HW1

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1 Problem 1

Because for $\forall X, L_n(X) = \frac{q_n(X)}{p_n(X)} \geq 0$, using Markov's Inequality, $P(\frac{q_n(X)}{p_n(X)} > M) \leq \frac{E(\frac{q_n(X)}{p_n(X)})}{M^2} = \frac{\int_{p_n(X)>0} \frac{q_n(X)}{p_n(X)} p_n(X)dx}{M^2} = \frac{\int_{p_n(X)>0} q_n(X)dx}{M^2} \leq \frac{1}{M^2}$.

So we have $P(L_n(X) > M) \leq \frac{1}{M^2}$ for $\forall n$, implying that $sup_n P(L_n(X) > M) < \frac{2}{M^2}$.

For $\forall \epsilon > 0$, take $M^* = \sqrt{\frac{2}{\epsilon}}$, Then $sup_n P(L_n(X) > M^*) < \epsilon$, the sequence is uniformly tight.

2 Problem 2

For $Beta(\theta, 1)$, $\bar{X_n} = \frac{\theta}{1+\theta}$ and $Var(X_n) = \frac{\theta}{(\theta+1)^2(\theta+2)})$.

By Central Limit Theorem, $\sqrt{n}(\bar{X_n} - \frac{\theta}{1+\theta}) \xrightarrow{d} N(0, Var(X_i))$, so $\sqrt{n}(\bar{X_n}(1+\theta) - \theta) = \sqrt{n}(\bar{X_n} - (1-\bar{X_n})\theta) \xrightarrow{d} N(0, (1+\theta)^2 Var(X_i))$.

By Law of Large Numbers, $\bar{X_n} \xrightarrow{p} \frac{\theta}{1+\theta}$, so $1 - \bar{X_n} \xrightarrow{p} \frac{1}{1+\theta}$.

Then by Slutsky's Lemma, $\sqrt{n}(\frac{\bar{X_n}}{1-\bar{X_n}} - \theta) \xrightarrow{d} N(0, Var(X_i)(1+\theta)^4)$, which is equivalent to $\sqrt{n}(\hat{\theta_n} - \theta) \xrightarrow{d} N(0, \frac{\theta(1+\theta)^2}{\theta+2})$

3 Problem 3

Let $U = X - E(X)$, and u be any non-negative number.

$P(X - E(X) \geq t) = P(U + u \geq t + u) \leq P((U+u)^2 \geq (t+u)^2)$. Using Markov inequality, we have $P((U+u)^2 \geq (t+u)^2) \leq \frac{E((U+u)^2)}{(t+u)^2} = \frac{(E(U+u))^2 + Var(U)}{(t+u)^2} = \frac{u^2+\sigma^2}{(t+u)^2} = f(u)$.

Optimizing this over $u \geq 0$, set $0 = f(u)^{'} = 2u(t+u)^2 - (u^2+\sigma^2)(2u+2t) = ut^2 + u^2t - u\sigma^2 - \sigma^2 t = (u+t)(tu - \sigma^2$. So $f(u)$ achieves maximum at $u = \frac{\sigma^2}{t}$. Max $f(u) = \frac{(\frac{\sigma^2}{t})^2 + \sigma^2}{t + (\frac{\sigma^2}{t})^2} = \frac{\sigma^2}{\sigma^2+t^2}$

Hence, $P(X - E(X) \geq t) \leq \frac{\sigma^2}{\sigma^2+t^2}$

4 Problem 4

4.1

Expectation does not converge. Let $X_n = X + \frac{1}{n}$. $E(g(X)) = \sum_{k=1}^{9} \frac{\lambda^k e^{-\lambda}}{k!}$ and $E(g(X_n)) = \sum_{k=0}^{9} \frac{\lambda^{k+\frac{1}{n}} e^{-\lambda}}{(k+\frac{1}{n})!}$. $E(g(X_n))$ doest not converge to $E(g(X))$ because at $x = 0$, $g(x_n) * f(x_n) = \frac{\lambda^{\frac{1}{n}} e^{-\lambda}}{\frac{1}{n}!} \rightarrow e^{-\lambda}$

4.2

$0 \leq g(x) = e^{-x^2} \leq 1$ is continuous everywhere. By Portmanteau theorem, convergence in distribution guarantees convergence in expectation.

4.3

$g(x)$ is clearly bounded. Pormanteau theorem states that if $g(x)$ is continuous everywhere, then $\sum_i g(X_n)f(X_n) \to \sum_i g(X)f(X)$. $\sum_i g(X_n)f(X_n) = \sum_{f(X_n)\neq 0} g(X_n)f(X_n) + \sum_{f(X_n)=0} h(X_n)f(X_n)$, where $h(X)$ could be any arbitrary function.This means that Pormanteau theorem works as long as $g(x)$ is continuous at the points where $f(x) \neq 0$. $g(x) = sgn(cos(x))$ is only noncontinuous at non-integers points, where $f(x) = 0$ under Poisson distribution. Hence expectation converges.

4.4

Expectation does not converge. Construct X_n in the following way:

$$P(X_n = k) = \begin{cases} \frac{1}{n} + (1 - \frac{1}{n})\frac{\lambda^n e^{-\lambda}}{n!} & \text{if } k = n \\ (1 - \frac{1}{n})\frac{\lambda^k e^{-\lambda}}{k!} & \text{if } k \in N, k \neq n \end{cases}$$

It's easy to check $f(X_n)$ is a valid density as $\sum_k P(X_n = k) = \frac{1}{n} + (1 - \frac{1}{n})\sum_k \frac{\lambda^k e^{-\lambda}}{k!} = \frac{1}{n} + 1 + \frac{1}{n} = 1$. And $X_n \xrightarrow{d} X$. $E(X) = \lambda$ but $E(X_n) = n\frac{1}{n} + (1 - \frac{1}{n})\sum_k \frac{\lambda^k e^{-\lambda}}{k!}k \to 1 + \lambda \neq \lambda$ as $n \to \infty$

5 Problem 5

5.1

$$P(X \leq x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 0, & \text{if } x < 0 \\ 1, & \text{otherwise} \end{cases}$$

$$P(X_n \leq x) = \begin{cases} \frac{\lfloor xn \rfloor}{n} & \text{if } 0 \leq x \leq 1 \\ 0, & \text{if } x < 0 \\ 1, & \text{otherwise} \end{cases}$$

Clearly, $\frac{xn-1}{n} \leq \frac{\lfloor xn \rfloor}{n} \leq \frac{xn}{n} = x$. As $n \to \infty$, $\frac{xn-1}{n} \to x$. Hence $\frac{\lfloor xn \rfloor}{n} \to x$. $P(X_n \leq x) \to P(X \leq x)$ as $n \to \infty$. We have $X_n \xrightarrow{d} X$.

5.2

For $\forall \epsilon > 0$, $P(|X_n - X| > \epsilon) \geq P(X_n > \epsilon + X) = \int_0^1 P(X_n > x + \epsilon)dx \geq \int_0^1 (1 - x - \epsilon)dx = x - x\epsilon - \frac{x^2}{2}|_0^1 = \frac{1}{2} - \epsilon$. This does not converge to 0 for $\epsilon < \frac{1}{2}$. Hence X_n does not converge to X in probability.