Equation de Korteweg-de Vries

$$\frac{\partial u(x,t)}{\partial t} + \varepsilon\, u(x,t)\,\frac{\partial u(x,t)}{\partial x} + \mu\,\frac{\partial^3 u(x,t)}{\partial x^3} = 0$$

On a :

(1) $$\frac{\partial u(x,t)}{\partial t} + \varepsilon u(x,t)\frac{\partial u(x,t)}{\partial x} + \mu \frac{\partial^3 u(x,t)}{\partial x^3} = 0$$

(2) on pose $\xi = x - ct$

$$u(x,t) = z(x-ct) = z(\xi)$$

$$\frac{\partial u}{\partial t} = \frac{\partial z}{\partial t} = \frac{dz}{d(x-ct)}\frac{\partial(x-ct)}{\partial t} = -c\frac{dz}{d\xi}$$

$$\frac{\partial u}{\partial x} = \frac{\partial z}{\partial x} = \frac{dz}{d(x-ct)}\frac{\partial(x-ct)}{\partial x} = \frac{dz}{d\xi}$$

$$\frac{\partial^3 u}{\partial x^3} = \left(\frac{\partial}{\partial x}\right)^3 u = \frac{d^3 z}{\partial \xi^3}$$

On substitue (2) à (1) et on obtient :

$$-c\frac{dz(\xi)}{d\xi} + \varepsilon z(\xi)\frac{dz(\xi)}{d\xi} + \mu\frac{d^3 z(\xi)}{d\xi^3} = 0$$

Transformation de la dérivée totale en dérivée partielle

$$C_1 = -c\int dz + \varepsilon \underbrace{\int dz\, z}_{\to \frac{1}{2}z^2} + \mu\int \frac{d^3 z(\xi)}{d^3\xi}$$

$$C_1 = -cz + \frac{\varepsilon}{2}z^2 + \mu\frac{d^2 z(\xi)}{d\xi^2}$$

$$C_1 = -cz + \mu \frac{d^2z}{d\varphi^2} + \frac{\varepsilon}{2} z^2$$

constante d'intégration

on multiplie par $\frac{dz}{d\varphi}$

$$C_1 \frac{dz}{d\varphi} = -cz\frac{dz}{d\varphi} + \frac{\varepsilon}{2} z^2 \frac{dz}{d\varphi} + \mu \frac{d^2z}{d\varphi^2} \cdot \frac{dz}{d\varphi}$$

$$C_1 \, dz = -cz\,dz + \frac{\varepsilon}{2} z^2 dz + \mu \frac{d^2z}{d\varphi^2} dz$$

on integre des deux cotés et on pose C_2 cste d'integration

$$C_1 \int dz = -c\int z\,dz + \frac{\varepsilon}{2}\int z^2 dz + \mu \int \frac{d^2z}{d\varphi^2} dz$$

$$C_1 z + C_2 = -\frac{c}{2} z^2 + \frac{\varepsilon}{6} z^3 + \frac{1}{2}\mu\left(\frac{dz}{d\varphi}\right)^2$$

Lorsque $x \longrightarrow \pm\infty$ on a $C_1 = C_2 = 0$ (Pourquoi ?)

à comprendre

On a donc

$$0 = -\frac{c}{2} z^2 + \frac{\varepsilon}{6} z^3 + \frac{1}{2}\mu\left(\frac{dz}{d\varphi}\right)^2$$

$$-\frac{1}{2}\mu\left(\frac{dz}{d\varphi}\right)^2 = -\frac{c}{2} z^2 + \frac{\varepsilon}{6} z^3$$

$$\left(\frac{dz}{d\varphi}\right)^2 = \frac{c}{\mu} z^2 - \frac{\varepsilon}{3\mu} z^3$$

$$\left(\frac{dz}{d\varphi}\right)^2 = z^2\left(\frac{c}{\mu} - \frac{\varepsilon}{3\mu} z\right)$$

Methode de la séparation des variables

$$\left(\frac{dz}{d\xi}\right)^2 = z^2\left(\frac{c}{\mu} - \frac{\varepsilon}{3\mu z}\right)$$

$$\frac{dz}{d\xi} = z\sqrt{\left(\frac{c}{\mu} - \frac{\varepsilon}{3\mu z}\right)}$$

$$\frac{d\varphi}{d\xi} = \varphi\sqrt{\left(\frac{c}{\mu} - \frac{\varepsilon}{3\mu}\varphi\right)}$$

$$\frac{d\varphi}{\varphi\sqrt{\frac{c}{\mu} - \frac{\varepsilon}{3\mu}\varphi}} = d\xi$$

$$\frac{dz}{z\sqrt{\frac{c}{\mu} - \frac{\varepsilon}{3\mu}z}} = d\xi$$

Soit $\begin{cases} z = \dfrac{3\mu}{\varepsilon} g^2 \\ dz = \dfrac{6\mu}{\varepsilon} g\,dg \end{cases}$

$$\frac{6\mu g\,dg}{\varepsilon\frac{3\mu}{\varepsilon}g^2\sqrt{\frac{c}{\mu} - g^2}} = d\xi$$

On integre des deux côtés

$$\int_0^g \frac{6\mu\eta\,d\eta}{\varepsilon\frac{3\mu}{\varepsilon}\eta^2\sqrt{\frac{c}{\mu} - \eta^2}} = \int_{\xi_0}^{\xi} d\xi$$

$$2\int_0^g \frac{d\eta}{\eta\sqrt{\frac{c}{\mu} - \eta^2}} = (\xi - \xi_0)$$

$$\int \frac{dx}{x\sqrt{a^2 - x^2}} = \frac{1}{a}\,\mathrm{sech}^{-1}\left(\frac{x}{a}\right)$$

Separation des variables

$$\frac{dy}{dx} = g(x)h(y)$$

$$\Rightarrow \frac{dy}{h(y)} = g(x)dx$$

$$g(\xi) = 1$$

$$h(\eta) = \eta\sqrt{\left(c - \frac{2}{3}\eta\right)}$$

$$2\int_0^g \frac{d\eta}{\eta\sqrt{\frac{c}{\mu}-\eta^2}} = (\xi - \xi_0) \qquad \int \frac{dx}{x\sqrt{a^2-x^2}} = \frac{1}{a}\,\text{sech}^{-1}\left(\frac{x}{a}\right)$$

$$2\left(-\frac{1}{\sqrt{\frac{c}{\mu}}}\,\text{sech}^{-1}\left(\frac{g}{\sqrt{\frac{c}{\mu}}}\right)\right) = (\xi - \xi_0)$$

$$\text{sech}^{-1}\left(\frac{g}{\sqrt{\frac{c}{\mu}}}\right) = \frac{(\xi - \xi_0)}{2}\cdot\sqrt{\frac{c}{\mu}}$$

$$g = \text{sech}\left(\frac{(\xi - \xi_0)}{2}\sqrt{\frac{c}{\mu}}\right)\cdot\sqrt{\frac{c}{\mu}}$$

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$$\boxed{\zeta = \frac{3\mu}{\varepsilon}g^2 = \frac{3c}{\varepsilon}\,\text{sech}^2\left(\frac{1}{2}\sqrt{\frac{c}{\mu}}(\xi - \xi_0)\right)}$$