Mathematics Exam Solutions

1 Elementary algebra

Problem 1.1. Simplify
$$\frac{x^{32}}{x^9 \cdot x^2} \cdot \frac{x^7}{x^2} = \frac{x^{39}}{x^{13}} = x^{26}$$

Problem 1.2. Solve for x:
$$8^2 \cdot 4^x \cdot 2^x = 8^4$$
$$2^6 \cdot 2^{2x} \cdot 2^x = 2^{12}$$
$$2^{3x} = 2^6$$
$$3 \cdot x = 6$$
$$x = 2$$

Problem 1.3. Calculate the missing value. If $\frac{x}{y}$ is 3, then $x^{-4}y^4 = \ldots$
$$x = 3 \cdot y$$
$$(3 \cdot y)^{-4} y^4 = 3^{-4} \cdot y^{-4} \cdot y^4 = \frac{1}{3^4} = \frac{1}{81} \approx 0.012$$

Problem 1.4. Calculate
$$\frac{\sqrt{4^{15}}}{\sqrt{16^7}} = \sqrt{\frac{4^{15}}{4^{14}}} = \sqrt{4} = 2$$

Problem 1.5. True or False (x and y and z are real numbers):

(a) $x + (y + z) = (y + x) + z$ TRUE

(b) $y(x + z) = xy + zy$ TRUE

(c) $x^{y+z} = x^z + x^y$ FALSE

(d) $\frac{x^z}{x^y} = x^{y-z}$ FALSE

Problem 1.6. Find the solution set for the inequality below:
$$\ln(x) \geq e$$
$$e^{\ln(x)} \geq e^e$$
$$x \geq e^e$$

2 Functions of one variable

Problem 2.1 (Based on SYD 2.5.6). The relationship between temperatures measured in Celsius and Fahrenheit is linear. 0°C is equivalent to 32°F and 100°C is the same as 212°F. Which temperature is measured by the same number on both scales?

Equations:

(a) $100 = 212 \cdot a + b$

(b) $0 = 32 \cdot a + b$

Thus $32 \cdot a = -b$ and:

$$100 = 212 \cdot a - 32 \cdot a = 180 \cdot a$$

Thus $b = -32 \cdot \frac{100}{180}$ and:

$$y = \frac{100}{180} \cdot x - 32 \cdot \frac{100}{180}$$

If $y = x$:

$$x - \frac{100}{180} \cdot x = -32 \cdot \frac{100}{180}$$
$$\frac{180}{100} \cdot x - x = -32$$
$$x = -40$$

Problem 2.2. Take the following function $f(x) = 3x - 12$. Find y if $f(y) = 0$.

$$f(y) = 3y - 12 = 0$$
$$3y = 12$$
$$y = 4$$

Problem 2.3. Find all values of x that satisfy:

$$9^{x^2-6x+2} = 81$$
$$9^{x^2-6x+2} = 9^2$$
$$x^2 - 6x + 2 = 2$$
$$x^2 - 6x = 0$$
$$x \cdot (x - 6) = 0$$

So $x_1 = 0$, $x_2 = 6$.

Problem 2.4. Solve the following problem. If the annual GDP growth of a country is 3%, how long does it take the economy to triple its GDP?

$$3 \cdot GDP = GDP \cdot 1.03^x$$
$$3 = 1.03^x$$
$$x = \log_{1.03}(3)$$
$$x = \frac{\ln 3}{\ln 1.03} \approx 37.167$$

Problem 2.5. Calculate the following value

$$\log_\pi \left(\frac{1}{\pi^5}\right)$$
$$\log_\pi \left(\pi^{-5}\right) = -5$$

3 Calculus

Problem 3.1. Calculate the following sum

$$\sum_{i=0}^{\infty}\left(\frac{1}{5^i}+0.3^i\right)=\sum_{i=0}^{\infty}\left(\frac{1}{5^i}\right)+\sum_{i=0}^{\infty}0.3^i$$

Applying the formula for infinite geometric series:

$$\sum_{i=0}^{\infty}\left(\frac{1}{5^i}\right)=\frac{5}{4}$$

$$\sum_{i=0}^{\infty}\left(\frac{3}{10}\right)^i=\frac{10}{7}$$

$$\sum_{i=0}^{\infty}\left(\frac{1}{5^i}+0.3^i\right)=\frac{5}{4}+\frac{10}{7}\approx 2.67857$$

Problem 3.2. Find the following limit

$$\lim_{x\to 5}\frac{x^2-25}{x-5}$$

$$\lim_{x\to 5}\frac{(x-5)(x+5)}{x-5}$$

$$\lim_{x\to 5}(x+5)=10$$

Problem 3.3. Find the slope of the function $f(x)=x^3-4$ at $(-2,-12)$.

$$f'(x)=3x^2$$

$$f'(-2)=12=m$$

$$y=m\cdot x$$

$$y-y_1=m\cdot(x-x_1)$$

$$y+12=12\cdot(x+2)$$

$$y=12x+12$$

Problem 3.4. Find the derivative of the following function:

$$f(x)=\frac{x^5+3}{x^2-1}$$

$$f'(x)=\frac{5x^4\cdot(x^2-1)-(x^5+3)\cdot 2x}{(x^2-1)^2}$$

Problem 3.5. Find the second derivative of the following function:

$$f(x)=x^9+3$$

$$f'(x)=9x^8$$

$$f''(x)=72x^7$$

Problem 3.6. Is the function $f(x)=\frac{1}{x}$ continuous at 0? Why?
It is not because division by zero is undefined therefore as x approaches 0 from the right, y approaches infinity and as x approaches 0 from the left, y approaches negative infinity.

Problem 3.7. Consider the following function. Find all of its local minima, local maxima or inflection points.

$$f(x) = 4x^3 - 12x$$
$$f'(x) = 12x^2 - 12 = 0$$
$$x^2 = 1$$
$$x = \pm 1$$

Possible local minima/maxima at $x = \pm 1$.

$$f''(x) = 24x$$

For $x = 1$, $f''(x) = 24$ so it's a local minimum.
For $x = -1$, $f''(x) = -24$ so it's a local maximum.
The function also has an inflection point at $x = 0$.

Problem 3.8. Let $f(x, y) = x^3 - y^2$. Calculate $f(2, 3)$

$$f(2, 3) = 8 - 9 = -1$$

Problem 3.9. Consider the following function: $f(x, y) = \ln(x - 3y)$. For what combinations of x and y is this function defined?

$$x - 3y > 0$$
$$x > 3y$$

Problem 3.10. Find the following partial derivative:

$$\frac{\partial}{\partial x}\left(x^5y^7 + \frac{x^2}{y^3}\right) = \left(5x^4y^7 + 2x \cdot \frac{1}{y^3}\right)$$

Problem 3.12. Solve the following constrained optimization problem using Lagrange's method: $\max x^2y^2$ s.t. $2x + y = 9$

$$L(x, y, \lambda) = f(x, y) - \lambda g(x, y) = x^2y^2 - \lambda(2x + y - 9)$$
$$\frac{\partial L}{\partial x} = 2xy^2 - 2\lambda = 0$$
$$\frac{\partial L}{\partial y} = 2yx^2 - \lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = 2x + y - 9 = 0$$
$$xy^2 = \lambda$$
$$2yx^2 = \lambda$$
$$y = 2x$$
$$x = \frac{9}{4}$$
$$y = \frac{9}{2}$$

4 Linear algebra

Problem 4.1. Take the following matrices:

$$A = \begin{bmatrix} 2 & 5 \\ 2 & 1 \\ 7 & 6 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 9 & 1 & 5 \end{bmatrix}$$

What is $B \cdot A$?

$$B \cdot A = \begin{bmatrix} 9 & 11 \\ 55 & 76 \end{bmatrix}$$

Problem 4.2. Take the following matrices:

$$A = \begin{bmatrix} 5 & 3 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 8 & 4 & 0 \\ 2 & 1 & 2 \end{bmatrix}$$

What is $A \cdot B$?

$$A \cdot B = \begin{bmatrix} 46 & 23 & 6 \\ 2 & 1 & 2 \\ 12 & 6 & 4 \end{bmatrix}$$

Problem 4.3. What is the transpose of the following matrix?

$$A = \begin{bmatrix} e & 93 & 4.7 \\ 2 & 6.1 & 4.22 \\ 4 & \pi & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} e & 2 & 4 \\ 93 & 6.1 & \pi \\ 4.7 & 4.22 & 0 \end{bmatrix}$$

Problem 4.4. Calculate the determinant of

$$A = \begin{bmatrix} 2 & 6 \\ 2 & 8 \end{bmatrix}$$

$$det(A) = 16 - 12 = 4$$

5 Probability theory

Problem 5.1. You run an experiment where you toss a dice two times. Each time you get either 1, 2, 3, 4, 5 or 6. What is the sample space of your experiment?

Sample space: $6 \cdot 6 = 36$.

Problem 5.2. Assume that in a certain country 0.1% of the population uses a certain drug. You have a way to test drug use, which will give you a positive result in 98% of the cases where the individual is indeed a drug user and a negative result in 99.7% of the cases where the individual doesn't use the drug. What is the probability that someone with a positive drug test is indeed a drug user?

Let's take the following events:

- A: positive drug test
- A^C: negative drug test
- B: drug user, $\mathrm{P}(B) = 0.001$
- B^C: not a drug user, $\mathrm{P}(B^C) = 0.999$

$\mathbf{P}(B|A) = \frac{P(A|B) \cdot P(B)}{P(A|B) \cdot P(B) + P(A|B^C) \cdot P(B^C)} = \frac{0.98 \cdot 0.001}{0.98 \cdot 0.001 + 0.003 \cdot 0.999} = 0.2464 \rightarrow 24.64\%$

Problem 5.3. You run an experiment in which you toss a dice 20 times and record how many times you ended up with a 1, 2, 3, 4, 5 or 6. Your random variable is the number of times you ended up with a 5. What is expected value of this random variable?

X = number of times we ended up with a 5

$$E(X) = 20 \cdot \frac{1}{6} \approx 3.33$$