Appendix: Construction of the Simulation Model

Excerpts from Vercammen, J. "Hedging Agricultural Commodities: A Structural Analysis", draft paper, July 2019.

Basic Assumptions

The spot market consists of a set of competitive farmers producing a homogeneous commodity and selling this commodity to a set of competitive merchants. There are two marketing years, and each year is divided into four quarters/seasons: $n \in \{\text{fall}, \text{winter}, \text{spring}, \text{summer}\}$. Harvesting of the commodity takes place in fall, which is the first quarter/season of the marketing year. Year 1 harvest in quarter 1 (Q1) occurs with certainty at level H, and year 2 harvest in Q5 is uncertain, with a 50 percent chance that it will be the same size as the year 1 harvest, and a 50 percent chance that it will be lower at level $H - L$.[1] The size of L relative to a pre-determined threshold value, L^*, determines whether the year 1 market will stock out in Q4 (i.e., $S_5 = 0$) or whether inventory will be carried over from year 1 to year 2 (i.e., $S_5 > 0$). Note that S_t denotes the level of stocks which leave quarter $t - 1$ and arrive at quarter t. By assumption, stocks are zero prior to year 1 harvest, and are also zero when the market ceases to operate at the end of Q8. Thus, $S_0 = S_9 = 0$.

The analysis is simplified by assuming that demand is stable across the eight seasons. Specifically, inverse demand in quarter t is given by $P_t = a - bX_t$ where P_t is the market price and X_t is the level of consumption. The merchants' cost of storing the marginal unit of the commodity from one quarter to the next consists of a physical storage cost and an opportunity cost of the capital that is tied up in the inventory. The capital cost should depend on the commodity's price but this linkage is ignored in order to simply the analysis. Instead, assume the marginal overall cost of storage is given by the increasing function $k_t = k_0 + k_1 S_t$. This specification ensures that marginal storage costs are highest in the fall quarter when stocks are at a maximum and gradually decline as the marketing year progresses. Merchants also receive a convenience yield

[1]The analysis is simplified considerably by assuming that the size of the harvest is exogenous and thus not dependent on the price of the commodity.

from owning the stocks rather than having to purchase on short notice. [2] Let $c_t = c_0 - c_1 S_t$ denote the marginal convenience yield for quarter t. This function is a decreasing function of stocks because the transaction cost associated with external procurement is assumed to be highest (lowest) when stocks are lowest (highest). Combined storage cost and convenience yield is referred to as the carrying cost. Let $m_t = k_t - c_t$ denote the marginal carrying cost for period t.

Competition amongst merchants ensures that the expected compensation for supplying storage, $E\{P_{t+1}\} - P_t$, is equal to the net cost of carry, m_t, provided that stocks are positive (i.e., no stock out). Substituting in the expressions for k_t and c_t, the supply of storage equation can be written as

$$E\{P_{t+1}\} - P_t = m_0 + m_s S_t \tag{1}$$

where $m_0 = k_0 - c_0$ and $m_1 = k_1 + c_1$. If the market has stocked out because merchants are moving from a high-priced pre-harvest quarter to a low-priced post-harvest quarter then equation 1 holds as an inequality rather than an equality. **?** defines the demand for storage by first noting that period t consumption, X_t, can be written as $X_t = S_{t-1} + H_t - S_t$. Inverse demand in quarter t can therefore be expressed as $P_t = a - b(S_{t-1} + H_t - S_t)$, and the demand for storage function is

$$P_{t+1} - P_t = [a - b(S_t + H_{t+1} - S_{t+1})] - [a - b(S_{t-1} + H_t - S_t)] \tag{2}$$

Equation (2) shows that $P_{t+1} - P_t$ is a decreasing function of S_t and thus represents a demand for storage. **?**) explains that higher stocks carried out of period t as measured by S_t is associated with an increase in P_t (since less is available for consumption in period t) and a decrease in P_{t+1} (since more is available for consumption in period $t+1$).

The set of equations which describe the market equilibrium can be written as

$$E\{P_{t+1}\} - P_t \begin{cases} = m_0 + m_1 S_t & \text{if } S_t > 0; \\ < m_0 + m_1 S_t & \text{if } S_t = 0. \end{cases} \tag{3}$$

[2] A standard explanation of convenience yield is that by having stocks on hand a firm can fill unexpected sales orders or create sales opportunities that would otherwise not be possible due to the high transaction costs associated with short-notice spot market transactions.

$$P_t = a - bX_t \tag{4}$$

$$S_t = S_{t-1} - X_t \tag{5}$$

Equation (3) is the supply of storage, equation (4) is quarterly demand for the commodity and equation (5) is the equation of motion, which ensures that for those quarters without a harvest ending stocks must equal beginning stocks minus consumption.

For the first four quarters uncertainty has yet to be resolved and thus storage and consumption decisions for these quarters are based on the expected Q5 (post-harvest) price. For this reason there is one set of values for stocks, consumption and price in Q1 through Q4. For the last four quarters each endogenous variable has two values corresponding to whether the year 2 harvest outcome is normal (H) or low ($H - L$). The linearity of the various functions imply that merchants' expectation of the Q5 price when making storage decisions in Q1 through Q4 is the straight average of the Q5 price with a normal year 2 harvest and the Q5 price with a low year 2 harvest.

To simplify the notation in the analysis below let $Z = m_0/b$ and $m = m_1/b$. That is, Z and m are the intercept and slope, respectively, of the carrying cost function, each normalized by the slope of the inverse demand schedule. As well, in some parts of the analysis it is useful to substitute S_0 for H in year 1 because by assumption the level of stocks that are carried out of quarter 0 and into quarter 1 is equal to the year 1 harvest. Finally, let $H_2 \in \{H, H - L\}$ denote the two alternative values for the level of harvest in year 2.

Solution for Quarters 1 through 4

In the first four quarters uncertainty has yet to be resolved and so there is just one equilibrium value for price (P_t), stocks (S_t) and consumption (X_t). First quarter consumption, X_1, is initially treated as a parameter rather than a variable. This allows the endogenous variables in equations (3) through (5) from Section 2 to be solved as functions of time, t, year 1 harvest, H, and first quarter consumption, X_1. Specifically, substitute $P_t = a - bX_t$ and $P_{t+1} = a - bX_{t+1}$ from equation (4) into equation (3) to obtain:

$$X_{t+1} = X_t - Z - mS_t \tag{6}$$

After substituting in equation (5) for S_t in equation (6), the following expression for X_t emerges:

$$X_t = -mS_{t-2} + (1+m)X_{t-1} - Z \qquad\qquad t = 2, 3, 4 \tag{7}$$

Equations (5) and (7) can now be solved iteratively to obtain a complete solution. Begin with

$$S_1 = S_0 - X_1 \tag{8}$$

Equation (7) with $t - 2$ gives

$$X_2 = -mS_0 + (1+m)X_1 - Z \tag{9}$$

From equation (5) it follows that $S_2 = S_1 - X_2$. Substituting in equations (8) and (9) gives

$$S_2 = (1+m)S_0 - (2+m)X_1 + Z \tag{10}$$

Equation (7) with $t = 3$ gives $X_3 = -mS_1 + (1+m)X_2 - Z$. Substituting in equations (8) and (9) gives

$$X_3 = -m(2+m)S_0 + (m + (1+m)^2)X_1 - (2+m)Z \tag{11}$$

From equation (5) it follows that $S_3 = S_2 - X_3$. Substituting in equations (10) and (11) gives

$$S_3 = (1 + 3m + m^2)S_0 - (3 + 4m + m^2)X_1 + (3+m)Z \tag{12}$$

Equation (7) with $t = 4$ gives $X_4 = -mS_2 + (1+m)X_3 - Z$. Substituting in equations (10) and (11) gives

$$X_4 = -m(1+m)(3+m)S_0 + (1 + 6m + 5m^2 + m^3)X_1 - (3 + 4m + m^2)Z \tag{13}$$

From equation (5) it follows that $S_4 = S_3 - X_4$. Substituting in equations (12) and (13) gives

$$S_4 = (1 + 6m + 5m^2 + m^3)S_0 - (4 + 10m + 6m^2 + m^3)X_1 + (3+m)(2+m)Z \tag{14}$$

The next step is to derive an expression for the equilibrium value of X_1 which up until now has been treated as a parameter. Begin by defining $\hat{R} \in \{0, R\}$ as an indicator variable which defines the level of carryover from year 1 to year 2. For $L \leq L^*$ the year 1 market stocks out and $\hat{R} = 0$. Conversely, for $L > L^*$ the market does not stock out and $\hat{R} = R$ units are carried over (the equilibrium value for R is derived below). This specification implies $S_4 = \hat{R}$ (i.e., stocks which are not consumed in the summer of year 1 are carried over to year 2). The solution value for X_1 as a function of $\hat{R}$ can now be obtained by setting equation (14) equal to $\hat{R}$ and then solving the resulting equation for X_1. The desired expression is

$$X_1^* = \frac{\gamma_1 H + \theta_0 Z - \hat{R}}{\theta_1} \tag{15}$$

where:

$$\gamma_1 = 1 + 6m + 5m^2 + m^3 \tag{16}$$

$$\theta_0 = (3 + m)(2 + m) \tag{17}$$

$$\theta_1 = 4 + 10m + 6m^2 + m^3 \tag{18}$$

To finalize the solution for the first four quarters note that expressions for P_1 through P_4 can be derived by substituting the above expressions for X_1 through X_4 into the demand schedule, $P_t = a - bX_t$.

Solution for Final 4 Quarters

There exists a critical value for L, call it L^*, which determines whether stock will be carried over from year 1 to year 2. Specifically, for $L \leq L^*$ the price appreciation that results from the reduced year 2 supply is insufficient to cover the marginal carrying cost, and so zero carry over (i.e., a year 1 stockout) is optimal. A positive carry over is optimal if the year 2 production shortfall is sufficiently large (i.e., $L > L^*$). The binary nature of the problem implies four possibilities: (1) a year 1 stockout and a normal year 2 harvest; (2) a year 1 stockout and a low year 2 harvest; (3) positive carry over and a normal year 2 harvest; and (4) a positive carry over and a low year 2 harvest.

The previous set of equations with some minor adjustments can be used solve the problem for the final 4 quarters. Year 2 always stocks out and so in this case $\hat{R} = 0$. If year 1 stocks out due to $L \leq L^*$ and if year 2 harvest is normal (i.e., $H_2 = H$) then the conditional solution for the final 4 quarters, P_t^N, S_t^N and X_t^N, will be identical to the solution for the first four quarters.[3] If year 1 stocks out due to $L \leq L^*$ and if year 2 harvest is low (i.e., $H_2 = H - L$) then the conditional solution for the final 4 quarters, P_t^L, S_t^L and X_t^L, can be determined by using the previous set of equations with $H_2 = H - L$ substituting for $S_0 \equiv H$. If year 1 does not stock out due to $L > L^*$ then the solution for the last four quarters can be determined by using the previous set of equations except now $H + R$ substitutes for S_0 when year 2 harvest is normal, and $R + H - L$ substitutes for S_0 when year 2 harvest is low. If these adjustments are substituted into equation (15) then expressions for Q5 consumption conditioned on the outcome for R can be written as[4]

$$X_{1,2}^N = \frac{\gamma_1(R + H) + \theta_0 Z}{\theta_1} \qquad \text{and} \qquad X_{1,2}^L = \frac{\gamma_1(R + H - L) + \theta_0 Z}{\theta_1} \tag{19}$$

What remains is solving for the equilibrium value of R, which is the level of carry over when $L > L^*$, and solving for L^*, which is the threshold loss that triggers carry over. To ensure there is no arbitrage when there is positive carry over it must the case that $E\{P_{1,2}\} = P_{4,1} + m_0 + m_1 S_{4,1}$ where E is the expectations operator. In other words, the supply of storage must hold across years as well as within a year when stocks are carried over across years, and it must also hold with respect to the expected pricing outcome as well as the actual pricing outcome.

To make this no arbitrage condition more explicit substitute R for $S_{4,1}$ because both variables describe the level of carryover. Next note that the linearity of the demand equation and the equal probability for the two alternative harvest outcomes imply that the expected price in

[3]For the final 4 quarters a superscripted "N" on a variable indicates that year 2 harvest was normal (i.e., $H_2 = H$) and a superscripted "L" indicates that the year 2 harvest was low (i.e., $H_2 = H - L$).

[4]To distinguish the final 4 quarters from the first four quarters let the first subscript on a variable denote the quarter and the second subscript denote the year. For example, $X_{3,2}$ refers to consumption in the third quarter of the second year.

the first quarter of year 2 can be expressed as $E(P_{1,2}) = a - \frac{b}{2}\left(X^N_{1,2} + X^L_{1,2}\right)$. Using equation (19) this expression can be rewritten as

$$E(P_{1,2}) = a - \frac{b}{\theta_1}\left(\gamma_1 H + \theta_0 Z - 0.5\gamma_1 L + \gamma_1 R\right) \tag{20}$$

To construct an expression for the price in the fourth quarter of year 1, $P_{4,1}$, it is useful to rewrite the inverse demand equation as $P_{4,1} = a - b(X_{4,1} - \gamma_1 X^*_{4,1}) + b\gamma_1 X^*_{4,1}$. It follows from equation (13) that

$$X_{4,1} - \gamma_1 X^*_{1,1} = \rho_0 S_0 + \rho_1 \tag{21}$$

where

$$\rho_0 = -m(1+m)(3+m) \qquad \text{and} \qquad \rho_1 = -(3 + 4m + m^2)Z \tag{22}$$

Also note that the $X^*_{1,1}$ term in equation (21) is given by equation (15). Thus, an alternative expression for $P_{4,1}$ can be written as

$$P_{4,1} = a - b(\rho_0 S_0 + \rho_1) + b\gamma_1 X^*_{1,1} \tag{23}$$

To complete the no arbitrage condition substitute the expression for $E(P_{1,2})$ from equation (20 and the expression for $P_{4,1}$ from equation (23) into the no arbitrage equation, $E\{P_{1,2}\} = P_{4,1} + m_0 + m_1 S_{4,1}$. Under the assumption that $L > L^*$ so that some carry over is optimal, the no-arbitrage equation can be solved for the equilibrium carry over:

$$R^* = \frac{1}{m_1}\left[b(\rho_0 S_0 + \rho_1) + \frac{b\gamma_1}{2\theta_1}L - m_0 - b(\gamma_1 + 1)\left(\frac{\gamma_1 H + \theta_0 Z}{\theta_1}\right)\right] \tag{24}$$

The equilibrium value for L^* can now be obtained as the value of L which makes equation (24) vanish since this ensures a corner solution outcome where the market is indifferent between zero and positive carry over.