The General Sieve Kernel and New Records in Lattice Reduction

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(Lattice) Sieving: What and Why?

Sieves:

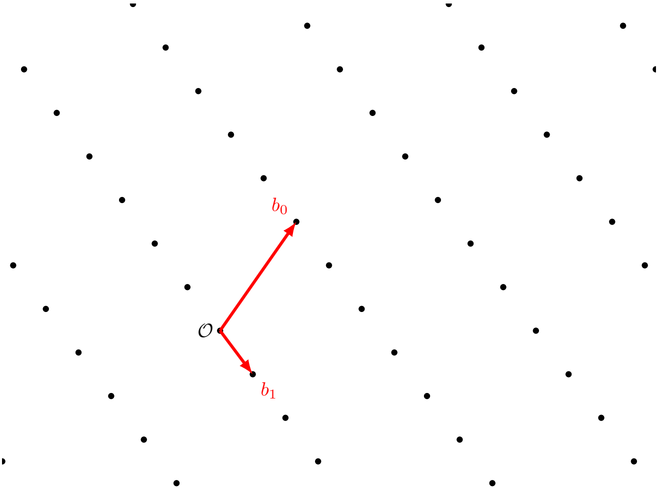
- take as input a description of a lattice (a “basis”),
- output a database of short vectors in that lattice.

Short vectors are critical in lattice reduction and in more general cryptanalysis.

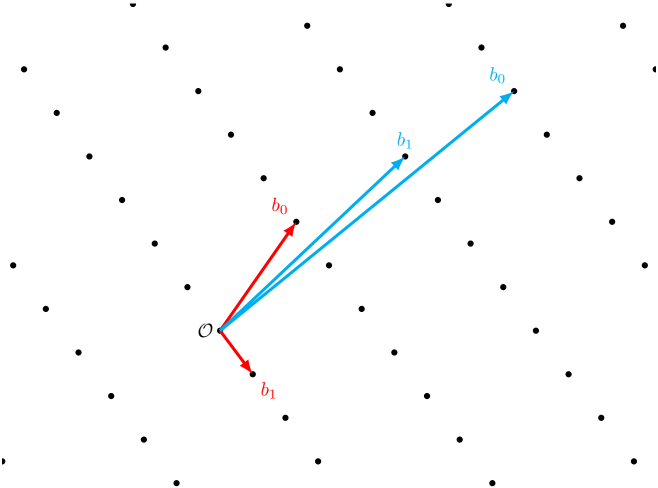
(Lattice) Sieving: What and Why?

A single exponential, in lattice dimension d, time and memory short vector finder.

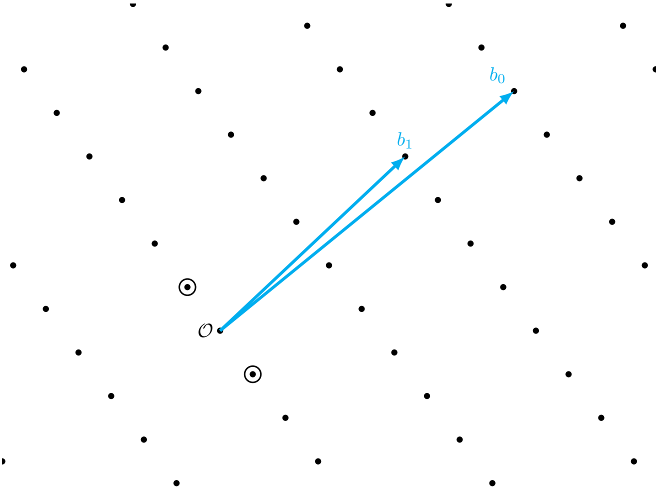
algorithm	time	memory
sieving	$\exp(\Theta(d))$	$\exp(\Theta(d))$
enumeration	$\exp(\Theta(d \log d))$	$\mathrm{poly}(d)$



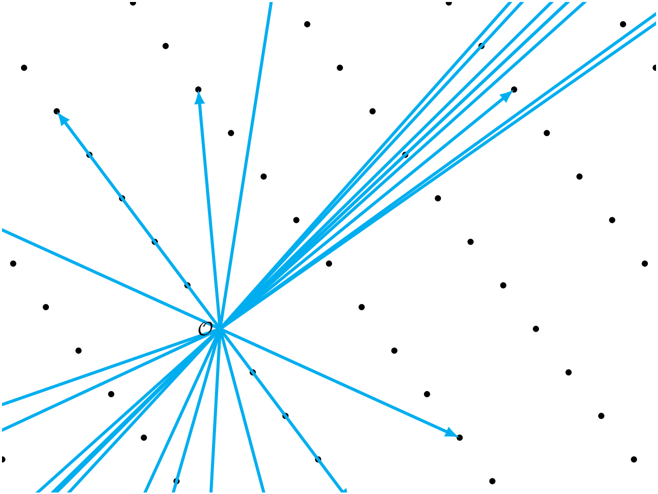
$\Lambda = \mathrm{Span}_{\mathbb{Z}}(b_0, \ldots, b_{d-1})$, $B = \{b_0, \ldots, b_{d-1}\} \subset \mathbb{R}^d$ basis



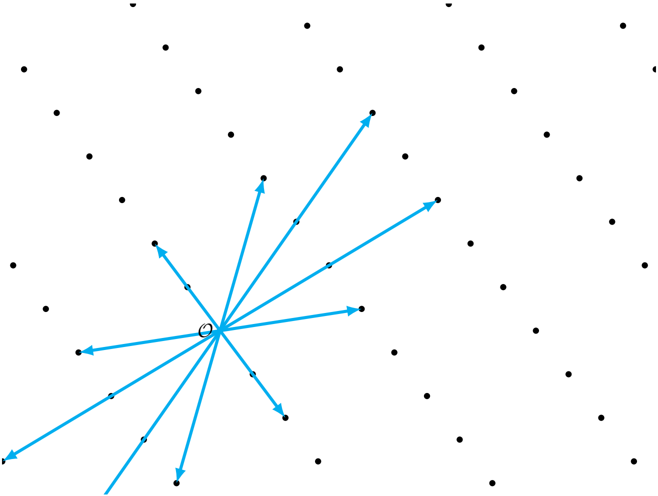
Good basis B, bad basis B



SVP: find $v \in \Lambda \setminus \{0\}$ such that $\|v\| \leq \|w\|$ for all $w \in \Lambda \setminus \{0\}$



Sieve: $v, w \in L \subset \Lambda$, if $\|v \pm w\| \leq \|v\|$, $v \leftarrow v \pm w$



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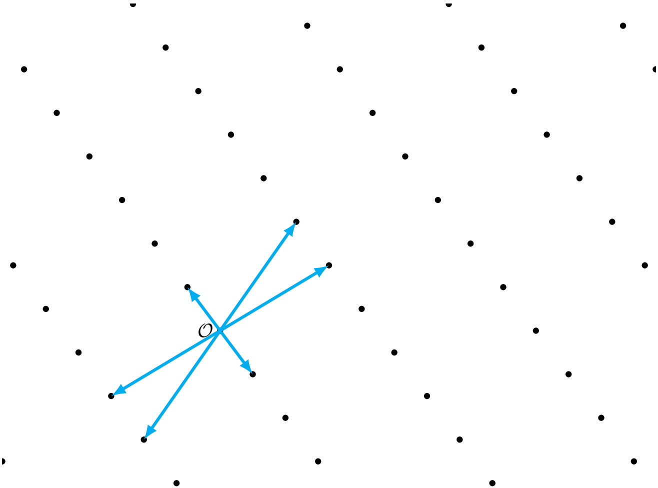

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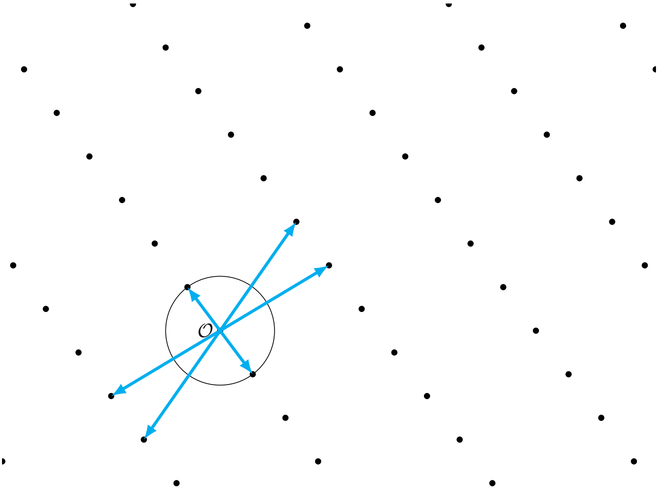

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Intuition

A sieve outputs many short vectors, but only the shortest is used.

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[1] L. Babai, On lovász' lattice reduction and the nearest lattice point problem, Combinatorica 6 (1986), no. 1, 1–13.

[2] Léo Ducas, Shortest vector from lattice sieving: A few dimensions for free, EUROCRYPT 2018, Part I (Jesper Buus Nielsen and Vincent Rijmen, eds.), LNCS, vol. 10820, Springer, Heidelberg, April / May 2018, pp. 125–145.

[3] Thijs Laarhoven and Artur Mariano, Progressive lattice sieving, Post-Quantum Cryptography – 9th International Conference, PQCrypto 2018 (Tanja Lange and Rainer Steinwandt, eds.), Springer, Heidelberg, 2018, pp. 292–311.

Intuition

A sieve outputs many short vectors, but only the shortest is used. Instead

- sieve in (projected) sublattices and lift[1] to the full lattice[2]
- sieve in sublattices to "seed" higher dimensional sieves[3]

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Intuition II

Recycle information between related lattices

and

Go beyond sieving as black box oracle for shortest vectors

Implicitly,

sieve $\leftrightarrow$ stateful machine

Contributions

In our paper we give

- a framework for treating sieves as stateful machines,
- an open source, $\{$documented, optimised, tweakable$\}$ implementation, `https://github.com/fplll/g6k`,
- a variety of new strategies for lattice reduction tasks.

Contributions

In our paper we give

- a framework for treating sieves as stateful machines,
- an open source, {documented, optimised, tweakable} implementation, `https://github.com/fplll/g6k`,
- a variety of new strategies for lattice reduction tasks.

We are therefore able to

- show sieving outperforms enumeration by low dimensions,
- show that SVP can be (slightly) amortised within BKZ,
- break a number of lattice challenge records.

The General Sieve Kernel, or
G6K (pronounced ʒe.si.ka)

A Simple Sieve

We use a grammar to define our sieving operations.

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$\text{Reset}_{0,d}$ S I_0

Key: sieve, lift, c_i = insert

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$$(b_0, b_1, b_2, \ldots, b_{d-3}, b_{d-2}, b_{d-1})$$

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A Progressive Sieve [LM18]

$\mathrm{Reset}_{0,1}\ (\mathrm{ER\ S})^{d-1}\ \mathrm{I}_0$

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$\mathrm{Reset}_{0,1}\ (\mathrm{ER\ S})^{d-1}\ \mathrm{I}_0$

$$({\color{red}b_0}, {\color{red}b_1}, b_2, \ldots, b_{d-3}, b_{d-2}, b_{d-1})$$

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A "Dimensions for Free" Sieve [Duc18]

$\mathrm{Reset}_{f,f+1}$ $(\mathrm{E}\,\mathrm{R}\,\mathrm{S})^{d-f-1}$ I_0 I_1 $\ldots$

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$$\left(b_0, b_1, \ldots, b_{f-1}, b_f, b_{f+1}, \ldots, b_{d-2}, \color{red}{b_{d-1}}\right)$$

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Principles, Sieves and Tweaks

G6K has three high level design principles, to

- recycle (short) vectors between lattices,
- lift vectors, on the fly, to higher dimensional lattices,
- decide the insertion position only *after* sieving.

We implement

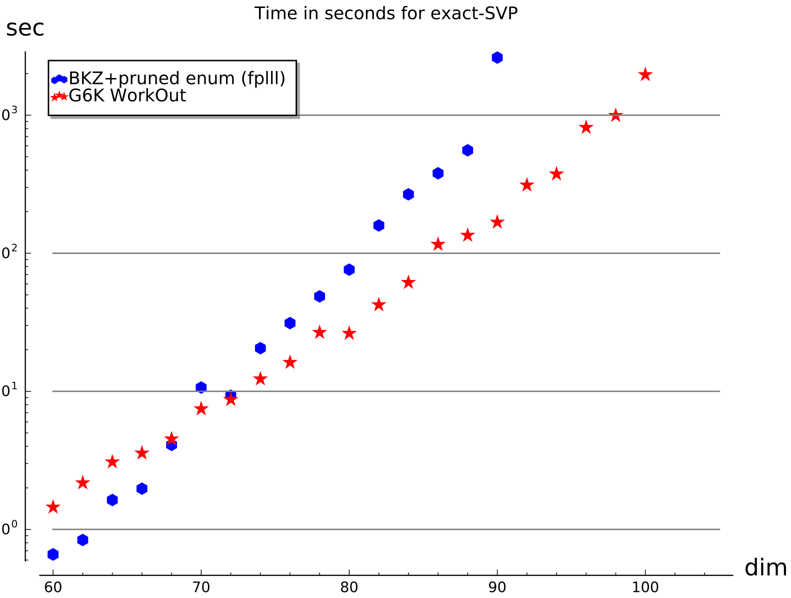
- `bgj1` [BGJ15] and `triple_sieve` [BLS16, HK17],

and make use of algorithmic tweaks

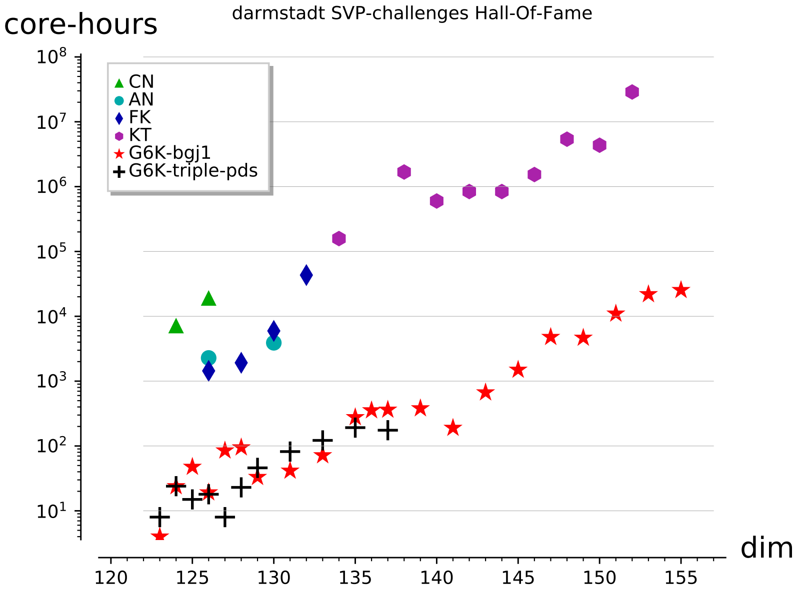
- `XOR-POPCNT`, non terminal insertion, opportunistic dimensions for free, a new database replacement condition...

Records

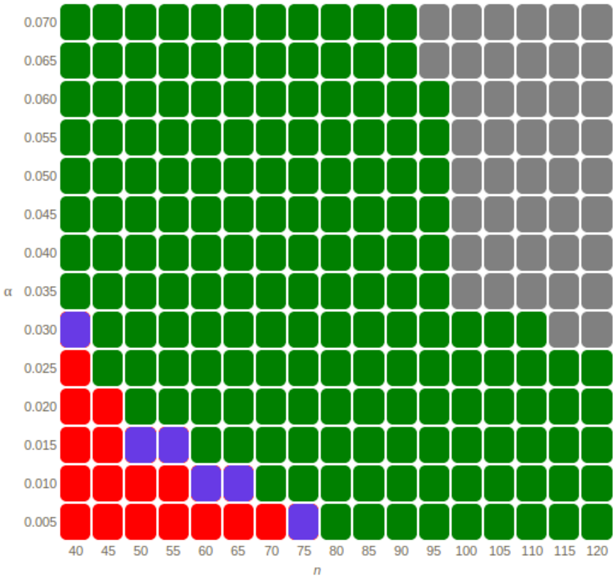
Exact SVP, `Workout` vs. Enumeration (FPLLL) [dt16]



Hermite SVP (Darmstadt Challenges), with `Workout`



LWE (Darmstadt Challenges), with `Pump&JumpBKZ`



Implementation

We have three layers

- c++: multithreaded, heavy operations (sieves, *db* updates)
- cython: middleware, basis maintainance
- python: control, tuning, monitoring

Thanks!

Questions?

L. Babai, *On lovász' lattice reduction and the nearest lattice point problem*, Combinatorica **6** (1986), no. 1, 1–13.

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