Diversification, protection of liability holders and regulatory arbitrage

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Abstract

Any solvency regime for financial institutions should be aligned with the two fundamental objectives of regulation: protecting liability holders and securing the stability of the financial system. From these objectives we derive two normative requirements for capital adequacy tests, called surplus and numéraire invariance, respectively. We characterize capital adequacy tests that satisfy surplus and numéraire invariance, establish an intimate link between these requirements, and highlight an inherent tension between the ability to meet them and the desire to give credit for diversification.

1 Introduction

One of the major advances in the regulation of financial institutions, be it banks or insurance companies, has been the introduction of risk-sensitive solvency regimes. Examples are the Basel Accord in the banking sector and Solvency 2 and the Swiss Solvency Test in the insurance sector. Let us recall the typical mathematical framework of risk-sensitive solvency regimes; see for instance Artzner, Delbaen, Eber & Heath [1], Föllmer & Schied [6] or, for an account in the spirit of this paper, Farkas, Koch-Medina & Munari [4]. *Capital positions* – assets net of liabilities – of financial institutions are assumed to belong to a

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space $\mathcal{X}$ of random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ representing the *future states of the economy*. At any state $\omega \in \Omega$, an institution with capital position $X \in \mathcal{X}$ will be able to meet its obligations whenever $X(\omega) \geq 0$ and will default whenever $X(\omega) < 0$. A financial institution is deemed to be adequately capitalized if its capital position belongs to a pre-specified subset $\mathcal{A}$ of $\mathcal{X}$, called the *acceptance set* or the *capital adequacy test*. Finally, *risk measures* describe the minimum cost of meeting the capital adequacy test by raising capital and investing it in a reference instrument, often assumed to be cash.

Capital adequacy tests are primarily an instrument of microprudential regulation, i.e. their main purpose is to help protect liability holders. At the same time, however, capital adequacy tests should ideally support or, at the very least, not undermine macroprudential regulation, whose objective is to secure the stability of the financial system. From this observation we derive two requirements for capital adequacy tests.

The first requirement follows directly from the objective of microprudential regulation: A capital adequacy test should be *surplus invariant*, i.e. for a financial institution with capital position X the size of the *surplus*

$$X^+ := \max\{X, 0\}\,,$$

which benefits only the institution's owners, should have no impact on whether the institution passes or fails the test. In other words, acceptability should only depend on the default option

$$X^- := \max\{-X, 0\}$$

which, in case of a company with limited liability, represents the difference between the contractual and the actual liability payment. Formally, this entails requiring that

$$X \in \mathcal{A},\ Y^- \leq X^- \implies Y \in \mathcal{A}\,.$$

The second requirement follows from the need not to undermine macroprudential regulation for instance by incentivizing regulatory arbitrage. In particular, in a harmonized global regulatory framework where each country applies the same capital adequacy test in its own currency, it should no be possible to pass from being unacceptable to being acceptable by merely changing to a different jurisdiction. Neglecting all practical difficulties associated with such a change, this can be viewed as a form regulatory arbitrage. To avoid the existence of such regulatory arbitrage opportunities, the test should be *numéraire invariant*, i.e. acceptability should not depend on the currency in which capital positions are expressed. The formal articulation of this requirement is as follows: For every change of numéraire, represented by a bounded random variable R that is strictly positive almost surely, we require that

$$X \in \mathcal{A} \implies RX \in \mathcal{A}\,.$$

While meeting the above requirements is reasonable, it is also reasonable — and in the interest of liability holders — that capital requirements give credit for diversification. This is because an institution that diversifies its risk exposures can reduce the costs related to holding capital, costs that are ultimately borne by liability holders. It is well-known that, for an acceptance set, the financial requirement of giving credit for diversification is captured by the mathematical property of *convexity*; see for instance Föllmer & Schied [5] or Frittelli & Rosazza-Gianin [7].

In this paper we characterize surplus-invariance and numéraire invariance for convex acceptance sets when the ambient space $\mathcal{X}$ is taken to be an L^p space for some $0 \leq p \leq \infty$. This will highlight an interesting and important tension that exists between the requirement that capital adequacy tests be surplus and numéraire invariant and that, at the same time, they give credit for diversification.

Section 3 contains our main results. After introducing and proving some basic properties of surplus-invariant acceptance sets, we turn to their characterization in case they are closed and convex. Here, closedeness refers to the usual topology on L^p if $0 \leq p < \infty$ and to the weak-star topology $\sigma(L^\infty, L^1)$ if $p = \infty$. We first provide in Theorem 3.7 a dual representation that relies on the fact that any closed, surplus-invariant acceptance set $\mathcal{A} \subset L^p$ can be recovered by taking the L^p closure of its restriction to L^∞. This allows us to obtain a "dual" representation even for the non locally convex case where $p < 1$. Armed with this representation we show in Theorem 3.10 that a closed, convex acceptance set $\mathcal{A}$ is surplus invariant if and only if we find a partition $\{A, B, C\}$ of Ω consisting of measurable sets such that

$$X \in \mathcal{A} \iff 1_A X \geq 0 \quad \text{and} \quad -1_B X^- \in \mathcal{D}_B\,, \tag{1}$$

where $\mathcal{D}_B$ is a closed, convex set that is tight, i.e. bounded in probability. The decomposition of Ω into the three classes of scenarios implied by the partition $\{A, B, C\}$ has a clear financial interpretation: A financial institution is adequately capitalized if and only if does not default at all on A, it defaults on B but in a "controlled" way, i.e. $-1_B X^-$ must belong to the tight set $\mathcal{D}_B$, and it is entirely unconstrained on C. Incidentally, Example 3.3 shows that the above characterization does not hold in case we equip L^∞ with its strong topology.

When applied to the special case of coherent acceptance sets, i.e. acceptance sets that are convex cones, we obtain a generalization to any L^p of the characterization by Koch-Medina, Moreno-Bromberg and Munari given in [8] of weak-star closed, coherent, surplus-invariant acceptance sets on L^∞. Indeed, it suffices to notice that a closed, convex, surplus-invariant acceptance set given by (1) is coherent if and only if $\mathbb{P}(B) = 0$. In other words, closed, coherent, capital adequacy tests are surplus-invariant if and only if we impose no restrictions on C and disallow any defaults in the stress scenarios, i.e. in the states belonging to A. The problem with these capital adequacy tests is that either $\mathbb{P}(A) < 1$

and they ignore what happens outside A, or $\mathbb{P}(A) = 1$ and the tests disallow defaults in every state of the world.

To obtain a better sense of just how controlled defaults must be on the scenarios belonging to B, we can exploit a characterization of tightness, given in Proposition 3.14, in terms of stochastic boundedness with respect to first order stochastic dominance. As a result we can show that there exists a random variable $X^* \in L^0$ such that

$$\mathrm{VaR}_\alpha(1_B X) \leq \mathrm{VaR}_\alpha(X^*) \ \text{ for all } X \in \mathcal{A},\ \alpha \in (0,1)\,,$$

where the *Value-at-Risk* of any $X \in L^0$ is defined as usual by

$$\mathrm{VaR}_\alpha(X) := \inf\{t \in \mathbb{R}\,;\ \mathbb{P}(X + t < 0) \leq \alpha\}\,.$$

It follows that, on B, the VaR of every acceptable position is rigidly controlled by the VaR of a single bounding random variable X^* for every level $\alpha \in (0,1)$.

Finally, in Section 4 we discuss numéraire invariance and show that it is intimately related to surplus invariance. Indeed, in Proposition 4.4 we show that a closed capital adequacy test is numéraire invariant if and only if it is a conical, surplus-invariant acceptance set. Hence, it follows that, under closedness, if we simultaneously require convexity and numéraire invariance we end up allowing no defaults on a measurable set A and imposing no constraints on the complement A^c.

This limited choice of convex capital adequacy tests that are either surplus or numéraire invariant reflects the tension we mentioned at the beginning of the introduction: Convexity reduces numéraire invariant capital adequacy tests to simple stress tests. Thus, if we want to avoid having states of the world in which financial institutions are completely unconstrained, we are left with a single possible test that is passed only by those institutions that will never default. This tension lessens considerably if we drop numéraire invariance while retaining surplus-invariance: Convexity now allows for tests where controlled defaults are possible and there are no uncontrolled scenarios. In Proposition 3.19 we provide a constructive way to define convex, surplus-invariant capital adequacy tests based on Expected Shortfall that have this property.

We finally mention that our work generalizes results obtained in Koch-Medina, Moreno-Bromberg & Munari [8] for coherent acceptance sets in the context of spaces of bounded positions in two ways. First, this paper goes beyond coherence and allows for convex acceptance sets and, second, we allow the ambient spaces to be any L_p space for $p \in [0, \infty]$. As discussed in detail in that paper, the notion of surplus-invariance is related to similar notions discussed by Staum [10] and by Cont, Deguest & He [3].

2 Financial positions and acceptance sets

Throughout this paper, $(\Omega, \mathcal{F}, \mathbb{P})$ will denote a fixed probability space. As usual, we will identify random variables that coincide almost surely (with respect to

$\mathbb{P}$). The indicator function of a set $A \in \mathcal{F}$ is denoted by 1_A. For any $0 \le p \le \infty$, the space $L^p := L^p(\Omega, \mathcal{F}, \mathbb{P})$ is endowed with the natural almost sure pointwise ordering, i.e. for $X, Y \in L^p$ we write $Y \ge X$ whenever $\mathbb{P}(Y \ge X) = 1$. A random variable $X \in L^p$ is said to be *positive* if $X \ge 0$ and *negative* if $X \le 0$. The *positive cone* and the *negative cone* are the closed, convex cones defined by

$$L^p_+ := \{X \in L^p\, ;\ X \ge 0\} \quad \text{and} \quad L^p_- := \{X \in L^p\, ;\ X \le 0\}\,.$$

For $\mathcal{A} \subset L^p$ we define $\mathcal{A}_+ := \mathcal{A} \cap L^p_+$ and $\mathcal{A}_- := \mathcal{A} \cap L^p_-$. Moreover, for any $X \in L^p$ we set $X^+ := \max\{X, 0\}$ and $X^- := \max\{-X, 0\}$. In particular, note that $X = X^+ - X^-$.

We always equip L^p with the usual topology when $0 \le p < \infty$. By contrast, if $p = \infty$, we will consider the weak-star topology $\sigma(L^\infty, L^1)$. Following this convention we write $\mathrm{cl}_p(\mathcal{A})$ for the strong closure if $0 \le p < \infty$ and for the weak-star closure if $p = \infty$.

We consider a one-period economy with dates $t = 0$ and $t = T$. At time $t = 0$, financial institutions issue liabilities and invest in assets. At time $t = T$ they receive the payoff of the assets and redeem their liabilities. Assets and liabilities are assumed to be denominated with respect to a fixed unit of account, e.g. a fixed currency, and to belong to L^p for some fixed $0 \le p \le \infty$. If $A \in L^p$ and $L \in L^p$ are positive random variables representing the terminal payoff of the institution's assets and liabilities, respectively, we will refer to the random variable $X := A - L \in L^p$ as the *capital position* of the financial institution. We will always assume the owners of the institution have *limited liability*, i.e. the institution will default at time $t = T$ whenever the payoff of the assets does not suffice to repay liabilities. A concern of regulators is the risk of financial institutions defaulting on their obligations and one of the key instruments they have to mitigate this risk is to require that financial institutions be *adequately capitalized*. Acceptance sets are used to formalize the process of testing for capital adequacy.

Recall that a non-empty, strict subset $\mathcal{A} \subset L^p$ is called an *acceptance set* or a *capital adequacy test* if it is *monotone*, i.e.

$$X \in \mathcal{A},\ Y \ge X \quad \Longrightarrow \quad Y \in \mathcal{A}.$$

Acceptance sets that are convex or coherent, i.e. convex cones, are of particular importance because they capture diversification.

3 Surplus invariance

Consider a financial institution with capital position $X \in L^p$. The positive random variable X^+ is called the *(owners') surplus* and the, also positive, random variable X^- is called the *(owners') option to default.* In case of a financial institution with limited liability, the surplus represents what belongs to the owners after liabilities have been settled and the option to default the amount by which the institution defaults. More precisely, X^- represents the difference between the contractual and the actual payment to liability holders. Since a capital adequacy test is designed to protect the interests of liability holders, it makes sense that an institution should not pass the test if its default profile is riskier than the default profile of an institution that has been deemed inadequately capitalized. Equivalently, if an institution has been deemed adequately capitalized, any institution with a less risky default profile should also pass the test. This leads to the notion of a "surplus invariant" acceptance set.

Definition 3.1 Let $p \in [0, \infty]$ and assume $\mathcal{A} \subset L^p$ is an acceptance set. We say that $\mathcal{A}$ is *surplus invariant* if acceptability does not depend on the surplus of a capital position, i.e.

$$X \in \mathcal{A},\ Y^- \leq X^- \implies Y \in \mathcal{A}\,.$$

We start by providing a list of useful alternative characterizations of surplus invariance, which will be used without explicit reference in the sequel.

Proposition 3.2 *Let $p \in [0, \infty]$ and consider an acceptance set $\mathcal{A} \subset L^p$. The following statements are equivalent:*

(a) $\mathcal{A}$ is surplus invariant;

(b) $X \in \mathcal{A}$ and $Y^- = X^-$ imply $Y \in \mathcal{A}$;

(c) $X \in \mathcal{A}$ implies $-X^- \in \mathcal{A}$;

(d) $X \in \mathcal{A}$ and $A \in \mathcal{F}$ imply $1_A X \in \mathcal{A}$.

Proof. It is clear that *(a)* implies *(b)*, which in turn implies *(c)*. Now, assume *(c)* holds and take $X \in \mathcal{A}$ and $A \in \mathcal{F}$. Since $1_A X \geq -X^-$, we immediately conclude that *(d)* is satisfied. Finally, assume *(d)* holds and take $X \in \mathcal{A}$. If $Y^- \leq X^-$, then $Y \geq 1_{\{X<0\}} X$ implying that $Y \in \mathcal{A}$ and showing that $\mathcal{A}$ is surplus invariant. ■

Remark 3.3 The characterization under *(b)* establishes the equivalence of the present definition of surplus invariance with the one introduced in Koch-Medina, Moreno-Bromberg, Munari [8].

The next result shows that a surplus-invariant acceptance set is fully determined by its negative part. A set $\mathcal{D} \subset L^p_-$ is said to be *solid (in L^p_-)* if it satisfies

$$X \in \mathcal{D},\ X \leq Y \leq 0 \implies Y \in \mathcal{D}\,.$$

Proposition 3.4 *Let $p \in [0, \infty]$. An acceptance set $\mathcal{A} \subset L^p$ is surplus invariant if and only if*

$$\mathcal{A} = \mathcal{D} + L^p_+ \tag{2}$$

for some solid set $\mathcal{D} \subset L^p_-$. In this case, $\mathcal{A}_- = \mathcal{D}$. The set $\mathcal{A}$ is convex if and only if $\mathcal{D}$ is convex. Moreover, $\mathcal{A}$ is closed if and only if $\mathcal{D}$ is closed.

Proof. It is clear that $\mathcal{A}$ is surplus invariant if it is given as in (2) for some solid set $\mathcal{D} \subset L^p_-$. On the other hand, if $\mathcal{A}$ is surplus invariant, then $\mathcal{A}_-$ is solid by monotonicity. Since $\mathcal{A}_- + L^p_+ \subset \mathcal{A}$ clearly holds, we only need to show the converse inclusion. Take $X \in \mathcal{A}$ and note that $-X^- \in \mathcal{A}$. Since $X = X^+ - X^-$, the claim follows. The equivalence between the convexity of $\mathcal{A}$ and that of $\mathcal{D}$ is obvious. Finally, it is clear that if $\mathcal{A}$ is closed so is $\mathcal{D}$. Conversely, by using the continuity of the operation corresponding to taking the negative part and by the monotonicity of $\mathcal{A}$, one can easily prove that $\mathcal{A}$ is closed whenever $\mathcal{D}$ is closed. ∎

3.1 Duality and surplus invariance

In this section we provide an external characterization of surplus-invariant acceptance sets in L^p, $0 \leq p \leq \infty$, by exploiting duality in $\big(L^\infty, \sigma(L^\infty, L^1)\big)$. This "dual" representation constitutes the key ingredient in the proof of the main result on the structure of surplus-invariant acceptance sets given later on.

The starting point is the following density lemma. In the sequel, for any $\mathcal{A} \subset L^p$ we will denote by $\mathcal{A}^\infty$ the set of all bounded elements of $\mathcal{A}$, i.e. we set

$$\mathcal{A}^\infty := \mathcal{A} \cap L^\infty .$$

Lemma 3.5 *Let $p \in [0, \infty)$ and assume $\mathcal{A} \subset L^p$ is a closed, surplus-invariant acceptance set. Then, we have*

$$\mathcal{A} = \mathrm{cl}_p(\mathcal{A}^\infty) .$$

Proof. Clearly, we only need to show the inclusion "$\subset$". To this end, take $X \in \mathcal{A}_-$ and note that $X_n := \max\{X, -n\}$ belongs to $\mathcal{A}^\infty$ for every $n \in \mathbb{N}$ and that $X_n \to X$ in L^p. It follows that $X \in \mathrm{cl}_p(\mathcal{A}^\infty_-)$. The statement now follows by Proposition 3.4. ∎

We now turn to describe the prototype of a closed, convex, surplus-invariant acceptance set in L^p. A map $\varphi : L^\infty \to \mathbb{R} \cup \{-\infty\}$ satisfying

(F1) $\varphi(Z) \leq 0$ for all $Z \in L^\infty_+$,

(F2) $\varphi(Z) > -\infty$ for some $Z \in L^\infty_+$,

(F3) φ is decreasing,

will be called a *decreasing floor function.*

Proposition 3.6 *Let $p \in [0, \infty]$ and take a decreasing floor function $\varphi : L^\infty \to \mathbb{R} \cup \{-\infty\}$. Then, the set*

$$\mathcal{A} = \{X \in L^p\,;\ -\mathbb{E}[X^- Z] \geq \varphi(Z),\ \forall Z \in L^\infty_+\} \tag{3}$$

is a closed, convex, surplus-invariant acceptance set in L^p.

Proof. Clearly $\mathcal{A}$ is convex and surplus invariant. It is also clear that $\mathcal{A}$ is weak-star closed if $p = \infty$. Hence, assume $p < \infty$ and consider a sequence (X_n) in $\mathcal{A}$ converging to X in L^p. We claim that $X \in \mathcal{A}$. To prove this, take $Z \in L^\infty_+$. If necessary passing to a suitable subsequence, it follows from Fatou's Lemma that

$$-\mathbb{E}[X^- Z] \geq -\liminf_{n\to\infty} \mathbb{E}[X_n^- Z] \geq \varphi(Z)\,.$$

Since Z was arbitrary, we conclude that $X \in \mathcal{A}$. ∎

The following result provides a representation for surplus-invariant acceptance sets in L^p by means of decreasing floor functions. Recall that the *(lower) support function* of a nonempty subset $\mathcal{C} \subset L^\infty$ is the superlinear and upper-semicontinuous map $\sigma_\mathcal{C} : L^1 \to \mathbb{R} \cup \{-\infty\}$ defined by

$$\sigma_\mathcal{C}(Z) := \inf_{X\in\mathcal{C}} \mathbb{E}[XZ]\,. \tag{4}$$

The effective domain of $\sigma_\mathcal{C}$, called the *barrier cone* of $\mathcal{C}$, is the convex cone

$$B(\mathcal{C}) := \{Z \in L^1\,;\ \sigma_\mathcal{C}(Z) > -\infty\}\,.$$

If $\mathcal{C}$ is a cone, then $Z \in B(\mathcal{C})$ if and only if $\sigma_\mathcal{C}(Z) = 0$. By Lemma 3.11 in Farkas, Koch-Medina, Munari [4], we have $B(\mathcal{C}) \subset L^1_+$ whenever $\mathcal{C}$ is a monotone set.

Theorem 3.7 *Let $p \in [0,\infty]$ and assume $\mathcal{A} \subset L^p$ is a closed, convex, surplus-invariant acceptance set. Then, there exists a decreasing floor function $\varphi : L^\infty \to \mathbb{R} \cup \{-\infty\}$ such that*

$$\mathcal{A} = \bigcap_{Z\in L^\infty_+} \{X \in L^p\,;\ -\mathbb{E}[X^- Z] \geq \varphi(Z)\}\,. \tag{5}$$

For φ we can always choose the restriction of $\sigma_{\mathcal{A}^\infty}$ to L^∞.

Proof. Clearly, $\mathcal{A}^\infty$ is a weak-star closed, convex, surplus-invariant acceptance set in L^∞. By surplus invariance, it is immediate to see that $\sigma_{\mathcal{A}^\infty} = \sigma_{\mathcal{A}^\infty_-}$ so that the restriction of $\sigma_{\mathcal{A}^\infty}$ to L^∞ fulfills (F1) and (F3). Moreover, by Theorem 4.1 in Koch-Medina, Moreno-Bromberg, Munari [8], we have

$$\mathcal{A}^\infty = \bigcap_{Z\in L^1_+} \{X \in L^\infty\,;\ -\mathbb{E}[X^- Z] \geq \sigma_{\mathcal{A}^\infty}(Z)\}\,. \tag{6}$$

If we denote by $\mathcal{B}$ the intersection in (5) for $p = \infty$ and $\varphi = \sigma_{\mathcal{A}^\infty}$, we see that $\mathcal{A}^\infty \subset \mathcal{B}$. We claim that $\mathcal{B} \subset \mathcal{A}^\infty$. To prove this, take $X \in \mathcal{B}$ and $Z \in L^1_+$. Since $Z_n := \min\{Z, n\} \in L^\infty_+$ for every $n \in \mathbb{N}$, we have

$$-\mathbb{E}[X^- Z_n] \geq \sigma_{\mathcal{A}^\infty}(Z_n) \geq \sigma_{\mathcal{A}^\infty}(Z)\,,$$

where we have used that $\sigma_{\mathcal{A}^\infty}$ is decreasing. We can now take the limit in the above equation, as the sequence $-X^- Z_n$ is monotone decreasing, i.e. we have

$$-\mathbb{E}[X^- Z] \geq -\liminf_{n\to\infty} \mathbb{E}[X^- Z_n] \geq \sigma_{\mathcal{A}^\infty}(Z).$$

Hence, $X \in \mathcal{A}^\infty$. Clearly, this implies that the restriction of $\sigma_{\mathcal{A}^\infty}$ to L^∞ fulfills (F2), for otherwise $\mathcal{A}^\infty$ would coincide with the entire L^∞. In conclusion, the assertion is established for $p = \infty$. For a general $p < \infty$, we simply note that $\mathcal{A} = \mathrm{cl}_p(\mathcal{A}^\infty)$ by Lemma 3.5. Hence, in light of Proposition 3.6, the claim follows directly from the representation (5) in the case $p = \infty$. ■

Remark 3.8 (i) Note that the representation (5) involves halfspaces generated by "functionals" in L^∞. In particular, if $p \geq 1$, this implies that closed, convex, surplus-invariant acceptance sets are automatically $\sigma(L^p, L^\infty)$-closed.

(ii) The preceding result holds for any space L^p, including the case $p < 1$. In spite of the structural lack of local convexity of these spaces, surplus invariance allows to provide an external characterization by using "duality" theory in L^∞ equipped with the weak-star topology. In this sense, the representation (5) should be compared with the bipolar representation on L^0 obtained in Brannath & Schachermayer [2] and generalized in Kupper & Svindland [9].

3.2 The structure of surplus invariance

In this section we prove the main results of the paper. For any $A \in \mathcal{F}$ and $\mathcal{A} \subset L^p$ we define the subset of L^p

$$1_A\mathcal{A} := \{1_A X\,;\ X \in \mathcal{A}\}\,.$$

In particular, we set

$$L^p(A) := 1_A L^p\,.$$

The corresponding positive and negative cones are denoted by $L^p_+(A)$ and $L^p_-(A)$, respectively. More precisely, we set

$$L^p_+(A) := L^p(A) \cap L^p_+ \quad \text{and} \quad L^p_-(A) := L^p(A) \cap L^p_-\,.$$

The following "decomposition" theorem is the key to understanding the structure of surplus-invariant capital adequacy tests. The proof relies on the exhaustion technique used in the proof of Théorème 2 in Yan [11].

Lemma 3.9 *Let $p \in [0, \infty]$ and assume $\mathcal{A} \subset L^p$ is a closed, convex, surplus-invariant acceptance set. Then, there exists a set $C \in \mathcal{F}$ such that*

(i) $Z = 0$ almost surely on C for every $Z \in B(\mathcal{A}^\infty)$;

(ii) $Z^* > 0$ almost surely on C^c for some $Z^* \in B(\mathcal{A}^\infty) \cap L^\infty$.

In particular, we have $\mathcal{A} = L^p(C) \oplus 1_{C^c}\mathcal{A}$.

Proof. We first prove that the class

$$\mathcal{G} := \{\{Z = 0\}\,;\ Z \in B(\mathcal{A}^\infty)\}$$

is closed under countable intersections. Indeed, consider a sequence (Z_n) in $B(\mathcal{A}^\infty)$ and take a sequence (α_n) of positive real numbers such that

$$\sum_{n\in\mathbb{N}} \alpha_n \left\|Z_n\right\|_1 \quad \text{and} \quad \sum_{n\in\mathbb{N}} \alpha_n \sigma_{\mathcal{A}^\infty}(Z_n)$$

both converge. The first condition implies that $\sum_n \alpha_n Z_n$ converges to some Z in L^1 and the second, by the upper semicontinuity of $\sigma_{\mathcal{A}^\infty}$, that $Z \in B(\mathcal{A}^\infty)$. Moreover,

$$\{Z = 0\} = \bigcap_{n\in\mathbb{N}} \{Z_n = 0\}\,.$$

It follows that $\mathcal{G}$ is closed under countable intersections. Take now a sequence (Z_n) in $B(\mathcal{A}^\infty)$ such that

$$\lim_{n\to\infty} \mathbb{P}(Z_n = 0) = \inf_{E\in\mathcal{G}} \mathbb{P}(E)\,.$$

Then, we find a suitable $Z^* \in B(\mathcal{A}^\infty)$ such that

$$\{Z^* = 0\} = \bigcap_{n\in\mathbb{N}} \{Z_n = 0\}\,.$$

In particular, $\{Z^* = 0\}$ attains the minimal probability over $\mathcal{G}$. We claim that

$$C := \{Z^* = 0\}$$

has the desired properties. To prove *(i)*, assume there exists $Z \in B(\mathcal{A}^\infty)$ with $Z > 0$ on a measurable subset E of C with nonzero probability. Then, $Z + Z^*$ would be an element of $B(\mathcal{A}^\infty)$ satisfying $\mathbb{P}(Z+Z^* = 0) \le \mathbb{P}(C \setminus E) < \mathbb{P}(C)$, in contrast with the minimality of C. Hence, *(i)* is satisfied. If $Z^* \in L^\infty$, then *(ii)* is also clearly satisfied. Otherwise, replace Z^* by $\min\{Z^*, 1\}$ which also belongs to $B(\mathcal{A}^\infty)$ since $\sigma_{\mathcal{A}^\infty}$ is decreasing.

To prove the last assertion, note that we only need to show that $L^p(C) \oplus 1_C\mathcal{A} \subset \mathcal{A}$. To this end, take $X \in \mathcal{A}$ and $Y \in L^p$. By *(i)* and surplus invariance, it follows that

$$-\mathbb{E}[(1_C Y + 1_{C^c} X)^- Z] = -\mathbb{E}[X^- 1_{C^c} Z] \ge \sigma_{\mathcal{A}^\infty}(Z)$$

for any $Z \in B(\mathcal{A}^\infty)$, where we have also used the representation obtained in Theorem 3.7. Hence, we can conclude the proof by noting that $1_C Y + 1_{C^c} X \in \mathcal{A}$ holds as a consequence of the same result. ∎

The preceding result has interesting financial implications. Indeed, acceptability can be described by requirements on the behaviour of capital positions on each of the "atoms" of a measurable partition $\{A, B, C\}$ of Ω: no defaults are allowed on A, a "controlled" default is allowed on B, and no requirements are imposed on C. The second condition will be made precise using the notion of tightness. Recall that a set $\mathcal{D} \subset L^p$ is *tight*, or *bounded in probability*, if for every $\varepsilon \in (0,1)$ there exists $M > 0$ such that $\mathbb{P}(-M < X < M) > \varepsilon$ for all $X \in \mathcal{D}$. In other words, tight sets are precisely those sets which are topologically bounded in L^0.

Theorem 3.10 *Let $p \in [0, \infty]$ and assume $\mathcal{A} \subset L^p$ is a closed, convex acceptance set. Then, $\mathcal{A}$ is surplus invariant if and only if there exists a measurable partition $\{A, B, C\}$ of Ω with $\mathbb{P}(C) < 1$, unique up to modifications on sets of nonzero probability, such that*

$$\mathcal{A} = L^p_+(A) \oplus (L^p_+(B) + \mathcal{D}_B) \oplus L^p(C)\,, \tag{7}$$

where $\mathcal{D}_B$ is a closed, convex, solid, tight subset of $L^p_-(B)$ such that for every measurable subset E of B there exists $X \in \mathcal{D}_B$ with $\mathbb{P}(1_E X < 0) > 0$.

Proof. The "if" implication is clear, hence we focus on the converse statement. First, take $C \in \mathcal{F}$ and $Z^* \in B(\mathcal{A}^\infty) \cap L^\infty$ as in Lemma 3.9. Since $\mathcal{A}_-$ is closed, convex and solid, so is the set $\mathcal{D} := 1_{C^c}\mathcal{A}_-$. Moreover, we claim that $\mathcal{D}$ is tight. Indeed, we would otherwise find $\varepsilon \in (0,1)$ such that for every $M > 0$ there exists $X_M \in \mathcal{D}$ with $\mathbb{P}(X_M < -M) > \varepsilon$. Choosing $\delta > 0$ such that

$$\mathbb{P}(C^c \setminus \{Z^* > \delta\}) > \mathbb{P}(C^c) - \frac{\varepsilon}{2}\,,$$

it follows that

$$\mathbb{P}(X_M < -M, Z^* > \delta) > \frac{\varepsilon}{2}$$

for every $M > 0$. However, since $X_M \in \mathcal{A}$ by surplus invariance, this would imply that

$$-M\delta\frac{\varepsilon}{2} \geq \mathbb{E}[X_M Z^*] \geq \sigma_{\mathcal{A}^\infty}(Z^*) > -\infty$$

for any $M > 0$, which is impossible. It follows that $\mathcal{D}$ is tight. To further decompose C^c, set

$$U := \operatorname*{ess\,inf}_{X \in \mathcal{D}} X$$

and define

$$A := \{U = 0\}, \quad B := C^c \setminus A, \quad \mathcal{D}_B := 1_B \mathcal{D}\,.$$

It is easy to check that $\{A, B, C\}$ is a measurable partition of Ω with the required properties. ∎

In Section 3.3 we will provide more insight into the type of constraints implied by the tightness of the set $\mathcal{D}_B$. The preceding result can be further sharpened in the context of coherent acceptance sets. In this case, acceptability reduces to disallowing defaults on a pre-specified set of stress scenarios.

Theorem 3.11 *Let $p \in [0,\infty]$ and assume $\mathcal{A}$ is a closed, coherent acceptance set. Then, $\mathcal{A}$ is surplus invariant if and only if there exists $A \in \mathcal{F}$ such that $\mathbb{P}(A) > 0$ and*

$$\mathcal{A} = L^p_+(A) \oplus L^p(A^c)\,.$$

Proof. The "if" implication is clear, hence we focus on the converse assertion. Consider the decomposition of $\mathcal{A}$ obtained in Theorem 3.10 and assume $\mathbb{P}(B) > 0$. In this case, by the same result, we would find a suitable $X \in \mathcal{A}$ such that $\mathbb{P}(1_B X < 0) > 0$. Since $\mathcal{A}$ is conical, it follows that $-n1_B X^- \in \mathcal{D}_B$ for each $n \in \mathbb{N}$. However, this would contradict the tightness of $\mathcal{D}_B$. In conclusion, we must have $\mathbb{P}(B) = 0$ and the assertion follows from Theorem 3.10. ∎

Remark 3.12 The previous result provides a generalization to any L^p space, $0 \leq p < \infty$, of the characterization of closed, coherent, surplus-invariant acceptance sets obtained in Koch-Medina, Moreno-Bromberg, Munari [8] in the context of spaces of bounded random variables. Note that, in contrast to [8], the separability of L^p is no longer required.

We conclude by showing that, with the clear exception of atomic spaces with a finite number of atoms, the decomposition obtained in Theorem 3.10 does not hold if we equip L^∞ with the norm topology.

Example 3.13 (L^∞_- with the strong topology) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space admitting an infinite partition (B_n) of Ω consisting of measurable sets of nonzero probability. This is equivalent to L^∞ being infinite dimensional. Define the increasing sequence (A_n) by setting

$$A_n := \bigcup_{k=1}^{n} B_k$$

for every $n \in \mathbb{N}$. By construction we have $\mathbb{P}(A_{n+1} \setminus A_n) = \mathbb{P}(B_{n+1}) > 0$ for each $n \in \mathbb{N}$ and $\bigcup_n A_n = \Omega$. For each $n \in \mathbb{N}$ the set

$$\mathcal{D}_n := L^\infty_-(A_n)$$

is norm-closed, convex and solid. Moreover, note that $(\mathcal{D}_n)$ is increasing. Hence, the union

$$\mathcal{D} := \bigcup_{n \in \mathbb{N}} \mathcal{D}_n$$

is easily seen to be a convex, solid subset of L^∞_-. It is also clear that $\mathcal{D}$ is not tight. Now, denote by $\overline{\mathcal{D}}$ the closure of $\mathcal{D}$ in the norm topology and consider the closed, convex, surplus invariant, monotone set

$$\mathcal{A} := \overline{\mathcal{D}} + L^\infty_+ \,.$$

Since $\mathbb{P}(X \geq -\frac{1}{2}) > 0$ holds for every $X \in \overline{\mathcal{D}}$, we see that $\overline{\mathcal{D}} \neq L^\infty_-$. In particular, $\mathcal{A}$ is an acceptance set. We claim that $\mathcal{A}$ cannot be decomposed as in Theorem 3.10. For assume we could write $\mathcal{A}$ in the standard form (7). In this case, we would have $\overline{\mathcal{D}} = \mathcal{D}_B \oplus L^p_-(C)$, where $\mathcal{D}_B$ is tight. Since $1_B\overline{\mathcal{D}}$ is not tight, we must have $\mathbb{P}(B) = 0$ so that $\overline{\mathcal{D}} = L^p_-(C)$. This would imply that $-1_{A_n} \in L^p_-(C)$, hence $\mathbb{P}(A_n) \leq \mathbb{P}(C)$, for any $n \in \mathbb{N}$. However, this would be possible only if $\mathbb{P}(C) = 1$, in contradiction to $\overline{\mathcal{D}} \neq L^\infty_-$.

3.3 Stochastic boundedness and surplus invariance

In the previous section we have proved that any closed, convex, surplus-invariant acceptance set $\mathcal{A} \subset L^p$ can be decomposed as

$$\mathcal{A} = L^p_+(A) \oplus (L^p_+(B) + \mathcal{D}_B) \oplus L^p(C)$$

for a suitable partition $\{A, B, C\}$ of Ω. In particular, the set $\mathcal{D}_B$ consists of acceptable default options and was shown to be tight, or bounded in probability. In this section we focus on the set $\mathcal{D}_B$ and show how to interpret this "boundedness" property from a capital adequacy perspective.

The distribution function of a random variable $X \in L^0$ will be denoted by F_X, i.e. we set $F_X(t) := \mathbb{P}(X \leq t)$ for every $t \in \mathbb{R}$. Recall that if X and Y are two random variables, X is said to be *(first order) stochastically preferred to* Y, denoted by $X \geq_{\mathrm{st}} Y$, if $F_X(t) \leq F_Y(t)$ for every $t \in \mathbb{R}$. A set $\mathcal{D} \subset L^0$ is *(first order) stochastically bounded* by a random variable X^* if every element of $\mathcal{D}$ is (first order) stochastically preferred to X^*, i.e. if $X \geq_{\mathrm{st}} X^*$ for all $X \in \mathcal{D}$. The following result establishes that, for a subset of L^0_-, tightness is equivalent to stochastic boundedness.

Proposition 3.14 *A set $\mathcal{D} \subset L^0_-$ is tight if and only if it is stochastically bounded.*

Proof. We show first the "if" implication. Assume $X^* \in L^0$ is a stochastic bound for $\mathcal{D}$. In particular, note that X^* must be negative. Now, for any $\varepsilon \in (0,1)$ we find $M > 0$ large enough to satisfy $\mathbb{P}(X^* > -M) > \varepsilon$. Since $\mathbb{P}(X > -M) \geq \mathbb{P}(X^* > -M)$ for all $X \in \mathcal{D}$, we conclude that $\mathcal{D}$ is tight.

To prove the converse implication, assume $\mathcal{D}$ is tight and consider the function $F : \mathbb{R}_+ \to [0,1]$ defined by setting

$$F(x) := \sup_{X \in \mathcal{D}} F_X(-x) \,.$$

It is clear that F is decreasing and satisfies

$$\lim_{x\to\infty} F(x) = 0\,. \tag{8}$$

The last assertion follows directly from the tightness of $\mathcal{D}$. The proof now reduces to showing that there exists a random variable $X^* \in L^0_-$ satisfying

$$F_{X^*}(-x) \geq F(x) \quad \text{for every } x > 0\,. \tag{9}$$

Now, assume first that there exists $M > 0$ such that $F(M) = 0$, i.e. the set $\mathcal{D}$ is uniformly bounded in L^∞. Then, the random variable $X^* := -M1_\Omega$ is easily seen to be a stochastic bound for $\mathcal{D}$.

Assume next that for every $M > 0$ we have $F(M) > 0$. Together with (8), this implies that for every $\varepsilon \in (0,1)$ there exists $M > 0$ such that

$$0 < \sup_{X\in\mathcal{D}} F_X(-M) \leq \varepsilon\,.$$

In this case, it is not difficult to show that we can find a countable partition (A_n) of Ω consisting of measurable sets with nonzero probability. Now, consider an increasing sequence (α_n) of strictly positive numbers such that $F(\alpha_n) < 1 - \sum_{k=1}^n \mathbb{P}(A_k)$ and set

$$X^* := -\sum_{n=1}^\infty \alpha_n 1_{A_n}\,.$$

We claim that (9) holds. To this end, take $x > 0$ and choose the smallest $n_0 \in \mathbb{N}$ such that $F(x) \geq 1 - \sum_{k=1}^{n_0} \mathbb{P}(A_k)$. Since F is decreasing, we have $\alpha_n \geq x$ for each $n \geq n_0$. Thus

$$F_{X^*}(-x) \geq \sum_{k=n_0}^\infty \mathbb{P}(A_k) = 1 - \sum_{k=1}^{n_0-1} \mathbb{P}(A_k) > F(x)\,,$$

proving (9) and concluding the proof of the proposition. ■

Remark 3.15 One could also prove that the function F defined in the above proof is right continuous and infer the existence of a random variable on a nonatomic probability space having $1-F$ as its distribution function. We prefer the above proof because it does not require us to consider random variables on a probability space that is different from our underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We proceed to apply the preceding characterization of tightness to complement the representation result obtained in Theorem 3.10. To this end, recall that the *Value-at-Risk* (VaR) of $X \in L^0$ at the level $\alpha \in (0,1)$ is defined as

$$\mathrm{VaR}_\alpha(X) := \inf\{t \in \mathbb{R}\,;\ \mathbb{P}(X + t < 0) \leq \alpha\}\,.$$

Moreover, the *Expected Shortfall* (ES) of X at level α is given by

$$\mathrm{ES}_\alpha(X) := \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\beta(X)d\beta\,.$$

The first part of the following simple lemma recalls a well-known characterization of first order stochastic dominance. The second part is an obvious corollary and is an equivalent formulation that first order stochastic dominance implies second order stochastic dominance.

Lemma 3.16 *For any random variables $X, Y \in L^0$, we have*

$$X \geq_{\mathrm{st}} Y \iff \mathrm{VaR}_\alpha(X) \leq \mathrm{VaR}_\alpha(Y) \;\; \textit{for every } \alpha \in (0,1)\,.$$

In particular,

$$X \geq_{\mathrm{st}} Y \implies \mathrm{ES}_\alpha(X) \leq \mathrm{ES}_\alpha(Y) \;\; \textit{for every } \alpha \in (0,1)\,.$$

The preceding lemma allows us to link the "boundedness" of the set of acceptable default profiles $\mathcal{D}_B$ to a stringent control of the VaR_α and ES_α of its elements for arbitrary $\alpha \in (0,1)$.

Theorem 3.17 *Let $p \in [0,\infty]$ and assume $\mathcal{A} \subset L^p$ is a closed, convex, surplus-invariant acceptance set with standard decomposition*

$$\mathcal{A} = L^p_+(A) \oplus (L^p_+(B) + \mathcal{D}_B) \oplus L^p(C)\,.$$

Then, there exists $X^* \in L^0_-$ such that

$$\mathrm{VaR}_\alpha(1_B X) \leq \mathrm{VaR}_\alpha(X^*) \;\; \textit{for every } X \in \mathcal{A},\; \alpha \in (0,1)\,.$$

In particular, if $X^* \in L^1_-$ then

$$\mathrm{ES}_\alpha(1_B X) \leq \mathrm{ES}_\alpha(X^*), \;\; \textit{for every } X \in \mathcal{A},\; \alpha \in (0,1)\,.$$

Proof. By Theorem 3.10, the set $\mathcal{D}_B$ is a tight subset of $L^p_-(B)$. Hence, Proposition 3.14 implies that $\mathcal{D}_B$ is stochastically bounded by some $X^* \in L^0_-$. The two assertions follow immediately from Lemma 3.16. ∎

The above result is also important since it provides a hint on how to *construct* closed, convex, surplus-invariant acceptance sets that allow "controlled" defaults but have no "uncontrolled" scenarios. First, we need the following lemma.

Lemma 3.18 *Let $p \in [1,\infty]$. Take $B \in \mathcal{F}$ and fix $X^* \in L^p_-$. Then, the set*

$$\mathcal{D}_B = \{X \in L^p_-(B)\,;\; \mathrm{ES}_\alpha(X) \leq \mathrm{ES}_\alpha(X^*),\; \forall \alpha \in (0,1)\}$$

is closed, convex and tight in L^p.

Proof. Since ES_α is continuous and convex on L^p, the set $\mathcal{D}_B$ is clearly closed and convex. To prove tightness, recall that $\mathrm{ES}_\alpha(X) \to \mathbb{E}[-X]$ as $\alpha \to 1$ for any $X \in L^p$. As a result, it follows that $\mathcal{D}_B$ is bounded in L^1 and hence, *a fortiori*, in L^0. ∎

Proposition 3.19 *Let $p \in [1,\infty]$ and consider a measurable partition $\{A,B\}$ of Ω. Moreover, take $X^* \in L^p_-$. Then, the set*

$$\mathcal{A} = L^p_+(A) \oplus (L^p_+(B) + \mathcal{D}_B)$$

where

$$\mathcal{D}_B = \{X \in L^p_-(B)\,;\ \mathrm{ES}_\alpha(X) \leq \mathrm{ES}_\alpha(X^*),\ \forall \alpha \in (0,1)\}$$

is a closed, convex, surplus-invariant acceptance set in L^p.

4 Numéraire invariance

Recall that we had expressed capital positions in a fixed unit of account, say some fixed currency. The process of changing the accounting currency can be described by means of a random variable R that is strictly positive almost surely, representing the exchange rate from the original into the new currency. For a capital position $X \in L^p$ expressed in the original currency, the random variable RX represents the position of the *same* company expressed in the new currency. In order that RX still belongs to L^p, we assume that R is in L^∞. For convenience, a random variable $R \in L^\infty_+$ which is strictly positive almost surely will be called a *rescaling factor*. Thus, rescaling factors are random variables that qualify to represent a change of unit of account.

The harmonization of solvency regimes across jurisdictions requires the specification of a common capital adequacy test $\mathcal{A} \subset L^p$ that can be performed in each jurisdiction in its respective currency — typically, the choices under discussion are test based on Value-at-Risk or Expected Shortfall. In such a situation a clear requirement must be that it should not be possible for an institution to go from being unacceptable to being acceptable by merely moving to another jurisdiction. This leads to the following notion of numéraire invariance.

Definition 4.1 Let $p \in [0,\infty]$. An acceptance set $\mathcal{A} \subset L^p$ is numéraire invariant if for every rescaling factor R we have

$$X \in \mathcal{A} \implies RX \in \mathcal{A}\,. \tag{10}$$

We start by showing that we can use a smaller or larger class of rescaling factors without changing the concept of numéraire invariance. Recall that a random variable $R \in L^\infty_+$ is said to be *bounded away from zero* if $\mathbb{P}(R > \varepsilon) = 1$ for some $\varepsilon > 0$.

Lemma 4.2 *Let $p \in [0,\infty]$ and assume $\mathcal{A} \subset L^p$ is a closed acceptance set. Then, $\mathcal{A}$ is numéraire invariant if and only if any of the following conditions are satisfied:*

(a) $X \in \mathcal{A}$ implies $RX \in \mathcal{A}$ for any $R \in L^\infty_+$;

(b) $X \in \mathcal{A}$ implies $RX \in \mathcal{A}$ for any $R \in L^\infty_+$ that is bounded away from zero.

Proof. Clearly, we only need to prove that *(b)* implies *(a)*. To this end, fix $X \in \mathcal{A}$ and take any $R \in L^\infty_+$. Moreover, consider the sequence with general term $R_n := R + \frac{1}{n}1_\Omega$. Since R_n is bounded away from zero for any $n \in \mathbb{N}$, we have $R_nX \in \mathcal{A}$. Hence, the limit RX must also belong to $\mathcal{A}$ by closedness, proving that *(a)* is satisfied. ■

Remark 4.3 The main reason for choosing a rescaling factor R to be an element of L^∞ is that for any capital position $X \in L^p$ we again have $RX \in L^p$. However, the following statement is true: a set $\mathcal{A} \subset L^p$ is numéraire invariant if and only if for every $R \in L^0_+$ we have

$$X \in \mathcal{A},\ RX \in L^p \implies RX \in \mathcal{A}\,.$$

Indeed this property, obviously, implies numéraire invariance, since for $R \in L^\infty_+$ we always have $RX \in L^p$, hence (a) in Lemma 4.2 is satisfied. To show the converse, note that for $R_n = \min\{R, n\}$ we have that $R_nX \in \mathcal{A}$ by numéraire invariance and that $R_nX \to RX$ in L^p by dominated convergence theorem.

As a result of the preceding lemma, every numéraire invariant acceptance set is also surplus invariant. Moreover, these two properties are equivalent for any conical acceptance set.

Proposition 4.4 *Let $p \in [0,\infty]$ and assume $\mathcal{A} \subset L^p$ is a closed acceptance set. Then, the following statements are equivalent:*

(a) $\mathcal{A}$ is numéraire invariant;

(b) $\mathcal{A}$ is conical and surplus invariant.

Proof. Assume first that $\mathcal{A}$ is numéraire invariant. That $\mathcal{A}$ is a cone is clear from the definition of numéraire invariance. Now, take $X \in \mathcal{A}$. Then, the previous lemma yields $X1_{X<0} \in \mathcal{A}$, which implies surplus invariance. Assume now that $\mathcal{A}$ is conical and surplus invariant. Take $X \in \mathcal{A}$ and recall that $-X^- \in \mathcal{A}$. For any rescaling factor R we have $-RX^- \geq -\left\|R\right\|_\infty X^-$. As a consequence of conicity and monotonicity, it follows that $-RX^- \in \mathcal{A}$. Hence $RX \in \mathcal{A}$, proving that $\mathcal{A}$ is numéraire invariant. ■

In light of the close link between numéraire and surplus invariance, we can use the preceding results to obtain the following representation of numéraire invariant acceptance sets.

Theorem 4.5 *Let $p \in [0,\infty]$ and assume $\mathcal{A}$ is a closed, convex acceptance set. Then, $\mathcal{A}$ is numéraire invariant if and only if there exists $A \in \mathcal{F}$ such that $\mathbb{P}(A) > 0$ and*

$$\mathcal{A} = L^p_+(A) \oplus L^p(A^c)\,.$$

Proof. By Proposition 4.4, the acceptance set $\mathcal{A}$ is surplus invariant and conical, hence coherent. Therefore, the claim follows at once from Theorem 3.11. ■

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