$$\begin{cases} f_0 = g_0 \\ f_k = g_k - \sum_{i=0}^{k-1} \frac{\langle g_k, f_i \rangle}{\langle f_i, f_i \rangle} f_i \end{cases}$$

Dowód indukcyjny po k

T_k: Po k-tym kroku dostajemy układ ortogonalny funkcji

Baza $(k=0)$:

Jedna funkcja tworzy układ ortogonalny

Krok $(T_k \Rightarrow T_{k+1})$:

$$f_{k+1} = g_{k+1} - \sum_{i=0}^{k} \frac{\langle g_k, f_i \rangle_N}{\langle f_i, f_i \rangle_N} f_i$$

Weźmy dowolne $j < k+1$. Pokażemy, że $\langle f_j, f_{k+1} \rangle_N = 0$

$$\langle f_j, f_{k+1} \rangle_N = \left\langle f_j,\ g_{k+1} - \sum_{i=0}^{k} \frac{\langle g_{k+1}, f_i \rangle_N}{\langle f_i, f_i \rangle_N} f_i \right\rangle_N$$

$\langle f+h, g \rangle = \langle f, g \rangle + \langle h, g \rangle$

$$= \langle f_i, g_{k+1} \rangle_N - \left\langle f_j, \sum_{i=0}^{k} \frac{\langle g_{k+1}, f_i \rangle_N}{\langle f_i, f_i \rangle_N} f_i \right\rangle_N$$

$$= \langle f_i, g_{k+1} \rangle_N - \sum_{i=0}^{k} \left\langle f_j, \frac{\langle g_{k+1}, f_i \rangle_N}{\langle f_i, f_i \rangle_N} f_i \right\rangle_N$$

$$= \langle f_i, g_{k+1} \rangle_N - \sum_{i=0}^{k} \frac{\langle g_{k+1}, f_i \rangle_N}{\langle f_i, f_i \rangle_N} \langle f_j, f_i \rangle_N$$

Wiemy, że $\langle f_j, f_i \rangle_N = 0$ dla $j \neq i$, czyli

$$\langle f_i, g_{k+1} \rangle_N - \sum_{i=0}^{k} \frac{\langle g_{k+1}, f_i \rangle_N}{\langle f_i, f_i \rangle_N} \langle f_j, f_i \rangle_N =$$

$$\langle f_i, g_{k+1} \rangle_N - \frac{\langle g_{k+1}, f_j \rangle_N}{\cancel{\langle f_j, f_j \rangle_N}} \cancel{\langle f_j, f_j \rangle_N} =$$

$$\langle f_i, g_{k+1} \rangle_N - \langle g_{k+1}, f_j \rangle_N =$$

$$\langle f_i, g_{k+1} \rangle_N - \langle f_j, g_{k+1} \rangle_N = 0$$