a)

$$[a_n, b_n] \supset [a_{n+1}, b_{n+1}] \overset{*}{\iff} a_n \le a_{n+1} < b_{n+1} \le b_n$$

$$b_n > a_n \wedge m_{n+1} = \frac{a_n + b_n}{2} \Rightarrow a_n < m_{n+1} < b_n$$

Przypadek 1

$$[a_{n+1}, b_{n+1}] = [m_{n+1}, b_n]$$

$$\overset{*}{} \quad a_n \le m_{n+1} < b_n \le b_n \quad \checkmark$$

Przypadek 2

$$[a_{n+1}, b_{n+1}] = [a_n, m_{n+1}]$$

$$\overset{*}{} \quad a_n \le a_n < m_{n+1} \le b_n \quad \checkmark$$

b)

$$|b_{n+1} - a_{n+1}| = \left| \frac{b_n - a_n}{2} \right|$$

$$d_0 = |a_0 - b_0| \qquad d_n = \frac{|a_n - b_n|}{2}$$

Pokażę, że

$$d_n = \frac{|a_0 - b_0|}{2^n}$$

Indukcja

1) $d_0 = \frac{|a_0 - b_0|}{2^0} = |a_0 - b_0|$ ✓

2) $d_n = \frac{|a_0 - b_0|}{2^n} \Rightarrow d_{n+1} = \frac{|a_0 - b_0|}{2^{n+1}}$

$$d_{n+1} = \frac{|a_{n+1} - b_{n+1}|}{2} = \frac{|a_n - b_n|}{2^2} = \frac{d_n}{2} = \frac{|a_0 - b_0|}{2^{n+1}} \quad \checkmark$$

c)

$$|e_n| \le \frac{b_0 - a_0}{2^{n+1}}$$

$$e_n = \alpha - m_{n+1}$$

$$m_{n+1} = \frac{a_n + b_n}{2}$$

Przypadek 1

$$[m_{n+1}, b_{n+1}]$$

Wiemy, że

$$m_{n+1} \le \alpha \le b_{n+1}$$

$$a_{n+1} \le m_{n+1} \le b_{n+1}$$

Szacując dostajemy

$$|e_n| = |\alpha - m_{n+1}| \le |b_{n+1} - a_{n+1}| = \frac{b_0 - a_0}{2^{n+1}}$$

Przypadek 1

$$[a_{n+1}, m_{n+1}]$$

Wiemy, że

$$a_{n+1} \le \alpha \le m_{n+1}$$

$$a_{n+1} \le m_{n+1} \le b_{n+1}$$

Szacując dostajemy

$$|e_n| = |\alpha - m_{n+1}| \le |m_{n+1} - a_{n+1}| \le |b_{n+1} - a_{n+1}| = \frac{b_0 - a_0}{2^{n+1}}$$

d) Tak, np. $x_0 = b_0$