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1. Firstly let us treat the contingent claims C and P as a single contingent claim, $C - P$. Then at their maturity T, we have

$$C_T = \max\{S_T - K, 0\} \text{ and } P_T = \max\{K - S_T, 0\}.$$

Then there are only two options, either $S_T > K$ or $S_T \leq K$. In both cases, $C - P$ gives payoff,

$$C_T - P_T = S_T - K.$$

We see that this contingent claim is replicable, since it corresponds to the strategy, ϕ, where $\alpha_t = 1, \beta_t = -\frac{K}{B_0}$ for all $t \in \{1, \cdots, T\}$, which also has payoff

$$1 \cdot S_T - \frac{K}{B_0} B_T = S_T - K$$

at time T (note that $B_0 = B_1 = \cdots = B_T$, since we have $r = 0$).

Then, as $C - T$ is replicable, and as we are in an arbitrage free market, by Theorem 4.4.1. we have that

$$C_t - P_t = \frac{1}{(1+r)^{T-t}} \mathbb{E}_{p*} \left[C - P \mid (S_s)_{s=0}^t \right] = V_t(\phi)$$

but since $r = 0$ by definition of the question,

$$C_t - P_t = \mathbb{E}_{p*} \left[C - P \mid (S_s)_{s=0}^t \right] = V_t(\phi)$$

By Lemma 3.3.17., since ϕ is self-financing, we have that

$$V_t^*(\phi) := \frac{V_t(\phi)}{(1+r)^t} = V_t(\phi) = \mathbb{E}_{p*} \left[C - P \mid (S_s)_{s=0}^t \right]$$

is a martingale, and hence,

$$\mathbb{E}_{p*} \left[C - P \mid (S_s)_{s=0}^t \right] = \max\{S_t - K, 0\} - \max\{K - S_t, 0\}.$$

Now, either $S_t > K$ (in which case $C_t = S_t - K$ and $P_t = 0$), or otherwise, $S_t \leq K$ (and hence $C_t = 0$ and $P_t = K - S_t$). Hence in either case,

$$\begin{aligned} C_t - P_t &= p(S_t - K) - (1-p)(K - S_t) \\ &= p(S_t - K) + (1-p)(S_t - K) \\ &= (S_t - K)(p + 1 - p) \\ &= S_t - K, \text{ as required.} \end{aligned}$$

2. We have that $B_0 = 1$ and $r = 0$, hence $B_1 = 1$, since $B_t = B_0(1+r)^t$. Moreover, we are given $K = 7$. Therefore plugging in these values into the bottom-most equation of the exercise sheet, we have

$$\beta_1 + \alpha_1^1 S_1^1 + \alpha_1^2 S_1^2 = \max\{7 - S_1^1, 0\}. \tag{0.1}$$

We now write this equation for all possible events in the same space:

$$\omega_1 \text{ occurs} \implies \beta_1 + 12\alpha_1^1 + 6\alpha_1^2 = \max\{7 - 12, 0\} = 0, \tag{0.2}$$

$$\omega_2 \text{ occurs} \implies \beta_1 + 6\alpha_1^1 + 10\alpha_1^2 = \max\{7 - 6, 0\} = 1, \tag{0.3}$$

$$\omega_3 \text{ occurs} \implies \beta_1 + 6\alpha_1^1 + 3\alpha_1^2 = \max\{7 - 6, 0\} = 1. \tag{0.4}$$

The question requires that (0.1) is always satisfied, hence we need it to be satsfied for all 3 events, hence (0.2), (0.3) and (0.4) need to be satisfied by some $\beta_1, \alpha_1^1, \alpha_1^2 \in \mathbb{R}$. Firstly, we subtract (0.4) from (0.3) which gives the equality

$$(1-1)\beta_1 + (6-6)\alpha_1^1 + (10-3)\alpha_1^2 = 1 - 1 \iff 7\alpha_1^2 = 0 \iff \alpha_1^2 = 0.$$

Then we subtract (0.2) from (0.3), and using that $\alpha_1^2 = 0$ we get:

$$(1-1)\beta_1 + (6-12)\alpha_1^1 + (10-6)\cdot 0 = 1 - 0 \iff -6\alpha_1^1 = 1 \iff \alpha_1^1 = -\frac{1}{6}.$$

Finally, we plug the values for $\alpha_1^1 and \alpha_1^2$ into 0.2, giving:

$$\beta_1 + 12 \cdot -\frac{1}{6} + 6 \cdot 0 = 0 \iff \beta_1 = 2.$$

Hence we have $\beta_1 = 2, \alpha_1^1 = -\frac{1}{6}, \alpha_1^2 = 0$ for the replicating strategy.

Note, apologies if this is a little pedantic, but in order to submit the coursework, the following tickbox below has to be checked; we would like to make it clear that both of us (Ventsi and Stephen) collaborated on these questions, as encouraged by this course.