23

a) $$a_n = \begin{cases} n, & n \text{ parzyste} \\ 1/n, & n \text{ nieparzyste} \end{cases}$$

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1}$$

$$= \sum_{k=0}^{\infty} 2k\, x^{2k} + \sum_{k=0}^{\infty} \frac{1}{2k+1} x^{2k+1}$$

$$= x \sum_{k=0}^{\infty} 2k\, x^{k-1} + \sum_{k=0}^{\infty} \left(\frac{s^{2k+1}}{2k+1} \right)_{s=0}^{s=x}$$

$$= x \sum_{k=0}^{\infty} (x^{2k})' + \sum_{k=0}^{\infty} \int_0^x s^{2k+2}\, ds$$

$$= x \left(\sum_{k=0}^{\infty} x^{2k} \right)' + \int_0^x \sum_{k=0}^{\infty} s^{2k+2}\, ds$$

$$= x \left(\sum_{k=0}^{\infty} (x^2)^k \right)' + \int_0^x s^2 \sum_{k=0}^{\infty} (s^2)^k\, ds$$

$$= x \left(\frac{1}{1-x^2} \right)' + \int_0^x s^2 \frac{1}{1-s^2}\, ds$$

$$= x \cdot \frac{2x}{(1-x^2)^2} + \int_0^x \frac{s^2 - 1 + 1}{1-s^2}\, ds$$

$$= \frac{2x^2}{(1-x^2)^2} + \int_0^x -1 + \frac{1}{1-s^2}\, ds$$

Rozkład na ułamki proste

$$\boxed{\begin{aligned} &\frac{1}{1-s^2} = \frac{1}{(1-s)(1+s)} = \frac{A}{1-s} + \frac{B}{1+s} \\ &\frac{1}{(1-s)(1+s)} = \frac{A(1+s) + B(1-s)}{(1-s)(1+s)} \\ &(A-B)s + (A+B) = 1 \\ &A = B = \tfrac{1}{2} \end{aligned}}$$

$$A(x) = \frac{2x^2}{(1-x^2)^2} - \int_0^x 1\, ds + \int_0^x \frac{1/2}{1-s} + \frac{1/2}{1+s}\, ds$$

$$= \frac{2x^2}{(1-x^2)^2} - s\Big|_0^x - \left(\frac{1}{2} \ln|1-s| \right)_0^x + \left(\frac{1}{2} \ln|1+s| \right)_0^x$$

$$= \frac{2x^2}{(1-x^2)^2} - x + \ln \sqrt{\left| \frac{1+x}{1-x} \right|}$$

$1\,(1 + x^1 + x^2 + \ldots) +$
$\frac{1}{2}(x^1 + x^2 + x^3 + \ldots) +$
$\frac{1}{3}(x^2 + x^3 + x^4 + \ldots) + \ldots$

b) $$\sum_{n=0}^{\infty} H_n x^n = \boxed{(1 + x + x^2 + \ldots) \sum_{n=1}^{\infty} \frac{1}{n} x^n}$$

$$= \frac{1}{1-x} \sum_{n=1}^{\infty} \int_0^x s^{n-1}\, ds = \frac{1}{1-x} \int_0^x \sum_{n=1}^{\infty} s^{n-1}\, ds$$

$$= \frac{1}{1-x} \int_0^x \frac{1}{1-s}\, ds = \frac{1}{1-x} \left(-\ln(1-s) \right)_0^x = \frac{\ln(1-x)}{x-1}$$