LIBOR Market Model Result Analysis

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1 Validate Volatility Assumption

After we generated LIBOR rates, we need to validate if the standard deviation of rates distribution still match the given volatility assumption.

1.1 Theory

1.1.1 Volatility Parametrization

Volatility formula,

$$\sigma_i(t) = \phi_i([a(T_{i-1} - t) + d]e^{-b(T_{i-1}-t)} + d) \tag{1}$$

where, $\sigma_i(t)$ is $\sigma(t, T_{i-1}, T_i)$. This form allows a humped shape in the graph of the instantaneous volatility

1.1.2 LIBOR Market Model Simulation

LIBOR Market Model,

$$dL_n(t_i) = \mu_n(t_i)L_n(t_i)dt + \sigma_n(t_i)L_n(t_i)dW_n(t_i) \tag{2}$$

Discrete form,

$$L_n(t_{i+1}) = L_n(t_i) + \mu_n(t_i)L_n(t)(t_{i+1} - t_i) + \sigma_n(t_i)L_n(t_i)\sqrt{t_{i+1} - t_i}Z_n^i \tag{3}$$

Integrate this equation,

$$L_n(t_{i+1}) = L_n(t_i)\exp\{(\mu_n(t_i) + \frac{1}{2}\sigma_n^2(t_i))(t_{i+1} - t_i) + \sigma_n(t_i)\sqrt{t_{i+1} - t_i}Z_n^i\} \tag{4}$$

1.1.3 The distribution of LIBOR rates

The rates from LIBOR Market Model follow a lognormal distrinbution, $log(L_n(t_{i+1})) \sim N\left(log(L_n(t_i)) + (\mu_n(t_i) + \frac{1}{2}\sigma_n^2(t_i))(t_{i+1} - t_i), \sigma_n^2(t_i)(t_{i+1} - t_i)\right)$. We need to validate if the variance of $log(L_n(t_{i+1}))$ distribution matches $\sigma_n^2(t_i)(t_{i+1} - t_i)$.

1.2 An Example

We would like to validate the volatility assumption of 3 Month length LIBOR rate starting from 9 Month to 12 Month with forward starting at 9 Month, we will write it as $L_{0.75}(0.75)$ for short.

1.2.1 Calculate the Volatility

We used $L_{0.75}(0.75)$ to simulate $L_{0.75}(0.5)$, and $L_{0.75}(0.5)$ is from simulate through $L_{0.75}(0.25)$. Finally, we have

$$L_n(t_{k+1}) \sim N\left(log(L_n(t_0)) + \sum_{i=1}^{k}(\mu_n(t_i) + \frac{1}{2}\sigma_n^2(t_i))(t_{i+1} - t_i), \sum_{i=1}^{k}\sigma_n^2(t_i)(t_{i+1} - t_i)\right) \tag{5}$$

In the volatility formula, t is 0.25, 0.5, 0.75, T_i is 1, 1.25, 1.5. In this example, we use $a = 0.19, b = 0.97, c = 0.08, d = 0.01$, we obtained $\sum_{i=1}^{k}\sigma_n^2(t_i)(t_{i+1} - t_i) = 0.017712$.

1.2.2 Calculate the Variance

We can take the log of the rates and apply sample variance formula to get the variance.

1.2.3 Chi-Square Test for the Variance

We can use a Chi-Square Test to validate if $\sigma^2 = \sigma_0^2$.
$H_0 : \sigma^2 = \sigma_0^2$
$H_a : \sigma^2 \neq \sigma_0^2$
Test Statistic: $T = \frac{N-1}{S^2/\sigma_0^2}$
where, N is the smaple size, S is the sample standard, σ_0 is the target standard deviation.
Critical region: Reject H_0 if $T < \chi^2_{\alpha/2,N-1}$ or $T > \chi^2_{1-\alpha/2,N-1}$

1.2.4 Result

Absoulate Error: $ABS(\hat{\sigma^2} - \sigma_0^2)$.
Percentage Error: $\frac{\hat{\sigma^2}-\sigma_0^2}{\sigma_0^2}$.

$N = 100$, Calculated Variance: 0.0147492, Target Variance: 0.017712, ABS Error: 0.00296284, Percentage Error: -0.167278

$N = 1000$, Calculated Variance: 0.0179749, Target Variance: 0.017712, ABS Error: 0.000262848, Percentage Error: 0.0148401

$N = 10000$, Calculated Variance: 0.0175451, Target Variance: 0.017712, ABS Error: 0.00016692, Percentage Error: -0.00942413

$N = 100000$, Calculated Variance: 0.0178083, Target Variance: 0.017712, ABS Error: 9.63033e-05, Percentage Error: 0.00543717

$N = 1000000$, Calculated Variance: 0.0177014, Target Variance: 0.017712, ABS Error: 1.06467e-05, Percentage Error: -0.000601098

$N = 100$, Test Statistic: 90.3411, Degree of Freedom: 99, $\chi^2_{0.025,99} = 73.361$, $\chi^2_{0.975,99} = 128.422$ The test statistic value of 90.3411 is within the range of $\left(\chi^2_{\alpha/2,N-1}, \chi^2_{1-\alpha/2,N-1}\right)$, so we cannot reject the null hypothesis.