

Z1

$$\|f\| = \sqrt{\sum_{k=0}^N p(x_k) f(x_k)^2}$$

I) $\|f\| = 0 \Rightarrow f = 0$

$$\sqrt{\sum_{k=0}^N \underset{>0}{p(x_k)} \underset{\geq 0}{f(x_k)^2}} = 0$$

suma nieujemnych wyrazów

II) $\|\alpha f\| = |\alpha| \|f\|$

$$\sqrt{\sum_{k=0}^N p(x_k) (\alpha f(x_k))^2} = |\alpha| \sqrt{\sum_{k=0}^N p(x_k) f(x_k)^2} = |\alpha| \|f\|$$

III) $\|g + f\| \leq \|g\| + \|f\|$

$$\sqrt{\sum_{k=0}^N p(x_k) (f(x_k) + g(x_k))^2} = \sqrt{\sum_{k=0}^N f(x_k)^2 p(x_k) + 2 \sum_{k=0}^N f(x_k) g(x_k) p(x_k) + \sum_{k=0}^N g(x_k)^2 p(x_k)} =$$

$$\sqrt{\sum_{k=0}^N f(x_k)^2 p(x_k) + 2 \sum_{k=0}^N f(x_k) g(x_k) p(x_k) + \sum_{k=0}^N g(x_k)^2 p(x_k)} \leq \sqrt{\sum_{k=0}^N f(x_k)^2 p(x_k) + 2 \sqrt{\sum_{k=0}^N f(x_k)^2 p(x_k)} \sqrt{\sum_{k=0}^N g(x_k)^2 p(x_k)} + \sum_{k=0}^N g(x_k)^2 p(x_k)}$$

$(f(x_k) \sqrt{p(x_k)}) \cdot (g(x_k) \sqrt{p(x_k)})$

$$\left(\sum_{i=1}^n u_i v_i \right)^2 \leq \left(\sum_{i=1}^n u_i^2 \right) \left(\sum_{i=1}^n v_i^2 \right)$$

$$\sqrt{\sum_{k=0}^N f(x_k)^2 p(x_k) + 2 \sqrt{\sum_{k=0}^N f(x_k)^2 p(x_k)} \sqrt{\sum_{k=0}^N g(x_k)^2 p(x_k)} + \sum_{k=0}^N g(x_k)^2 p(x_k)} = \sqrt{\left(\sqrt{\sum_{k=0}^N f(x_k)^2 p(x_k)} + \sqrt{\sum_{k=0}^N g(x_k)^2 p(x_k)} \right)^2} = \sqrt{\sum_{k=0}^N f(x_k)^2 p(x_k)} + \sqrt{\sum_{k=0}^N g(x_k)^2 p(x_k)}$$

$\|f\| + \|g\|$