Inductively Defined Sets

- one can define sets inductively via inference rules of form

$$\frac{premise_1 \quad \dots \quad premise_n}{conclusion}$$

  meaning: if all premises are satisfied, then one can conclude
- example: the set of even numbers

$$\frac{}{0 \in Even} \qquad\qquad \frac{x \in Even}{x + 2 \in Even}$$

- the inference rules describe what is contained in the set
- this can be modeled as formula

$$0 \in Even \land (\forall x.\ x \in Even \longrightarrow x + 2 \in Even)$$

- nothing else is in the set (this is not modeled in the formula!)

Inductively Defined Sets, Continued

- the set of even numbers

$$\frac{}{0 \in Even} \qquad\qquad \frac{x \in Even}{x + 2 \in Even}$$

- membership in the set can be proved via inference trees
- example: $4 \in Even$, proved via inference tree

$$\cfrac{\cfrac{\overline{0 \in Even}}{2 \in Even}}{4 \in Even}$$

- proving that something is not in the set is more difficult:
  show that no inference tree exists
- example: $3 \notin Even$, $-2 \notin Even$

Inductively Defined Sets and Grammars

- inference rules are similar to grammar rules
- example
  - the context-free grammar

$$S \to aSab \mid b \mid TaS \qquad\qquad T \to TT \mid \epsilon$$

  - is modeled via the inference rules

$$\frac{w \in S}{awab \in S} \qquad \frac{}{b \in S} \qquad \frac{w \in T \quad u \in S}{wau \in S}$$

$$\frac{w \in T \quad u \in T}{wu \in T} \qquad \frac{}{\epsilon \in T}$$

- in the same way, inference trees are similar to derivation trees

Inductively Defined Sets: Monotonicity

- inference rules of inductively defined sets must be monotone, it is forbidden to negatively refer to the currently defined set
- ill-formed example

$$\frac{}{0 \in Bad} \qquad\qquad \frac{0 \in Bad}{0 \notin Bad}$$

- one of the problems: the corresponding formula can be contradictory

$$0 \in Bad \wedge (0 \in Bad \longrightarrow 0 \notin Bad)$$

- allowed example: we define Odd, and negatively refer to previously defined $Even$

$$\frac{x \notin Even}{x \in Odd}$$

Inductively Defined Sets: Structural Induction

- example: the set of even numbers

$$\frac{}{0 \in Even} \qquad\qquad \frac{x \in Even}{x + 2 \in Even}$$

- inductively defined sets give rise to a structural induction rule
- induction rule for $Even$, written again as inference rule:

$$\frac{y \in Even \quad P(0) \quad \forall x. P(x) \longrightarrow P(x+2)}{P(y)}$$

where P is an arbitrary property; alternatively as formula

$$\forall y.\, y \in Even \longrightarrow \underbrace{P(0)}_{base} \longrightarrow \underbrace{(\forall x. P(x) \longrightarrow P(x+2))}_{step} \longrightarrow P(y)$$