# Clarifying Eichler's proof of the Baker-Campbell-Hausdorff formula's existence

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## 1 Introduction

In this text we will build up the required knowledge to illustrate and understand Martin Eichler's elegant proof of the Baker-Campbell-Hausdorff formula's existence [1]. Namely, we prove via induction that for all $A, B \in \mathfrak{gl}(N, \mathbb{R})$ we have $D \in \mathfrak{gl}(N, \mathbb{R})$ such that

$$\exp(A)\exp(B) = \exp(D), \text{ with}$$

$$D = \sum_{n\in\mathbb{N}} F_n(A, B) \tag{1.1}$$

where for all $n, F_n(A, B)$ is a homogeneous Lie polynomial of degree $n$ involving exclusively $A$ and $B$. We define homogeneity in the next definition and Lie polynomials later.

## 2 Homogeneous polynomials

**Definition 2.1.** Consider a $k$-variate $N \times N$ matrix polynomial, $G \in \mathfrak{gl}(N, \mathbb{R})[X_1, \cdots, X_k]$. $G$ is *homogeneous of degree* $n$, if the sum of exponents of $X_1, \cdots, X_k$ is equal to $n$ in each summand.

**Example 2.2.** $G_1(X_1, X_2, X_3) = 2X_1X_2{X_3}^3 + 3{X_1}^5 + 4X_1X_2{X_1}^3$ is a 3-variate homogeneous polynomial of degree 5 since the exponents of $X_1, X_2$ and $X_3$ in both summands sum to 5, but $G_2(X_1, X_2, X_3) = 2X_1X_2{X_3}^2 + 3{X_1}^5 + 4X_1X_2{X_1}^3$ is not homogeneous since the first summand is of degree 4 and the other is of degree 5.

**Lemma 2.3.** Given a commutative constant $r$ and a homogeneous polynomial $G$ of degree $n$ in $k$ variables we have that $G(rX_1, rX_2, \cdots, rX_k) = r^n G(X_1, X_2, \cdots, X_k)$.

*Proof.* $G(rX_1, rX_2, \cdots, rX_k)$ is homogeneous of degree $n$, so by definition we can express it as

$$G(rX_1, rX_2, \cdots, rX_k) = \sum_{i=1}^{l} B_i\Big((rX_1)^{a_{i1}}(rX_2)^{a_{i2}} \cdots (rX_k)^{a_{ik}}\Big)$$

such that for all $i \in \{1, \cdots, l\}$ $\sum_{j=1}^{k} a_{ij} = n$ and $B_i$ are constants.

Hence we can factor out $r$ from the right hand-side and we have

$$G(rX_1, \cdots, rX_k) = \sum_{i=1}^{l} r^{a_{i1}+\cdots+a_{ik}} B_i\Big(X_1^{a_{i1}} \cdots X_k^{a_{ik}}\Big) = r^n \sum_{i=1}^{l} B_i\Big(X_1^{a_{i1}} \cdots x_k^{a_{ik}}\Big).$$

Now note that the summation on the right-hand side is precisely in the form of a homogeneous polynomial involving $X_1, X_2, \cdots, X_k$ and we have the required result:

$$G(rX_1, rX_2, \cdots, rX_k) = r^n G(X_1, X_2, \cdots, X_k).$$

$\square$

# 3 What can we derive from the direct expression of $D$?

In this section, we take $D$ to represent the input of the exponential function such that it equals the product of two other exponential functions with two separate (but not necessarily distinct) arguments in $\mathfrak{gl}(N, \mathbb{R})$. In equation (1.1), $D$ has precisely this meaning, in the context of $A$ and $B$; whenever we talk of $D$, it need not necessarily involve $A$ and $B$, the variables will depend on the context.

Moreover, by the 'direct' expression of $D$ we mean precisely equation (3.1).

Before that though, let us set some more notation which will be used throughout this section:

1. Let $G(A, B) = \exp(A)\exp(B) - I = A + B + AB + \frac{A^2}{2} + \frac{B^2}{2} + \cdots$. [1] Note that this is a matrix polynomial, since $A, B \in \mathfrak{gl}(N, \mathbb{R})$.
2. For all polynomials $G(X_1, \cdots, X_k)$, let $G(X_1, \cdots, X_k)_n$ be the polynomial formed by summing all $n$-degree summands of $G(X_1, \cdots, X_k)$.
3. $F_n(A, B)$ as defined in the following proposition.

**Proposition 3.1.** We can express $D$ as a sum of homogeneous $n$-degree matrix polynomials, namely $D = \sum_{n=1}^{\infty} F_n(A, B)$, where $F_n(A, B) = \sum_{i=1}^{n} -(-1)^i \frac{1}{i} G(A, B)^i{}_n$ for each $n$.

*Proof.* We begin by expressing $D$ 'directly':

$$\begin{aligned} D &= \ln(I + G(A, B)) = G(A, B) - \frac{1}{2}G(A, B)^2 + \frac{1}{3}G(A, B)^3 - \cdots {}^2 \\ D &= \sum_{n=1}^{\infty} -(-1)^n \frac{1}{n} G(A, B)^n. \end{aligned} \tag{3.1}$$

Then notice that all the terms of $G(A, B)^n$ have degree $k$ for some $k \in \{n, n+1, n+2, \cdots\}$. Moreover, given any $m$ and $n \geq m$, there are finitely many terms of degree $n$ in $G(A, B)^m$.

Hence, given any $n$, we only need to sum the (finitely many) $n$-degree terms from each of $G(A, B)_1, \cdots, G(A, B)_n$ (which are also clearly finitely many) and hence this is computable. Precisely, let $F_n(A, B) = \sum_{i=1}^{n} -(-1)^i \frac{1}{i} G(A, B)^i{}_n$.

[1] This expression is defined in such a way as to simplify the expansion of $\log(\exp(A)\exp(B))$.
[2] This result was proven in the MATH426 lecture notes.

Then clearly summing $F_n(A,B)$ over all $n$ gives precisely $D$, and we have

$$D = \sum_{n=1}^{\infty} F_n(A,B).$$

$\square$

**Corollary 3.2.** $F_1(A,B) = A + B$ and $F_2(A,B) = \frac{1}{2}[A,B]$ [3]. [4]

While we can compute all of $D$ directly via iteration, the algebra gets very messy, very fast; instead we can make the stronger and much more useful claim that $D$ is a summation of homogeneous Lie polynomials. This gives rise to a calculation involving only Lie brackets; precisely the Baker-Campbell-Hausdorff formula.

**Corollary 3.3.** For all $A \in \mathfrak{gl}(N,\mathbb{R})$ and $n \geq 2$, given commutative constants $\lambda, \mu \in \mathbb{R}$,

$$F_n(\lambda A, \mu A) = 0.$$

*Proof.* We know that $\exp(A)\exp(B) = \exp(A+B)$ when $AB = BA$ [5]. Clearly $A$ commutes with itself (since the notation $A^2$ is unambiguous) and since $\lambda$ and $\mu$ are commutative constants,

$$\exp(\lambda A)\exp(\mu A) = \exp(\lambda A + \mu A).$$

Recalling how $D$ is defined in this section (see first sentence of the section), we see that $D = \lambda A + \mu A$ and hence $D$ has a (potentially) non-zero term of degree 1. In other words, $F_1(\lambda A, \mu A) = \lambda A + \mu A$ and recalling that via propostion 3.1, $D = \sum_{n=1}^{\infty} F_n(\lambda A, \mu A)$, we notice that

$$\sum_{n=2}^{\infty} F_n(\lambda A, \mu A) = 0.$$

This can only happen if $F_n(\lambda A, \mu A) = 0$ for all $n \geq 2$ since given $k, l \in \{2, \cdots\}, k \neq l$, we can always find some $A \in \mathfrak{gl}(N,\mathbb{R})$ such that $F_k(\lambda A, \mu A) - F_l(\lambda A, \mu A) \neq 0$. Hence we have proven the result. $\square$

# 4 Lie polynomials

Briefly, Lie polynomials are polynomials, in say $A_1, \cdots, A_k$, which are formed by iteratively taking Lie brackets of $A_1, \cdots, A_k$. In particular, we are mostly interested in 2-variate Lie polynomials.

This brief summary is useful for giving us an idea of what we are working with, but in order to give a concrete definition, we will consider a bijection between Lie monomials and binary trees.

**Definition 4.1.** A *binary tree*[6] is a finite structure which can be constructed via the following method:

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[3] $[A,B] = AB - BA$.

[4] This computation was shown in the MATH426 notes.

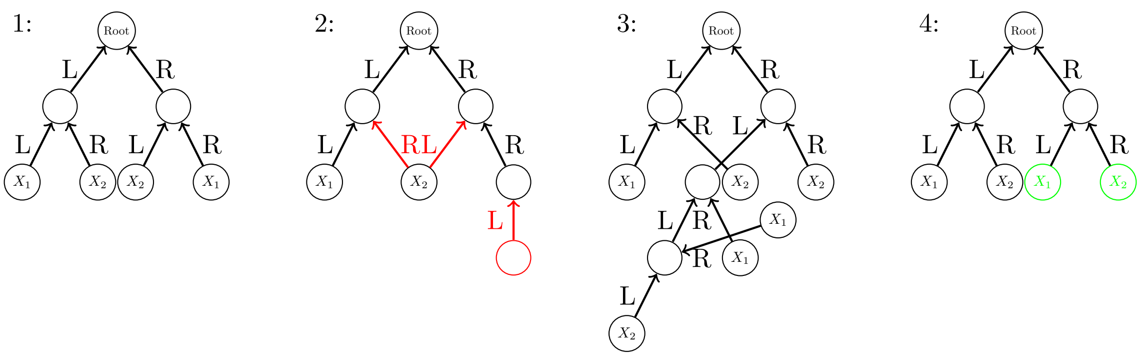
[5] This result was proven in the MATH426 lecture notes without the use of the Baker-Campbell-Hausdorff formula.

[6] Binary trees are a concept of graph theory, but to define them we only need to understand basic concepts of graph theory; namely what vertices and directed edges mean.

1. Start by defining a vertex (we refer to it as the root vertex). We treat it as the generating vertex.
2. Add $k \in \{0, 2\}$ new vertices. We call these vertices, generated vertices.
3. If $k = 2$, draw 2 directed edges pointing from each of the generated vertices to the generating vertex[7], labeling one, "$L$" and the other, "$R$". If $k = 0$, do nothing.
4. Repeat steps two and three $k$ times, this time treating each generated vertex as the generating vertex.
5. We have now constructed the whole tree[8] and in this last step we label it. Consider each vertex which has no edge pointing into it; we call these the outer-most vertices. We label each outer-most vertex by either $X_1$ or $X_2$ such that any two vertices which have edges pointing to a common (non-outer-most) vertex have different labels.

Let us denote the set of all uniquely labeled binary trees by $T$ in this section.

**Example 4.2.** Below are pictorial diagrams of examples and counter-examples of binary trees:




1. This <u>is a binary tree</u> since except for the root vertex each vertex has 1 edge going out of it and each vertex has either 2 edges pointing into it or the vertex is labeled.
2. This <u>is not a binary tree</u> for many reasons. Firstly, the two red edges coming out of $B$ contradict our definition of a binary tree. Secondly the bottom red edge is the only one going into a vertex but each vertex must have $k \in \{0, 2\}$ edges going into it. Thirdly, the red vertex should be labeled since it has no edges pointing into it.
3. This graph has little resemblence to a binary tree, but upon inspection we notice it does not contradict our definition. Therefore this <u>is a binary tree</u>.
4. Similarly to the first graph, this <u>is a binary tree</u>, but note that it has a different labelling in the green vertices; the one adjacent to edge $L$ is labeled $X_1$ as opposed to $X_2$ as in graph 1 and vice-versa. Therefore those are two distinct binary trees.

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[7]It is standard by most definitions of binary trees for the edges to point in the opposite direction, however this direction is more useful to use in defining Lie polynomials, as we will see shortly.

[8]Note that this process terminates by itself when we generate $k = 0$ new vertices enough times, which must happen at some point since the binary tree is a finite structure by definition.

**Definition 4.3.** The set of *degree* $-\infty$ $N \times N$ *Lie monomials* is the set containing only the 0 $N \times N$ matrix, $0_{N\times N}$.

**Definition 4.4.** The set of *degree 0* $N \times N$ *Lie monomials* is the set $\mathfrak{gl}(N, \mathbb{R}) \setminus \{0_{N\times N}\}$ [9].

**Definition 4.5.** The set of $N \times N$ *2-variate Lie monomials* [10] *in* $A, B \in \mathfrak{gl}(N, \mathbb{R})$ is defined as the set of those elements that are uniquely constructed by each $t \in T$ in the following way:

1. Re-label all labeled vertices from $X_1$ and $X_2$ to $A$ and $B$ respectively. [11]
2. Consider the set of vertices which are unlabeled and connected to two labeled vertices; if there are none skip to step 5. Each such vertex is connected to one vertex, say $l$, via edge $L$ and to another vertex, say $r$, via edge $R$; we label that vertex $[l, r]$ where the square brackets are Lie brackets.
3. We repeat step 2 until the root vertex is labeled.
4. Our Lie monomial is given by the labeling of the root vertex.

Moreover, we define the monomial's degree to equal the number of labeled vertices after step 1.

**Remark 4.6.** Note that since each $t \in T$ constructs a unique Lie monomial of degree $n \geq 1$, and since we have defined the Lie monomials to have no other elements, then the two sets are bijective, that is there exists a one-to-one mapping between the two.

**Remark 4.7.** The definition for 2-variate Lie monomials can be adapted for $k$-variate Lie monomials in terms of $A_1, \cdots, A_k$ by altering step 5 in definition 4.1 so that instead of labeling by $X_1$ and $X_2$, we label by $X_1, \cdots, X_k$[12] and by altering step 1 of definition 4.5 so that we re-label from $X_i$ to $A_i$ for all $i \in \{1, \cdots, k\}$. However, for our purpouses we only need 2-variate Lie monomials.

**Definition 4.8.** The set of *Lie monomials* is the union of the set of degree $-\infty$, degree 0 and degree $n \geq 1$ Lie monomials.

**Definition 4.9.** The set of *Lie polynomials in terms of* $A, B \in \mathfrak{gl}(N, \mathbb{R})$ is the set of linear combinations of Lie monomials in $A$ and $B$.

**Definition 4.10.** A Lie polynomial is *homogeneous of degree* $n$ if it is the linear combination of Lie monomials exclusively of degree $n$.

**Example 4.11.** Here are some examples of monomials and polynomials:

1. The identity $N$ by $N$ matrix is a Lie monomial of degree 0.
2. The $N$ by $N$ zero matrix is of degree $-\infty$.
3. Let us construct the Lie monomial given by binary tree 1 from above. Firstly we re-label $X_1$ and $X_2$ to $A$ and $B$ respectively, and then applying step 2 twice we have the labeling

[9] We do not define the degree $-\infty$ and 0 monomials in relation to a variable since they are unaffected by variables.

[10] From now on, in order to save space, 'whenever we write "Lie monomials", we mean, "$N \times N$ 2-variate Lie monomials"; similarly for Lie polynomials.

[11] Note that we do not consider the root vertex as labeled.

[12] If (for say a 3-variate polynomial) we omit this alteration, then we lose the bijection between the set of binary trees and Lie monomials, since a tree with a pair of vertices labeled $A$ and $B$ sharing a common vertex can be labeled as $X_1, X_2$ or $X_1, X_3$ respectively and hence correspond to at least 2 distinct Lie monomials.

$[[A, B], [B, A]]$ for the root vertex and hence our Lie monomial. The method gives $n = 3$ and hence the degree of this monomial is 3.

4. Similarly, from binary tree 3, we construct the monomial $[[A, B], [[[B, A], A], B]]$ with $n = 5$.
5. $3[[A, B], [B, A]] + 2[[A, B], [[[B, A], A], B]]$ is a non-homogeneous Lie polynomial; note that it is a linear combination of the Lie monomials from 2 and 3 which have different degrees.

**Lemma 4.12.** If we know a degree $n$ homogeneous matrix polynomial to be a Lie ploynomial, we know it must have been a homogeneous $n$ degree Lie polynomial.

*Proof.* Consider a homogeneous, degree $m$ Lie polynomial on $A, B \in \mathfrak{gl}(N, \mathbb{R})$; it is, by definition, the summation of some number of Lie brackets applied to $m$ total copies of $A$ and $B$. Equivalently re-stated, $m$ variables are multiplied by each other exactly once in various orders and are then added/subtracted from each other. In other words, this is a linear combination of matrix monomials each with exponent $m$; this is precisely a homogeneous matrix polynomial of degree $m$ by definition and we see that $m = n$. □

**Proposition 4.13.** If $F_i(A, B), F_j(A, B)$ and $F_k(A, B)$ are each homogeneous Lie polynomials, then

$$F(A, B) = F_i\Big(F_j(A, B), F_k(A, B)\Big)$$

is a Lie polynomial.

*Proof.* Let $G_i(A, B), G_j(A, B)$ and $G_k(A, B)$ represent any general monomial that makes up $F_i(A, B), F_j(A, B)$ and $F_k(A, B)$ respectively. It is easy to see that by proving the result for all possible monomials of $F_i(A, B)$, $G_i(A, B)$, we also prove the result for the homogeneous polynomial $F_i(A, B)$ since a summation of homogeneous $i$-degree Lie polynomials is an $i$-degree homogeneous Lie polynomial.

Less intuitively, this relationship also holds for $j$ and $k$. For $i \geq 2$, the monomial $G_i(A, B)$ consists entirely of Lie bracket operations. We know that $[A, C] + [A, D] + [B, C] + [B, D] = [A + B, C + D]$[13] and so given any homogeneous Lie polynomials $F_j(A, B)$ and $F_k(A, B)$, we can express $G_i\big(F_j(A, B), F_k(A, B)\big)$ as a sum of monomials of $F_j$ and $F_k$, namely $G_j$ and $G_k$.

Therefore we prove the result for $G_i(A, B), G_j(A, B)$ and $G_k(A, B)$: consider the binary trees corresponding to each monomial, say $t_i, t_j$ and $t_k$. We first consider $t_i$. Remove all the vertices labeled $A$ in $t_i$ and replace them with that many copies to the tree $t_j$ (with each's root positioned in place of the removed vertices). Similarly replace $t_i$'s vertices labeled $B$ with the trees $t_k$. Then it is easy to see that we construct a binary tree and hence it corresponds to a Lie polynomial (more specifically monomial), as required.

This procedure is applied across all monomials of the Lie polynomials $F_i, F_j$ and $F_k$ and summing the resulting Lie polynomials gives a Lie polynomial, say $F(A, B) = F_i\Big(F_j(A, B), F_k(A, B)\Big)$, as required. □

[13] This result was stated in the MATH426 notes.

## 5 Eichler's proof of the existence of the Baker-Campbell-Hausdorff formula

Finally, in this section we prove the result in the first line of the text, but first we need to address some formalities.

**Definition 5.1.** Given two expressions, say $G_1(A,B)$ and $G_2(A,B)$, we write the relation $G_1(A,B) \sim G_2(A,B)$ to mean that $G_1(A,B) - G_2(A,B)$ gives a Lie polynomial.

Note that $G_1(A,B)$ is a Lie polynomial if and only if $G_1(A,B) \sim 0$.

**Lemma 5.2.** $\big(\exp(A)\exp(B)\big)\exp(C) = \exp(A)\big(\exp(B)\exp(C)\big)$ for all $A, B, C \in \mathfrak{gl}(N,\mathbb{R})$.

*Proof.* $\exp(A) \in GL(N,\mathbb{R})$ for all $A \in \mathfrak{gl}(N,\mathbb{R})$[14]; as $GL(N,\mathbb{R})$ is associative, the equality holds. □

**Theorem 5.3** (Existence of the Baker-Campbell-Haussdorff formula)**.** For all $A, B \in \mathfrak{gl}(N,\mathbb{R})$, $\exp(A)\exp(B) = \exp(D)$ with

$$D = \sum_{n\in\mathbb{N}} F_n(A,B)$$

where for all $n$, $F_n(A,B)$ is a homogeneous Lie polynomial of degree $n$ exclusively in $A$ and $B$. [1]

*Proof.* We already established in proposition 3.1 that $F_n$ is a homogeneous matrix polynomial of degree $n$ for all $n$. Therefore we only need to prove that $F_n$ is a Lie polynomial, and it follows that it is precisely a homogeneous Lie polynomial of degree $n$, by lemma 4.12.

We begin by considering consider the equality $\exp(W) = \big(\exp(A)\exp(B)\big)\exp(C) = \exp(A)\big(\exp(B)\exp(C)\big)$. We apply proposition 3.1 twice and get:

$$\begin{aligned}
\exp(W) &= \exp\left(\sum_{j=1}^{\infty} F_j(A,B)\right)\exp(C) = \exp(A)\exp\left(\sum_{j=1}^{\infty} F_j(B,C)\right)\\
\exp(W) &= \exp\left(\sum_{i=1}^{\infty} F_i\left(\sum_{j=1}^{\infty} F_j(A,B), C\right)\right) = \exp\left(\sum_{i=1}^{\infty} F_i\left(A, \sum_{j=1}^{\infty} F_j(B,C)\right)\right)\\
\implies {}^{15}W &= \sum_{i=1}^{\infty} F_i\left(\sum_{j=1}^{\infty} F_j(A,B), C\right) = \sum_{i=1}^{\infty} F_i\left(A, \sum_{j=1}^{\infty} F_j(B,C)\right)
\end{aligned}$$

We now prove the statement by induction: let $X, Y \in \mathfrak{gl}(N,\mathbb{R})$. In corollary 5.2 we showed that $F_1(X,Y) = X + Y$ and $F_2(X,Y) = \frac{1}{2}[X,Y]$. Those are of homogeneous Lie polynomials of degree 1[16] and 2 respectively by definition.

Now, assume that for all $X$ and $Y$, $F_k(X,Y)$ is a homogeneous $k$-degree Lie polynomial for all natural numbers $k < n$. Now, showing the result for $n$, implies the same for all $n$, as we require.

[14] This result was proven in the MATH426 lecture notes.

[15] This implication holds because we showed in the MATH426 notes that exp is invertible.

[16] The monomial $A$ corresponds to the binary tree consisting of 1 vertex labeled $A$ and similarly for $B$.

By proposition 4.13, we know that for $i < n$, $j < n$, $F_i\big(F_j(A,B),C\big)$ and $F_i\big(A,F_j(B,C)\big)$[17] give Lie polynomials, and we have our result for all those values of $i$ and $j$. Let us denote all such $n$-degree Lie polynomials on the left hand-side as $F_L$ and $F_R$ for the right hand-side.

Now we need to consider the other possible combinations of $i$ and $j$ resulting in $n$-degree homogeneous polynomials. We can safely ignore any expressions of $W$ involving either $i \geq n+1$ or $j \geq n+1$, since we have already shown that $F_i(A,B)$ is an $i$-degree homogeneous matrix polynomial in proposition 3.1, and if that expression has an equivalent Lie polynomial, it would also need to be of degree $i \geq n+1$ as per lemma 4.12. Moreover, applying $F_j$ to it cannot decrease its degree, and so it has degree at least $n+1$.

Similarly for $j$.

Hence the only expressions which are unaccounted for (and hence have a chance of being Lie polynomials) are $F_1\big(F_n(A,B),C\big) + F_n\big(F_1(A,B),C\big)$ on the left hand-side and $F_1\big(A,F_n(B,C)\big) + F_n\big(A,F_1(B,C)\big)$ on the right hand-side.

Comparing (possible) Lie polynomial terms of degree $n$ on the left and right hand-side of $W$, we have

$$F_n(A,B) + \cancel{C} + F_n(A+B,C) + F_L = \cancel{A} + F_n(B,C) + F_n(A,B+C) + F_R.$$

Here, we use the $\sim$ relation introduced in definition 5.1, which allows us to cancel $F_L$ and $F_R$ since we know them to be Lie polynomials[18], hence

$$F_n(A,B) + F_n(A+B,C) \sim F_n(A,B+C) + F_n(B,C). \tag{5.1}$$

Now we remember that we are trying to prove that $F_n(A,B)$ is a Lie polynomial, and note that it is enough to show that $F_n(A,B) \sim 0$; we proceed to do so from the equation above algebraically; from here on out, we write $F$ instead $F_n$ to save space.

We begin by inserting $C = -B$ into (5.1), and after applying corollary 3.3, we have

$$\begin{aligned} & F(A,B) + F(A+B,B) \sim F(A,0) + F(B,-B) \\ \implies & F(A,B) + F(A+B,B) \sim \cancel{F(1\cdot A, 0\cdot A)} + \cancel{F(1\cdot B, -1\cdot B)} \\ \implies & F(A,B) + F(A+B,B) \sim 0 \\ & \implies F(A,B) \sim -F(A+B,-B). \end{aligned} \tag{5.2}$$

Next we insert $A = -B$ in (5.1) and similarly apply corollary 3.3, giving

$$\begin{aligned} \cancel{F(-B,B)} + \cancel{F(0,C)} &\sim F(-B,B+C) + F(B,C) \\ \implies 0 &\sim F(-B,B+C) + F(B,C) \\ \implies F(B,C) &\sim -F(-B,B+C). \end{aligned}$$

We now write $A$ instead of $B$ and $B$ instead of $C$ in the equation above, and we have

$$F(A,B) \sim -F(-A,A+B). \tag{5.3}$$

[17] Note that $A$ and $C$ are homogeneous Lie polynomials of degree 1, and for $i,j < k$, so are $F_i$ and $F_j$.
[18] See one line below the definition.

We now consider $F(A,B)$ and apply (5.3) to it:

$$F(A,B) \sim -F(-A, A+B).$$

Then we apply (5.2) to the right hand-side expression above, and we have:

$$-F(-A, A+B) \sim -\Big( -F\big( -A + A + B, -(A+B)\big)\Big) = F(B, -A-B).$$

We again apply (5.3) to the right hand-side and since we know the equations to be homogeneous matrix polynomials, we can apply lemma 2.3, giving

$$F(B, -A-B) \sim -F\big( -B, B + (-A-B)\big) = -F(-B-A) \overset{2.3}{=} -(-1)^n F(B,A),$$

and hence we have

$$F(A,B) \sim -(-1)^n F(B,A). \tag{5.4}$$

We now proceed by inserting $C = -\frac{1}{2}B$ in (5.1) and cancelling the right-most term via corollary 3.3:

$$\begin{aligned} F(A,B) + F\Big(A+B, -\frac{1}{2}B\Big) &\sim F\Big(A, \frac{1}{2}B\Big) + \cancel{F\Big(B, -\frac{1}{2}B\Big)} \\ \implies F(A,B) &\sim F\Big(A, \frac{1}{2}B\Big) - F\Big(A+B, -\frac{1}{2}B\Big) \end{aligned} \tag{5.5}$$

Now apply $A = -\frac{1}{2}B$ to (5.1) and cancelling similarly via corollary 3.3 we have:

$$\begin{aligned} \cancel{F\Big(-\frac{1}{2}A, B\Big)} + F\Big(\frac{1}{2}B, C\Big) &\sim F\Big(-\frac{1}{2}B, B + C\Big) + F(B,C) \\ \implies F(B,C) &\sim F\Big(\frac{1}{2}B, C\Big) - F\Big(-\frac{1}{2}B, B+C\Big), \end{aligned}$$

and re-labeling with $A$ instead $B$ and $B$ instead of $C$, we have

$$F(A,B) \sim F\Big(\frac{1}{2}A, B\Big) - F\Big(-\frac{1}{2}A, A+B\Big). \tag{5.6}$$

Now we apply (5.5) to the right hand-side of (5.6):

$$F(A,B) \sim F\Big(\frac{A}{2}, \frac{B}{2}\Big) - F\Big(\frac{A}{2} + B, -\frac{B}{2}\Big) - F\Big(-\frac{A}{2}, \frac{A}{2} + \frac{B}{2}\Big) + F\Big(\frac{A}{2} + B, -\frac{A}{2} - \frac{B}{2}\Big)$$

Applying (5.2) to the second term on the right hand-side, (5.3) to the third term and (5.2) to the fourth term gives:

$$F(A,B) \sim F\Big(\frac{A}{2}, \frac{B}{2}\Big) + F\Big(\frac{A}{2} + \frac{B}{2}, \frac{B}{2}\Big) + F\Big(\frac{A}{2}, \frac{B}{2}\Big) - F\Big(\frac{B}{2}, \frac{A}{2} + \frac{B}{2}\Big).$$

Now we again apply lemma 2.3 due to $F$'s homogeneity, and we have

$$\begin{aligned} F(A,B) &\sim \Big(\frac{1}{2}\Big)^n F(A,B) + \Big(\frac{1}{2}\Big)^n F(A+B,B) + \Big(\frac{1}{2}\Big)^n F(A,B) - \Big(\frac{1}{2}\Big)^n F(B, A+B) \\ \implies F(A,B) &\sim 2(2^{-n})F(A,B) + 2^{-n}F(A+B,B) - 2^{-n}F(B,A+B) \\ \implies (1 - 2^{-n+1})F(A,B) &\sim 2^{-n}F(A+B,B) - 2^{-n}F(B,A+B). \end{aligned}$$

Now we apply (5.4) to the right-most term and factor the right side:

$$\begin{aligned}
(1-2^{-n+1})F(A,B) &\sim 2^{-n}F(A+B,B) - 2^{-n}\big(-(-1)^n F(A+B,B)\big)\\
(1-2^{-n+1})F(A,B) &\sim \big(2^{-n}+2^{-n}(-1)^n\big)F(A+B,B)\\
(1-2^{-n+1})F(A,B) &\sim 2^{-n}\big(1+(-1)^n\big)F(A+B,B).
\end{aligned}$$

In the case that $n$ is odd, we see that the coefficient of $F(A,B)$ is non zero while the coefficient of $F(A+B,B)$ becomes precisely 0 since $1+(-1)^n = 1-1 = 0$, and:

$$(1-2^{-n+1})F(A,B) \sim 0 \implies F(A,B) \sim 0, \text{ as required.}$$

Now assume $n$ is even for the other case; then we simplify $1+(-1)^n = 1+1 = 2$, and insert $A-B$ instead of $A$:

$$(1-2^{-n+1})F(A-B,B) \sim 2^{-n}(2)F(A,B),$$

then we apply (5.2) to the left hand-side and simplify the right hand-side, giving:

$$-(1-2^{-n+1})F(A,-B) \sim 2^{-n+1}F(A,B).$$

Then, we switch the left and right hand-sides and multiply both sides by $2^{n-1}$:

$$\cancel{2^{n-1}2^{-n+1}}F(A,B) \sim -2^{n-1}(1-2^{-n+1})F(A,-B) = (-2^{n-1}+\cancel{2^{n-1}2^{-n+1}})F(A,-B),$$

which simplifies to

$$F(A,B) \sim (1-2^{n-1})F(A,-B). \tag{5.7}$$

Finally, we apply (5.7) to the right hand side of the above equation and we have:

$$(1-2^{n-1})F(A,-B) \sim (1-2^{n-1})^2F(A,B)$$

and hence,

$$F(A,B) \sim (1-2^{n-1})^2F(A,B).$$

As we have already established $F_1$ and $F_2$ to be Lie polynomials, we know that here $n \geq 3$, and so $(1-2^{n-1})^2 \neq 1$, meaning that in the following relation, the left hand-side coefficient is non-zero:

$$\begin{aligned}
\Big(1-(1-2^{n-1})^2\Big)&F(A,B) \sim 0\\
&\implies F(A,B) \sim 0, \text{ as required.}
\end{aligned}$$

□

# References


[1] Eichler, M. (1968). A new proof of the Baker-Campbell-Hausdorff formula. *J. Math. Soc. Japan*, 20:23–25. 1, 7