1. (a) We begin with the left hand-side and arrive at the right:

$$
\begin{aligned}
&\mathbb{E}[(\mathbb{E}[X \mid Y])^2] \overset{2.3.25}{=} \mathbb{E}[F(Y)\mathbb{E}[X \mid Y]] \overset{2.3.23(c)}{=} \mathbb{E}[\mathbb{E}[F(Y)X \mid Y]] \\
\overset{Tower}{=}&\mathbb{E}[F(Y)X] \overset{2.3.25}{=} \mathbb{E}[\mathbb{E}[X \mid Y]X] = \mathbb{E}[X\mathbb{E}[X \mid Y]],
\end{aligned}
$$

where F is some function such that $\mathbb{E}[X \mid Y] = F(Y)$.

Note that we can apply the tower property to $\mathbb{E}[\mathbb{E}[F(Y)X \mid Y]]$ because we are given a realisation of Y as a condition of the probability, and hence we can treat $F(Y)$ simply as a constant, so we can think of the expression instead as $\mathbb{E}[\mathbb{E}[AX \mid Y]]$ for a constant A, and then it is clearer that the tower propert applies.

(b) We begin with the right hand-side and arrive at the left:

$$
\begin{aligned}
&\mathbb{E}[(X - \mathbb{E}[X \mid Y])^2] + \mathbb{E}[(\mathbb{E}[X \mid Y] - f(Y))^2] \\
=&\mathbb{E}[X^2 - 2X\mathbb{E}[X \mid Y] + \mathbb{E}[X \mid Y]^2] + \mathbb{E}[\mathbb{E}[X \mid Y]^2 - 2f(Y)\mathbb{E}[X \mid Y] + f(Y)^2] \\
\overset{2.3.7}{=}&\mathbb{E}[X^2] - 2\mathbb{E}[X\mathbb{E}[X \mid Y]] + \mathbb{E}[\mathbb{E}[X \mid Y]^2] + \\
&\mathbb{E}[\mathbb{E}[X \mid Y]^2] - 2\mathbb{E}[f(Y)\mathbb{E}[X \mid Y]] - \mathbb{E}[f(Y)^2] \\
\overset{q1(a)}{=}&\mathbb{E}[X^2] - 2\mathbb{E}[\mathbb{E}[X \mid Y]^2] + 2\mathbb{E}[\mathbb{E}[X \mid Y]^2] - 2\mathbb{E}[f(Y)\mathbb{E}[X \mid Y]] + \mathbb{E}[f(Y)^2] \\
\overset{2.3.23(c)}{=}&\mathbb{E}[X^2] - 2\mathbb{E}[\mathbb{E}[X \mid Y]^2] + 2\mathbb{E}[\mathbb{E}[X \mid Y]^2] - \mathbb{E}[\mathbb{E}[2f(Y)X \mid Y]] + \mathbb{E}[f(Y)^2] \\
\overset{Tower}{=}&\mathbb{E}[X^2] - 2\mathbb{E}[\mathbb{E}[X \mid Y]^2] + 2\mathbb{E}[\mathbb{E}[X \mid Y]^2] - 2f(Y)\mathbb{E}[X] + \mathbb{E}[f(Y)^2] \\
\overset{2.3.23(c)}{=}&\mathbb{E}[X^2] - 2\mathbb{E}[\mathbb{E}[X \mid Y]^2] + 2\mathbb{E}[\mathbb{E}[X \mid Y]^2] - 2\mathbb{E}[f(Y)X] + \mathbb{E}[f(Y)^2] \\
=&\mathbb{E}[X^2] - 2\mathbb{E}[f(Y)X] + \mathbb{E}[f(Y)^2] \\
\overset{2.3.7}{=}&\mathbb{E}[X^2 - 2f(Y)X + f(Y)^2] \\
=&\mathbb{E}[(X - f(Y))^2],
\end{aligned}
$$

which is precisely the left hand-side.

Notice that again, as in the first part of the question, when we applied the tower law, we are given a realisation of Y as a condition in the probability, and hence we can treat $f(Y)$ as being a constant.

(c) We firstly consider the equality in part (b) and write,

$$\mathbb{E}[(\mathbb{E}[X \mid Y] - f(Y))^2] = \mathbb{E}[(X - f(Y))^2] - \mathbb{E}[(X - \mathbb{E}[X \mid Y])^2],$$

and hence, we re-write the inequality as follows:

$$\begin{aligned} &\mathbb{E}[(X - \mathbb{E}[X \mid Y])^2] \leq \mathbb{E}[(X - f(Y))^2] \\ \iff &0 \leq \mathbb{E}[(X - f(Y))^2] - \mathbb{E}[(X - \mathbb{E}[X \mid Y])^2] \\ \iff &0 \leq \mathbb{E}[(\mathbb{E}[X \mid Y] - f(Y))^2], \end{aligned}$$

and the last inequality clearly holds as the last expectation is an expectation of squares, which are all greater than or equal to 0, and hence the expectation is greater than or equal to 0 too. The last inequality holds if and only if the first holds and hence we have proven the inequality that was required.

Moreover, we see that $\mathbb{E}[(\mathbb{E}[X \mid Y] - f(Y))^2] = 0$ precisely when $\mathbb{E}[X \mid Y] - f(Y) = 0$ for all values of Y, or equivalently, when $f(Y) = \mathbb{E}[X \mid Y]$ with probability holds.

The last inequality is an equality if and only if the first inequality is an equality, and hence we have proven the statement.

2. (a) This call option would be cheap because we are operating under the assumption that everyone predicts this security to have a price close to £10 in 3 months, namely the seller of the call option. Hence the seller would expect that there is a low probability that the security exceeds £15 in 3 months and hence a low probability that we'd exercise our option of buying the security. Therefore, the seller thinks they are almost definitely not going to make a loss from the call option and would hence be willing to sell it cheaply.

   (b) The precise optimal strategy for making money with call options in this scenario depends on exactly what the price of the options are given a particular strike price. However, without deeply analysing numbers, it is safe to say that buying call options with a strike price a decent amount higher than £10 would be fairly cheap for similar reasons as in 2(a).

   (c) Assuming that we have all the £75,000 today, and that we do not gain any more money to invest in the 3 months leading up to the maturity, there is no reason to buy call options with a strike price higher than £9, since we would also have to pay £1 for the options itself, whic would me more expensive than buying the security today, directly for £10. On the other hand, buying any option with strike price less than £9 for £1 would yield higher profits than buying the security directly.

   In particular, given a strike price £K and a price of the call option, £P, it would be beneficial to buy any option with values K, P such that $K + P < 10$.

   However, realistically, it is doubtful that one would be able to find many such options, since the sellers expect the security to stay around £10, and would therefore set $K + P$ to around 10, and since they would be in a disadvantageous position, by definition of options, they would feel the need to increase the price even further to account for the risk they are taking.