Reinforcement Learning
Intelligent Systems Series
Lecture 4 (Part 1)

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Slides adapted from David Silver, Deepmind

MPI for Intelligent Systems, Tübingen, Germany

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Markov Decision Processes

Markov Process (Reminder)

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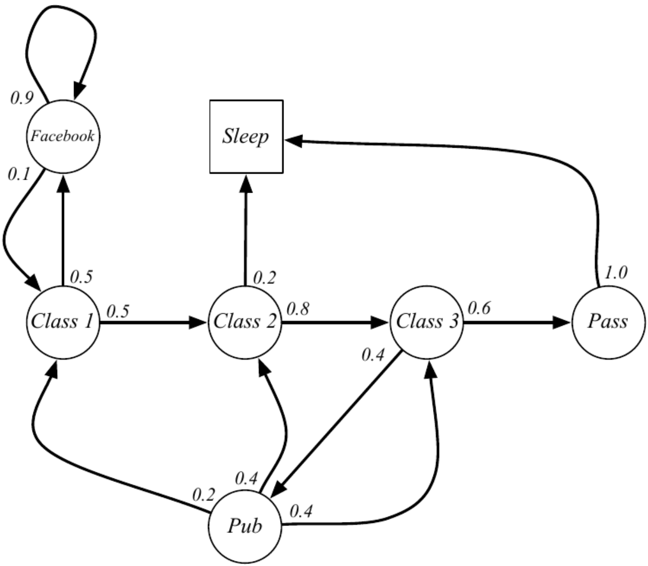
Definition (Markov Process/ Markov Chain)

A *Markov Process* (or *Markov Chain*) is a tuple $(\mathcal{S}, \mathcal{P})$

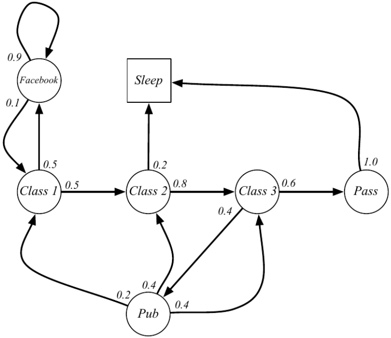
- $\mathcal{S}$ is a (finite) set of states
- $\mathcal{P}$ is a state transition 0 probability matrix,

$$P_{ss'} = P(S_{t+1} = s' \mid S_t = s)$$

Example: Student Markov Chain



Example: Student Markov Chain Transition Matrix



$$\mathcal{P} = \begin{array}{c} \\ C1 \\ C2 \\ C3 \\ Pass \\ Pub \\ FB \\ Sleep \end{array} \begin{array}{c} \begin{array}{ccccccc} C1 & C2 & C3 & Pass & Pub & FB & Sleep \end{array} \\ \left[\begin{array}{ccccccc} & 0.5 & & & & 0.5 & \\ & & 0.8 & & & & 0.2 \\ & & & 0.6 & 0.4 & & \\ & & & & & & 1.0 \\ 0.2 & 0.4 & 0.4 & & & & \\ 0.1 & & & & & 0.9 & \\ & & & & & & 1 \end{array}\right] \end{array}$$

Markov Reward Process

A Markov reward process is a Markov chain with values.

Definition (MRP)

A *Markov Reward Process* is a tuple $(\mathcal{S}, \mathcal{P}, \mathcal{R}, \gamma)$

- $\mathcal{S}$ is a finite set of states
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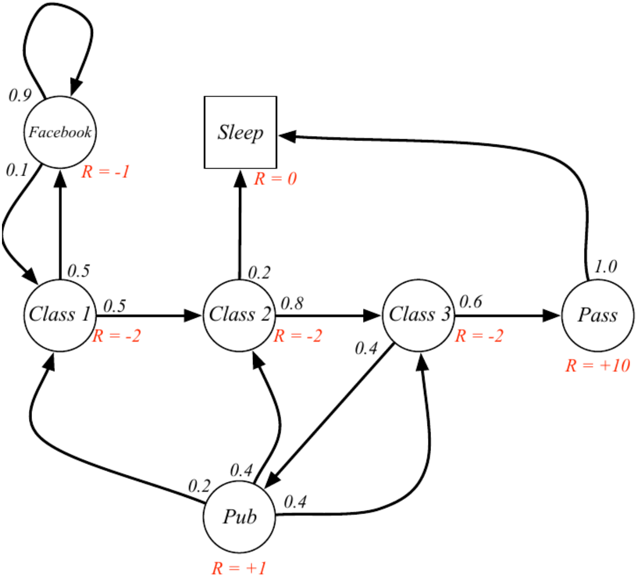
- $\mathcal{R}$ is a reward function, $\mathcal{R}_s = \mathbb{E}[R_{t+1} | S_t = s]$
- γ is a discount factor, $\gamma \in [0, 1]$

Note that the reward can be stochastic ($\mathcal{R}_s$ is in expectation)

Example: Student MRP



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Return (end of Reminder)

Definition

The *return* G_t is the total discounted reward from time-step t.

$$G_t = R_{t+1} + \gamma R_{t+2} + \cdots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

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- The discount $\gamma \in [0, 1]$ devaluates future rewards: reward R after $k + 1$ time-steps is counted as $\gamma^k R$.
- Extreme cases:
  - γ close to 0 leads to immidate reward maximization only
  - γ close to 1 leads to far-sighted evaluation

Value Function

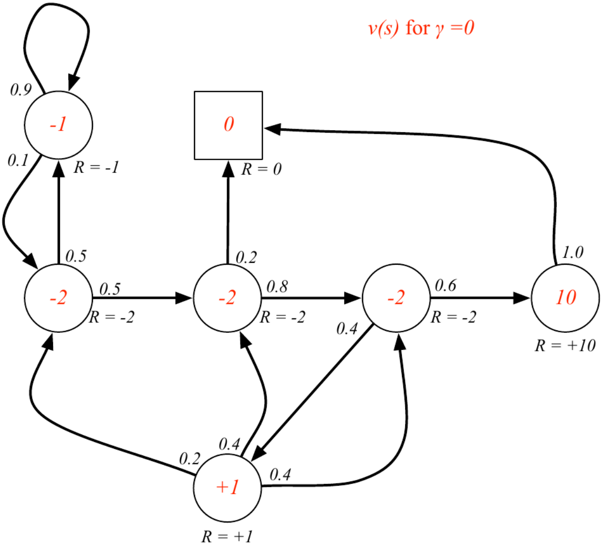
The value function describes the value of a state (in the stationary state)

Definition

The state *value function* $v(s)$ of an MRP is the expected return starting from state s

$$v(s) = \mathbb{E}[G_t | S_t = s]$$

Example: Value Function for Student MRP



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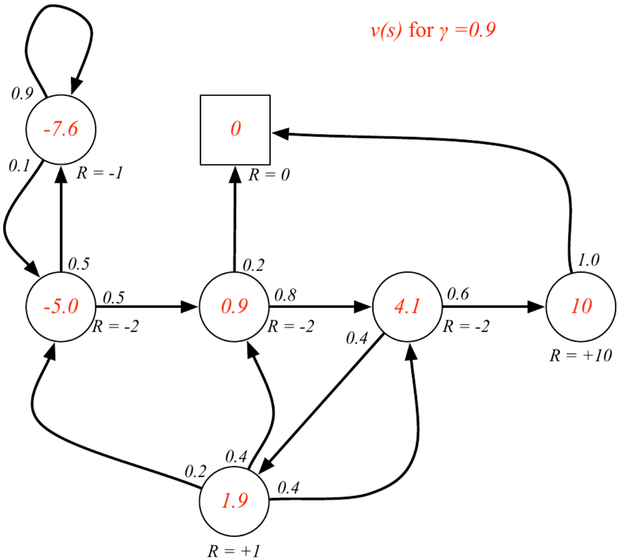

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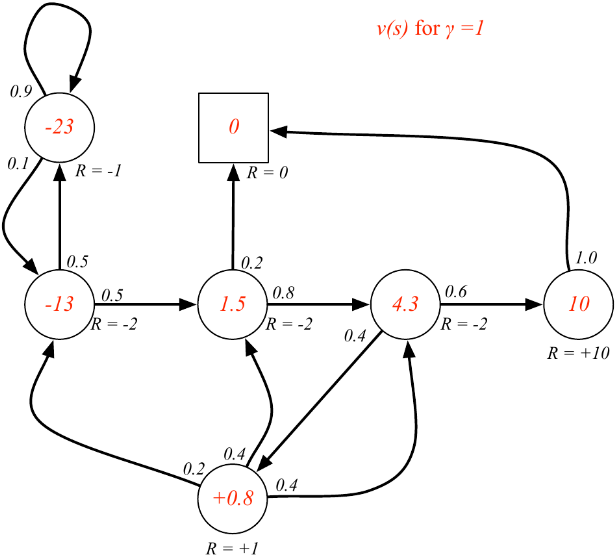

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Bellman Equation (MRP) I

Idea: Make value computation recursive by tearing apart contributions from:

- immediate reward
- and from discounted future rewards

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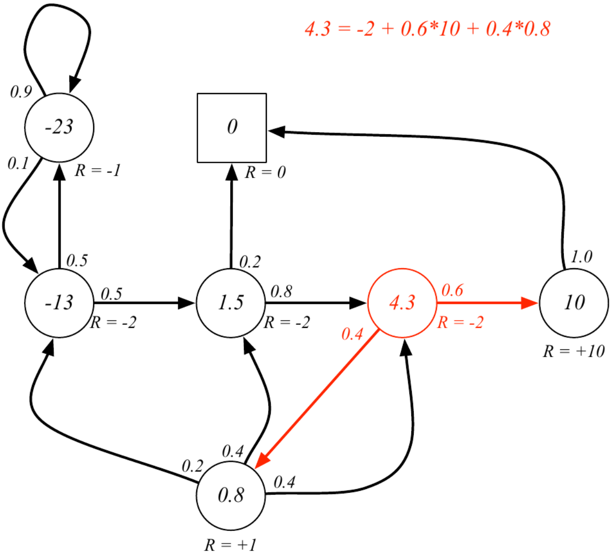
Mh... need Expectation over S_{t+1}

Use transition matrix to get probabilities of succeeding state:

$$v(s) = \mathcal{R}_s + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'} v(s')$$

Example: Bellman Equation for Student MRP



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Bellman Equation (MRP) II

Bellman equations in matrix form:

$$v = \mathcal{R} + \gamma \mathcal{P} v$$

where $v \in \mathbb{R}^{|S|}$ and $\mathcal{R}$ are vectors

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The Bellman equation can be solved directly:

$$v = (\mathbb{I} - \gamma \mathcal{P})^{-1} \mathcal{R}$$

- computational complexity is $O(|S|^3)$

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Definition (MDP)

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$$\mathcal{P}^a_{ss'} = P(S_{t+1} \mid S_t, A_t = a)$$

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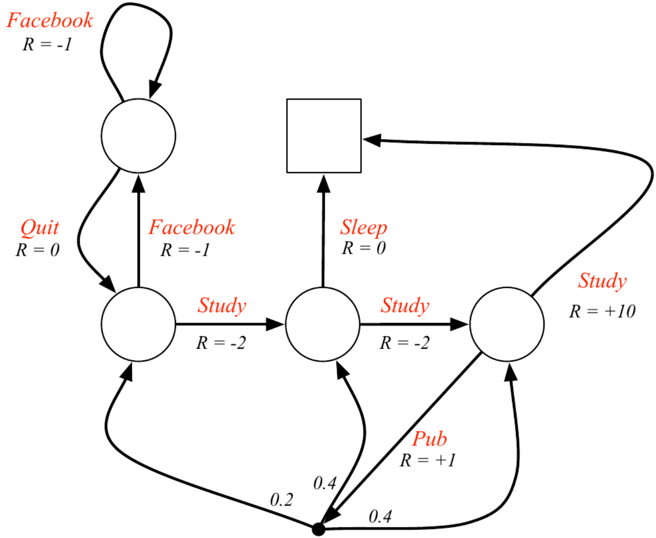
Example: Student MDP



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How to model decision taking?

The agent has a action function called policy.

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- Implication: policies are stationary (independent of time)

An MDP with a given policy turns into a MRP:

$$\mathcal{P}^{\pi}_{ss'} = \sum_{a \in \mathcal{A}} \pi(a|s) \mathcal{P}^{a}_{ss'}$$

$$\mathcal{R}^{\pi}_{s} = \sum_{a \in \mathcal{A}} \pi(a|s) \mathcal{R}^{a}_{s}$$

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How much does choosing a different action change the value?

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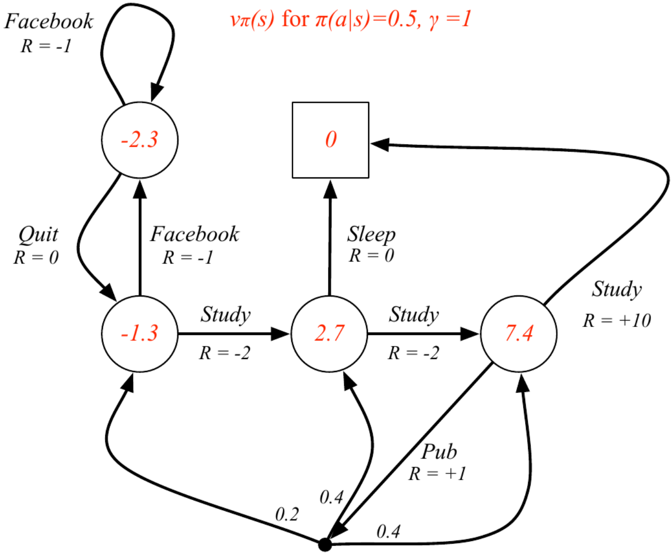
Definition

The *action-value* function $q_\pi(s, a)$ of an MDP is the expected return when starting from state s, taking action a, and then following policy π.

$$q_\pi(s, a) = \mathbb{E}[G_t | S_t = s, A_t = a]$$

Example: State-Value function for Student MDP



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Bellman Expectation Equation

Recall: Bellman Equation: decompose expected reward into immediate reward plus discounted value of successor state,

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The action-value function can be similarly decomposed,

$$q_\pi(s, a) = \mathbb{E}_\pi[R_{t+1} + \gamma q_\pi(S_{t+1}, A_{t+1}) | S_t = s, A_t = a]$$

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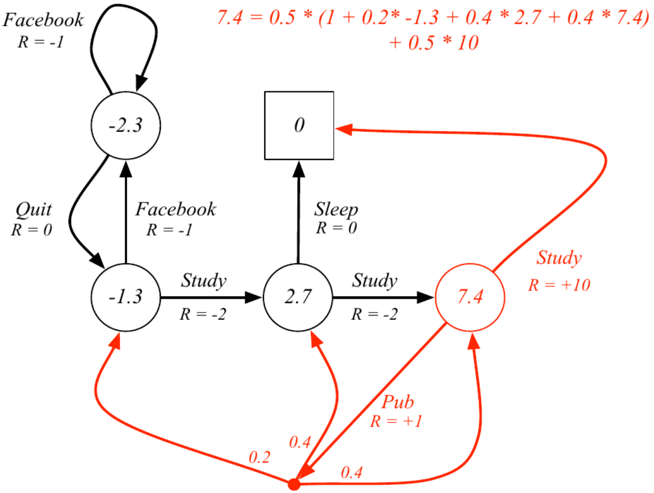
Example: Bellman update for v in Student MDP



from David Silver

$$\pi(a|s) = 0.5$$

Explicit solution for v_π

Since a policy induces a MRP v_π can be diretly computed (as before)

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Since a policy induces a MRP v_π can be diretly computed (as before)

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But do we want v_π?

We want to find the optimal policy and its value function!

Optimal Value Function

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What does it mean?

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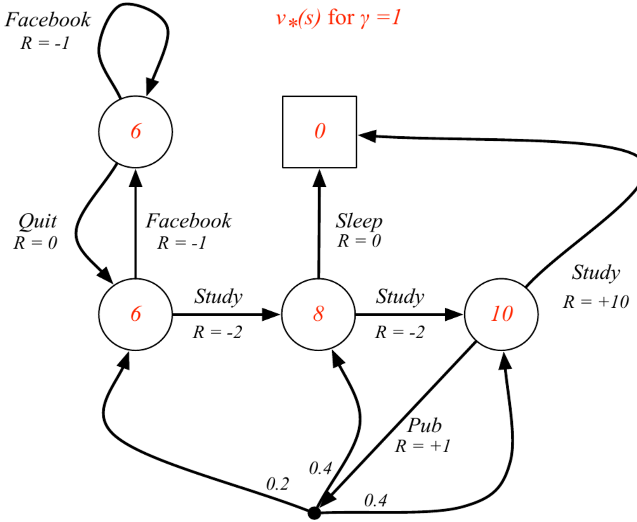
What does it mean?

- v_* specifies the best possible performance in an MDP
- Knowing v_* solves the MDP (how? we will see...)

Example: Optimal Value Function v_* in Student MDP



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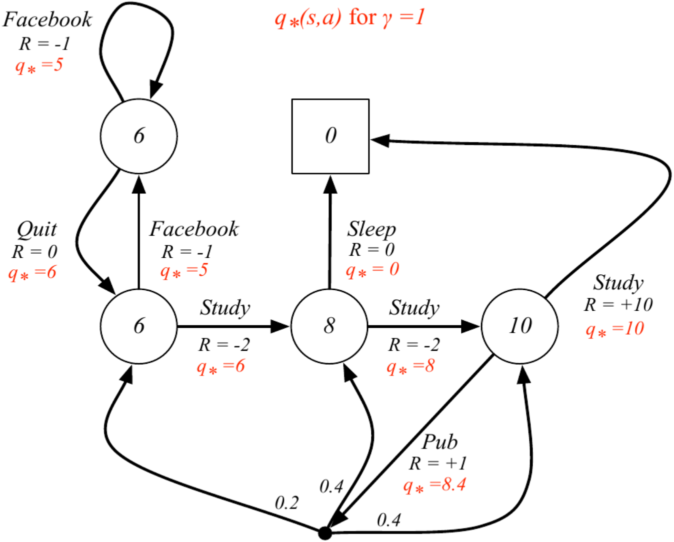
Example: Optimal State Function q_* in Student MDP



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Optimal Policy

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Define a partial ordering over policies

$$\pi \geq \pi' \text{ if } v_\pi(s) \geq v_{\pi'}(s), \forall s$$

Theorem

For any Markov Decision Process

- *There exists an optimal policy π_* that is better than or equal to all other policies, $\pi_* \geq \pi, \forall \pi$*
- *All optimal policies achieve the optimal state-value function, $v_{\pi_*}(s) = v_*(s)$*
- *All optimal policies achieve the optimal action-value function, $q_{\pi_*}(s, a) = q_*(s, a)$*

Finding an Optimal Policy

Given the optimal action-value function q_*:

Finding an Optimal Policy

Given the optimal action-value function q_*: the optimal policy is given by maximizing it.

$$\pi_*(a|s) = [\![a = \arg\max_{a \in \mathcal{A}} q_*(s, a)]\!]$$

$[\![\cdot]\!]$ is Iverson bracket: 1 if *true*, otherwise 0.

- There is always a deterministic optimal policy for any MDP
- If we know $q_*(s, a)$, we immediately have the optimal policy (greedy)

Bellman Equation for optimal value functions

Also for the optimal value functions we can use Bellmans optimality equations:

$$v_*(s) = \max_{a \in \mathcal{A}} q_*(s, a)$$

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Solving the Bellman Optimality Equation

Bellman Optimality Equation is non-linear

- No closed form solution (in general)
- Many iterative solution methods
  - Value Iteration
  - Policy Iteration
  - Q-learning
  - SARSA

Part 2

David Silver's Lecture 3 . . .