Pokażmy, że $rząd(Q_n) \geq n+1 \iff Q_n(f) = \int_a^b L_n^f(x)\,dx$

- $Q_n(f) = \int_a^b L_n^f(x)\,dx \Rightarrow rząd(Q_n) \geq n+1$

$$Q_n(w) = \int_a^b L_n^w(x)\,dx = \int_a^b w(x)\,dx$$

$$\boxed{w \in \Pi_n \Rightarrow L_n^w(w) = w}$$

Wiemy, że $\forall_{w \in \Pi_n} \left(Q_n(w) = \int_a^b w(x)\,dx \right) \Rightarrow rząd(Q_n) \geq n+1$

czyli $rząd(Q_n) \geq n+1$

- $rząd(Q_n) \geq n+1 \Rightarrow Q_n(f) = \int_a^b L_n^f(x)\,dx$

$$Q_n := \sum_{k=0}^{n} A_k f(x_k)$$

Weźmy dowolny $\lambda_j(x) = \prod_{\substack{i=0 \\ i \neq j}}^{n} \frac{x - x_i}{x_j - x_i} \in \Pi_n$

$rząd(Q_n) \geq n+1 \Rightarrow Q_n(\lambda_j) = \int_a^b \lambda_j(x)\,dx$

$$Q_n(\lambda_j) = \sum_{k=0}^{n} A_k \lambda_j(x_k) = A_j \lambda_j(x_j) = A_j$$

$$\boxed{\lambda_j(x) = \prod_{\substack{i=0 \\ i \neq j}}^{n} \frac{x - x_i}{x_j - x_i} \qquad \forall_{x_k \neq x_j} \lambda_j(x_k) = 0} \qquad \boxed{\lambda_j(x_j) = 1}$$

Skoro $Q_n(\lambda_j) = A_j \Rightarrow A_j = \int_a^b \lambda_j(x)\,dx$.

$$Q_n(f) := \sum_{k=0}^{n} f(x_k) A_k = \sum_{k=0}^{n} f(x_k) \int_a^b \lambda_k(x)\,dx =$$

$$= \sum_{k=0}^{n} \int_a^b f(x_k)\lambda_k(x)\,dx = \int_a^b \underbrace{\sum_{k=0}^{n} f(x_k)\lambda_k(x)}_{\text{Lagrange}}\,dx =$$

$$= Q_n(f) = \int_a^b L_n^f(x)\,dx$$

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