}$

$a = \frac{2\pi - 6}{-3}$

$\therefore a = -\frac{2\pi - 6}{3}, -\frac{4\pi - 6}{3}, -\frac{6\pi - 6}{3} \ldots$

$= -\frac{2n\pi - 6}{3}$, where $n = 1, 2, 3 \ldots \infty$ $(n \in \mathbb{N})$

or (alternative), $-\frac{n\pi - 6}{3}$ where $n = 2, 4, 6, 8 \ldots \infty$

Math 256 Week 09 Assignment - written

1. (12 pts) Morphine is cleared from the body at a rate proportional to the amount present with a rate constant k given in units of hour^{-1}. A patient is given an injection of m_{inj} mg morphine immediately after surgery. The subsequent doses occur every τ hours and are $0.5m_{inj}$ mg morphine. This continues indefinitely. The patient has no morphine in her system before the first injection.

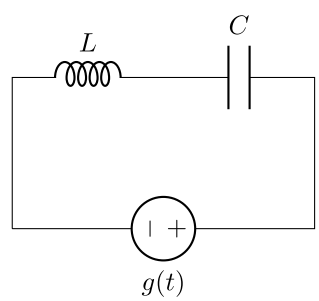
   (a) (4 pts) Treating each injection as an instantaneous event, write down an Initial Value Problem to model the quantity of morphine $m(t)$ in the patient's body as a function of time.

   (b) (3 pts) Solve the IVP. Express the solution in terms of the Heaviside functions $u(t - n\tau)$ where $n = 0, 1, 2, \ldots$

   (c) (3 pts) Sketch the graph of the solution. You can use a graphing tool to get an idea of what it looks like but the sketch should be hand-drawn.

   (d) (2 pts - this is a more challenging question) After a long time, the value of the solution immediately before an injection and immediately after an injection asymptotically approach constant values m_{min}^{∞} and m_{max}^{∞}.
   For treatment purposes, you would choose m_{inj} and τ so that m_{max}^{∞} is below the level at which the drug is toxic and m_{min}^{∞} is above the level at which the drug starts having the desired effect.
   Calculate $m_{\text{max}}^{\infty} = \lim_{p\to\infty} m(p\tau^+)$ where p is an integer and τ^+ is ever so slightly bigger than τ. $m_{\text{min}}^{\infty} = \lim_{p\to\infty} m(p\tau^-)$ can be calculated by subtracting $0.5m_{inj}$ from m_{max}^{∞}. Note: the formula for a geometric series might be useful.

2. (7 pts) When an ODE needs to be solved repeatedly with different inhomogenous terms, it is often more efficient to calculate the *Transfer Function* and use convolution to get the solution for each inhomogeneous term. For example, suppose you have an LC circuit with a power source ($g(t)$) that can be set to deliver different voltage profiles, like a decaying voltage ($g(t) = e^{-3t}$) or an AC source ($g(t) = \sin(\omega t)$).



   Instead of solving
   $$y'' + 2y' + 2y = g(t), \quad y(0) = 0, \quad y'(0) = 0$$
   separately for each $g(t)$, we can solve
   $$h'' + 2h' + 2h = \delta(t), \quad h(0) = 0, \quad h'(0) = 0$$
   and then use convolution to solve for $y(t)$:
   $$y(t) = \int_0^t g(t - z)h(z)dz$$

   (a) (3 pts) Calculate $h(t)$ using Laplace transforms.

   (b) (4 pts) For $g(t) = e^{-4t}$, use $h(t)$ and convolution to calculate the solution $y(t)$. You should find yourself needing to calculate $\int_0^t e^{3z} \sin(z)dz$. Feel free to use WolframAlpha.com for this.

1. (12 pts) Morphine is cleared from the body at a rate proportional to the amount present with a rate constant k given in units of hour^{-1}. A patient is given an injection of m_{inj} mg morphine immediately after surgery. The subsequent doses occur every τ hours and are $0.5m_{inj}$ mg morphine. This continues indefinitely. The patient has no morphine in her system before the first injection.

(a) (4 pts) Treating each injection as an instantaneous event, write down an Initial Value Problem to model the quantity of morphine $m(t)$ in the patient's body as a function of time.

(b) (3 pts) Solve the IVP. Express the solution in terms of the Heaviside functions $u(t - n\tau)$ where $n = 0, 1, 2, \ldots$

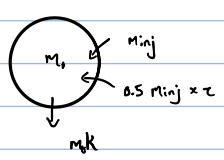
(c) (3 pts) Sketch the graph of the solution. You can use a graphing tool to get an idea of what it looks like but the sketch should be hand-drawn.

(d) (2 pts - this is a more challenging question) After a long time, the value of the solution immediately before an injection and immediately after an injection asymptotically approach constant values m^{∞}_{min} and m^{∞}_{max}.

For treatment purposes, you would choose m_{inj} and τ so that m^{∞}_{max} is below the level at which the drug is toxic and m^{∞}_{min} is above the level at which the drug starts having the desired effect.

Calculate $m^{\infty}_{\text{max}} = \lim_{p\to\infty} m(p\tau^{+})$ where p is an integer and τ^{+} is ever so slightly bigger than τ. $m^{\infty}_{\text{min}} = \lim_{p\to\infty} m(p\tau^{-})$ can be calculated by subtracting $0.5m_{inj}$ from m^{∞}_{max}. Note: the formula for a geometric series might be useful.

1.



$m(0) = 0$

a) $L[y] := m(t)$ $\quad m(t)k \quad$ $m(0) = m_{inj}$ $\quad t = 0, \tau$

$m(t)k \rightarrow \frac{dm(t)}{dt} = -km(t)$

$m'(t) + km(t) = 0 \qquad m(0) = m_{inj}$

b) $m' + km = 0$

$Ys - y(0) + kY = 0$

$Y(s+k) = m_{inj}$

$Y = \frac{m_{inj}}{k+s}$

$Y^{-1} = e^{-kt} \cdot m_{inj}$

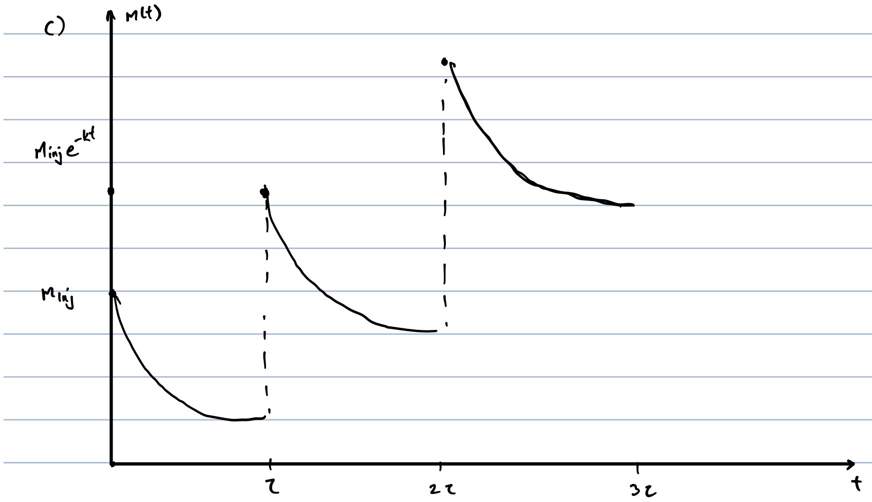
$m(t) = m_{inj} e^{-kt}$

$m(t) = m_{inj} e^{-kt}$

$= m_{inj} e^{-kt} \cdot u(t-\tau)$

$= m_{inj} \left(e^{-kt \cdot u(t-\tau)} + 1 \right)$

c)



$M_{inj}e^{-kt}$ $\quad M(\tau^+) = M_{inj}(e^{-kt}+1)$ $\quad M(2\tau^+) = M_{inj}(e^{-kt}+e^{-2kt}+1)$

d) $m' + km = u(t)$, $n=0$ $\quad = u(t-\tau)$, $n=1$

$= \frac{e^{-t}}{s}$ $\quad = \frac{e^{-\tau T}}{s}$

$m=2$
$k=3$

$Y = \frac{sM_{inj}+1}{sk+s^2}$ $\quad Y = \frac{sM_{inj}+e^{-s}}{ks+s^2}$

$Y^{-1} = M_{inj}(e^{-kt}+1)$ $\quad Y^{-1} = M_{inj}(e^{kt}+e^{-2kt}+1)$

$M(\tau^+) = M_{inj}(e^{-kt}+1)$ $\quad M(2\tau^+) = M_{inj}(e^{-kt}+e^{-2kt}+1)$

$M(n\tau^+) = M_{inj}(e^{-kt}+e^{-2kt}+\dots e^{-nkt})$

$= M_{inj}\, e^{-kt}\left(\frac{1}{1-e^{-kt}}\right)$

$M_{max}(t\to\infty) = M(n\tau^+) = M_{inj}(e^{-kt}+e^{-2kt}\dots$

$= M_{inj}\left(\frac{1}{1-e^{-kt}}\right)$

$M_{min} = M_{inj}\left(\frac{1}{1-e^{-kt}}\right) - 0.5M_{inj}$

Instead of solving

$$y'' + 2y' + 2y = g(t), \quad y(0) = 0, \quad y'(0) = 0$$

separately for each $g(t)$, we can solve

$$h'' + 2h' + 2h = \delta(t), \quad h(0) = 0, \quad h'(0) = 0$$

and then use convolution to solve for $y(t)$:

$$y(t) = \int_0^t g(t-z)h(z)dz$$

(a) (3 pts) Calculate $h(t)$ using Laplace transforms.

(b) (4 pts) For $g(t) = e^{-4t}$, use $h(t)$ and convolution to calculate the solution $y(t)$. You should find yourself needing to calculate $\int_0^t e^{3z}\sin(z)dz$. Feel free to use WolframAlpha.com for this.

a) $Y s^2 - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0} + 2(Ys - y(0)) + 2Y(s) = e$

$+ 2Ys - \cancel{2y(0)}^{0} + 2Y$

$Y(s^2 + 2s + 2) = 1$

$$Y = \frac{1}{s^2+2s+2}$$

$$= \frac{1}{(s+1)^2+1}$$

$Y^{-1} = e^{-1} \cdot \sin(t)$

$$h(t) = \frac{\sin(t)}{e^t}$$

b) $g = e^{-4t}$ $\quad y'' + 2y' + 2y = e^{-4t} \quad y(0) = 0 \quad y'(0) = 0$

$y(t) = (g \cdot h)$

$= \int_0^t g(t-z)h(z)dz \qquad \int_0^t e^{3z}\sin(z)dz$

$$= \int_0^t e^{-4(t-z)} \frac{\sin(z)}{e^z} dz$$

$$= \int_0^t e^{-4t+4z} \cdot e^{-z} \cdot \sin(z) dz$$

$$= \int_0^t e^{-4t} \cdot e^{3z} \sin(z) dz$$

$$= e^{-4t} \int_0^t e^{3z} \sin(z) dz$$

$$= e^{-4t} \cdot \frac{-e^{3t}\cos(t) + 3e^{3t}\sin(t) + 1}{10}$$

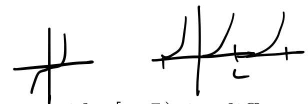
$$= e^{-7t}\frac{\cos(t) + 3e^{-t}\sin(t) + e^{-4t}}{10}$$

Math 256 Week 10 Assignment - written

Using online tools
You may use an online tool (OT) to calculate the integrals requiring integration by parts without showing work. Use $\stackrel{\text{OT}}{=}$ to indicate which steps were carried out this way. It is strongly recommended that you check your Fourier series using Desmos or equivalent. And simplify the OT's answers!

Summary
In this assignment, you will calculate three different Fourier series that all represent the same function, $f(x)$, which is initially only defined on the interval $[0, L)$:

$$f(x) = x^3 \qquad 0 \le x < L$$



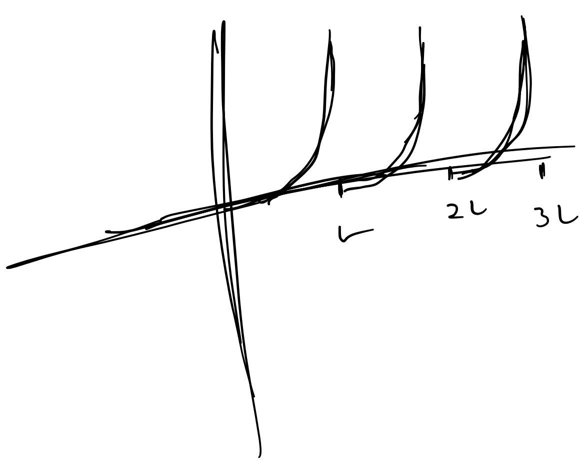
The reason they will be different is that you will define the function outside $[0, L)$ in different ways for different effects.

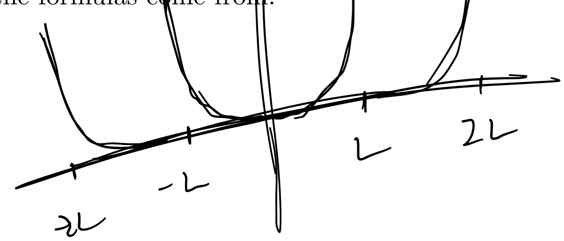
Periodic extension to get the Fourier series
We can only calculate Fourier series for periodic functions on the whole real line and the simplest way to ensure that our function is periodic is to define a periodic extension $f_{pe}(x)$ to be the same as $f(x)$ on $[0, L)$ and, outside $[0, L)$ define it so that it simply repeats exactly the shape seen on $[0, L)$ every L. The resulting function has period L and its Fourier series will consist of a constant term and sine and cosine functions.

Even periodic extension to get the Fourier Cosine series
If we are a bit pickier about what kind of Fourier series we want, say a Fourier series that includes only the even function terms (constant and cosine terms), then we want to define the even periodic extension. This means first defining $f_{epe}(x) = f(x)$ for $x \in [0, L)$ and then extending it to the interval $(-L, 0)$ so that $f_{epe}(x) = f(-x)$ for $x \in (-L, 0)$. We continue to define $f_{epe}(x)$ outside $(-L, L)$ by repeating the shape on $(-L, L)$ every $2L$. Finally, because $f_{epe}(x)$ is even by construction, we can be sure that $\lim_{x\to L} f_{epe}(x) = f_L$ exists, and so we can define $F_{epe}((2n+1)L) = f_L$ for n any integer. This extension is therefore a continuous function for all x.

Odd periodic extension to get the Fourier Sine series
Finally, if we want a Fourier series that includes only the odd function terms (sine terms), then we define the odd periodic extension. This means first defining $f_{ope}(x) = f(x)$ for $x \in [0, L)$ and then extending it to the interval $(-L, 0)$ so that $f_{ope}(x) = -f(-x)$ for $x \in (-L, 0)$. We continue to define $f_{ope}(x)$ outside $[-L, L)$ by repeating the shape on $[-L, L)$ every $2L$. Finally, we define $F_{ope}((2n+1)L) = 0$ for n any integer.

Caution
When calculating the FS for F_{pe}, be careful to use the period L. For f_{epe} and f_{ope}, you have to use the period $2L$ because the mirroring step forces you to double the period. The most common errors will arise from using an incorrect period or not using the correct versions of the coefficient formulas for the given case which can be avoided by understanding where the formulas come from.





For all questions in this assignment, consider a function defined only on $[0, L)$ with formula $f(x) = x^3$. Even though the formula is well-defined outside that interval, we are not giving f any values out there.

1. (4 pts) Define $f_{pe}(x)$ as described above. It can be tricky to define such extensions properly using formulas so use a sketch (i.e. sketch required, formula optional). Your sketch should include at least $x \in [-2L, 2L]$, with labelled tick marks on both axes for any notable values, and should indicate which side of any jumps the value the function takes using filled/empty dots.

2. (7 pts) Find the Fourier series for $f_{pe}(x)$ as defined in the previous part.

3. (3 pts) Define $f_{epe}(x)$ as described above. It can be tricky to define such extensions properly using formulas so use a sketch (i.e. sketch required, formula optional). Your sketch should include at least $x \in [-3L, 3L]$, with labelled tick marks on both axes for any notable values, and should indicate which side of any jumps the value the function takes using filled/empty dots.

4. (7 pts) Find the Fourier series for $f_{epe}(x)$ as defined in the previous part.

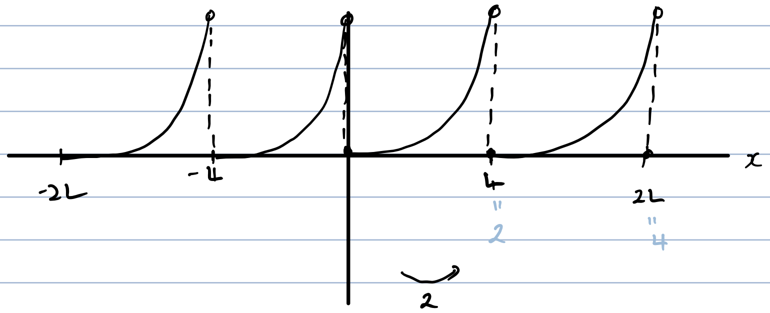
5. (4 pts) Define $f_{ope}(x)$ as described above. It can be tricky to define such extensions properly using formulas so use a sketch (i.e. sketch required, formula optional). Your sketch should include at least $x \in [-3L, 3L]$, with labelled tick marks on both axes for any notable values, and should indicate which side of any jumps the value the function takes using filled/empty dots.

6. (5 pts) Find the Fourier series for $f_{ope}(x)$ as defined in the previous part.

1. (4 pts) Define $f_{pe}(x)$ as described above. It can be tricky to define such extensions properly using formulas so use a sketch (i.e. sketch required, formula optional). Your sketch should include at least $x \in [-2L, 2L]$, with labelled tick marks on both axes for any notable values, and should indicate which side of any jumps the value the function takes using filled/empty dots.

2. (7 pts) Find the Fourier series for $f_{pe}(x)$ as defined in the previous part.

1.



Assume $L = 2$

$\therefore T = 2$

$A_0 = \frac{1}{T}\int_0^T f(t)\,dt$

$a_n = \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2n\pi}{T}t\right)dt$

$b_n = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2n\pi}{T}t\right)dt$

2. $FS\{f(t)\} = A_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi t) + \sum_{n=1}^{\infty} b_n \sin(n\pi t)$

$A_0 = \frac{1}{2}\int_0^2 x^3\,dx = \frac{1}{2}\left[\frac{x^4}{4}\Big|_0^2\right] = 2$

$a_n = \frac{2}{2}\int_0^2 x^3 \cos(\pi n x)\,dx$

$= 1\int_0^2 x^3\cos(\pi n x)\,dx$

$\overset{\text{or}}{=} \frac{4\pi n(2\pi^2 n^2 - 3)\sin(2\pi n) + 6(2\pi^2 n^2 - 1)\cos(2\pi n) + 6}{\pi^4 n^4}$

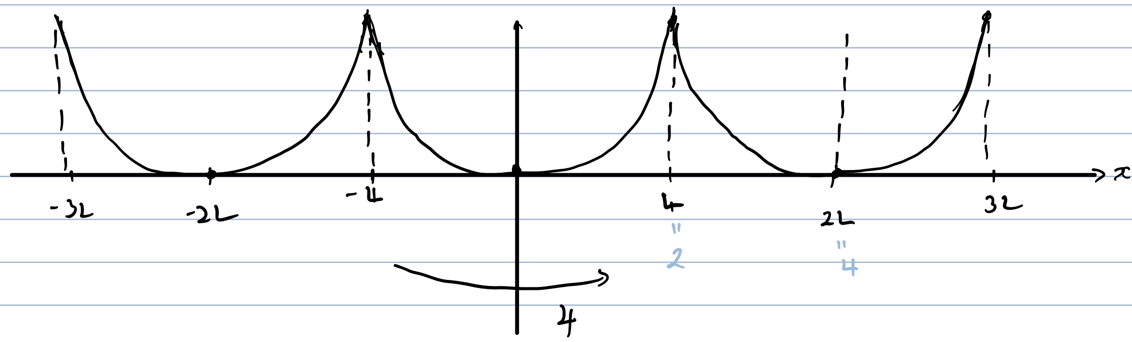
$b_n = \frac{2}{2}\int_0^2 x^3 \sin(\pi n x)$

$= 1\int_0^2 x^3 \sin(\pi n x)$

$\overset{\text{or}}{=} \frac{6(2\pi^2 n^2 - 1)\sin(2\pi n) + 2\pi n(6 - 4\pi^2 n^2)\cos(2\pi n)}{\pi^4 n^4}$

$FS\{f(x)\} = 2 + \sum_{n=1}^{\infty} \frac{4\pi n(2\pi^2 n^2 - 3)\sin(2\pi n) + 6(2\pi^2 n^2 - 1)\cos(2\pi n) + 6}{\pi^4 n^4}\cos(\pi n x) + \sum_{n=1}^{\infty} \frac{6(2\pi^2 n^2 - 1)\sin(2\pi n) + 2\pi n(6 - 4\pi^2 n^2)\cos(2\pi n)}{\pi^4 n^4}\sin(\pi n x)$

3.



4. $FS\{f(t)\} = A_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{2}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{2}\right)$

Assume $L = 2$
$\therefore T = 4$

$A_0 = \frac{1}{4}\int_{-2}^{2} x^3\,dx = \frac{1}{2}\int_0^2 x^3\,dt = \left[\frac{x^4}{4}\Big|_0^2\right]\cdot\frac{1}{2} = 2$

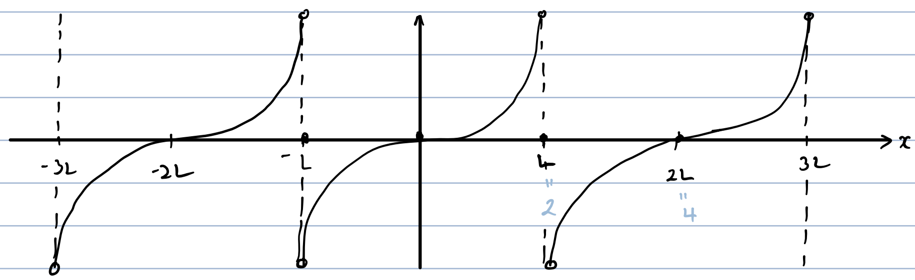
$a_n = {}^4/_4 \int_0^2 x^3 \cos\left(\frac{n\pi x}{2}\right)dx$ $\quad b_n = 0$ as integrand is odd

$= 1\int_0^2 x^3\cos\left(\frac{n\pi x}{2}\right)dx$

$\overset{OT}{=} \frac{16\left(\pi n(\pi^2 n^2 - 6)\sin(\pi n) + 3(\pi^2 n^2 - 2)\cos(\pi n) + 6\right)}{\pi^4 n^4}$

$FS\{f(x)\} = 2 + \sum_{n=1}^{\infty} \frac{16\left(\pi n(\pi^2 n^2 - 6)\sin(\pi n) + 3(\pi^2 n^2 - 2)\cos(\pi n) + 6\right)}{\pi^4 n^4}\cdot\cos\left(\frac{n\pi x}{2}\right)$

5.



Assume $L = 2$
$\therefore T = 4$

6. $FS\{f(t)\} = A_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{2}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{2}\right)$

$A_0 = 0$ as integrand is odd

$a_n = 0$ $\quad$ "

$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin\left(\frac{2n\pi}{T}t\right)dt$

$= 1\int_0^2 x^3\sin\left(\frac{n\pi x}{2}\right)dt$

$\overset{OT}{=} \frac{48(\pi^2 n^2 - 2)\sin(\pi n) - 16\pi n(\pi^2 n^2 - 6)\cos(\pi n)}{\pi^4 n^4}$

$FS\{f(x)\} = \sum_{n=1}^{\infty} \frac{48(\pi^2 n^2 - 2)\sin(\pi n) - 16\pi n(\pi^2 n^2 - 6)\cos(\pi n)}{\pi^4 n^4}\cdot\sin\left(\frac{n\pi x}{2}\right)$

Math 256 Week 11-12 Assignment - written

1. (17 pts) When baking potatoes, the main limiting factor is getting the temperature of the entire potato up to 100°C. By cutting a potato into smaller pieces, you can deliver heat more rapidly. Although the analysis described here can be carried out with similar conclusions for spherical and even ellipsoidal potatoes, given the constraints of the course content, we will consider cube-like chunks of potato that are perfectly insulated on four sides and exposed to an oven temperature of 190°C on the top and bottom surface. This geometry leads to uniform temperatures in the x and y directions and only variation in the z direction. The temperature of the potato $u(z,t)$ at height z changes according to the Heat Equation
$$u_t = Du_{zz}.$$
Initially, the temperature of the potato is 20°C everywhere from bottom ($z = 0$) to top ($z = L$). The oven is kept at a constant temperature 190°C so that the bottom and top surfaces of the potato chunk are always 190°C.

   (a) (1 pt) Express the initial condition in the usual mathematical format.

   (b) (1 pt) Express the boundary conditions (BCs) in the usual mathematical format.

   (c) (1 pt) Your BC in the previous part ought to be inhomogeneous. However, by defining a new temperature scale Delsius that uses the same size increment as Celsius but is zero when the Celsius scale is 190°C, we can convert the BC into a homogeneous one. Define $v(x,t) = u(x,t) - 190$ and write down a PDE, IC, and BC for $v(x,t)$.

   (d) (2 pts) Express the general solution to the PDE for $v(x,t)$ in terms of the correct type of Fourier series (leaving the arbitrary constants unspecified for now).

   (e) (5 pts) Find values for the arbitrary coefficients in your general solution $v(x,t)$ to get the particular solution for the IC.

   (f) (1 pt) Convert back to Celsius and write down the particular solution $u(x,t)$ to the original PDE, BC, and IC.

   (g) (1 pt) The temperature at all points in the potato go from the initial 20°C to 190°C as $t \to \infty$. The part of the potato chunk at $x = L/2$ is the slowest to heat up. Write down the temperature at the middle of the chunk, $u(L/2,t)$.

   (h) (3 pts) Which of the Fourier terms (with a non-zero coefficient) decays slowest? Ignore the others and use that slowest mode to find an expression for the time at which $u(L/2,t) = 100$.

   (i) (2 pts) You initially cut the chunks into pieces of size $L = 4$ cm. From experience, you know it will take 40 minutes for these to cook. But you're in a rush. Is it worth taking another 20 minutes to cut them into pieces of size $L = 2$ cm? A naive assumption might be that pieces half the size would take half the time to cook so it would not be any faster to do the extra cutting (40 minutes cooking versus 20 minutes cutting + 20 minutes cooking). Use your answer to the previous part to estimate the cooking time more accurately and make a more reliable decision. Justify your answer.

$z = x$

a. $u(x,0) = 20$

b. $u(0,t) = 190$

$u(L,t) = 190$

c. $v(x,t) = u(x,t) - 190$ $\qquad v_t = v_{xx}$

$v(x,0) = -170$ $\qquad v = Ax + B - 190$

$v(0,t) = 0$ $\qquad v(0,t) = B - 190$ $\qquad v(L,t) = AL$

$v(L,t) = 0$ $\qquad 0 =$ $\qquad 0 =$

$B = 190$ $\qquad A = 0$

d. $V(x,t) = A_0 + Bx + \sum_{n=1}^{\infty} a_n \cos(w_n x) e^{\lambda_n t} + \sum_{n=1}^{\infty} b_n \sin(w_n x) e^{\lambda_n t}$

use sin to satisfy BC at 0 & L = 0 $\quad \therefore A_0,\ Bx,\ \sum_{n=1}^{\infty} a_n \cos(w_n x) e^{\lambda_n t} = 0$

$v(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{\lambda_n t}$ $\qquad w_n = \frac{n\pi}{L}$, $\lambda_n = \frac{-Dn^2\pi}{L^2}$

e. $V(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$,

$-170 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$, $\quad V(x,0) = f(x) = -170$

$b_n = \frac{2}{2L} \int_{-L}^{L} f_{ope} \sin\left(\frac{n\pi x}{L}\right) dx$

$= \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

$= \frac{2}{L} \int_0^L -170 \sin\left(\frac{n\pi x}{L}\right) dx$

$= \frac{2}{L} \left[\frac{-170 \cos\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}} \Big|_0^L \right]$

$= \frac{340\left(\cos\left(\frac{\pi n}{L}\right) - 1\right)}{\pi n}$

$b_n = \frac{340\left((-1)^n - 1\right)}{n\pi}$ $\begin{cases} \frac{-2(340)}{n\pi} \ (-680), & n_{odd} \\ 0, & n_{even} \end{cases}$

$= \frac{340L\left(\cos\left(\frac{\pi n}{L}\right) - 1\right)}{\pi n}$

$= \sum_{n_{odd}=1}^{\infty} \frac{-680}{\pi n} \sin\left(\frac{\pi n x}{L}\right) e^{\frac{-Dn^2\pi^2 t}{L^2}}$

f. $V(x,t) = u(x,t) - 190$

$u(x,t) = \sum_{n_{odd}=1}^{\infty} -\frac{680}{\pi n} \sin\left(\frac{\pi n x}{L}\right) e^{\frac{-Dn^2\pi^2 t}{L^2}} + 190$, $\quad u(x,0) = 20$

$u(0,t)\ //\ u(L,t) = 190$

g. $u(\frac{L}{2}, t) = \sum_{n_{odd}=1}^{\infty} -\frac{680}{\pi n} \sin\left(\frac{\pi n}{L} \cdot \frac{L}{2}\right) e^{-\frac{Dn^2\pi^2 t}{L^2}} + 190$

$= \sum_{n_{odd}=1}^{\infty} -\frac{680}{\pi n} \sin\left(\frac{\pi n}{2}\right) e^{-\frac{Dn^2\pi^2 t}{L^2}} + 190$

h. $100 = \sum_{n_{odd}=1}^{\infty} -\frac{680}{\pi n} \sin\left(\frac{\pi n}{2}\right) e^{-\frac{Dn^2\pi^2 t}{L^2}} + 190$

approx. $n = 5$

$-90 = \frac{-680}{5\pi} \sin\left(\frac{5\pi}{2}\right) e^{-\frac{D(5)^2\pi^2 t}{L^2}}$

$\ln\left(\frac{-90(5\pi)}{-680} \cdot \frac{1}{\sin(\frac{5\pi}{2})}\right) = -\frac{5D\pi^2 t}{L^2}$

$0.73188 = -\frac{5D\pi^2 t}{L^2}$

$t = \left|\frac{0.73188 L^2}{-5^2 D\pi^2}\right| = \frac{0.732 L^2}{25\pi^2 D}$

i. $u(x,t) = \sum_{n_{odd}=1}^{\infty} -\frac{680}{\pi n} \sin\left(\frac{\pi n x}{L}\right) e^{-\frac{Dn^2\pi^2 t}{L^2}} + 190$

approx $n = 97$ from last.

Decay lowest → faster.

$-90 = \frac{-680}{99\pi} \sin\left(\frac{97\pi}{2}\right) e^{-\frac{D(100)^2\pi^2 t}{L^2}}$

$\ln\left(\frac{-90(97\pi)}{-686} \cdot \sin\left(\frac{97\pi}{2}\right)\right) = \frac{-D(100)^2\pi^2 t}{L^2}$

$3.688 =$

$t = \frac{3.688 L^2}{D(97)^2\pi^2}$

$t_{L=4} = \frac{59.014}{(97)^2\pi^2 D}$ $\quad t_{L=2} = \frac{14.7535}{(97)^2\pi^2 D}$

Yes, it's worth taking 20min to cut the pieces into $L = 2$cm as the time needed to reach 100°C at $\frac{L}{2}$ for two differ by a factor more than two $t_{L=4} > 2t_{L=2}$, thus, easily compensating the time spent for cutting the pieces (20min in this case).