Recurrence and transience of discrete Markov chains and Pólya's result for random walks

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Abstract

This dissertation is about discrete Markov chains, focusing on George Pólya's result that 1 and 2 dimensional random walks are certain to return to their starting point while those of higher dimensions are not. We cover key notions of discrete Markov chains, most notably recurrence and null-recurrence, followed by the local limit theorem, which allows us to analyse infinite chains (e.g. walks). Then we build up to Pólya's theorem by introducing walks and combinatoric results essential for its proof, and then prove the theorem exhaustively. Finally, we show that despite the 1 and 2 dimensional walks' certainty of return, the expected time is infinite.

Contents

0 Introduction

In this text, I will assume that the reader has completed the compulsary probability courses MATH103 and MATH230 from years 1 and 2.

For the majority of probabilistic results and definitions up to section 5, I have used Grimmett and Stirzaker (1982) (in sections 1 through 3) and Borovkov (2013) (in sections 2 through 5), and Kozdron (1998) for Pólya's theorem on the recurrence of walks in n-dimensions, Theorem 6.6. In all cases, I have read their proofs, understood their arguments and argued them in my own words, almost always finding myself clarifying steps or algebra that they had glanced over.

All other references I have cited serve as justification to back up statements I have made.

In all instances where a source has informed a result/proof, or the source has been used to justify a statement, this is stated in the particular place where that occurred. Note, that this does not apply to definitions, as often they are commonly agreed upon (and in cases when they are not, it is mentioned) and constantly citing definitions' sources would clutter up the text.

1 Notation and basics

Taking a first look at stochastic processes, it might be useful to begin by disambiguating between some terminology which is sometimes mistakenly used interchangeably.

Definition 1.1. A *stochastic process*, X, is an indexed family (most often by time, $t \in T$) of random variables, $\{X_t\}_{t\in T}$, and we will think of its *state space, I,* as the set of all possible values X_t can take across all $t \in T$.

This first definition is mostly for the sake of setting the notation that will be used in this text. Otherwise, general stochastic processes will be of little interest to us in this text, and we will instead be focusing on stochastic processes obeying the Markov property.

Definition 1.2. A stochastic process, X, with index set $T \subseteq \mathbb{Z}_+$[1] and discrete state space I satisfies the *Markov property*, or *Markov condition*, if we have that

$$\mathbb{P}(X_t = x_t \mid X_{t-1} = x_{t-1}) = \mathbb{P}(X_t = x_t \mid X_{t-1} = x_{t-1}, X_{t-2} = x_{t-2}, \dots, X_0 = x_0)$$

[1] Define $\mathbb{Z}_+ := \{r \in \mathbb{Z} : r \geq 0\}$.

for all $t \in T$ and for all $x_0, \ldots, x_t$. That is, X_t only depends on X_{t-1}.

For stochastic processes with no restriction on the index set or state space we only briefly discuss the idea of this definition.

For the index set, the obstacle is when it is uncountable, since for some $t \in T$ we cannot find a 'closest' $s \in T$ (which is what Definition 1.2 depends on). We overcome this by tweaking the idea of the definition as follows: given any $s, t \in T$, such that $s < t$, we have that

$$\mathbb{P}(X_t \mid X_s) = \mathbb{P}(X_t \mid X_s = x_s, X_r = x_r) \text{ for all } r < s \in T.$$

The obstacle we encounter with continuous state spaces is that the probability of a random variable being equal to any particular state is 0, and hence the Markov property, as written above, would hold for all random processes trivially.

Definition 1.3. A *Markov process* is a stochastic process which satisfies the Markov property.

Definition 1.4. A *Markov chain* is a Markov process with index set, $T = \mathbb{Z}_+$, and a continuous state space, I.

Definition 1.5. A *discrete Markov chain* is a Markov process with index set, $T = \mathbb{Z}_+$, and discrete state space, I.

In this text, we focus exclusively on discrete Markov chains, which we will commonly denote as X, indexed by $T = \mathbb{Z}_+$ and with state space I. Often we take $I = \mathbb{Z}^d$ with $d \in \{1, 2, \ldots\}$.

Definition 1.6. *The state space of a discrete Markov chain,* X, can be defined as the set

$$I = \{i : \exists t \in T \text{ such that } \mathbb{P}(X_t = i) > 0\}.$$

Definition 1.7. A discrete Markov chain, X, is (time) *homogeneous* if

$$\mathbb{P}(X_1 = i \mid X_0 = j) = \mathbb{P}(X_{t+1} = i \mid X_t = j)$$

for all $t \in T$ and $i, j \in I$. In other words, the transition probability of any given state in X is independent of time.

It may be useful to consider a few examples.

Remark 1.8. Recall that a binomial distribution with n trials and probability of success p is denoted by $B(n, p)$ and is formed by taking the sum of n Bernoulli trials giving 1 with probability p (success) and 0 with probability $q = 1 - p$ (failure).

Example 1.9. Let $\{X_t \mid t \in \mathbb{Z}_+\}$ be a family of i.i.d. random variables distributed by $B(1, \frac{1}{2})$. Clearly this is a stochastic process, since this is an indexed family of random variables, but it is also a Markov process, since it satisfies the Markov property; the random variable's distribution is entirely independent of X, so

$$\begin{aligned}&\mathbb{P}(X_t = x_t) = \mathbb{P}(X_t = x_t \mid X_{t-1} = x_{t-1})\\ =&\mathbb{P}(X_t = x_t \mid X_{t-1} = x_{t-1}, X_{t-2} = x_{t-2}, \dots, X_0 = x_0).\end{aligned}$$

Moreover, this is a discrete Markov chain since $T = \mathbb{Z}_+$ and $I = \{0, 1\}$ (by Definition 1.6), which is discrete.

Finally, this is also a homogeneous discrete Markov chain, since X_t is distributed identically for all t.

While this example fits our framework quite well, it is a very trivial example and of little interest to us.

Example 1.10. Let

$$X_0, X_1 \sim B\left(1, \frac{1}{2}\right), \text{ and } X_t \sim B\left(1, \frac{X_{t-1} + X_{t-2}}{2}\right)$$

for all $t \in \mathbb{Z}_+$ such that $t \geq 2$. Clearly, this is again a stochastic process, however it does not satisfy the Markov property: assume we have $X_1 = 0$, then if $X_0 = 0$,

$$(X_2 \mid X_1 = 0) \sim B(1, 0) \implies \mathbb{P}(X_2 = 0 \mid X_1 = 0) = 1,$$

but if $X_0 = 1$, then

$$(X_2 \mid X_1 = 0) \sim B(1, \frac{1}{2}) \implies \mathbb{P}(X_2 = 0 \mid X_1 = 0) = \frac{1}{2}.$$

As this is not a Markov process we cannot say much more about it, since our homogeneity definition only applies for Markov chains.

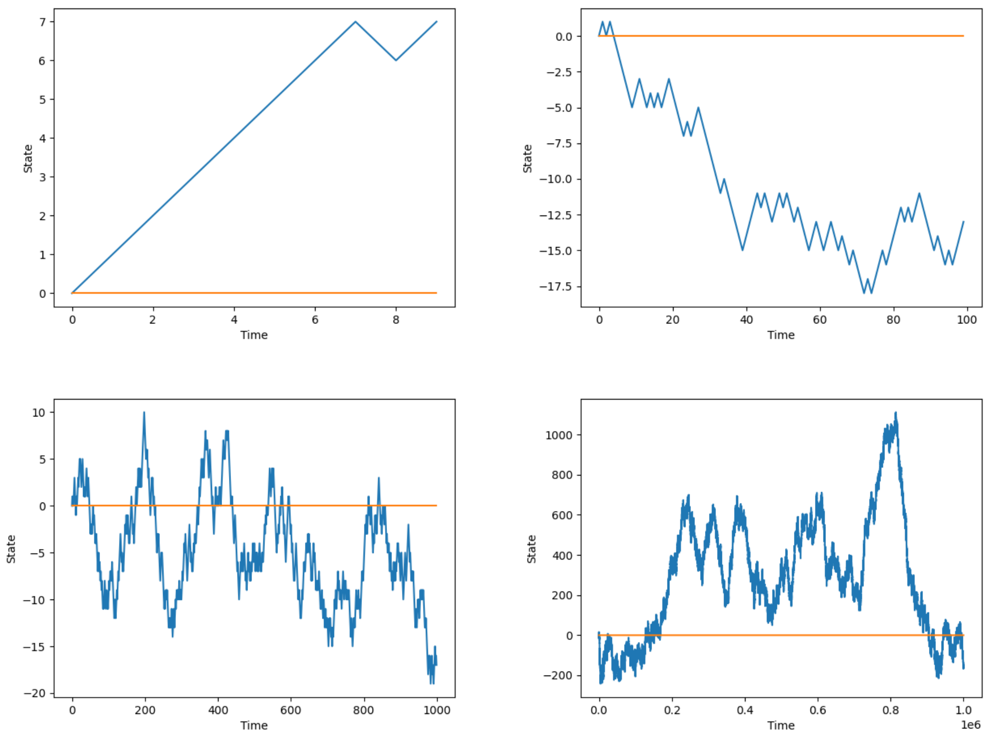
Example 1.11. Let $X_0 = 0$. Then, for all $t \in \mathbb{Z}_+$ such that $t \geq 1$, let $X_t = X_{t-1} - 1$ with probability $\frac{1}{2}$, and let $X_t = X_{t-1} + 1$ with probability $\frac{1}{2}$. Again, this is clearly a stochastic process, and the distribution of X at each t only depends on X_{t-1}, hence it satisfies the Markov property.

By Definition 1.6, its state space, I, is the set $I = \mathbb{Z}$, which is discrete, and since T is discrete too, X is a discrete Markov chain.

Finally, this chain is homogeneous since for all t, X_t does not depend on time, only on the state before it.

This is a fundamental discrete Markov chain, and is called the 1-dimensional symmetric random walk. We will discuss walks extensively later in this text.

Example 1.12. Here are 4 plots of 4 different 1-dimensional symmetric random walks, showing the chains up to time 9, 99, 999 and 999999 respectively (an orange line along the x-axis is also plotted for reference):



The Python code that was used to generate and plot these walks can be found in Appendix A.

For the rest of this text, homogeneity will be assumed for all further chains, unless stated otherwise. We do this because in many cases they more accurately model real life scenarios than non-homogeneous chains. This is also quite convenient for us, because these chains are more well-behaved and hence we can make more general statements about them and the notation is cleaner.

With this in mind, we introduce the following notation for all $t \in \mathbb{Z}_+$:

$$p_{ij}(t) := \mathbb{P}(X_{s+t} = i \mid X_s = j) \text{ and, } p_{ij} = \sum_{t \in \mathbb{N}} p_{ij}(t), \text{ for some } s \in T. \tag{1.1}$$

Note that $p_{ij}(t)$ is independent of s and hence only makes sense for homogeneous chains. Most often, when we write out $p_{ij}(t)$ fully, we set $s = 0$ for simplicity.

Also note that we define $\mathbb{N} = \{1, 2, 3, \dots\}$, and hence p_{ij} is a summation excluding $t = 0$.

Theorem 1.13 (Chapman-Kolmogorov equations)**.**

$$p_{ij}(t + r) = \sum_{k \in I} p_{ik}(t) p_{kj}(r).$$

This theorem and its proof can be found in e.g. Grimmett and Stirzaker (1982) as Theorem 6.1.7 on page 215, and the proof given here clarifies those ideas.

Proof. Writing out $p_{ij}(t + r)$ fully, we have

$$p_{ij}(t + r) = \mathbb{P}(X_{t+r} = i \mid X_0 = j).$$ [2]

On the right hand side of the expression in the theorem, we have a summation over the state space, so it may be a good idea to express $p_{ij}(t+r)$ as such a summation too:

$$p_{ij}(t + r) = \sum_{k \in I} \left(\mathbb{P}(X_{t+r} = j, X_t = k \mid X_0 = i) \right).$$

Now, recall that by definition,

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \implies \mathbb{P}(A \cap B) = \mathbb{P}(B)\mathbb{P}(A \mid B).$$

Then conditioning both sides on C, we have that

$$\mathbb{P}(A \cap B \mid C) = \mathbb{P}(B \mid C)\mathbb{P}(A \mid B \cap C), \text{ and hence,}$$

$$p_{ij}(t + r) = \sum_{k \in I} \left(\mathbb{P}(X_t = k \mid X_0 = i)\mathbb{P}(X_{t+r} = j \mid X_t = k, X_0 = i) \right).$$

[2] Here we take X_{t+r} and X_0 but as X is homogeneous, we can take X_{s+t+r} and X_s for any s.

Now using the Markov property, we can simplify the expression to

$$p_{ij}(t+r) = \sum_{k \in I} \left(\mathbb{P}(X_t = k \mid X_0 = i) \mathbb{P}(X_{t+r} = j \mid X_t = k) \right),$$

and by homogeneity, $\mathbb{P}(X_{t+r} = j \mid X_t = k) = \mathbb{P}(X_r = j \mid X_0 = k)$, giving

$$p_{ij}(t+r) = \sum_{k \in I} p_{ik}(t) p_{kj}(r)$$

□

Now let us introduce the notation,

$$f_{ij}(t) := \mathbb{P}(X_t = i, X_{t-1} \neq i, X_{t-2} \neq i, \ldots, X_1 \neq i \mid X_0 = j).$$

Note that by homogeneity, adding some $s \in \mathbb{N}$ to all indexing integers on the right hand-side does not change the probability.

Moreover, we define,

$$f_{ij} := \sum_{t \in \mathbb{N}} f_{ij}(t).$$

2 Reccurence and transience

Definition 2.1. A state $i \in I$ is *recurrent* (or *persistent*) if and only if $f_{ii} = 1$. Otherwise the state is *transient*.

Remark 2.2. This definition aligns with our intuition: assume a state is recurrent, the term 'recurrent state' carries the implicaton that we must necessarily return to this state at some point, hence the probability that we return for the first time at some point in time must be 1, and hence the summation equals 1.

On the other hand, assume that the summation equals 1. Then that implies that we return for the first time at some point in time with probability 1, and hence the state recurs.

The proof of the following theorem (and hence its corollary) relies on generating functions, so we define that before stating the theorem:

Definition 2.3. The *generating function* of some indexed set, $\{a_n\}_{n \in \mathbb{N}}$ with $a_i \in \mathbb{R}$ $\forall i \in \mathbb{N}$, is the function

$$G_a(z) = \sum_{i \in \mathbb{N}} a_i z^i \text{ over } z \in \mathbb{R} \text{ such that } G_a(z) \text{ converges.}$$

Theorem 2.4. A state $i \in I$ in a chain X, is recurrent if and only if

$$p_{ii} = \sum_{t \in \mathbb{N}} p_{ii}(t) = \infty.$$

Theorem 2.4, Corollary 2.5 and their proofs can be found in Borovkov (2013) as Theorem 13.2.1 on page 395, and the proof given here clarifies those ideas.

Proof. We start by writing out $p_{ii}(t)$ as it is defined in equation (1.1), taking $s = 0$ for simplicity (X is homogeneous so we can take any value of s):

$$p_{ii}(t) = \mathbb{P}(X_t = i \mid X_0 = i).$$

Recall the law of total conditional probability, which states that given any countable partition of the sample space, say $\beta = \{B_1, B_2, \ldots\}$, we have

$$\mathbb{P}(A \mid C) = \sum_{B_k \in \beta} \left(\mathbb{P}(B_k \mid C)\mathbb{P}(A \mid B_k \cap C)\right). \tag{2.1}$$

Now consider the partition,

$$\{(X_s = i, X_{s-1} \neq i, \ldots, X_1 \neq i) : s \in \{1, \ldots, t\}\} \cup \{(X_t \neq i, X_{t-1} \neq i, \ldots, X_1 \neq i)\}.$$

This covers all possibilities, and all events are mutually exclusive, hence this is a partition. Note that for all $s \in \{1, \ldots, t\}$,

$$\mathbb{P}\left((X_s = i, X_{s-1} \neq i, \ldots, X_1 \neq i) \mid (X_0 = i)\right) =: f_{ii}(s),$$

which is precisely $\mathbb{P}(B_k \mid C)$ from 2.1 when we apply the law of total probability to $p_{ii}(t)$. Moreover, we can apply the Markov property to the $\mathbb{P}(A \mid B_k \cap C)$ term. This gives

$$\begin{aligned}
p_{ii}(t) =& f_{ii}(1)\mathbb{P}(X_t = i \mid X_1 = i) + f_{ii}(2)\mathbb{P}(X_t = i \mid X_2 = i) + \ldots \\
&+ f_{ii}(t-1)\mathbb{P}(X_t = i \mid X_{t-1} = i) + f_{ii}(t)\mathbb{P}(X_t = i \mid X_t = i) \\
&+ \mathbb{P}(X_t \neq i, X_{t-1} \neq i, \ldots, X_1 \neq i)\mathbb{P}(X_t = i \mid X_t \neq i) \\
=& f_{ii}(1)p_{ii}(t-1) + f_{ii}(2)p_{ii}(t-2) + \ldots + f_{ii}(t-1)p_{ii}(1) + f_{ii}(t) \cdot 1 \\
&+ \mathbb{P}(X_t \neq i, X_{t-1} \neq i, \ldots, X_1 \neq i) \cdot 0 \\
=& f_{ii}(1)p_{ii}(t-1) + f_{ii}(2)p_{ii}(t-2) + \ldots + f_{ii}(t-1)p_{ii}(1) + f_{ii}(t).
\end{aligned} \tag{2.2}$$

Treating $p_{ii}(t)$ and $f_{ii}(t)$ as series indexed by $t \in \mathbb{N}$, we look to their generating functions: (we ommit the $t = 0$ term)

$$G_{p_{ii}}(z) = \sum_{t \in \mathbb{N}} p_{ii}(t)z^t, \text{ and, } G_{f_{ii}}(z) = \sum_{t \in \mathbb{N}} f_{ii}(t)z^t.$$

We multiply (2.2) by z^t on both sides and we have:

$$p_{ii}(t)z^t = f_{ii}(1)p_{ii}(t-1)z^t + f_{ii}(2)p_{ii}(t-2)z^t + \ldots + f_{ii}(t-1)p_{ii}(1)z^t + f_{ii}(t)z^t$$

Now we sum over $t \in \mathbb{N}$ and clearly the left hand-side becomes precisely $G_{p_{ii}}(z)$, and we have

$$G_{p_{ii}}(z) = p_{ii}(1)z + p_{ii}(2)z^2 + p_{ii}(3)z^3 + p_{ii}(4)z^4 + \ldots,$$

and writing out each summand individually, we have:

$$\begin{aligned}
p_{ii}(1)z &= zf_{ii}(1) \text{ (since } p_{ii}(0) = 1),\\
p_{ii}(2)z^2 &= z^2 f_{ii}(1)p_{ii}(1) + z^2 f_{ii}(2),\\
p_{ii}(3)z^3 &= z^3 f_{ii}(1)p_{ii}(2) + z^3 f_{ii}(2)p_{ii}(1) + z^3 f_{ii}(3),\\
p_{ii}(4)z^4 &= z^4 f_{ii}(1)p_{ii}(3) + z^4 f_{ii}(2)p_{ii}(2) + z^4 f_{ii}(3)p_{ii}(1) + z^4 f_{ii}(4),\\
&\vdots\\
p_{ii}(t)z^t &= z^t f_{ii}(1)p_{ii}(t-1) + z^t f_{ii}(2)p_{ii}(t-2) + \ldots + z^t f_{ii}(t-1)p_{ii}(1) + z^t f_{ii}(t),\\
&\vdots
\end{aligned}$$

Hence a clear pattern emerges: summing all of the expressions above and bracketing terms with equal values of t for $f_{ii}(t)$, we have

$$\begin{aligned}
G_{p_{ii}}(z) =&(zf_{ii}(1) + z^2 f_{ii}(1)p_{ii}(1) + \ldots + z^t f_{ii}(1)p_{ii}(t-1) + \ldots)\\
&+ (z^2 f_{ii}(2) + z^3 f_{ii}(2)p_{ii}(1) + \ldots + z^{t+1} f_{ii}(2)p_{ii}(t-1) + \ldots)\\
&+ (z^3 f_{ii}(3) + z^4 f_{ii}(3)p_{ii}(1) + \ldots + z^{t+2} f_{ii}(3)p_{ii}(t-1) + \ldots)\\
&\vdots\\
&+ (z^t f_{ii}(t) + z^{t+1} f_{ii}(t)p_{ii}(1) + \ldots + z^{t+t-1} f_{ii}(t)p_{ii}(t-1) + \ldots)\\
&\vdots
\end{aligned}$$

Now we factorise each bracket by $z^t f_{ii}(t)$ for $t \in \mathbb{N}$ and we have

$$\begin{aligned}
G_{p_{ii}}(z) =&zf_{ii}(1)(1 + zp_{ii}(1) + z^2 p_{ii}(2) + \ldots + z^{t-1}p_{ii}(t-1) + \ldots)\\
&+z^2 f_{ii}(2)(1 + zp_{ii}(1) + z^2 p_{ii}(2) + \ldots + z^{t-1}p_{ii}(t-1) + \ldots)\\
&+z^3 f_{ii}(3)(1 + zp_{ii}(1) + z^2 p_{ii}(2) + \ldots + z^{t-1}p_{ii}(t-1) + \ldots)\\
&\quad\vdots\\
&+z^t f_{ii}(t)(1 + zp_{ii}(1) + z^2 p_{ii}(2) + \ldots + z^{t-1}p_{ii}(t-1) + \ldots)\\
&\quad\vdots
\end{aligned}$$

Now notice that

$$(1 + zp_{ii}(1) + z^2 p_{ii}(2) + \ldots + z^{t-1} p_{ii}(t-1) + \ldots) = (1 + G_{p_{ii}}(z)).$$

Hence we have,

$$\begin{aligned} G_{p_{ii}}(z) =& z f_{ii}(1)(1 + G_{p_{ii}}(z)) + z^2 f_{ii}(2)(1 + G_{p_{ii}}(z)) + z^3 f_{ii}(3)(1 + G_{p_{ii}}(z)) + \ldots \\ &+ z^t f_{ii}(t)(1 + G_{p_{ii}}(z)) \\ =& (1 + G_{p_{ii}}(z))(z f_{ii}(1) + z^2 f_{ii}(2) + z^3 f_{ii}(3) + \ldots + z^t f_{ii}(t)) \end{aligned}$$

and now notice that we have exactly $G_{f_{ii}}(z)$ on the right hand-side, hence

$$G_{p_{ii}}(z) = (1 + G_{p_{ii}}(z)) G_{f_{ii}}(z) \iff G_{f_{ii}}(z) = \frac{G_{p_{ii}}(z)}{1 + G_{p_{ii}}(z)}. \tag{2.3}$$

Now, taking the inverse of both sides of the right implication, we have,

$$\begin{aligned} \frac{1}{G_{f_{ii}}(z)} &= \frac{1 + G_{p_{ii}}(z)}{G_{p_{ii}}(z)} = \frac{1}{G_{p_{ii}}(z)} + 1 \\ \iff \frac{1}{G_{p_{ii}}(z)} &= \frac{1}{G_{f_{ii}}(z)} - 1 = \frac{1 - G_{f_{ii}}(z)}{G_{f_{ii}}(z)} \\ \iff G_{p_{ii}}(z) &= \frac{G_{f_{ii}}(z)}{1 - G_{f_{ii}}(z)}. \end{aligned} \tag{2.4}$$

Notice that for $z = 1$, $G_{p_{ii}}(z) = p_{ii}$ and $G_{f_{ii}}(z) = f_{ii}$.

We now recall what we are trying to prove. We begin with the converse direction of the if and only if statement:

$\Longleftarrow$: We will assume that $p_{ii} = \infty$ and aim to show that i is recurrent, or in other words that $f_{ii} = 1$ (by the definition of recurrence). As $z \uparrow 1$,

$$G_{p_{ii}}(z) \to p_{ii} = \infty \implies G_{f_{ii}}(z) = \frac{G_{p_{ii}}(z)}{1 + G_{p_{ii}}(z)} \to 1.$$

Since for $z < 1$, $G_{f_{ii}}(z) < f_{ii}$, we have that $f_{ii} = 1$ and hence i is recurrent.

$\Longrightarrow$: We assume that $f_{ii} = 1$ and aim to show that $p_{ii} = \infty$. As $z \uparrow 1$, $G_{f_{ii}}(z) \to f_{ii} = 1$ and hence

$$G_{p_{ii}}(z) = \frac{G_{f_{ii}}(z)}{1 - G_{f_{ii}}(z) \to \infty}.$$

Since for $z < 1$, $G_{p_{ii}}(z) < p_{ii}$, we have that $p_{ii} = \infty$, as required. $\square$

Corollary 2.5. For a transient state, $i \in I$,

$$f_{ii} = \frac{p_{ii}}{1 + p_{ii}} \text{ and } p_{ii} = \frac{f_{ii}}{1 - f_{ii}}.$$

Proof. This comes directly from equations (2.3) and (2.4) when we set $z = 1$. $\square$

Definition 2.6. The *time of the first occurence* (also known as *first hitting time*) of i in the chain X is

$$T_i(X) = \min\{t \geq 1 : X_t = i\}.$$

For example, the probability that the first recurrence of i is at time t, for some $t \in T$, can be expressed as

$$\mathbb{P}(T_i(X) = t \mid X_0 = i) = f_{ii}(t).$$

We write $T_i(X) = \infty$ if the state i in the chain X is never reached.

Definition 2.7. The *expected recurrence time* of i, μ_i, is

$$\mu_i := \mathbb{E}(T_i(X) \mid X_0 = i).$$

If i is transient then by convention, we have

$$\mu_i = \infty,$$

and if i is recurrent then by the definition of the expectation and by the fact that $\mathbb{P}(T_i(X) = t \mid X_0 = i) = f_{ii}(t)$, we have

$$\mu_i = \mathbb{E}(T_i(X) \mid X_0 = i) = \sum_{t \in \mathbb{N}} t\mathbb{P}(T_i(X) = t \mid X_0 = i) = \sum_{t \in \mathbb{N}} t f_{ii}(t).$$

Definition 2.8. A recurrent state, i, is called *null* recurrent if $\mu_i = \infty$, otherwise it is *positive recurrent* (or *non-null).*

Remark 2.9. The idea of null recurrence sounds paradoxical since recurrence implies certainty of return and an infinite expected time of recurrence implies uncertainty of return. This is an instance where intuition misleads us, as we see in the following example.

Example 2.10. Assume there exists some Markov chain with state i such that $p_{ii}(t) = \frac{c}{t}$, for some constant $c \in (0, 1]$, for all $t \geq 2$ and $p_{ii}(1) = 0$. Then i is recurrent, by Theorem 2.4 and by the divergence of the harmonic series (and hence the harmonic series multiplied by a constant).

Then, we consider that

$$\begin{aligned} f_{ii}(t) :&= \mathbb{P}(X_t = i, X_{t-1} \neq i, X_{t-2} \neq i, \ldots, X_2 \neq i, X_1 \neq i \mid X_0 = j) \\ &= p_{ii}(t) \cdot (1 - p_{ii}(t-1)) \cdot (1 - p_{ii}(t-2)) \ldots (1 - p_{ii}(2)) \cdot (1 - 0) \\ &= \frac{c}{t} \cdot \left(1 - \frac{c}{t-1}\right) \cdot \left(1 - \frac{c}{t-2}\right) \cdots \left(1 - \frac{c}{2}\right) \cdot 1. \end{aligned}$$

Then taking the maximal value for c, $c = 1$, for all fractions but the first gives smaller product than $f_{ii}(t)$, namely:

$$\begin{aligned} f_{ii}(t) &\geq \frac{c}{t}\left(1 - \frac{1}{t-1}\right)\left(1 - \frac{1}{t-2}\right)\left(1 - \frac{1}{t-3}\right) \cdots \left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{2}\right) \cdot 1 \\ &= \frac{c}{t} \cdot \frac{t-2}{t-1} \cdot \frac{t-3}{t-2} \cdot \frac{t-4}{t-3} \cdots \frac{2}{3} \cdot \frac{1}{2} \cdot 1. \end{aligned}$$

We see that the numerator of each fraction past the first cancels with the denominator of the next, and we have:

$$= \frac{c}{t(t-1)} > \frac{c}{t^2}.$$

Then we see that,

$$\mu_i = \sum_{t \in \mathbb{N}} t f_{ii}(t) > \sum_{t \in \mathbb{N}} t \frac{c}{t^2} = c \sum_{t \in \mathbb{N}} \frac{1}{t} = \infty,$$

and hence the state is both recurrent and with infinite expected time of recurrence.

There is a problem, however, in that we simply assume that such a Markov chain exists; it may not necessarily be the case. The idea of this example is to show that a chain could possess the numerical qualities so that it is null recurrent.

Remark 2.11. Later in this text we have more concrete examples of Markov chains with a null recurrent state. (In fact all of their states are null recurrent!)

Theorem 2.12. A recurrent state, i, is null if and only if $p_{ii}(t) \to 0$ as $t \to \infty$. Otherwise, it is positive recurrent.

Corollary 2.13. For a null recurrent state, i, we have that $p_{ij}(t) \to 0$ as $t \to \infty$ for all $j \in I$.

We will use Theorem 2.12 and Corollary 2.13 without proof in this text; those statements and their proofs can be found in Grimmett and Stirzaker (1982) as Theorem 6.2.9. on page 222, with its proof on page 232.

Definition 2.14. The *period* of a state i is defined by the function $d : I \to \mathbb{N}$ such that $d(i) = \gcd(t : p_{ii}(t) > 0)$. State i is *aperiodic* if $d(i) = 1$ and *periodic* if $d(i) \geq 2$.

Example 2.15. Recall Example 1.11, where we defined the 1-dimensional symmetric random walk. Let us try to find the period of this walk. Firstly, in case that we have $X_t = X_{t-1} - 1$, let us refer to this as a left step, and if we have $X_t = X_{t-1} + 1$, let us refer to this as a right step.

Let us consider some odd t; then we have some number $t_l \leq t$ of left steps and $t_r \leq t$ or right steps. Let us assume that $t_r \leq t_l$. Then since clearly, a right step cancels a left step (and vice-versa), over time t we have moved $t - 2t_r$ units to the left. Since t is odd and $2t_r$ is even (and an odd number minus an even number gives an odd number) we have move by an odd number of steps, and hence having started at 0, it is impossible to be at 0 at time t.

We proceed symmetrically if $t_l \leq t_r$. Hence, if t is odd, then $p_{00}(t) = 0$. This is enough to conclude that the period of 0 is at least 2.

Now suppose that t is even, and let $t = 2d$. Then we have

$$\mathbb{P}(\text{First } d \text{ steps are all to the left}) = \left(\frac{1}{2}\right)^d = \mathbb{P}\left(\text{Last } d \text{ steps are all to the right}\right),$$

which would bring us back to 0 at at time t.

This is one way of returning to 0 in t steps, hence

$$p_{00}(t) \geq \left(\frac{1}{2}\right)^{2d} = \left(\frac{1}{2}\right)^t > 0,$$

hence the period of the 0 state in the walk is precisely 2.

Definition 2.16. A chain, X, is *recurrent* (or *persistent*) if and only if all of its states are recurrent; the chain is *transient* if and only if it is not recurrent.

3 State communication

Definition 3.1. In a chain X, the state $i \in I$ *communicates* with $j \in I$, written $i \to j$, if $p_{ij}(t) > 0$ for some $t \in T$. Moreover, if $i \to j$ and $j \to i$, then i and j *intercommunicate*, written $i \leftrightarrow j$.

Lemma 3.2. If $i \leftrightarrow j$ then there exist $t_1, t_2 \in T$ such that $p_{ij}(t_1 + k(t_1 + t_2)) > 0$ and $p_{ji}(t_2 + k(t_1 + t_2)) > 0$ for all $k \in \mathbb{Z}_+$.

Proof. $i \leftrightarrow j$ implies that there exist t_1 and t_2 such that $p_{ij}(t_1) > 0$ and $p_{ji}(t_2) > 0$. Then, the Chapman-Kolmogorv equation (applied twice) implies that

$$p_{ij}(t_1 + (t_1 + t_2)) \geq p_{ij}(t_1)p_{jj}(t_1 + t_2) \geq p_{ij}(t_1)p_{ji}(t_2)p_{ij}(t_1) \text{ and}$$

$$p_{ji}(t_2 + (t_2 + t_1)) \geq p_{ji}(t_2)p_{ii}(t_2 + t_1) \geq p_{ji}(t_2)p_{ij}(t_1)p_{ji}(t_2),$$

or more generally,

$$\begin{aligned} p_{ij}(t_1 + k(t_1 + t_2)) &\geq p_{ij}(t_1) \left(p_{ji}(t_2)p_{ij}(t_1)\right)^k \text{ and} \\ p_{ji}(t_2 + k(t_2 + t_1)) &\geq p_{ji}(t_2) \left(p_{ij}(t_1)p_{ji}(t_2)\right)^k. \end{aligned}$$

Now since $p_{ij}(t_1) > 0$ and $p_{ji}(t_2) > 0$ and since the product of strictly positive numbers is strictly positive, we have that $p_{ij}(t_1+k(t_1+t_2)) > 0$ and $p_{ji}(t_2+k(t_1+t_2)) > 0$ for all $k \in \mathbb{Z}_+$ by the above inequalities. $\square$

Definition 3.3. A set of states, $I_0 \subseteq I$ within a chain X is called

irreducible if $i \leftrightarrow j$ for all $i, j \in I_0$.

Theorem 3.4. Let I_0 be irreducible and pick any $i, j \in I_0$, then the following hold:

1. i is transient $\iff$ j is transient,
2. i is null recurrent $\iff$ j is null recurrent,
3. i has preiod d $\iff$ j has period d.

Note that an equivalent re-wording of this theorem is that for any irreducible set of states, I_0, the type of a state $i \in I_0$ determines the type of any state in I_0.

This theorem and a partial proof can be found in Grimmett and Stirzaker (1982) as Theorem 6.3.2 on page 224; here we give a more extensive proof.

Proof. I_0 is irreducible, so for any $i, j \in I_0$, we have $t_1 > 0$ and $t_2 > 0$ such that $p_{ij}(t_1) > 0$ and $p_{ji}(t_2) > 0$. Then the Chapman-Kolmogorov equation (applied twice) implies that for all $r \in T$,

$$p_{ii}(t_1 + t_2 + r) \geq p_{ij}(t_1 + r)p_{ji}(t_2) \geq p_{ij}(t_1)p_{jj}(r)p_{ji}(t_2).$$

Let $\alpha = p_{ij}(t_1)p_{ji}(t_2) > 0$, hence

$$p_{ii}(t_1 + t_2 + r) \geq \alpha p_{jj}(r), \tag{3.1}$$

or equivalently,

$$p_{ii}(r) \leq \alpha p_{jj}(r - t_1 - t_2)^3 \text{ and}, \tag{3.2}$$

$$p_{jj}(r) \leq \frac{1}{\alpha} p_{ii}(t_1 + t_2 + r). \tag{3.3}$$

Similarly applying the Chapman-Kolmogorov equation to $p_{jj}(t_1 + t_2 + r)$ gives the following by symmetry:

$$p_{jj}(t_1 + t_2 + r) \geq \alpha p_{ii}(r), \tag{3.4}$$

or equivalently,

$$p_{ii}(r) \geq \frac{1}{\alpha} p_{jj}(t_1 + t_2 + r). \tag{3.5}$$

$$p_{jj}(r) \geq \alpha p_{ii}(r - t_1 - t_2) \text{ and} \tag{3.6}$$

Then by (3.2) and (3.5) we have the inequality

$$\frac{1}{\alpha} p_{jj}(t_1 + t_2 + r) \geq p_{ii}(r) \geq \alpha p_{jj}(r - t_1 - t_2) \tag{3.7}$$

$$\implies \frac{1}{\alpha} \sum_{r \in T} p_{jj}(t_1 + t_2 + r) \geq \sum_{r \in T} p_{ii}(r) \geq \alpha \sum_{r \in T} p_{jj}(r - t_1 - t_2), \tag{3.8}$$

and by (3.3) and (3.6) we have the inequality

$$\frac{1}{\alpha} p_{ii}(t_1 + t_2 + r) \geq p_{jj}(r) \geq \alpha p_{ii}(r - t_1 - t_2) \tag{3.9}$$

$$\implies \frac{1}{\alpha} \sum_{r \in T} p_{ii}(t_1 + t_2 + r) \geq \sum_{r \in T} p_{jj}(r) \geq \alpha \sum_{r \in T} p_{ii}(r - t_1 - t_2). \tag{3.10}$$

[3]For any states $i, j \in I, p_{ij}(t) = 0$ for all $t \notin T$.

1. Assume i is transient. Equivalently,

$$\sum_{r\in T} p_{ii}(r) < \infty,$$

which implies that

$$\infty > \sum_{r\in T} p_{ii}(r) \geq \sum_{r\in T} p_{ii}(t_1+t_2+r)^4 \implies \sum_{r\in T} p_{ii}(t_1+t_2+r) < \infty.$$

Hence (since multiplying by constants does not affect inequalities involving infinity) (3.10) gives that

$$\infty > \frac{1}{\alpha}\sum_{r\in T} p_{ii}(t_1+t_2+r) \geq \sum_{r\in T} p_{jj}(r) \iff j \text{ is transient.}$$

Now we assume j is transient for the converse, and symmetrically (this time using inequality (3.8)) we have that i is transient as required.

2. Assume that i is recurrent with period d_i. Firstly let us show recurrence for j. Recurrence of i implies that

$$\sum_{r\in T} p_{ii}(r) = \infty.$$

Since $T = \mathbb{Z}_+$,

$$\infty = \sum_{r\in T} p_{ii}(r) = \sum_{r\in T} p_{ii}(r-t_1-t_2).$$

Hence (since multiplying by constants does not affect equalities involving infinity), inequality (3.10) gives

$$\sum_{r\in T} p_{jj}(r) \geq \alpha \sum_{r\in T} p_{ii}(r-t_1-t_2) = \infty \iff j \text{ is recurrent.}$$

Now we assume j is recurrent for the converse, and symmetrically (this time using inequality (3.8)) we have that i is recurrent as required.

Now turning our attention to periodicity, we set $r = 0$ in inequality (3.1), and we have

$$p_{ii}(t_1+t_2) \geq \alpha \times 1 > 0 \text{ (since } \alpha > 0).$$

Now let j have period d_j, hence $p_{jj}(d_j) > 0$. By (3.9),

$$p_{ii}(t_1+t_2+d_j) \geq p_{jj}(d_j) > 0.$$

[4] For any states $i, j \in I, p_{ij}(t) = 0$ for all $t \notin T$.

Moreover, we know that state i has preiod d_i, then by definition $d_i \mid t_1 + t_2$, hence

$$d_i \mid t_1 + t_2 + d_j \implies d_i \mid d_j \text{ and } d_i \leq d_j.$$

By symmetry (instead using inequalities (3.4) and (3.7)), we have that

$$d_i \geq d_j.$$

Hence $d_i = d_j$, as required.

3. Assume i is null recurrent, that is i is recurrent, but

$$\lim_{r\to\infty} (p_{ii}(r)) = 0.$$

Then since t_1 and t_2 are constants, we also have that

$$\lim_{r\to\infty} (p_{ii}(t_1 + t_2 + r)) = 0 \text{ and } \lim_{r\to\infty} (p_{ii}(r - t_1 - t_2)) = 0.$$

Hence, by inequality (3.9), we have that as $r \to \infty$,

$$0 \geq p_{jj}(r) \geq 0 \iff p_{jj}(r) = 0.$$

Since we have already proved that recurrence of i implies recurrence of j, j is null recurrent.

$\square$

Example 3.5. Consider again the symmetric 1-dimensional walk from Examples 1.11 and 2.15. We see that any state, say i, in the walk communicates with any other state, say j. Assume without loss of generality that $i < j$. Then starting from i we can get to j in $j - i$ units of time by taking $j - i$ right steps.

Similarly we can get from j to i in $j - i$ units of time by taking $j - i$ left steps. Hence the two states intercommunicate, i.e. $i \leftrightarrow j$. Clearly also, $i \leftrightarrow i$ since we have $i \to j \to i$.

As this argument applies for any i and j, we have that all states in the walk intercommunicate. Hence the set of all states in the walk forms an irreducible set of states.

Moreover, since by Example 2.15 we have already established that the state 0 has period 2, we can apply Theorem 3.4, and we have that all states in the walk have the same period (since the whole walk is irreducible), and hence the whole chain has period 2.

4 The local limit theorem

We momentarily take a step away from Markov chains, and consider the following general results regarding limits to infinity. Since our chains are of iterative nature, it is clear that such results are useful to us.

Proposition 4.1 (Stirling's approximation)**.** As $n \to \infty$,

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n;$$

precisely, $\sim$ denotes that the ratio of the two expressions tends to 1.

This is a well known result of combinatorics (sometimes also referred to as Stirling's formula) and we will omit its proof, which can be found in Feller (1967).

Theorem 4.2 fits Proposition 4.1 in the framework of probability. That result will be instrumental for the proofs of some big results relating to walks in the next section.

Theorem 4.2 (The local limit theorem)**.** Consider a binomial random variable [5], $Y_n \sim B(n,p)$ with $p \in (0,1)$ a constant. As $k \to \infty$ and $n-k \to \infty$, we have the approximation

$$\mathbb{P}\left(\frac{Y_n}{n} = \frac{k}{n} = p^*\right) \sim \frac{1}{\sqrt{2\pi n p^*(1-p^*)}} e^{-nH(p*)}, \text{ where}$$

$$H\left(p^*\right) = p^* \ln\left(\frac{p^*}{p}\right) + \left(1-p^*\right)\ln\left(\frac{1-p^*}{1-p}\right).$$

This theorem and its proof can be found in Borovkov (2013) as Theorem 5.2.1 on page 109, and the proof given here clarifies those ideas.

Proof. Firstly we note that

$$\mathbb{P}\left(\frac{Y_n}{n} = \frac{k}{n} = p^*\right) = \mathbb{P}\left(Y_n = k\right),$$

hence we work with the right hand-side, since we can use the binomial probability mass function and write out $\binom{n}{k}$ to get:

$$\mathbb{P}\left(Y_n = k\right) = \binom{n}{k} p^k (1-p)^{n-k} = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}.$$

[5] See Remark 1.8 for a summary of Binomial random variables.

Then we apply Stirling's approximation to $n!$, $k!$ and $(n-k)!$ and we let $k \to \infty$ and $(n-k) \to \infty$ (which also imply that $n \to \infty$). To save ourselves some space, let the symbol $\sim^*$ mean that the expression left of the symbol can be approximated by the expression right of the symbol, when $k \to \infty$ and $(n-k) \to \infty$. Then we have:

$$\begin{aligned}
\mathbb{P}\left(Y_n = k\right) &\sim^* \frac{\sqrt{2\pi n}(\frac{n}{e})^n}{\sqrt{2\pi k}(\frac{k}{e})^k \sqrt{2\pi(n-k)}\left(\frac{(n-k)}{e}\right)^{(n-k)}} p^k (1-p)^{n-k} \\
&= \frac{\sqrt{2\pi n}}{\sqrt{2\pi k}\sqrt{2\pi(n-k)}} \frac{n^n e^{-n}}{k^k (n-k)^{n-k} e^{-k-n+k}} p^k (1-p)^{n-k} \\
&= \sqrt{\frac{n}{2\pi k(n-k)}} \frac{n^n}{k^k (n-k)^{n-k}} p^k (1-p)^{n-k}.
\end{aligned}$$

Now, we multiply the fraction in the square root by n^{-1} on top and bottom and we apply $\exp(\ln(\ ))$ to the rest of the expression (which we can do because $\exp(\ln(x)) = x$ for all $x \in \mathbb{R}$, $x > 0$, and one can check that each multiple, and hence the whole equation, is indeed strictly positive):

$$\mathbb{P}\left(Y_n = k\right) \sim^* \sqrt{\frac{1}{2\pi \frac{k}{n} n (1 - \frac{k}{n})}} \exp\left(\ln\left(\frac{n^n}{k^k (n-k)^{n-k}} p^k (1-p)^{n-k}\right)\right).$$

We now use that

$$n^n = n^{k+n-k} = n^k n^{n-k} \text{ and } \ln((ab)^c) = c\ln(ab) = c(\ln(a) + \ln(b)):$$

$$\begin{aligned}
\mathbb{P}\left(Y_n = k\right) \sim^* &\sqrt{\frac{1}{2\pi n \frac{k}{n}(1 - \frac{k}{n})}} \exp\left(\ln\left(\frac{n^k n^{n-k}}{k^k (n-k)^{n-k}} p^k (1-p)^{n-k}\right)\right) \\
= &\sqrt{\frac{1}{2\pi n \frac{k}{n}(1 - \frac{k}{n})}} \exp\left(\ln\left(\left(\frac{n}{k}\right)^k \left(\frac{n}{n-k}\right)^{n-k} p^k (1-p)^{n-k}\right)\right) \\
= &\sqrt{\frac{1}{2\pi n \frac{k}{n}(1 - \frac{k}{n})}} \exp\left(\ln\left(\left(\frac{k}{n}\right)^{-k} \left(\frac{n-k}{n}\right)^{-(n-k)} p^k (1-p)^{n-k}\right)\right) \\
= &\sqrt{\frac{1}{2\pi n \frac{k}{n}(1 - \frac{k}{n})}} \exp\left(-k\ln\left(\frac{k}{n}\right) - (n-k)\ln\left(1 - \frac{k}{n}\right)\right. \\
&\left. + k\ln(p) + (n-k)\ln(1-p)\right).
\end{aligned}$$

We now insert $p^* = \frac{k}{n}$ and we multiply the expression in exp() by n on top and bottom and we have,

$$\begin{aligned}\mathbb{P}\left(Y_n = k\right) \sim^* \sqrt{\frac{1}{2\pi n p^*(1-p^*)}} \exp(&-k\ln(p^*) - (n-k)\ln(1-p^*) \\ &+ k\ln(p) + (n-k)\ln(1-p))\end{aligned}$$

$$\begin{aligned}= \sqrt{\frac{1}{2\pi n p^*(1-p^*)}} \exp\Bigg(n\bigg(&-\frac{k}{n}\ln(p^*) - \left(1-\frac{k}{n}\right)\ln(1-p^*) \\ &+ \frac{k}{n}\ln(p) + \left(1-\frac{k}{n}\right)\ln(1-p)\bigg)\Bigg)\end{aligned}$$

$$\begin{aligned}= \sqrt{\frac{1}{2\pi n p^*(1-p^*)}} \exp(n(&-p^*\ln(p^*) - (1-p^*)\ln(1-p^*) \\ &+ p^*\ln(p) + (1-p^*)\ln(1-p))).\end{aligned}$$

We now factorise, and use that

$$\ln(a) - \ln(b) = \ln\left(\frac{a}{b}\right),$$

to see that,

$$\begin{aligned}\mathbb{P}\left(Y_n = k\right) \sim^* \sqrt{\frac{1}{2\pi n p^*(1-p^*)}} \exp(n(&p^*(-\ln(p^*) + \ln(p)) \\ &+ (1-p^*)(-\ln(1-p^*)\ln(1-p)))\end{aligned}$$

$$\begin{aligned}&= \sqrt{\frac{1}{2\pi n p^*(1-p^*)}} \exp\left(n\left(p^*\left(\ln\left(\frac{p}{p^*}\right)\right)\right.\right. \\ &\left.\left.+ (1-p^*)\left(\ln\left(\frac{1-p}{1-p^*}\right)\right)\right)\right) \\ &= \sqrt{\frac{1}{2\pi n p^*(1-p^*)}} \exp\left(nH(p^*)\right).\end{aligned}$$

Hence, as $k \to \infty$ and $(n-k) \to \infty$, we have the approximation

$$\mathbb{P}\left(Y_n = k\right) \sim \frac{1}{\sqrt{2\pi n p^*(1-p^*)}} e^{-nH(p*)}.$$

$\square$

5 Discrete random walks

Definition 5.1. The *discrete random walk in n-dimensions* is a discrete Markov chain, X, with index set, $T = \mathbb{Z}_+$ and state space, $I = \mathbb{Z}^n$ where for all $i \in I$, we have that

$$X_t = i = (a_1, a_2, \ldots, a_n) \implies X_{t+1} = (b_1, b_2, \ldots, b_n),$$

where for all $m \in \{1, \ldots, n\}$, we have constants, $p_m, q_m \in [0, 1]$, such that

$$p_m := \mathbb{P}(b_m = a_m + 1) \text{ and } q_m := \mathbb{P}(b_m = a_m - 1),$$

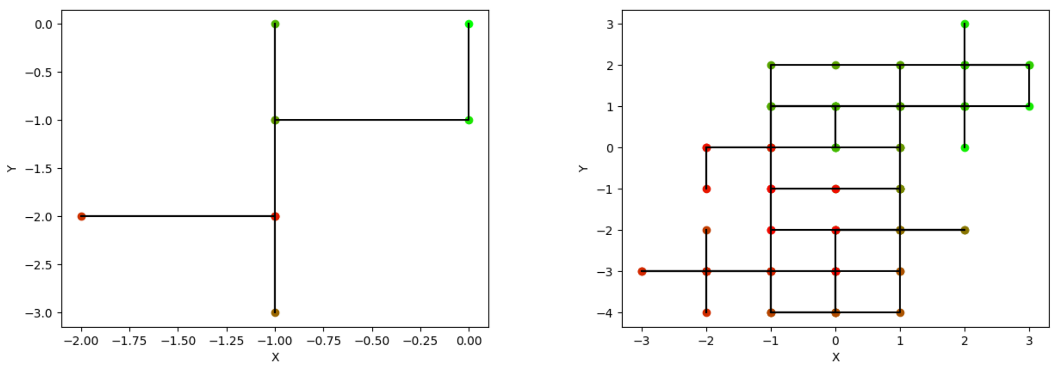
with

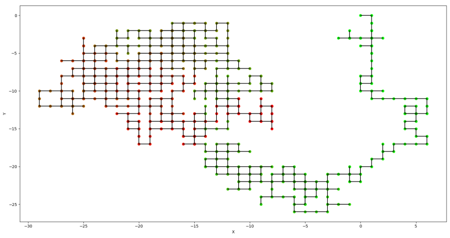
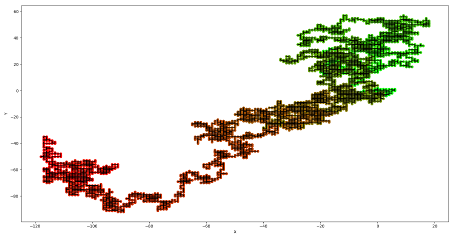
$$\sum_{m \in \{1,\ldots,n\}} (p_m + q_m) = 1.$$

Whenever we refer to p_m, q_m, they are defined as in Definition 5.1.

While some authors will choose to define it differently (e.g. see the beginning of Chapter 13.3 (page 398) in Borovkov (2013)), our definition for the random walk in n dimensions is the standard one.

Example 5.2. Here are 4 colour-coded plots of 4 different 2-dimensional symmetric random walks, showing the walks up to time 9, 99, 999 and 9999 respectively. The state of the walk at time $t = 0$ is represented by the bright green colour, and the colour apporaches bright red as we apporach time $t = 9$, $t = 99$, $t = 999$ and $t = 9999$ respectively.



The Python code that was used to generate and plot these walks can be found in Appendix B.

Definition 5.3. For an n-dimensional walk, a *path* of length t is denoted S_t, where S_t is defined such that $S_t = X_t - X_0$. Equivalently, $X_t = X_0 + S_t$.

Definition 5.4. For an n-dimensional walk, a *step* is an element of the set

$$\begin{aligned}&\big\{(a_1, \ldots, a_n) \mid \big[a_k = \pm 1, a_l = 0 \text{ for all } l \in \{1, \ldots, n\} \setminus \{k\}\big] \ \forall\, k \in \{1, \ldots, n\}\big\}\\ &= \big\{(1,0,0,\ldots,0), (0,1,0,\ldots,0), \ldots, (0,0,0,\ldots,1)\big\}.\end{aligned}$$

In other words, $S \in \{(\pm 1, 0, \ldots, 0), (0, \pm 1, \ldots, 0), \ldots, (0, 0, \ldots, \pm 1)\}$. Note that S is such that $X_0 + S = X_1$.

Note that a path of length t can be expressed a summation of t steps.

Definition 5.5. A random walk in n-dimensions, X, is *symmetric* if $p_m = q_m = \frac{1}{2n}$ for all $m \in \{1, \ldots, n\}$.

Theorem 5.6. The 1-dimensional random walk X forms a recurrent Markov chain if and only if X is symmetric.

This theorem and its proof can be found in Borovkov (2013) as Theorem 13.3.1 on page 399, and the proof given here clarifies those ideas.

Proof. Firstly, we consider the case that $p \in \{0, 1\}$; then clearly the chain can never return to its starting point, since the chain moves with certainty in one direction and hence it is not recurrent. This obeys our theorem.

Hence assume $p \in (0, 1)$, which implies that the entire state space of the walk forms an irreducible Markov chain (that is for all $i, j \in I, i \leftrightarrow j$) since $p_{ij}(t)$ can be expressed solely as a product of $p \neq 0$ and $q = 1 - p \neq 0$, which is clearly strictly positive.

Now, by Theorem 3.4 if a particular state is recurrent, then all states are recurrent and by Theorem 2.4, we have that a state $i \in I$ is recurrent $\iff \sum_{t \in \mathbb{N}} p_{ii}(t) = \infty$.

Combining these two statements, we see that "X is symmetric" $\iff \sum_{t \in \mathbb{N}} p_{00}(t) = \infty$ (we take the arbitrary point $i = 0$ for simplicity's sake). Hence we have reduced the statement in the theorem to: $p = \frac{1}{2}$ [6] $\iff \sum_{t \in \mathbb{N}} p_{00}(t) = \infty$.

We prove the statement by proving that $p = \frac{1}{2} \implies \sum_{t \in \mathbb{N}} p_{00}(t) = \infty$ and $p \neq \frac{1}{2} \implies \sum_{t \in \mathbb{N}} p_{00}(t) \neq \infty$.

Now we remember that the sum of steps, S_t is defined such that $X_t = X_0 + S_t$. $p_{00}(t)$ means that $X_t = X_0$ and re-arranging the equation, we get that $S_t = 0$. Hence we have that

$$\begin{aligned} p_{00}(t) &= \mathbb{P}(S_t = 0) \\ \implies \sum_{t \in \mathbb{N}} p_{00}(t) &= \sum_{t \in \mathbb{N}} \mathbb{P}(S_t = 0). \end{aligned} \tag{5.1}$$

Clearly since we can only move in steps 1 or -1 this means that we must have an equal number of 1 steps and -1 steps at time t.

We transform the random variable S_t into a random variable where instead of moving by 1 and -1, we instead move by 1 and 0, respectively; we call this new random variable Y_t. Note that Y_t is the sum of t Bernoulli random variables with probability of succes $p = 1 - q$.

We are interested in $\mathbb{P}(S_t = 0)$, and we notice that

$$\mathbb{P}(S_t = 0) = \mathbb{P}\left(Y_t = \frac{t}{2}\right) = \mathbb{P}\left(\frac{Y_t}{t} = \frac{1}{2}\right).$$

We can now apply Theorem 4.2 to the right-most expression to get that as $t \to \infty$[7]:

$$\mathbb{P}(S_t = 0) = \mathbb{P}\left(\frac{Y_t}{t} = \frac{1}{2}\right) \sim \frac{1}{\sqrt{2\pi t \frac{1}{2}(1 - \frac{1}{2})}} e^{-tH(\frac{1}{2})},$$

$$\text{with, } H\left(\frac{1}{2}\right) = \frac{1}{2} \ln\left(\frac{\frac{1}{2}}{p}\right) + \left(1 - \frac{1}{2}\right) \ln\left(\frac{1 - \frac{1}{2}}{q}\right).$$

[6] Note that since $p = 1 - q$, we have that $p = \frac{1}{2} \iff q = \frac{1}{2}$

[7] If we look back to Theorem 4.2, we notice that we require two conditions: $k \to \infty$ and $n - k \to \infty$. Since here we are taking $n = t$ and $k = \frac{t}{2}$, $t \to \infty$ implies the two conditions and is hence sufficient.

Simplifying, we have,

$$\begin{aligned}
\mathbb{P}(S_t = 0) &\sim \frac{1}{\sqrt{\frac{\pi t}{2}}} \exp\left(-t\left(\frac{1}{2}\ln\left(\frac{1}{2p}\right) + \frac{1}{2}\ln\left(\frac{1}{2q}\right)\right)\right) \\
&= \frac{1}{\sqrt{\frac{\pi t}{2}}} \exp\left(-t\ln\left(\frac{1}{(4pq)^{\frac{1}{2}}}\right)\right) = \frac{1}{\sqrt{\frac{\pi t}{2}}} \exp\left(\ln\left(\frac{1}{(4pq)^{\frac{-t}{2}}}\right)\right) \\
&= \frac{1}{\sqrt{\frac{\pi t}{2}}} \exp\left(\ln\left((4pq)^{\frac{t}{2}}\right)\right) = \frac{1}{\sqrt{\frac{\pi t}{2}}} (4pq)^{\frac{t}{2}}.
\end{aligned}$$

Let us define $\beta(p) = 4pq$, hence as $t \to \infty$:

$$\mathbb{P}(S_t = 0) \sim \frac{\beta(p)^{\frac{t}{2}}}{\sqrt{\frac{\pi t}{2}}} \tag{5.2}$$

We notice here, that in the interval $[0, 1], \beta(p)$ has one maximum at $p = \frac{1}{2}$ and $\beta(\frac{1}{2}) = 1$. For all other values of $p \in [0, 1], \beta(p) < 1$.

We are now ready to prove the two statements we set out to prove in the beginning:

1. Assume $p = \frac{1}{2}$. Then,

$$\sum_{t \in \mathbb{N}} p_{00}(t) = \sum_{t \in \mathbb{N}} \mathbb{P}(S_t = 0).$$

As $t \to \infty$,

$$\frac{p_{00}(t)}{\frac{\beta(p)^{\frac{t}{2}}}{\sqrt{\frac{\pi t}{2}}}} \to 1$$

since the denominator is an approximation of the numerator (see (5.2)) and hence

$$\sum_{t \in \mathbb{N}} p_{00}(t) = \infty \iff \sum_{t \in \mathbb{N}} \frac{\beta(p)^{\frac{t}{2}}}{\sqrt{\frac{\pi t}{2}}} = \infty.$$

We now notice that $p = \frac{1}{2}$ implies that $\beta(p) = 1$ and hence,

$$\sum_{t \in \mathbb{N}} \frac{\beta(p)^{\frac{t}{2}}}{\sqrt{\frac{\pi t}{2}}} = \sum_{t \in \mathbb{N}} \frac{1}{\sqrt{\frac{\pi t}{2}}} = \sum_{t \in \mathbb{N}} \frac{1}{\sqrt{\frac{\pi}{2}}} \frac{1}{\sqrt{t}} = \frac{1}{\sqrt{\frac{\pi}{2}}} \sum_{t \in \mathbb{N}} \frac{1}{\sqrt{t}}. \tag{5.3}$$

By comparing our right-most summation (excluding the constant) to the Harmonic series, which diverges, we see that

$$\left[\sum_{t\in\mathbb{N}} \frac{1}{\sqrt{t}} > \sum_{t\in\mathbb{N}} \frac{1}{t} = \infty\right] f \implies \left[\sum_{t\in\mathbb{N}} \frac{1}{\sqrt{t}} = \infty\right] \iff$$
$$\left[\frac{1}{\sqrt{\frac{\pi}{2}}} \sum_{t\in\mathbb{N}} \frac{1}{\sqrt{t}} = \sum_{t\in\mathbb{N}} \frac{\beta(p)^{\frac{t}{2}}}{\sqrt{\frac{\pi t}{2}}} = \infty\right] \iff \left[\sum_{t\in\mathbb{N}} p_{00}(t) = \infty\right], \text{ as required.}$$

2. Assume $p \neq \frac{1}{2}$. Then $\beta(p) < 1$, and we again have,

$$\sum_{t\in\mathbb{N}} p_{00}(t) = \infty \iff \sum_{t\in\mathbb{N}} \frac{\beta(p)^{\frac{t}{2}}}{\sqrt{\frac{\pi t}{2}}} = \infty, \text{ or equivalently,}$$

$$\sum_{t\in\mathbb{N}} p_{00}(t) < \infty \iff \sum_{t\in\mathbb{N}} \frac{\beta(p)^{\frac{t}{2}}}{\sqrt{\frac{\pi t}{2}}} < \infty.$$

We now use the ratio test for convergence to check whether the right summation converges by evaluating the t^{th} and $(t+1)^{\text{th}}$ terms:

$$\lim_{t\to\infty} \left(\frac{\beta(p)^{\frac{t+1}{2}}}{\sqrt{\frac{\pi(t+1)}{2}}} \frac{\sqrt{\frac{\pi t}{2}}}{\beta(p)^{\frac{t}{2}}} \right) = \lim_{t\to\infty} \left(\frac{\beta(p)^{\frac{t+1}{2}}}{\beta(p)^{\frac{t}{2}}} \frac{\sqrt{\frac{\pi t}{2}}}{\sqrt{\frac{\pi(t+1)}{2}}} \right)$$
$$= \lim_{t\to\infty} \left(\beta(p)^{\frac{1}{2}} \sqrt{\frac{t}{t+1}} \right) = \lim_{t\to\infty} (\beta(p)^{\frac{1}{2}}) = \beta(p)^{\frac{1}{2}} < 1.$$

As the limit is strictly less than one, the summation converges by the ratio test, and hence

$$\sum_{t\in\mathbb{N}} p_{00}(t) < \infty, \text{ as required.}$$

□

6 Pólya's result for the recurrence of walks in n dimensions

Before we are able to prove Pólya's result, we first have to lay some combinatorical groundwork.

Lemma 6.1. The number of unordered selections, without repetition, of k_1 objects A_1, k_2 objects A_2, ..., k_m objects A_m, where $A_1, A_2, \dots, A_m$ are distinct objects, out of a total of $n = \sum_{i=1}^{m} k_i$ objects is given by the multinomial coefficient

$$\binom{n}{k_1, k_2, \dots, k_m} := \frac{(n)!}{k_1!k_2!\dots k_m!}.$$

I first encountered multinomials on page 13 in Kozdron (1998), where the above notation was written, but the term 'monomial' was not used and the equality was not justified. The proof of this lemma was written independently of any source, but if needed for justification, one can find a proof in chapter 1, section 9, page 32 of Berge (1971).

More information about multinomials can be found in section 26.4 of Olver et al. (2010).

Proof. We can count the total number of selections, by procedurally considering the number of selections for A_1 out of the n objects, then the number of selections for A_2 out of the reamining non-A_1 objects, that is $n - k_1$ objects, and continuing similarly until we have covered all objects A_i, $\forall i \in \{1, 2, \dots, m\}$, and then multiplying all of those numbers by each other.

The number of unordered selections, without repetition, of k_1 objects A_1 out of n objects is $\binom{n}{k_1}$.

Then we have $n - k_1$ remaining objects and, similarly, the unordered selection (without repetition) of k_2 objects A_2 from those is $\binom{n-k_1}{k_2}$, leaving us with $n-k_1-k_2$ objects.

Then we have $n-k_1-k_2$ remaining objects and, similarly, the unordered selection (without repetition) of k_2 objects A_2 from those is $\binom{n-k_1-k_2}{k_3}$, leaving us with $n - k_1 - k_2 - k_3$ objects.

We proceed similarly until we reach k_m, where the possible selections of k_m objects A_m out of

$$\begin{aligned} & n - (k_1 + k_2 + \dots + k_{m-1}) \\ =& (k_1 + k_2 + \dots + k_{m-1} + k_m) - (k_1 + k_2 + \dots + k_{m-1}) = k_m \end{aligned}$$

total objects is $\binom{k_m}{k_m} = 1$.

Then multiplying, we have the following expression for the total number of selections:

$$\binom{n}{k_1} \cdot \binom{n-k_1}{k_2}\binom{n-k_1-k_2}{k_3} \cdots \binom{n-(k_1+k_2+\ldots+k_{m-1})}{k_m}$$
$$= \frac{(n)!}{k_1!(n-k_1)!} \cdot \frac{(n-k_1)!}{k_2!(n-k_1-k_2)!} \cdots \frac{n-(k_1+k_2+\ldots+k_{m-1})}{k_m!0!}.$$

We then see that the right factorial expression in each fraction's denominator (but the last) cancels the numerator of the next, hence we have

$$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{(n)!}{k_1!k_2!\ldots k_m!}.$$

□

Remark 6.2. An alternative way to think of $\binom{n}{k_1,k_2,\ldots,k_m}$ is the number of ways of placing n distinct objects into boxes labeled 1 to m, with k_i objects in box i for all $i \in \{1, 2, \ldots, m\}$.

This way of thinking about multinomials is talked about in Olver et al. (2010) in the beginning of 26.4(i) on page 620.

Lemma 6.3. Given a multinomial $\binom{n}{k_1,k_2,\ldots,k_m}$, with $n = md - l$, where $d \in \mathbb{N}$ and $l \in \{0, 1, \ldots, m-1\}$, the multinomial is maximised when (without loss of generality)

$$k_i = \left\lfloor \frac{n}{m} \right\rfloor \text{ for all } i \in \{1, \ldots, m\}, i \leq l \text{ and } k_j = \left\lceil \frac{n}{m} \right\rceil \text{ for all } j \in \{1, \ldots, m\}, j > l.$$

In other words,

$$\binom{n}{\underbrace{\left\lfloor \frac{n}{m} \right\rfloor, \ldots, \left\lfloor \frac{n}{m} \right\rfloor}_{l \text{ times}}, \underbrace{\left\lceil \frac{n}{m} \right\rceil, \ldots, \left\lceil \frac{n}{m} \right\rceil}_{m-l \text{ times}}} \geq \binom{n}{k_1, k_2, \ldots, k_m},$$

for all $k_1, k_2, \ldots, k_n$ such that $\sum_{r=1}^{m} k_r = n$.

This lemma was motivated by an argument made on page 16 in Kozdron (1998), where the statement of Lemma 6.3 was vaguely alluded to. The proof below is my own work.

Proof. Re-writing the multinomials in the inequality above as fractions, we aim to show that

$$\frac{(n)!}{\lceil\frac{n}{m}\rceil!\dots\lceil\frac{n}{m}\rceil!\cdot\lfloor\frac{n}{m}\rfloor!\dots\lfloor\frac{n}{m}\rfloor!}\geq\frac{(n)!}{k_1!k_2!\dots k_m!}.$$

As both fractions are positive, we can take the reciprocal of each and flip the inequality sign to get the equivalent statement:

$$\left\lceil\frac{n}{m}\right\rceil!\dots\left\lceil\frac{n}{m}\right\rceil!\cdot\left\lfloor\frac{n}{m}\right\rfloor!\dots\left\lfloor\frac{n}{m}\right\rfloor!\leq k_1!k_2!\dots k_m!. \tag{6.1}$$

We aim to arrive at (6.1). Given any $k_1,\dots,k_m$, we can express $k_i=\lceil\frac{n}{m}\rceil+\alpha_i$ for all $i\in\{0,1,\dots,l\}$ and $k_j=\lceil\frac{n}{m}\rceil+\alpha_j$ for all $j\in\{l+1,l+2,\dots,m\}$, where

$$\alpha_i\in\left\{-\left\lceil\frac{n}{m}\right\rceil,-\left\lceil\frac{n}{m}\right\rceil+1,\dots,0,\dots,n-\left\lceil\frac{n}{m}\right\rceil\right\}=:X\text{ and}$$
$$\alpha_j\in\left\{-\left\lfloor\frac{n}{m}\right\rfloor,-\left\lfloor\frac{n}{m}\right\rfloor+1,\dots,0,\dots,n-\left\lfloor\frac{n}{m}\right\rfloor\right\}=:Y.$$

Notice that since $l\lceil\frac{n}{m}\rceil+(m-l)\lfloor\frac{n}{m}\rfloor=n$,

$$\sum_{i=1}^{l}\alpha_i+\sum_{j=l+1}^{m}\alpha_j=0$$

and we need that

$$\sum_{r=1}^{m}k_r=n.$$

Using the above notation, we can write,

$$k_1!k_2!\dots k_m!=\left(\left\lceil\frac{n}{m}\right\rceil+\alpha_1\right)!\dots\left(\left\lceil\frac{n}{m}\right\rceil+\alpha_l\right)!\cdot\left(\left\lfloor\frac{n}{m}\right\rfloor+\alpha_{l+1}\right)!\dots\left(\left\lfloor\frac{n}{m}\right\rfloor+\alpha_m\right)!.$$

A trivial case is when all α_i and α_j are zero; then the right hand-side above is precisely the left hand side of (6.1), and hence the statement is satisfied. Now assume there is at least one non-zero α_i or α_j. If it has positive value, then we must have at least one other value α_i or α_j which is negative, and vice-versa.

Let $\{\beta_1,\beta_2,\dots,\beta_s\}=\{\alpha_i\in X\mid\alpha_i>0\}$, $\{\gamma_1,\gamma_2,\dots,\gamma_t\}=\{\alpha_i\in X\mid\alpha_i<0\}$, $\{\delta_1,\delta_2,\dots,\delta_u\}=\{\alpha_j\in Y\mid\alpha_j>0\}$ and $\{\epsilon_1,\epsilon_2,\dots,\epsilon_v\}=\{\alpha_j\in Y\mid\alpha_j<0\}$, where $s,t,u,v\in\mathbb{Z}$; we know that at least one of the first and third sets is non-empty and that at least one of the second and fourth sets is non-empty. Hence we can write $k_1!k_2!\dots k_m!$ as

$$\left\lceil\frac{n}{m}\right\rceil!\dots\left\lceil\frac{n}{m}\right\rceil!\cdot\left\lfloor\frac{n}{m}\right\rfloor!\dots\left\lfloor\frac{n}{m}\right\rfloor!$$
$$\cdot\frac{\left(\prod_{r=1}^{\beta_1}(\lceil\frac{n}{m}\rceil+r)\dots\prod_{r=1}^{\beta_s}(\lceil\frac{n}{m}\rceil+r)\right)\left(\prod_{r=1}^{\delta_1}(\lfloor\frac{n}{m}\rfloor+r)\dots\prod_{r=1}^{\delta_u}(\lfloor\frac{n}{m}\rfloor+r)\right)}{\left(\prod_{r=\gamma_1}^{-1}(\lceil\frac{n}{m}\rceil+r)\dots\prod_{r=\gamma_t}^{-1}(\lceil\frac{n}{m}\rceil+r)\right)\left(\prod_{r=\epsilon_1}^{-1}(\lfloor\frac{n}{m}\rfloor+r)\dots\prod_{r=\epsilon_v}^{-1}(\lfloor\frac{n}{m}\rfloor+r)\right)}.$$

Then, looking at the fraction, we see that any multiple in the numerator is greater than any multiple in the denominator; namely, comparing a minimal multiple in the numerator with a maximal multiple in the denominator we have that

$$\left\lfloor \frac{n}{m} \right\rfloor + 1 > \left\lceil \frac{n}{m} \right\rceil - 1.$$

Therefore, as all multiples are positive,

$$M := \frac{\left(\prod_{r=1}^{\beta_1}(\lceil \frac{n}{m} \rceil + r) \dots \prod_{r=1}^{\beta_s}(\lceil \frac{n}{m} \rceil + r)\right)\left(\prod_{r=1}^{\delta_1}(\lfloor \frac{n}{m} \rfloor + r) \dots \prod_{r=1}^{\delta_u}(\lfloor \frac{n}{m} \rfloor + r)\right)}{\left(\prod_{r=\gamma_1}^{-1}(\lceil \frac{n}{m} \rceil + r) \dots \prod_{r=\gamma_t}^{-1}(\lceil \frac{n}{m} \rceil + r)\right)\left(\prod_{r=\epsilon_1}^{-1}(\lfloor \frac{n}{m} \rfloor + r) \dots \prod_{r=\epsilon_v}^{-1}(\lfloor \frac{n}{m} \rfloor + r)\right)} > 1,$$

and hence,

$$\begin{aligned} k_1!k_2!\dots k_m! &= \left\lceil \frac{n}{m} \right\rceil! \dots \left\lceil \frac{n}{m} \right\rceil! \cdot \left\lfloor \frac{n}{m} \right\rfloor! \dots \left\lfloor \frac{n}{m} \right\rfloor! \cdot M \\ &\geq \left\lceil \frac{n}{m} \right\rceil! \dots \left\lceil \frac{n}{m} \right\rceil! \cdot \left\lfloor \frac{n}{m} \right\rfloor! \dots \left\lfloor \frac{n}{m} \right\rfloor!, \end{aligned}$$

as required. □

Proposition 6.4 (The Chu-Vandermonde identity)**.** For all $k, m, n \in \mathbb{Z}_+$ with $m \geq n$[8], we have the identity

$$\sum_{i=0}^{k} \binom{n}{i}\binom{m-n}{k-i} = \binom{m}{k}.$$

This is a result of combinatorics and hence we will not prove it here; the statement and its proof can be found on page 20 in Aigner (2007).

However, we will look at a special case result of this proposition:

Corollary 6.5. Consider the special case where $m = 2n$ and $k = n$. Then we have that

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}.$$

[8]We can actually take $m, n \in \mathbb{C}$, but restrict the statement to $m, n \in \mathbb{Z}_+$ since we do not need the more general statement.

Proof. Setting $m = 2n$ and $k = n$, gives us the following expression for the Chu-Vandermonde identity:

$$\sum_{i=0}^{n} \binom{n}{i}\binom{2n-n}{n-i} = \binom{2n}{n}.$$

Recall that the binomial coefficient has the property that $\binom{a}{b} = \binom{a}{a-b}$ for $a, b \in \mathbb{Z}_+, b \leq a$. Hence, the identity can be simplified to

$$\sum_{i=0}^{n} \binom{n}{i}\binom{n}{i} = \sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}, \text{ as required.}$$

□

What follows is George Pólya's result, which appeared in Pólya (1921). Note that it is a generalised statement (with respect to dimensions) of the forward implication of Theorem 5.6.

Theorem 6.6. [Pólya's theorem on the recurrence of walks in n-dimensions] Let X be an n-dimensional symmetric walk. If $n \leq 2$ then the walk is recurrent and if $n \geq 3$ then the walk is transient.

This theorem and an outline for the main ideas of the proof can be found in Kozdron (1998) on page 9 for the same definition of the walk as we use here. The proof given here clarifies and vastly expands on that proof.

Proof. X is symmetrical so for a walk of any dimension, n, $p_m = q_m = \frac{1}{2n} > 0$, hence the walk is an irreducible Markov chain (since non-zero probability for movement in all directions implies all states intercommunicate), so by Theorem 3.4, showing recurrence or transience for a single point, say $i_0 = (0, \ldots, 0) \in \mathbb{Z}^n$, implies recurrence or transience, respectively, for the whole chain (or walk in our case).

Moreover, by Theorem 2.4, we have that a state $i \in I$ is recurrent if and only if $\sum_{t\in\mathbb{N}} p_{ii}(t) = \infty$, and since "$i \in I$ is not recurrent" $\iff$ "$i \in I$ is transient" we have that $i \in I$ is transient if and only if $\sum_{t\in\mathbb{N}} p_{ii}(t)$ is finite.

By combining the above statements, we see that deducing whether $\sum_{t\in\mathbb{N}} p_{00}(t) = \infty$ or whether it is finite is equivalent to deducing whether the walk is recurrent or transient respectively; we check the former conditions in this proof. Note that in an n-dimensional walk, the expression $p_{00}(t)$ refers to the probability of the return to the 0 element in $\mathbb{R}^n$ after t steps, for some $t \in \mathbb{N}$.

Finally we note that, in any dimension, assuming we have $t_1, t_2, \ldots, t_n$ steps of values $(1, 0, \ldots, 0)$, $(0, 1, \ldots, 0)$, $\ldots$, $(0, 0, \ldots, 1)$ respectively, then X is recurrent if and only if we also have $t_1, t_2, \ldots, t_n$ steps of values $(-1, 0, \ldots, 0)$, $(0, -1, \ldots, 0)$, $\ldots$, $(0, 0, \ldots, -1)$ respectively. In that case we have a total number of $2(t_1 + t_2 + \ldots + t_n)$ steps, which is an even number. Hence given an odd number of steps, t, then $p_{00}(t) = 0$. Therefore

$$\sum_{t \in \mathbb{N}} p_{00}(t) = \sum_{t \in \mathbb{N}} p_{00}(2t).$$

We proceed on a case-by-case basis, starting with $n = 1$; this walk is recurrent by Theorem 5.6, as required.

Next, we proceed with $n = 2$. Note that return to $(0, 0)$ in $2t$ steps is equivalent to taking t_1 $(1, 0)$ steps, t_1 $(-1, 0)$ steps, $t - t_1$ $(0, 1)$ steps and $t - t_1$ $(0, -1)$ steps, for some $t_1 \in \mathbb{Z}_+$, $t_1 \leq t$.

By Lemma 6.1, the number of the total unique combinations of t_1 $(1, 0)$ steps, t_1 $(-1, 0)$ steps, $t - t_1$ $(0, 1)$ steps and $t - t_1$ $(0, -1)$ steps out of a total possible $2t$ steps is given by

$$\binom{2t}{t_1, t_1, t - t_1, t - t_1} := \frac{(2t)!}{t_1!t_1!(t - t_1)!(t - t_1)!}.$$

Then, summing the above expression over all values of $t_1 \in \{0, 1, \ldots, t\}$ gives us the total number of paths ending (and starting) at (0,0). Since X is symmetric, any particular path of length $2t$ has a probability $(\frac{1}{4})^{2t}$ of ocurring. Hence the number of unique paths multiplied by that probability gives us precisely the probability of returning to $(0, 0)$ in $2t$ steps, namely,

$$p_{00}(2t) = \left(\frac{1}{4}\right)^{2t} \sum_{t_1=0}^{t} \frac{(2t)!}{t_1!t_1!(t - t_1)!(t - t_1)!}.$$

Then we can re-write the right hand-side as,

$$\left(\frac{1}{4}\right)^{2t} \sum_{t_1=0}^{t} \frac{(2t)!t!t!}{t!t!t_1!t_1!(t - t_1)!(t - t_1)!} = \left(\frac{1}{4}\right)^{2t} \sum_{t_1=0}^{t} \frac{(2t)}{!t!t!} \frac{t!t!}{t_1!t_1!(t - t_1)!(t - t_1)!}.$$

As the first fraction is not in terms of t_1, we can take it out of the summation:

$$\left(\frac{1}{4}\right)^{2t} \frac{(2t)}{!t!t!} \sum_{t_1=0}^{t} \frac{t!t!}{t_1!t_1!(t - t_1)!(t - t_1)!}.$$

Now, writing $t = 2t - t$ and simplifying the expression in the summation, makes it clear that we can re-write the expressions as binomial coefficients:

$$\left(\frac{1}{4}\right)^{2t} \frac{(2t)}{!t!(2t-t)!} \sum_{t_1=0}^{t} \left(\frac{t!}{t_1!(t-t_1)!}\right)^2 = \left(\frac{1}{4}\right)^{2t} \binom{2t}{t} \sum_{t_1=0}^{t} \binom{t}{t_1}^2.$$

By Corollary 6.5, we have that

$$p_{00}(2t) = \left(\frac{1}{4}\right)^{2t} \binom{2t}{t}\binom{2t}{t} = \left(\frac{1}{4}\right)^{2t} \binom{2t}{t}^2. \tag{6.2}$$

We now look at the convergence/divergence of the summation of $p_{00}(t)$ over all t:

$$\begin{aligned}\sum_{t\in\mathbb{N}} p_{00}(t) = \sum_{t\in\mathbb{N}} p_{00}(2t) &= \sum_{t\in\mathbb{N}} \left(\left(\frac{1}{4}\right)^{2t} \binom{2t}{t}^2\right) \\ &= \sum_{t\in\mathbb{N}} \left(\left(\frac{1}{4}\right)^{2t} \left(\frac{(2t)!}{(t)!(2t-t)!}\right)^2\right) = \sum_{t\in\mathbb{N}} \left(\left(\frac{1}{4}\right)^{2t} \left(\frac{(2t)!}{t!^2}\right)^2\right).\end{aligned}$$

Then we refer to Stirling's approximation, Proposition 4.1, which gives us the approximation

$$\sum_{t\in\mathbb{N}} \left(\left(\frac{1}{4}\right)^{2t} \left(\frac{(2t)!}{t!^2}\right)^2\right) \sim \sum_{t\in\mathbb{N}} \left(\left(\frac{1}{4}\right)^{2t} \left(\frac{\sqrt{2\pi(2t)}\left(\frac{2t}{e}\right)^{2t}}{\left(\sqrt{2\pi t}\left(\frac{t}{e}\right)^{t}\right)^2}\right)^2\right).$$

Clearly one is infinite if and only if the other is infinite, hence checking the divergence of the right hand-side is equivalent to checking the divergence of the left hand-side; we do the former:

$$\begin{aligned}\sum_{t\in\mathbb{N}} \left(\left(\frac{1}{4}\right)^{2t} \left(\frac{\sqrt{2\pi(2t)}\left(\frac{2t}{e}\right)^{2t}}{\left(\sqrt{2\pi t}\left(\frac{t}{e}\right)^{t}\right)^2}\right)^2\right) &= \sum_{t\in\mathbb{N}} \left(\left(\frac{1}{4}\right)^{2t} \left(\frac{\sqrt{4\pi t}\left(\frac{2t}{e}\right)^{2t}}{2\pi t\left(\frac{t}{e}\right)^{2t}}\right)^2\right) \\ = \sum_{t\in\mathbb{N}} \left(\left(\frac{1}{4}\right)^{2t} \frac{4\pi t\left(\frac{2t}{e}\right)^{4t}}{4\pi^2 t^2\left(\frac{t}{e}\right)^{4t}}\right) &= \sum_{t\in\mathbb{N}} \left(\left(\frac{1}{4}\right)^{2t} \frac{(2)^{4t}\left(\frac{t}{e}\right)^{4t}}{\pi t\left(\frac{t}{e}\right)^{4t}}\right) \\ = \sum_{t\in\mathbb{N}} \left(\frac{1}{\pi t}\left(\frac{1}{4}\right)^{2t} 2^{4t}\right) &= \sum_{t\in\mathbb{N}} \left(\frac{1}{\pi t}\left(\frac{1}{4}\right)^{2t} 4^{2t}\right) = \sum_{t\in\mathbb{N}} \frac{1}{\pi t} = \infty.\end{aligned}$$

Equivalently,

$$\sum_{t\in\mathbb{N}} p_{00}(t) = \left(\frac{1}{4}\right)^{2t} \sum_{t\in\mathbb{N}} \left(\left(\frac{(2t)!}{t!^2}\right)^2\right) = \infty,$$

and hence the 2 dimensional walk is recurent.

We proceed to the $n = 3$ case: Recall that for any $t \in \mathbb{N}$, return to $(0,0,0)$ in $2t-1$ steps (i.e. an odd number of steps) is impossible. Returning in $2t$ steps occurs if and only if we have taken t_1 $(1,0,0)$ steps, t_1 $(-1,0,0)$ steps, t_2 $(0,1,0)$ steps, t_2 $(0,-1,0)$ steps, $t-(t_1+t_2)$ $(0,0,1)$ steps and $t-(t_1+t_2)$ $(0,0,-1)$ steps for some $t_1, t_2 \in \mathbb{Z}_+$.

Hence, via Lemma 6.1, the total possible number of ways of returning to $(0,0,0)$ is

$$\binom{2t}{t_1, t_1, t_2, t_2, t-(t_1+t_2), t-(t_1+t_2)} := \frac{(2t)!}{t_1!t_1!t_2!t_2!\,(t-(t_1+t_2))!\,(t-(t_1+t_2))!}.$$

Each path of $2t$ steps has a probability of $(\frac{1}{6})^{2t}$ of occurring, and so the probability multiplied by the number of paths as described above over all values of t_1 and t_2 such that $0 \le t_1 + t_2 \le t$ is equivalent to the probability of returning to $(0,0,0)$ in $2t$ steps, namely:

$$p_{00}(t) = p_{00}(2t) = \sum_{0 \le t_1+t_2 \le t} \left(\left(\frac{1}{6}\right)^{2t} \frac{(2t)!}{t_1!t_1!t_2!t_2!\,(t-(t_1+t_2))!\,(t-(t_1+t_2))!} \right).$$

Now, introducing dummy variables and re-arranging, we get:

$$\begin{aligned}
&\sum_{0 \le t_1+t_2 \le t} \left(\left(\frac{1}{6}\right)^{2t} \frac{(2t)!t!t!}{t!t!t_1!t_1!t_2!t_2!\,(t-(t_1+t_2))!\,(t-(t_1+t_2))!} \right) \\
&= \left(\frac{1}{6}\right)^{2t} \frac{(2t)!}{t!t!} \sum_{0 \le t_1+t_2 \le t} \left(\frac{t!t!}{t_1!t_1!t_2!t_2!\,(t-(t_1+t_2))!\,(t-(t_1+t_2))!} \right) \\
&= \left(\frac{1}{2}\right)^{2t} \left(\frac{1}{3}\right)^{2t} \frac{(2t)!}{t!(2t-t)!} \sum_{0 \le t_1+t_2 \le t} \left(\frac{t!t!}{t_1!t_1!t_2!t_2!\,(t-(t_1+t_2))!\,(t-(t_1+t_2))!} \right) \\
&= \left(\frac{1}{2}\right)^{2t} \binom{2t}{t} \sum_{0 \le t_1+t_2 \le t} \left(\left(\frac{1}{3}\right)^{2t} \frac{t!t!}{t_1!t_1!t_2!t_2!\,(t-(t_1+t_2))!\,(t-(t_1+t_2))!} \right) \\
&= \left(\frac{1}{2}\right)^{2t} \binom{2t}{t} \sum_{0 \le t_1+t_2 \le t} \left(\left(\left(\frac{1}{3}\right)^{t} \frac{t!}{t_1!t_2!\,(t-(t_1+t_2))!} \right)^2 \right) = p_{00}(t). \qquad (6.3)
\end{aligned}$$

Since $t \in \mathbb{N}$, there are exactly 3 possible cases: given some t, there is some $d \in \mathbb{N}$ such that either $t = 3d$, $t = 3d-1$ or $t = 3d-2$. Firstly we consider the case when $t = 3d$; then by Lemma 6.3 we have that $p_{00}(t)$ is less than or equal to

$$\left(\frac{1}{2}\right)^{2t}\binom{2t}{t}\sum_{0\leq t_1+t_2\leq t}\left(\left(\left(\frac{1}{3}\right)^{t}\frac{t!}{\lceil\frac{t}{3}\rceil!\lceil\frac{t}{3}\rceil!\lceil\frac{t}{3}\rceil!}\right)\left(\left(\frac{1}{3}\right)^{t}\frac{t!}{t_1!t_2!\,(t-(t_1+t_2))!}\right)\right)$$
$$=\left(\frac{1}{2}\right)^{2t}\left(\frac{1}{3}\right)^{t}\binom{2t}{t}\frac{t!}{\lceil\frac{t}{3}\rceil!\lceil\frac{t}{3}\rceil!\lceil\frac{t}{3}\rceil!}\sum_{0\leq t_1+t_2\leq t}\left(\left(\frac{1}{3}\right)^{t}\frac{t!}{t_1!t_2!\,(t-(t_1+t_2))!}\right)$$
$$=\left(\frac{1}{12}\right)^{t}\binom{2t}{t}\frac{t!}{\lceil\frac{t}{3}\rceil!\lceil\frac{t}{3}\rceil!\lceil\frac{t}{3}\rceil!}\sum_{0\leq t_1+t_2\leq t}\left(\left(\frac{1}{3}\right)^{t}\frac{t!}{t_1!t_2!\,(t-(t_1+t_2))!}\right). \tag{6.4}$$

We now refer to Remark 6.2, which states that $\binom{n}{k_1,k_2,\ldots,k_m} := \frac{(n)!}{k_1!k_2!\ldots k_m!}$ counts the number of ways of placing n distinct objects into boxes labeled 1 through m, with k_i objects in box i for all $i \in \{1, 2, \ldots, m\}$.

Now we treat the scenario as a random variable whereby we procedurally place each one of n distinct objects into one of the labeled boxes, with equal probability, $\frac{1}{m}$, for each box. Then clearly, any particular final arrangement of the n objects has a probability $\frac{1}{m}^n$ of occurring.

Summing each outcome multiplied by its probability, should result in 1, since the scenario above is a random variable. In this case, as each outcome has equal probability, we have that

$$\sum_{0<k_1+k_1+\ldots+k_m\leq n}\left(\frac{1}{m}^n\frac{(n)!}{k_1!k_2!\ldots k_m!}\right) = 1.$$

We see that for $n = t, m = 3, k_1 = t_1, k_2 = t_2$ and $k_3 = t_3$, this is precisely the summation in equation (6.4) and hence, for all t such that $t = 3d$, we have that

$$p_{00}(t) \leq \left(\frac{1}{12}\right)^{t}\binom{2t}{t}\frac{t!}{\lceil\frac{t}{3}\rceil!\lceil\frac{t}{3}\rceil!\lceil\frac{t}{3}\rceil!}.$$

Similarly, for the case where $t = 3d-1$, we have that $p_{00}(t)$ is less than or equal to

$$\left(\frac{1}{12}\right)^{t}\binom{2t}{t}\sum_{0\leq t_1+t_2\leq t}\left(\left(\left(\frac{1}{3}\right)^{t}\frac{t!}{\lfloor\frac{t}{3}\rfloor!\lceil\frac{t}{3}\rceil!\lceil\frac{t}{3}\rceil!}\right)\left(\left(\frac{1}{3}\right)^{t}\frac{t!}{t_1!t_2!\,(t-(t_1+t_2))!}\right)\right)$$
$$\leq\left(\frac{1}{12}\right)^{t}\binom{2t}{t}\frac{t!}{\lfloor\frac{t}{3}\rfloor!\lceil\frac{t}{3}\rceil!\lceil\frac{t}{3}\rceil!},$$

and for the case where $t = 3d + 2$, we have that $p_{00}(t)$ is less than or equal to

$$\left(\frac{1}{12}\right)^t \binom{2t}{t} \sum_{0 \le t_1 + t_2 \le t} \left(\left(\left(\frac{1}{3}\right)^t \frac{t!}{\lfloor \frac{t}{3} \rfloor! \lfloor \frac{t}{3} \rfloor! \lceil \frac{t}{3} \rceil!} \right) \left(\left(\frac{1}{3}\right)^t \frac{t!}{t_1! t_2! \left(t - (t_1 + t_2)\right)!} \right) \right)$$
$$\le \left(\frac{1}{12}\right)^t \binom{2t}{t} \frac{t!}{\lfloor \frac{t}{3} \rfloor! \lfloor \frac{t}{3} \rfloor! \lceil \frac{t}{3} \rceil!}.$$

Hence,

$$\begin{aligned}\sum_{t \in \mathbb{N}} p_{00}(t) \le \sum_{t \in 3\mathbb{N}} &\left(\left(\frac{1}{2}\right)^{2t} \binom{2t}{t} \left(\frac{1}{3}\right)^t \frac{t!}{\lceil \frac{t}{3} \rceil! \lceil \frac{t}{3} \rceil! \lceil \frac{t}{3} \rceil!} \right) \\ + \sum_{t \in 3\mathbb{N}-1} &\left(\left(\frac{1}{2}\right)^{2t} \binom{2t}{t} \left(\frac{1}{3}\right)^t \frac{t!}{\lfloor \frac{t}{3} \rfloor! \lceil \frac{t}{3} \rceil! \lceil \frac{t}{3} \rceil!} \right) \\ + \sum_{t \in 3\mathbb{N}-2} &\left(\left(\frac{1}{2}\right)^{2t} \binom{2t}{t} \left(\frac{1}{3}\right)^t \frac{t!}{\lfloor \frac{t}{3} \rfloor! \lfloor \frac{t}{3} \rfloor! \lceil \frac{t}{3} \rceil!} \right).\end{aligned}$$ [9]

Note that the first of the three summations is at least as big as the other two, and hence we can consider the greater expression,

$$3 \sum_{t \in \mathbb{N}} \left(\left(\frac{1}{2}\right)^{2t} \binom{2t}{t} \left(\frac{1}{3}\right)^t \frac{t!}{\lceil \frac{t}{3} \rceil! \lceil \frac{t}{3} \rceil! \lceil \frac{t}{3} \rceil!} \right). \tag{6.5}$$

Showing that (6.5) is finite implies that $\sum_{t \in \mathbb{N}} p_{00}(t)$ is finite too, and hence that the 3-dimensional walk is transient.

We begin by simplifying the expression:

$$3 \sum_{t \in \mathbb{N}} \left(\left(\frac{1}{4}\right)^t \left(\frac{1}{3}\right)^t \frac{(2t)!}{t!(2t-t)!} \frac{t!}{\left(\lceil \frac{t}{3} \rceil!\right)^3} \right) = 3 \sum_{t \in \mathbb{N}} \left(\left(\frac{1}{12}\right)^t \frac{(2t)!}{t!} \left(\left\lceil \frac{t}{3} \right\rceil! \right)^{-3} \right),$$

and we apply Stirling's approximation (Proposition 4.1), giving:

$$\begin{aligned}&3 \sum_{t \in \mathbb{N}} \left(\left(\frac{1}{12}\right)^t \frac{(2t)!}{t!} \left(\left\lceil \frac{t}{3} \right\rceil! \right)^{-3} \right) \\ \sim &3 \sum_{t \in \mathbb{N}} \left(\left(\frac{1}{12}\right)^t \frac{\sqrt{2\pi(2t)} \left(\frac{2t}{e}\right)^{2t}}{\sqrt{2\pi t} \left(\frac{t}{e}\right)^t} \left(\sqrt{2\pi \left\lceil \frac{t}{3} \right\rceil} \left(\frac{\lceil \frac{t}{3} \rceil}{e} \right)^{-3\lceil \frac{t}{3} \rceil} \right) \right).\end{aligned}$$

[9] Here we define $m\mathbb{N} + l := \{m + l, 2m + l, \ldots\}$ for all $m, l \in \mathbb{N}$

As in the case for $n = 2$, the summation is finite if and only if its approximation is finite; we show that the approximation is finite:

$$
\begin{aligned}
&3\sum_{t\in\mathbb{N}}\left(\left(\frac{1}{12}\right)^t \frac{\sqrt{2}\sqrt{2\pi t}\left(\frac{4t^2}{e^2}\right)^t}{\sqrt{2\pi t}\left(\frac{t}{e}\right)^t}\left(\sqrt{2\pi\left\lceil\frac{t}{3}\right\rceil}\right)^{-3}\left(\frac{\left\lceil\frac{t}{3}\right\rceil}{e}\right)^{-3\left\lceil\frac{t}{3}\right\rceil}\right)\\
=&3\sum_{t\in\mathbb{N}}\left(\left(\frac{1}{12}\right)^t\sqrt{2}\left(\frac{4t}{e}\right)^t\frac{1}{2\pi\left\lceil\frac{t}{3}\right\rceil\sqrt{2}\sqrt{\pi\left\lceil\frac{t}{3}\right\rceil}}\left(\frac{\left\lceil\frac{t}{3}\right\rceil}{e}\right)^{-3\left\lceil\frac{t}{3}\right\rceil}\right)\\
=&3\sum_{t\in\mathbb{N}}\left(\left(\frac{t}{3e}\right)^t\frac{1}{2}\frac{1}{\pi\left\lceil\frac{t}{3}\right\rceil\sqrt{\pi\left\lceil\frac{t}{3}\right\rceil}}\left(\frac{\left\lceil\frac{t}{3}\right\rceil}{e}\right)^{-3\left\lceil\frac{t}{3}\right\rceil}\right)\\
=&\frac{3}{2}\sum_{t\in 3\mathbb{N}}\left(\left(\frac{t}{3e}\right)^t\left(\frac{e}{\left\lceil\frac{t}{3}\right\rceil}\right)^{3\left\lceil\frac{t}{3}\right\rceil}\frac{1}{\left(\pi\left\lceil\frac{t}{3}\right\rceil\right)^{\frac{3}{2}}}\right).
\end{aligned}
$$

Now notice that $t = 3d - 2$, $t = 3d - 1$, or $t = 3d$ and in each of those cases we have (respectively):

1. $\left\lceil\frac{t}{3}\right\rceil = \left\lceil\frac{3d-2}{3}\right\rceil = \left\lceil d - \frac{2}{3}\right\rceil = d$ (since $d \in \mathbb{N}$),
2. $\left\lceil\frac{t}{3}\right\rceil = \left\lceil\frac{3d-1}{3}\right\rceil = \left\lceil d - \frac{1}{3}\right\rceil = d$ and
3. $\left\lceil\frac{t}{3}\right\rceil = \left\lceil\frac{3d}{3}\right\rceil = \lceil d\rceil = d$.

Hence,

$$
\begin{aligned}
\frac{3}{2}\sum_{t\in 3\mathbb{N}}\left(\left(\frac{t}{3e}\right)^t\left(\frac{e}{\left\lceil\frac{t}{3}\right\rceil}\right)^{3\left\lceil\frac{t}{3}\right\rceil}\frac{1}{\left(\pi\left\lceil\frac{t}{3}\right\rceil\right)^{\frac{3}{2}}}\right) =&\frac{3}{2}\sum_{d\in\mathbb{N}}\left(\left(\frac{d}{e}\right)^{3d}\left(\frac{e}{d}\right)^{3d}\frac{1}{(\pi d)^{\frac{3}{2}}}\right)\\
=&\frac{3}{2\pi^{\frac{3}{2}}}\sum_{d\in\mathbb{N}}\left(\left(\frac{1}{d}\right)^{\frac{3}{2}}\right).
\end{aligned}
$$

We now use the Cauchy condensation test (justification for this test can be found in Rudin (1976)) to analyse,

$$
\sum_{t\in\mathbb{N}}\left(\frac{1}{d}\right)^{\frac{3}{2}}.
$$

The test states that for a non-increasing, non-negative real number sequence, $f(d)$,

$$
\sum_{d\in\mathbb{N}} f(d) \text{ converges if and only if } \sum_{t\in\mathbb{N}} 2^d f\left(2^d\right) \text{ converges.}
$$

This test is applicable to our series by taking $f(d) = (\frac{1}{d})^{\frac{3}{2}}$, since the sequence of summands $(\frac{1}{d})^{\frac{3}{2}}$ is non-increasing (in fact decreasing) as d increases, and non-negative. We analyse the Cauchy condensation:

$$\sum_{d\in\mathbb{N}} 2^d \left(\frac{1}{2^d}\right)^{\frac{3}{2}} = \sum_{d\in\mathbb{N}} \left(\frac{1}{2^d}\right)^{\frac{1}{2}} = \sum_{d\in\mathbb{N}} \left(\frac{1}{2}\right)^{\frac{d}{2}} = \sum_{d\in\mathbb{N}} \left(\frac{1}{\sqrt{2}}\right)^{d}.$$

This is a geometric series, and hence we have the closed form expression:

$$\sum_{d\in\mathbb{N}} \left(\frac{1}{\sqrt{2}}\right)^{d} = \frac{1}{1-\sqrt{2}} < \infty,$$

or equivalently, by the Cauchy condensation test,

$$\sum_{d\in\mathbb{N}} \left(\frac{1}{d}\right)^{\frac{3}{2}} < \infty \iff \frac{3}{2\pi^{\frac{3}{2}}} \sum_{d\in\mathbb{N}} \left(\frac{1}{d}\right)^{\frac{3}{2}} < \infty$$

Recall that

$$\frac{3}{2\pi^{\frac{3}{2}}} \sum_{d\in\mathbb{N}} \left(\left(\frac{1}{d}\right)^{\frac{3}{2}}\right) = 3\sum_{t\in\mathbb{N}} \left(\left(\frac{1}{12}\right)^{t} \frac{\sqrt{2\pi(2t)}\left(\frac{2t}{e}\right)^{2t}}{\sqrt{2\pi t}\left(\frac{t}{e}\right)^{t}} \left(\sqrt{2\pi\left\lceil\frac{t}{3}\right\rceil}\left(\frac{\lceil\frac{t}{3}\rceil}{e}\right)^{-3\lceil\frac{t}{3}\rceil}\right)\right)$$
$$< \infty,$$

and since clearly Stirling's approximation of an expression is finite if and only if the expression itself is finite, we have that

$$3\sum_{t\in\mathbb{N}} \left(\left(\frac{1}{2}\right)^{2t} \binom{2t}{t} \left(\frac{1}{3}\right)^{t} \frac{t!}{\lceil\frac{t}{3}\rceil!\,\lceil\frac{t}{3}\rceil!\,\lceil\frac{t}{3}\rceil!}\right) < \infty,$$

which is precisely equation (6.5). Hence the walk is transient.

For random walks of dimension $n = 4$, given any $t \in \mathbb{N}$, return to $(0,0,0,0)$ in $2t-1$ steps (i.e. an odd number of steps) is impossible, and return in $2t$ steps occurs if and only if we have taken t_1 $(1,0,0,0)$ steps, t_1 $(-1,0,0,0)$ steps, t_2 $(0,1,0,0)$ steps, t_2 $(0,-1,0,0)$ steps, t_3 $(0,0,1,0)$ steps, t_3 $(0,0,-1,0)$ steps, $t-(t_1+t_2+t_3)$ $(0,0,0,1)$ steps and $t-(t_1+t_2+t_3)$ $(0,0,0,-1)$ steps for some $t_1, t_2, t_3 \in \mathbb{Z}_+$.

This implies that by returning to $(0,0,0,0)$ in 4 dimensions, we have also returned in 3 dimensions, hence as the 3-dimensional walk is transient, so is the 4-dimensional.

In fact all walks of dimension $n \geq 3$ are transient and hence the theorem is proven. $\square$

Remark 6.7. Let us take the next page or so to explore a seeming paradox arising from Pólya's theorem, Theorem 6.6; namely we will justify that it is not actually a paradox. The 'paradox' goes as follows:

"We can re-formulate the 3-dimensional random walk starting at $(0, 0, 0)$, with respect to time t, as a Markov process in $\mathbb{Z}^3$, where a 1-dimensional walk starting at 0, with respect to a time t_1, controls the first entry of our 3-dimensional vector, and and a 2-dimensional walk starting at $(0, 0)$, with respect to time t_2, controls the second and third entries of our vector. Finally, we also need to stipulate that at each time t, we generate a Bernoulli random variable which gives 0 with probability $\frac{1}{3}$, in which case we increment t_1, and 1 with probability $\frac{2}{3}$, in which case we increment t_2.

It is easy to see that this re-formulation is indeed equivalent to the 3-dimensional random walk. Moreover, it is easy to see that the 3-dimensional walk returns to $(0, 0, 0)$ if and only if the 1-dimensional walk returns to 0 at some time $t_1 = t_{1_r}$ and the 2-dimensional walk returns to $(0, 0)$ at some time $t_2 = t_{2_r}$ such that $t_1 = t_{1_r}$.

Now, Pólya's theorem states that the 1 and 2-dimensional random walks are recurrent; this implies that if we consider the 1-dimensional random walk, then it must return to 0 with probability 1, and hence once this occurs, say at some time $t_1 = t_{1_{r_1}}$, we are again in the same position as at time $t_1 = 0$, and hence, at this point, return to 0 for a second time occurs with probability 1, at some time $t_1 = t_{1_{r_2}} > t_{1_{r_1}}$, as well. Iterating this argument shows us that return to 0 for the m^{th} time occurs with probability 1, at some time $t_1 = t_{1_{r_m}} > t_{1_{r_{m-1}}}$, for all $m \in \mathbb{N}$.

We now only consider snapshots of the 2-dimensional walk whenever we have $t_1 \in \{t_{1_{r_1}}, t_{1_{r_2}}, \ldots, t_{1_{r_m}}\}$. The 1 and 2-dimensional random walks are independent from each other, and hence this forms a completely random and unbiased sample of realisations of the 2-dimensional random walk. Hence, if we let $m \to \infty$, we have an infinite set of completely random values of the 2-dimensional random walk and hence its recurrence implies that, with probability 1, the state $(0, 0)$ is in this set; without loss of generaility, assume this happens at time $t_2 = t_{2_r}$.

In other words, there is a probability of 1 that there exists this time t_{2_r}, during which the 2-dimensional walk is at the state $(0, 0)$, and moreover, we have defined that during t_{2_r}, we have $t_1 \in \{t_{1_{r_1}}, t_{1_{r_2}}, \ldots\}$; hence during this time, the 1-dimensional walk is at the state 0. We have already established that this happens if and only if the 3-dimensional random walk is at $(0, 0, 0)$, hence we have that the probability of the 3-dimensional random walk recurring is 1.

However, this contradicts Pólya's theorem, which states that the 3-dimensional random walk is transient."

The above argument is wrong since there is an incorrect assumption made, which is that the 1 and 2-dimensional walks are independent; let $t = 1$, then if $t_1 = 0$, $\mathbb{P}(t_2 = 0) = 0$ and $\mathbb{P}(t_2 = 1) = 1$, since we must have $t_1 + t_2 = t$, however if $t_1 = 1$, then clearly $\mathbb{P}(t_2 = 0) = 1$ and $\mathbb{P}(t_2 = 1) = 0$. Therefore the walks are not independent

7 Null-recurrence of the 1 and 2 dimensional walks

In the final section of this text, we explore some interesting consequences of Pólya's theorem, Theorem 6.6, relating to null recurrence.

Lemma 7.1. The symmetric 1-dimensional random walk, X, is null recurrent.

Proof. We saw in Theorem 5.6 that $p_{00}(t) = \mathbb{P}(S_t = 0)$, and that as $t \to \infty$,

$$\mathbb{P}(S_t = 0) \sim \frac{\beta(p)^{\frac{t}{2}}}{\sqrt{\frac{\pi t}{2}}},$$

where $\beta(p) = 4p(1-p)$. Since X is symmetric, and so $p = \frac{1}{2}$, we have

$$\beta(p) = 4\left(\frac{1}{2}\right)\left(1 - \frac{1}{2}\right) = 1.$$

Hence as $t \to \infty$,

$$\frac{1}{\sqrt{\frac{\pi t}{2}}} \to 0,$$

and as $p_{00}(t) \sim \frac{1}{\sqrt{\frac{\pi t}{2}}}$,

$$p_{00}(t) \to 0,$$

since clearly Stirling's approximation of an expression tends to 0 if and only if the expression itself tends to 0.

Hence by Theorem 2.12, the state 0 is null recurrent. In Theorem 3.4, we showed that the classification of a single state is equivalent to the classification of the chain. Therefore X is null recurrent. $\square$

Example 7.2. Let us consider some computer simulations of 1 dimensional walks. Here we use Python to define a function which generates n 1-dimensional walks starting at 0, each of which terminates at its first recurrence to 0. The 'avg' function defines the mean of a list of numbers.

We take a sample of 1000 walks and output the list of numbers corresponding to the steps it took to return to 0 in each walk (as a sanity check, all of these number should be even), the average number of steps it took to return to 0 across the 1000 walks, the maximum number of steps that it took to return to 0 among the 1000 walks and the proportion of walks which recurred in 2 steps (this is again a sanity check; if our model is correct then, of course, this number should be close to 0.5).

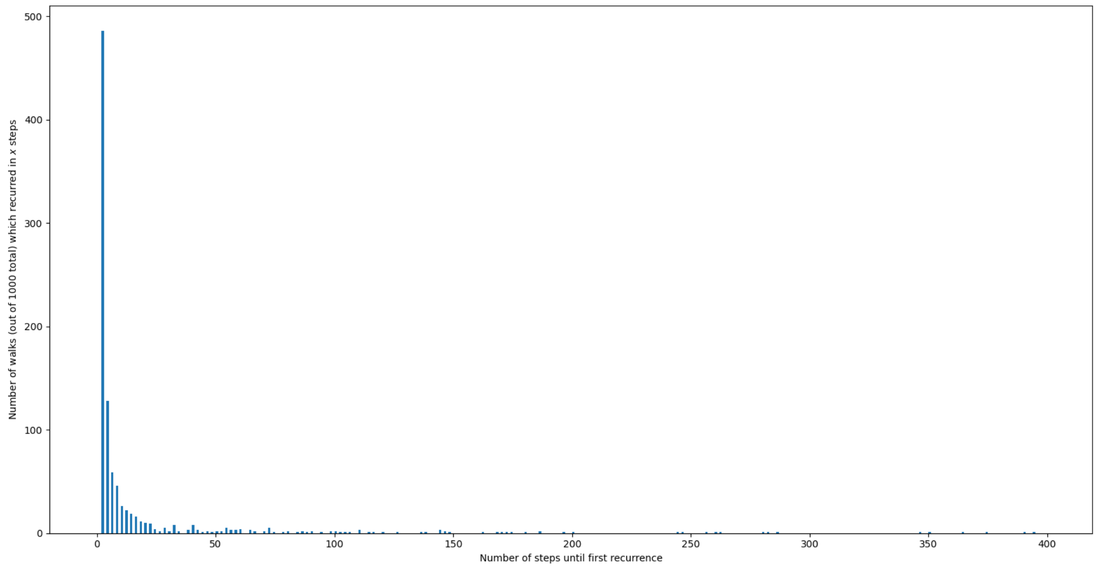
The Python code is given in Appendix C.

We ran this code 20 times, and all results can be found at `https://pastebin.com/PKX5e7EV`.

Firstly, we see that the numbers of steps it took to return to 0 are even, which is a good sign, but more notablty, most walks return to 0 pretty quickly, however the outliers take a significantly longer time. Moreover, the averages vary drastically between the 20 executions of the code, the maximum number of steps it took to return to 0 vary even more drastically and the proportion of walks which recurred in 2 steps consistently hover around 0.5, which again fits our sanity check.

We also ran a Python similar code (which can be found in Appendix D) which used the same function as above to generate 1000 1-dimensional random walks until their first return to 0, and then plotted all of those on the histogram below.

Note that we did have walks which took more than 400 steps to return, however they have been omitted from the gaph, since extending the x-axis makes the histogram unclear.



Lemma 7.3. The symmetric 2-dimensional random walk, X, is null recurrent.

Proof. In Theorem 6.6 we saw that for all even values of t, where $t = 2d$ for some $d \in \mathbb{N}$, by equation (6.2) we have

$$p_{00}(2d) = \left(\frac{1}{4}\right)^{2d} \binom{2d}{d}^2 = \left(\frac{1}{4}\right)^{2d} \frac{(2d!)^2}{(d!)^4},$$

and applying Stirling's approximation, Proposition 4.1, gives

$$p_{00}(2d) \sim \left(\frac{1}{4}\right)^{2d} \frac{\left(\sqrt{2\pi 2d}\left(\frac{2d}{e}\right)^{2d}\right)^2}{\left(\sqrt{2\pi d}\left(\frac{d}{e}\right)^{d}\right)^4} = \frac{(4\pi d)\left(\frac{2d}{e}\right)^{4d}}{4^{2d}(4\pi^2 d^2)\left(\frac{d}{e}\right)^{4d}} = \frac{2^{4d}\left(\frac{d}{e}\right)^{4d}}{4^{2d}\pi d\left(\frac{d}{e}\right)^{4d}} = \frac{1}{(2)^{2d}\pi d}.$$

Clearly, Stirling's approximation of an expression tends to 0 if and only if the expression itself tends to 0. Hence, we see that clearly,

$$\lim_{d\to\infty}\left(\frac{1}{(2)^{2d}\pi d}\right) = 0,$$

or in other words,

$$\lim_{d\to\infty}\left(p_{00}(2d)\right) = 0.$$

Since $p_{00}(t) = 0$ for all odd values of t, and we have shown that this probability also tends to 0 for all even values of t, we have that $p_{00}(t) \to 0$ as $t \to \infty$ and hence the state 0 is null recurrent by Theorem 2.12 and moreover, the whole 2-dimensional walk is null recurrent by Theorem 3.4. $\square$

In Remark 2.11 we promised to give some concrete examples of chains entirely consisting of null-recurrent states; we have now fulfilled this promise.

Example 7.4. We again consider computer simulations, this time for the 2 dimensional walk. Similarly, we use python to generate 2-dimensional random walks starting at $(0, 0)$, which again terminate upon returning to $(0, 0)$.

In this case, the code only simulates 1 walk at a time (as opposed 1000 for 1-dimension walks). This is because while the 2-dimensional walk most often returns to $(0, 0)$ quite quickly like the 1-dimensional walk, the outlier cases of the 2-dimensional walk take significantly longer. While experimenting with the code, the program often had to be manually terminated. In one instance, the code was left to run for nearly 24 hours to no avail; note that Python can simulate hundreds of thousands of steps per second.

We ran the code 20 times, each time giving the number of steps it took a single walk to first return to $(0, 0)$, the results are listed here: `https://pastebin.com/WHRWb7iK`. You will notice that there are only 19 results; in one case the code was left to run for several hours, but it was still not long enough for the walk to return to $(0, 0)$.

The code used to simulate this can be found in Appendix E.

8 Conclusion

In this dissertation, we first introduced discrete Markov chains and built up a solid foundation of understanding about them, most notably what it means for states to be classified as either recurrent, null-recurrent or transient and how this affects the classification of the whole chain. Next, we used Stirling's approximation to prove the local limit theorem, which is essential for analysing infinite random processes, e.g. random walks.

We then introduced random walks, along with the notions of paths, steps and symmetricity and proved that recurrence and symmetricity are equivalent in 1 dimension. Next, we begun building up to Pólya's theorem, by introducing notions of combinatorics, essential for the proof of the theorem, and proceeded to prove it exhaustively, utilising the knowledge we had built up. Following its proof, we took a page in correcting an intuitive but wrong argument seeming to contradict the theorem. In the final section we prove that the 1 and 2 dimensional walks are not only recurrent, but null-recurrent, and we try to demonstrate how this manifests in reality.

A Plotting four 1-dimensional symmetric random walks in Python

```
import random
from  matplotlib import pyplot as plt
import numpy as np

def walkGenerator(T):
    i = 0
    walk = [0]
    t = 1
    while t < T:
        d = random.randint(0,1)
        if d == 0:
            d = -1
        i = i + d
        walk.append(i)
        t=t+1
    return walk

def plotWalk(T):
    plt.plot(list(range(0,(T))), walkGenerator(T), [0]*T)
    plt.xlabel("Time")
    plt.ylabel("State")
    plt.show()

plotWalk(10)
plotWalk(100)
plotWalk(1000)
plotWalk(1000000)
```

B Plotting four 2-dimensional symmetric random walks in Python

```
import random
from  matplotlib import pyplot as plt
import numpy as np

def twoDwalkGenerator(T):
    i = np.array([0,0])
    walkX = [0]
    walkY = [0]
    t = 1
    while t < T:
        d = random.randint(0,3)
        if d == 0:
            d = np.array([-1,0])
        elif d == 1:
            d = np.array([1,0])
        elif d == 2:
            d = np.array([0,-1])
        else:
            d = np.array([0,1])
        i = i + d
        walkX.append(i[0])
        walkY.append(i[1])
        t=t+1
    return walkX, walkY

def plotTwoDWalk(T):
    Walk = twoDwalkGenerator(T)
    X = list(Walk[0])
    Y = list(Walk[1])
    gradUp = np.arange(0, (1+ 1/T), 1/T)
    gradDown = 1 - gradUp
    for i in range(0,T):
        plt.scatter(X[i], Y[i], color = (gradUp[i],gradDown[i],0))
    plt.plot(X,Y, color = (0,0,0))
    plt.xlabel("X")
    plt.ylabel("Y")
    plt.show()

plotTwoDWalk(10)
plotTwoDWalk(100)
plotTwoDWalk(1000)
plotTwoDWalk(10000)
```

C Simulating 1-dimensional symmetric random walks until first recurrence in Python

```
import random

def timesOfRec(n):
    j = 1
    tTaken = []
    while j <= n:
        i=0
        i0 = random.randint(0,1)
        if (i0 == 0):
            i = -1
        else:
            i = 1
        t = 1
        while i != 0:
            d0 = random.randint(0,1)
            if (d0 == 0):
                d = -1
            else:
                d = 1
            i = i + d
            t = t + 1
        tTaken.append(t)
        j = j + 1
    return tTaken

def avg(l):
    return sum(l)/len(l)

sample = timesOfRec(1000)
print(sample)
print(avg(sample))
print(max(sample))
print(sample.count(2)/len(sample))
```

D Plotting a histogram of the number of steps (up to 400) it took 1000 walks to first return to 0 in Python

```python
import random
from  matplotlib import pyplot as plt
import numpy as np

def timesOfRec(n):
    j = 1
    tTaken = []
    while j <= n:
        i=0
        i0 = random.randint(0,1)
        if (i0 == 0):
            i = -1
        else:
            i = 1
        t = 1
        while i != 0:
            d0 = random.randint(0,1)
            if (d0 == 0):
                d = -1
            else:
                d = 1
            i = i + d
            t = t + 1
        tTaken.append(t)
        j = j + 1
    return tTaken

data = timesOfRec(1000)

bins = np.arange(0,400, 1)
plt.hist(data, bins=bins, alpha=1)
plt.xlabel("Number of steps until first recurrence")
plt.ylabel("Number of walks (out of 1000 total) which recurred in
                                 $x$ steps")
plt.show()
```

E Simulating 2-dimensional symmetric random walks until first recurrence in Python

```python
import random
import numpy as np

def timesOfRec(n):
    j = 1
    tTaken = []
    while j <= n:
        i= np.array([0,0])
        i0 = random.randint(0,3)
        if (i0 == 0):
            i = np.array([-1,0])
        elif(i0 == 1):
            i = np.array([1,0])
        elif(i0 == 2):
            i = np.array([0,-1])
        else:
            i = np.array([0,1])
        t = 1
        while i[0] != 0 or i[1] != 0:
            d0 = random.randint(0,3)
            if (d0 == 0):
                d = np.array([-1,0])
            elif(d0 == 1):
                d = np.array([1,0])
            elif(d0 == 2):
                d = np.array([0,-1])
            else:
                d = np.array([0,1])
            i = i + d
            t = t + 1
        tTaken.append(t)
        j = j + 1
    return tTaken

print(timesOfRec(1))
```

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