The Ancient Secrets of Computer Vision

Previously On

the

Ancient Secrets of Computer Vision

Say we want affine transformation

- How many knowns do we get with one match? **mA = n**
  - $n_x = a_{00}*m_x + a_{01}*m_y + a_{02}*1$
  - $n_y = a_{10}*m_x + a_{11}*m_y + a_{12}*1$
  - Solve **M a = b**
    - $M^T M a = M^T b$
    - $a = (M^T M)^{-1} M^T b$
  - Still works if overdetermined
    - WHY???

$$
\begin{array}{ccc}
\mathbf{M} & \mathbf{a} & \mathbf{b}
\end{array}
$$

$$
\begin{bmatrix}
m_{x1} & m_{y1} & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & m_{x1} & m_{y1} & 1 \\
m_{x2} & m_{y2} & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & m_{x2} & m_{y2} & 1 \\
m_{x3} & m_{y3} & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & m_{x3} & m_{y3} & 1 \\
 & & \dots & & & \\
 & & \dots & & &
\end{bmatrix}
\begin{bmatrix}
a_{00} \\ a_{01} \\ a_{02} \\ a_{10} \\ a_{11} \\ a_{12}
\end{bmatrix}
=
\begin{bmatrix}
n_{x1} \\ n_{y1} \\ n_{x2} \\ n_{y2} \\ n_{x3} \\ n_{y3}
\end{bmatrix}
$$

Linear least squares

Want to minimize squared error: $|| b - M a ||^2$ =

$b^Tb - 2a^TM^Tb + a^TM^TMa$

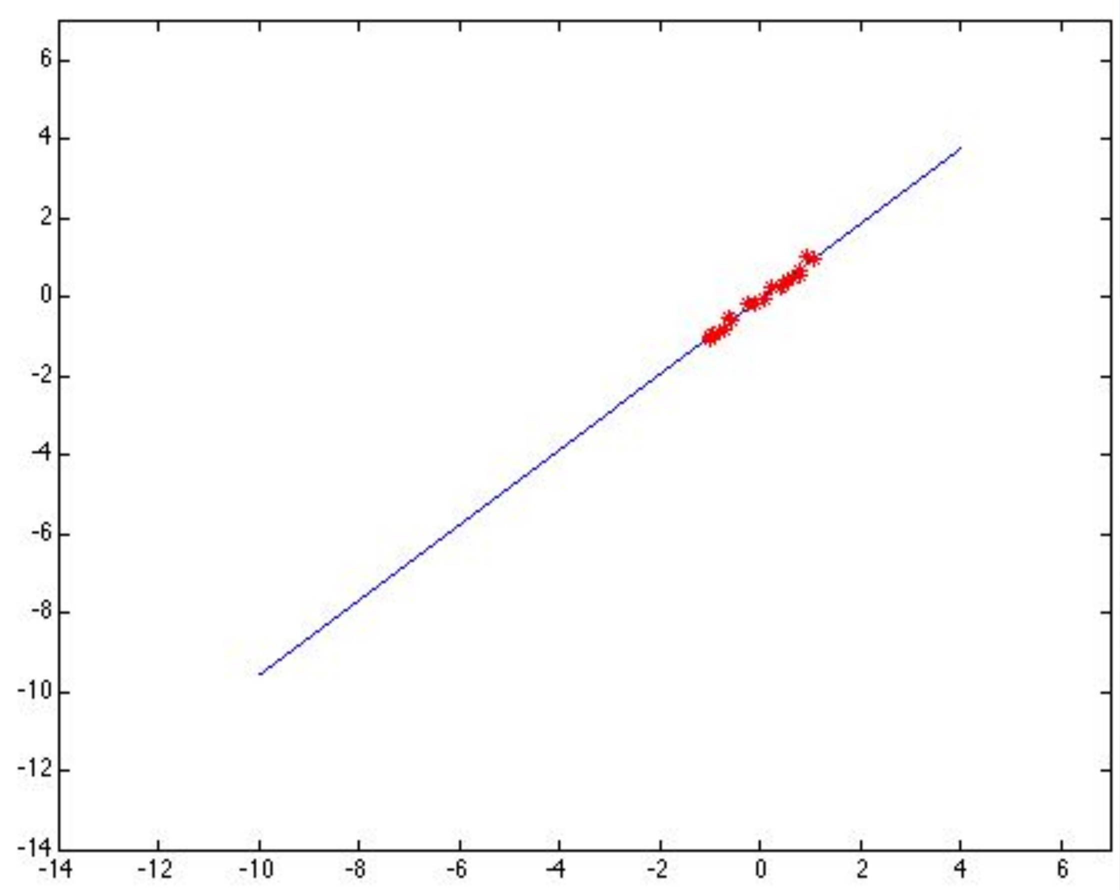
This is convex and minimized when gradient = 0. So we take the derivative wrt a and set = 0.

$-M^Tb + (M^TM)a = 0$

$(M^TM)a = M^Tb$

$a = (M^TM)^{-1}M^Tb$ yay!!!!

So how does least squares do?

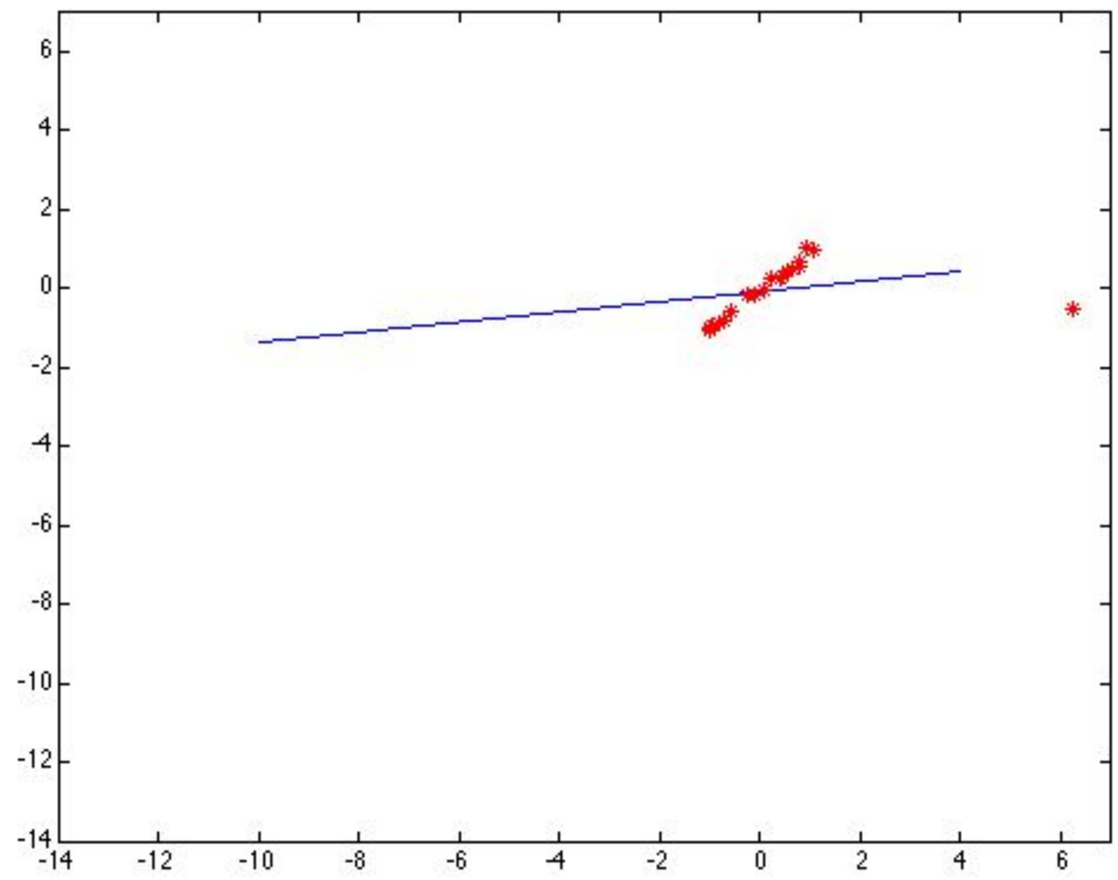


So how does least squares do?

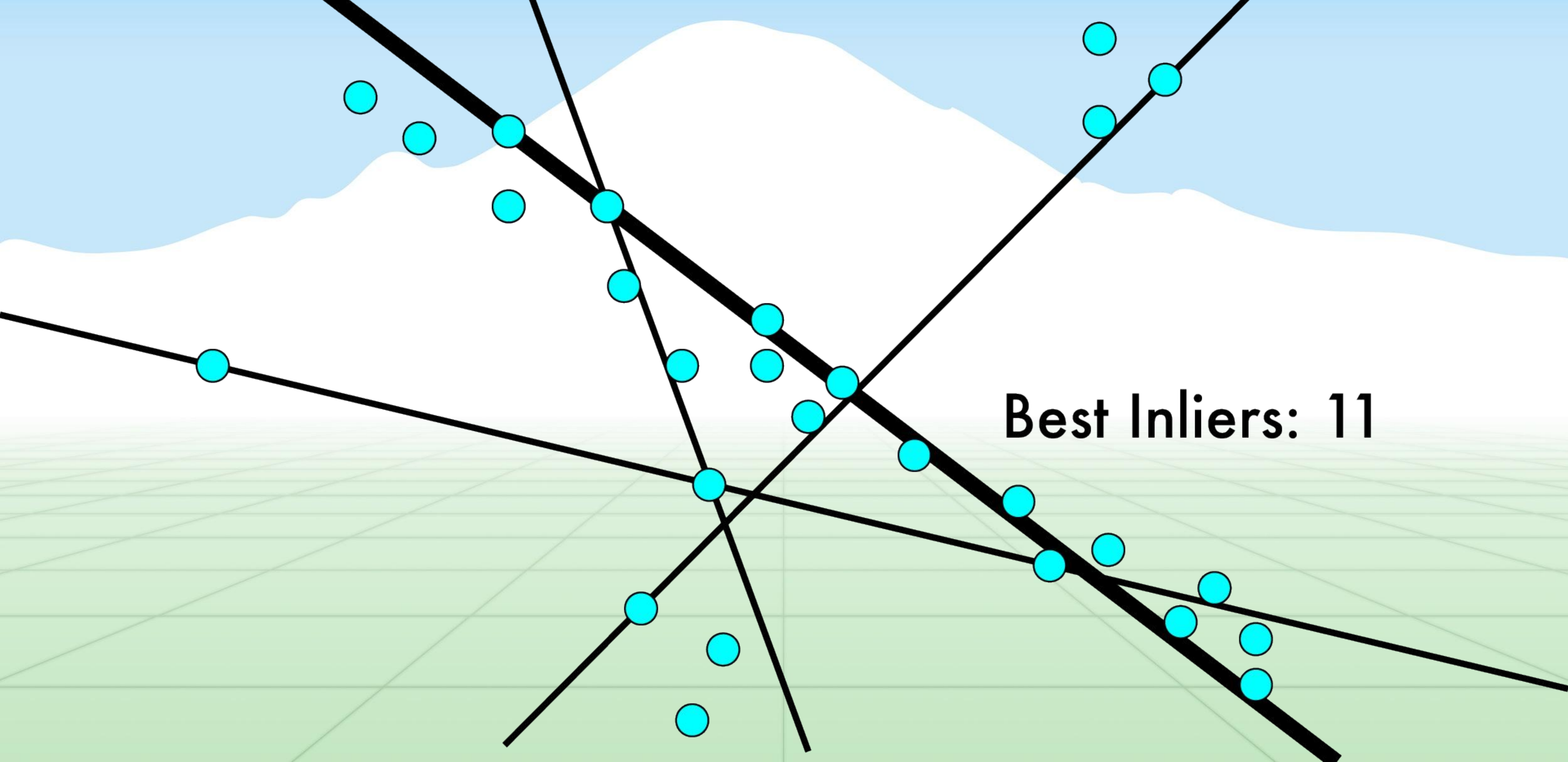
Error based on **squared** residual

Very scared of being wrong, even for just one point

Very bad at handling outliers in data



RANSAC: RANdom SAmple Consensus



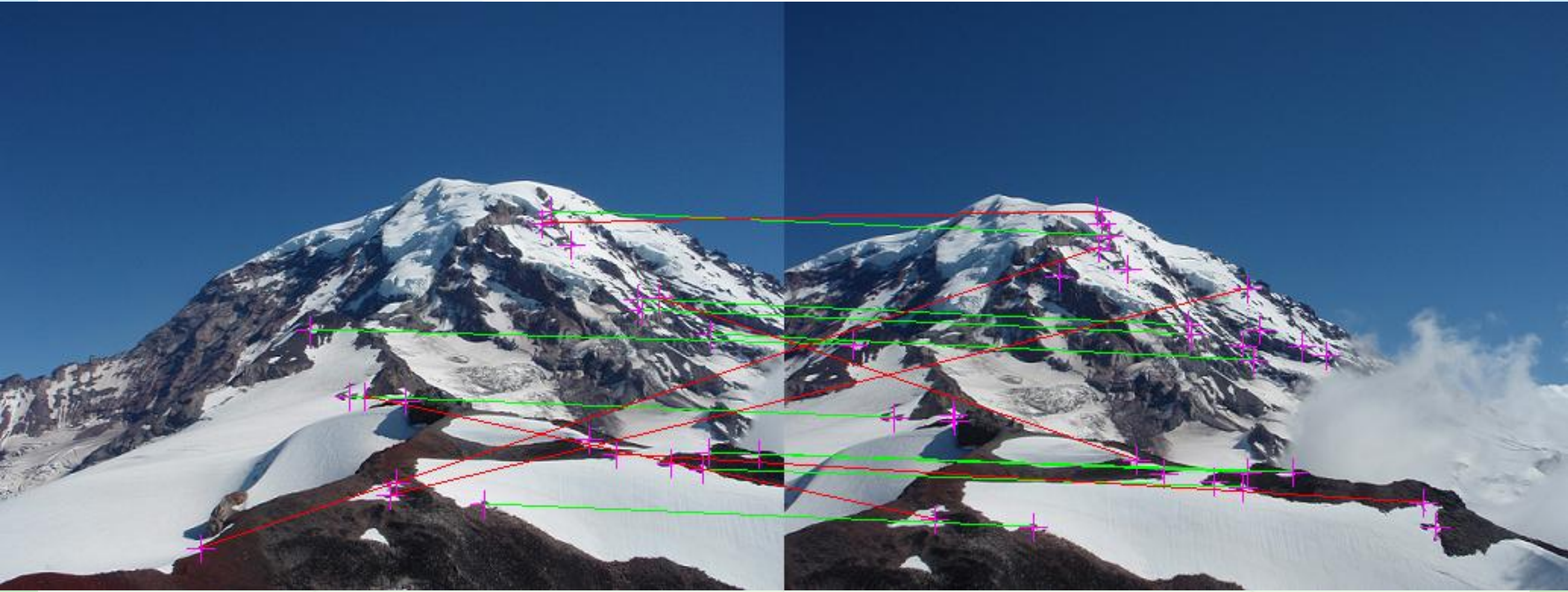
RANSAC: RANdom SAmple Consensus

- Parameters: data, model, n points to fit model, k iterations, t threshold, d “good” fit cutoff

```
bestmodel = None
bestfit = INF
While i < k:
     sample = draw n random points from data
     Fit model to sample
     inliers = data within t of model
     if inliers > bestfit:
          Fit model to all inliers
          bestfit = fit
          bestmodel = model
          if inliers > d:
               return model
return bestmodel
```

RANSAC: RANdom **SA**mple **C**onsensus

- Works well even with extreme noise.



We want projective (homography)

- New matrix equations:

$$
\underbrace{\begin{bmatrix}
m_{x1} & m_{y1} & 1 & 0 & 0 & 0 & -m_{x1}n_{x1} & -m_{y1}n_{x1} \\
0 & 0 & 0 & m_{x1} & m_{y1} & 1 & -m_{x1}n_{y1} & -m_{y1}n_{y1} \\
m_{x2} & m_{y2} & 1 & 0 & 0 & 0 & -m_{x2}n_{x2} & -m_{y2}n_{x2} \\
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m_{x3} & m_{y3} & 1 & 0 & 0 & 0 & -m_{x3}n_{x3} & -m_{y3}n_{x3} \\
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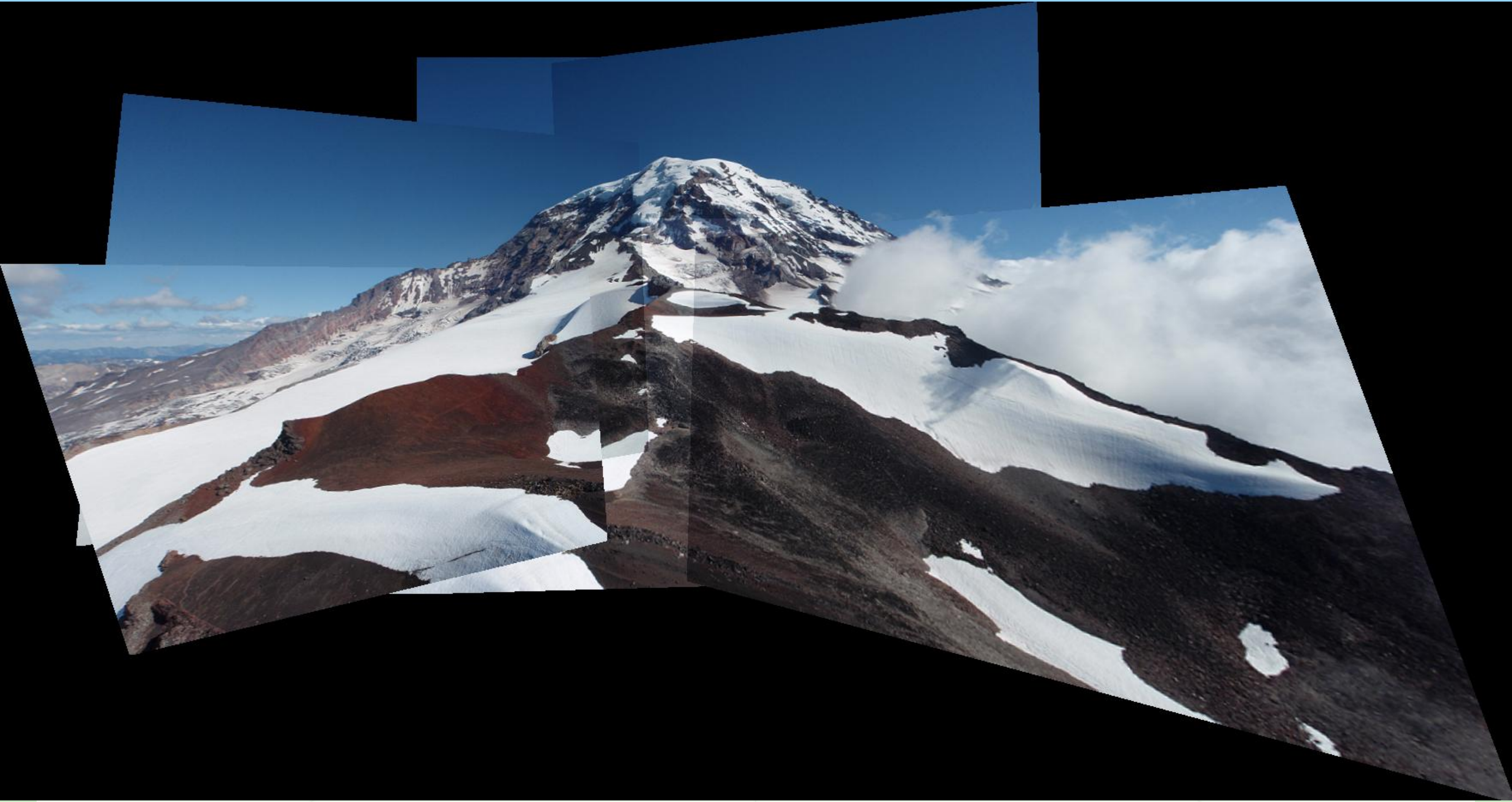
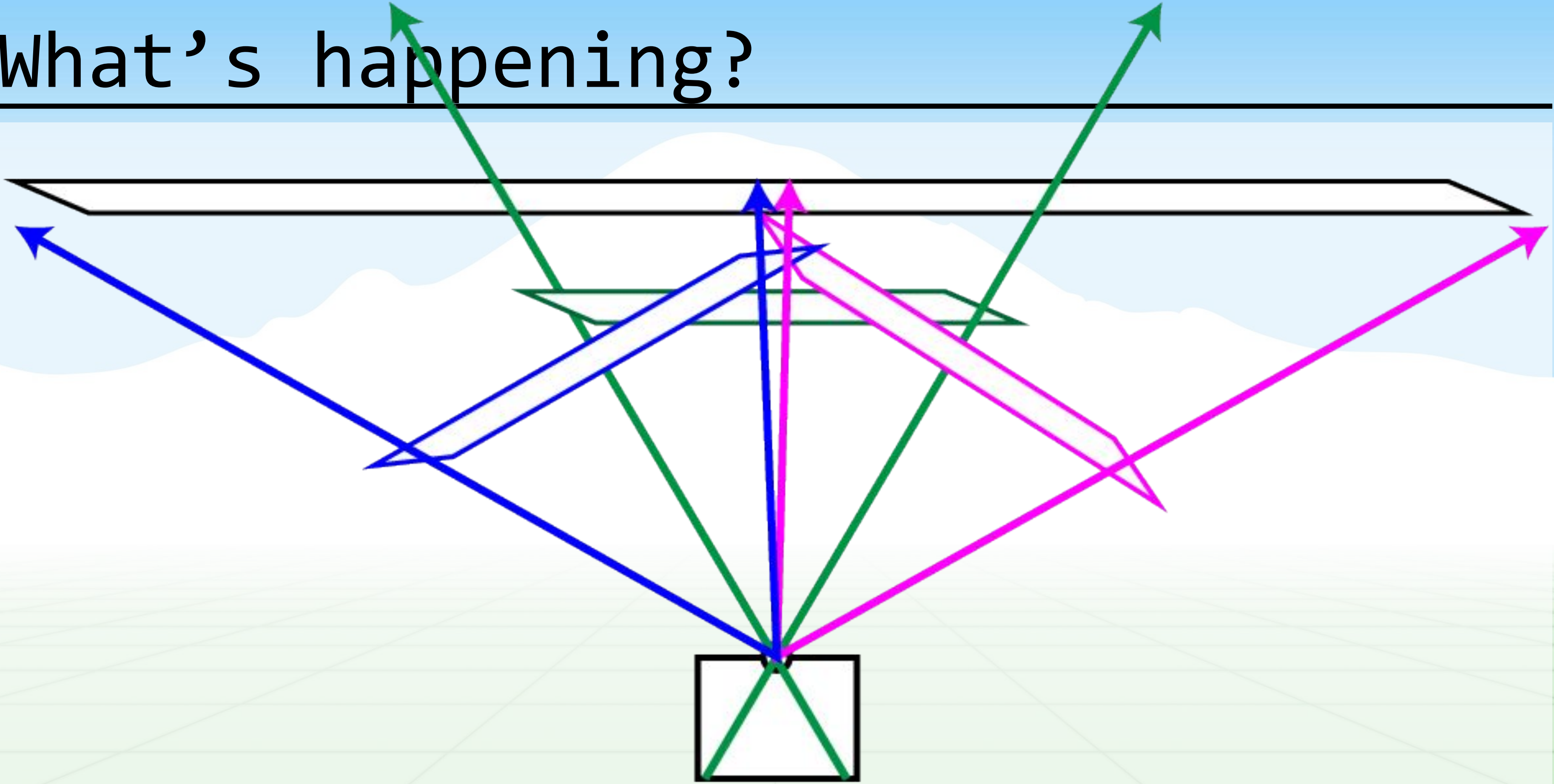
m_{x4} & m_{y4} & 1 & 0 & 0 & 0 & -m_{x4}n_{x4} & -m_{y4}n_{x4} \\
0 & 0 & 0 & m_{x4} & m_{y4} & 1 & -m_{x4}n_{y4} & -m_{y4}n_{y4}
\end{bmatrix}}_{\mathbf{M}}
\underbrace{\begin{bmatrix} h_{00} \\ h_{01} \\ h_{02} \\ h_{10} \\ h_{11} \\ h_{12} \\ h_{20} \\ h_{21} \end{bmatrix}}_{\mathbf{a}}
=
\underbrace{\begin{bmatrix} n_{x1} \\ n_{y1} \\ n_{x2} \\ n_{y2} \\ n_{x3} \\ n_{y3} \\ n_{x4} \\ n_{y4} \end{bmatrix}}_{\mathbf{b}}
$$

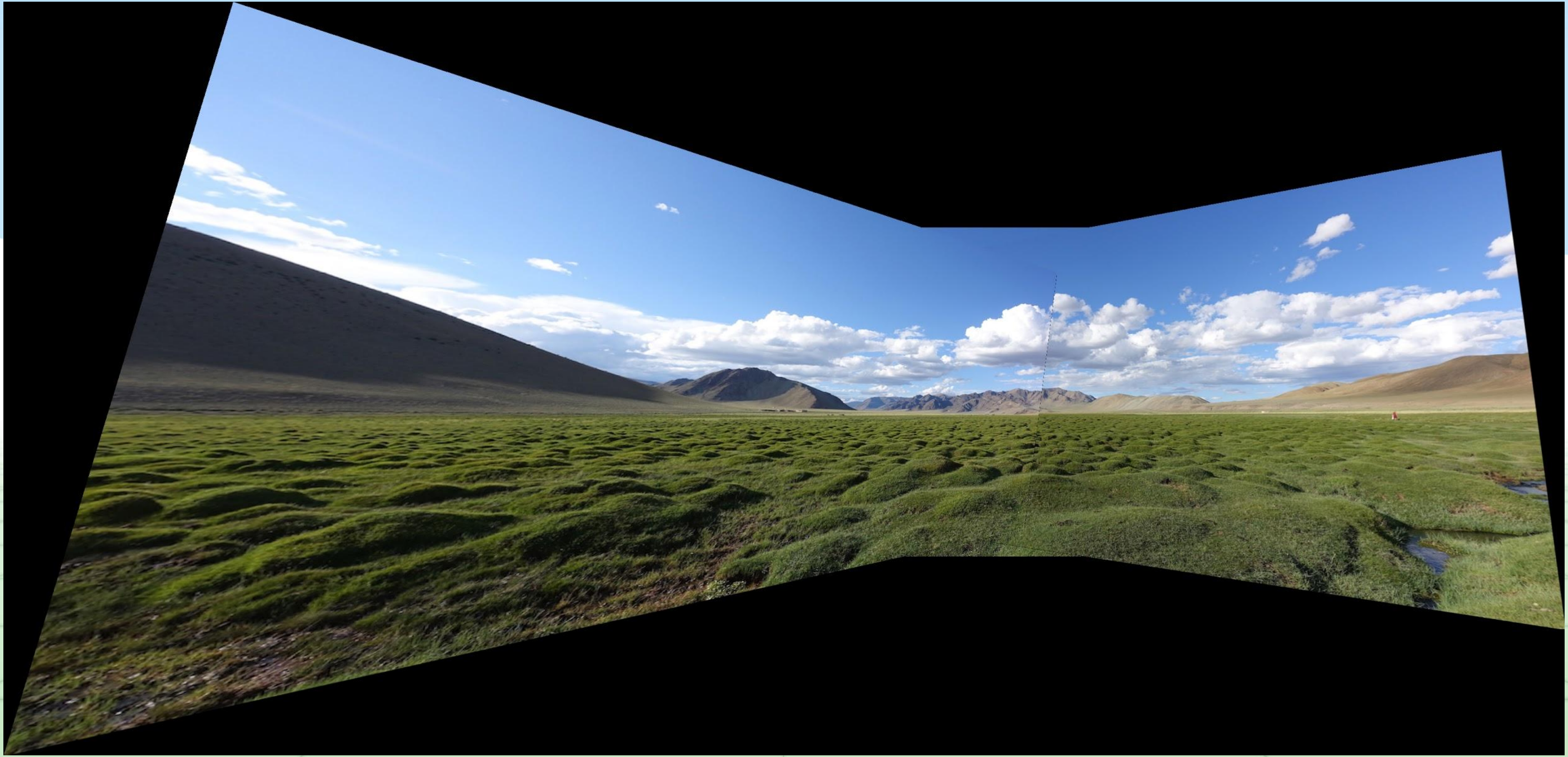
- Same procedure, Solve **M a = b**
  - Exact if #rows of M = 8
  - Least squares if #rows of M > 8



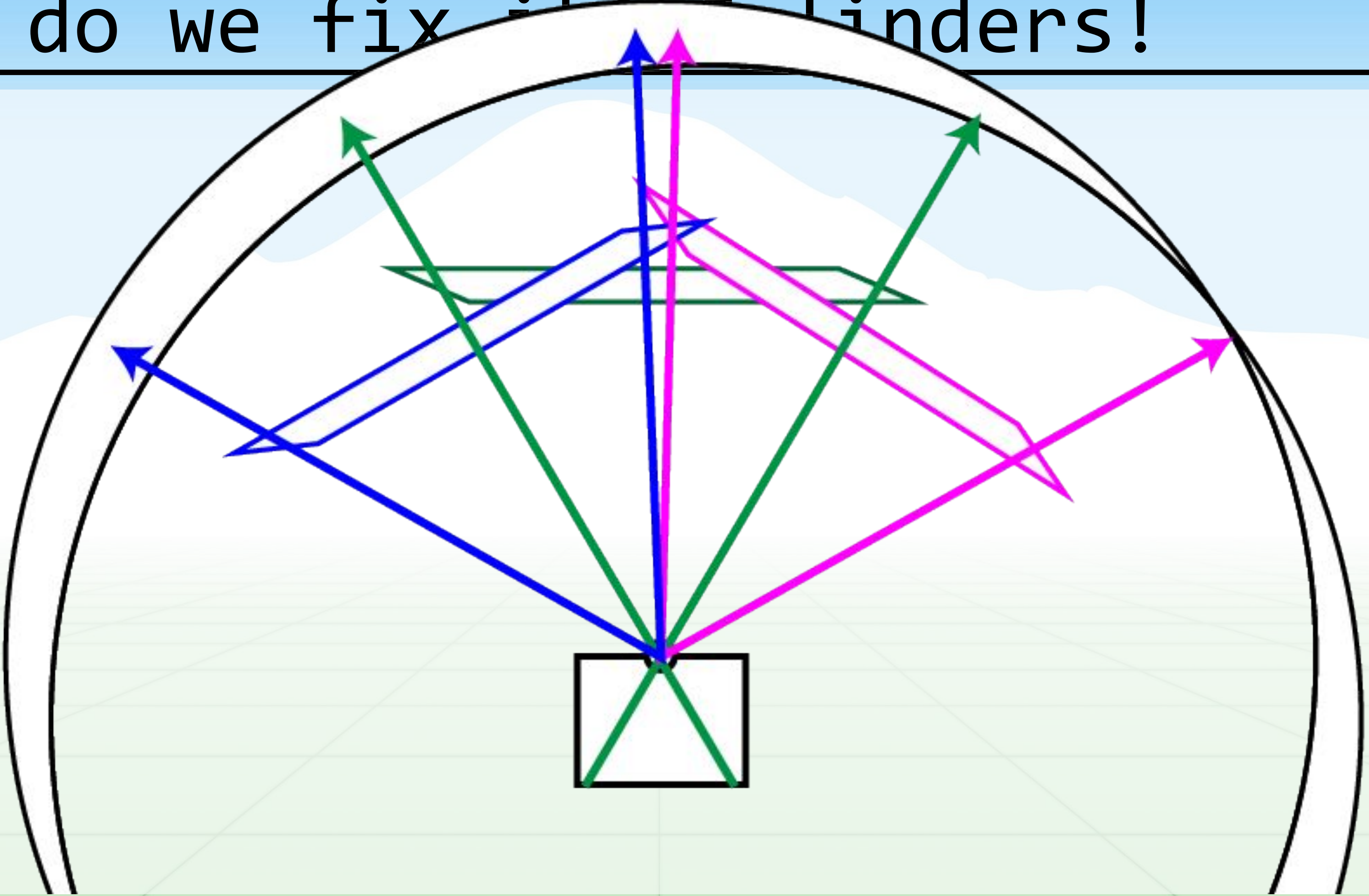
What's happening?



Very bad for big panoramas!



How do we fix it? Cylinders!



How do we fix it? Cylinders!

Calculate angle and height:

$\theta = (x - xc) / f$

$h = (y - yc) / f$

Find unit cylindrical coords:

$X' = \sin(\theta)$

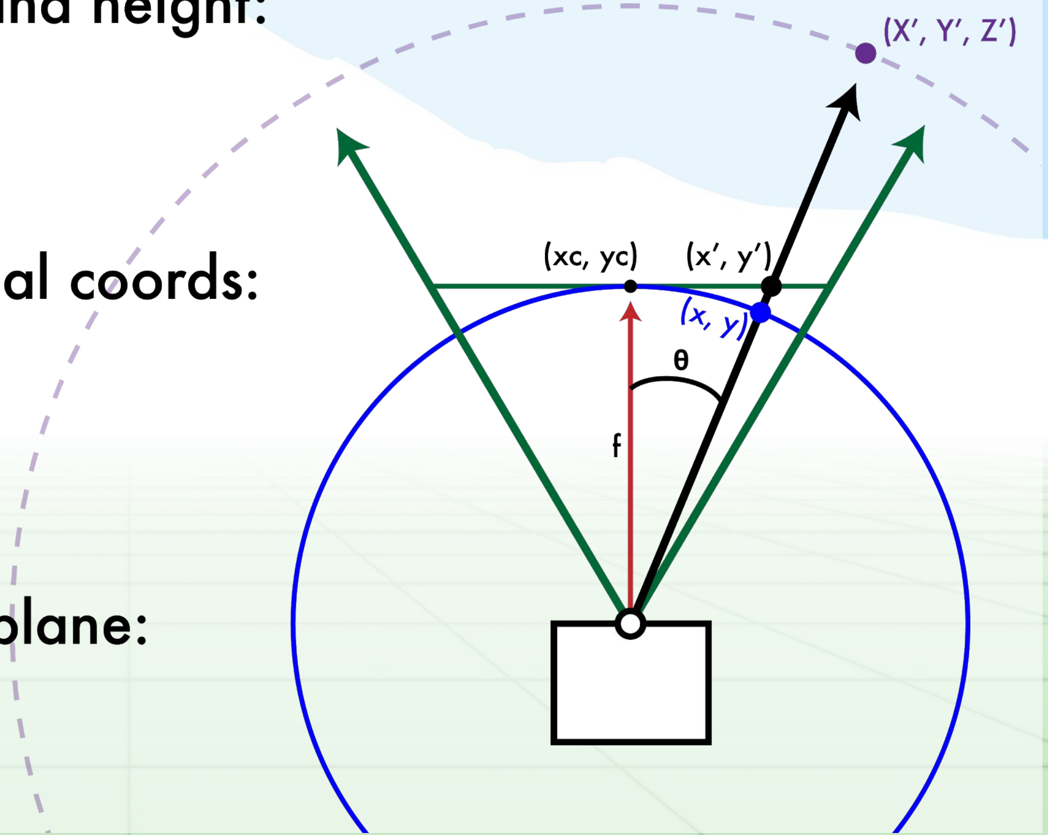
$Y' = h$

$Z' = \cos(\theta)$

Project to image plane:

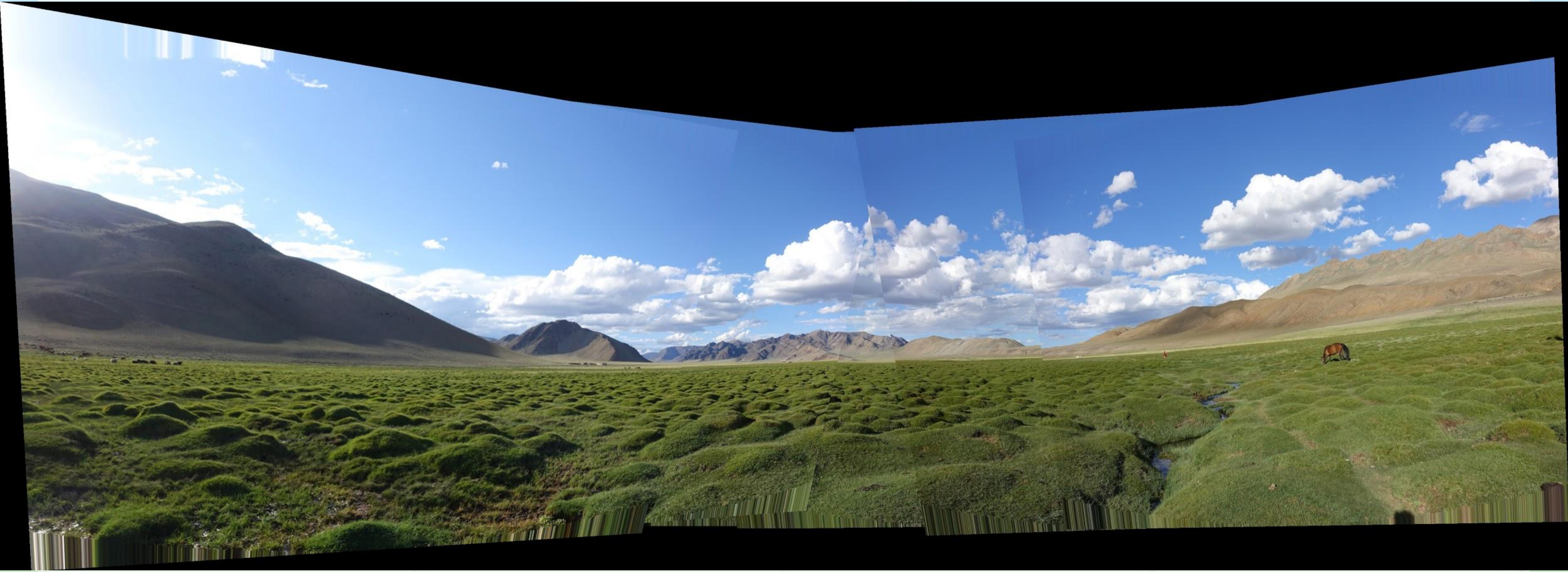
$x' = f\, X'/Z' + xc$

$y' = f\, Y'/Z' + yc$



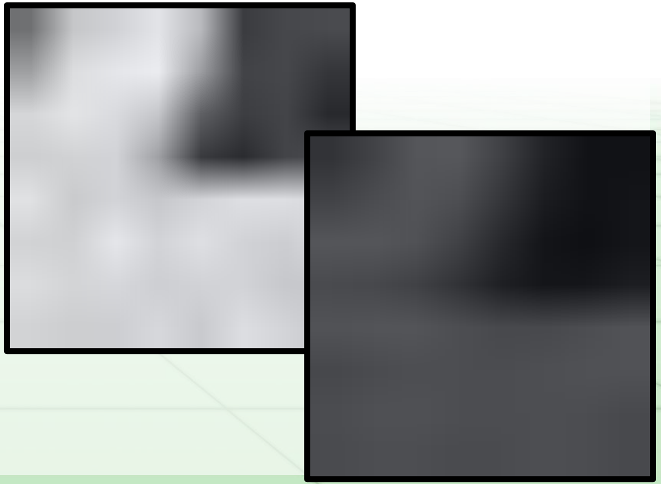
Does it work?

Does it work? Yay!

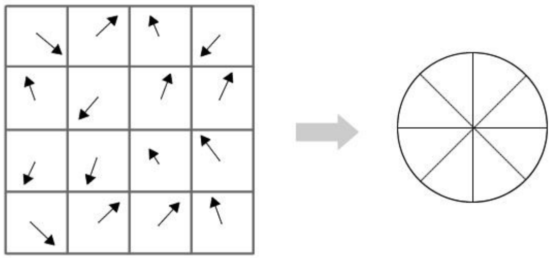


THIS IS AS GOOD AS IT GETS!

- Not so fast...
- Our descriptor has some issues:
  - Not invariant to lighting changes!
- Want features invariant to lighting



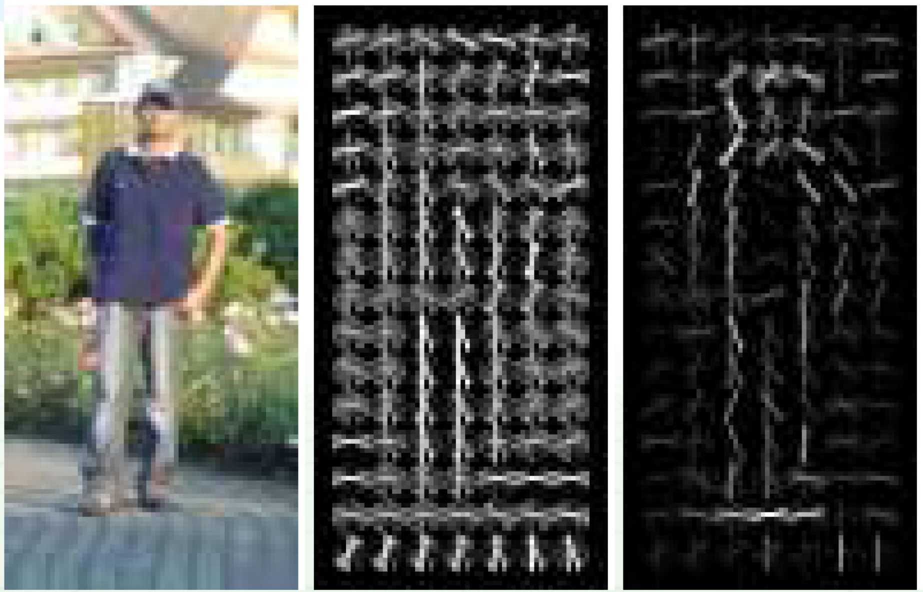
Histogram of Oriented Gradients (HOG)



- Dalal and Triggs 2005
- Better image descriptor
  - Compute gradients
  - Bin gradients
  - Aggregate blocks (4x4, 16x16 cells)
  - Normalize gradient magnitudes
- Not reliant on magnitude, just direction
  - Invariant to some lighting changes
- Train SVM to recognize people

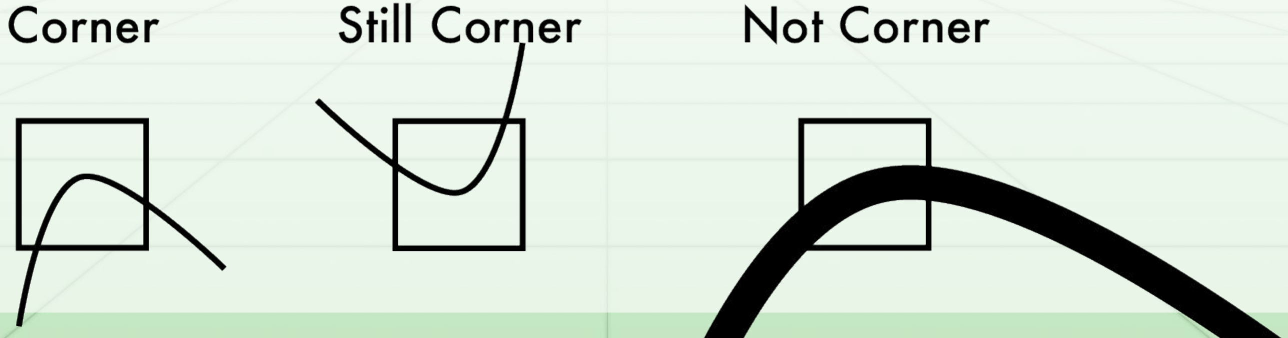
Binning from http://aishack.in/tutorials/sift-scale-invariant-feature-transform-features/

Histogram of Oriented Gradients (HOG)



THIS IS AS GOOD AS IT GETS!

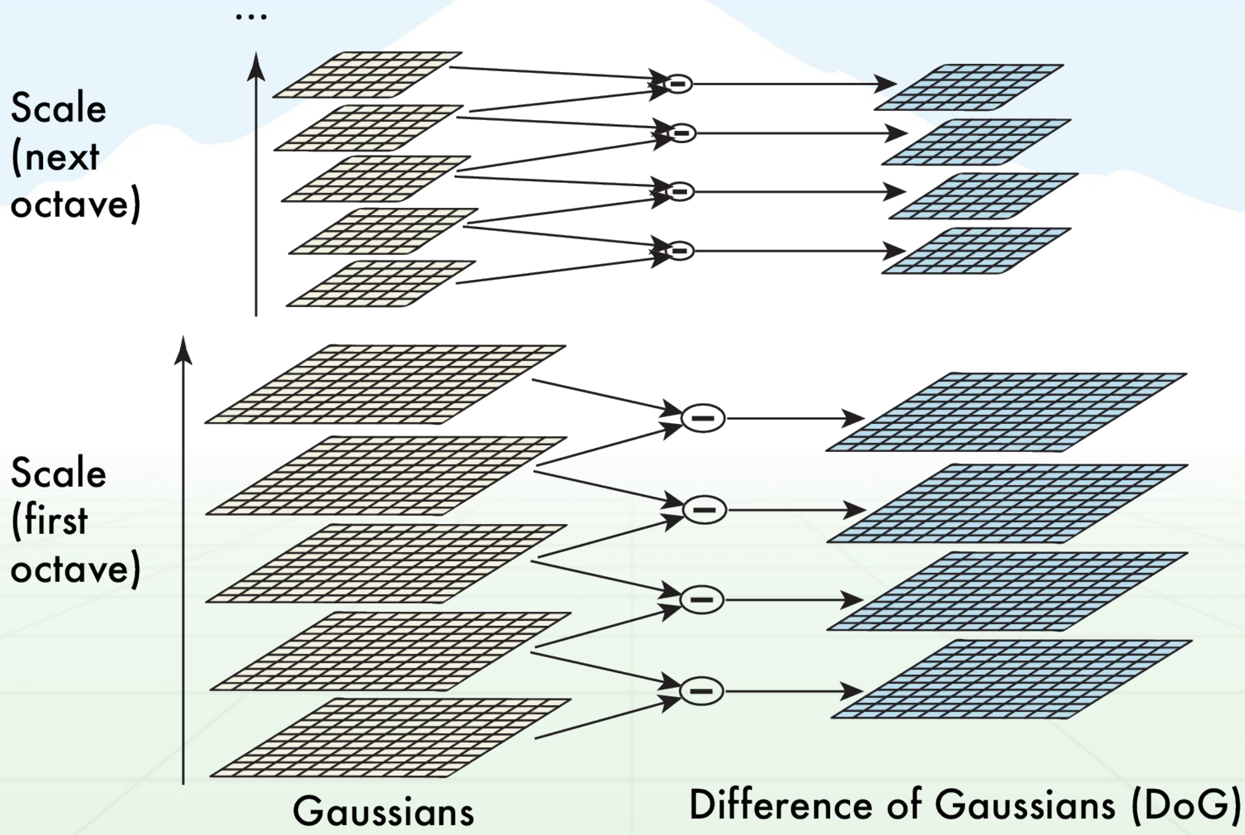
- Not so fast…
- Harris and has some issues:
  - Corners detection is rotation invariant
  - Harris not invariant to scale
- Descriptors are also hard
  - Just looking at pixels is not rotation invariant!
  - HOG also not rotation invariant



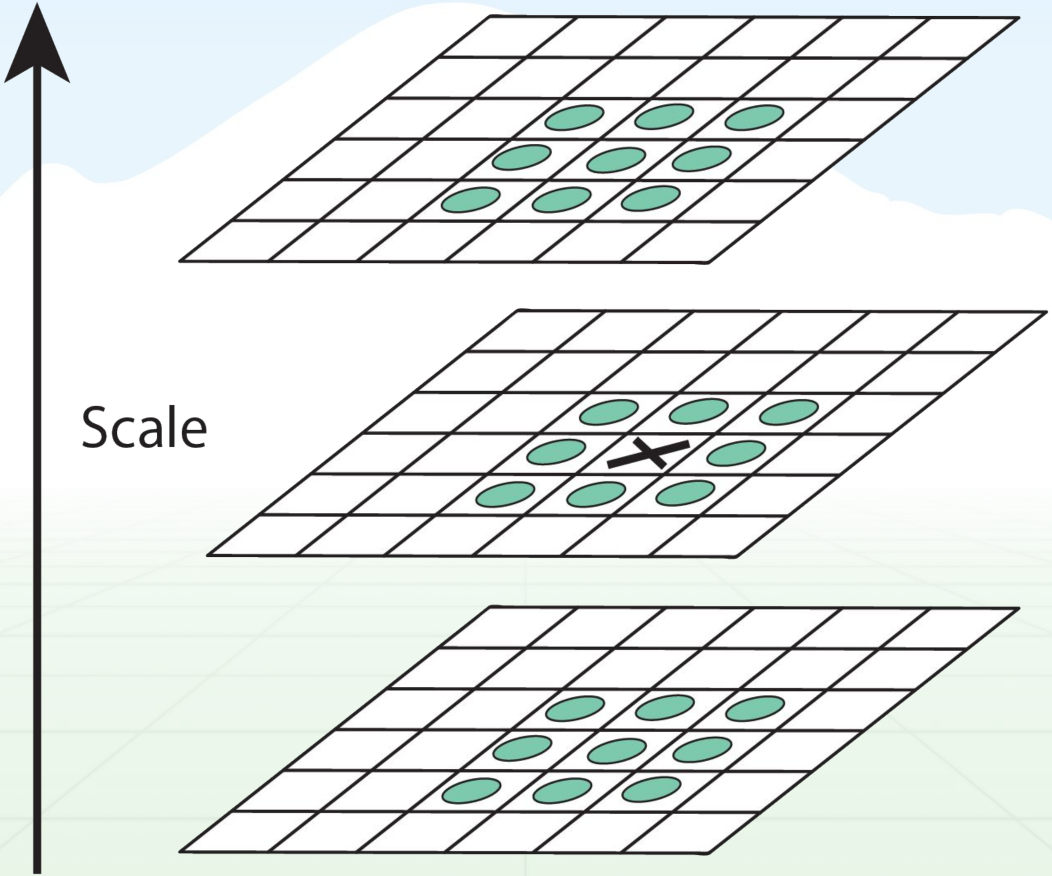
Want features invariant to scaling, rotation, etc.

- Scale Invariant Feature Transform (SIFT)
  - Lowe et al. 2004, many images from that paper
- Get scale-invariant response map
- Find keypoints
- Extract rotation-invariant descriptors
  - Normalize based on orientation
  - Normalize based on lighting

Extract DoG features at multiple scales



Find local-maxima in location and scale



Throw out weak responses and edges

- Estimate gradients
  - Similar to before, look at nearby responses
  - Not whole image, only a few points! Faster!
  - Throw out weak responses
- Find cornery things
  - Same deal, structure matrix, use det and trace information

$$\frac{\mathrm{Tr}(\mathbf{H})^2}{\mathrm{Det}(\mathbf{H})} < \frac{(r+1)^2}{r}$$

Find main orientation of patches

- Look at weighted histogram of nearby gradients
    - Any gradient within 80% of peak gets its own descriptor
        - Multiple keypoints per pixel
    - Descriptors are normalized based on main orientation

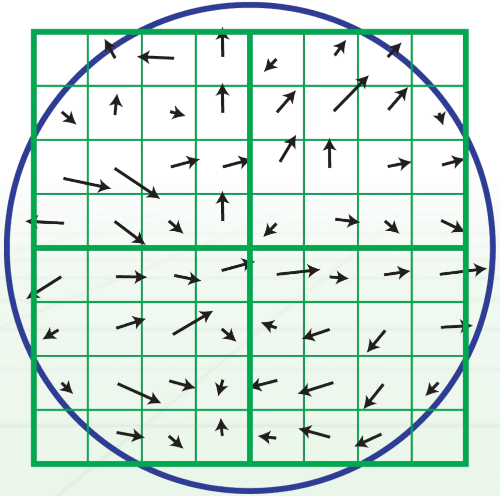
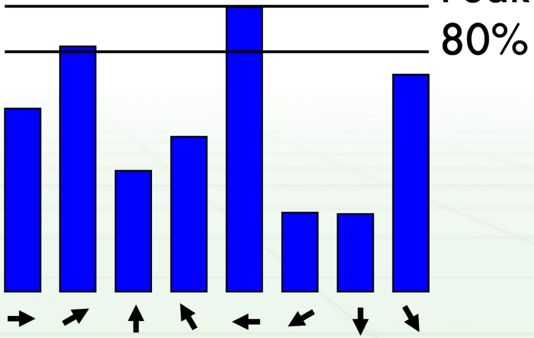
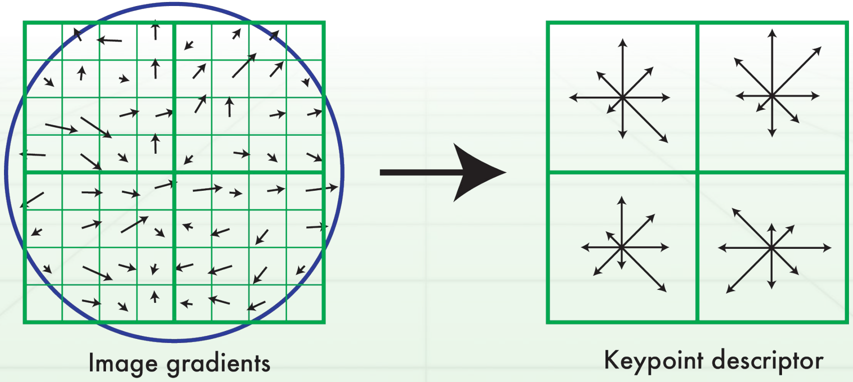


Image gradients



Keypoints are normalized gradient histograms

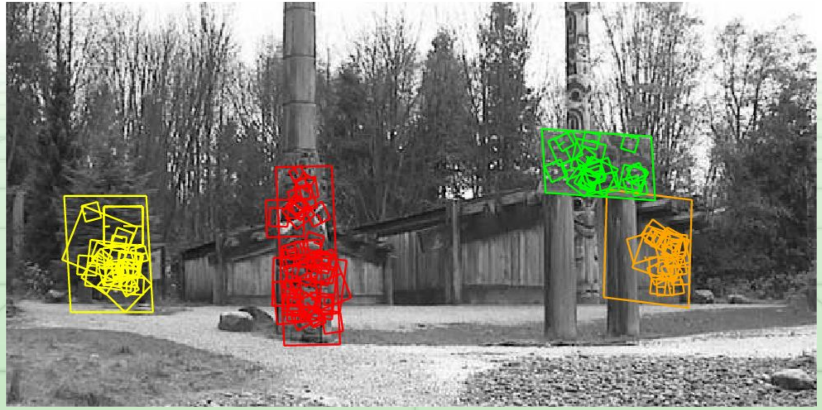
- Divide into subwindows (2x2, 4x4)
- Bin gradients within subwindow, get histogram
    - Normalize to unit length
    - Clamp at maximum .2
    - Normalize again
    - Helps with lighting changes!



SIFT is great!

- Find good keypoints, describe them
- Finding objects, recognition, panoramas, etc.

SIFT is great!



Chapter Eight

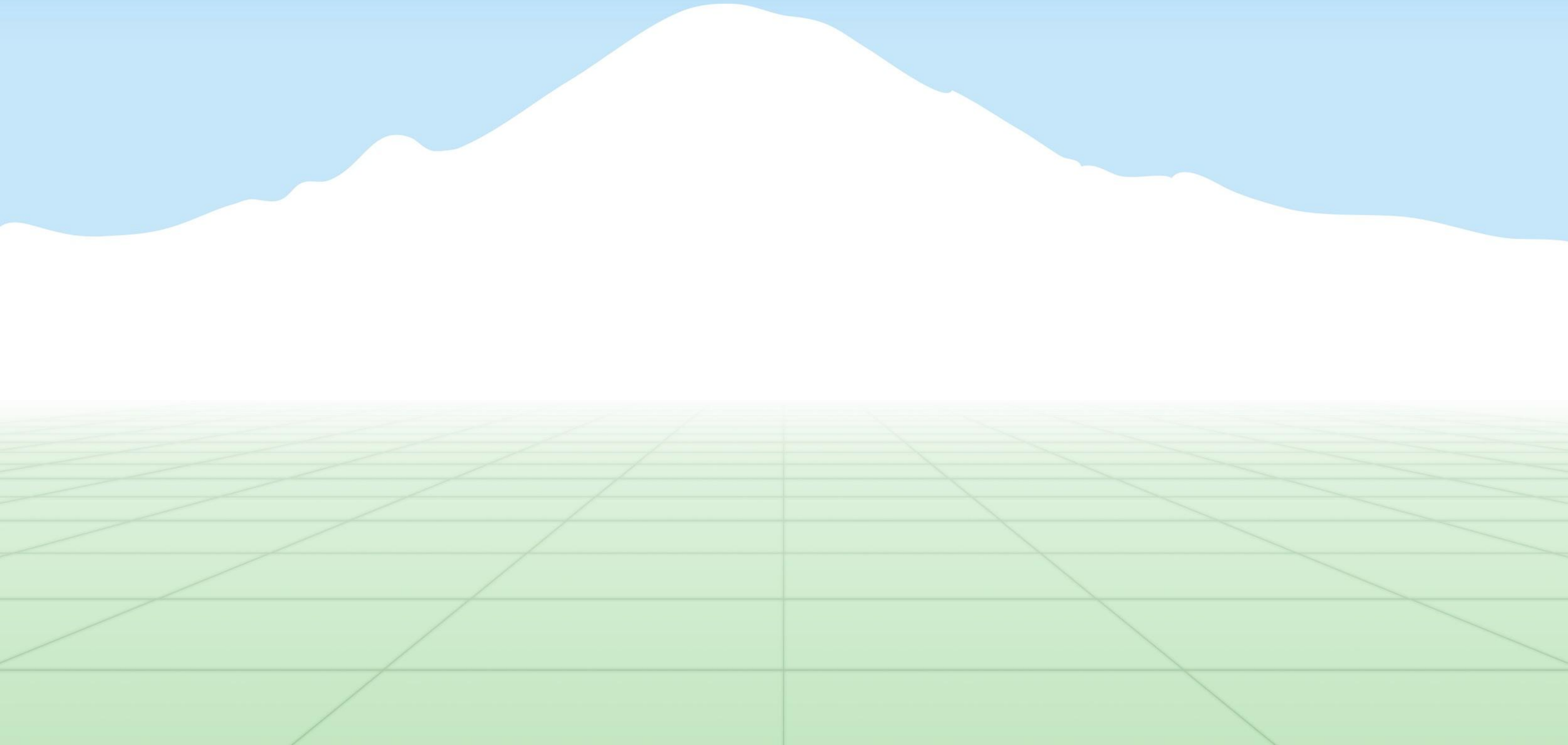
Optical Flow

Topics:

- Sparse* Optical Flow (Lucas-Kanade, 1981)
- Dense Optical Flow (Farneback, 2003)

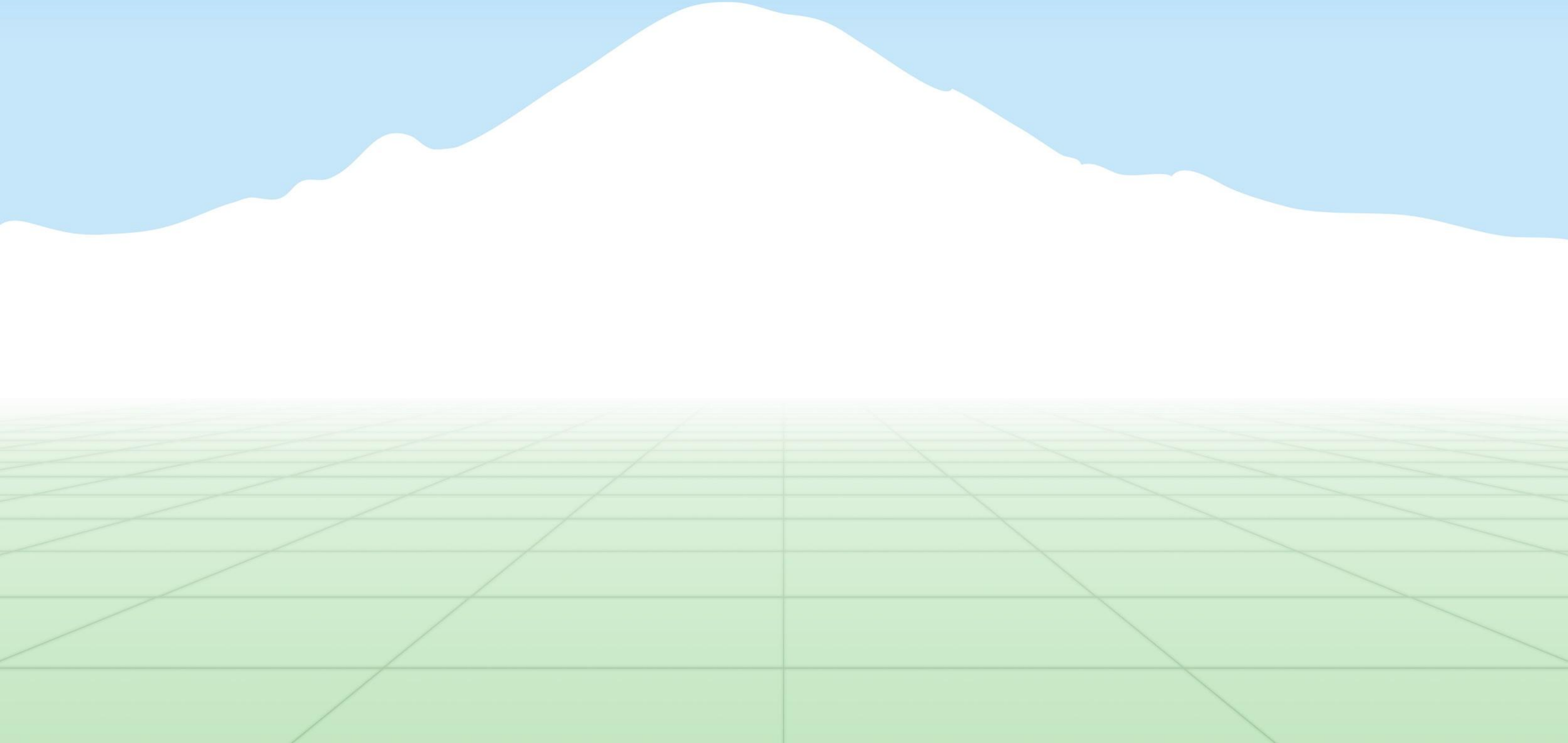
*LK can technically be dense.

What is Optical Flow?

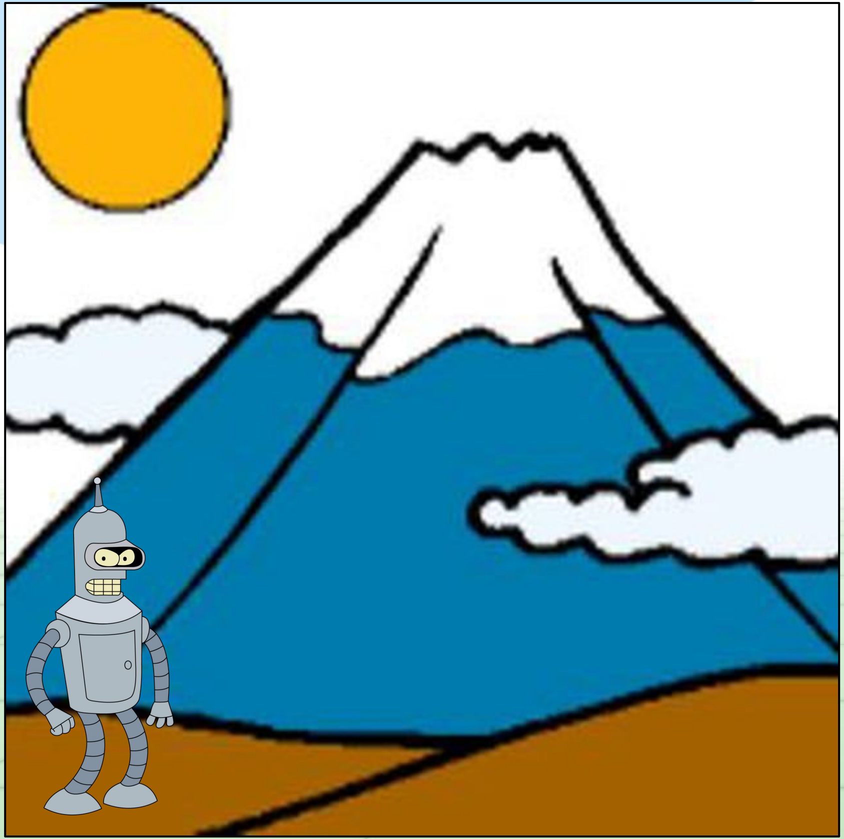


What is Optical Flow?

Movement

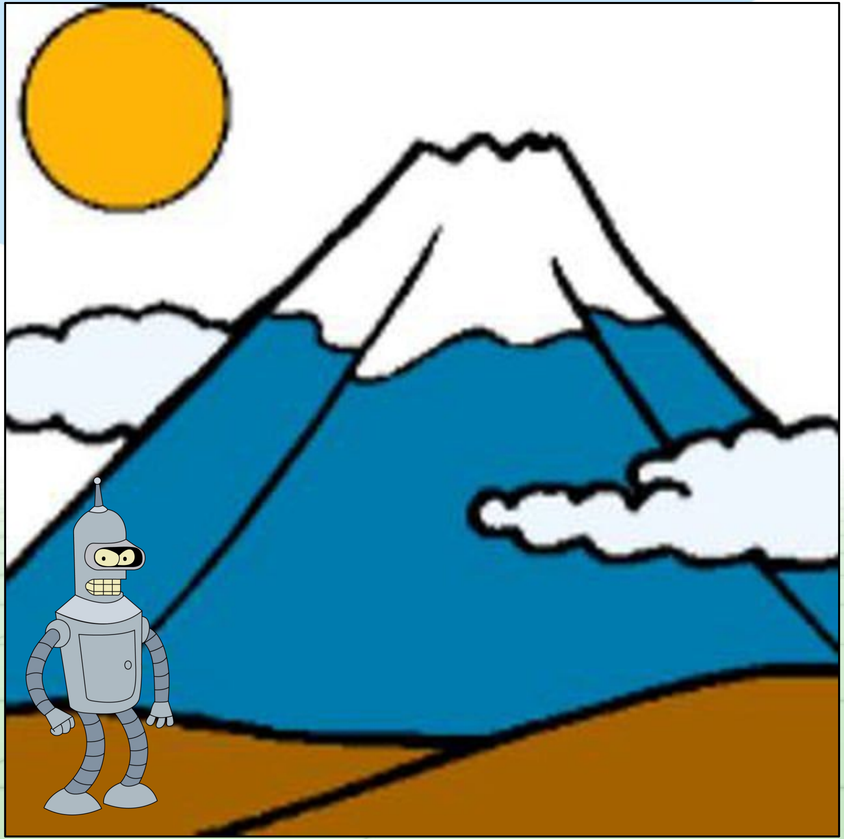


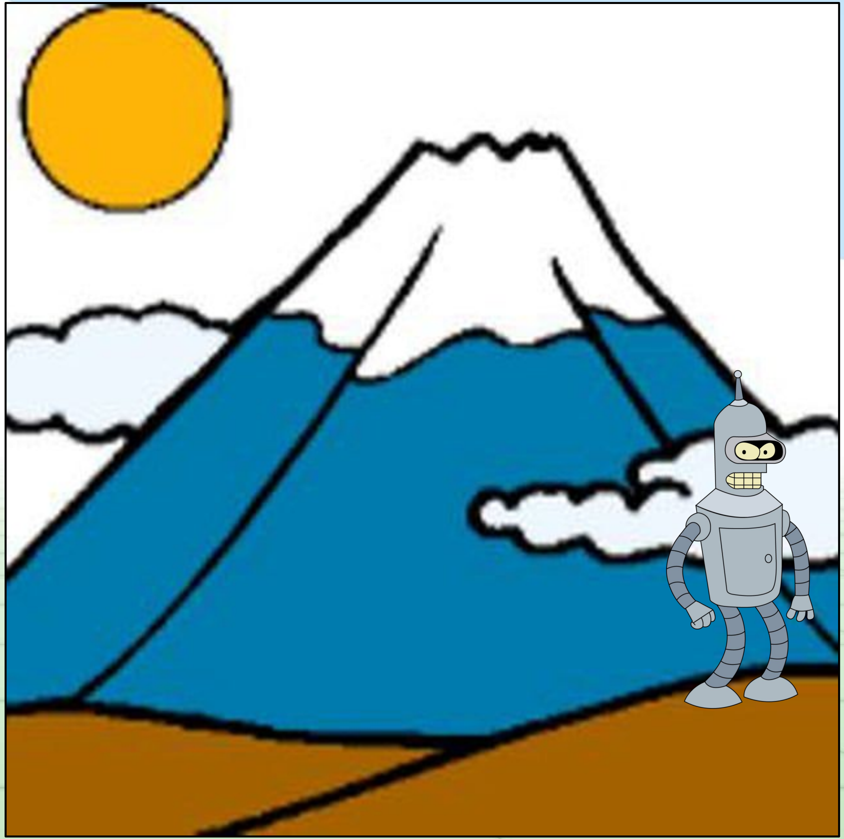
What is Optical Flow? **Movement**



What is Optical Flow?

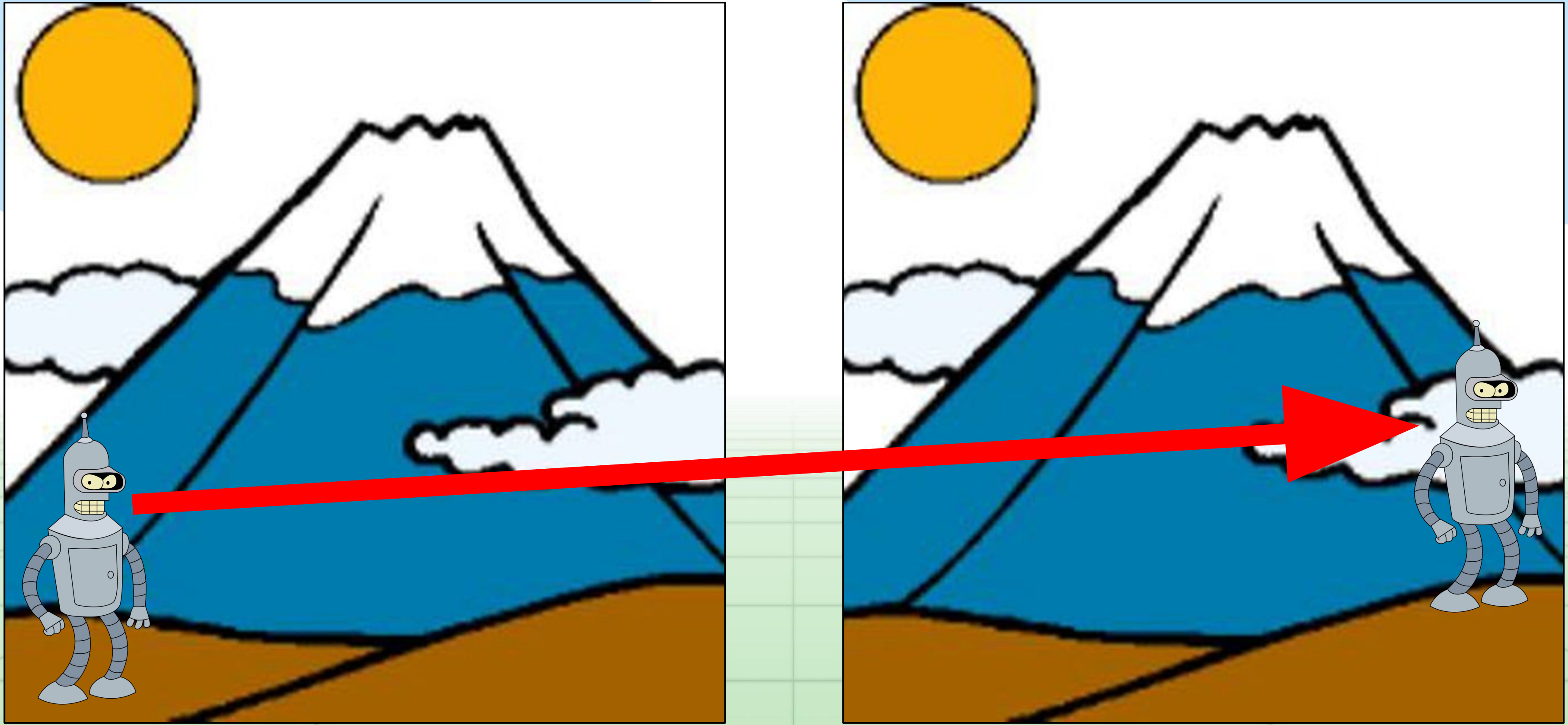
Movement



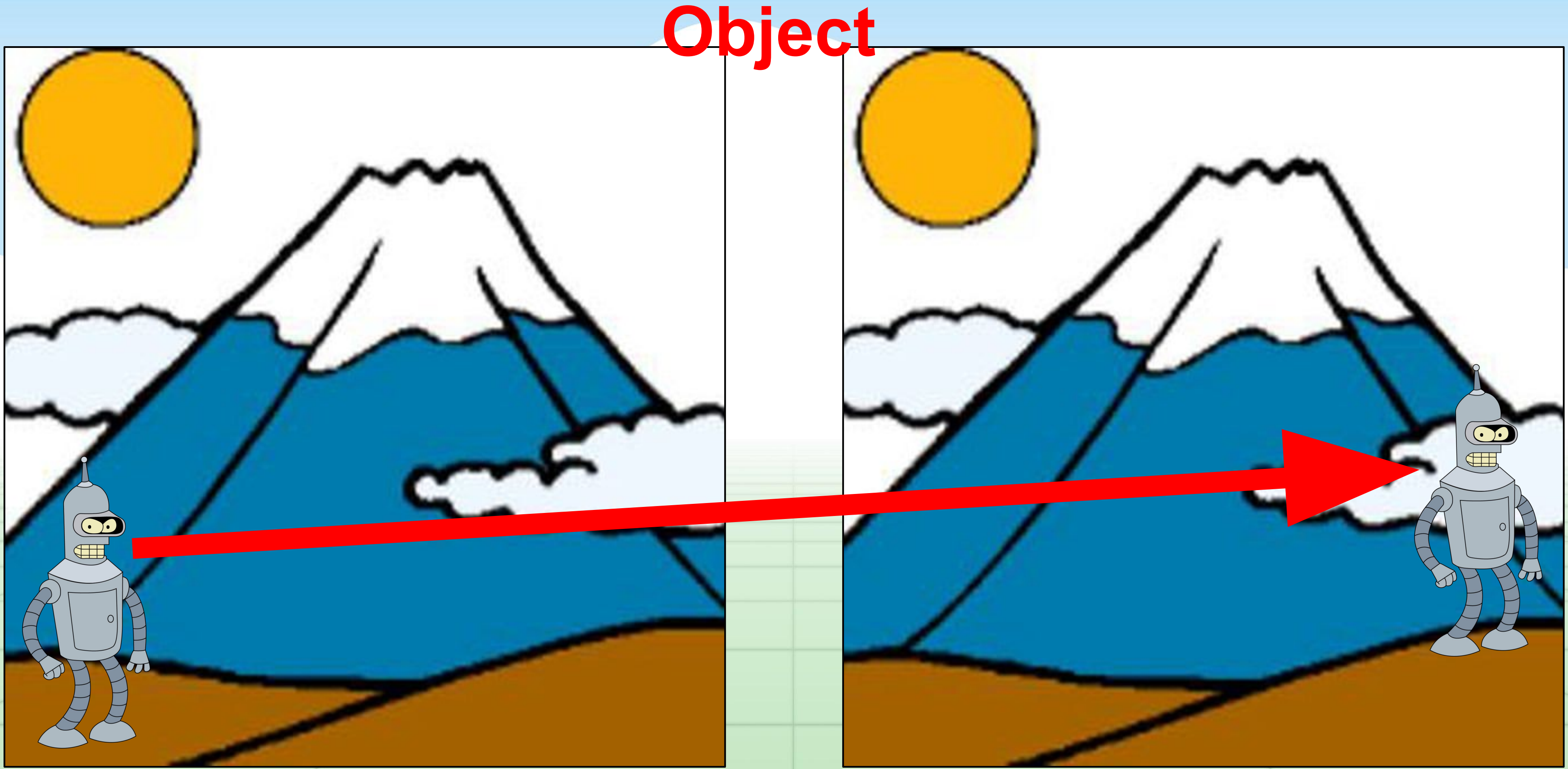


What is Optical Flow?

Movement



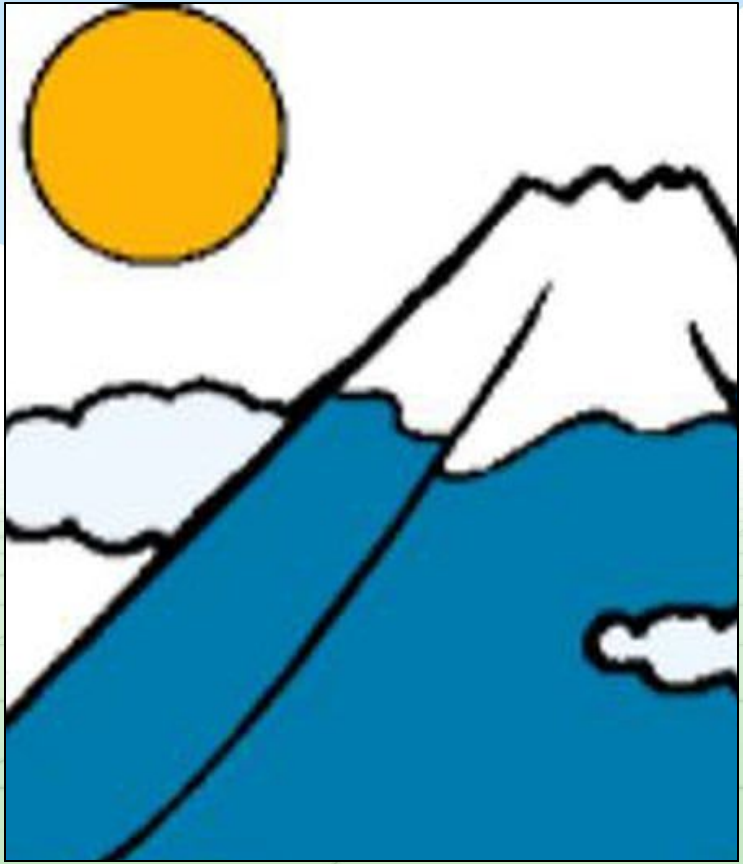
What is Optical Flow?

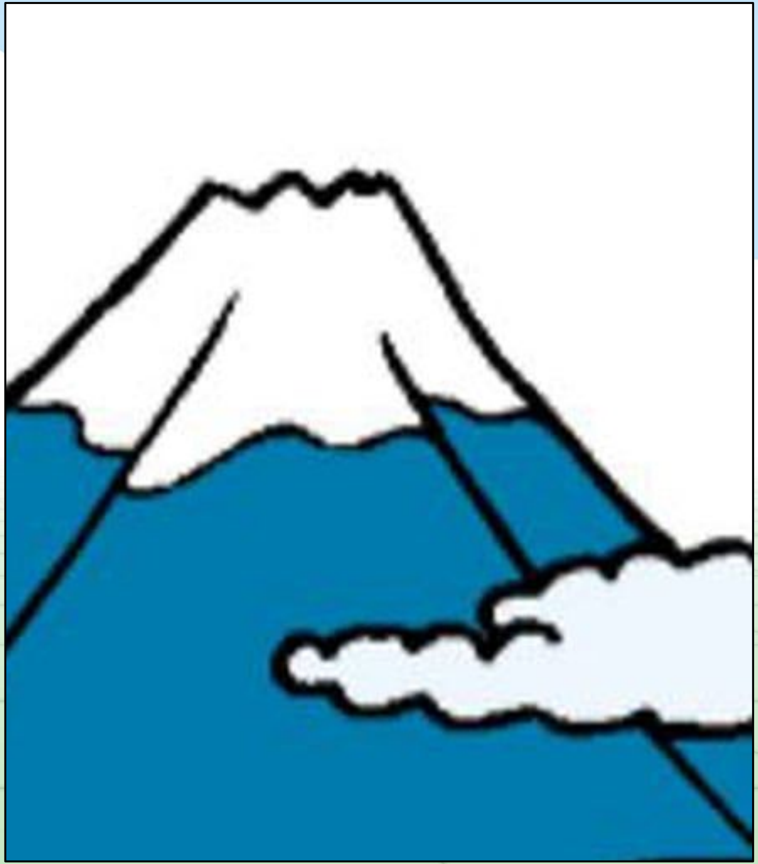


What is Optical Flow?

Movement

Pan

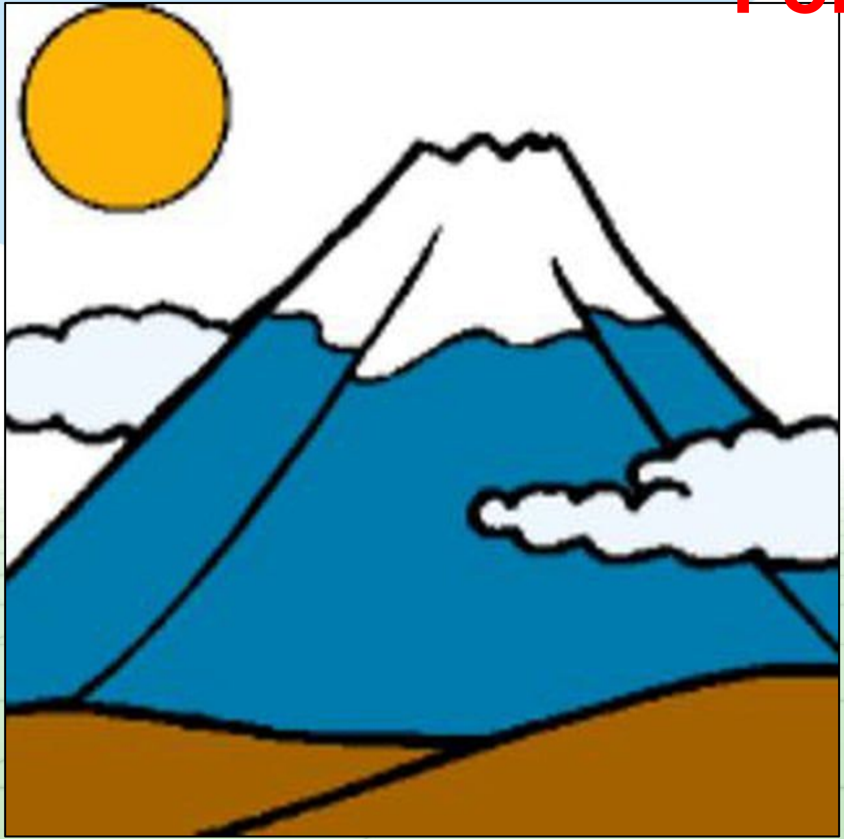


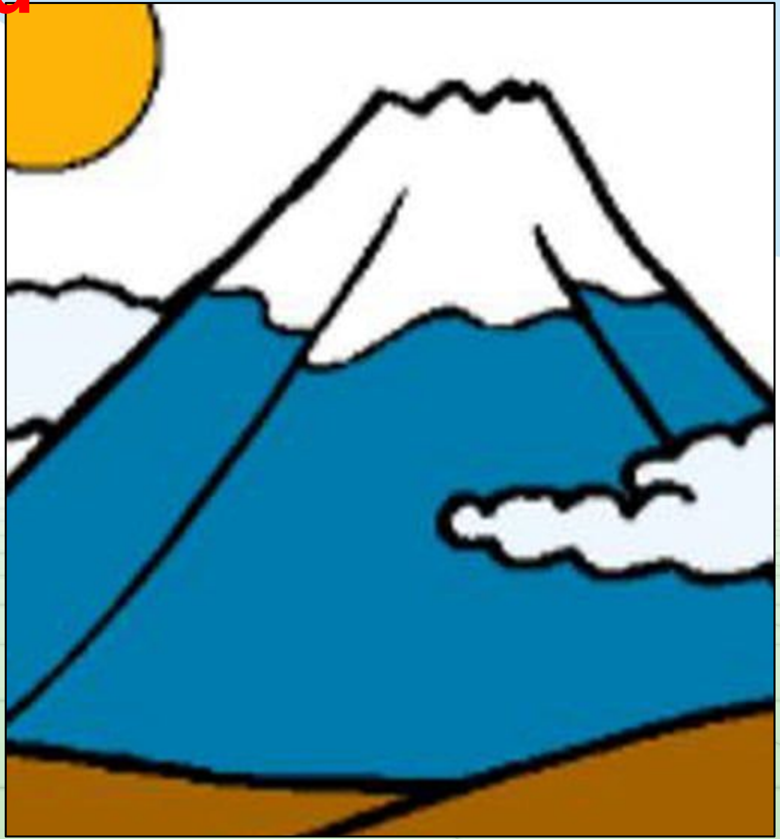


What is Optical Flow?

Movement

Forward



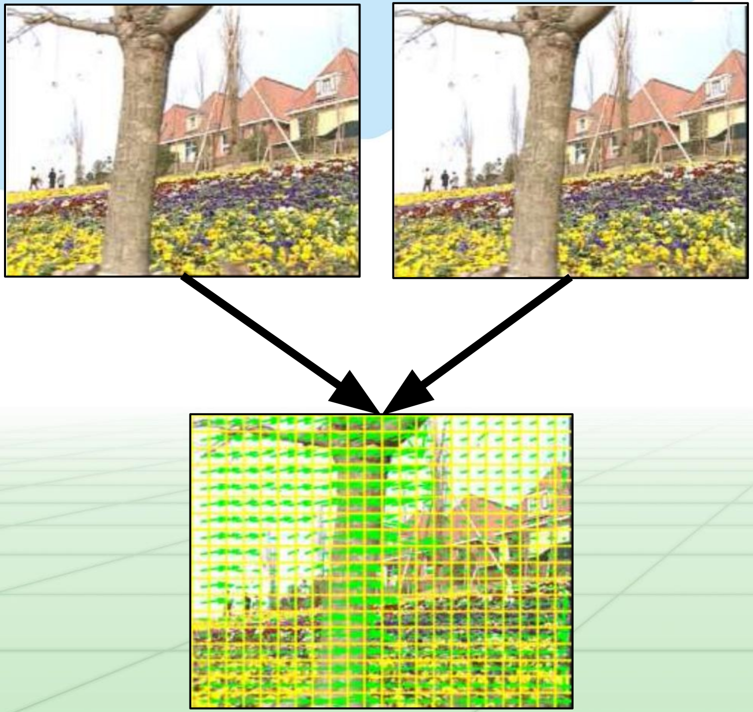


What is Optical Flow? **Movement**

Why do we want Optical Flow?

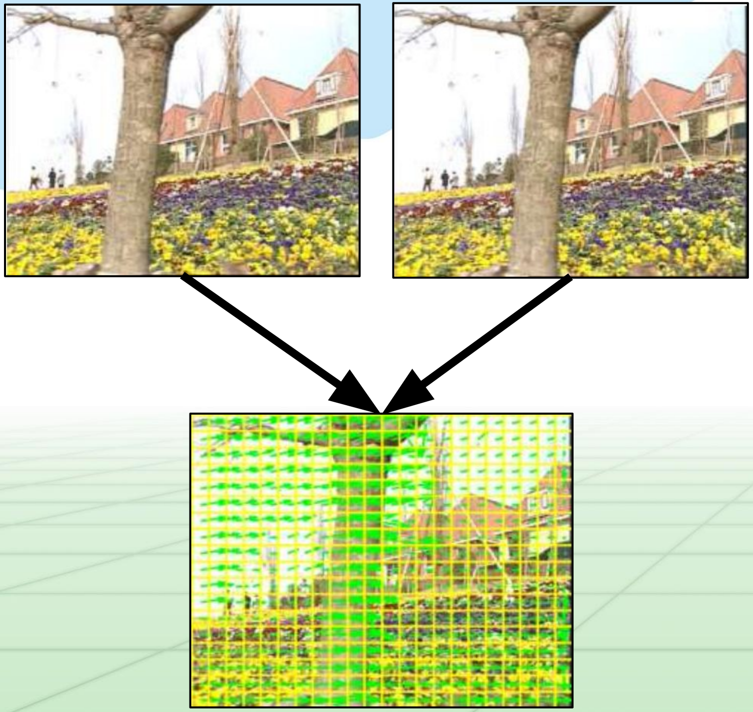
Why do we want Optical Flow?

Motion Estimation



Why do we want Optical Flow?

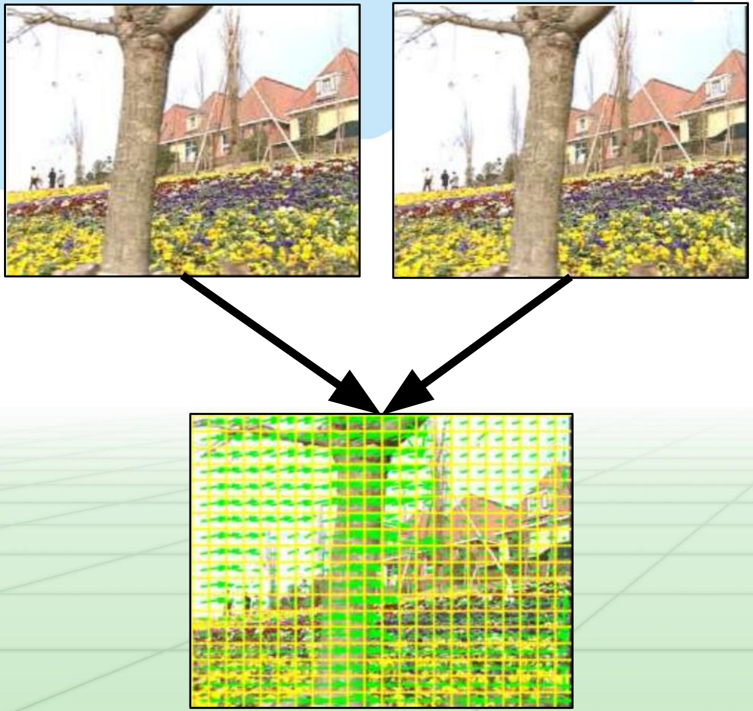
Motion Estimation



Object Tracking

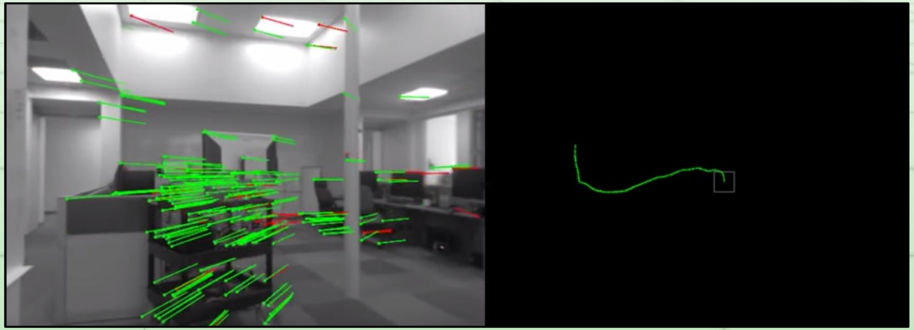
Why do we want Optical Flow?

Motion Estimation



Object Tracking

Visual Odometry

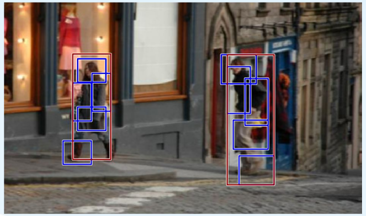


How do we find the flow in an image?

Feature Matching

Previously: Features!

- Highly descriptive local regions
- Ways to describe those regions
- Useful for:
  - Matching
  - Recognition
  - Detection



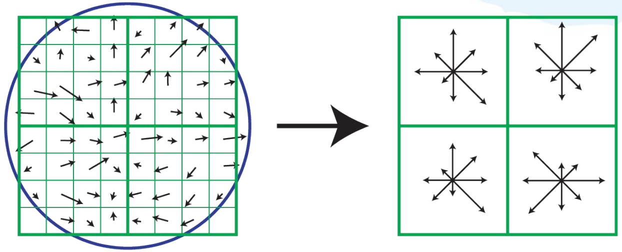
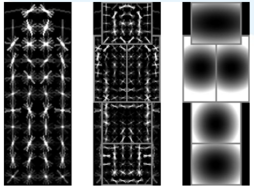
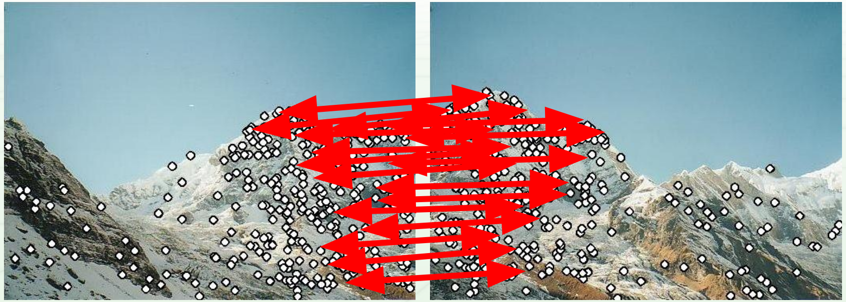


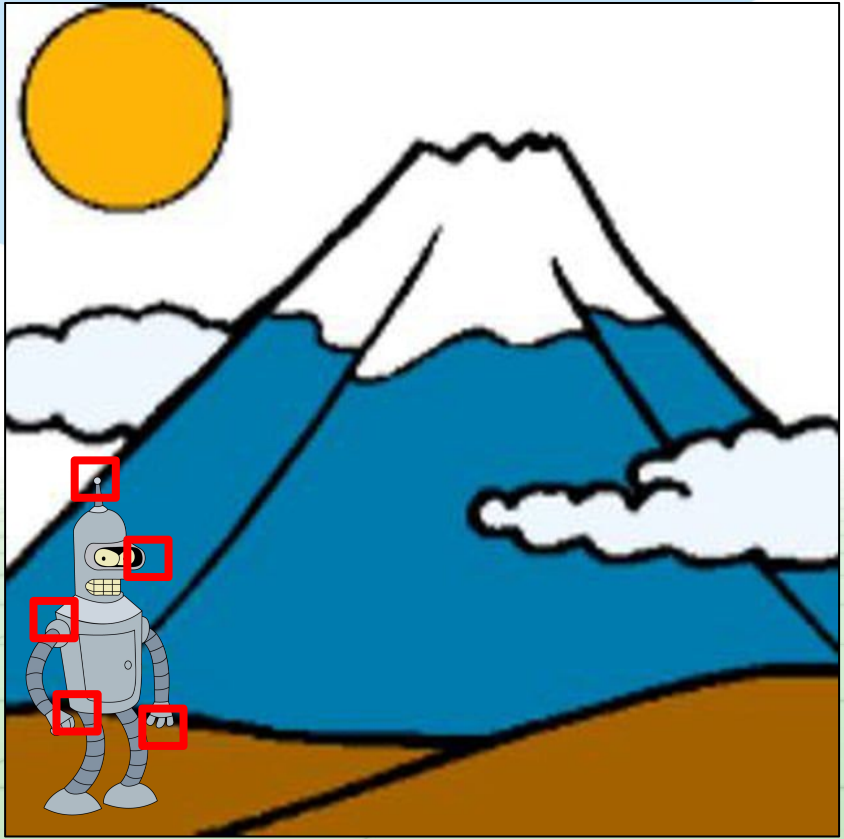
Image gradients

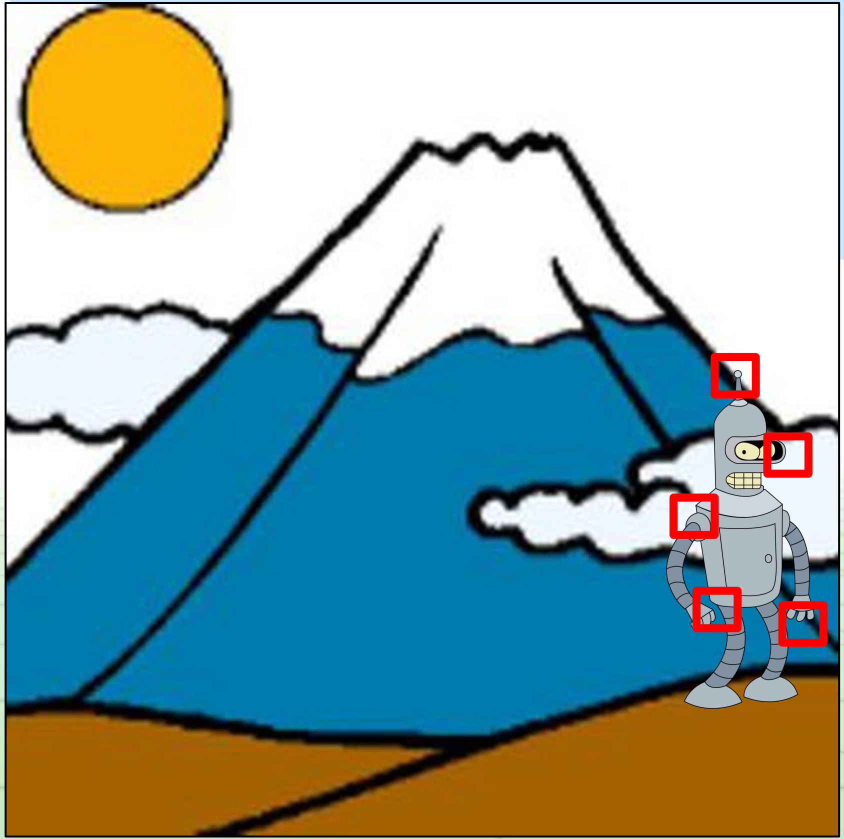
Keypoint descriptor



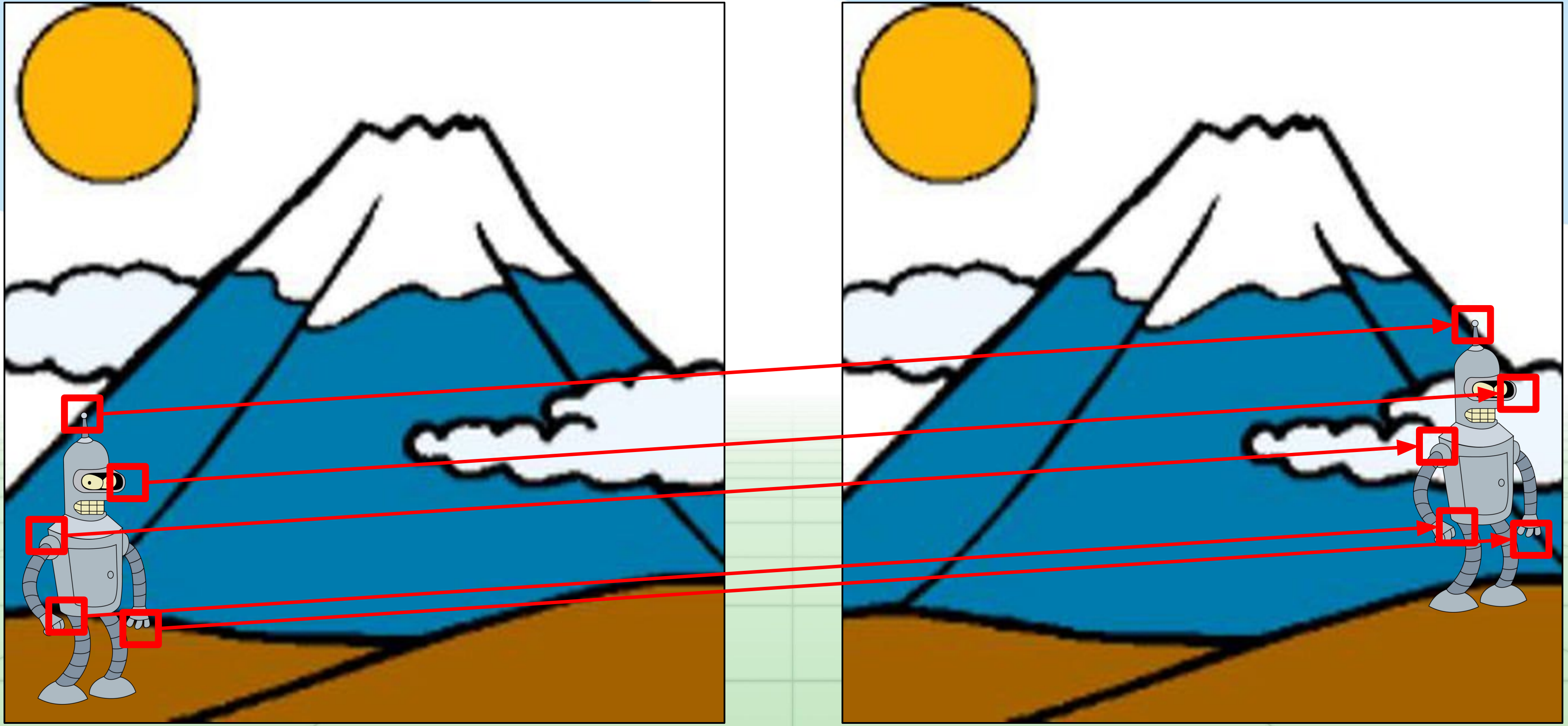


Feature Matching

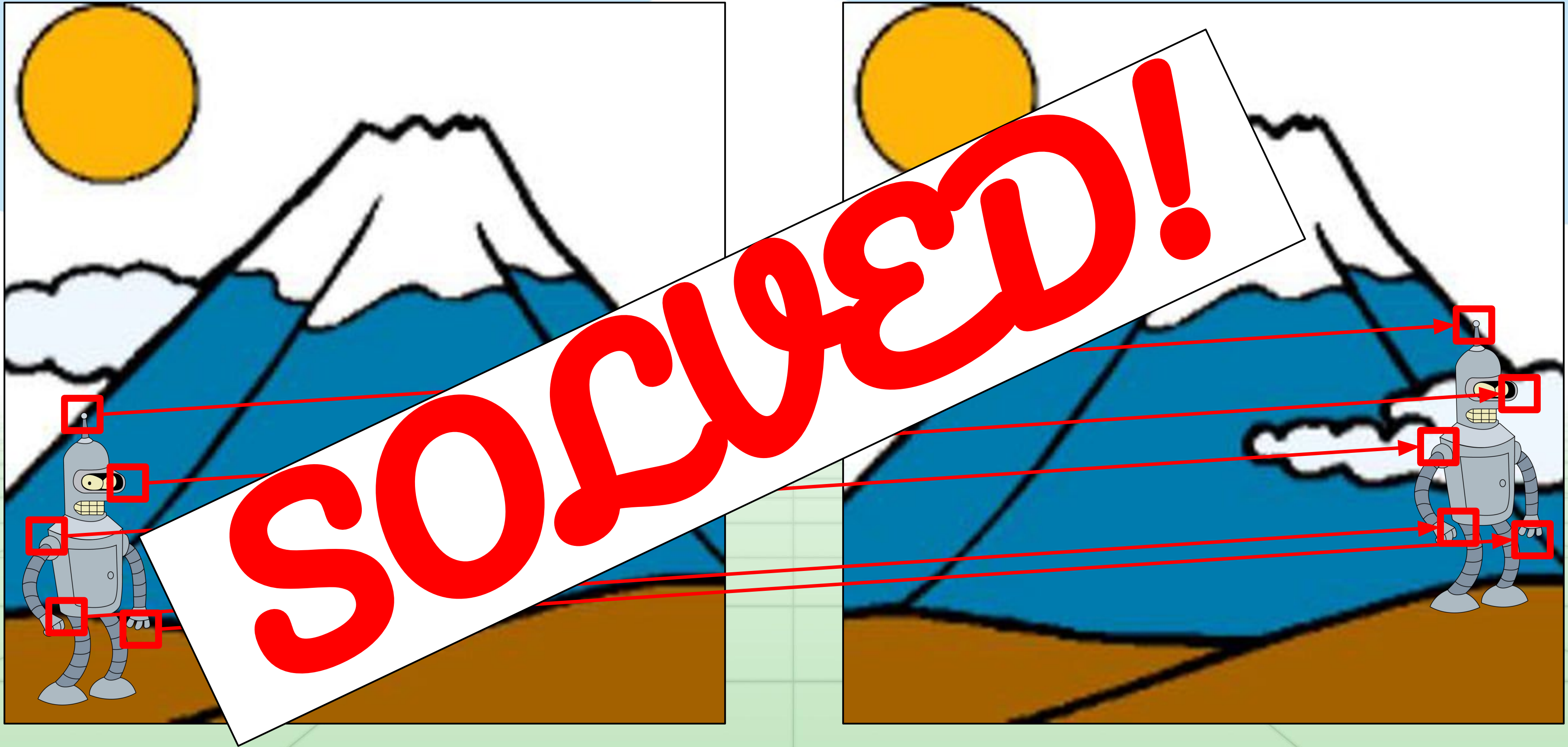




Feature Matching



Feature Matching



Feature Matching

Disadvantages:

Feature Matching

Disadvantages:

-Sparse!

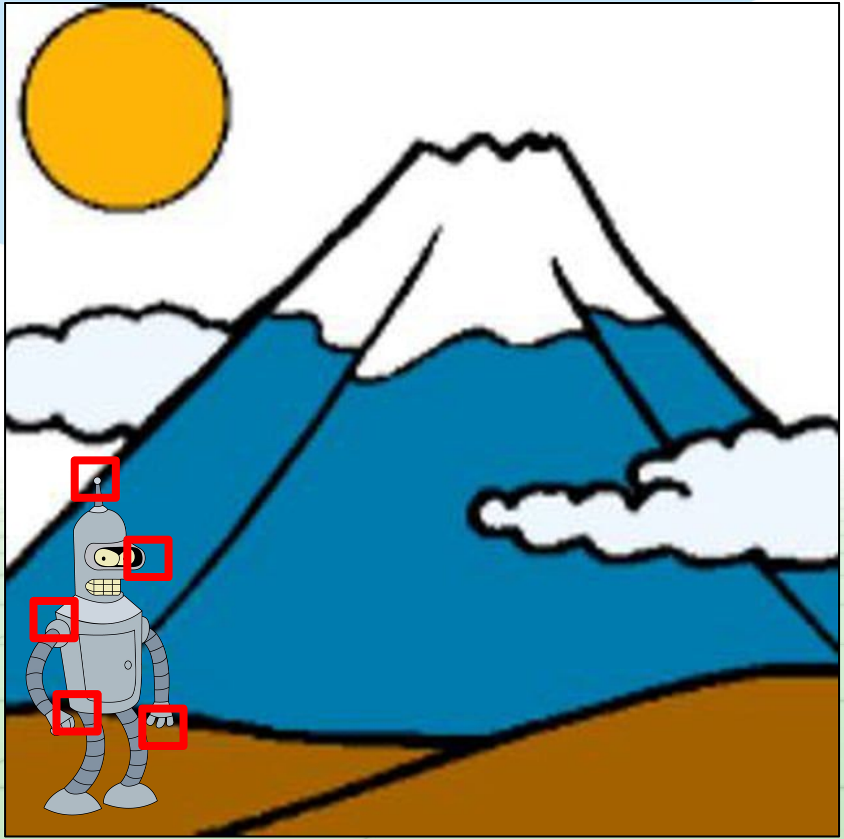
Feature Matching

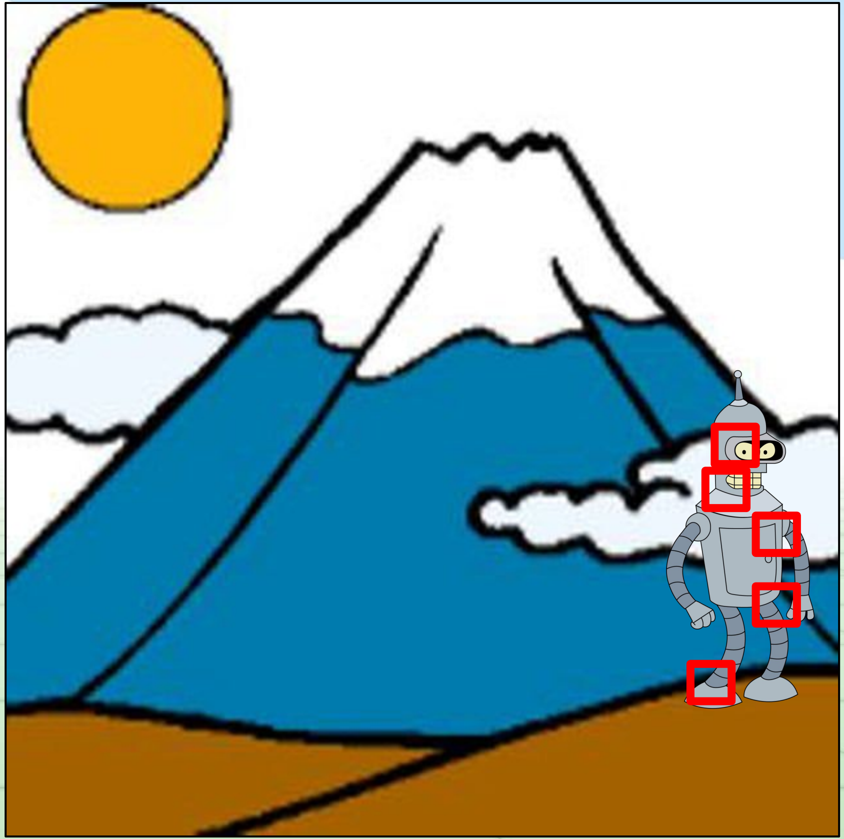
Disadvantages:

-Sparse!

-Feature alignment not exact

Feature Matching





Feature Matching

Disadvantages:

-Sparse!

-Feature alignment not exact

-Low accuracy

Feature Matching

Disadvantages:

-Sparse!

-Feature alignment not exact

-Low accuracy

Advantages:

Feature Matching

Disadvantages:

-Sparse!

-Feature alignment not exact

-Low accuracy

Advantages:

-Scale/rotation invariant

-*kinda* lighting invariant

-Can handle large movements

Feature Matching

Disadvantages:

-Sparse!

-Feature alignment not exact

-Low accuracy

Advantages:

-Scale/rotation invariant

hting

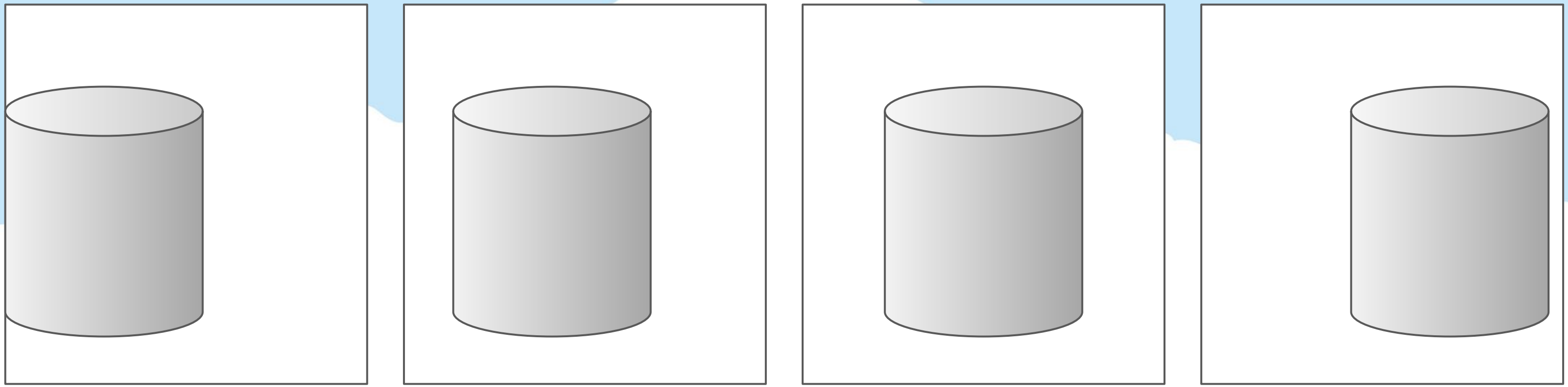
large

movements

Overall: Doesn't work very well for Optical Flow

What do we do instead?

An observation...



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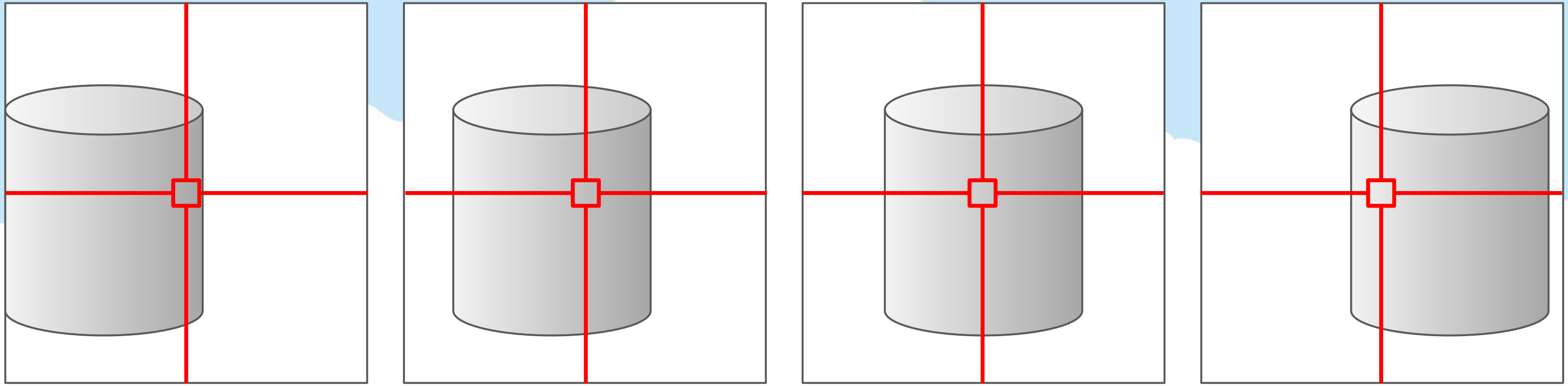

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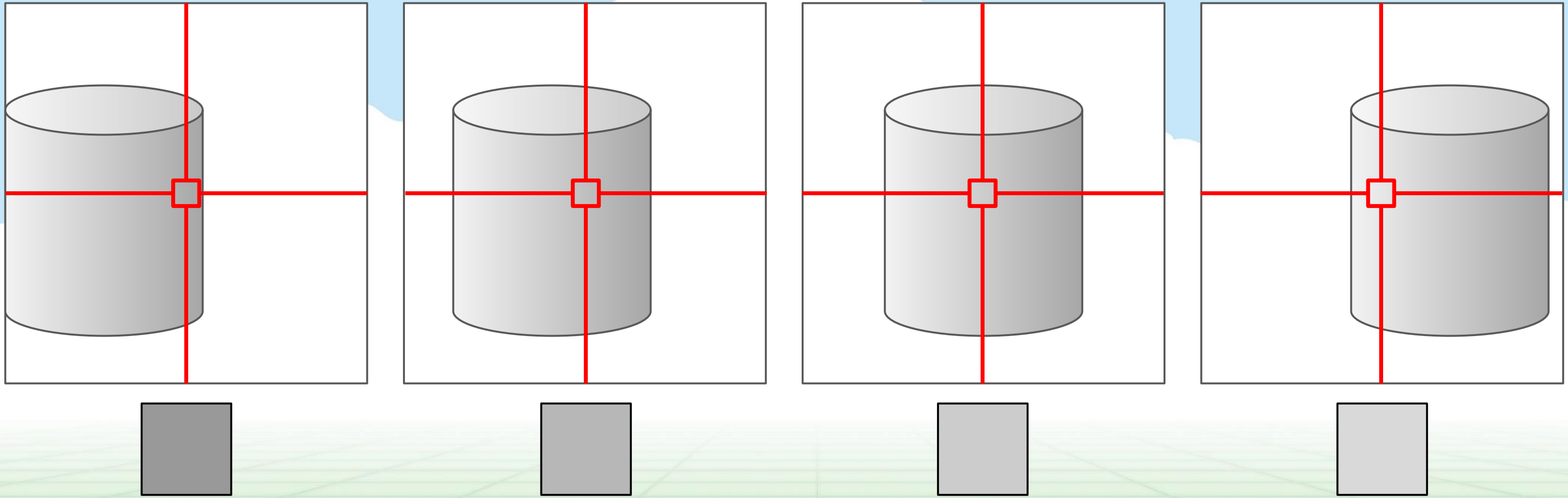

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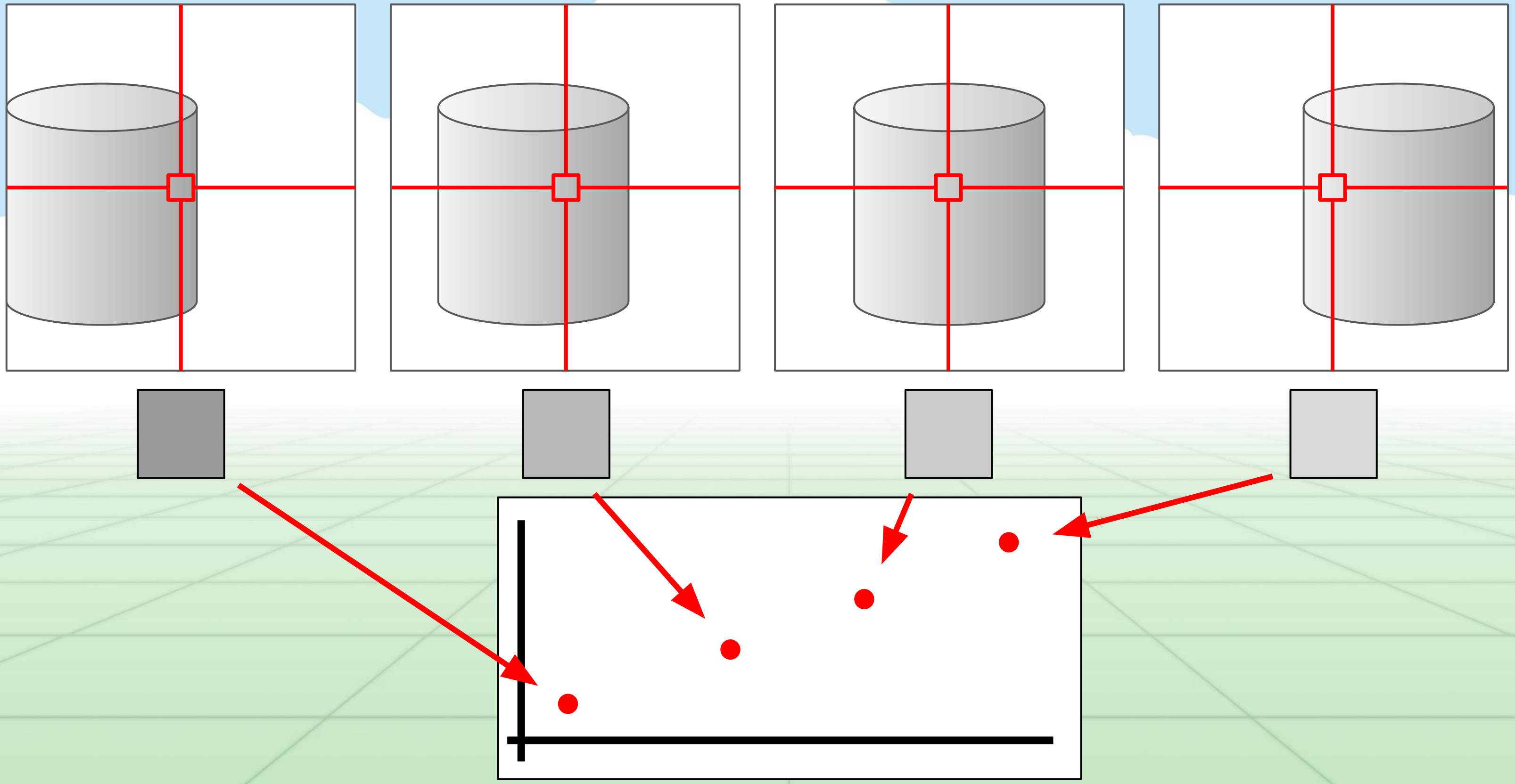

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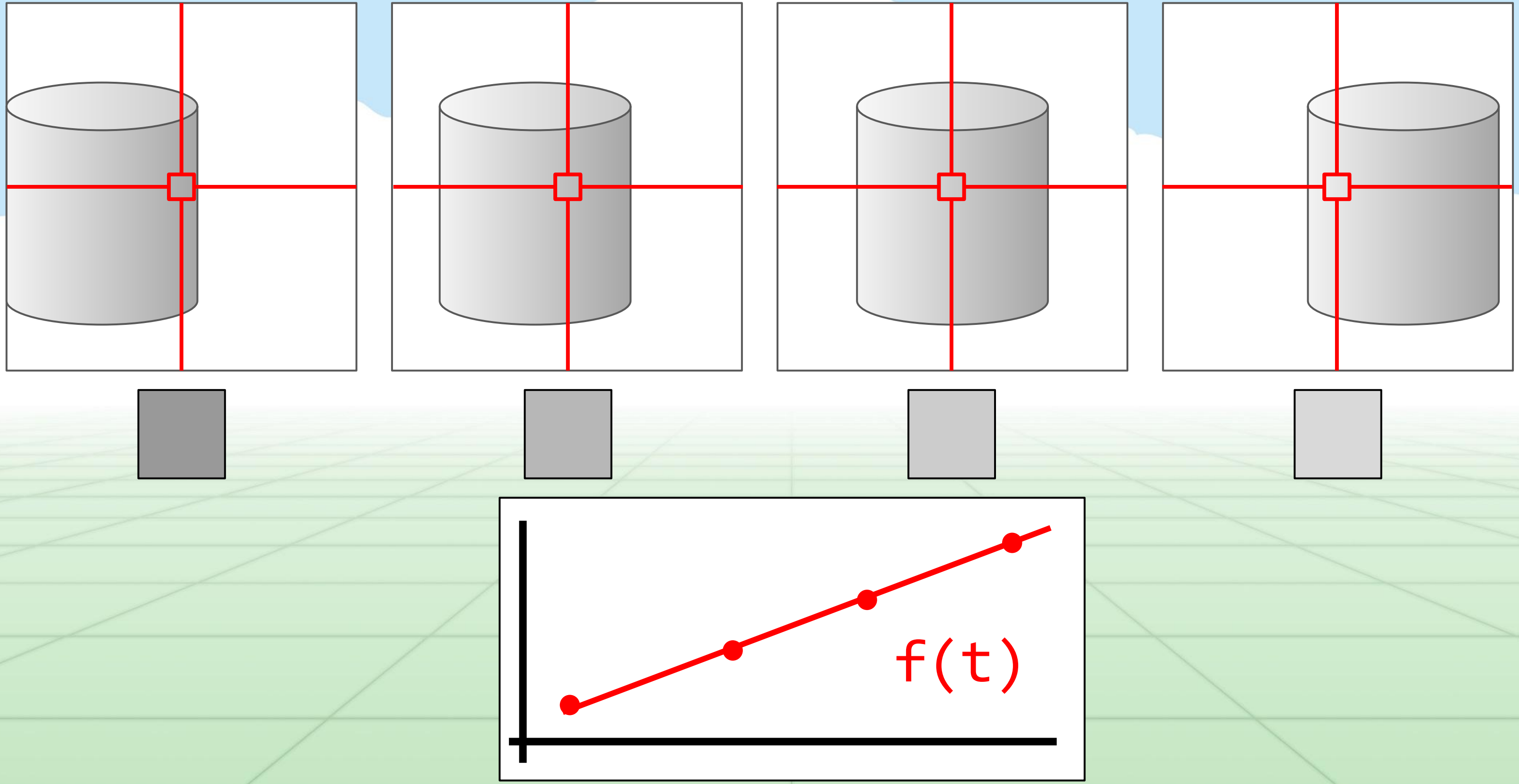
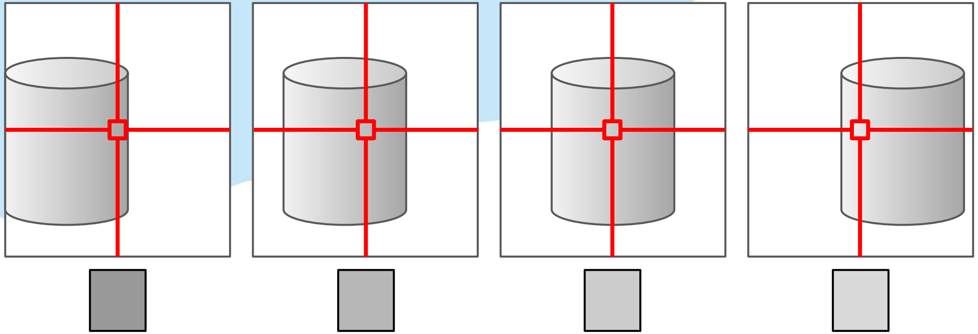
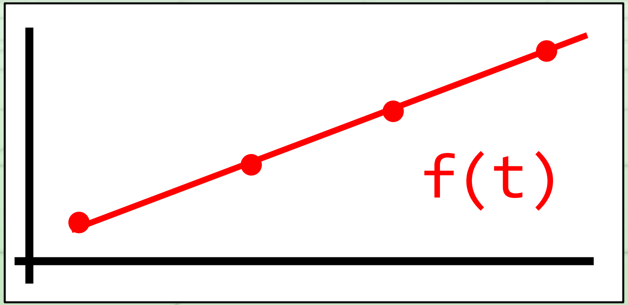


An observation...





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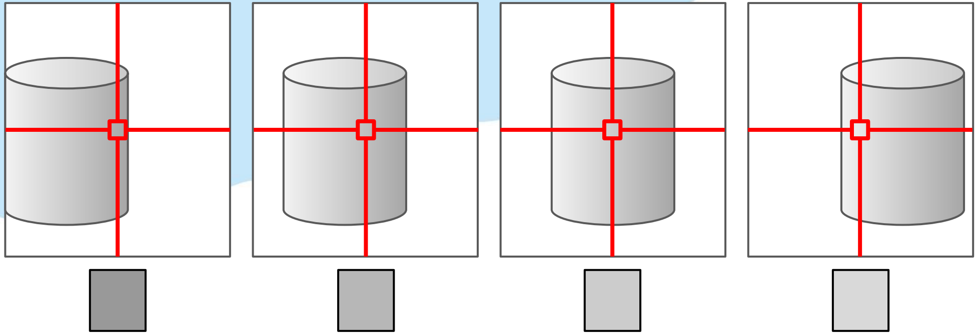

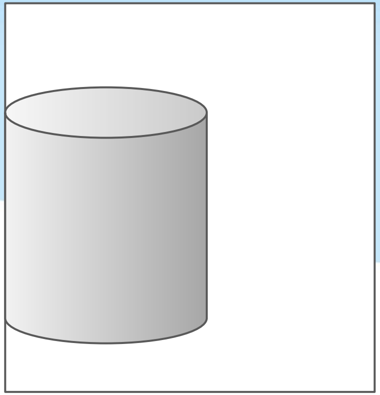

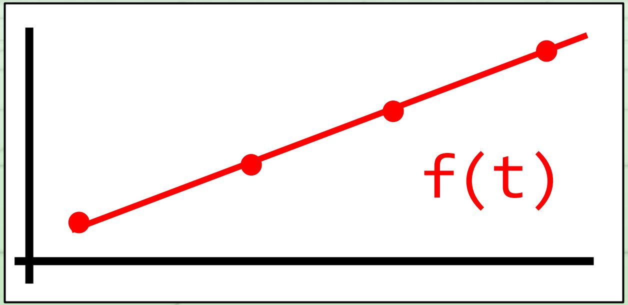

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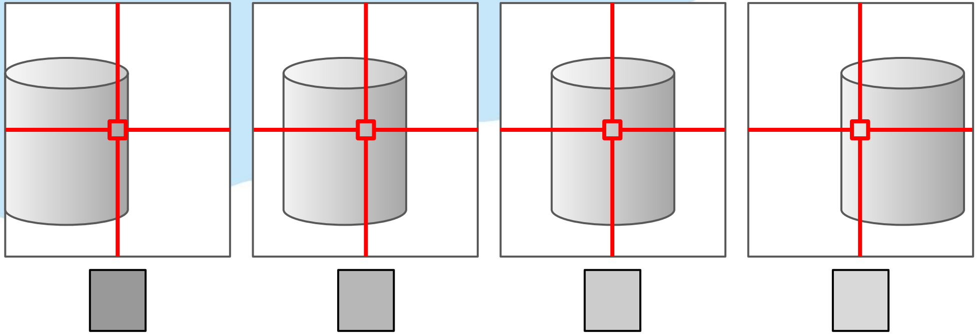

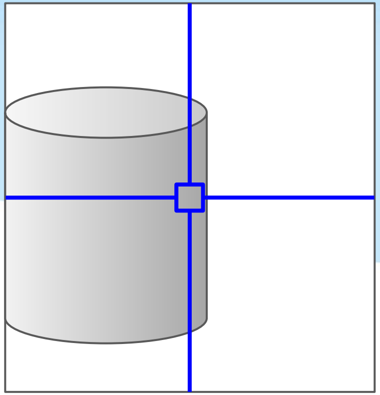

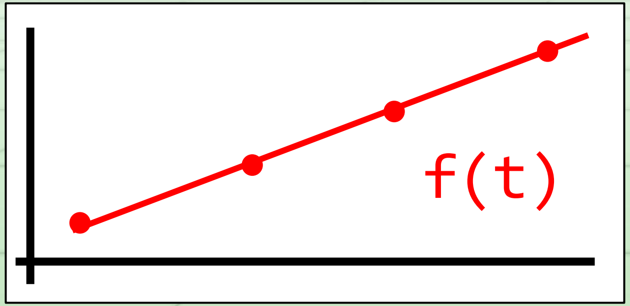

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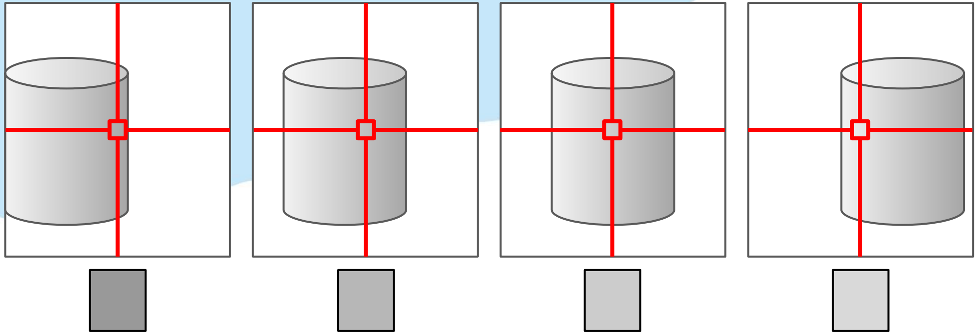

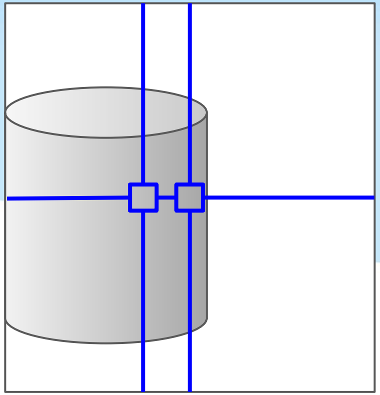

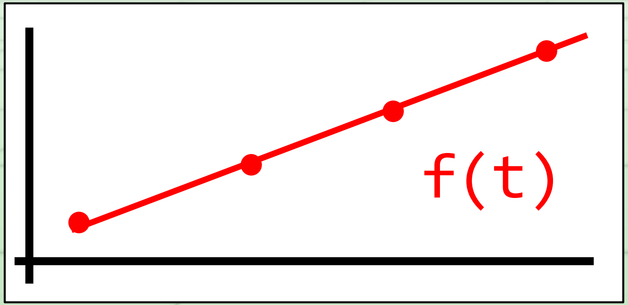

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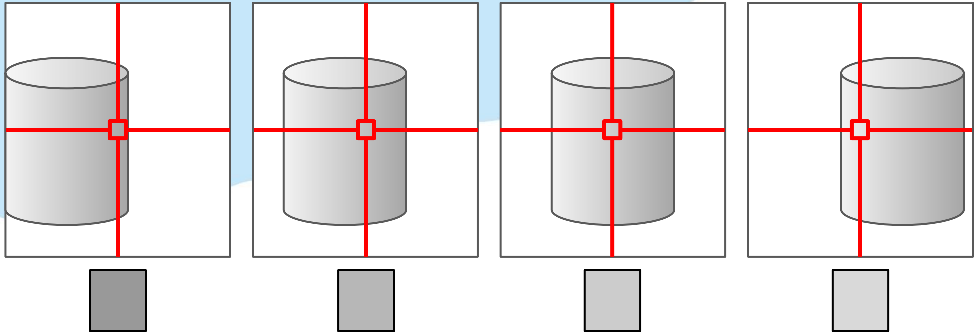

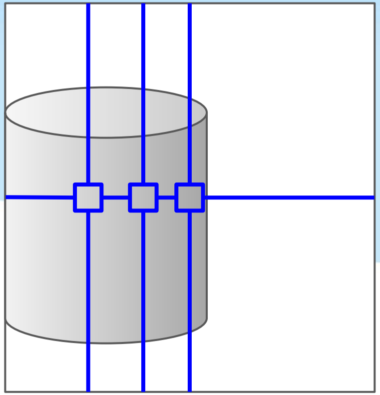

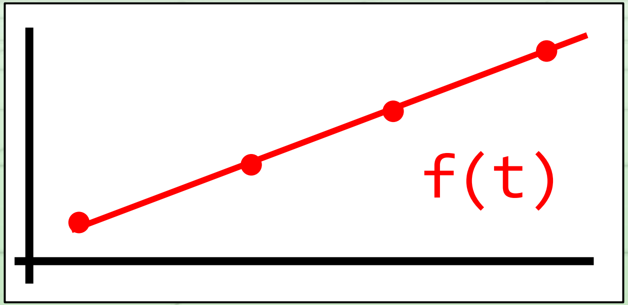

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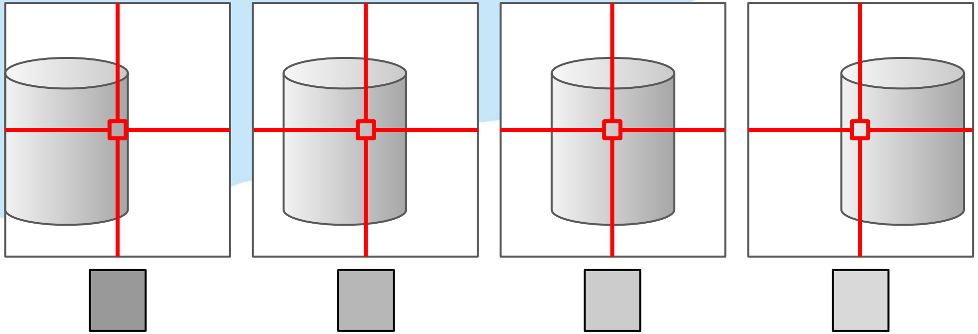

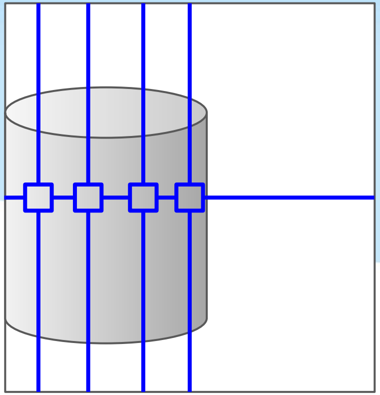

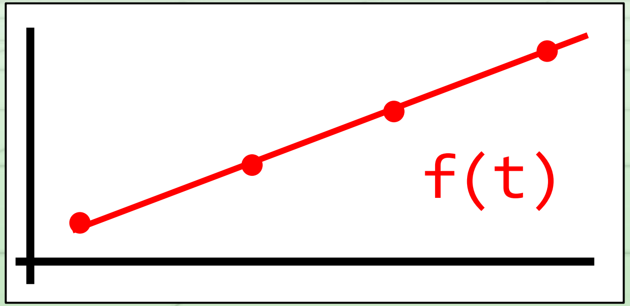

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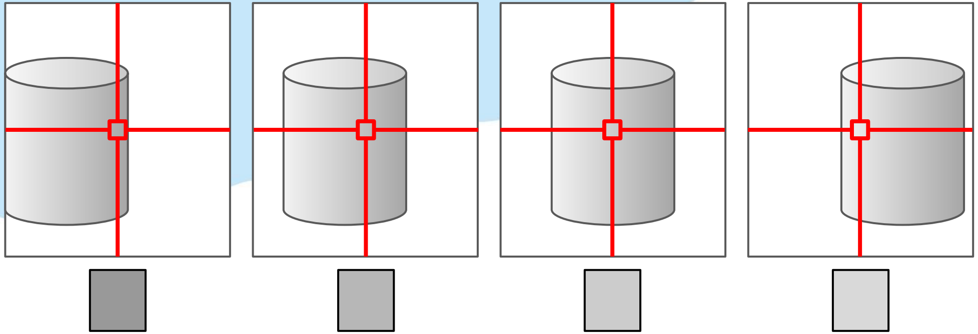

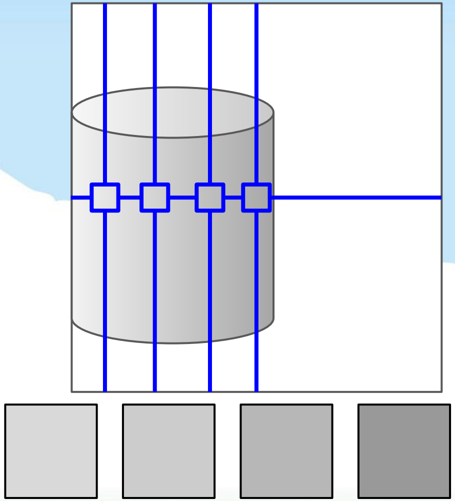

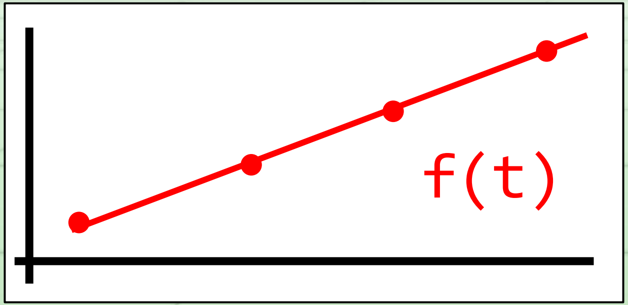

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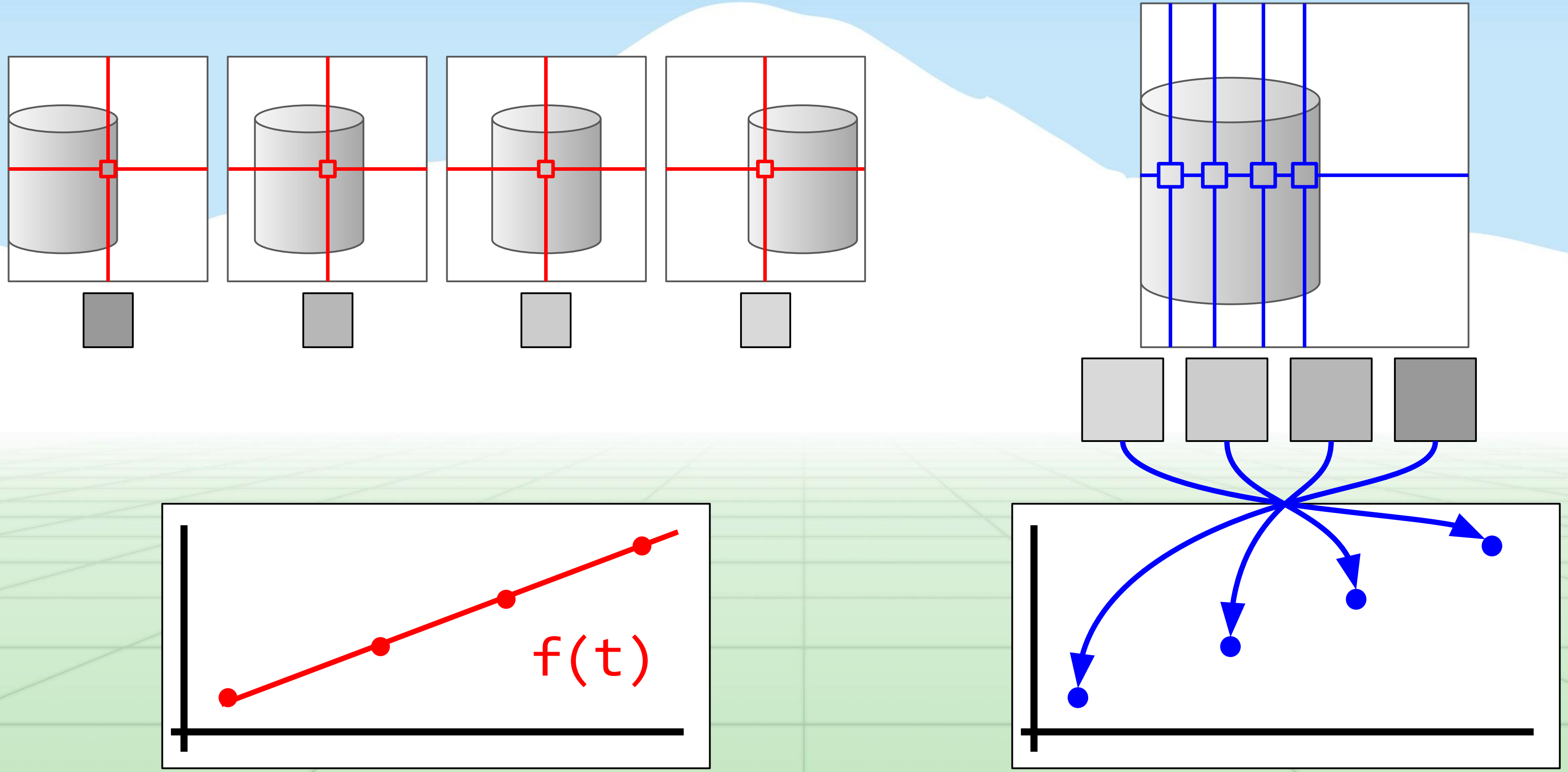

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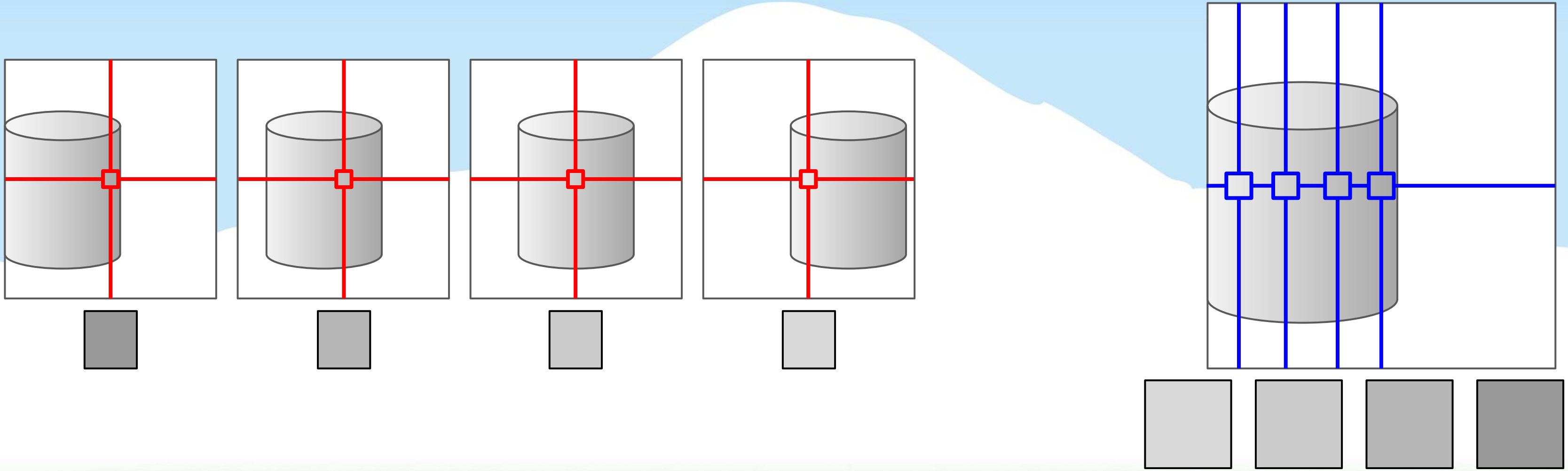

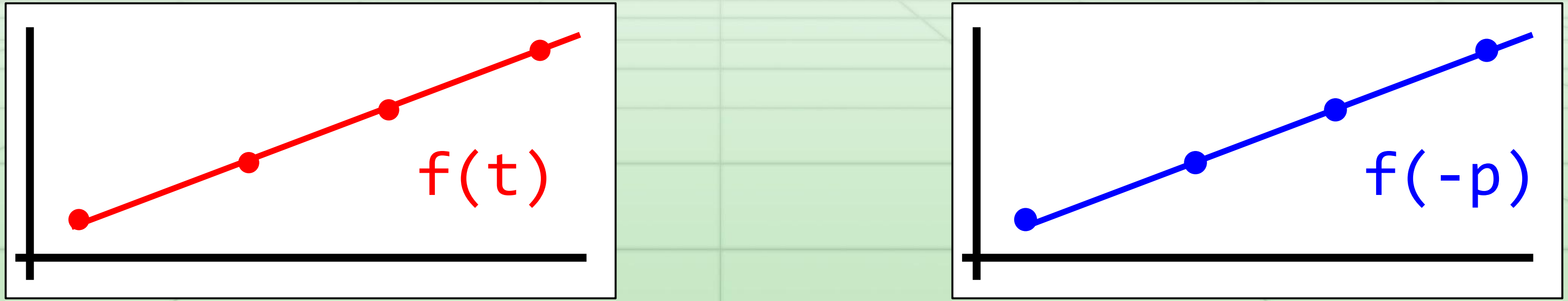

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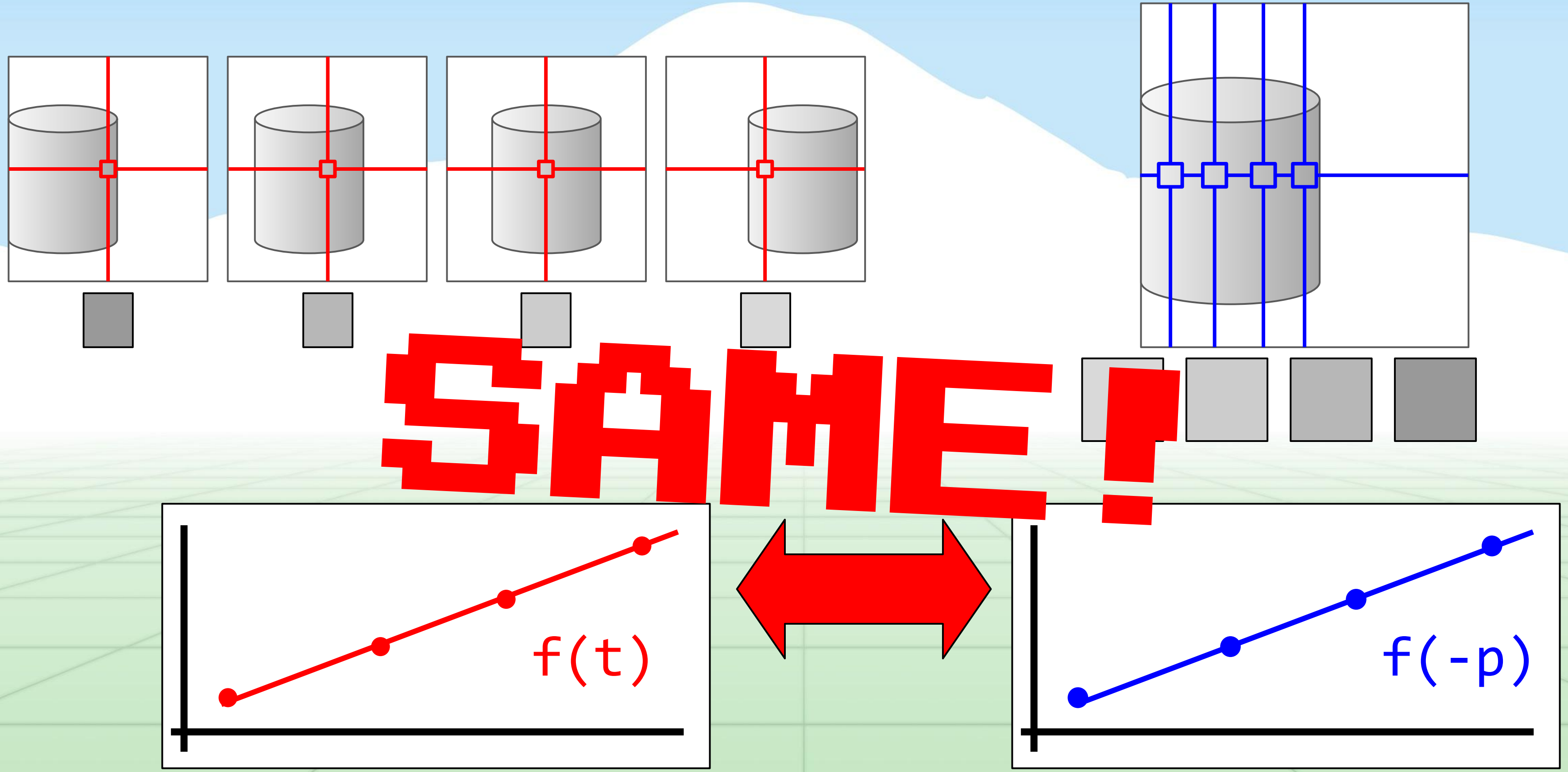

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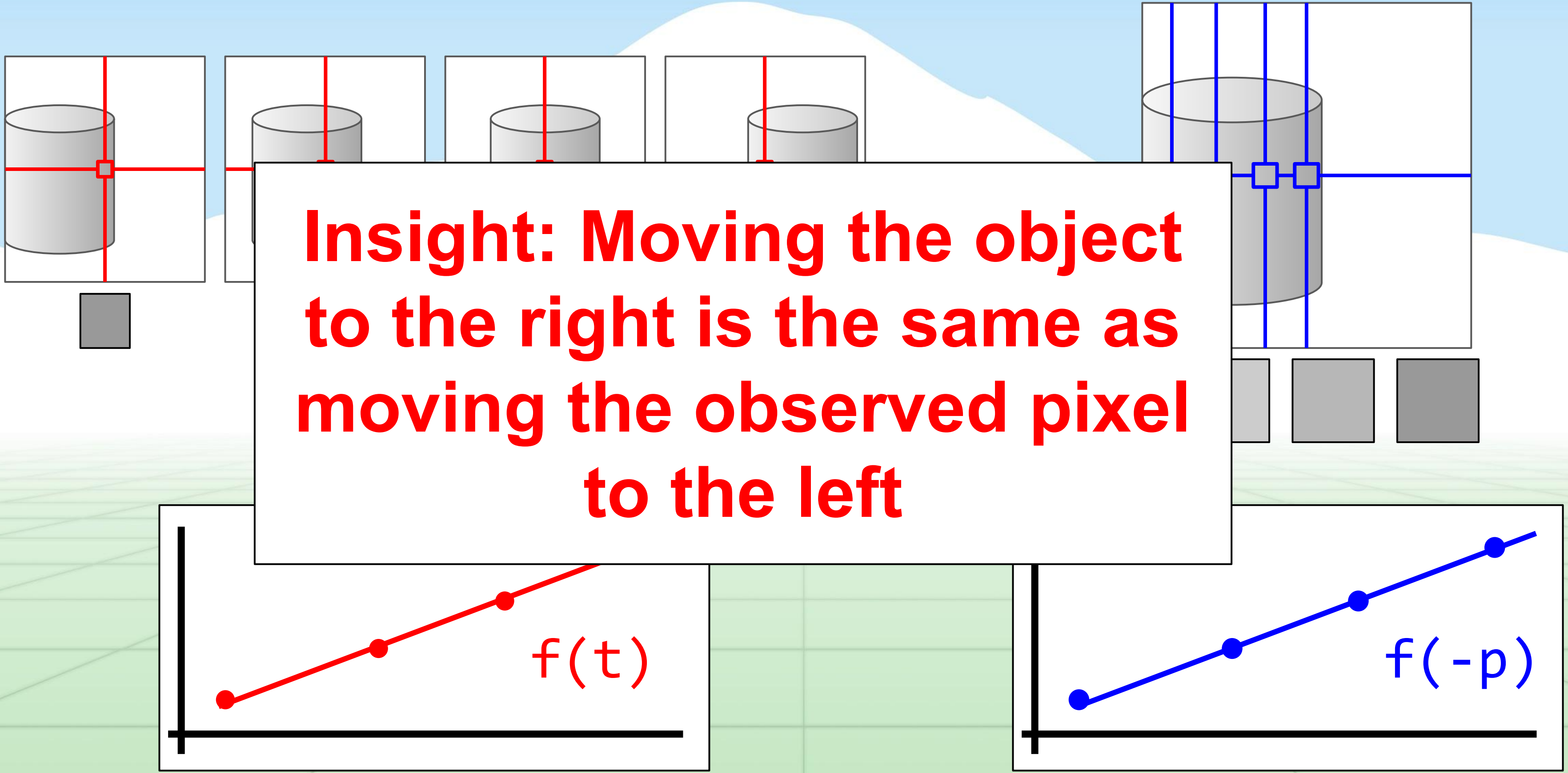


Insight: Moving the object to the right is the same as moving the observed pixel to the left

Lucas-Kanade Optical Flow

Can we use this relationship to compute optical flow?

Lucas-Kanade Optical Flow

Can we use this relationship to compute optical flow?

B.D. Lucas, T. Kanade, “An Image Registration Technique with an Application to Stereo Vision”, in Proceedings of Image Understanding Workshop, 1981, pp. 121-130.

Lucas-Kanade Optical Flow

Can we use this relationship to compute optical flow?

You will be implementing the LK optical flow algorithm for the homework

B.D. Lucas, T. Kanade, “An Image Registration Technique with an Application to Stereo Vision”, in Proceedings of Image Understanding Workshop, 1981, pp. 121-130.

Lucas-Kanade Optical Flow

Assuming the object moves by $\mathbf{\Delta p}$ over Δt

$$f(\mathbf{p} - \mathbf{\Delta p}, t) = f(\mathbf{p}, t + \Delta t)$$

Lucas-Kanade Optical Flow

Assuming the object moves by $\mathbf{\Delta p}$ over Δt

$$f(\mathbf{p} - \mathbf{\Delta p}, t) = f(\mathbf{p}, t + \Delta t)$$

Observation point

Lucas-Kanade Optical Flow

Assuming the object moves by $\Delta\mathbf{p}$ over Δt

$$f(\mathbf{p} - \Delta\mathbf{p},\ t) = f(\mathbf{p},\ t + \Delta t)$$

The movement @**p**

Lucas-Kanade Optical Flow

Assuming the object moves by $\Delta\mathbf{p}$ over Δt

$$f(\mathbf{p} - \Delta\mathbf{p}, t) = f(\mathbf{p}, t + \Delta t)$$

First image

Second image

Lucas-Kanade Optical Flow

Assuming the object moves by $\Delta \mathbf{p}$ over Δt

$$f(\mathbf{p} - \Delta \mathbf{p},\ t) = f(\mathbf{p},\ t + \Delta t)$$

Brightness

Lucas-Kanade Optical Flow

Assuming the object moves by $\mathbf{\Delta p}$ over Δt

$$f(\mathbf{p} - \mathbf{\Delta p}, t) = f(\mathbf{p}, t + \Delta t)$$

Approximate with a first-order Taylor expansion

Lucas-Kanade Optical Flow

Assuming the object moves by $\Delta \mathbf{p}$ over Δt

$$f(\mathbf{p} - \Delta \mathbf{p}, t) = f(\mathbf{p}, t + \Delta t)$$

Approximate with a first-order Taylor expansion

What in the doodely is a first-order Taylor expansion?

Quick & Dirty: Taylor Expansion

Approximate complex function with n^{th}-order polynomial

Quick & Dirty: Taylor Expansion

Approximate complex function with n^{th}-order polynomial

$$f(x) \approx \Sigma_i \ a_i x^i$$

Quick & Dirty: Taylor Expansion

Approximate complex function with n^th^-order polynomial

$$f(x) \approx \Sigma_i \; a_i x^i$$

For first-order i=1

$$\approx a_1 x^1 + a_0 x^0$$

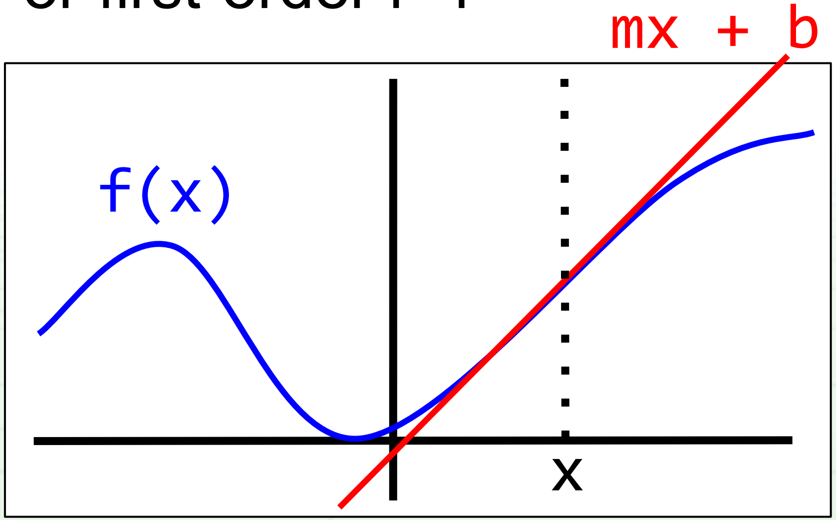
$$\approx mx + b$$

Quick & Dirty: Taylor Expansion

Approximate complex function with n^{th}-order polynomial

$$f(x) \approx \Sigma_i \; a_i x^i$$

For first-order i=1



$$\approx a_1 x^1 + a_0 x^0$$

$$\approx mx + b$$

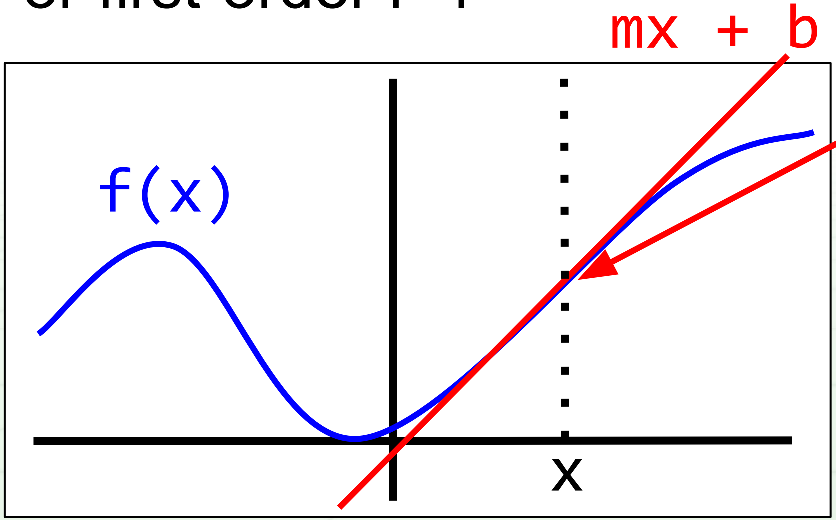
Quick & Dirty: Taylor Expansion

Approximate complex function with n^{th}-order polynomial

$$f(x) \approx \Sigma_i \ a_i x^i$$

For first-order i=1

Equal at x



$$\approx a_1 x^1 + a_0 x^0$$

$$\approx mx + b$$

Lucas-Kanade Optical Flow

Assuming the object moves by $\mathbf{\Delta p}$ over Δt

$$f(\mathbf{p} - \mathbf{\Delta p}, t) = f(\mathbf{p}, t + \Delta t)$$

Approximate with a first-order Taylor expansion

$$m(\mathbf{p} - \mathbf{\Delta p}) + b \approx f(\mathbf{p}, t + \Delta t)$$

Lucas-Kanade Optical Flow

Assuming the object moves by $\mathbf{\Delta p}$ over Δt

$$f(\mathbf{p} - \mathbf{\Delta p}, t) = f(\mathbf{p}, t + \Delta t)$$

Approximate with a first-order Taylor expansion

$$m(\mathbf{p} - \mathbf{\Delta p}) + b \approx f(\mathbf{p}, t + \Delta t)$$

$$m\mathbf{p} - m\mathbf{\Delta p} + b \approx f(\mathbf{p}, t + \Delta t)$$

Lucas-Kanade Optical Flow

$$m\mathbf{p} - m\Delta\mathbf{p} + b \approx f(\mathbf{p}, t + \Delta t)$$

Out of space

Lucas-Kanade Optical Flow

$$m\mathbf{p} - m\Delta\mathbf{p} + b \approx f(\mathbf{p}, t + \Delta t)$$

$$m\mathbf{p} - m\Delta\mathbf{p} + b - f(\mathbf{p}, t) \approx f(\mathbf{p}, t + \Delta t) - f(\mathbf{p}, t)$$

Subtract $f(\mathbf{p}, t)$
from both sides

Lucas-Kanade Optical Flow

$$\mathbf{mp} - \mathrm{m}\boldsymbol{\Delta}\mathbf{p} + \mathrm{b} \approx \mathrm{f}(\mathbf{p}, \mathrm{t} + \boldsymbol{\Delta}\mathrm{t})$$

$$\mathbf{mp} - \mathrm{m}\boldsymbol{\Delta}\mathbf{p} + \mathrm{b} - \mathrm{f}(\mathbf{p}, \mathrm{t}) \approx \mathrm{f}(\mathbf{p}, \mathrm{t} + \boldsymbol{\Delta}\mathrm{t}) - \mathrm{f}(\mathbf{p}, \mathrm{t})$$

Recall $\mathrm{f}(\mathrm{x}, \mathrm{t}) = \mathrm{mx} + \mathrm{b}$

Lucas-Kanade Optical Flow

$$m\mathbf{p} - m\Delta\mathbf{p} + b \approx f(\mathbf{p}, t + \Delta t)$$

$$m\mathbf{p} - m\Delta\mathbf{p} + b - f(\mathbf{p}, t) \approx f(\mathbf{p}, t + \Delta t) - f(\mathbf{p}, t)$$

Recall $f(x, t) = mx + b$

Plugging $m\mathbf{p} + b$ in for $f(\mathbf{p}, t)$ cancels out!

Lucas-Kanade Optical Flow

$$m\mathbf{p} - m\Delta\mathbf{p} + b \approx f(\mathbf{p},\ t + \Delta t)$$

$$\cancel{m\mathbf{p}} - m\Delta\mathbf{p} + \cancel{b} - \cancel{f(\mathbf{p},\ t)} \approx f(\mathbf{p},\ t + \Delta t) - f(\mathbf{p},\ t)$$

Recall $f(x,\ t) = mx + b$

Plugging $m\mathbf{p} + b$
in for $f(\mathbf{p},\ t)$
cancels out!

Lucas-Kanade Optical Flow

$$m\mathbf{p} - m\Delta\mathbf{p} + b \approx f(\mathbf{p}, t + \Delta t)$$

$$m\mathbf{p} - m\Delta\mathbf{p} + b - f(\mathbf{p}, t) \approx f(\mathbf{p}, t + \Delta t) - f(\mathbf{p}, t)$$

Recall $f(\mathbf{x}, t) = m\mathbf{x} + b$

$$-m\Delta\mathbf{p} \approx f(\mathbf{p}, t + \Delta t) - f(\mathbf{p}, t)$$

Lucas-Kanade Optical Flow

$$m\mathbf{p} - m\Delta\mathbf{p} + b \approx f(\mathbf{p},\ t + \Delta t)$$

$$m\mathbf{p} - m\Delta\mathbf{p} + b - f(\mathbf{p},\ t) \approx f(\mathbf{p},\ t + \Delta t) - f(\mathbf{p},\ t)$$

Recall $f(\mathbf{x},\ t) = m\mathbf{x} + b$

$$-m\Delta\mathbf{p} \approx f(\mathbf{p},\ t + \Delta t) - f(\mathbf{p},\ t)$$

These we know

Lucas-Kanade Optical Flow

$$m\mathbf{p} - m\Delta\mathbf{p} + b \approx f(\mathbf{p}, t + \Delta t)$$

$$m\mathbf{p} - m\Delta\mathbf{p} + b - f(\mathbf{p}, t) \approx f(\mathbf{p}, t + \Delta t) - f(\mathbf{p}, t)$$

Recall $f(\mathbf{x}, t) = m\mathbf{x} + b$

$$-m\Delta\mathbf{p} \approx f(\mathbf{p}, t + \Delta t) - f(\mathbf{p}, t)$$

This is what we want to know

These we know

Lucas-Kanade Optical Flow

$$m\mathbf{p} - m\Delta\mathbf{p} + b \approx f(\mathbf{p}, t + \Delta t)$$

$$m\mathbf{p} - m\Delta\mathbf{p} + b - f(\mathbf{p}, t) \approx f(\mathbf{p}, t + \Delta t) - f(\mathbf{p}, t)$$

Recall $f(\mathbf{x}, t) = m\mathbf{x} + b$

What is this?!

$$-m\Delta\mathbf{p} \approx f(\mathbf{p}, t + \Delta t) - f(\mathbf{p}, t)$$

This is what we want to know

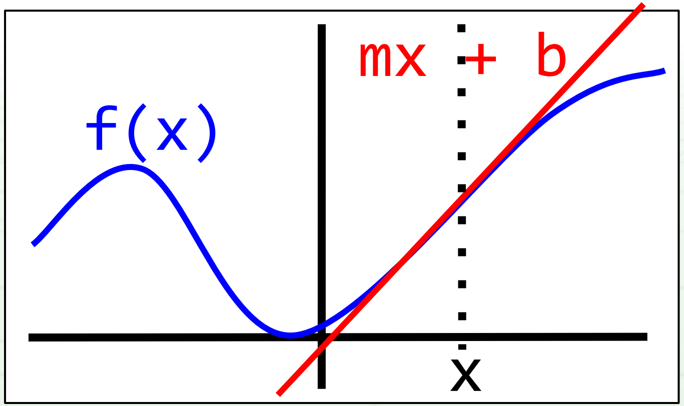
These we know

Lucas-Kanade Optical Flow

$$m\mathbf{p} - m\Delta\mathbf{p} + b \approx f(\mathbf{p}, t + \Delta t)$$

$$m\mathbf{p} - m\Delta\mathbf{p} + b - f(\mathbf{p}, t) \approx f(\mathbf{p}, t + \Delta t) - f(\mathbf{p}, t)$$

Recall $f(\mathbf{x}, t) = m\mathbf{x} + b$



$$-m\Delta\mathbf{p} \approx f(\mathbf{p}, t + \Delta t) - f(\mathbf{p}, t)$$

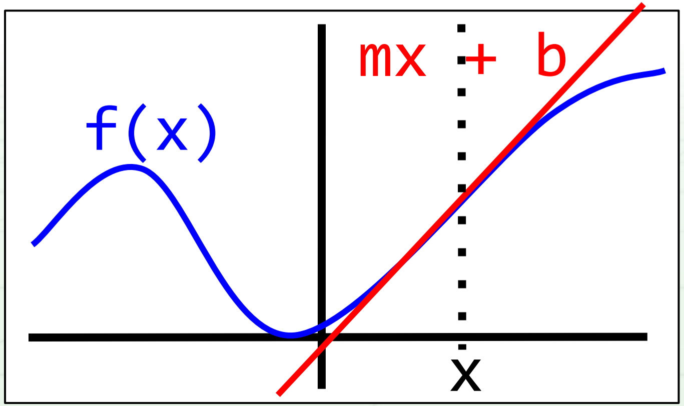
Slope

Lucas-Kanade Optical Flow

$$m\mathbf{p} - m\Delta\mathbf{p} + b \approx f(\mathbf{p}, t + \Delta t)$$

$$m\mathbf{p} - m\Delta\mathbf{p} + b - f(\mathbf{p}, t) \approx f(\mathbf{p}, t + \Delta t) - f(\mathbf{p}, t)$$

Recall $f(\mathbf{x}, t) = m\mathbf{x} + b$



$$-m\Delta\mathbf{p} \approx f(\mathbf{p}, t + \Delta t) - f(\mathbf{p}, t)$$

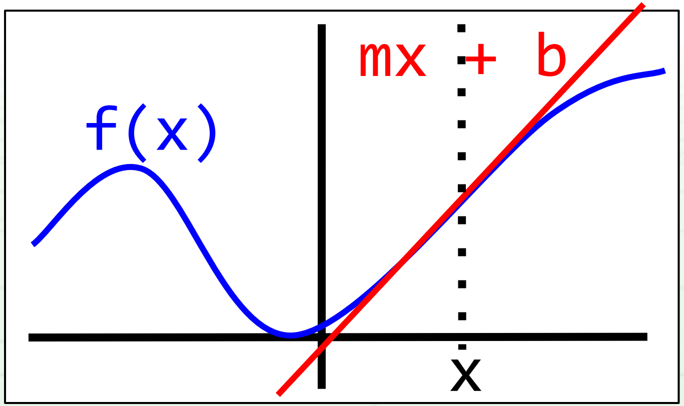
Slope = derivative

Lucas-Kanade Optical Flow

$$m\mathbf{p} - m\Delta\mathbf{p} + b \approx f(\mathbf{p}, t + \Delta t)$$

$$m\mathbf{p} - m\Delta\mathbf{p} + b - f(\mathbf{p}, t) \approx f(\mathbf{p}, t + \Delta t) - f(\mathbf{p}, t)$$

Recall $f(\mathbf{x}, t) = m\mathbf{x} + b$

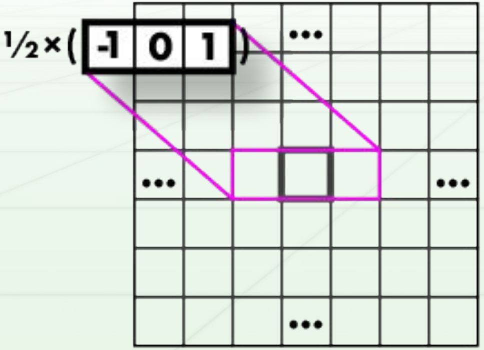


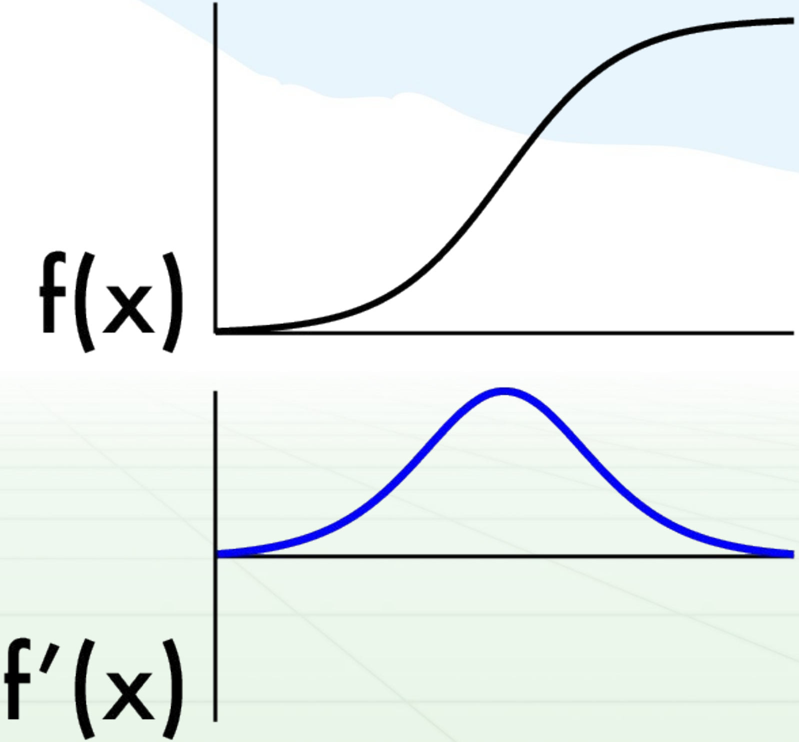
$$-m\Delta\mathbf{p} \approx f(\mathbf{p}, t + \Delta t) - f(\mathbf{p}, t)$$

Slope = derivative, aka gradient,
We know how to do gradients!

Previously: Image derivatives

- Recall:
  - $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$
- We don't have an "actual" Function, must estimate
- Possibility: set h = 2
- What will that look like?



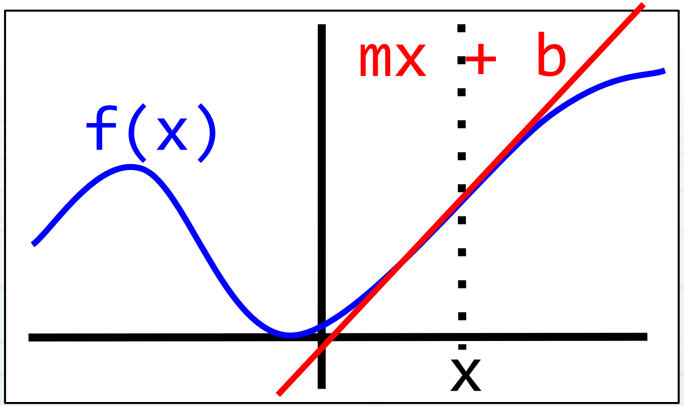


Lucas-Kanade Optical Flow

$$m\mathbf{p} - m\Delta\mathbf{p} + b \approx f(\mathbf{p}, t + \Delta t)$$

$$m\mathbf{p} - m\Delta\mathbf{p} + b - f(\mathbf{p}, t) \approx f(\mathbf{p}, t + \Delta t) - f(\mathbf{p}, t)$$

Recall $f(\mathbf{x}, t) = m\mathbf{x} + b$



$$-m\Delta\mathbf{p} \approx f(\mathbf{p}, t + \Delta t) - f(\mathbf{p}, t)$$

$$m = d\mathbf{p} = \begin{bmatrix} dx & 0 \\ 0 & dy \end{bmatrix}$$

Lucas-Kanade Optical Flow

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$$m\mathbf{p} - m\Delta\mathbf{p} + b - f(\mathbf{p}, t) \approx f(\mathbf{p}, t + \Delta t) - f(\mathbf{p}, t)$$

Recall $f(\mathbf{x}, t) = m\mathbf{x} + b$

$$-m\Delta\mathbf{p} \approx f(\mathbf{p}, t + \Delta t) - f(\mathbf{p}, t)$$

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Lucas-Kanade Optical Flow

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Let's drop the vector notation

Lucas-Kanade Optical Flow

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$$-d\mathbf{p}\Delta\mathbf{p} \approx f(\mathbf{p}, t + \Delta t) - f(\mathbf{p}, t)$$

$$-dx\Delta x + -dy\Delta y \approx f((x,y), t + \Delta t) - f((x,y), t)$$

Lucas-Kanade Optical Flow

$$-dx\Delta x + -dy\Delta y \approx f((x,y), t + \Delta t) - f((x,y), t)$$

Lucas-Kanade Optical Flow

$$-dx\Delta x + -dy\Delta y \approx f((x,y),\ t + \Delta t) - f((x,y),\ t)$$

Let’s clean up the notation a bit

$$dx * u + dy * v = I_t[x,y] - I_{t+\Delta t}[x,y]$$

Lucas-Kanade Optical Flow

$$-dx\Delta x + -dy\Delta y \approx f((x,y),\ t + \Delta t) - f((x,y),\ t)$$

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Lucas-Kanade Optical Flow

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Solve for u,v

Lucas-Kanade Optical Flow

$$-dx\Delta x + -dy\Delta y \approx f((x,y),\ t + \Delta t) - f((x,y),\ t)$$

Let's clean up the notation a bit

$$dx*u + dy*v = I_t[x,y] - I_{t+\Delta t}[x,y]$$

Solve for u,v

1 equation, 2 unknowns

Lucas-Kanade Optical Flow

$$-dx\Delta x + -dy\Delta y \approx f((x,y),\ t + \Delta t) - f((x,y),\ t)$$

Let’s clean up the notation a bit

$$dx{*}u + dy{*}v = I_t[x,y] - I_{t+\Delta t}[x,y]$$

Solve for u,v

1 equation, 2 unknowns

IMPOSSIBLE!

Lucas-Kanade Optical Flow

Let's back up

$$dx*{\color{red}u} + dy*{\color{red}v} = I_t[x,y] - I_{t+\Delta t}[x,y]$$

Lucas-Kanade Optical Flow

Let's back up

$$dx*u + dy*v = I_t[x,y] - I_{t+\Delta t}[x,y]$$

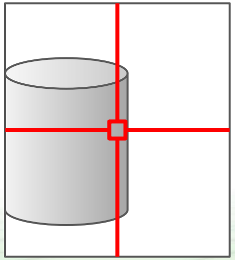
What does this *mean*?

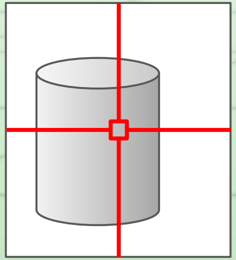
Lucas-Kanade Optical Flow

$$dx*u + dy*v = I_t[x,y] - I_{t+\Delta t}[x,y]$$

Lucas-Kanade Optical Flow

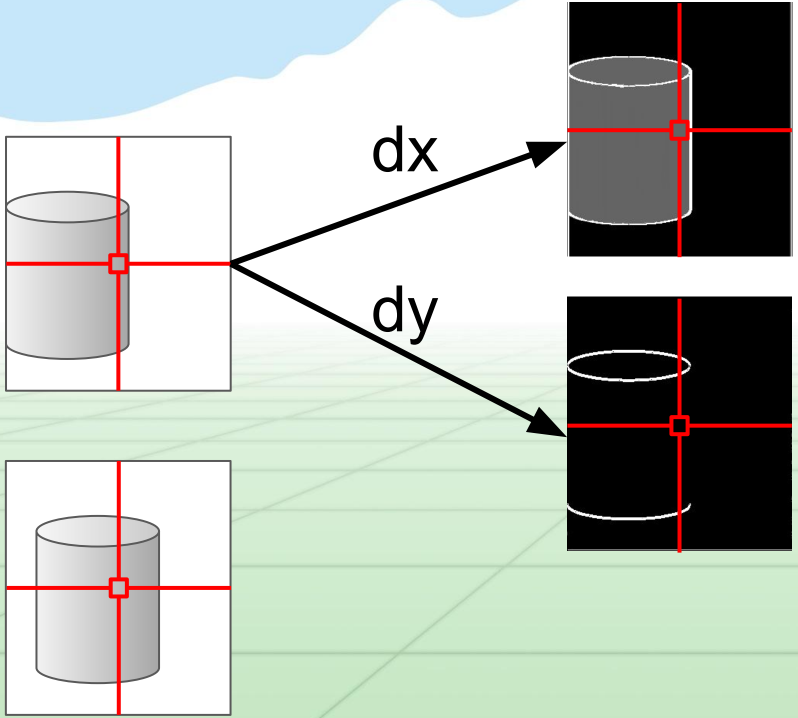
$$dx*u + dy*v = I_t[x,y] - I_{t+\Delta t}[x,y]$$





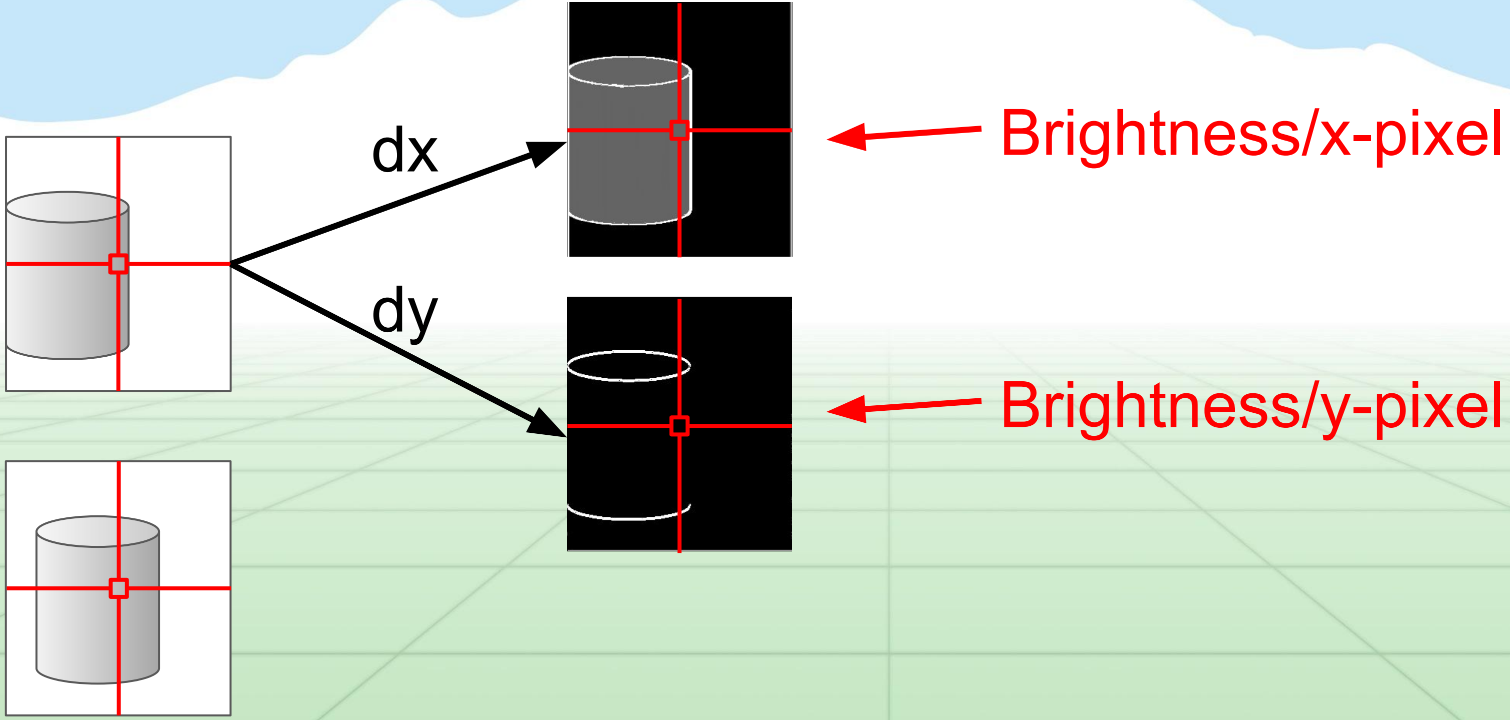
Lucas-Kanade Optical Flow

$$dx*u + dy*v = I_t[x,y] - I_{t+\Delta t}[x,y]$$



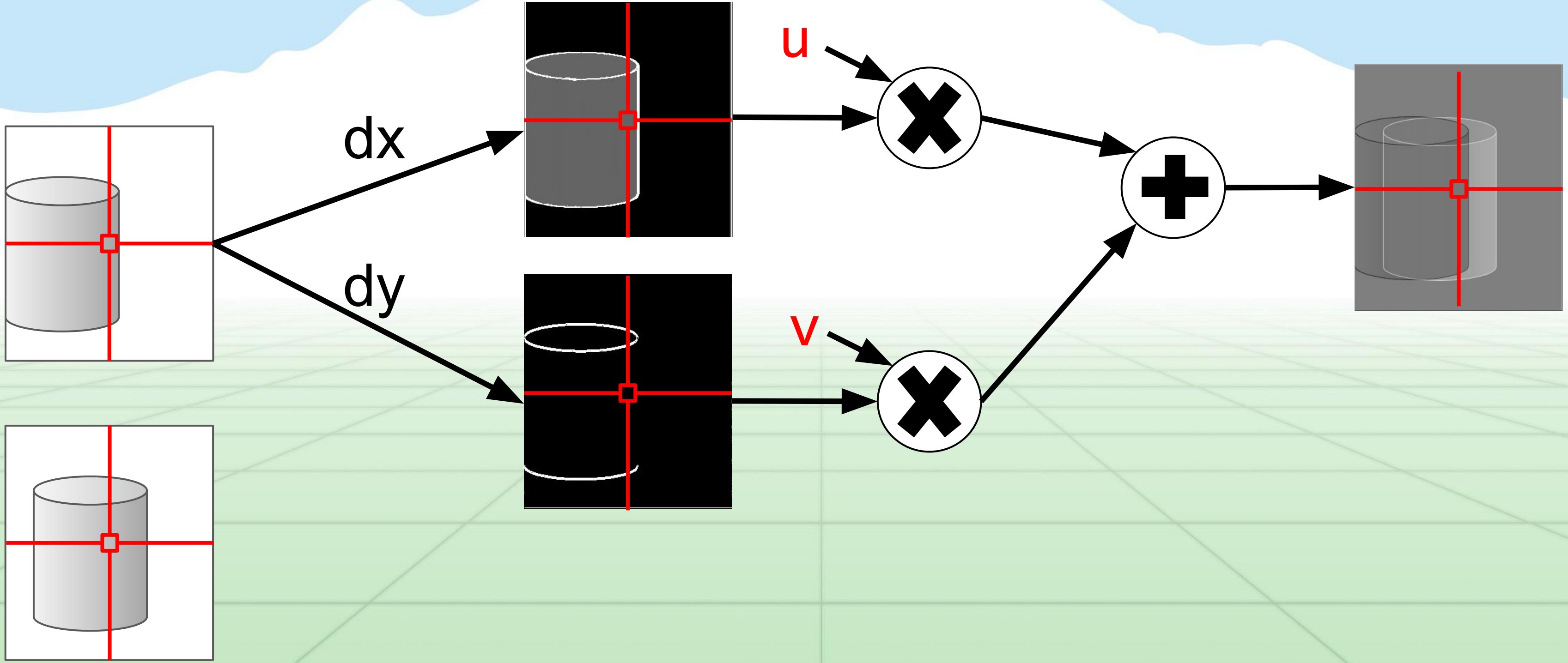
Lucas-Kanade Optical Flow

$$dx*u + dy*v = I_t[x,y] - I_{t+\Delta t}[x,y]$$



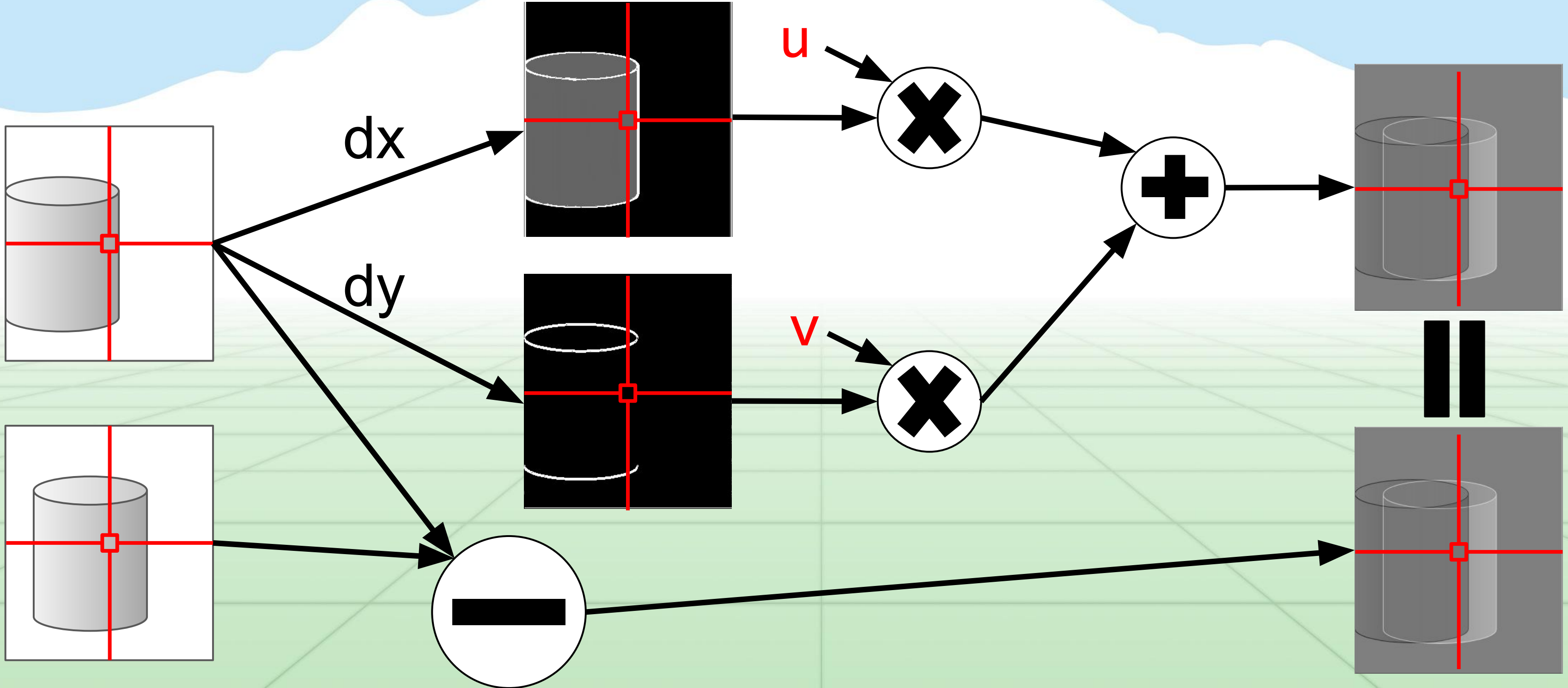
Lucas-Kanade Optical Flow

$$dx*u + dy*v = I_t[x,y] - I_{t+\Delta t}[x,y]$$



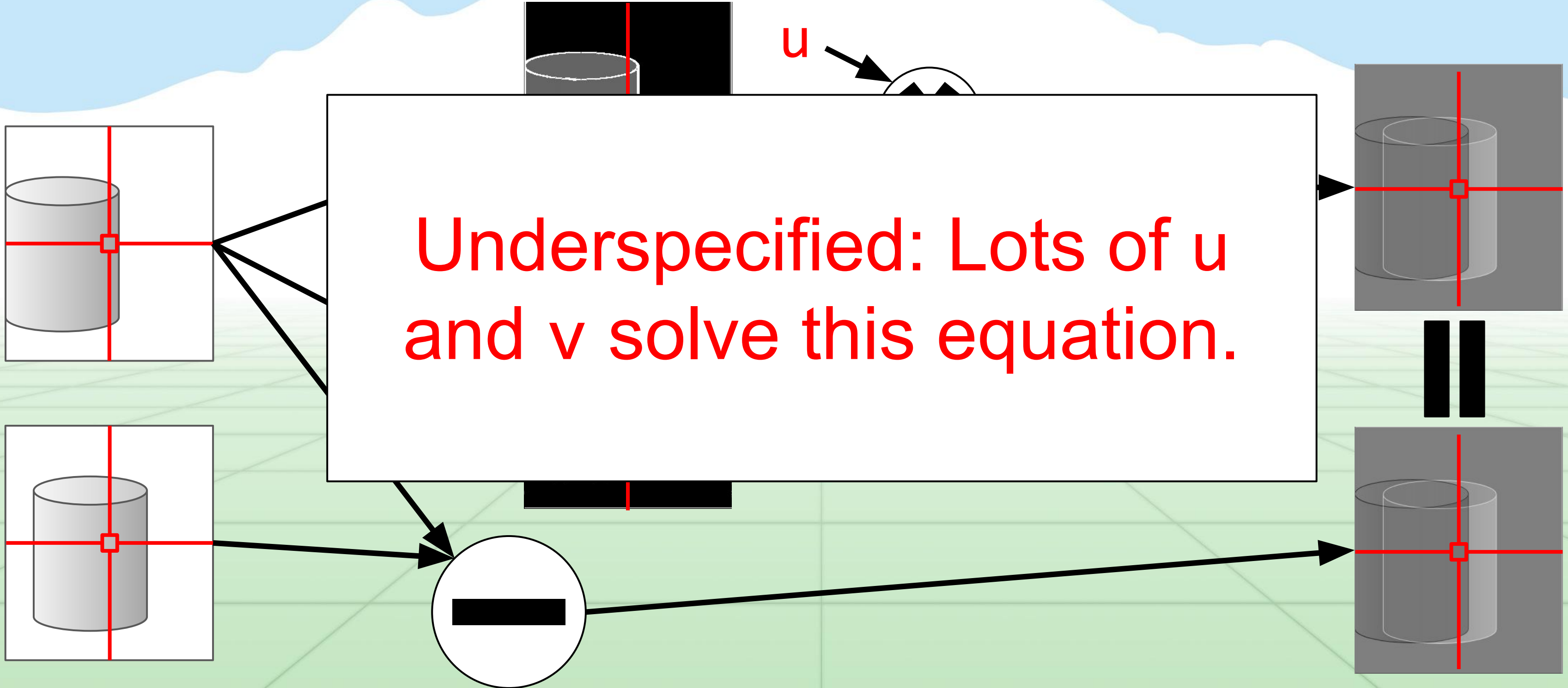
Lucas-Kanade Optical Flow

$$dx*\color{red}{u} + dy*\color{red}{v} = I_t[x,y] - I_{t+\Delta t}[x,y]$$



Lucas-Kanade Optical Flow

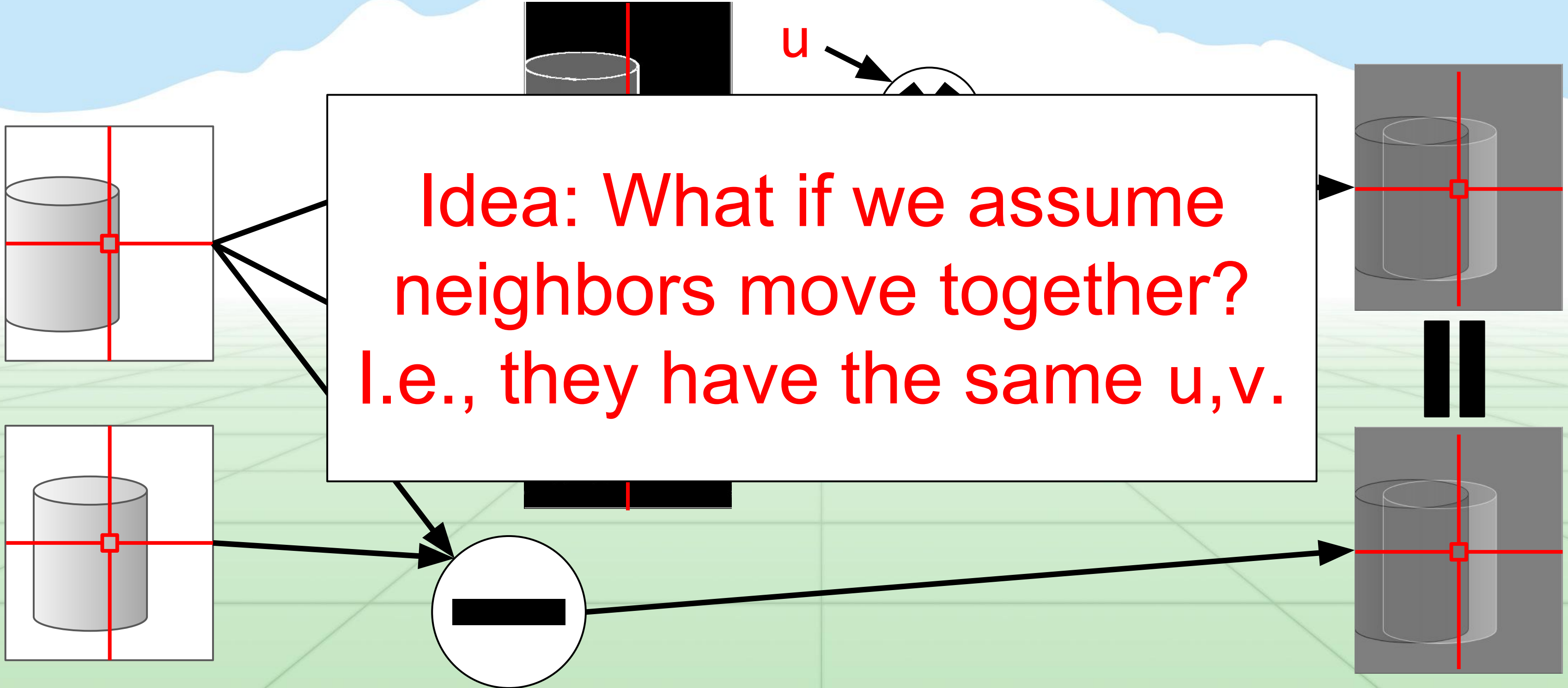
$$dx*{\color{red}u} + dy*{\color{red}v} = I_t[x,y] - I_{t+\Delta t}[x,y]$$



Underspecified: Lots of u and v solve this equation.

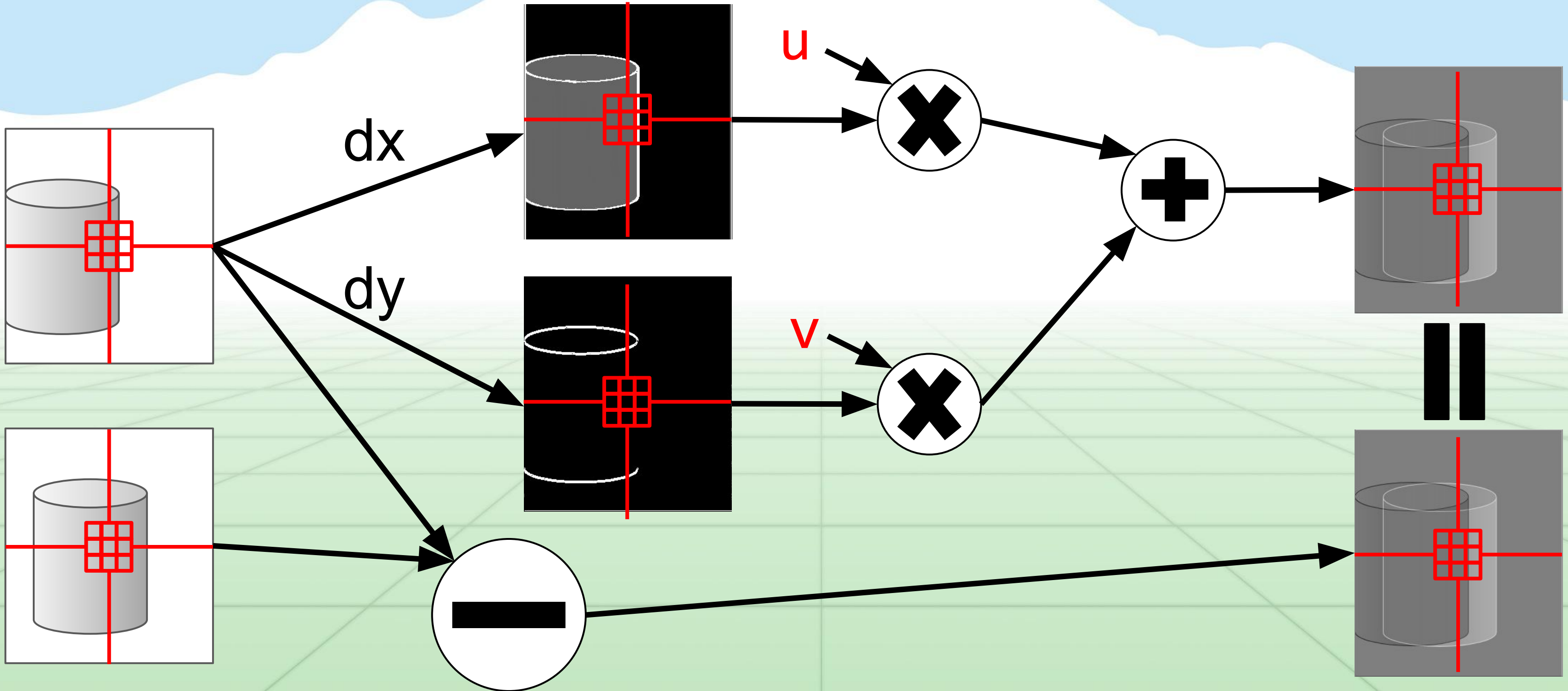
Lucas-Kanade Optical Flow

$$dx*u + dy*v = I_t[x,y] - I_{t+\Delta t}[x,y]$$



Lucas-Kanade Optical Flow

$$dx * u + dy * v = I_t[x,y] - I_{t+\Delta t}[x,y]$$



Lucas-Kanade Optical Flow

$$dx_1 * u + dy_1 * v = I_t[x_1, y_1] - I_{t+\Delta t}[x_1, y_1]$$

$$dx_2 * u + dy_2 * v = I_t[x_2, y_2] - I_{t+\Delta t}[x_2, y_2]$$

$$\ldots$$

$$dx_9 * u + dy_9 * v = I_t[x_9, y_9] - I_{t+\Delta t}[x_9, y_9]$$

9 equations, 2 unknowns

Lucas-Kanade Optical Flow

$$dx_1 * \color{red}{u} + dy_1 * \color{red}{v} = I_t[x_1, y_1] - I_{t+\Delta t}[x_1, y_1]$$

$$dx_2 * \color{red}{u} + dy_2 * \color{red}{v} = I_t[x_2, y_2] - I_{t+\Delta t}[x_2, y_2]$$

$$\ldots$$

$$dx_9 * \color{red}{u} + dy_9 * \color{red}{v} = I_t[x_9, y_9] - I_{t+\Delta t}[x_9, y_9]$$

OVERSPECIFIED!

9 equations, 2 unknowns

Lucas-Kanade Optical Flow

$$S = \begin{bmatrix} dx_1 & dy_1 \\ dx_2 & dy_2 \\ \multicolumn{2}{c}{\cdots} \\ dx_9 & dy_9 \end{bmatrix} \quad \mathbf{\Delta p} = \begin{bmatrix} u \\ v \end{bmatrix} \quad T = \begin{bmatrix} I_t[x_1,y_1] - I_{t+\Delta t}[x_1,y_1] \\ I_t[x_2,y_2] - I_{t+\Delta t}[x_2,y_2] \\ \cdots \\ I_t[x_9,y_9] - I_{t+\Delta t}[x_9,y_9] \end{bmatrix}$$

$$S\mathbf{\Delta p} = T$$

Lucas-Kanade Optical Flow

$$S = \begin{bmatrix} dx_1 & dy_1 \\ dx_2 & dy_2 \\ \multicolumn{2}{c}{\cdots} \\ dx_9 & dy_9 \end{bmatrix} \quad \Delta\mathbf{p} = \begin{bmatrix} u \\ v \end{bmatrix} \quad T = \begin{bmatrix} I_t[x_1,y_1] - I_{t+\Delta t}[x_1,y_1] \\ I_t[x_2,y_2] - I_{t+\Delta t}[x_2,y_2] \\ \cdots \\ I_t[x_9,y_9] - I_{t+\Delta t}[x_9,y_9] \end{bmatrix}$$

$$S\Delta\mathbf{p} = T$$

Least-squares solution

$$||S\Delta\mathbf{p} - T||^2 = 0$$

$$\Delta\mathbf{p} = (S^T S)^{-1} S^T T$$

Lucas-Kanade Optical Flow

$$S = \begin{bmatrix} dx_1 & dy_1 \\ dx_2 & dy_2 \\ \multicolumn{2}{c}{\cdots} \\ dx_9 & dy_9 \end{bmatrix} \qquad \mathbf{\Delta p} = \begin{bmatrix} u \\ v \end{bmatrix} \qquad T = \begin{bmatrix} I_t[x_1,y_1] - I_{t+\Delta t}[x_1,y_1] \\ I_t[x_2,y_2] - I_{t+\Delta t}[x_2,y_2] \\ \cdots \\ I_t[x_9,y_9] - I_{t+\Delta t}[x_9,y_9] \end{bmatrix}$$

$$S\mathbf{\Delta p} = T$$

What if this isn't invertible?

$$||S\mathbf{\Delta p} - T||^2 = 0$$

$$\mathbf{\Delta p} = (S^TS)^{-1}S^TT$$

Lucas-Kanade Optical Flow

$$S = \begin{bmatrix} dx_1 & dy_1 \\ dx_2 & dy_2 \\ \multicolumn{2}{c}{\cdots} \\ dx_9 & dy_9 \end{bmatrix}$$

$$\begin{matrix} - I_{t+\Delta t}[x_1,y_1] \\ - I_{t+\Delta t}[x_2,y_2] \\ \cdots \\ - I_{t+\Delta t}[x_9,y_9] \end{matrix}$$

$$||S\mathbf{\Delta p} - T||^2 = 0$$

What if this isn't invertible?

$$\mathbf{\Delta p} = (S^TS)^{-1}S^TT$$

E.G., no structure around (x,y)

Lucas-Kanade Optical Flow

Testing for invertibility: $\mathbf{\Delta p} = (S^{T}S)^{-1}S^{T}T$

Lucas-Kanade Optical Flow

Testing for invertibility: $\Delta\mathbf{p} = (S^TS)^{-1}S^TT$

S^TS is symmetric so it can be decomposed as

$$S^TS = U\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}U^T$$

Lucas-Kanade Optical Flow

Testing for invertibility: $\boldsymbol{\Delta p} = (S^TS)^{-1}S^TT$

S^TS is symmetric so it can be decomposed as

$$S^TS = U\begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix}U^T$$

$$\det(S^TS - \boldsymbol{\lambda I}) = 0$$

Lucas-Kanade Optical Flow

Testing for invertibility: $\mathbf{\Delta p} = (S^TS)^{-1}S^TT$

S^TS is symmetric so it can be decomposed as

$$S^TS = U\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}U^T$$

$$\det(S^TS - \boldsymbol{\lambda}\mathbf{I}) = 0$$

$$\begin{vmatrix} \boldsymbol{\Sigma}_i (dx_i)^2 - \boldsymbol{\lambda} & \boldsymbol{\Sigma}_i (dx_i)(dy_i) \\ \boldsymbol{\Sigma}_i (dx_i)(dy_i) & \boldsymbol{\Sigma}_i (dy_i)^2 - \boldsymbol{\lambda} \end{vmatrix} = 0$$

Lucas-Kanade Optical Flow

Testing for invertibility: $\mathbf{\Delta p} = (S^T S)^{-1} S^T T$

$S^T S$ is symmetric so it can be decomposed as

$$S^T S = U \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} U^T$$

$$\det(S^T S - \boldsymbol{\lambda} \mathbf{I}) = 0$$

$$\begin{vmatrix} \Sigma_i (dx_i)^2 - \lambda & \Sigma_i (dx_i)(dy_i) \\ \Sigma_i (dx_i)(dy_i) & \Sigma_i (dy_i)^2 - \lambda \end{vmatrix} = 0$$

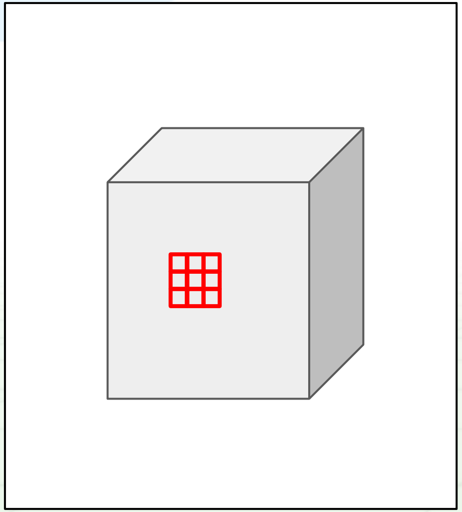
If either λ_1 or λ_2 are ≤ 0, S is not invertible and there is no solution!

Lucas-Kanade Optical Flow

How to pick (x,y)? What makes a good feature?

Lucas-Kanade Optical Flow

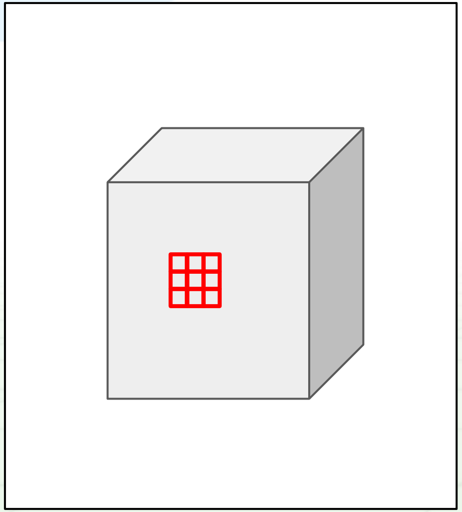
How to pick (x,y)? What makes a good feature?



Lucas-Kanade Optical Flow

How to pick (x,y)? What makes a good feature?

Bad

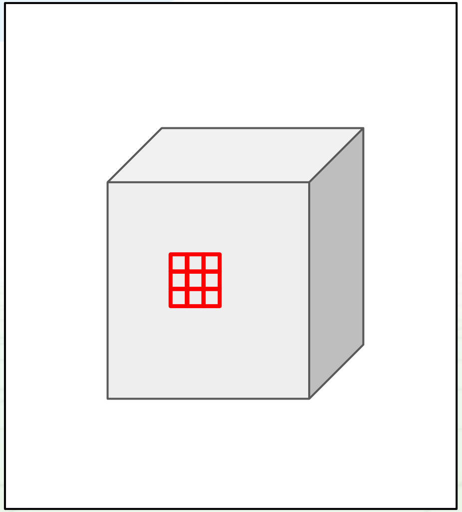


No gradient

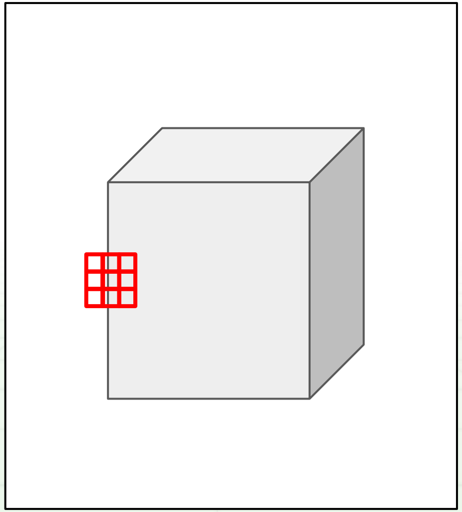
Lucas-Kanade Optical Flow

How to pick (x,y)? What makes a good feature?

Bad



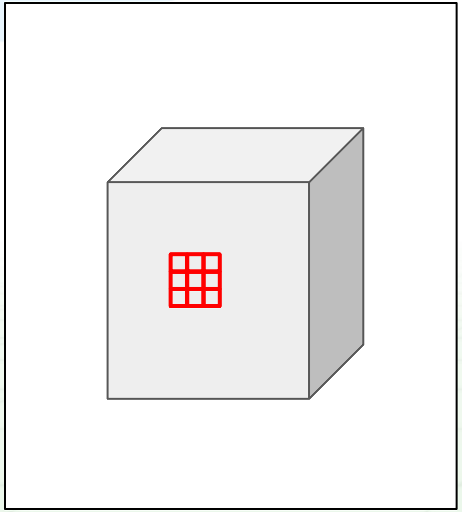
No gradient



Lucas-Kanade Optical Flow

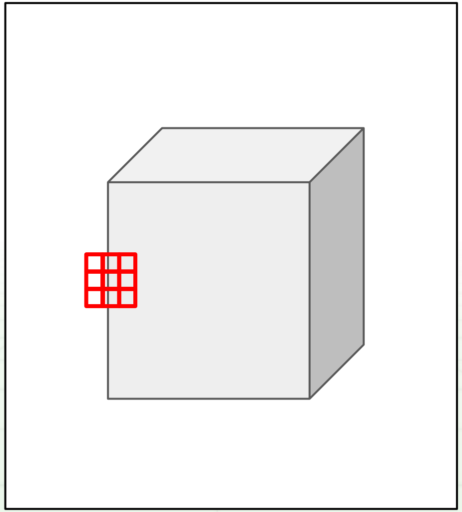
How to pick (x,y)? What makes a good feature?

Bad



No gradient

Slightly better

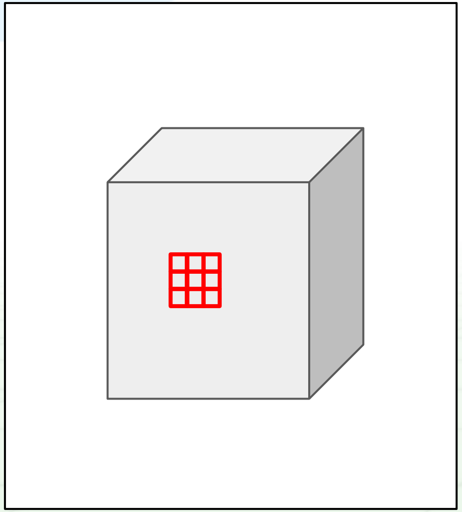


Can't detect
vertical motion

Lucas-Kanade Optical Flow

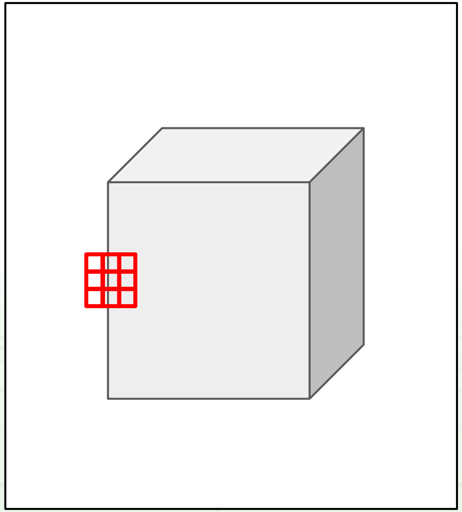
How to pick (x,y)? What makes a good feature?

Bad

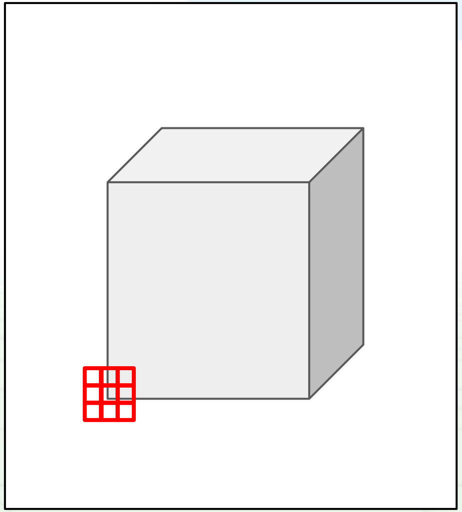


No gradient

Slightly better



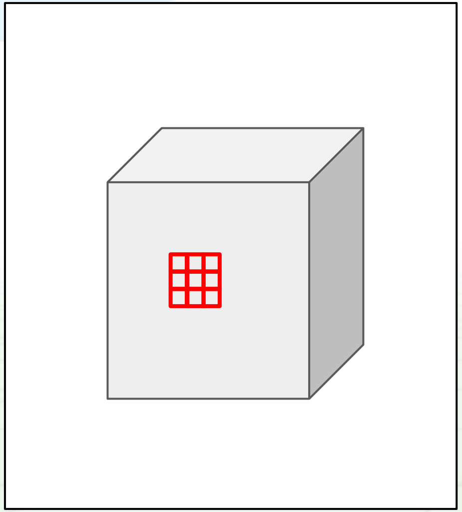
Can’t detect
vertical motion



Lucas-Kanade Optical Flow

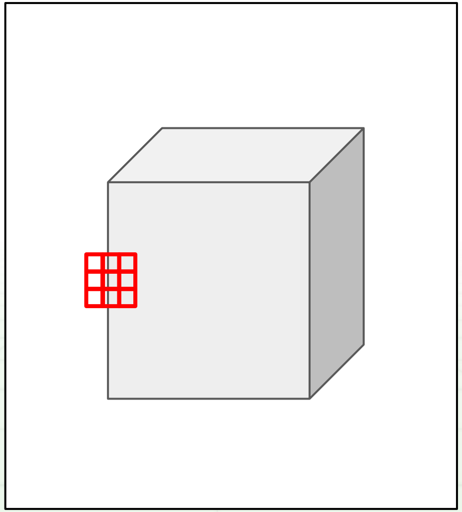
How to pick (x,y)? What makes a good feature?

Bad



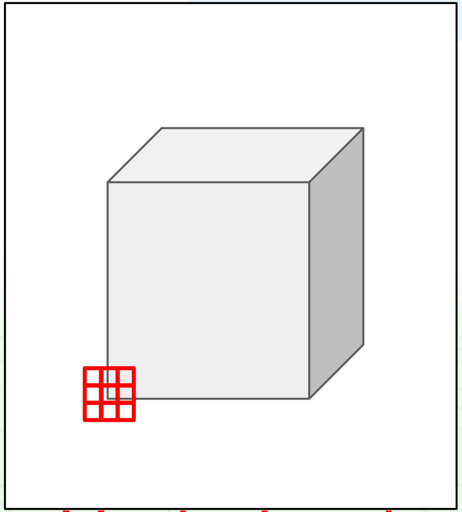
No gradient

Slightly better



Can’t detect vertical motion

Best

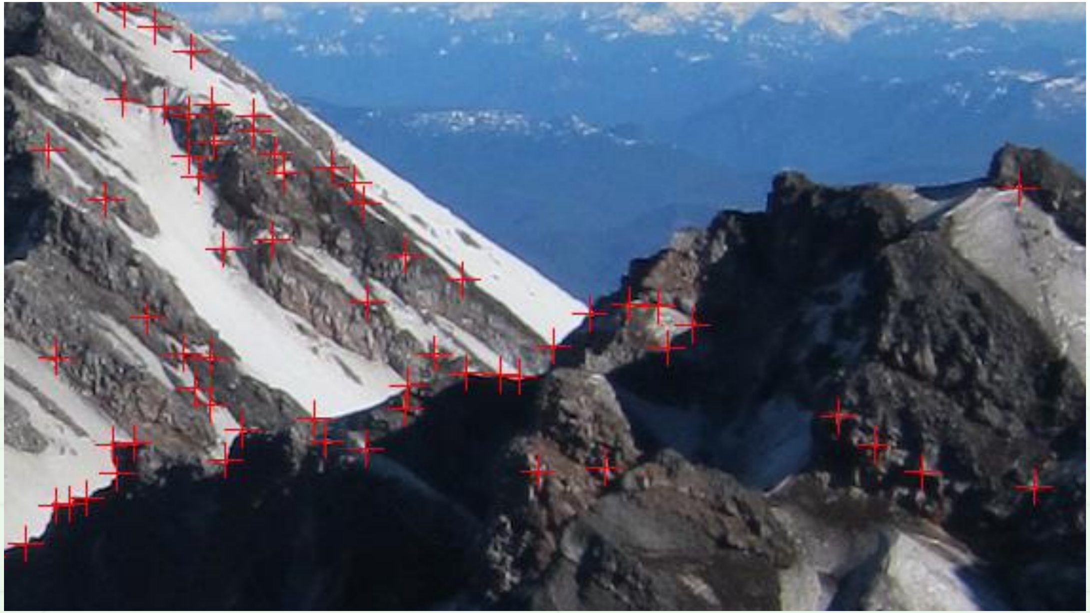


Vertical and horizontal motion

Lucas-Kanade Optical Flow

How to pick (x,y)? What makes a good feature?

<u>CORNERS!</u>



Lucas-Kanade Optical Flow

Demo

Lucas-Kanade Optical Flow

When does LK fail?

Lucas-Kanade Optical Flow

When does LK fail?

-Lighting changes

-Large movement

-Specularities

-No good features

-Aperture Problem

Lucas-Kanade Optical Flow

When does LK fail?

-Lighting changes

-Large movement

-Specularities

-No good features

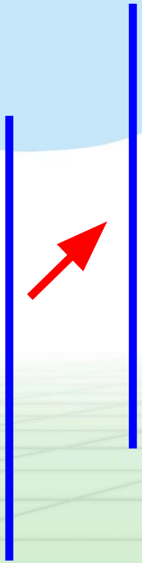
-Aperture Problem

Lucas-Kanade Optical Flow

Aperture Problem

Lucas-Kanade Optical Flow

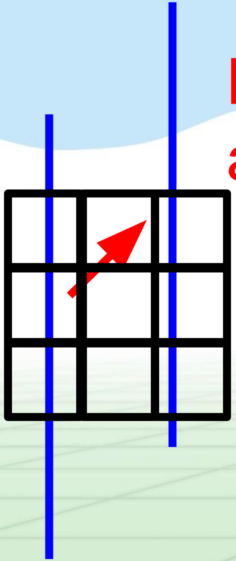
Aperture Problem



Line moving up
and to the right

Lucas-Kanade Optical Flow

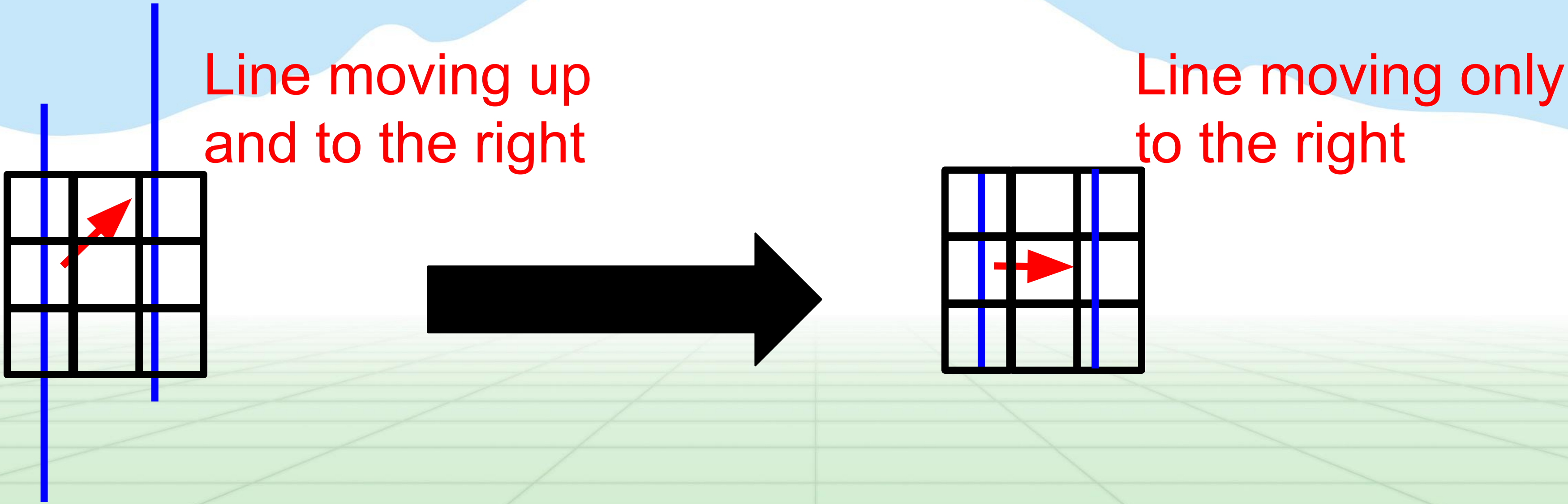
Aperture Problem



Line moving up
and to the right

Lucas-Kanade Optical Flow

Aperture Problem



Lucas-Kanade Optical Flow

Aperture Problem - Barber pole illusion

Lucas-Kanade Optical Flow

Aperture Problem - Barber pole illusion

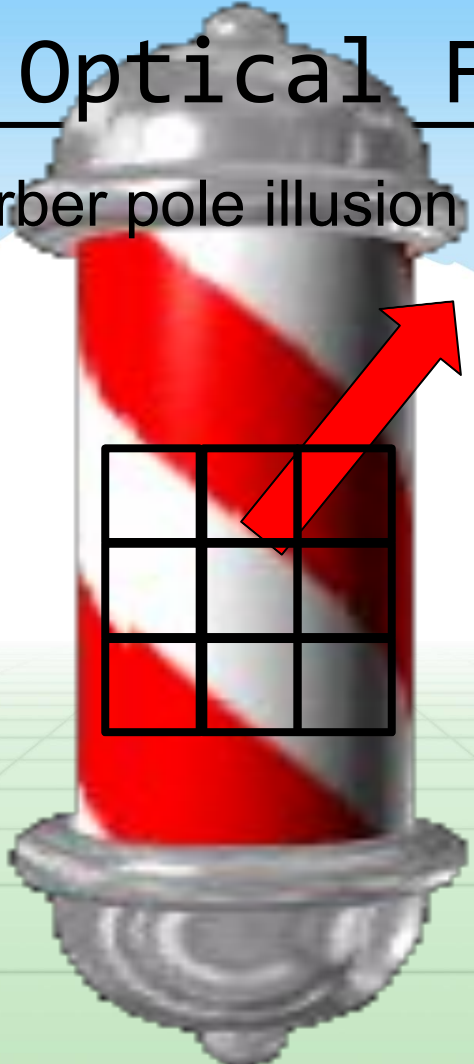
Lucas-Kanade Optical Flow

Aperture Problem - Barber pole illusion

Actual movement to the right

Lucas-Kanade Optical Flow

Aperture Problem - Barber pole illusion



Homework 3

Comes out tomorrow, due a week from today.

Implement Lucas-Kanade.

Come up with some *minor* way to improve it and test it.

Improving on LK: Iterative LK

$$dx_1 * u + dy_1 * v = I_t[x,y] - I_{t+\Delta t}[x,y]$$

Improving on LK: Iterative LK

$$dx_1 * u + dy_1 * v = I_t[x,y] - I_{t+\Delta t}[x,y]$$

Propagate dx_1, dy_1
forward and re-estimate

$$dx_2 * u + dy_2 * v = I_t[x,y] - I_{t+\Delta t}[x + dx_1, y + dy_1]$$

Improving on LK: Iterative LK

$$dx_1 * u + dy_1 * v = I_t[x,y] - I_{t+\Delta t}[x,y]$$

Propagate dx_1, dy_1 forward and re-estimate

$$dx_2 * u + dy_2 * v = I_t[x,y] - I_{t+\Delta t}[x + dx_1, y + dy_1]$$

Propagate dx_2, dy_2 forward and re-estimate

...

$$dx_n * u + dy_n * v = I_t[x,y] - I_{t+\Delta t}[x + dx_{n-1}, y + dy_{n-1}]$$

Improving on LK: Image Pyramids

Improving on LK: Image Pyramids

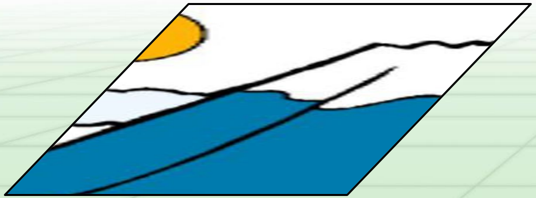


Image 1

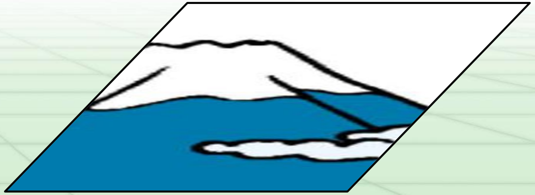


Image 2

Improving on LK: Image Pyramids

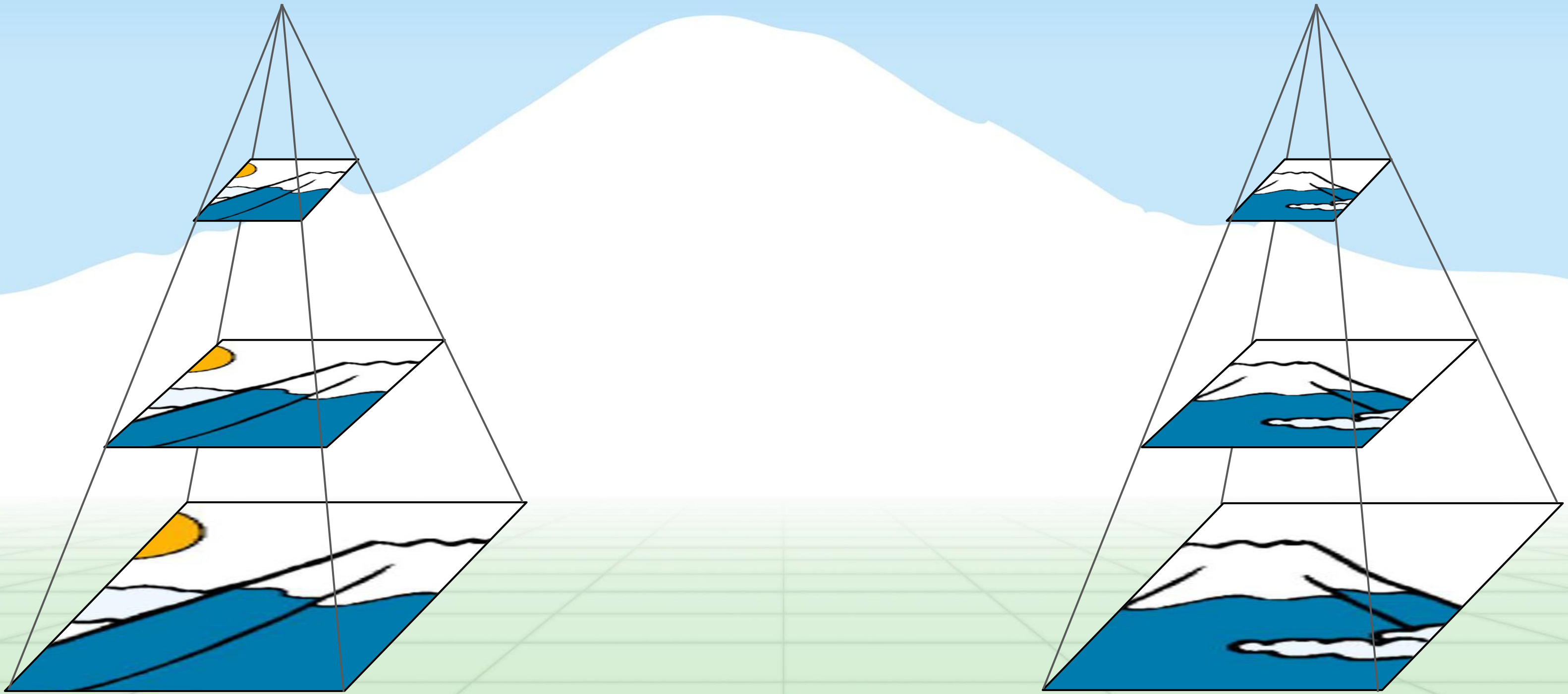
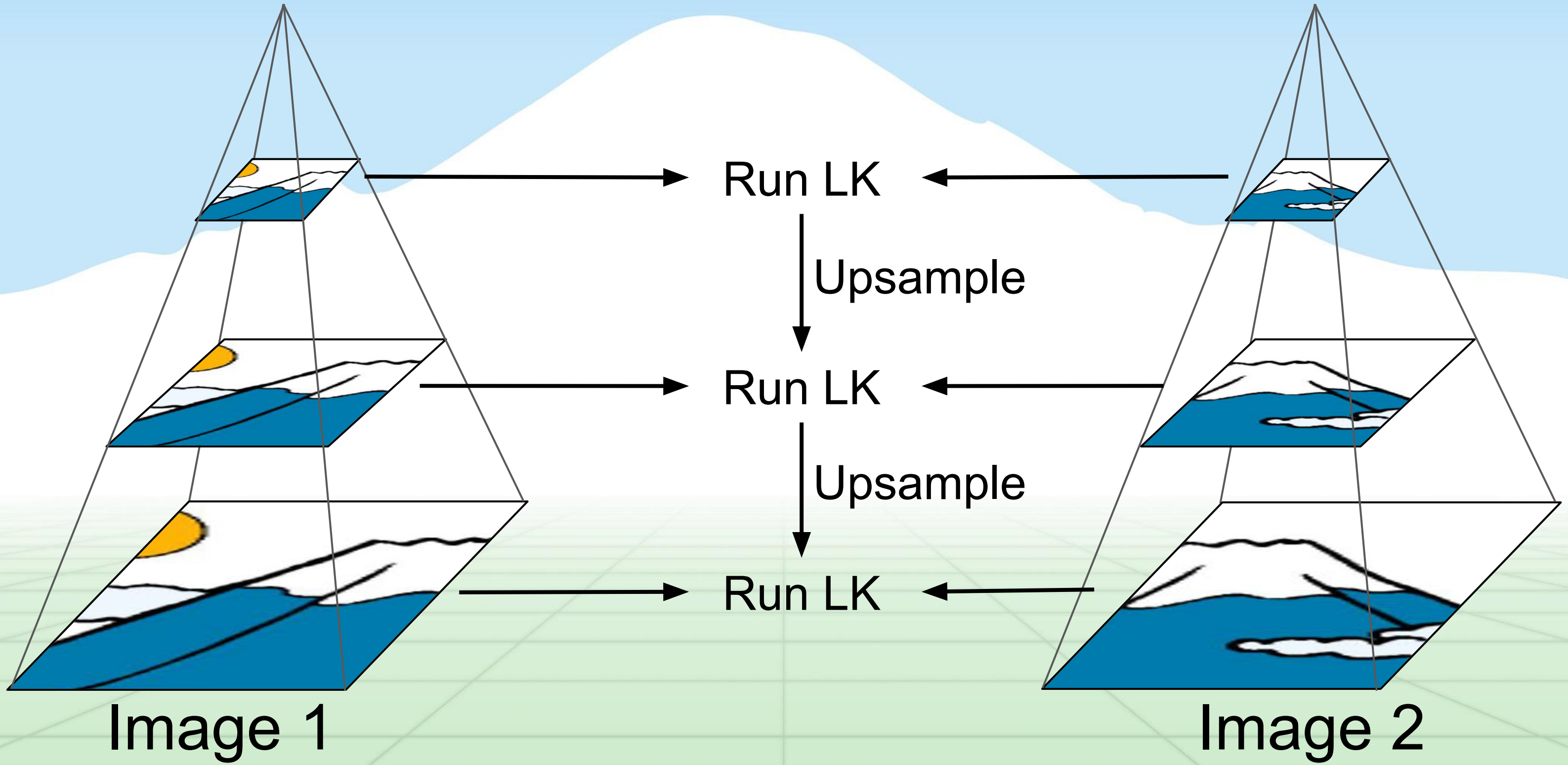


Image 1 Image 2

Improving on LK: Image Pyramids



Optical Flow

Sparse vs Dense

Optical Flow

Sparse vs Dense

Compute flow only for specific features

Compute flow for all pixels

Optical Flow

Sparse vs Dense

Compute flow only for specific features

Compute flow for all pixels

Lucas-Kanade is technically a dense algorithm, but in practice only works on good feature points, i.e., it’s sparse.

Is there a way to do dense optical flow?

Farnebäck Optical Flow

Farnebäck, Gunnar. "Two-frame motion estimation based on polynomial expansion." *Scandinavian conference on Image analysis*. Springer, Berlin, Heidelberg, 2003.

Farnebäck Optical Flow

Recall: We assumed the object moves by $\Delta\mathbf{p}$ over Δt

$$f(\mathbf{p} - \Delta\mathbf{p},\ t) = f(\mathbf{p},\ t + \Delta t)$$

Farnebäck Optical Flow

Recall: We assumed the object moves by $\mathbf{\Delta p}$ over Δt

$$f(\mathbf{p} - \mathbf{\Delta p}, t) = f(\mathbf{p}, t + \Delta t)$$

In LK we approximate this with a first-order Taylor Expansion

Farnebäck Optical Flow

Recall: We assumed the object moves by $\Delta \mathbf{p}$ over Δt

$$f(\mathbf{p} - \Delta \mathbf{p}, t) = f(\mathbf{p}, t + \Delta t)$$

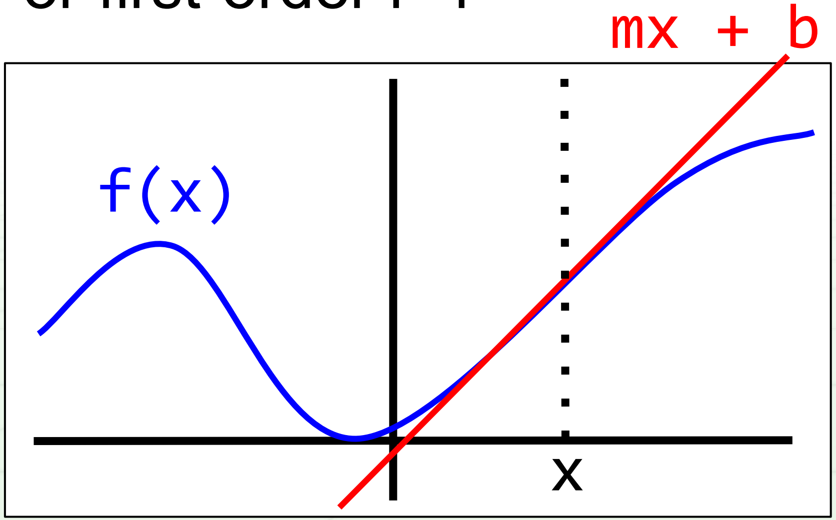
Instead, let's approximate with a second-order Taylor Expansion

Quick & Dirty: Taylor Expansion

Approximate complex function with n^{th}-order polynomial

$$f(x) \approx \Sigma_i \; a_i x^i$$

For first-order i=1



$$\approx a_1 x^1 + a_0 x^0$$

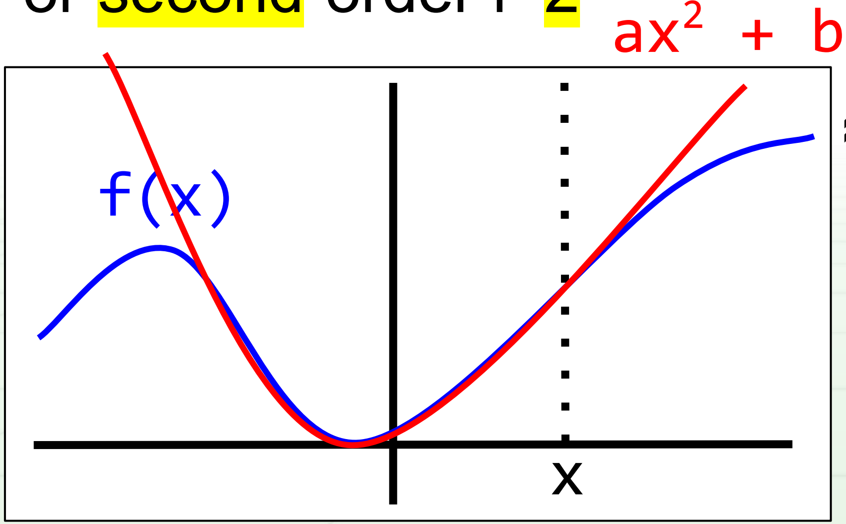
$$\approx mx + b$$

Quick & Dirty: Taylor Expansion

Approximate complex function with n^{th}-order polynomial

$$f(x) \approx \Sigma_i \; a_i x^i$$

For second-order i=2



$$\approx a_1 x^2 + a_1 x^1 + a_0 x^0$$

$$\approx ax^2 + bx + c$$

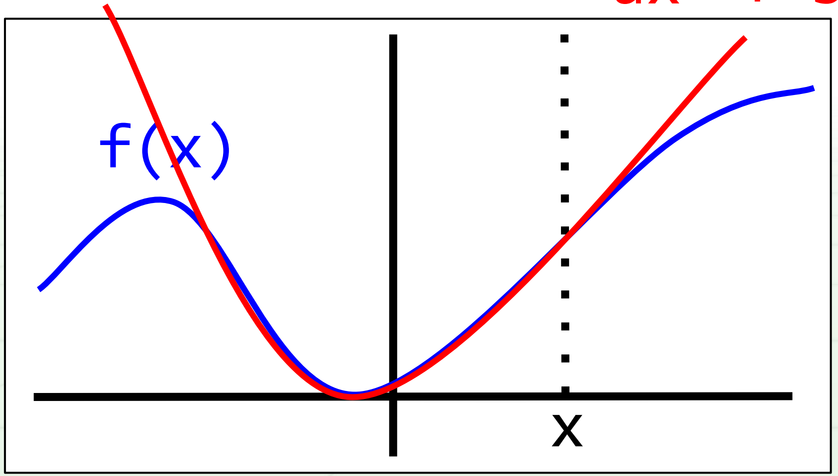
Quick & Dirty: Taylor Expansion

Approximate complex function with n^{th}-order polynomial

$$f(x) \approx \Sigma_i \; a_i x^i$$

For second-order i=2

$ax^2 + bx + c$



$$\approx a_1 x^2 + a_1 x^1 + a_0 x^0$$

$$\approx ax^2 + bx + c$$

Vector notation

$$\approx \mathbf{x}^T A \mathbf{x} + b\mathbf{x} + c$$

Farnebäck Optical Flow

Recall: We assumed the object moves by $\mathbf{\Delta p}$ over Δt

$$f(\mathbf{p} - \mathbf{\Delta p}, t) = f(\mathbf{p}, t + \Delta t)$$

Instead, let's approximate with a second-order Taylor Expansion

Farnebäck Optical Flow

Recall: We assumed the object moves by $\mathbf{\Delta p}$ over Δt

$$f(\mathbf{p} - \mathbf{\Delta p}, t) = f(\mathbf{p}, t + \Delta t)$$

Instead, let's approximate with a second-order Taylor Expansion

$$(\mathbf{p}-\mathbf{\Delta p})^T A_t (\mathbf{p}-\mathbf{\Delta p}) + b_t (\mathbf{p}-\mathbf{\Delta p}) + c_t = \mathbf{p}^T A_{t+\Delta t} \mathbf{p} + b_{t+\Delta t} \mathbf{p} + c_{t+\Delta t}$$

Farnebäck Optical Flow

Recall: We assumed the object moves by $\mathbf{\Delta p}$ over Δt

$$f(\mathbf{p} - \mathbf{\Delta p}, t) = f(\mathbf{p}, t + \Delta t)$$

Instead, let's approximate with a second-order Taylor Expansion

$$(\mathbf{p}-\mathbf{\Delta p})^T A_t (\mathbf{p}-\mathbf{\Delta p}) + b_t (\mathbf{p}-\mathbf{\Delta p}) + c_t = \mathbf{p}^T A_{t+\Delta t} \mathbf{p} + b_{t+\Delta t} \mathbf{p} + c_{t+\Delta t}$$

Poly parameters from first image

Poly parameters from second image

Farnebäck Optical Flow

Recall: We assumed the object moves by $\mathbf{\Delta p}$ over Δt

$$f(\mathbf{p} - \Delta\mathbf{p}, t) = f(\mathbf{p}, t + \Delta t)$$

Instead, let's approximate with a second-order Taylor Expansion

$$(\mathbf{p}-\Delta\mathbf{p})^T A_t(\mathbf{p}-\Delta\mathbf{p}) + b_t(\mathbf{p}-\Delta\mathbf{p}) + c_t = \mathbf{p}^T A_{t+\Delta t}\mathbf{p} + b_{t+\Delta t}\mathbf{p} + c_{t+\Delta t}$$

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Recall: We assumed the object moves by $\mathbf{\Delta p}$ over Δt

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Farnebäck Optical Flow

$$b_{t+\Delta t} = b_t - 2A_t\mathbf{\Delta p}$$

Farnebäck Optical Flow

$$b_{t+\Delta t} = b_t - 2A_t\mathbf{\Delta p}$$

Solve for $\mathbf{\Delta p}$

$$\mathbf{\Delta p} = -\tfrac{1}{2}(A_t)^{-1}(b_{t+\Delta t} - b_t)$$

Farnebäck Optical Flow

$$b_{t+\Delta t} = b_t - 2A_t\mathbf{\Delta p}$$

Solve for $\mathbf{\Delta p}$

$$\mathbf{\Delta p} = -½(A_t)^{-1}(b_{t+\Delta t} - b_t)$$

A, b, and c can be solved pointwise using least squares over a small neighborhood like before

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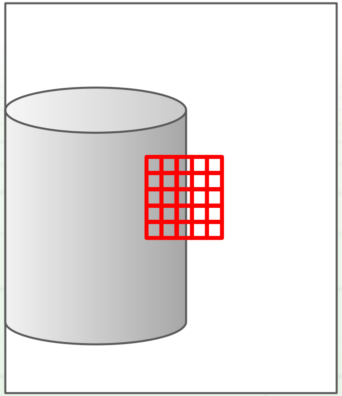

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A, b, and c can be solved pointwise using least squares over a small neighborhood like before

$$S = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \multicolumn{2}{c}{\cdots} \\ x_k & y_k \end{bmatrix} \quad T = \begin{bmatrix} I[x_1,y_1] \\ I[x_2,y_2] \\ \cdots \\ I[x_k,y_k] \end{bmatrix}$$

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$$SAS^T + bS^T + c = T$$

$$||SAS^T + bS^T + c - T||^2 = 0$$

Farnebäck Optical Flow

$$b_{t+\Delta t} = b_t - 2A_t\Delta \mathbf{p}$$

Solve for $\Delta \mathbf{p}$

$$\Delta \mathbf{p} = -½(A_t)^{-1}(b_{t+\Delta t} - b_t)$$

A, b, and c can be solved pointwise using least squares over a small neighborhood like before

$$S = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \multicolumn{2}{c}{\cdots} \\ x_k & y_k \end{bmatrix} \quad T = \begin{bmatrix} I[x_1,y_1] \\ I[x_2,y_2] \\ \cdots \\ I[x_k,y_k] \end{bmatrix}$$

$$SAS^T + bS^T + c = T$$

$$||SAS^T + bS^T + c - T||^2 = 0$$

Quadratic Regression

Farnebäck Optical Flow

$$b_{t+\Delta t} = b_t - 2A_t\Delta \mathbf{p}$$

Solve for $\Delta \mathbf{p}$

$$\Delta \mathbf{p} = -\tfrac{1}{2}(A_t)^{-1}(b_{t+\Delta t} - b_t)$$

Same problem as before:
What if it's not invertible?

Farnebäck Optical Flow

$$b_{t+\Delta t} = b_t - 2A_t\mathbf{\Delta p}$$

Let’s try a different approach:

Farnebäck Optical Flow

$$b_{t+\Delta t} = b_t - 2A_t\mathbf{\Delta p}$$

Let's try a different approach:

$$A_t\mathbf{\Delta p} = -\tfrac{1}{2}(b_{t+\Delta t} - b_t)$$

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$$b_{t+\Delta t} = b_t - 2A_t\mathbf{\Delta p}$$

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$$\boxed{A_t = A_{t+\Delta t}}$$

$$A_t\mathbf{\Delta p} = -½(b_{t+\Delta t} - b_t)$$

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$$b_{t+\Delta t} = b_t - 2A_t\Delta \mathbf{p}$$

Let's try a different approach:

$$A_t\Delta \mathbf{p} = -\tfrac{1}{2}(b_{t+\Delta t} - b_t)$$

$\boxed{A_t = A_{t+\Delta t}}$

In reality not equal

$\boxed{A_t \neq A_{t+\Delta t}}$

Farnebäck Optical Flow

$$b_{t+\Delta t} = b_t - 2A_t\Delta\mathbf{p}$$

Let's try a different approach:

$$A_t\Delta\mathbf{p} = -\tfrac{1}{2}(b_{t+\Delta t} - b_t)$$

$$\boxed{A_t = A_{t+\Delta t}}$$

In reality not equal

$$\boxed{A_t \neq A_{t+\Delta t}}$$

Average together

$$\boxed{A = \tfrac{1}{2}(A_t + A_{t+\Delta t})}$$

Farnebäck Optical Flow

$$b_{t+\Delta t} = b_t - 2A_t\mathbf{\Delta p}$$

Let's try a different approach:

$$A\mathbf{\Delta p} = -\tfrac{1}{2}(b_{t+\Delta t} - b_t)$$

Farnebäck Optical Flow

$$b_{t+\Delta t} = b_t - 2A_t\Delta \mathbf{p}$$

Let's try a different approach:

$$A\Delta \mathbf{p} = -\tfrac{1}{2}(b_{t+\Delta t} - b_t)$$

$$\boxed{\Delta b = -\tfrac{1}{2}(b_{t+\Delta t} - b_t)}$$

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$$b_{t+\Delta t} = b_t - 2A_t\Delta\mathbf{p}$$

Let's try a different approach:

$$A\Delta\mathbf{p} = \Delta b$$

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Assume neighbors move together (just like LK)

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$$b_{t+\Delta t} = b_t - 2A_t\mathbf{\Delta p}$$

Let's try a different approach:

$$A\mathbf{\Delta p} = \Delta b$$

Assume neighbors move together (just like LK)

$$A_i\mathbf{\Delta p} = \Delta b_i$$

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$$b_{t+\Delta t} = b_t - 2A_t\mathbf{\Delta p}$$

Let's try a different approach:

$$A\mathbf{\Delta p} = \mathbf{\Delta}b$$

Assume neighbors move together (just like LK)

$$A_i\mathbf{\Delta p} = \mathbf{\Delta}b_i$$

Apply least squares

$$\mathbf{\Sigma}_i\ w_i||A_i\mathbf{\Delta p} - \mathbf{\Delta}b_i||^2$$

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$$b_{t+\Delta t} = b_t - 2A_t\mathbf{\Delta p}$$

Let’s try a different approach:

$$A\mathbf{\Delta p} = \Delta b$$

Assume neighbors move together (just like LK)

$$A_i\mathbf{\Delta p} = \Delta b_i$$

Apply least squares

$$\Sigma_i \; w_i||A_i\mathbf{\Delta p} - \Delta b_i||^2$$

Gaussian weight (so we care more about the center of the neighborhood than the edges)

Farnebäck Optical Flow

Still assuming $\Delta\mathbf{p}$ is constant over the neighborhood, but not really true…

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Relax and assume instead $\Delta\mathbf{p}$ is an affine function in the neighborhood.

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Relax and assume instead $\Delta\mathbf{p}$ is an affine function in the neighborhood.

$$\Delta\mathbf{p} = \begin{bmatrix} a_1 + a_2x + a_3y + a_7x^2 + a_8xy \\ a_4 + a_5x + a_6y + a_7xy + a_8y^2 \end{bmatrix}$$

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Still assuming $\Delta\mathbf{p}$ is constant over the neighborhood, but not really true…

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$$\Delta\mathbf{p} = S\mathbf{d}$$

$$S = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 & x^2 & xy \\ 0 & 0 & 0 & 1 & x & y & xy & y^2 \end{bmatrix}$$

$$\mathbf{d} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix}$$

Farnebäck Optical Flow

$$\boldsymbol{\Sigma}_i\ w_i||A_i\boldsymbol{\Delta}\mathbf{p} - \boldsymbol{\Delta}b_i||^2$$

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$$\boldsymbol{\Sigma}_i \; w_i ||A_i \boldsymbol{\Delta p} - \boldsymbol{\Delta} b_i||^2$$

$$\xrightarrow{\text{Plug in } S\mathbf{d} \text{ for } \boldsymbol{\Delta p}.}$$

$$\boldsymbol{\Sigma}_i \; w_i ||A_i S\mathbf{d} - \boldsymbol{\Delta} b_i||^2$$

Farnebäck Optical Flow

$$\Sigma_i \; w_i||A_i\Delta\mathbf{p} - \Delta b_i||^2$$

Plug in S**d** for Δ**p**.

$$\Sigma_i \; w_i||A_iS\mathbf{d} - \Delta b_i||^2$$

Least squares solve for **d**

$$\mathbf{d} = \left(\Sigma_i w_i (S_i)^T (A_i)^T S_i\right)^{-1} \Sigma_i w_i (S_i)^T (A_i)^T \Delta b_i$$

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Same inverse problem, but in practice less of an issue

Farnebäck Optical Flow

Demo