$$B(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}\,dt \;,\; p>0,\, q>0$$

a)

$$B(p,q+1) = \int_0^1 t^{p-1}(1-t)^q\,dt = \frac{1}{p}\int_0^1 (t^p)'(1-t)^q\,dt = \underbrace{\frac{1}{p}\Big[t^p(1-t)^q\Big]_0^1}_{0} - \frac{1}{p}\int_0^1 t^p(1-t)^{q-1}(-q)\,dt$$

$$= \frac{q}{p}\int_0^1 t^p(1-t)^{q-1}\,dt = \frac{q}{p}\int_0^1 \Big(t^{p-1} - \underset{(t^p - t^{p-1} + t^{p-1})}{t^{p-1}(1-t)}\Big)(1-t)^{q-1}\,dt = \frac{q}{p}\left(\int_0^1 t^{p-1}(1-t)^{q-1}\,dt - \int_0^1 t^{p-1}(1-t)^q\,dt\right)$$

$$= \frac{q}{p}B(p,q) - \frac{q}{p}B(p,q+1)$$

$$B(p,q+1) = \frac{q}{p}B(p,q) - \frac{q}{p}B(p,q+1)$$

$$B(p,q+1) + \frac{q}{p}B(p,q+1) = \frac{q}{p}B(p,q)$$

$$\frac{p+q}{p}B(p,q+1) = \frac{q}{p}B(p,q)$$

$$B(p,q+1) = \frac{q}{p+q}B(p,q)$$

b)

(*) Pokażmy, że $B(p,q) = B(q,p)$

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt = \left|\begin{matrix} 1-t = x \\ -dt = dx\end{matrix}\right| = -\int_1^0 (1-x)^{p-1}x^{q-1}\,dx = \int_0^1 (1-x)^{p-1}x^{q-1}\,dx = B(q,p)$$

$$B(p,q+1) + B(p+1,q) \overset{(*)}{=} B(p,q+1) + B(q,p+1) \overset{a)}{=} \frac{q}{p+q}B(p,q) + \frac{p}{q+p}B(q,p) \overset{(*)}{=} B(p,q)\underbrace{\left(\frac{q}{p+q} + \frac{p}{q+p}\right)}_{=1} = B(p,q)$$