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CSCI 3104, Algorithms **Due January 29, 2021**

Problem Set 2 (50 points) **Spring 2021, CU-Boulder**

Advice 1: For every problem in this class, you must justify your answer: show how you arrived at it and why it is correct. If there are assumptions you need to make along the way, state those clearly.
Advice 2: Verbal reasoning is typically insufficient for full credit. Instead, write a logical argument, in the style of a mathematical proof.

Instructions for submitting your solution:

- The solutions **should be typed** and we cannot accept hand-written solutions. Here's a short intro to Latex.
- You should submit your work through **Gradescope** only.
- The easiest way to access Gradescope is through our Canvas page. There is a Gradescope button in the left menu.
- Gradescope will only accept **.pdf** files.
- It is vital that you match each problem part with your work. Skip to 1:40 to just see the matching info.

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1. The following problems are a review of logarithm and exponent topics.

   (a) Solve for x.

   i. $3^{2x} = 81$

   ii. $3(5^{x-1}) = 375$

   iii. $\log_3 x^2 = 4$

   (b) Solve for x.

   i. $x^2 - x = \log_5 25$

   ii. $\log_{10}(x+3) - \log_{10} x = 1$

   (c) Answer each of the following with a TRUE or FALSE.

   i. $a^{\log_a x} = x$

   ii. $a^{\log_b x} = x$

   iii. $a = b^{\log_b a}$

   iv. $\log_a x = \frac{\log_b x}{\log_b a}$

   v. $\log b^m = m \log b$

Solution:

(ai)

$$\begin{aligned} (3^2)^x &= 81 \\ 9^x &= 81 \\ x &= \log_9 81 \\ &= \mathbf{2} \end{aligned}$$

(aii)

$$\begin{aligned} 3(5^{x-1}) &= 375 \\ \frac{5^x}{5} &= 125 \\ 5^x &= 625 \\ x &= \log_5 625 \\ &= \mathbf{4} \end{aligned}$$

a(iii)

$$\begin{aligned} \log_3 x^2 &= 4 \\ 3^4 &= x^2 \\ x &= \sqrt{81} \\ &= \pm\mathbf{9} \end{aligned}$$

(bi)

$$\begin{aligned} x^2 - x &= \log_5 25 \\ x^2 - x &= 2 \\ x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0 \\ x &= \mathbf{-1, 2} \end{aligned}$$

(bii)

$$\begin{aligned} \log_{10}(x+3) - \log_{10} x &= 1 \\ \log_{10} \frac{(x+3)}{x} &= 1 \\ 10 &= \frac{x+3}{x} \\ 10x &= x+3 \\ 9x &= 3 \\ x &= \mathbf{\frac{1}{3}} \end{aligned}$$

(ci) **TRUE**, a's will cancel out
(cii) **FALSE**
(ciii) **TRUE**, b's will cancel out
(civ) **TRUE**
(cv) **TRUE**

2. Compute the following limits at infinity. Show all work and justify your answer.

   (a) $\lim\limits_{x\to\infty} \dfrac{3x^3+2}{9x^3-2x^2+7}$

   (b) $\lim\limits_{x\to\infty} \dfrac{x^3}{e^{x/2}}$

   (c) $\lim\limits_{x\to\infty} \dfrac{\ln x^4}{x^3}$

Solution:

(a) Using LH:

$$\lim_{x\to\infty} \frac{3x^3+2}{9x^3-2x^2+7} \qquad \Rightarrow \lim_{x\to\infty} \frac{\frac{1}{x^3}(3x^3+2)}{\frac{1}{x^3}(9x^3-2x^2+7)}$$

$$\lim_{x\to\infty} \frac{3+\frac{2}{x^3}}{9-\frac{2}{x}+\frac{7}{x^3}} \qquad \Rightarrow \lim_{x\to\infty} 3+\frac{2}{x^3} = 3$$

$$\lim_{x\to\infty} 9-\frac{2}{x}+\frac{7}{x^3} = 9$$

$$\lim_{x\to\infty} \frac{3x^3+2}{9x^3-2x^2+7} = \frac{9}{3} = \mathbf{\frac{1}{3}}$$

(b) Using LH:

$$\lim_{x\to\infty} \frac{x^3}{e^{x/2}} \qquad \Rightarrow \lim_{x\to\infty} \frac{3x^2}{\frac{1}{2}e^{x/2}}$$

$$\lim_{x\to\infty} \frac{6x^2}{e^{x/2}} \qquad \Rightarrow \lim_{x\to\infty} \frac{12x}{\frac{1}{2}e^{x/2}}$$

$$\lim_{x\to\infty} \frac{24x}{e^{x/2}} \qquad \Rightarrow \lim_{x\to\infty} \frac{24}{\frac{1}{2}e^{x/2}} = \mathbf{0}$$

(c) Using LH:

$$\lim_{x\to\infty} \frac{\ln x^4}{x^3} \qquad \Rightarrow \lim_{x\to\infty} \frac{4\ln x}{x^3}$$

$$\lim_{x\to\infty} \frac{\frac{4}{x}}{3x^2} \qquad \Rightarrow \lim_{x\to\infty} \frac{4}{3x^3} = \mathbf{0}$$

3. Compute the following limits at infinity. Show all work and justify your answer.

   (a) For real numbers $m, n > 0$ compute $\lim_{x\to\infty} \frac{x^m}{e^{nx}}$

   (b) What does this tell us about the rate at which e^{nx} approaches infinity relative to x^m? A brief explanation is fine for this part.

   (c) For real numbers $m, n > 0$ compute $\lim_{x\to\infty} \frac{(\ln x)^n}{x^m}$

   (d) What does this tell us about the rate at which $(\ln x)^n$ approaches infinity relative to x^m? A brief explanation is fine for this part.

Solution:

(3a) Using LH:

$$\lim_{x\to\infty} \frac{x^m}{e^{nx}} \qquad \Rightarrow \lim_{x\to\infty} \frac{mx^{m-1}}{ne^{nx}}$$
$$\lim_{x\to\infty} \frac{m(m-1)x^{m-2}}{n^2e^{nx}} \qquad \Rightarrow \lim_{x\to\infty} \frac{m(m-1)(m-2)x^{m-3}}{n^3e^{nx}}$$
$$\lim_{x\to\infty} \frac{m(m-1)(m-2)...(3)x^2}{n^{m-2}e^{nx}} \qquad \Rightarrow \lim_{x\to\infty} \frac{m(m-1)(m-2)...(3)(2)x^1}{n^{m-1}e^{nx}}$$
$$\lim_{x\to\infty} \frac{m(m-1)(m-2)...(3)(2)(1)}{n^m e^{nx}} \qquad \Rightarrow \lim_{x\to\infty} \frac{m!}{n^m e^{nx}}$$
$$m! \lim_{x\to\infty} \frac{1}{n^m e^{nx}} = \mathbf{0}$$

(3b) e^{nx} approaches infinity faster than x^m due to the fact that $\lim_{x\to\infty} \frac{x^m}{e^{nx}} = 0$ for all (arbitrary) real numbers of m, n > 0. We can say that x^m is $o(e^{nx})$.

(3c) Using LH:

$$\lim_{x\to\infty} \frac{(\ln x)^n}{x^m} \qquad \Rightarrow \lim_{x\to\infty} \frac{((n/x)(\ln x)^{n-1}}{mx^{m-1}}$$
$$\lim_{x\to\infty} \frac{n(\ln x)^{n-1}}{mx^m} \qquad \Rightarrow \lim_{x\to\infty} \frac{((1/x)n(n-1)(\ln x)^{n-2}}{m^2x^{m-1}}$$
$$\lim_{x\to\infty} \frac{n(n-1)(\ln x)^{n-2}}{m^2x^m} \qquad \Rightarrow \lim_{x\to\infty} \frac{n!}{m^n x^m}$$
$$\frac{n!}{m^n} \lim_{x\to\infty} \frac{1}{x^m} = \mathbf{0}$$

Side Work:

$$\frac{((n/x)(\ln x)^{n-1}}{mx^{m-1}} = \frac{n(\ln x)^{n-1}}{xmx^{m-1}} = \frac{n(\ln x)^{n-1}}{xm\frac{x^m}{x^1}} = \frac{n(\ln x)^{n-1}}{mx^m}$$
$$\frac{1/x}{x^{m-1}} = \frac{1}{x} * \frac{1}{x^{m-1}} = \frac{1}{x*x^{m-1}} = \frac{1}{x*x^m/x^1} = \frac{1}{x^m}$$

(3d) x^m approaches infinity faster than $(\ln x)^n$ due to the fact that $\lim_{x\to\infty} \frac{(\ln x)^n}{x^m} = 0$ for all (arbitrary) real numbers of m, n > 0. We can say that $(\ln x)^n$ is $o(x^m)$.

4. The problems in this question deal with the Root and the Ratio Tests. Determine the convergence or divergence of the following series. State which test you used.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$

   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$

   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^n$

Solution:

(4a) Using the Root Test:

$$\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n} \qquad \Rightarrow \lim_{n\to\infty} \left|\frac{e^{2n}}{n^n}\right|^{1/n}$$

$$\lim_{n\to\infty} \frac{e^{2n/n}}{n^{n/n}} \qquad \Rightarrow \lim_{n\to\infty} \frac{e^2}{n}$$

$$e^2 \lim_{n\to\infty} \frac{1}{n} = \mathbf{0} < 1, \textbf{converges}$$

(4b) Using the Ratio Test:

$$\sum_{n=0}^{\infty} \frac{2^n}{n!} \qquad \Rightarrow \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}$$

$$\lim_{n\to\infty} \frac{2^n 2n!}{2^n (n+1)!} \qquad \Rightarrow \lim_{n\to\infty} \frac{2n!}{(n+1)!}$$

$$\lim_{n\to\infty} \frac{2}{n+1} \qquad \Rightarrow 2 \lim_{n\to\infty} \frac{1}{n+1} = \mathbf{0} < 1, \textbf{converges}$$

(4c) Using the Ratio Test:

$$\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n} \qquad \Rightarrow \lim_{n\to\infty} \frac{(n+1)^2 \cdot 2^{n+2}}{3^{n+1}} \cdot \frac{3^n}{n^2 2^{n+1}}$$

$$\lim_{n\to\infty} \frac{(n+1)^2 \cdot 2^n 2^2}{3^n \cdot 3^1} \cdot \frac{3^n}{n^2 2^n 2^1} \qquad \Rightarrow \lim_{n\to\infty} \frac{2(n+1)^2}{3n^2}$$

$$\lim_{n\to\infty} \frac{2n^2+4n+2}{3n^2} \qquad \Rightarrow \frac{1}{3} \lim_{n\to\infty} \frac{2n^2+4n+2}{n^2}$$

$$\lim_{n\to\infty} \frac{2n^2}{n^2} = 2, \lim_{n\to\infty} \frac{4n}{n^2} = 0, \lim_{n\to\infty} \frac{2}{n^2} = 0$$

$$\frac{1}{3}(2+0+0) = \mathbf{\frac{2}{3}} < 1, \textbf{converges}$$

(4d) Using the Root Test:

$$\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^n \qquad \Rightarrow \lim_{n\to\infty} \left|\left(\frac{\ln n}{n}\right)^n\right|^{1/n}$$

$$\lim_{n\to\infty} \frac{(\ln n)^{n/n}}{n^{n/n}} \qquad \Rightarrow \lim_{n\to\infty} \frac{\ln n}{n}$$

$$\text{using L'H, } \lim_{n\to\infty} \frac{1/n}{1} = \mathbf{0} < 1, \textbf{converges}$$