$X \perp Y$

$X \sim \text{Gamma}(b, p) \qquad U = X + Y$

$Y \sim \text{Gamma}(b, q) \qquad V = \frac{X}{X+Y}$

Przechodzimy z (X,Y) na (Z,V)

$$G(x,y) := \left(\frac{x}{x+y}, x+y\right)$$

$$G^{-1}(v,u) = (uv, u - uv)$$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial v} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial u} \end{vmatrix} = \begin{vmatrix} u & v \\ -u & 1-v \end{vmatrix} = u$$

Jako że X, Y są niezależne to ich gęstość jest iloczynem gęstości brzegowych.

$$f(x,y) = \frac{b^{p+q}}{\Gamma(p)\,\Gamma(q)}\, x^{p-1} y^{q-1} e^{-b(x+y)}$$

$$g(v,u) = \frac{b^{p+q}}{\Gamma(p)\,\Gamma(q)}\, u^{p+q-1} v^{p-1} (1-v)^{q-1} e^{-bu}$$

$$\begin{cases} 0 < x \\ 0 < y \end{cases} \Rightarrow \begin{cases} 0 < uv \\ 0 < u - uv \end{cases} \Rightarrow \begin{cases} u \in (0, \infty) \\ v \in (0,1) \end{cases}$$

b) $$g_u(u) = \frac{b^{p+q}}{\Gamma(p)\,\Gamma(q)}\, u^{p+q-1} e^{-bu} \int_0^1 v^{p-1} (1-v)^{q-1}\, dv$$

$$= \frac{b^{p+q}}{\Gamma(p)\,\Gamma(q)}\, u^{p+q-1} e^{-bu}\, B(p,q) = \frac{b^{p+q} u^{p+q-1} e^{-bu}}{\cancel{\Gamma(p)\,\Gamma(q)}} \; \frac{\cancel{\Gamma(p)\,\Gamma(q)}}{\Gamma(p+q)}$$

$$= \frac{b^{p+q} u^{p+q-1} e^{-bu}}{\Gamma(p+q)} \sim \text{Gamma}(b, p+q)$$

c) $$g_v(v) = \frac{b^{p+q}}{\Gamma(p)\,\Gamma(q)}\, v^{p-1} (1-v)^{q-1} \int_0^\infty u^{p+q-1} e^{-bu}\, du$$

$$= \frac{b^{p+q}}{\Gamma(p)\,\Gamma(q)}\, v^{p-1} (1-v)^{q-1} \int_0^\infty u^{p+q-1} e^{-bu}\, du = \left[\begin{array}{l} t = bu \\ u = \frac{t}{b} \\ dt = b\, du \end{array}\right]$$

$$= \frac{b^{p+q}}{\Gamma(p)\,\Gamma(q)}\, v^{p-1} (1-v)^{q-1} \int_0^\infty b^{-(p+q-1)} t^{q+p-1} e^{-t}\, \frac{dt}{b}$$

$$= \frac{\cancel{b^{p+q}}}{\Gamma(p)\,\Gamma(q)}\, v^{p-1} (1-v)^{q-1}\, \cancel{b^{-(p+q)}} \int_0^\infty t^{q+p-1} e^{-t}\, dt$$

$$= \frac{v^{p-1} (1-v)^{q-1}}{\Gamma(p)\,\Gamma(q)}\, \Gamma(p+q) = \frac{1}{B(p,q)}\, v^{p-1} (1-v)^{q-1}$$

a) $g_v(v)\, g_u(u) = g(u,v) \Rightarrow U \perp V$

$$g(v,u) = \frac{b^{p+q}}{\Gamma(p)\,\Gamma(q)}\, u^{p+q-1} v^{p-1} (1-v)^{q-1} e^{-bu}$$

$$g_v(v)\, g_u(u) = \frac{1}{B(p,q)}\, v^{p-1} (1-v)^{q-1}\, \frac{b^{p+q} u^{p+q-1} e^{-bu}}{\Gamma(p+q)}$$

$$= \frac{v^{p-1} (1-v)^{q-1}}{\Gamma(p)\,\Gamma(q)}\, \cancel{\Gamma(p+q)}\, \frac{b^{p+q} u^{p+q-1} e^{-bu}}{\cancel{\Gamma(p+q)}}$$

$$= \frac{b^{p+q}}{\Gamma(p)\,\Gamma(q)}\, u^{p+q-1} v^{p-1} (1-v)^{q-1} e^{-bu} = g(v,u)$$