THE UNIVERSITY OF BRITISH COLUMBIA
STAT 321 - Stochastic Signals and Systems
Assignment 3

Due: **Wednesday, November 16 at 9:00pm**.

- Submit your assignment online in the ***pdf format*** under module "Assignments". You can either typeset your solutions or scan a handwritten copy.
- Assignments are to be completed individually.
- Define notation for events and random variables, and include all steps of your derivations. Writing down the final answer will not be sufficient to receive full marks.
- Please make sure your submission is clear and neat. It is a student's responsibility to ensure that the submitted file is in good order (e.g., not corrupted and contains what you intend to submit).
- **Late submission penalty**: 1% per hour or fraction of an hour. (In the event of technical issues with submission, you can email your assignment to the instructor to get a time stamp but submit on Canvas as soon as it becomes possible to make it available for grading.)

1. Let $X \sim U\left(-\frac{1}{2}, +\frac{1}{2}\right)$, i.e.

$$f_X(x) = \begin{cases} 1\ , & x \in \left(-\frac{1}{2}, +\frac{1}{2}\right) \\ 0\ , & \text{otherwise} \end{cases}$$

   Determine the pdf, $f_Y(y)$, of Y, where $y = x^2$.

2. We have 2 types of binary sources. Each source outputs one bit, i.e. '0' or '1', independently at each time instant: for a source of type A, the probabilities are $P(0) = 0.5, P(1) = 0.5$ whereas for a type B source, the probabilities are $P(0) = 0.9, P(1) = 0.1$.

   Someone gives us a box containing 5 unlabelled, identical-looking sources, 4 of which are of type A and 1 of type B. We choose one of the 5 sources at random and observe its output at 4 time instants. Suppose that the four observed bits are all 0's, what is the probability that we chose a type A source?

3. The probability of a sunny day in Vancouver is $\frac{1}{3}$. You listen to 4 independent weather forecasts, each of which has an error probability of $\frac{1}{5}$. What is the probability that it will be sunny if all 4 forecasts are for a sunny day?

4. Alice and Bob are playing a coin-tossing game. They start and continue to toss a fair coin until 'HT' or 'TT' appears. If 'HT' appears first, Alice wins; if 'TT' appears first, Bob wins. Determine the probability that Alice wins.

5. A total of n balls are tossed into 10 bins as follows: each ball is tossed independently and has an equal probability of 0.1 of falling into Bin $i, i = 1, 2, \ldots, 10$.

   Let Y denote the number of balls in Bin 1. Let Z denote the total number of balls in Bins 6-10.

   (a) Determine the pmf of Y. *Hint: Think binomial distribution.*
   (b) Determine the pmf of Z.
   (c) Determine the conditional pmf $p(z|y)$. Clearly specify the range of y and z.
   (d) Determine the conditional pmf $p(y|z)$. Clearly specify the range of y and z.

6. A certain transmission system has 2 transmission modes. In Mode 1, the transmitted signal X is equally likely to be $+1$ or -1. In Mode 2, the transmitted signal X is 0. The 2 transmission modes are equally likely to be used. Note that $X \in \{-1, 0, +1\}$.

   Let the received (observed) signal be $Y = X + Z$ where the noise $Z \sim U(-1, +1)$ is independent of X.

   (a) Determine the conditional pdf $f_{Y|M}(y|M = 1)$, i.e. the conditional pdf of Y given Mode 1.

   (b) Determine the conditional pdf $f_{Y|M}(y|M = 2)$, i.e. the conditional pdf of Y given Mode 2.

   (c) Determine the optimal (i.e. minimum probability of error) decoding rule for choosing the mode of the system, i.e. for a given value y, specify whether we should choose Mode 1 or Mode 2. Your answer should take the following form: choose Mode 1 if y is a value in this (or these) interval(s); otherwise, choose Mode 2.

   (d) Determine the probability of error for the decoder in Part (c).

321 ASN 3 Peter Kim, 18693002.

1. $X \sim U(-\tfrac{1}{2}, +\tfrac{1}{2})$

$$f_X(x) = \begin{cases} 1, & x \in (-\tfrac{1}{2}, \tfrac{1}{2}) \\ 0, & \text{otw} \end{cases}$$

$y = x^2$

$\pm\sqrt{y} = x \rightarrow g^{-1}(x) = \pm\sqrt{x} \rightarrow g^{-1}(y) = \pm\sqrt{y}$

can't be neg as sqrt

$f_Y(y) = f_X(g^{-1}(y)) \cdot \left|\frac{d}{dy} g^{-1}(y)\right|$ where $g^{-1}(y) = \pm\sqrt{y}$ with $0 \le y \le \tfrac{1}{4}$

for $g^{-1}(y) = \sqrt{y}$:

$f_Y(y) = 1 \cdot \left|\frac{1}{2\sqrt{y}}\right|$ for $0 \le y \le \tfrac{1}{4}$

for $g^{-1}(y) = -\sqrt{y}$:

$f_Y(y) = 1 \cdot \left|\frac{-1}{2\sqrt{y}}\right|$ for $0 \le y \le \tfrac{1}{4}$

$$\therefore f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} + \frac{1}{2\sqrt{y}} & \text{for } 0 \le y \le \tfrac{1}{4} \\ 0 & \text{otw} \end{cases} = \boxed{\begin{cases} \frac{1}{\sqrt{y}} & \text{for } 0 \le y \le \tfrac{1}{4} \\ 0 & \text{otw} \end{cases}}$$

2. Two sources A & B

$$A = \begin{cases} 0 & \text{wp } 0.5 \\ 1 & \text{wp } 0.5 \end{cases} \qquad B = \begin{cases} 0 & \text{wp } 0.9 \\ 1 & \text{wp } 0.1 \end{cases}$$

4A 1B

$P(A) = \tfrac{4}{5}$

$P(B) = \tfrac{1}{5}$

$P(A=0) = 0.5$ for 4 instances, $P(A=0)_4 = 0.5^4$. Similarly, $P(B=0)_4 = 0.9^4$

$P(A=1) = 0.5$ $P(A=1)_4 = 0.5^4$ $P(B=1)_4 = 0.1^4$

$X \rightarrow 0$ for 4 instance, $P(A|X) = \dfrac{P(X|A)P(A)}{P(X|A)P(A) + P(X|B)P(B)}$

$P(X|A) = 0.5 \rightarrow P(X|A)_4 = 0.5^4$

$P(X|B) = 0.9 \rightarrow P(X|A)_4 = 0.9^4$

$$= \frac{0.5^4 \cdot \tfrac{4}{5}}{0.5^4 \cdot \tfrac{4}{5} + 0.9^4 \cdot \tfrac{1}{5}}$$

$$= \boxed{0.2759}$$

3. S - Sunny S^c - Not Sunny Ws - Forecast Sunny Ws^c - Forecast Not Sunny

$P(S) = 1/3$ $P(S^c) = 2/3$

$P(Ws \mid S) = 4/5$ $P(Ws \mid S^c) = 1/5$ $P(Ws^c \mid S) = 1/5$ $P(Ws^c \mid S^c) = 4/5$

$P(Ws \mid S)$ for 4 ⫫ forecast $= (4/5)^4$ $P(Ws^c \mid S)$ " $= (1/5)^4$

$P(Ws \mid S^c)$ " $= (1/5)^4$ $P(Ws^c \mid S^c)$ " $= (4/5)^4$

$$P(S \mid Ws_4) = \frac{P(Ws_4 \mid S)P(S)}{P(Ws_4 \mid S)P(S) + P(Ws_4 \mid S^c)P(S^c)} = \frac{(4/5)^4(1/3)}{(4/5)^4(1/3) + (1/5)^4(2/3)} = \boxed{0.99225}$$

4. Fair Coin $\begin{cases} H, & w.p\ 0.5 \\ T, & w.p\ 0.5 \end{cases}$ Alice & Bob.
{HT} {TT} → winning condition.

Observation : If H comes up at any point, alice will win. → Bob can only with if the first two tosses yields to tails. { THHT , HTT , THT ... }
* Thus, if H is found during first two tosses, Alice win.

$\underline{HH \quad HT \quad TH} \quad TT \quad \rightarrow \quad P(H_1 \cup H_2) = 3/4$ $\boxed{\therefore P(\text{Alice win}) = 0.75}$
$= P(\text{Alice Win})$

5. $n \rightarrow$ # of balls tossed. 10 bin in total $P(\text{falling to } bin_i) = 0.1$

$Y \sim$ # balls in bin_1 $Z \sim$ # balls in $bin_{6\sim10}$

a) $Bin(n, 0.1) \rightarrow p_Y(y) = \binom{n}{y}(0.1)(1-0.1)^{n-y} = \boxed{\binom{n}{y}(0.1)^y(0.9)^{n-y}}$

$\binom{n}{z}(0.5)^n$

b) $Bin(z, 0.5)$, where $0.5 = P(\text{falling to } bin_{6\sim10}) \rightarrow p_Z(z) = \binom{n}{z}(0.5)^z(1-0.5)^{n-z} = \boxed{\binom{n}{z}(0.5)^z(0.5)^{n-z}}$

c) Given y, $n \Rightarrow n-y$ $\therefore Bin(n-y, 5/9)$ where $5/9 = P(\text{falling to } bin_{6\sim10} \text{ without } bin_1)$
$\rightarrow p(z \mid y) = \binom{n-y}{z}(5/9)^z(1-5/9)^{n-y-z}$ $0 < z < n-y$

d) Given Z, $n \Rightarrow n-Z$ $\therefore \text{Bin}(n-Z, \frac{1}{5})$ where $\frac{1}{9} = P(\text{falling to bin, without bin}_{6\sim10})$

$\rightarrow p(y|z) = \underline{\binom{n-z}{y}(\frac{1}{5})^y(1-\frac{1}{5})^{n-z-y}}$ $\quad 0 < y < n-z$

6. $X_1 = \begin{cases} 1 & \text{wp } 0.5 \\ -1 & \text{wp } 0.5 \end{cases}$ signal, mode $\quad X_2 = 0 \quad P(X_1 \text{ used}) = P(X_2 \text{ used}) = 0.5 \quad Y = X + Z$

$Z \sim U(-1, +1)$

a) Mode 1, $X = \begin{cases} 1 & \text{wp } 0.5 \\ -1 & \text{wp } 0.5 \end{cases}$ $\quad Z = \begin{cases} \frac{1}{2} & -1 \le z \le 1 \\ 0 & \text{otw} \end{cases}$ $\quad \frac{1}{1-(-1)}$

$f_{Y|M}(y|M=1) = f_{Y|X}(y|X=1)P_X(x=1) + f_{Y|X}(y|X=-1)P_X(x=-1)$

$= \frac{1}{2} f_{Y|X}(y|X=1) + \frac{1}{2} f_{Y|X}(y|X=-1)$

$f_{Y|X}(y|X=1)$:

$X = 1 \rightarrow Y = 1 + Z$

$Y \sim U(0, 2) \rightarrow Y = \begin{cases} \frac{1}{2-0} & 0 \le Y \le 2 \\ 0 & \text{otw} \end{cases}$

$f_{Y|X}(y|X=-1)$:

$X = -1 \rightarrow Y = -1 + Z$

$Y \sim U(-2, 0) \rightarrow Y = \begin{cases} \frac{1}{0-(-2)} & -2 \le Y \le 0 \\ 0 & \text{otw} \end{cases}$

$f_{Y|M}(y|M=1) = \begin{cases} \frac{1}{2} \cdot \frac{1}{2} & \text{for } 0 \le Y \le 2 \\ \frac{1}{2} \cdot \frac{1}{2} & \text{for } -2 \le Y \le 0 \\ 0 & \text{otw} \end{cases} = \boxed{\begin{cases} \frac{1}{4} & -2 \le y \le 2 \\ 0 & \text{otw} \end{cases}}$

b) Mode 2, $X = 0$ $\quad Z = \begin{cases} \frac{1}{2} & -1 \le z \le 1 \\ 0 & \text{otw} \end{cases}$

$f_{Y|M}(y|M=2) = f_{Y|X}(y|X=0)\,P_X(x=0) = f_{Y|X}(y|X=0)$: $\quad X = 0 \rightarrow Y = Z \rightarrow Y = \begin{cases} \frac{1}{2} & -1 \le Y \le 1 \\ 0 & \text{otw} \end{cases}$ $\quad Y \sim U(-1, +1)$

$= \boxed{\begin{cases} \frac{1}{2} & -1 \le y \le 1 \\ 0 & \text{otw} \end{cases}}$

c) Min Prob. Error $\rightarrow$ MAP , Mode $= \begin{cases} 1 & \text{wp } 0.5 \\ 2 & \text{wp } 0.5 \end{cases}$

$f_{Y|M}(y|M=1) = \begin{cases} 1/4, & -2 \le y \le 2 \\ 0, & \text{otw} \end{cases}$ $\quad f_{Y|M}(y|M=2) = \begin{cases} 1/2, & -1 \le y \le 1 \\ 0, & \text{otw} \end{cases}$

$\hat{M}_{MAP}(y) = \boxed{\begin{cases} 2 & \text{for } -1 \le y \le 1 \\ 1 & \text{otw.} \end{cases}}$ as Mode 2 has a lower probability of error when chosen.

d) $P_e = 1 - P_{correct}$

$= 1 - \{ P(-2 \le y \le 2 | M=1) + P(-1 \le y \le 1 | M=1) \}$

$= 1 - (1/4 + 1/2)$

$= 1/4$

$= 0.25$

1-4