

Equation de Korteweg-de Vries

$$\frac{\partial u(x,t)}{\partial t} + c u(x,t) \frac{\partial u(x,t)}{\partial x} + \mu \frac{\partial^3 u(x,t)}{\partial x^3} = 0$$

On a :

$$(1) \quad \frac{\partial u(x,t)}{\partial t} + \varepsilon u(x,t) \frac{\partial u(x,t)}{\partial x} + \mu \frac{\partial^3 u(x,t)}{\partial x^3} = 0$$

$$(2) \quad \text{on pose } \xi = x - ct$$

$$u(x,t) = z(x-ct) = z(\xi)$$

$$\left\{ \begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial z}{\partial t} = \frac{dz}{d(x-ct)} \frac{\partial (x-ct)}{\partial t} = -c \frac{dz}{d\xi} \\ \frac{\partial u}{\partial x} &= \frac{\partial z}{\partial x} = \frac{dz}{d(x-ct)} \frac{\partial (x-ct)}{\partial x} = \frac{dz}{d\xi} \\ \frac{\partial^3 u}{\partial x^3} &= \left( \frac{\partial}{\partial x} \right)^3 u = \frac{d^3 z}{d\xi^3} \end{aligned} \right.$$

On substitue (2) à (1) et on obtient :

$$-c \frac{dz(\xi)}{d\xi} + \varepsilon z(\xi) \frac{dz(\xi)}{d\xi} + \mu \frac{d^3 z(\xi)}{d\xi^3} = 0$$

Transformation de la dérivée totale en dérivée partielle

$$C_1 = -c \int dz + \varepsilon \overset{\frac{1}{2} z^2}{\int dz z} + \mu \int \frac{d^3 z(\xi)}{d\xi^3}$$

$$C_1 = -c z + \frac{\varepsilon}{2} z^2 + \mu \frac{d^2 z(\xi)}{d\xi^2}$$

$$C_1 = -c_3 + \mu \frac{d^2 z}{d\varphi^2} + \frac{\varepsilon}{2} z^2$$

constante d'intégration

on multiplie par  $\frac{dz}{d\varphi}$

$$C_1 \frac{dz}{d\varphi} = -c_3 \frac{dz}{d\varphi} + \frac{\varepsilon}{2} z^2 \frac{dz}{d\varphi} + \mu \frac{d^2 z}{d\varphi^2} \cdot \frac{dz}{d\varphi}$$

$$C_1 dz = -c_3 dz + \frac{\varepsilon}{2} z^2 dz + \mu \frac{d^2 z}{d\varphi^2} dz$$

on intègre des deux côtés et on pose  $C_2$  constante d'intégration

$$C_1 \int dz = -c \int z dz + \frac{\varepsilon}{2} \int z^2 dz + \mu \int \frac{d^2 z}{d\varphi^2} dz$$

$$C_1 z + C_2 = -\frac{c}{2} z^2 + \frac{\varepsilon}{6} z^3 + \frac{1}{2} \mu \left( \frac{dz}{d\varphi} \right)^2$$

Lorsque  $x \rightarrow \pm \infty$  on a  $C_1 = C_2 = 0$

(Pourquoi?)  
↓  
à comprendre

On a donc

$$0 = -\frac{c}{2} z^2 + \frac{\varepsilon}{6} z^3 + \frac{1}{2} \mu \left( \frac{dz}{d\varphi} \right)^2$$

$$-\frac{1}{2} \mu \left( \frac{dz}{d\varphi} \right)^2 = -\frac{c}{2} z^2 + \frac{\varepsilon}{6} z^3$$

$$\left( \frac{dz}{d\varphi} \right)^2 = \frac{c}{\mu} z^2 - \frac{\varepsilon}{3\mu} z^3$$

$$\left( \frac{dz}{d\varphi} \right)^2 = z^2 \left( \frac{c}{\mu} - \frac{\varepsilon}{3\mu} z \right)$$

Méthode de  
la séparation  
des variables

$$\left(\frac{dz}{d\varphi}\right)^2 = z^2 \left(c - \frac{E}{3}z\right)$$

$$\frac{dz}{d\varphi} = z \sqrt{c - \frac{E}{3}z}$$

$$\frac{d\varphi}{dz} = \frac{1}{z \sqrt{c - \frac{E}{3}z}}$$

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$$\text{soit } \begin{cases} z = \frac{3\mu}{E} g^2 \\ dz = \frac{6\mu}{E} g dg \end{cases}$$

$$\frac{6\mu g dg}{E \frac{3\mu}{E} g^2 \sqrt{\frac{c}{\mu} - g^2}} = d\varphi$$

On intègre des deux côtés

$$\int_0^g \frac{6\mu n dn}{E \frac{3\mu}{E} n^2 \sqrt{\frac{c}{\mu} - n^2}} = \int_{\varphi_0}^{\varphi} d\varphi$$

$$2 \int_0^g \frac{dn}{n \sqrt{\frac{c}{\mu} - n^2}} = (\varphi - \varphi_0)$$

$$\int \frac{dx}{x \sqrt{a^2 - x^2}} = \frac{1}{a} \operatorname{sech}^{-1}\left(\frac{x}{a}\right)$$

séparation  
des variables

$$\frac{dy}{dx} = g(x)h(y)$$

$$\Rightarrow \frac{dy}{h(y)} = g(x)dx$$

$$g(x) = 1$$

$$h(y) = \sqrt{c - \frac{E}{3}y}$$

$$2 \int_0^g \frac{d\eta}{\eta \sqrt{\frac{c}{\mu} - \eta^2}} = (\psi - \psi_0) \quad \int \frac{dx}{x \sqrt{a^2 - x^2}} = \frac{1}{a} \operatorname{sech}^{-1}\left(\frac{x}{a}\right)$$

$$2 \left( -\frac{1}{\sqrt{\frac{c}{\mu}}} \operatorname{sech}^{-1}\left(\frac{g}{\sqrt{\frac{c}{\mu}}}\right) \right) = (\psi - \psi_0)$$

$$\operatorname{sech}^{-1}\left(\frac{g}{\sqrt{\frac{c}{\mu}}}\right) = \frac{(\psi - \psi_0)}{2} \cdot \sqrt{\frac{c}{\mu}}$$

$$g = \operatorname{sech}\left(\frac{(\psi - \psi_0)}{2} \sqrt{\frac{c}{\mu}}\right) \cdot \sqrt{\frac{c}{\mu}}$$

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$$\boxed{z = \frac{3\mu}{\varepsilon} g^2 = \frac{3c}{\varepsilon} \operatorname{sech}^2\left(\frac{1}{2} \sqrt{\frac{c}{\mu}} (\psi - \psi_0)\right)}$$