

Equation de Korteweg-de Vries

$$\frac{\partial u(x,t)}{\partial t} + c u(x,t) \frac{\partial u(x,t)}{\partial x} + \mu \frac{\partial^3 u(x,t)}{\partial x^3} = 0$$

On a :

$$(1) \quad \frac{\partial u(x,t)}{\partial t} + \varepsilon u(x,t) \frac{\partial u(x,t)}{\partial x} + \frac{\partial^3 u(x,t)}{\partial x^3} = 0$$

$$(2) \quad \text{on pose } \varphi = x - ct$$

$$u(x,t) = z(x-ct) = z(\varphi)$$

$$\left\{ \begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial z}{\partial t} = \frac{dz}{d(x-ct)} \frac{\partial(x-ct)}{\partial t} = -c \frac{dz}{d\varphi} \\ \frac{\partial u}{\partial x} &= \frac{\partial z}{\partial x} = \frac{dz}{d(x-ct)} \frac{\partial(x-ct)}{\partial x} = \frac{dz}{d\varphi} \\ \frac{\partial^3 u}{\partial x^3} &= \left(\frac{\partial}{\partial x} \right)^3 u = \frac{d^3 z}{d\varphi^3} \end{aligned} \right.$$

On substitue (2) à (1) et on obtient :

$$-c \frac{dz(\varphi)}{d\varphi} + \varepsilon z(\varphi) \frac{dz(\varphi)}{d\varphi} + \frac{d^3 z(\varphi)}{d\varphi^3} = 0$$

Transformation de la dérivée totale en dérivée partielle

$$C_1 = -c \int dz + \varepsilon \overset{\frac{1}{2} z^2}{\int dz z} + \int \frac{d^3 z(\varphi)}{d\varphi^3}$$

$$C_1 = -c z + \frac{\varepsilon}{2} z^2 + \frac{d^2 z(\varphi)}{d\varphi^2}$$

$$\text{constante d'intégration} \Rightarrow C_1 = -C_3 + \frac{d^2 z}{d\varphi^2} + \frac{\varepsilon}{2} z^2$$

on multiplie par $\frac{dz}{d\varphi}$

$$C_1 \frac{dz}{d\varphi} = -C_3 \frac{dz}{d\varphi} + \frac{\varepsilon}{2} z^2 \frac{dz}{d\varphi} + \frac{d^2 z}{d\varphi^2} \cdot \frac{dz}{d\varphi}$$

$$C_1 dz = -C_3 dz + \frac{\varepsilon}{2} z^2 dz + \frac{d^2 z}{d\varphi^2} dz$$

on intègre des deux côtés et on pose C_2 constante d'intégration

$$C_1 \int dz = -C_3 \int dz + \frac{\varepsilon}{2} \int z^2 dz + \int \frac{d^2 z}{d\varphi^2} dz$$

$$C_1 z + C_2 = -\frac{C_3}{2} z^2 + \frac{\varepsilon}{6} z^3 + \frac{1}{2} \left(\frac{dz}{d\varphi} \right)^2$$

Lorsque $x \rightarrow \pm \infty$ on a $C_1 = C_2 = 0$

(Pourquoi?)
↓
à comprendre

On a donc

$$0 = -\frac{C_3}{2} z^2 + \frac{\varepsilon}{6} z^3 + \frac{1}{2} \left(\frac{dz}{d\varphi} \right)^2$$

$$-\frac{1}{2} \left(\frac{dz}{d\varphi} \right)^2 = -\frac{C_3}{2} z^2 + \frac{\varepsilon}{6} z^3$$

$$\left(\frac{dz}{d\varphi} \right)^2 = C_3 z^2 - \frac{\varepsilon}{3} z^3$$

$$\left(\frac{dz}{d\varphi} \right)^2 = z^2 \left(C_3 - \frac{\varepsilon}{3} z \right)$$

Méthode de la séparation des variables

$$\left(\frac{dz}{d\varphi}\right)^2 = z^2 \left(c - \frac{\varepsilon}{3} z\right)$$

$$\frac{dz}{d\varphi} = z \sqrt{c - \frac{\varepsilon}{3} z}$$

$$\frac{d\eta}{d\varphi} = \eta \sqrt{c - \frac{\varepsilon}{3} \eta}$$

$$d\eta = \left(\eta \sqrt{c - \frac{\varepsilon}{3} \eta}\right) d\varphi$$

Séparation des variables

$$\frac{dy}{dx} = g(x)h(y)$$

$$\Rightarrow \frac{dy}{h(y)} = g(x)dx$$

$$g(x) = 1$$

$$h(\eta) = \eta \sqrt{c - \frac{\varepsilon}{3} \eta}$$

On intègre des 2 côtés

$$\frac{d\eta}{\eta \sqrt{c - \frac{\varepsilon}{3} \eta}} = d\varphi$$

$$\int \frac{d\eta}{\eta \sqrt{c - \frac{\varepsilon}{3} \eta}} = \int_{\varphi_0}^{\varphi} d\varphi$$

Soit
$$\begin{cases} \eta = \frac{3}{\varepsilon} g^2 \\ d\eta = \frac{6}{\varepsilon} g dg \end{cases}$$

$$\int \frac{dx}{x \sqrt{a^2 - x^2}} = \frac{1}{a} \operatorname{sech}^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{6g dg}{\varepsilon \frac{3}{\varepsilon} g^2 \sqrt{c - g^2}} = (\varphi - \varphi_0)$$

$$2 \int \frac{dg}{g \sqrt{c - g^2}} = (\varphi - \varphi_0)$$

cte d'intégration

$$2 \left(-\frac{1}{\sqrt{c}} \operatorname{sech}^{-1}\left(\frac{g}{\sqrt{c}}\right) \right) + \text{cte} = (\varphi - \varphi_0)$$

$$\operatorname{sech}^{-1}\left(\frac{g}{\sqrt{c}}\right) = \frac{(\varphi - \varphi_0) + \text{cte}}{2} \cdot \sqrt{c}$$

$$g = \operatorname{sech}\left(\frac{(\varphi - \varphi_0) + \text{cte}}{2} \sqrt{c}\right) \sqrt{c}$$

$$g = \operatorname{sech}\left(\frac{(\psi - \xi_0) + cste}{2} \sqrt{c}\right) \sqrt{c}$$

$$z = \eta = \frac{3}{\varepsilon} g^2 = \frac{3c}{\varepsilon} \operatorname{sech}^2\left(\frac{1}{2}((\psi - \xi_0) + cste) \sqrt{c}\right)$$

$$((\psi - \xi_0) + cste) \sqrt{c} = (\xi - \xi_0) \frac{\sqrt{c}}{\sqrt{\mu}}$$

$$(\xi - \xi_0) \sqrt{c} + \sqrt{c} cste = (\xi - \xi_0) \frac{\sqrt{c}}{\sqrt{\mu}}$$

$$(\xi - \xi_0) + cste = \frac{(\xi - \xi_0)}{\sqrt{\mu}}$$

$$cste = \frac{(\xi - \xi_0)}{\sqrt{\mu}} - (\xi - \xi_0)$$

$$cste = \frac{(\xi - \xi_0) \sqrt{\mu} (\xi - \xi_0)}{\sqrt{\mu}}$$

avec
on obtient

$$cste = \frac{(\xi - \xi_0)(1 - \sqrt{\mu})}{\sqrt{\mu}}$$

$$z = \frac{3c}{\varepsilon} \operatorname{sech}^2\left(\frac{1}{2} \sqrt{\frac{c}{\mu}} (\xi - \xi_0)\right)$$