$$\boxed{\begin{array}{l} X \sim \chi^2(n),\ Y \sim \chi^2(k) \\ Z = \frac{X}{Y} \cdot \frac{k}{n} \end{array}}$$

X, Y są niezależne, więc

$$f_{\binom{X}{Y}}(x,y) = \left(\frac{\left(\frac{1}{2}\right)^{n/2}}{\Gamma\left(\frac{n}{2}\right)} x^{\frac{n}{2}-1} e^{-\frac{x}{2}} \right) \left(\frac{\left(\frac{1}{2}\right)^{k/2}}{\Gamma\left(\frac{k}{2}\right)} y^{\frac{k}{2}-1} e^{-\frac{y}{2}} \right) = \frac{e^{-(x+y)/2}}{2^{(n+k)/2}\, \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{k}{2}\right)} x^{\frac{n}{2}-1} y^{\frac{k}{2}-1}$$

Przechodzimy z (x,y) na (z,v)

$$\begin{cases} G\binom{x}{y} := \left(\frac{xk}{yn}, y \right) \\ z = \frac{xk}{yn}, \quad v = y \end{cases} \Rightarrow \quad G^{-1}(z,v) = \Big(\underbrace{\frac{nzv}{k}}_{x(z,v)}, \underbrace{v}_{y(z,v)} \Big), \qquad J = \begin{vmatrix} x'_z & x'_v \\ y'_z & y'_v \end{vmatrix} = \begin{vmatrix} \frac{nv}{k} & \frac{nz}{k} \\ 0 & 1 \end{vmatrix} = \frac{nv}{k}$$

$$f_{\binom{Z}{V}}(z,v) = f_{\binom{X}{Y}}\left(G^{-1}(z,v) \right) |J| = \frac{e^{-\left(\frac{nzv}{k} + v\right)/2}}{2^{(n+k)/2}\, \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{k}{2}\right)} \left(\frac{nzv}{k}\right)^{\frac{n}{2}-1} v^{\frac{k}{2}-1} \frac{nv}{k} = \frac{\frac{n}{k} \left(\frac{nz}{k}\right)^{\frac{n}{2}-1}}{2^{(n+k)/2}\, \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{k}{2}\right)} e^{-\frac{nz+k}{2k} v} v^{\left(\frac{k+n}{2} - 2\right)}$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{\binom{Z}{V}}(z,v)\, dv = \int_0^{\infty} \frac{\frac{n}{k} \left(\frac{nz}{k}\right)^{\frac{n}{2}-1}}{2^{(n+k)/2}\, \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{k}{2}\right)} e^{-\frac{nz+k}{2k} v} v^{\left(\frac{k+n}{2} - 1\right)} dv =$$

$$= \frac{\frac{n}{k} \left(\frac{nz}{k}\right)^{\frac{n}{2}-1}}{2^{(n+k)/2}\, \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{k}{2}\right)} \int_0^{\infty} v^{\left(\frac{k+n}{2} - 1\right)} e^{-\frac{nz+k}{2k} v} dv, \quad A := \frac{\frac{n}{k} \left(\frac{nz}{k}\right)^{\frac{n}{2}-1}}{2^{(n+k)/2}\, \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{k}{2}\right)}$$

Wiemy, że $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt$ stąd

$$f_Z(z) = A \int_0^{\infty} v^{\left(\frac{k+n}{2} - 1\right)} e^{-\frac{nz+k}{2k} v} dv = \begin{bmatrix} t = \frac{nz+k}{2k} v \\ dt = \frac{nz+k}{2k} dv \\ dv = \frac{2k}{nz+k} dt \\ v = \frac{2k}{nz+k} t \end{bmatrix} = A \int_0^{\infty} \left(\frac{2k}{nz+k} t \right)^{\left(\frac{k+n}{2} - 1\right)} e^{-t} \frac{2k}{nz+k} dt$$

$$= A \left(\frac{2k}{nz+k}\right)^{\frac{k+n}{2}} \int_0^{\infty} t^{\left(\frac{k+n}{2} - 1\right)} e^{-t} dt = A \left(\frac{2k}{nz+k}\right)^{\frac{k+n}{2}} \Gamma\left(\frac{k+n}{2}\right)$$

Z definicji $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

$$f_Z(z) = \frac{\frac{n}{k} \left(\frac{nz}{k}\right)^{\frac{n}{2}-1} \left(\frac{2k}{nz+k}\right)^{\frac{k+n}{2}}}{2^{(n+k)/2}} \frac{\Gamma\left(\frac{k+n}{2}\right)}{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{k}{2}\right)} = \frac{\frac{n}{k} \left(\frac{nz}{k}\right)^{\frac{n}{2}-1} \left(\frac{2k}{nz+k}\right)^{\frac{k+n}{2}}}{2^{(n+k)/2}\, B\left(\frac{n}{2}, \frac{k}{2}\right)}$$

$$= \frac{\frac{nz}{k} \left(\frac{nz}{k}\right)^{\frac{n}{2}-1} \left(\frac{2k}{nz+k}\right)^{\frac{k+n}{2}}}{2^{(n+k)/2} \cdot z B\left(\frac{n}{2}, \frac{k}{2}\right)} = \frac{\left(\frac{nz}{k}\right)^{\frac{n}{2}} \left(\frac{2k}{nz+k}\right)^{\frac{k+n}{2}}}{2^{(n+k)/2} \cdot z B\left(\frac{n}{2}, \frac{k}{2}\right)} = \sqrt{\frac{\left(\frac{nz}{k}\right)^{n} \left(\frac{2k}{nz+k}\right)^{k+n}}{2^{n+k}}} \Big/ z B\left(\frac{n}{2}, \frac{k}{2}\right)$$

$$= \sqrt{\frac{(nz)^n\ \cancel{2^{n+k}}\ k^{\cancel{n}+k}}{\cancel{k^n}\ \cancel{2^{n+k}}\ (nz+k)^{n+k}}} \Big/ z B\left(\frac{n}{2}, \frac{k}{2}\right) = \sqrt{\frac{(nz)^n \cdot k^k}{(nz+k)^{n+k}}} \Big/ z B\left(\frac{n}{2}, \frac{k}{2}\right) \Rightarrow \quad Z \sim F(n,k)$$