NASIC problem specification

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1 Introduction

Notation:

- I: num. airfields
- J: num. tasks
- L: num. aircraft types
- $b(i, l)$: num. l-type aircraft available at airfield i
- $B(i)$: max. num. aircraft able to be dispatched from airfield i
- $A(l)$: total num. l-type aircraft available
- $m(j, l)$: how much l-type aircraft contribute to task j
- $M(j)$: how much task j needs
- $r(l, l', j)$: how much coverage l'-type aircraft provide for l-type aircraft executing task j
- $R(i, j, l)$: how much coverage each l-type aircraft from airfield i executing task j needs
- $C(i, j, l)$: Range of l-type aircraft from airfield i executing task l
- $h(l)$: Range of l-type aircraft
- $N(i, j)$: set of airfields from which an aircraft can cover another aircraft executing task j from airfield i
- $Q(i, j, l, i')$: set of aircraft types in airfield i' that can cover an l-type aircraft from airfield i executing task j

2 Constraints

- **Limited cover availability:** Only certain aircrafts from certain airfields are available for cover

$$y_{i,l,i',j',l'} \text{ only for } i \in N(i', j') \text{ and } l \in Q(i', j', l', i) \tag{1}$$

- **Limited airfield and type resource:** The sum of l-type primary and cover aircrafts dispatched from a airfield i cannot exceed the number of l-type aircrafts stationed there

$$\sum_j x_{i,j,l} + \sum_{i',j',l'} y_{i,l,i',j',l'} \leq b(i, l) \qquad \forall i, l \tag{2}$$

- **Limited airfield resource:** The sum of any type of primary and cover aircrafts dispatched from a airfield i cannot exceed the total number aircrafts stationed there

$$\sum_{j,l} x_{i,j,l} + \sum_{l,i',j',l'} y_{i,l,i',j',l'} \leq B(i) \qquad \forall i \tag{3}$$

- **Limited type resource:** The sum of l-type primary and cover aircrafts dispatched from all airfield cannot exceed the total number of l-type aircrafts

$$\sum_{i,j} x_{i,j,l} + \sum_{i,i',j',l'} y_{i,l,i',j',l'} \leq A(l) \qquad \forall l \tag{4}$$

- **Tasks must be fulfilled:** The sum of contributions of all aircrafts dispatched to fulfill task j must exceed the resource needed by that task

$$\sum_{i,l} m(j,l) x_{i,j,l} \geq M(j) \qquad \forall j \tag{5}$$

- **Aircraft range is limited:** We can not dispatch aircrafts for tasks j if the necessary range is beyond the maximum range of these aircrafts

$$h(l) < C(i,j,l) \Rightarrow x_{i,j,l} = 0 \qquad \forall i,j,l \tag{6}$$

- **Cover must be provided:** The sum of cover contributions to must exceed the cover needed.

$$\sum_{i',l'} r(l,l',j) y_{i',l',i,j,l} \geq R(i,j,l) x_{i,j,l} \qquad \forall i,j,l \tag{7}$$

3 Instance ensembles

Simplifications:

- Set $A(l) = \sum_i b(i,l)$
- Set $B(i) = \sum_l b(i,l)$
- Have $y_{i,l,i',j',l'}$ only when $(i,l) \in K_2(i',j',l')$
- Have $x_{i,j,l}$ only when $l \in K_1(i,j)$
- $r(l,l',j) = r(j) = 1$
- $R(i,j,l) = R(i,j) = 1$

Oversimplifications:

- $L = 1$
- I small
- J small
- $m(j,l) = 1$
- $M(j)$ small
- $b(i,l) = b(i)$
- (Maybe) $K_2(i',j',l') = I \times L = I \times \{1\}$

- (Maybe) $K_1(i,j) = L$

Remaining:

- I, J, L
- $b(i,l)$
- $m(j,l)$, $M(j)$
- $r(l,l',j)$, $R(i,j,l)$
- $N(i,j)$, $C(i,j,l)$, $h(l)$, $Q(i,j,l,i') \Rightarrow K_1, K_2$

Reduced problem: $\mathbf{x} = \{x_{i,j}\}$, $\mathbf{y} = \{y_{i,i',j'}\}$

- Constraint (1) always fulfilled
- Constraint (2) $\forall i$: $\sum_j x_{i,j} + \sum_{i',j'} y_{i,i',j'} \leq b(i,l)$
- (3) $\Leftrightarrow$ (2)
- (4) implies (2)
- Constraint (5) $\forall j$: $\sum_i x_{i,j} \geq M(j)$
- Constraint (6) always fulfilled
- Constraint (7) $\forall i,j$: $\sum_{i'} y_{i',i,j} \geq x_{i,j}$

(For all sums, $j \in [J]$, $i \in [I]$.)

- $I = 2, 3, 4$
- $J = I, 2I, 3I$
- $M(j) = 1, 2, 3$
- $b(i) = 1, 2, \ldots, 5$

Is this hard?

4 QUBO

4.1 Reduced problem

Variables

- $x_{i,j} \in \{0, 1, \ldots, b(i)\}$
- $y_{i,i',j'}\{0, 1, \ldots, b(i)\}$

1. Binary representation of $x_{i,j}$ and $y_{i,i',j'}$:

$$x_{i,j} = \sum_{\alpha=0}^{D_i-1} 2^\alpha x_{i,j,\alpha} + \sum_{\alpha=1}^{b(i)-2^{D_i}+1} \alpha x_{i,j,D_i-1+\alpha}$$

$$y_{i,i',j'} = \sum_{\alpha=0}^{D_i-1} 2^\alpha y_{i,i',j',\alpha} + \sum_{\alpha=1}^{b(i)-2^{D_i}+1} \alpha y_{i,i',j',D_i-1+\alpha}$$

with $D_i = \log_2 b(i) \in \mathbb{N}$. QUBO contribution due to the unary representation of the second term

$$C_{\text{one}} = \sum_{ij} \left(\sum_{\alpha=1}^{b(i)-2^{D_i}+1} x_{i,j,D_i-1+\alpha} - 1 \right)^2 + \sum_{ii'j} \left(\sum_{\alpha=1}^{b(i)-2^{D_i}+1} y_{i,i',j,D_i-1+\alpha} - 1 \right)^2$$

2. Unary representation of $x_{i,j}$ and $y_{i,i',j'}$:

$$x_{i,j} = \sum_{\alpha=0}^{b(i)} \alpha x_{i,j,\alpha}$$
$$y_{i,j} = \sum_{\alpha=0}^{b(i)} \alpha y_{i,i',j,\alpha}$$

QUBO contribution to ensure a single value of each variable

$$C_{\text{one}} = \sum_{ij} \left(\sum_{\alpha=0}^{b(i)} x_{i,j,\alpha} - 1 \right)^2 + \sum_{ii'j} \left(\sum_{\alpha=0}^{b(i)} y_{i,i',j,\alpha} - 1 \right)^2$$

- Incorporation of constraint (2):

$$0 \leq \underbrace{b(i) - \sum_j x_{i,j} - \sum_{i',j'} y_{i,i',j'}}_{=:z_i^1} \leq b(i)$$

1. Binary representation of the slack variable z_i^1:

$$z_i^1 = \sum_{\alpha=0}^{D_i-1} 2^\alpha z_{i,\alpha}^1 + \sum_{\alpha=1}^{b(i)-2^{D_i}+1} \alpha z_{i,D_i-1+\alpha}^1$$

QUBO contribution due to the unary representation of the second term

$$C_{2,\text{one}} = \sum_i \left(\sum_{\alpha=1}^{b(i)-2^{D_i}+1} z_{i,D_i-1+\alpha}^1 - 1 \right)^2$$

2. Unary representation of the slack variable z_i^1:

$$z_i^1 = \sum_{\alpha=0}^{b(i)} \alpha z_{i,\alpha}^1$$

QUBO contribution in case of unary representation

$$C_{2,\text{one}} = \sum_i \left(\sum_{\alpha=0}^{b(i)} z_{i,\alpha}^1 - 1 \right)^2$$

QUBO contribution

$$C_2 = \sum_i \left(b(i) - \sum_j x_{i,j} - \sum_{i',j'} y_{i,i',j'} - z_i^1 \right)^2$$

- Incorporation of constraint (5):

$$0 \leq \underbrace{M(j) - \sum_i x_{i,j}}_{=:z_j^2} \leq M(j)$$

1. Binary representation of the slack variable z_j^2:

$$z_j^2 = \sum_{\alpha=0}^{D_j-1} 2^\alpha z_{j,\alpha}^2 + \sum_{\alpha=1}^{M(j)-2_j^D+1} \alpha z_{j,D_j-1+\alpha}^2$$

with $D_j = \log_2 M(j) \in \mathbb{N}$. QUBO contribution due to the unary representation of the second term

$$C_{5,\text{one}} = \sum_j \left(\sum_{\alpha=1}^{M(j)-2^{D_j}+1} z_{j,D_i-1+\alpha}^2 - 1 \right)^2$$

2. Unary representation of the slack variable z_j^2:

$$z_j^2 = \sum_{\alpha=0}^{M(j)} \alpha z_{j,\alpha}^2$$

QUBO contribution in case of unary representation

$$C_{5,\text{one}} = \sum_j \left(\sum_{\alpha=0}^{M(j)} z_{j,\alpha}^2 - 1 \right)^2$$

QUBO contribution

$$C_5 = \sum_j \left(M(j) - \sum_i x_{i,j} - z_j^2 \right)^2$$

- Incorporation of constraint (7):

$$0 \le \underbrace{\sum_{i'} y_{i',i,j} - x_{i,j}}_{=:z_{i,j}^3} \le \underbrace{\sum_i b(i) - 1}_{=:Z^3}$$

1. Binary representation of the slack variable $z_{i,j}^3$:

$$z_{i,j}^3 = \sum_{\alpha=0}^{D-1} 2^\alpha z_{i,j,\alpha}^3 + \sum_{\alpha=1}^{Z^3-2^D+1} \alpha z_{i,j,D-1+\alpha}^3$$

with $D = \log_2 Z^3 \in \mathbb{N}$. QUBO contribution due to the unary representation of the second term

$$C_{7,\text{one}} = \sum_{ij} \left(\sum_{\alpha=1}^{Z^3-2^D+1} z_{i,j,D-1+\alpha}^3 - 1 \right)^2$$

2. Unary representation of the slack variable z_j^2:

$$z_{i,j}^3 = \sum_{\alpha=0}^{Z^3} \alpha z_{i,j,\alpha}^3$$

QUBO contribution in case of unary representation

$$C_{7,\text{one}} = \sum_{ij} \left(\sum_{\alpha=0}^{Z^3} z_{i,j,\alpha}^3 - 1 \right)^2$$

QUBO contribution

$$C_7 = \sum_{ij} \left(\sum_{i'} y_{i',i,j} - x_{i,j} - z_{ij}^3 \right)^2$$

4.1.1 Estimation of the number of binary variables

1. **Binary representation** For the binary representation of $x_{i,j}$, we have following number of binary variables

$$
\begin{aligned}
N_x(i) &= \log_2 b(i) + \underbrace{b(i)}_{<2^{D_i+1}} - 2^{D_i} + 2 \\
&< \log_2 b(i) + 2^{D_i+1} - 2^{D_i} + 2 \\
&= \log_2 b(i) + 2^{D_i} + 2
\end{aligned}
$$

   Analogously we get

$$
\begin{aligned}
N_y(i) &< \log_2 b(i) + 2^{D_i} + 2 \\
N_z^1(i) &< \log_2 b(i) + 2^{D_i} + 2 \\
N_z^2(j) &< \log_2 M(j) + 2^{D_j} + 2 \\
N_z^3 &< \log_2 \left(\sum_i b(i) - 1 \right) + 2^D + 2
\end{aligned}
$$

   Hence, the total number of variables reads

$$
\begin{aligned}
N_{\text{total}}^{\text{bin. rep.}} &= \sum_i \big(N_x(i) + N_y(i) + N_z^1(i) \big) + \sum_j N_z^2(j) + \sum_{ij} N_z^3 \\
&< 3 \sum_i \big(\log_2 b(i) + 2^{D_i} \big) + \sum_j \big(\log_2 M(j) + 2^{D_j} \big) + IJ \log_2 \left(\sum_i b(i) - 1 \right) + IJ 2^D + 2IJ + 8
\end{aligned}
$$

2. **Unary representation** For the unary representation of all variables, the total number of variables reads

$$
\begin{aligned}
N_{\text{total}}^{\text{un. rep.}} &= \sum_i (b(i) + b(i) + b(i)) + \sum_j M(j) + \sum_{ij} \left(\sum_{i'} b(i') - 1 \right) \\
&= 3 \sum_i (b(i)) + \sum_j M(j) + IJ \left(\sum_{i'} b(i') - 1 \right)
\end{aligned}
$$

$\Rightarrow$ For small problems, the unary representation is favorable (see figure 1).

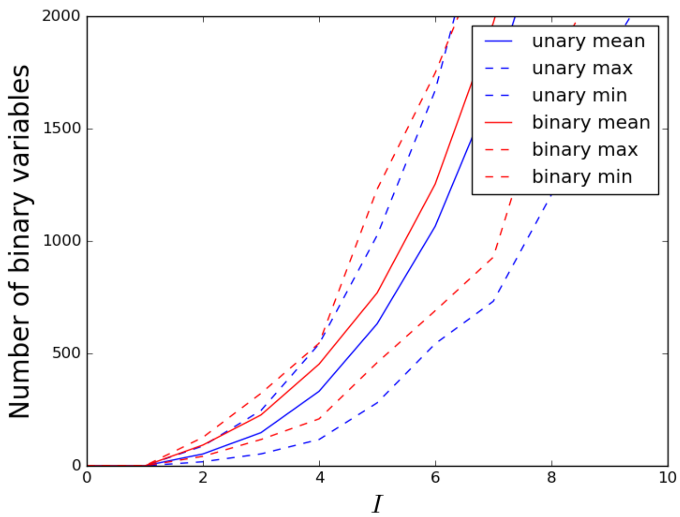

Figure 1: Scaling of the problem size with I by setting $J = 2I$ and using 1000 random samples for $M(j) \in \{1, 2, 3\}$ and $b(i) \in \{1, \ldots, 5\}$