



# Mathematical Programs with Equilibrium Constraints: Enhanced Fritz-John Conditions, new Constraint Qualifications and improved exact Penalty Results

Christian Kanzow, Alexandra Schwartz – September 14th, 2010

www.mathematik.uni-wuerzburg.de/am2.html

Research partially supported by a grant from the Elite-Network of Bavaria

#### Outline

#### Introduction

Standard Nonlinear Programs

Mathematical Programs with Equilibrium Constraints

New Fritz-John Result and new Constraint Qualifications

New Fritz-John Result New Constraint Qualifications Relations to Existing Results

Application to Penalty Functions



# Standard Nonlinear Problem (NLP)

We refer to the following optimization problem als NLP:

min 
$$f(x)$$
 subject to  $g_i(x) \le 0 \quad \forall i = 1, \dots, m,$   
 $h_i(x) = 0 \quad \forall i = 1, \dots, p$  (1)

with  $f, g_i, h_i : \mathbb{R}^n \to \mathbb{R}$  continuously differentiable. Its feasible set is denoted by  $\mathcal{X}$  and the set of active inequalities in a point  $x \in \mathcal{X}$  by

$$I_g(x) = \{i \mid g_i(x) = 0\}.$$



#### Standard Fritz-John Conditions

#### Theorem (Standard Fritz-John conditions)

Let  $x^*$  be a local minimum of (1). Then  $x^*$  is a Fritz-John point, i.e. there are multipliers  $\alpha \geq 0, \lambda \geq 0$ , and  $\mu$  such that  $(\alpha, \lambda, \mu) \neq 0$  and

$$\alpha \nabla f(x^*) + \sum_{i \in I_g(x^*)} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^p \mu_i \nabla h_i(x^*) = 0.$$



#### Standard Fritz-John Conditions

#### Theorem (Standard Fritz-John conditions)

Let  $x^*$  be a local minimum of (1). Then  $x^*$  is a Fritz-John point, i.e. there are multipliers  $\alpha \geq 0, \lambda \geq 0$ , and  $\mu$  such that  $(\alpha, \lambda, \mu) \neq 0$  and

$$\alpha \nabla f(x^*) + \sum_{i \in I_g(x^*)} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^p \mu_i \nabla h_i(x^*) = 0.$$

- ▶ If  $\alpha = 0$ , no information about the objective function is included.
- ▶ If  $\alpha \neq 0$ , we may assume  $\alpha = 1$  without loss of generality.





#### Karush-Kuhn-Tucker Point

#### Definition (KKT Points)

Let  $x^*$  be feasible for (1). Then  $x^*$  is called a *KKT point* if there are multipliers  $\lambda \geq 0$  and  $\mu$  such that

$$\nabla f(x^*) + \sum_{i \in I_g(x^*)} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^p \mu_i \nabla h_i(x^*) = 0.$$



#### Karush-Kuhn-Tucker Point

#### Definition (KKT Points)

Let  $x^*$  be feasible for (1). Then  $x^*$  is called a *KKT point* if there are multipliers  $\lambda \geq 0$  and  $\mu$  such that

$$\nabla f(x^*) + \sum_{i \in I_g(x^*)} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^p \mu_i \nabla h_i(x^*) = 0.$$

- ▶ However, not every local minimum is a KKT point.
- ▶ Additional conditions on the constraints are needed.





# Standard Constraint Qualifications I

A point  $x^*$  feasible for (1) is said to satisfy

▶ linear independence constraint qualification (LICQ), if there are no multipliers  $\lambda$  and  $\mu$  such that  $(\lambda, \mu) \neq 0$  and

$$\sum_{i \in I_g(x^*)} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^p \mu_i \nabla h_i(x^*) = 0,$$

# Standard Constraint Qualifications I

A point  $x^*$  feasible for (1) is said to satisfy

▶ linear independence constraint qualification (LICQ), if there are no multipliers  $\lambda$  and  $\mu$  such that  $(\lambda, \mu) \neq 0$  and

$$\sum_{i \in I_g(x^*)} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^p \mu_i \nabla h_i(x^*) = 0,$$

▶ Mangasarian-Fromovitz constraint qualification (MFCQ), if there are no multipliers  $\lambda \geq 0$  and  $\mu$  such that  $(\lambda, \mu) \neq 0$  and

$$\sum_{i \in I_g(x^*)} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^p \mu_i \nabla h_i(x^*) = 0,$$



# Standard Constraint Qualifications I

A point  $x^*$  feasible for (1) is said to satisfy

▶ linear independence constraint qualification (LICQ), if there are no multipliers  $\lambda$  and  $\mu$  such that  $(\lambda, \mu) \neq 0$  and

$$\sum_{i \in I_g(x^*)} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^p \mu_i \nabla h_i(x^*) = 0,$$

▶ Mangasarian-Fromovitz constraint qualification (MFCQ), if there are no multipliers  $\lambda \geq 0$  and  $\mu$  such that  $(\lambda, \mu) \neq 0$  and

$$\sum_{i \in I_g(x^*)} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^p \mu_i \nabla h_i(x^*) = 0,$$

► Abadie constraint qualification (ACQ), if

$$T_{\mathcal{X}}(x^*) = L_{NLP}(x^*).$$



# Standard Constraint Qualifications II

The relations between these constraint qualifications are as follows:

$$\mathsf{LICQ} \Longrightarrow \mathsf{MFCQ} \Longrightarrow \mathsf{ACQ}$$

# Standard Constraint Qualifications II

The relations between these constraint qualifications are as follows:

$$LICQ \Longrightarrow MFCQ \Longrightarrow ACQ$$

#### Theorem (KKT conditions)

Let  $x^*$  be a local minimum of (1) that satisfies ACQ. Then  $x^*$  is a KKT point.



# Mathematical Program with Equilibrium Constraints (MPEC)

#### An MPEC is an optimization problem of the form

$$\begin{aligned} \min f(x) & \text{ subject to } & g_i(x) \leq 0 & \forall i = 1, \dots, m, \\ & h_i(x) = 0 & \forall i = 1, \dots, p, \\ & G_i(x) \geq 0 & \forall i = 1, \dots, I, \\ & H_i(x) \geq 0 & \forall i = 1, \dots, I, \\ & G_i(x) \cdot H_i(x) = 0 & \forall i = 1, \dots, I \end{aligned}$$

with  $f, g_i, h_i, G_i, H_i : \mathbb{R}^n \to \mathbb{R}$  continuously differentiable. The corresponding feasible set is denoted by X.



# Mathematical Program with Equilibrium Constraints (MPEC)

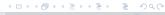
#### An MPEC is an optimization problem of the form

$$\begin{aligned} \min f(x) & \text{ subject to } & g_i(x) \leq 0 & \forall i = 1, \dots, m, \\ & h_i(x) = 0 & \forall i = 1, \dots, p, \\ & G_i(x) \geq 0 & \forall i = 1, \dots, I, \\ & H_i(x) \geq 0 & \forall i = 1, \dots, I, \\ & G_i(x) \cdot H_i(x) = 0 & \forall i = 1, \dots, I \end{aligned}$$

with  $f, g_i, h_i, G_i, H_i : \mathbb{R}^n \to \mathbb{R}$  continuously differentiable. The corresponding feasible set is denoted by X.

- ▶ The constraints G and H are called complementarity constraints.
- ▶ One also speaks of a mathematical problem with complementarity constraints (MPCC).





#### Notation: Index Sets

We define the following index sets for an arbitrary  $x \in X$ :

$$I_{g}(x) := \{i \mid g_{i}(x) = 0\},\$$

$$I_{00}(x) := \{i \mid G_{i}(x) = 0, H_{i}(x) = 0\},\$$

$$I_{0+}(x) := \{i \mid G_{i}(x) = 0, H_{i}(x) > 0\},\$$

$$I_{+0}(x) := \{i \mid G_{i}(x) > 0, H_{i}(x) = 0\}.$$

- $I_g(x)$  consists of the active inequalities.
- ▶  $I_{00}(x)$ ,  $I_{0+}(x)$ , and  $I_{+0}(x)$  form a partition of the complementarity conditions.





#### Standard Fritz-John Conditions are too weak for MPECs

#### Consider the 2-dimensional MPEC

$$\min f(x)$$
 subject to  $x_1 \ge 0, x_2 \ge 0, x_1x_2 = 0.$ 

All feasible points are of the form  $(a,0)^T$  or  $(0,a)^T$  with  $a \ge 0$ . Consider a feasible point  $x = (a,0)^T$ . Then we have

$$0 \cdot \nabla f(x) - 0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} - a \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ a \end{pmatrix} = 0,$$

i.e. x is a Fritz-John point.



# Standard Fritz-John Conditions are too weak for MPECs

Consider the 2-dimensional MPEC

$$\min f(x)$$
 subject to  $x_1 \ge 0, x_2 \ge 0, x_1x_2 = 0.$ 

All feasible points are of the form  $(a,0)^T$  or  $(0,a)^T$  with  $a \ge 0$ . Consider a feasible point  $x = (a,0)^T$ . Then we have

$$0 \cdot \nabla f(x) - 0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} - a \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ a \end{pmatrix} = 0,$$

i.e. x is a Fritz-John point.

#### **Theorem**

Let x be feasible for (2). Then x is a Fritz-John point.



# KKT conditions are too strong for MPECs

#### Consider the 3-dimensional MPEC

$$\min x_1+x_2-x_3 \ \text{ subject to } -4x_1+x_3\leq 0,$$
 
$$-4x_2+x_3\leq 0,$$
 
$$x_1\geq 0, x_2\geq 0, x_1x_2=0.$$

The global minimum  $(0,0,0)^T$  is not a KKT point since elementary calculations yield that

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \lambda_1 \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix} - \lambda_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \lambda_4 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \mu_1 \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

is only possible if  $\lambda_3 + \lambda_4 = -2$ , i.e. at least one of them has to be negative.



# Standard CQs are usually not satisfied

Consider again the 2-dimensional MPEC

min 
$$f(x)$$
 subject to  $x_1 \ge 0, x_2 \ge 0, x_1x_2 = 0$ 

and a feasible point  $(a,0)^T$  with  $a \ge 0$ . Then we have

$$-0\cdot \begin{pmatrix} 1\\0 \end{pmatrix} - a\cdot \begin{pmatrix} 0\\1 \end{pmatrix} + 1\cdot \begin{pmatrix} 0\\a \end{pmatrix} = 0,$$

i.e. MFCQ is violated in x.

# Standard CQs are usually not satisfied

Consider again the 2-dimensional MPEC

min 
$$f(x)$$
 subject to  $x_1 \ge 0, x_2 \ge 0, x_1x_2 = 0$ 

and a feasible point  $(a,0)^T$  with  $a \ge 0$ . Then we have

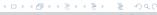
$$-0\cdot \begin{pmatrix} 1\\0 \end{pmatrix} - a\cdot \begin{pmatrix} 0\\1 \end{pmatrix} + 1\cdot \begin{pmatrix} 0\\a \end{pmatrix} = 0,$$

i.e. MFCQ is violated in x.

#### **Theorem**

Let x be feasible for (2). Then both LICQ and MFCQ are violated in x.





Standard Fritz-John conditions are too weak KKT conditions are often violated

Standard constraint qualifications are too strong





Standard Fritz-John conditions are too weak

KKT conditions are often violated

Standard constraint
qualifications are too
strong

Special MPEC
constraint
qualifications have
been introduced, e.g.
MPEC-MFCQ,
MPEC-ACQ



Standard Fritz-John conditions are too weak

KKT conditions are often violated

l

Weaker stationarity concepts have been developed, e.g. M-stationarity

qualifications are too
strong

Special MPEC
constraint

Standard constraint

qualifications have been introduced, e.g. MPEC-MFCQ, MPEC-ACQ



Standard Fritz-John conditions are too weak



Preliminary result by Jane Ye

KKT conditions are often violated



Weaker stationarity concepts have been developed, e.g. M-stationarity

Standard constraint qualifications are too strong

U
Special MPEC

Special MPEC constraint qualifications have been introduced, e.g. MPEC-MFCQ, MPEC-ACQ



#### MPEC Constraint Qualifications

A point  $x^*$  feasible for (2) is said to satisfy

▶ *MPEC-MFCQ*, if there are no multipliers  $\lambda \geq 0$  and  $\mu, \gamma, \nu$  such that  $(\lambda, \mu, \gamma, \nu) \neq 0$  and

$$\sum_{i:g_i(x^*)=0} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^{p} \mu_i \nabla h_i(x^*) - \sum_{i:G_i(x^*)=0} \gamma_i \nabla G_i(x^*) - \sum_{i:H_i(x^*)=0} \nu_i \nabla H_i(x^*) = 0.$$

#### MPEC Constraint Qualifications

A point  $x^*$  feasible for (2) is said to satisfy

▶ MPEC-MFCQ, if there are no multipliers  $\lambda \geq 0$  and  $\mu, \gamma, \nu$  such that  $(\lambda, \mu, \gamma, \nu) \neq 0$  and

$$\sum_{i:g_i(x^*)=0} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^p \mu_i \nabla h_i(x^*) - \sum_{i:G_i(x^*)=0} \gamma_i \nabla G_i(x^*) - \sum_{i:H_i(x^*)=0} \nu_i \nabla H_i(x^*) = 0.$$

▶ MPEC-ACQ if  $T_X(x^*) = L_{MPEC}(x^*)$ , where

$$\begin{split} T_X(x^*) &:= \{d \mid \exists \{t_k\} \downarrow 0, \{x^k\} \to_X x^* : \frac{x^k - x^*}{t_k} \to d\}, \\ L_{MPEC}(x^*) &:= \{d \mid \nabla g_i(x^*)^T d \leq 0 \ (i \in I_g(x^*)), \\ \nabla h_i(x^*)^T d = 0 \ (i = 1, \dots, p), \\ \nabla G_i(x^*)^T d = 0 \ (i \in I_{0+}(x^*)), \\ \nabla H_i(x^*)^T d = 0 \ (i \in I_{+0}(x^*)), \\ \nabla G_i(x^*)^T d \geq 0, \nabla H_i(x^*)^T d \geq 0 \ (i \in I_{00}(x^*)), \\ (\nabla G_i(x^*)^T d)(\nabla H_i(x^*)^T d) = 0 \ (i \in I_{00}(x^*)). \end{split}$$

# Weaker Stationarity Concept for MPECs

#### Definition (M-stationarity)

A vector  $x^*$  feasible for (2) is called *M-stationary* if there are multipliers  $\lambda \geq 0$  and  $\mu, \gamma, \nu$  such that

$$\nabla f(x^*) + \sum_{i:g_i(x^*)=0} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^{p} \mu_i \nabla h_i(x^*) - \sum_{i:G_i(x^*)=0} \gamma_i \nabla G_i(x^*) - \sum_{i:H_i(x^*)=0} \nu_i \nabla H_i(x^*) = 0$$

and either  $\gamma_i > 0$ ,  $\nu_i > 0$  or  $\gamma_i \nu_i = 0$  for all  $i \in I_{00}(x^*)$ .

# Weaker Stationarity Concept for MPECs

#### Definition (M-stationarity)

A vector  $x^*$  feasible for (2) is called *M-stationary* if there are multipliers  $\lambda \geq 0$  and  $\mu, \gamma, \nu$  such that

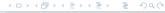
$$\nabla f(x^*) + \sum_{i:g_i(x^*)=0} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^{p} \mu_i \nabla h_i(x^*) - \sum_{i:G_i(x^*)=0} \gamma_i \nabla G_i(x^*) - \sum_{i:H_i(x^*)=0} \nu_i \nabla H_i(x^*) = 0$$

and either  $\gamma_i > 0$ ,  $\nu_i > 0$  or  $\gamma_i \nu_i = 0$  for all  $i \in I_{00}(x^*)$ .

#### **Theorem**

Let  $x^*$  be a local minimum of (2) that satisfies MPEC-ACQ. Then  $x^*$  is M-stationary.





#### New Fritz-John Result

#### Theorem (MPEC Fritz-John conditions)

Let  $x^*$  be a local minimum of (2). Then there are multipliers  $\alpha \geq 0, \lambda \geq 0$ , and  $\mu, \gamma, \nu$  such that  $(\alpha, \lambda, \mu, \gamma, \nu) \neq 0$ ,

$$\alpha \nabla f(x^*) + \sum_{i:g_i(x^*)=0} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^p \mu_i \nabla h_i(x^*) - \sum_{i:G_i(x^*)=0} \gamma_i \nabla G_i(x^*) - \sum_{i:H_i(x^*)=0} \nu_i \nabla H_i(x^*) = 0,$$

and either  $\gamma_i > 0$ ,  $\nu_i > 0$  or  $\gamma_i \nu_i = 0$  for all  $i \in I_{00}(x^*)$ .

#### New Fritz-John Result

#### Theorem (MPEC Fritz-John conditions)

Let  $x^*$  be a local minimum of (2). Then there are multipliers  $\alpha \geq 0, \lambda \geq 0$ , and  $\mu, \gamma, \nu$  such that  $(\alpha, \lambda, \mu, \gamma, \nu) \neq 0$ ,

$$\alpha \nabla f(x^*) + \sum_{i:g_i(x^*)=0} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^{P} \mu_i \nabla h_i(x^*) - \sum_{i:G_i(x^*)=0} \gamma_i \nabla G_i(x^*) - \sum_{i:H_i(x^*)=0} \nu_i \nabla H_i(x^*) = 0,$$

and either  $\gamma_i > 0$ ,  $\nu_i > 0$  or  $\gamma_i \nu_i = 0$  for all  $i \in I_{00}(x^*)$ . If  $(\lambda, \mu, \gamma, \nu) \neq 0$ , there is a sequence  $\{x^k\} \to x^*$  such that for all  $k \in \mathbb{N}$  we have  $f(x^k) < f(x^*)$  and

$$\lambda_i > 0 \Longrightarrow \lambda_i g_i(x^k) > 0$$
 and  $\mu_i \neq 0 \Longrightarrow \mu_i h_i(x^k) > 0$ ,  $\gamma_i \neq 0 \Longrightarrow \gamma_i G_i(x^k) < 0$  and  $\nu_i \neq 0 \Longrightarrow \nu_i H_i(x^k) < 0$ .



# MPEC-generalized quasinormality

#### Definition

A vector  $x^*$  feasible for (2) is said to satisfy MPEC generalized quasinormality, if there are no multipliers  $\lambda \geq 0$  and  $\mu, \gamma, \nu$  such that  $(\lambda, \mu, \gamma, \nu) \neq 0$ ,

$$\sum_{i:g_i(x^*)=0} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^{P} \mu_i \nabla h_i(x^*) - \sum_{i:G_i(x^*)=0} \gamma_i \nabla G_i(x^*) - \sum_{i:H_i(x^*)=0} \nu_i \nabla H_i(x^*) = 0$$

and either  $\gamma_i > 0$ ,  $\nu_i > 0$  or  $\gamma_i \nu_i = 0$  for all  $i \in I_{00}(x^*)$  and there is a sequence  $\{x^k\} \to x^*$  such that for all  $k \in \mathbb{N}$ 

$$\lambda_i > 0 \Longrightarrow \lambda_i g_i(x^k) > 0$$
 and  $\mu_i \neq 0 \Longrightarrow \mu_i h_i(x^k) > 0$ ,  $\gamma_i \neq 0 \Longrightarrow \gamma_i G_i(x^k) < 0$  and  $\nu_i \neq 0 \Longrightarrow \nu_i H_i(x^k) < 0$ .



# MPEC-generalized pseudonormality

#### Definition

A vector  $x^*$  feasible for (2) is said to satisfy MPEC generalized pseudonormality, if there are no multipliers  $\lambda \geq 0$  and  $\mu, \gamma, \nu$  such that  $(\lambda, \mu, \gamma, \nu) \neq 0$ ,

$$\sum_{i:g_i(x^*)=0} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^{P} \mu_i \nabla h_i(x^*) - \sum_{i:G_i(x^*)=0} \gamma_i \nabla G_i(x^*) - \sum_{i:H_i(x^*)=0} \nu_i \nabla H_i(x^*) = 0$$

and either  $\gamma_i > 0$ ,  $\nu_i > 0$  or  $\gamma_i \nu_i = 0$  for all  $i \in I_{00}(x^*)$  and there is a sequence  $\{x^k\} \to x^*$  such that for all  $k \in \mathbb{N}$ 

$$\sum_{i:g_i(x^*)=0} \lambda_i g_i(x^k) + \sum_{i=1}^p \mu_i h_i(x^k) - \sum_{i:G_i(x^*)=0} \gamma_i G_i(x^k) - \sum_{i:H_i(x^*)=0} \nu_i H_i(x^k) > 0.$$

# MPEC-generalized MFCQ

#### Definition

A vector  $x^*$  feasible for (2) is said to satisfy MPEC generalized MFCQ, if there are no multipliers  $\lambda \geq 0$  and  $\mu, \gamma, \nu$  such that  $(\lambda, \mu, \gamma, \nu) \neq 0$ ,

$$\sum_{i:g_i(x^*)=0} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^p \mu_i \nabla h_i(x^*) - \sum_{i:G_i(x^*)=0} \gamma_i \nabla G_i(x^*) - \sum_{i:H_i(x^*)=0} \nu_i \nabla H_i(x^*) = 0$$

and either  $\gamma_i > 0$ ,  $\nu_i > 0$  or  $\gamma_i \nu_i = 0$  for all  $i \in I_{00}(x^*)$ .





# Relations to Existing Results

MPEC-MFCQ

WPEC generalized MFCQ

MPEC generalized pseudonormality

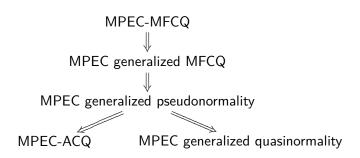
MPEC-ACQ

MPEC generalized quasinormality





#### Relations to Existing Results



#### **Theorem**

Let  $x^*$  be a local minimum of (2) that satisfies MPEC generalized quasinormality. Then  $x^*$  is M-stationary.





# Definition of the Penalty Function

Consider the penalty function

$$P_{\alpha}(x) := f(x) + \alpha \left[ \sum_{i=1}^{m} \mathsf{dist}_{(-\infty,0]} g_{i}(x) + \sum_{i=1}^{p} \mathsf{dist}_{\{0\}} h_{i}(x) + \sum_{i=1}^{l} \mathsf{dist}_{C}(G_{i}(x), H_{i}(x)) \right],$$

where distances are measured in the  $I_1$ -norm and

$$C = \{(a, b) \in \mathbb{R}^2 \mid a \ge 0, b \ge 0, ab = 0\}.$$

# Definition of the Penalty Function

Consider the penalty function

$$P_{\alpha}(x) := f(x) + \alpha \left[ \sum_{i=1}^{m} \mathsf{dist}_{(-\infty,0]} g_{i}(x) + \sum_{i=1}^{p} \mathsf{dist}_{\{0\}} h_{i}(x) + \sum_{i=1}^{l} \mathsf{dist}_{C}(G_{i}(x), H_{i}(x)) \right],$$

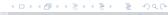
where distances are measured in the  $l_1$ -norm and

$$C = \{(a, b) \in \mathbb{R}^2 \mid a \ge 0, b \ge 0, ab = 0\}.$$

Its explicit representation is

$$P_{\alpha}(x) = f(x) + \alpha \left[ \sum_{i=1}^{m} \max\{g_{i}(x), 0\} + \sum_{i=1}^{p} |h_{i}(x)| + \sum_{i=1}^{l} \max\{-G_{i}(x), -H_{i}(x), -(G_{i}(x) + H_{i}(x)), \min\{G_{i}(x), H_{i}(x)\}\} \right].$$





#### **Exactness Result**

#### Theorem (Exactness of the Penalty Function)

Let  $x^*$  be a local minimum of (2) that satisfies MPEC generalized pseudonormality. Then  $P_{\alpha}$  is exact in  $x^*$ , i.e. there is an  $\bar{\alpha}>0$  such that  $x^*$  is an unconstrained local minimum of  $P_{\alpha}$  for all  $\alpha>\bar{\alpha}$ .



#### **Exactness Result**

#### Theorem (Exactness of the Penalty Function)

Let  $x^*$  be a local minimum of (2) that satisfies MPEC generalized pseudonormality. Then  $P_{\alpha}$  is exact in  $x^*$ , i.e. there is an  $\bar{\alpha}>0$  such that  $x^*$  is an unconstrained local minimum of  $P_{\alpha}$  for all  $\alpha\geq\bar{\alpha}$ .

If instead distances are measured in the  $l_{\infty}$ -norm, one obtains

$$\tilde{P}_{\alpha}(x) = f(x) + \alpha \left[ \sum_{i=1}^{m} \max\{g_{i}(x), 0\} + \sum_{i=1}^{p} |h_{i}(x)| + \sum_{i=1}^{l} |\min\{G_{i}(x), H_{i}(x)\}| \right],$$

which has the same exactness properties as  $P_{\alpha}$  thanks to the equivalence of norms in finite dimensional spaces.





#### Conclusion

#### All in all we have

- presented a new Fritz-John result specialized for MPECs,
- derived some new constraint qualifications guaranteeing M-stationarity of local minima,
- used one of these new CQs to prove exactness of two penalty functions.



#### Conclusion

#### All in all we have

- presented a new Fritz-John result specialized for MPECs,
- derived some new constraint qualifications guaranteeing M-stationarity of local minima,
- used one of these new CQs to prove exactness of two penalty functions.

# Thank you very much for your attention!



