Internship Report

Jean Vereecke

École Normale Supérieure de Rennes (France)

Under the supervision of:
Prof. Dr. Barbara Gentz
University of Bielefeld (Germany)

August 27, 2023

Contents

Introduction

The aim of this internship report is to understand the Adler equation connected with random noise. To do this, we'll learn about things like continuous-time stochastic processes, Brownian motion, Itô integrals, and stochastic differential equations.
Because there's a lot to cover in these areas, most of the details about stochastic processes are in a separate section. At first, we'll focus on explaining Brownian motion and some of its features.
After that, we'll talk about Itô integrals and stochastic differential equations in the next part. This is important groundwork for the last section, where we'll discuss stochastic processes and study the Adler equation.
In the additional section, you can find the step-by-step instructions for simulation, helpful pictures, and extra information that adds to the main points in the report.

Notations

- Let $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_+^*$ denote respectively the set of all real numbers, the set of all nonnegative numbers and the set of all positive numbers. Let $\mathbb{N}$ and $\mathbb{N}^*$ denote respectively the set of all nonnegative integers and the set of all positive integers.
- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.
- Let $(E, \mathcal{E})$ be a mesurable space and let $I \subset \mathbb{R}_+$ denote an interval.
  - Let E^I be the set of all applications from I to E.
  - Let us denote $\mathcal{E}^I = \mathcal{E}^{\otimes I}$. In the case that $(E, \mathcal{E}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, we will denote $\mathcal{B}^I = (\mathcal{B}(\mathbb{R}))^I$.
  - For $n \in \mathbb{N}^*$ and all choices of $t = (t_1, \dots, t_n) \in I^n$, let us define the map $\pi_t : E^I \ni f \mapsto (f(t_1), \dots, f(t_n)) \ni E^n$.
  - For $n \in \mathbb{N}^*$, for $j \in [\![1, n]\!]$, $\varphi_j^{(n)} : E^n \ni (x_1, \dots, x_n) \mapsto (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \in E^{n-1}$.
- For $f, g \in \left(\mathbb{R}_+^*\right)^{\mathbb{R}_+^*}$, if $\ln f(\epsilon) \underset{\epsilon \to 0}{\sim} \ln g(\epsilon)$, we denote $f(\epsilon) \underset{\epsilon \to 0}{\asymp} g(\epsilon)$.

1 Brownian motion

In this section, we will define the notion of Brownian motion. In order to do so, we will need some notions of stochastic processes that are recalled in the appendix in section C.

1.1 Definition and existence of Brownian motion

In this subsection, we will define Brownian motion and prove its existence. Nowadays, Brownian motion is defined as an $\mathbb{R}$-valued stochastic process $(W_t)_{t \in \mathbb{R}_+}$ with the following properties:

$W_0 = 0$ $\mathbb{P}$-almost surely ;	$(W_t)_{t \in \mathbb{R}_+}$ is a Gaussian process ;	$(W_t)_{t \in \mathbb{R}_+}$ has indepedent increments ;
$(W_t)_{t \in \mathbb{R}_+}$ have stationary increments[1] ;	for all $t \geq 0$, $W_t \sim \mathcal{N}(0, t)$;	$(W_t)_{t \in \mathbb{R}_+}$ have continuous sample paths.

We will define it as follows in definition 1.

Definition 1 (Brownian motion in $\mathbb{R}$) — An $\mathbb{R}$-valued centred Gaussian process $X = (X_t)_{t \in \mathbb{R}_t}$ with covariance function

$$\forall s, t \in \mathbb{R}_+, \sigma(s,t) = s \wedge t$$

and continuous sample paths is called a (standard) Brownian motion.

Figure 1 shows, for a Brownian motion on $\mathbb{R}$ and fixed choice of $\omega \in \Omega$, the corresponding graph of the maps $t \mapsto X(\omega)$.

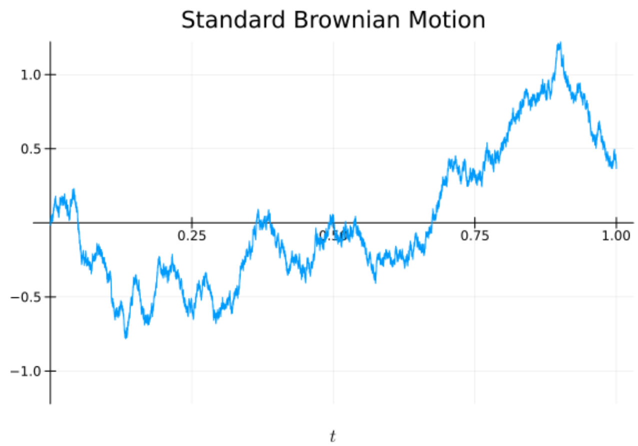


Figure 1: A sample path of a Brownian motion

Then we have this result thanks to theorem 63 and 66 (the proof is in the appendix).

Corollary 2 (Brownian motion exists) — A centred Gaussian process with the covariance function of Brownian motion and continuous sample paths exists *ie* Brownian motion exists.

Since we will need to work with filtrations and adapted processes, we need to be more precise with which Brownian motion we will need. Let $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$ be a filtration of $\mathcal{F}$. We will henceforth define what is an $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$-Brownian motion.

Definition 3 ($\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$-Brownian motion) — A centred Gaussian process W with continuous sample paths and covariance function σ which satisfies for all $s, t \in \mathbb{R}_+$, $\sigma(s,t) = s \wedge t$ is called a $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$-Brownian motion if and only if

(i) W is $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$-adapted ;

(ii) W has independent increments with respect to $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$.

Remark 1. For any Brownian motion W, if $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$ is $\{\mathcal{F}_t^W\}_{t \in \mathbb{R}_+}$ or $\{\mathcal{F}_{t+}^W\}_{t \in \mathbb{R}_+}$, W is an $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$-Brownian motion.

1.2 Construction of the Brownian motion

In the previous subsection, we have seen that Brownian motion exists as a Gaussian process with continuous sample paths. However, if we want to establish important properties of the Brownian motion, we will need a different approach. Thus we will need to construct the Brownian motion in an other way. We will propose two different ways to do so.

1.2.1 Wiener space and the Lévy-Ciesielski construction of Brownian motion

In this subsubsection, we will present the Lévy-Ciesielski construction of Brownian motion. We will first construct a Brownian motion on the time interval $I = [0, 1]$ and extend it to $I = \mathbb{R}_+$ afterwards.
One of the first problem we can see with the previous result is the fact that we constructed the Brownian motion W as an $\mathbb{R}^{\mathbb{R}_+}$-valued random element and the sets $\{W \in \mathcal{C}^0(\mathbb{R}_+, \mathbb{R})\}$ and $\{W \in \mathcal{C}_0^0(\mathbb{R}_+, \mathbb{R})\}$ are no events.
In order to make these sets events, we will equip $\mathcal{C}^0([0, 1], \mathbb{R})$ with a new σ-algebra $\mathfrak{C}^{[0,1]} = \sigma(\{\pi_t, t \in [0, 1]\})$.

Definition 4 (Definition of $\mathfrak{C}^{[0,1]}$) — We will work with $\mathcal{C}^0([0, 1], \mathbb{R})$ equipped with the σ-algebra $\mathfrak{C}^{[0,1]} = \sigma(\{\pi_t, t \in [0, 1]\})$.

For a stochastic process X with continuous sample paths, theorem 54 garantees that $X : \Omega \to \mathcal{C}([0, 1], \mathbb{R})$ is an $\mathcal{C}([0, 1], \mathbb{R})$-valued random element ie is $\mathcal{F}$-$\mathfrak{C}^{[0,1]}$-measurable. For any probability measure $\mathbb{Q}$ on $(\mathcal{C}^0([0, 1], \mathbb{R}), \mathfrak{C}^{[0,1]})$, we obtain the canonical model of a stochastic process with distribution $\mathbb{Q}$ by choosing $(\mathcal{C}([0, 1], \mathbb{R}), \mathfrak{C}^{[0,1]}, \mathbb{Q})$ as probability space and the evaluation maps $(\pi_t)_{t\in[0,1]}$ as our stochastic process on this probability space. We can do the same for $\mathcal{C}_0^0([0, 1], \mathbb{R})$.

Let us now introduce the Wiener measure and the Wiener space before we turn to our first constructive proof of existence for Brownian motion.

Definition 5 (Wiener measure) — Let W be a Brownian motion, defined on a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then the induced measure $\mathbb{P}\overset{-1}{W}$ on $(\mathcal{C}^0(\mathbb{R}_+, \mathbb{R}), \mathfrak{C}^{\mathbb{R}_+})$ is called the Wiener measure and the resulting probability space $\left(\mathcal{C}(\mathbb{R}_+, \mathbb{R}), \mathfrak{C}^{\mathbb{R}_+}, \mathbb{P}\overset{-1}{W}\right)$ is called the Wiener space.

To continue our construction, we will need some properties of Hilbert spaces that are recalled in the appendix. We now define the Haar basis of $\mathcal{L}^2(\mathbb{P})$, a representation of which can be seen with figure 11.

Definition 6 (Haar basis in $\mathcal{L}^2([0, 1])$) — The Haar basis of $\mathcal{L}^2([0, 1])$ is defined by $f_0 = \mathbb{1}_{[0,1]}$ and for $n \in \mathbb{N}$ and $k \in [\![1, 2^{n-1}]\!]$, $f_{n,k} = 2^{\frac{n-1}{2}} \left(\mathbb{1}_{\left[\frac{2k-2}{2^n}, \frac{2k-1}{2^n}\right[} - \mathbb{1}_{\left[\frac{2k-1}{2^n}, \frac{2k}{2^n}\right[}\right)$.

As given in the name, the Haar basis is a basis for the $\mathcal{L}^2([0, 1])$-space.

Lemma 7 (Haar basis in $\mathcal{L}^2([0, 1])$ is a complete orthonormal basis) — The family $\mathfrak{F} = \{f_0\} \cup \{f_{n,k} : n \in \mathbb{N}, k \in [\![1, 2^{n-1}]\!]\}$ is a complete orthonormal basis of the Hilbert space $\mathcal{L}^2([0, 1])$.

From Parseval's identity, we obtain the following corollary.

Corollary 8 (Parseval's identity with the Haar basis) —

$$\forall g_1, g_2 \in \mathcal{L}^2([0,1]), \langle g_1, g_2 \rangle = \langle g_1, f_0 \rangle \langle g_2, f_0 \rangle + \sum_{n=1}^{+\infty} \sum_{k=1}^{2^{n-1}} \langle g_1, f_{n,k} \rangle \langle g_2, f_{n,k} \rangle.$$

We continue our construction by introducing the integrals of the elements f of the Haar basis. Let us set

$$F_0 : [0,1] \ni t \mapsto \int_0^t f_0 \in \mathbb{R} \qquad \text{and} \qquad \forall n \in \mathbb{N}^*, k \in [\![1, 2^{n-1}]\!], \quad F_{n,k} : [0,1] \ni t \mapsto \int_0^t f_{n,k} \in \mathbb{R}.$$

A representation of the integrated Haar basis can be seen with figure 12. Let us now define a sequence of stochastic processes to approach the Brownian motion. Let $\{\xi_0\} \cup \{\xi_{n,k} : n \in \mathbb{N}, k \in [\![1, 2^{n-1}]\!]\}$ be a family of iid and standard normally distributed random variables on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We define then for all $N \in \mathbb{N}^*$ a stochastic process $W^{(N)}$ by setting

$$\forall t \in [0,1], \forall \omega \in \Omega, \quad W_t^{(N)}(\omega) = F_0(t)\xi_0(\omega) + \sum_{n=1}^{N} \sum_{k=1}^{2^{n-1}} F_{n,k}(t)\xi_{n,k}(\omega),$$

that is represented in figure 2.

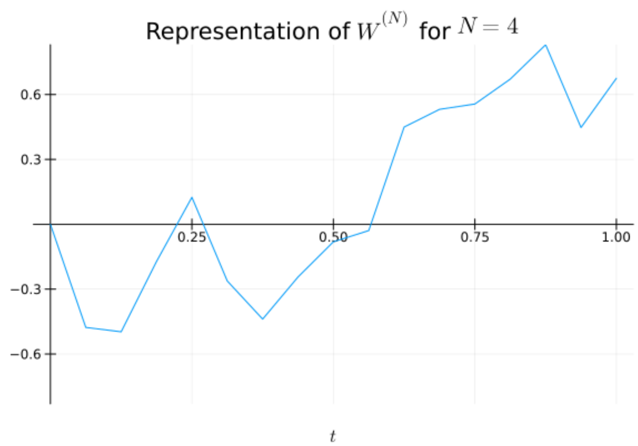


Figure 2: Representation of the $W^{(N)}$

Lemma 9 ($W^{(N)}$ is a centred Gaussian process with continuous sample paths.) — We have the following properties.

(a) $W^{(N)}$ has continuous sample paths.

(b) $W^{(N)}$ is a centred Gaussian process.

Proof (Lemma 9). The first point is easily obtained as F_0 and the $F_{n,k}$s are continuous, the second one is also easily obtained as ξ_0 and the $\xi_{n,k}$s are centred normal variables. □

Let us prove that $W^{(N)}$ converges towards a Brownian motion. Before we specify in which sense the convergence is supposed to be, let us show that the covariances of $W^{(N)}$ indeed converge to those of the Brownian motion. Actually, this will be the case for any complete orthonormal basis of $\mathcal{L}^2([0,1])$ and does not depend on our choice of working with the Haar basis.

Lemma 10 (Convergence of covariances) — Let $(h_n)_{n\in\mathbb{N}}$ be a complete orthonormal basis of $\mathcal{L}^2([0,1])$ and define for all $t \in [0,1]$ $H_n(t) = \int_0^t h_n s$. Assume that $\{\xi_n : n \in \mathbb{N}\}$ is a family of iid standard Gaussian random variables. Then

$$\forall s,t \in [0,1], \qquad \lim_{N\to+\infty} \mathbb{E}\left[\left(\sum_{n=0}^{N} H_n(s)\xi_n\right)\left(\sum_{n=0}^{N} H_n(t)\xi_n\right)\right] = s \wedge t.$$

Proof (Lemma 10). Let $s,t \in [0,1]$,

$$\mathbb{E}\left[\left(\sum_{n=0}^{N} H_n(s)\xi_n\right)\left(\sum_{n=0}^{N} H_n(t)\xi_n\right)\right] = \sum_{n=0}^{N} H_n(s)H_n(t) \xrightarrow[n\to+\infty]{} \sum_{n=0}^{+\infty} H_n(s)H_n(t) = s \wedge t,$$

thanks to Parseval's identity. □

We want now to precise in which sense $W^{(N)}$ could converge towards the Brownian motion. Since we want to preserve the fact that $W^{(N)}$ has continuous sample paths, we will show the following result in the appendix.

Lemma 11 (The approximations converge uniformly) — There exists a set $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that for all $\omega \in \Omega_0$, the sequence of continuous functions $\left(W^{(N)}(\omega)\right)_{N\in\mathbb{N}}$ converges uniformly towards a function $W(\omega) : [0,1] \to \mathbb{R}$.

Remark 2. Since the convergence is uniform, $W(\omega) : [0,1] \to \mathbb{R}$ is a continous function for all $\omega \in \Omega_0$.

What we can do in order to have a Brownian motion, is to restrict the probability space to Ω_0 by exchanging $(\Omega, \mathcal{F}, \mathbb{P})$ and $\left(\Omega_0, \mathcal{F}|_{\Omega_0}, \mathbb{P}_{|\mathcal{F}|_{\Omega_0}}\right)$ and verify that, on this new probability space, W is a Brownian motion.

Lemma 12 (W is a Brownian motion on $[0,1]$) — The stochastic process W as constructed in this section has the following properties

(a) W is centred ;

(b) W is a Gaussian process ;

(c) for all $s,t \in [0,1]$, $\mathrm{Cov}(W_s, W_t) = s \wedge t$.

Thus is W is a Brownian motion on $[0,1]$.

Proof (Theorem 12). For any $d \in \mathbb{N}^*$ and $t_1, \ldots, t_d \in [0,1]$, $\left(W_{t_1}^{(N)}, \ldots, W_{t_d}^{(N)}\right) \Rightarrow (W_{t_1}, \ldots, W_{t_d})$ thus $(W_{t_1}, \ldots, W_{t_d})$ is a centred Gaussian vector and its covariance matrix is $(t_i \wedge t_j)_{i,j\in[\![1,d]\!]}$. Hence W verifies the three conditions and then we can conclude that W is a Brownian motion. □

Let us now extend the Brownian motion on $\mathbb{R}_+$ thanks to the following theorem based on the scaling behaviour for Brownian motion (whose proof is in the appendix).

Theorem 13 (Extension of the Brownian motion on $\mathbb{R}_+$) — For all $t \in \mathbb{R}_+$, set $\overline{W_t} = (1+t)W_{1/(1+t)} - W_1$. Then $\overline{W}$ is a Brownian motion.

This first construction gives us an algorithm in order to compute a Brownian motion. The algorithm is in subsection A.1 and a representation of what is obtained can be found in figure 3.

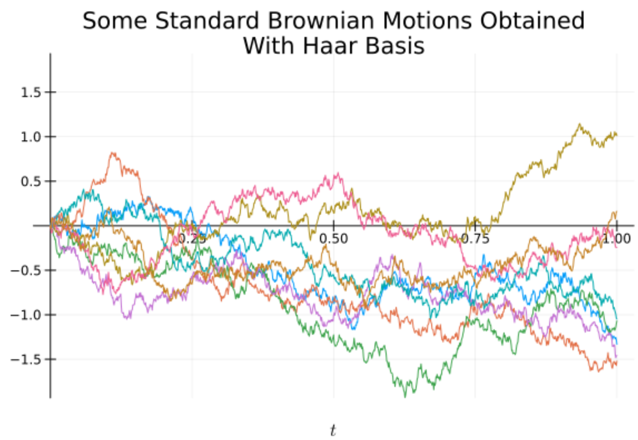


Figure 3: 8 sample paths of the Brownian motion obtained thanks to the previous construction principle

1.2.2 Donsker's invariance principle

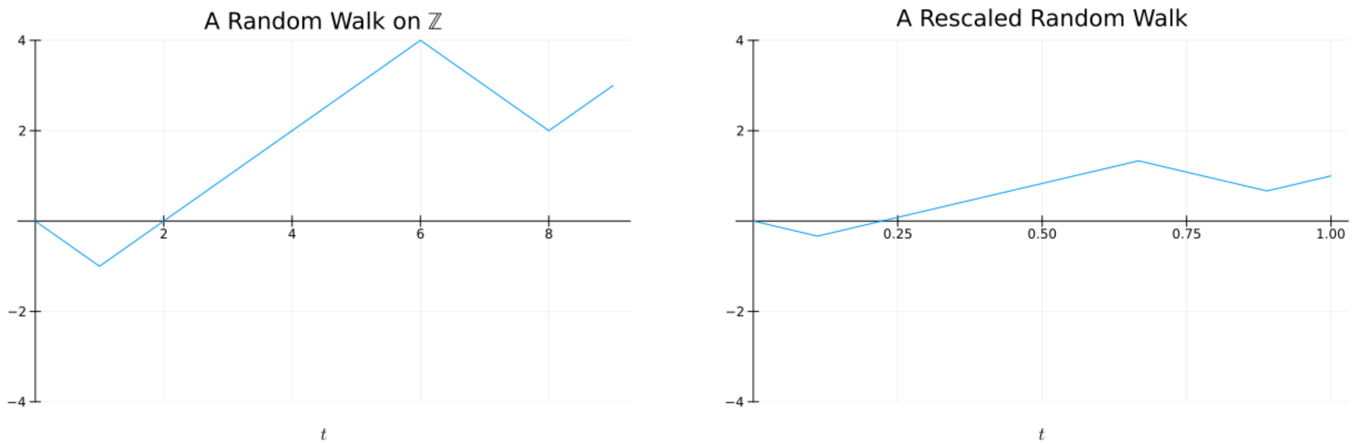
We will give another construction of the Brownian motion based on the random walks. Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of iid centred random variable with variance 1 $(\Omega, \mathcal{F}, \mathbb{P})$ and for all $n \in \mathbb{N}^*$ set $S_n = \sum_{i=1}^{n} X_i$. Figure 4a shows the polygonal path obtained by linearly interpolating between the points of realization of this realisation of this simple random walk. For each $\omega \in \Omega$, the linear interpolation is the graph of a continuous function $[0, n] \to \mathbb{R}$. By the central limit theorem, we know that:

$$\frac{1}{\sqrt{n}} S_n \Rightarrow Z \sim \mathcal{N}(0, 1).$$

Figure 4b shows what happens when we rescale the whole polygonal path with n^{-1} in time and $n^{-1/2}$ in space instead of only rescaling the end point. For fixed $\omega \in \Omega$, the rescaled polygonal path is the piecewise linear function

$$\forall t \in [0,1], \quad W_t^{(n)} = \frac{1}{\sqrt{n}} \left[S_{\lfloor tn \rfloor} + (tn - \lfloor tn \rfloor) X_{\lfloor tn \rfloor + 1} \right].$$

The first think we want is to be assured that the processes $W^{(n)}$ are really random variables. Thus comes lemma 14



(a) The random walk (plutôt à part avec points puis reliés)

(b) The rescaled random walk (plutôt à part avec différents n)

Figure 4: Illustration of the process with X_is being Rademacher variable: figure (4a) shows the initial random walk and figure (4b) shows the rescaled random walk

Lemma 14 ($W^{(n)}$ is a random variable) — $W^{(n)}$ is an $\mathcal{C}([0,1],\mathbb{R})$-valued random element *ie* $\mathcal{F}$-$\mathfrak{C}^{[0,1]}$-measurable.

Proof (Lemma 14). For all $t \in [0,1]$, $W_t^{(n)} = \frac{1}{\sqrt{n}}\left[S_{\lfloor tn \rfloor} + (tn - \lfloor tn \rfloor)X_{\lfloor tn \rfloor + 1}\right]$ is $\mathcal{F}$-$\mathcal{B}(\mathbb{R})$-measurable, then by theorem 54, $W^{(n)}$ is $\mathcal{F}$-$\mathcal{B}^{[0,1]}|_{\mathcal{C}([0,1],\mathbb{R})}$-measurable *ie* $W^{(n)}$ is $\mathcal{F}$-$\mathfrak{C}^{[0,1]}$-measurable. □

We have (the proof is admitted) that the finite-dimensional distributions of $W^{(n)}$ converge towards the finite-dimensional distributions of a Brownian motion.

Theorem 15 (Convergence of finite-dimensional distributions) — Let $d \in \mathbb{N}$ and $0 \leq t_1 < \cdots < t_d \leq 1$. Then,

$$\left(W_{t_1}^{(n)}, \ldots, W_{t_d}^{(n)}\right) \Rightarrow \left(W_{t_1}, \ldots, W_{t_d}\right),$$

where W denotes a Brownian motion.

The following is devoted to the following results on convergence of the random polygonal paths to the sample paths of a Brownian motion in the space of continuous functions, in the sense of definition 16.

Definition 16 (Weak convergence of probability measures on metric spaces) — A sequence $(\mu_n)_{n\in\mathbb{N}}$ of probability measures on a metric space (E,d) equipped with its Borel σ-field $\mathcal{B}(E)$ is called *weakly convergent towards a probability measure μ on* $(E,\mathcal{B}(E))$ if and only if

$$\forall f \in \mathcal{C}_b(E,\mathbb{R}), \quad \lim_{n\to+\infty} \int f \, \mathrm{d}\mu_n = \int f \, \mathrm{d}\mu,$$

where $\mathcal{C}_b(E,\mathbb{R})$ denotes the vector space of all functions $f : E \to \mathbb{R}$ which are continuous and bounded.

If we view $W^{(n)}$ and W as $\mathcal{C}([0,1])$-random elements, then their distributions are probability measures on $\left(\mathcal{C}([0,1],\mathbb{R}),\mathfrak{C}^{[0,1]}\right)$. Note that we always consider $\mathcal{C}([0,1],\mathbb{R})$ as a metric space, with the metric d_∞. In order to be able to use definition 16, we need to work with the Borel σ-field instead of $\mathfrak{C}^{[0,1]}$. However, as the following lemma shows (whose proof is in the appendix), the two of them coincide.

Lemma 17 (The σ-field generated by the Borel sets in $(\mathcal{C}([0,1],\mathbb{R}^d),d_\infty)$) — Let d_∞ denote the distance induced by the supremum norm on $\mathcal{C}([0,1],\mathbb{R})$ and denote by $\mathcal{B}_\infty=\mathcal{B}(\mathcal{C}([0,1],\mathbb{R}^d))$ the Borel sets generated by the open sets in the metric space $(\mathcal{C}([0,1],\mathbb{R}^d),d_\infty)$. Then

$$\mathcal{B}_\infty=\mathfrak{C}^{[0,1]}.$$

Now we are ready to state the main result of this subsection whose proof will be admitted.

Theorem 18 (Donsker's invariance principle) — Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of iid random variables such that $\mathbb{E}X_i=0$ and $\mathbb{V}X_i=1$ and define the polygonal paths $W^{(n)}$. Let $\mathbb{P}_n=\mathbb{P}\overset{-1}{W^{(n)}}$ denote the distribution of $W^{(n)}$. Then the sequence of probability measures $(\mathbb{P}_n)_{n\in\mathbb{N}}$ converges weakly towards a probability measure $\mathbb{P}_\star$ on $(\mathcal{C}([0,1],\mathbb{R}),\mathfrak{C}^{[0,1]})$.
On the probability space $(\mathcal{C}([0,1],\mathbb{R}),\mathfrak{C}^{[0,1]},\mathbb{P}_\star)$, the evaluations maps $(\pi_t)_{t\in[0,1]}$ define a stochastic process which is a Brownian motion.

Remarks 3. 1. The probability measure $\mathbb{P}_\star$ is the Wiener measure.

2. Note that we automatically obtain a stochastic process with continuous sample paths, as the measure of the sequence are only defined on $(\mathcal{C}([0,1],\mathbb{R}),\mathfrak{C}^{[0,1]})$.
3. Note that, just like in the central limit theorem, the limiting process W does not depend on the explicit choice of the distribution of the random variables X_i, only on their mean and their variance.

This construction gives us an other algorothm in order to compute a Brownian motion. The algorithm is in subsection A.2 an a representation of what is obtained can be found in figure 5.

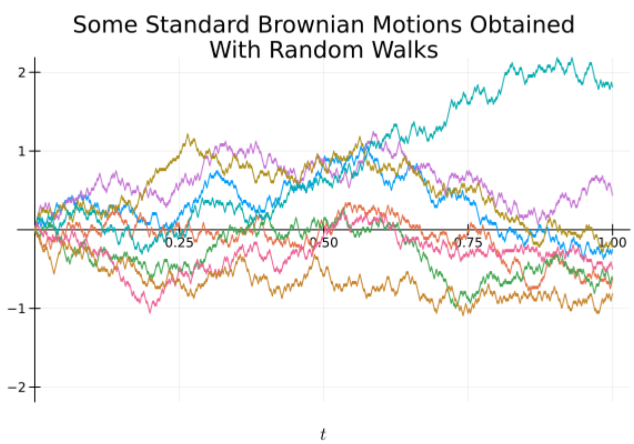


Figure 5: 8 sample paths of the Brownian motion obtained thanks to the previous construction principle

1.3 Properties of the Brownian motion

In this subsection, we will enumerate some properties of the Brownian motion. Some of them will be put in the appendix. One of the first interesting properties of the Brownian motion is the transformations you can apply to it that still give you a Brownian motion. The proof of the following result is in the appendix.

Theorem 19 (Transformations of Brownian motion) — Let W be a Brownian motion.

(a) The stochastic process $-W$ is a Brownian motion.

(b) For any $\lambda > 0$, the stochastic process $(\sqrt{\lambda}W_{t/\lambda})_{t\in\mathbb{R}_+}$ is a Brownian motion.

(c) For any $s \geq 0$, the stochastic process $(W_{t+s} - W_s)_{t\in\mathbb{R}_+}$ is a Brownian motion.

(d) There exists $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that on the probability space $\left(\Omega_0, \mathcal{F}|_{\Omega_0}, \mathbb{P}|_{\mathcal{F}|_{\Omega_0}}\right)$, the stochastic process $(tW_{1/t})_{t\in\mathbb{R}_+}$ (with the convention $0W_{1/0} = 0$) is a Brownian motion.

Remark 4. In theorem 19, the last property is slightly more involved since we need to restrict to the set of all ω such that $t \mapsto tW_{1/t}$ is continuous in $t = 0$.

Next, let us convince ourselves that Brownian motion has independent increments in the sense of definition 88 in theorem 20 (whose proof is in the appendix).

Theorem 20 (Brownian motion has independent increments) — A Brownian motion W has independent increments (with respect to $\mathcal{F}^W$).

Thanks to the previous result and theorem 91, we can show the following property.

Corollary 21 (Brownian motion is a Markov process) — A Brownian motion is a Markov process.

We already have thanks to theorem 20 that Brownian motion has independent increments but it is also true with respect to the filtration $\left\{\mathcal{F}^W_{t+}\right\}_{t\in\mathbb{R}_+}$ as can be seen in theorem 22, what is of interest as $\left\{\mathcal{F}^W_{t+}\right\}_{t\in\mathbb{R}_+}$ is right-continuous.
The next theorem proves that for Brownian motion the distinction between the two filtrations $\{\mathcal{F}^W_t\}_{t\in\mathbb{R}_+}$ and $\{\mathcal{F}^W_{t+}\}_{t\in\mathbb{R}_+}$ are not that dramatic as the important results we proved in the previous subsection still hold!

Theorem 22 (Brownian motion has independent increments with respect to $\left\{\mathcal{F}^W_{t+}\right\}_{t\in\mathbb{R}_+}$) — A Brownian motion W has independent increments with respect to $\left\{\mathcal{F}^W_{t+}\right\}_{t\in\mathbb{R}_+}$. Consequently a Brownian motion is a $\left\{\mathcal{F}^W_{t+}\right\}_{t\in\mathbb{R}_+}$-Markov process.

We can show the following consequences of theorem 22. The first one's proof is in the appendix.

Corollary 23 (Brownian motion changes sign infinitely often during the time interval $[0, \epsilon]$) — Let $\epsilon > 0$, then for $\mathbb{P}$-almost all $\omega \in \Omega$, the sample path $t \mapsto W_t(\omega)$ changes sign infinitely many times during the times interval $[0, \epsilon]$ ie for $\mathbb{P}$-almost all $\omega \in \Omega$, there exists a sequence of times $\epsilon > t_0 > t_2 > \cdots > 0$ such that for all $k \in \mathbb{N}$ $W_{2k} < 0$ and $W_{2k+1} > 0$.

Corollary 24 (Brownian motion is an $\{\mathcal{F}_{t+}^W\}_{t\in\mathbb{R}_+}$-Markov process) — A Brownian motion is an $\{\mathcal{F}_{t+}^W\}_{t\in\mathbb{R}_+}$-Markov process.

Proof (Corollary 24). This result is a direct consequence of theorem 22 and theorem 91. □

We might want to observe a Brownian until a certain time decided by a stopping time, however it can be useful to observe to observe it starting at a certain time decided by the same stopping, thus we will need to observe how we can restart a Brownian motion. Let us do the following for the rest of the subsection:

(a) set $\mathcal{F}_t = \mathcal{F}_{t+}^W$ for all $t \geq 0$, we will work exclusively with the resulting right-continuous filtration $\{\mathcal{F}_t\}_{t\geq 0}$;

(b) let τ be an $\{\mathcal{F}_t\}_{t\geq 0}$-stopping time such that $\mathbb{P}(\tau < +\infty) > 0$;

(c) we restrict the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $(\Omega', \mathcal{F}', \mathbb{P}')$ with

- $\Omega' = \{\tau < +\infty\} \in \mathcal{F}$;
- $\mathcal{F}' = \mathcal{F}|_{\{\tau<+\infty\}}$;
- $\mathbb{P}' = \mathcal{P}|_{\mathcal{F}|_{\{\tau<+\infty\}}}$.

Remark 5. (a) Under these assumptions, W_τ is a random variable on $(\Omega', \mathcal{F}', \mathbb{P}')$ as can be seen as follows. For any $A \in \mathcal{B}(\mathbb{R})$,

$$C = \{W_\tau \in A\} \cap \{\tau < +\infty\}$$

satisfies $C \in \mathcal{F}_\tau \subset \mathcal{F}$ by lemma 75 as well $C \subset \{\tau < +\infty\}$. Hence $C \in \mathcal{F}'$ which proves the $\mathcal{F}'$-$\mathcal{B}(\mathbb{R})$-measurability of W_τ.

(b) As an immediate consequence we obtain that $(W_{\tau+t} - W_\tau)_{t\geq 0}$ is a stochastic process with continuous sample paths on $(\Omega', \mathcal{F}', \mathbb{P}')$.

(c) τ is a stopping time for the trace filtration.

Now we are ready to restart a Brownian motion at a random time, provided this random time is a stopping time.

Theorem 25 (Restarting Brownian motion in (τ, W_τ)) — The stochastic process $W^{(\tau)} = (W_{\tau+t} - W_\tau)_{t\geq 0}$ is a Brownian motion which is independent of $\mathcal{F}_\tau$ with respect to $\mathbb{P}'$.

One of the consequence of theorem 25 is corollary 26 which is admitted.

Corollary 26 (Strong Markov property) — Let τ be an $\{\mathcal{F}_t\}_{t\geq 0}$-stopping time. Set $\mathcal{F}^{\geq\tau} = \sigma(\{W_{\tau+t} : t \geq 0\})$. Then,

$$\forall A \in \mathcal{F}^{\geq\tau}, \mathbb{P}(A|\mathcal{F}_\tau) = \mathbb{P}(A|W_\tau)\ \mathbb{P}\text{-almost surely on } \{\tau < +\infty\}.$$

We say that W satisfies the strong Markov property.

Remark 6. The assumption "on $\{\tau < +\infty\}$" of all statements can be avoided by adding a cemetery state Δ to the state space and setting $W_{+\infty} = \Delta$. Then $\mathbb{W}$ is defined everywhere on Ω.

As we proved it earlier for classic Brownian motion, we have the following result.

Lemma 27 ($\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$-Brownian motion as Markov process and martingale) — An $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$-Brownian motion is an $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$-Markov process and an $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$-martingale.

Proof (Lemma 27). It is a direct consequence of the fact that $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$-Brownian motion has independent increments. □

2 Itô integral and stochastic differential equations

In this section, we'll delve into Itô integrals and stochastic differential equations (SDEs). These concepts are vital for grasping how randomness affects dynamic systems. Itô integrals handle the integration of stochastic processes, while SDEs describe how these processes change over uncertain times. Let us start by construction the Itô integral. We will only do the one-dimensional case, but everything can be generalized to a d-dimensionsional space.

2.1 Construction of the Itô integral for elementary functions

As a first step, we introduce the stochastic integral for elementary functions which are step functions in time.

Definition 28 (Elementary functions) — A function $h : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is called an elementary function if and only if there exists a constant $K \in \mathbb{N}^*$, a partition $0 = t_0 < \cdots < t_K$ of $\mathbb{R}_+$ and $\mathcal{F}_{t_k}$-$\mathcal{B}(\mathbb{R})$-measurable random variables for all $k \in [\![0, K-1]\!]$, $h_k : \Omega \to \mathbb{R}$ such that

$$\forall t \in \mathbb{R}_+, \forall \omega \in \Omega, \quad h(t,\omega) = \sum_{k=0}^{K-1} h_k(\omega)\mathbb{1}_{]t_k,t_{k+1}]}(t). \tag{1}$$

We denote by $\mathcal{V}_0$ the vector space of all elementary functions.

Note that an elementary functions can be viewed as a progressively measurable stochastic process (see defintion 68 if needed). For elementary integrands, the stochastic integral can be defined in a natural way as follows.

Definition 29 (Itô integral for elementary functions) — For an elementary function $h \in \mathcal{V}_0$ with representation (1), the Itô integral is defined as

$$
\begin{aligned}
&\forall t \in \mathbb{R}_+, \forall \omega \in \Omega, \quad \int_0^t h(s,\omega)\mathrm{d}W_s(\omega) = \sum_{k=0}^{K-1} h_k(\omega)[W_{t_{k+1}\wedge t}(\omega) - W_{t_k \wedge t}(\omega)] \;; \\
&\forall \omega \in \Omega, \qquad\qquad \int_0^{+\infty} h(s,\omega)\mathrm{d}W_s(\omega) = \sum_{k=0}^{K-1} h_k(\omega)[W_{t_{k+1}}(\omega) - W_{t_k}(\omega)].
\end{aligned}
$$

One can wonder: why is there this strange assumption that h_k is $\mathcal{F}_{t_k}$-measurable in the definition of elementary functions? The choice of where we take the measurability of the h_k will have an influence on the property of the stochastic integral. For more information, see 13.

We postpone the discussion of properties of the Itô integral including its being well-defined (which will be proven thanks to proposition 102) until we have extended the definition beyond elementary functions. For the time being, we will need only one property which can be viewed as an isometry between Hilbert spaces.

Lemma 30 (Itô isometry for integrals of elementary functions) — Let the elementary function h from (1) be such that $h_k \in \mathcal{L}^2(\mathbb{P})$ for all $k \in [\![0, K-1]\!]$. Then,

$$\mathbb{E}\left[\left(\int_0^\infty h(t)\mathrm{W}_t\right)^2\right] = \int_0^\infty \mathbb{E}[h(t)^2]\,\mathrm{d}t. \tag{2}$$

In other words, the $\mathcal{L}^2(\mathbb{P})$-norm of the stochastic integral $\int_0^\infty h(t)\ \mathrm{d}W_t$ equals the $\mathcal{L}^2(\lambda \otimes \mathbb{P})$-norm of the integrand $h : \mathbb{R}_+ \times \Omega \to \mathbb{R}$.

Proof (Lemma 30). We have

$$\mathbb{E}\left[\left(\int_0^{+\infty} h(t)\,\mathrm{d}W_t\right)\right] = \sum_{k,l=1}^{K-1} E\left[h_k h_l (\Delta W_k)(\Delta W_l)\right]$$

where $\Delta W_k = W_{t_{k+1}} - W_{t_k}$.
For $k < l$, $h_k h_l(\Delta W_k)$ is $\mathcal{F}_{t_l}$-measurable whereas $\Delta W_l \perp\!\!\!\perp \mathcal{F}_{t_l}$, and thus only the term with $k = l$ contributes to the sum. As h_k is $\mathcal{F}_{t_k}$-measurable and $\Delta W_k \perp\!\!\!\perp \mathcal{F}_{t_k}$, we know that $E\left[h_k^2(\Delta W_k)^2\right] = E\left[h_k^2\right]\Delta t_k$. Thus we have

$$\mathbb{E}\left[\left(\int_0^{+\infty} h(t)\,\mathrm{d}W_t\right)\right] = \sum_{k=1}^{K-1} E\left[h_k^2\right]\Delta t_k = \int_0^\infty \mathbb{E}[h(t)^2]\,\mathrm{d}t.$$

What gives us the conclusion. □

2.2 Extension of the Itô integral to $\mathcal{L}^2$ integrands

Using the Itô isometry, we will be able to extend definition 29 to the following class $\mathcal{V}$ of functions $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}$:

Definition 31 (The class of $\mathcal{L}^2$-integrands) — Let $\mathcal{V}$ denote the class of all functions $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ such that

(i) For all $t \geq 0$, $[0,t] \times \Omega \ni (s,\omega) \mapsto f(s,\omega)$ is $\mathcal{B}([0,t]) \times \mathcal{F}_t$-$\mathcal{B}(\mathbb{R})$-measurable *ie* seen as a stochastic process, f is progressively measurable with respect to $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$;

(ii) $\int_0^{+\infty} \mathbb{E}[f(t,\cdot)^2] \, \mathrm{d}t < +\infty$.

We also introduce the notation $\mathcal{V}_T$ for $T > 0$ which we will use to denote the class of all functions $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ such that

(i') For all $t \in [0,T]$, $[0,t] \times \Omega \ni (s,\omega) \mapsto f(s,\omega)$ is $\mathcal{B}([0,t]) \times \mathcal{F}_t$-$\mathcal{B}(\mathbb{R})$-measurable ;

(ii') $\int_0^{T} \mathbb{E}[f(t,\cdot)^2] \, \mathrm{d}t < +\infty$.

To extend the definition of the Itô integral, we need to show that any $f \in \mathcal{V}$ can be approximated by a sequence of elementary functions $h^{(n)}$ in $\mathcal{L}^2(\lambda|_{\mathbb{R}_+} \times \mathbb{P})$ (whose proof in in the appendix).

Theorem 32 (Approximating $\mathcal{L}^2$-integrands by elementary functions) — For any $f \in \mathcal{V}$, there exists a sequence $(h^{(n)})_{n\in\mathbb{N}}$ of elementary functions $h^{(n)} \in \mathcal{V}_0$ for all $n \in \mathbb{N}$ such that

$$\int_0^{+\infty} \mathbb{E}[(f(t) - h^{(n)}(t))^2] \, \mathrm{d}t \xrightarrow[n\to+\infty]{} 0. \tag{3}$$

If we can prove theorem 32, then the Itô isometry (2) for elementary functions will allow to define the stochastic integral of f by setting

$$\int_0^{+\infty} f(t) \, \mathrm{d}W_t = \lim_{n\to+\infty} \int_0^{+\infty} h^{(n)}(t) \, \mathrm{d}W_t. \tag{4}$$

where the limit will exist in $\mathcal{L}^2$.

Remarks 7. (a) If $(f(t,\cdot))_{t\in\mathbb{R}_+}$ is progressively measurable, then the stochastic process $(f(t,\cdot))_{t\in\mathbb{R}_+}$ is obviously also $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$-$\mathcal{B}(\mathbb{R})$-measurable and $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$-adapted, then there exists a progressively measurable modification of $(f(t,\cdot))_{t\in\mathbb{R}_+}$, see theorem T46 of [Mey66].

(b) Obviously, theorem 32 also holds for $f \in \mathcal{V}_T$ *ie* we can find a sequence $(h^{(n)})_{n\in\mathbb{N}}$ of elementary functions $h^{(n)} \in \mathcal{V}_0$ for all $n \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \int_0^{T} \mathbb{E}[(f(t) - h^{(n)}(t))^2] \, \mathrm{d}t = 0.$$

It simply doesn't matter which values $h^{(n)}(t,\omega)$ takes for $t > T$, so we don't even need to introduce elementary function on $[O,T] \times \Omega$ but we can work with $\mathcal{V}_0$ as introduced in definition 28.

Now we can combine the Itô isometry for elementary functions (lemma 30) and theorem 32 to define the Itô integral for arbitrary integrands $f \in \mathcal{V}$.

Lemma 33 (Existence of the Itô integral) — Let $f \in \mathcal{V}$ and $(h^{(n)})_{n\in\mathbb{N}}$ be a sequence of elementary functions $h^{(n)} \in \mathcal{V}_0$ such that (3) holds. Then $\lim\limits_{n\to+\infty} \displaystyle\int_0^{\infty} h^{(n)}(t,\omega)\, \mathrm{d}W_t(\omega)$ exists in $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$.

Proof (Lemma 33). By (3) we know that

$$\|f - h^{(n)}\| \xrightarrow[n\to+\infty]{} 0.$$

Hence $(h^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{L}^2(\mathbb{R}_+ \times \Omega, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}, \lambda|_{\mathbb{R}_+} \otimes \mathbb{P})$. Since $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ is complete, it suffices to show that the sequence of stochastic integrals is a Cauchy sequence in $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$. By the Itô isometry,

$$\begin{aligned}
\left\| \int_0^{+\infty} h^{(n)}(t)\, \mathrm{d}W_t - \int_0^{+\infty} h^{(m)}(t)\, \mathrm{d}W_t \right\|^2_{\mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})} &= \left\| \int_0^{+\infty} [h^{(n)}(t) - h^{(m)}(t)]\, \mathrm{d}W_t \right\|^2_{\mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})} \\
&= \left\| h^{(n)} - h^{(m)} \right\|^2_{\mathcal{L}^2(\mathbb{R}_+\times\Omega, \mathcal{B}(\mathbb{R}_+)\otimes\mathcal{F}, \lambda|_{\mathbb{R}_+}\otimes\mathbb{P})},
\end{aligned}$$

the property of being a Cauchy sequence carries over from $(h^{(n)})_{n\in\mathbb{N}}$ to the sequence of stochastic integrals. Therefore, $(\int_0^{+\infty} h^{(n)}(t)\, \mathrm{d}W_t)_{n\in\mathbb{N}}$ converges in $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$.
In the proof we used the fact that the Itô integral for elementary functions is linear (see lemma 103). ☐

Definition 34 (Itô integral for $\mathcal{L}^2$-integrands) — Let $f \in \mathcal{V}$ and $(h^{(n)})_{n\in\mathbb{N}}$ a sequence of elementary functions $h^{(n)} \in \mathcal{V}_0$ such that (3) holds. We define the Itô integral of f by setting

$$\int_0^{+\infty} f(t,\omega)\, \mathrm{d}W_t(\omega) = \lim_{n\to+\infty} \int_0^{+\infty} h^{(n)}(t,\omega)\, \mathrm{d}W_t(\omega)$$

where the limit is to be understood as convergence in $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$.

For any interval $I \subset \mathbb{R}_+$, we define

$$\int_I f(t)\, \mathrm{d}W_t = \int_0^{+\infty} \mathbb{1}_I(t) f(t)\, \mathrm{d}W_t.$$

If I has endpoints $a \in \mathbb{R}_+$ and $b \in]0, +\infty]$ with $a < b$, we also write $\int_I f(t)\, \mathrm{d}W_t = \int_a^b f(t)\, \mathrm{d}W_t$.

2.3 Simple properties of the Itô integral

First of all, we need to prove that the integral is well-defined. For practical reasons, we will first show that the stochastic integral is well-defined for integrands in $\mathcal{V}_0$ and then establish that the integral is linear as a function of integrands from $\mathcal{V}_0$ before showing that the integral is also well-defined for integrands from $\mathcal{V}$.

Proposition 35 (The Itô integral for $\mathcal{L}^2$-integrands is well-defined) — The Itô integral is well-defined in the sense that for $f \in \mathcal{V}$, the integral $\displaystyle\int_0^{+\infty} f(t)\,\mathrm{d}W_t$ does not depend on the choice of the approximating sequence $(h^{(n)})_{n\in\mathbb{N}}$ in $\mathcal{V}_0$.

Proof (Proposition 35). Suppose that there are two sequences $(h^{(n)})_{n\in\mathbb{N}}$ and $(g^{(n)})_{n\in\mathbb{N}}$ in $\mathcal{V}_0$ that converge towards f in $\mathcal{L}^2(\mathbb{R}_+ \times \Omega, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}, \lambda|_{\mathbb{R}_+} \otimes \mathbb{P})$. Using lemmas 103 and 30, we find that

$$\left\| \int_0^{+\infty} h^{(n)}\,\mathrm{d}W_t - \int_0^{+\infty} g^{(n)}\,\mathrm{d}W_t \right\|^2_{\mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})} = \|h^{(n)} - g^{(n)}\|_{\mathcal{L}^2(\mathbb{R}_+\times\Omega,\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{F},\lambda|_{\mathbb{R}_+}\otimes\mathbb{P})} \xrightarrow[n\to+\infty]{} 0,$$

therefore $\int_0^{+\infty} f\,\mathrm{d}W_t$ does not depend on the choice of the approximating sequence. □

Now we are ready to provide a list of useful properties of Itô integrals. First of all, let us remark that by construction, the Itô isometry extends to integrals with integrands from $\mathcal{V}$. This is a direct consequence of lemma 30 and definition 34.

Corollary 36 (Itô isometry for $\mathcal{L}^2$-integrands) — For any $f \in \mathcal{V}$,

$$\mathbb{E}\left[\left(\int_0^{+\infty} f(t)\,\mathrm{d}W_t\right)^2\right] = \int_0^{+\infty} \mathbb{E}[f(t)^2]\,\mathrm{d}t.$$

The following whose proof is in the annex gives us simple properties of the Itô integral

Theorem 37 (Simple properties of the Itô integral for $\mathcal{L}^2$-integrands) — Let $f, g \in \mathcal{V}$ and $c \in \mathbb{R}$. Then the following assertions hold.

(a) splitting of integrals:

$$\forall 0 \le s \le u \le t < +\infty, \int_s^t f(v)\,\mathrm{d}W_v = \int_s^u f(v)\,\mathrm{d}W_v + \int_u^t f(v)\,\mathrm{d}W_v\ ; \tag{5}$$

(b) linearity:

$$\int_0^{+\infty} [cf(t) + g(t)]\,\mathrm{d}W_t = c\int_0^{+\infty} f(t)\,\mathrm{d}W_t + \int_0^{+\infty} g(t)\,\mathrm{d}W_t\ ; \tag{6}$$

(c) expectation:

$$\mathbb{E}\left[\int_0^{+\infty} f(t)\,\mathrm{d}W_t\right] = 0\ ; \tag{7}$$

(d) covariance of stochastic integrals/isometry:

$$\mathbb{E}\left[\left(\int_0^{+\infty} f(t)\,\mathrm{d}W_t\right)\left(\int_0^{+\infty} g(t)\,\mathrm{d}W_t\right)\right] = \int_0^{+\infty} \mathbb{E}[f(t)g(t)]\,\mathrm{d}t\ , \tag{8}$$

where both integrals on the lefthand side are taken with respect to the same Brownian motion.

As we will discuss below, the definition of the Itô integral can easily be extended to integrands f satisfying the same measurability assumptions as before but a weaker integrability assumptions. In fact, it is sufficient to assume that

$$\mathbb{P}\left(\forall t \in \mathbb{R}_+, \int_0^t f(s,\omega)^2 \mathrm{d}s < +\infty\right) = 1. \tag{9}$$

The stochastic integral can then be defined as the limit in probability of integrals of elementary functions. Here a word of warning is due: for such f, those of the above properties of the stochastic integral which involve expectations may fail.

Definition 38 (The class $\mathcal{W}$ of integrands) — Let $\mathcal{W}$ be the class of all functions $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ such that

(i) for all $t \in \mathbb{R}_+$, $[0,t] \times \Omega \ni (s,\omega) \mapsto f(s,\omega)$ is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-$\mathcal{B}(\mathbb{R})$-measurable ie the stochastic process $(f(t,\cdot))_{t\in\mathbb{R}_+}$ is progressively measurable with respect to $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$;

(ii) the integrability condition (9) holds.

We will only briefly describe how to construct the Itô integral for intergands from $\mathcal{W}$. Suppose $f \in \mathcal{W}$. Then there exists a sequence $(h^{(n)})_{n\in\mathbb{N}}$ of elementary functions with $h^{(n)} \in \mathcal{W} \cap \mathcal{V}_0$ and for all $t \in \mathbb{R}_+$, $\int_0^t |h^{(n)} - f|^2 \to 0$ in probability. One can show that for such a sequence of elementary functions the stochastic integrals $\int_0^t h^{(n)}(s)\, \mathrm{d}W_s$ converge in probability towards a random variable an $n \to \infty$ and that the limit depends only in $f \in \mathcal{W}$ but not on the sequence $(h^{(n)})_{n\in\mathbb{N}}$. Thus the stochastic integral of f can be defined as

$$\forall t \in \mathbb{R}_+, \quad I_t(f) = \int_0^t f(s)\, \mathrm{d}W_s = \lim_{n\to+\infty} \int_0^t h^{(n)}(s)\, \mathrm{d}W_s,$$

where the limit is to be understood as convergence in probability.

Remarks 8. (a) There exists a version of the Itô integral for $f \in \mathcal{W}$ which has continuous sample paths, see section 2.3 of [McK69].

(b) In general, for $f \in \mathcal{W}$ the martingale property does not hold. For an exemple show that the martingale property may fail in this case, see [Li17]. Nevertheless, for $f \in \mathcal{W}$ the stochastic integral still defines a local martingale, see section 3.2 of [KS91].

For Itô integrals with integrand $f \in \mathcal{W}$, we do a similar assumption as before *ie* from now on, if $f \in \mathcal{W}$ we will always assume that the filtration is complete and that we are working with a version of $\left(\int_0^t f(s)\, \mathrm{d}W_s\right)_{t\in\mathbb{R}_+}$ which is a martingale with continuous sample paths.

2.4 Itô integral as a stochastic process

In this subsection, we consider the Itô integral as a function of the upper integration limit. We show that the resulting stochastic process is a martingale provided the integrand is square-integrable in the sense of definition 31, we extend the class of possible integrands by weakening the integrability assumption in definition 31, and finally, we present the Itô formula which serves as a replacement for the chain rule in calculus. We will assume that W is an $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$-Brownian motion. Let $f \in \mathcal{V}$ and set

$$\forall t \in \mathbb{R}_+, \quad I_t(f) = \int_0^t f(s)\,\mathrm{d}W_s.$$

We are interested in the properties of the Itô integral $\int_0^t f(s)\,\mathrm{d}W_s$ as a function of the upper integration limit *ie* we will analyse the stochastic process $(I_t(f))_{t\in\mathbb{R}_+}$.

We would like to prove that the stochastic process associated with the Itô integral is a $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$-martingale with continuous sample paths, it is unfortunately, this is not necessarily true because the process is defined only as an $\mathcal{L}^2$-limit. In order to make it nonetheless work w will do two things:

- firstly, we will complete the filtration we consider,
- secondly, we will consider a version of our stochastic process that has the property we want.

In this case, we obtain the following theorem whose proof is admitted.

Theorem 39 (The Itô integral is a martingale) — Let $f \in \mathcal{V}$ and assume that the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ is complete. Then there exists a version J of $(I_t(f))_{t\in\mathbb{R}_+}$ which is an $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$-martingale with continuous sample paths.

If the filtration is complete, only the continuity of sample paths actually recquire to distinguish between $(I_t(f))_{t\in\mathbb{R}_+}$ and the version J. Thus, we can rephrase theorem 39 as follows.

Theorem 40 (The Itô integral is a martingale - version 2) — Let $f \in \mathcal{V}$ and assume that the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ is complete. Then $(I_t(f))_{t\in\mathbb{R}_+}$ is an $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$-martingale which possesses a version with continuous sample paths.

From now on, if $f \in \mathcal{V}$ we will always assume that the filtration is complete and that we are working with a version of $\left(\int_0^t f(s)\,\mathrm{d}W_s\right)_{t\in\mathbb{R}_+}$ which is a martingale with continuous sample paths.

2.5 Algorithm for the computation of the Itô integral

An question we could ask ourselves after we have seen how the Itô integral was constructed is the question of the numerical computation of the algorithm. One of the easiest ways to compute an Itô integral is to compute it as we would do with the usual integral using the Euler method. Then we could write something like

$$\int_0^T f(\omega,t)\,\mathrm{d}W_t(\omega) \simeq \sum_{k=0}^{n-1} f(\omega,t_k)(W_{t_{k+1}}(\omega) - W_{t_k}(\omega)).$$

We could then write an algorithm taking three entries : an already computed random function `f`, an already computed Brownian motion `W` and an array of times `t`. The algorithm would initiate a number `integral` to `0.0` and then do a loop on `i` going from the second index of `t` to the last such that at each step `integral` is incremented by the value `f(t[i-1]) * (W(t[i]) - W(t[i-1]))`. We can find just after the code written in the

Julia language.

```
function ito_integral(W, f, t)
    integral = 0.0
    for i in 2:length(t)
        dW = W(t[i]) - W(t[i-1])
        integral += f(t[i-1]) * dW
    end
    return integral
end
```

For instance, this is what we obtain if we want to compute a sample path of $t \mapsto \int_0^t W_s \, \mathrm{d}W_s$.

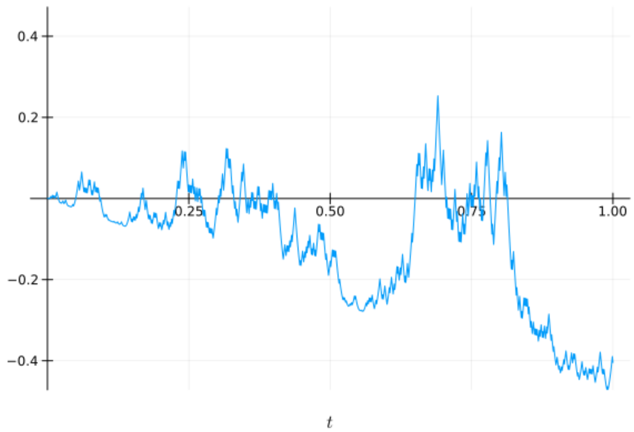


Figure 6: A sample path of $t \mapsto \int_0^t W_s \, \mathrm{d}W_s$

2.6 Itô's formula

It is not easy to guess stichastic integrals. This should not come as a surprise. After all, in general we cannot easily guess the value of a Riemann integral from its definition. But for Riemann integrals, we have the fundamental theroem of calculus and the chain rule as important tools for calculating these integrals. For stochastic integrals, we are at a disadvantage as there is no differentiation theory. But we do have a replacement for the chain rule, the so-called Itô's formula. In order to use Itô's formula, we will need the following definition.

Definition 41 (Itô processes) — An $\mathbb{R}$-valued stochastic process X is called an Itô process if and only if X can be represented as

$$\forall t \in \mathbb{R}_+, \quad X_t = X_0 + \int_0^t u(s)\,\mathrm{d}s + \int_0^t v(s)\,\mathrm{d}W_s, \tag{10}$$

where

(i) $v \in \mathcal{W}$;

(ii) the stochastic proccess $(u(t,\cdot))_{t\in\mathbb{R}_+}$ is progressively measurable with respect to $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ and

$$\mathbb{P}\left(\forall t \geq 0, \int_0^t |u|^2 < +\infty\right) = 1.$$

If X is an Itô process with representation (10), we also write as a shorthand

$$\mathrm{d}X_t = u\,\mathrm{d}t + v\,\mathrm{d}W_t.$$

The following theorem, whose proof is admitted, provides the analogue of the chain rule of calculus for stochastic integrals.

Theorem 42 (Itô's formula in dimension 1) — Let X be an Itô process with representation (10), let $T \in \mathbb{R}_+^*$. Assume that $g : \mathbb{R} \times [0,T] \to \mathbb{R}$ is continuous and has continuous partial derivatives $g_t = \partial_t g, g_x = \partial_x g, g_{xx} = \partial_{xx}^2 g$. Then $Y_t = g(X_t, t)$ is given by

$$Y_t = g(X_0, 0) + \int_0^t \left[g_t(X_s, s) + g_x(X_s, s)u(s) + \frac{1}{2}g_{xx}(X_s, s)v(s)^2\right] \mathrm{d}s + \int_0^t g_x(X_s, s)v(s)\,\mathrm{d}W_s \tag{11}$$

for all $t \in \mathbb{R}_+$.

Remarks 9. (a) Denote (10) briefly as $\mathrm{d}X_t = u\,\mathrm{d}t + v\,\mathrm{d}W_t$, Itô's formula can be written as

$$\mathrm{d}Y_t = g_t\,\mathrm{d}t + g_t\,\mathrm{d}X_t + \frac{1}{2}g_{xx}(\mathrm{d}X_t)^2,$$

where $(\mathrm{d}X_t)^2$ is calculated according to the scheme

$$(\mathrm{d}t)^2 = (\mathrm{d}t)(\mathrm{d}W_t) = (\mathrm{d}W_t)(\mathrm{d}t) = 0, (\mathrm{d}W_t)^2 = \mathrm{d}t. \tag{12}$$

On the one hand, this scheme can be considered as a rule of thumb which garantees that the formal calculation based on (12) gives the correct results (11) when we return to the integral representation. On the other hand, we will see in the proof of Itô's formula that this scheme has an interpretation as vanishing contributions to an approximation of lefthand side of 10.

(b) Itô's formula differs from the chain rule of calculus by the term $\frac{1}{2}g_{xx}(\mathrm{d}X_t)^2$. This term is called the Itô correction. Note that the correction is not present if the transformation g is linear in x.

2.7 Stochastic differential equations and solutions

We want to be able to write something of the following form

$$\forall t \in [0,T], \quad \dot{x}_t = f(x_t, t) + F(x_t, t)\dot{W}_t, \tag{13}$$

where $T > 0$ and $\dot{W}_t$ denotes white noise, the generalised derivative of Brownian motion. We do so by intepreting (13) as an integral equation. Hence we consider the stochastic differential equation (SDE)

$$\forall t \in [0,T], \quad X_t = X_0 + \int_0^t f(X_s, s)\,\mathrm{d}s + \int_0^t F(X_s, s)\,\mathrm{d}W_s, \tag{14}$$

where W is a Brownian motion and $\int_0^t F(X_s, s)\,\mathrm{d}W_s$ is the Itô integral of F. Using our notation for Itô processes, we write

$$\forall t \in [0,T], \quad \mathrm{d}X_t = f(X_t, t)\,\mathrm{d}t + F(X_t, t)\,\mathrm{d}W_t. \tag{15}$$

for (14). When working with a given initial condition x_0, we specify $X_0 = x_0$ in addition by writing

$$\forall t \in [0,T], \quad \mathrm{d}X_t = f(X_t, t)\,\mathrm{d}t + F(X_t, t)\,\mathrm{d}W_t, \qquad X_0 = x_0. \tag{16}$$

Note that we allow for x_0 being a random variable.
As a motivation, we first look at simple examples, before addressing the question what we actually mean when we speak of a solution of an SDE. Then we will establish the basic existence and uniqueness result.

2.7.1 Set-up and the concept of a strong solution

Let us formulate the assumptions which will guarantee that all terms in the SDE (16) are well-defined. We assume that W is a Brownian motion and that

$$f : \mathbb{R} \times [0,T] \to \mathbb{R} \quad \text{and} \quad F : \mathbb{R} \times [0,T] \to \mathbb{R}$$

do not depend on $\omega \in \Omega$. Such a dependence will only enter via $x = X_s(\omega)$ in (14). We call f the drift coefficient and F the diffusion coefficient.
The initial condition needs to be independent of all increments of the Brownian motion. This is equivalent to assuming that X_0 is independent of W.
These considerations lead to the following assumptions:

(i) Each of the functions $f : \mathbb{R} \times [0,T] \to \mathbb{R}$ and $F : \mathbb{R} \times [0,T] \to \mathbb{R}$ is jointly measurable in $(x,t) \in \mathbb{R} \times [0,T]$ *ie* f is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}([0,T])$-$\mathcal{B}(\mathbb{R})$-measurable and F is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}([0,T])$-$\mathcal{B}(\mathbb{R})$-measurable.

(ii) The initial condition x_0 is independent to W.

(iii) The filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ is given by the completion of $\sigma(x_0, \mathcal{F}_{t+}^W)$ by all subsets of sets of $\mathbb{P}$-measure zero.

Now we are ready to formulate what we mean by a (strong) solution of the SDE (16).

Definition 43 (Strong solution) — A stochastic process $X = (X_t)_{t\in[0,T]}$ is called a (strong) solution of the SDE (16) if and only if

(i) X is $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted ;

(ii) X has continuous sample paths ;

(iii) The functions

$$\overline{f} : [0,T] \times \Omega \ni (t,\omega) \mapsto f(X_t(\omega),t) \in \mathbb{R} \quad \text{and} \quad \overline{F} : [0,T] \times \Omega \ni (t,\omega) \mapsto F(X_t(\omega),t) \in \mathbb{R}$$

satisfy the integrability conditions

$$\mathbb{P}\left(\int_0^T \|\overline{f}(s,\omega)\| \mathrm{d}s < +\infty\right) = 1 \quad \text{and} \quad \mathbb{P}\left(\int_0^T \|\overline{F}(s,\omega)\|^2 \mathrm{d}s < +\infty\right) = 1.$$

(iv) For any $t \in [0,T]$, (14) holds with probability 1.

2.7.2 Existence and uniqueness of strong solutions

The following theorem states the standard result on existence and uniqueness of a strong solution, its proof is admitted.

Theorem 44 (Existence and uniqueness of strong solutions) — Assume that there exists a constant $K > 0$ such that the following holds for all $t \in [0,T]$ and all $x, y \in \mathbb{R}^n$:

- Lipschitz condition:

$$\|f(x,t) - f(y,t)\| + \|F(x,t) - F(y,t)\| \leq K\|x - y\| \; ; \tag{17}$$

- linear-growth condition:

$$\|f(x,t)\| + \|F(x,t)\| \leq K(1 + \|x\|). \tag{18}$$

Then the SDE (16) has a $\mathbb{P}$-almost surely (pathwise) unique strong solution X. Here uniqueness means that for any two solutions X and Y,

$$\mathbb{P}\left(\sup_{0} \|X_t - Y_t\| > 0\right) = 0 \tag{19}$$

As in the deterministic case, there exists counter-examples proving that the two conditions cannot be weakened too much.

3 Adler Equation

We finally arrive to the main part of this internship report: in this section, we will study the Adler equation, after having studied a particular example and searching conditions for SDEs driven by a potential to have a stationary measure.

3.1 A first example

Before jumping introducing and studying the Adler equation, we will start with a first example of a SDE driven by a potential with a small noise.
Let W be a $\mathbb{R}$-valued Brownian motion, X_0 be a random variable with finite varianz and independent of W. Let $\alpha > 0$ and $\sigma > 0$ (where σ is little) be two positive numbers und let $T > 0$. Let us study the following SDE with $X_0 \in L^2$ as an initial condition:

$$\forall\, t \in [0,T], \quad \mathrm{d}X_t = -\nabla U(X_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t, \tag{20}$$

where $U : \mathbb{R} \ni x \mapsto \alpha\frac{x^2}{2} \in \mathbb{R}$ is represented in figure 7.

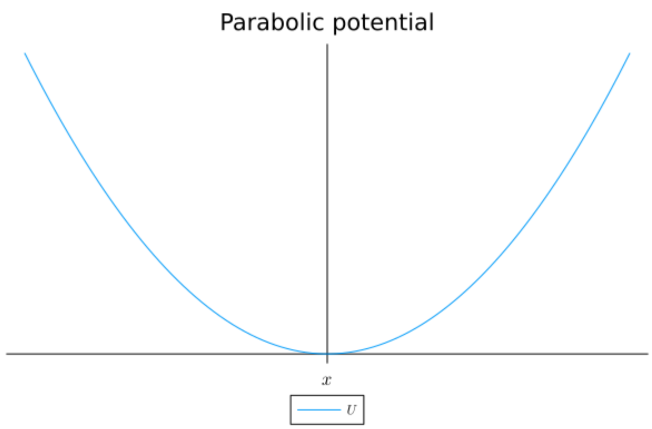


Figure 7: Representation of the parabolic potential U.

In the deterministic case *ie* in the study of the ODE $\dot{x}_t = -\nabla U(x_t)$, we know the exact behaviour of x_t in such a case: starting from the point x_0 the solution will tend as an exponential to 0 as t goes towards infinity. We want to study what will happen if we add to this process a small random perturbation.
Let us start by solving the SDE (20), let us show that there exists a unique solution and give its expression in the following result.

Result 45 — The SDE (20) has a $\mathbb{P}$-almost surely strong solution given by

$$\forall t \in [0,T], \quad X_t = \mathrm{e}^{-\alpha t}\left(X_0 + \sigma\int_0^t \mathrm{e}^{\alpha s}\,\mathrm{d}W_s\right). \tag{21}$$

Proof. The functions $f : \mathbb{R} \times [0,T] \ni (x,t) \mapsto -\alpha x \in \mathbb{R}$ and $F : \mathbb{R} \times [0,T] \ni (x,t) \mapsto \sigma \in \mathbb{R}$ are measurable and for all $t \in [0,T]$ and all $x, y \in \mathbb{R}$, the following conditions hold:

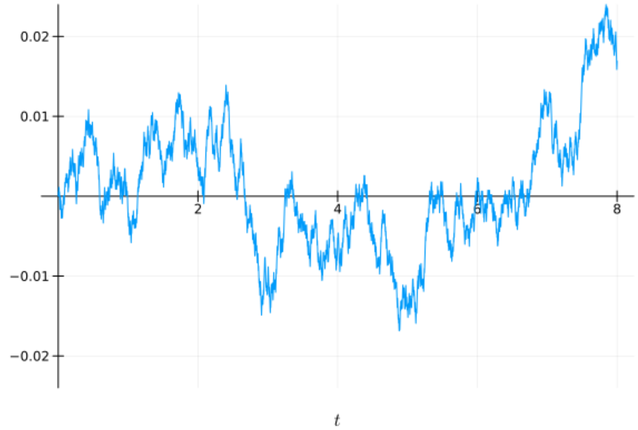


Figure 8: One sample path of the process given by (20) with $\alpha = 1$ and $\sigma = 0.1$

- (Lipschitz-condition) $\|f(x,t) - f(y,t)\| + \|F(x,t) - F(y,t)\| \leqq \alpha\|x - y\|$,
- (linear-growth-condition) $\|f(x,t)\| + \|F(x,t)\| = \alpha\|x\| + \sigma \leq (\alpha \vee \sigma)(1 + \|x\|)$.

Thus there exists a $\mathbb{P}$-almost-surely unique strong solution X to the SDE (20) with $X_0 = X_0$. What is this solution? Let $Y = (\mathrm{e}^{\alpha t} X_t)_{t\in[0,T]}$. Thanks to Itô's formula we can affirm Y follows the following SDE:

$$\begin{aligned}\mathrm{d}Y_t &= \alpha \mathrm{e}^{\alpha t} X_t \,\mathrm{d}t + \mathrm{e}^{\alpha t}\,\mathrm{d}X_t = \alpha \mathrm{e}^{\alpha t} X_t \,\mathrm{d}t + (-\alpha X_t \,\mathrm{d}t + \sigma \mathrm{d}W_t)\mathrm{e}^{\alpha t} \\ &= \mathrm{e}^{\alpha t} \sigma \mathrm{d}W_t.\end{aligned}$$

That's why we have:

$$\forall t \in [0,T], Y_t = \mathrm{e}^{\alpha t} X_t \iff \begin{cases} \forall t \in [0,T],\ \mathrm{d}Y_t = \sigma \mathrm{e}^{\alpha t}\,\mathrm{d}W_t \\ Y_0 = X_0 \end{cases}$$
$$\iff \forall t \in [0,T],\ Y_t = X_0 + \sigma \int_0^t \mathrm{e}^{\alpha s}\,\mathrm{d}W_s \iff \forall t \in [0,T],\ X_t = \mathrm{e}^{-\alpha t}\left(X_0 + \sigma \int_0^t \mathrm{e}^{\alpha s}\,\mathrm{d}W_s\right). \qquad \square$$

Then the unique solution of (20) is:

$$\forall t \in [0,T],\ X_t = \mathrm{e}^{-\alpha t}\left(X_0 + \sigma \int_0^t \mathrm{e}^{\alpha s}\,\mathrm{d}W_s\right).$$

Let us calculate the expectancy and the variance of the process X. Let $t \in [0,T]$, thanks to (21) we can affirm we have:

$$\mathbb{E}X_t = \mathrm{e}^{-\alpha t}\left(\mathbb{E}X_0 + \sigma\mathbb{E}\left[\int_0^t \mathrm{e}^{\alpha s}\,\mathrm{d}W_s\right]\right) = \mathrm{e}^{-\alpha t}\,\mathbb{E}X_0 \xrightarrow[t\to+\infty]{} 0 \tag{22}$$

and

$$\mathbb{V}X_t \overset{X_0 \perp\!\!\!\perp \mathbb{W}}{=} \mathrm{e}^{-2\alpha t}\left(\mathbb{V}X_0 + \sigma^2\mathbb{V}\left[\int_0^t \mathrm{e}^{\alpha s}\,\mathrm{d}W_s\right]\right) = \mathrm{e}^{-2\alpha t}\left(\mathbb{V}X_0 + \sigma^2\frac{\mathrm{e}^{2\alpha t} - 1}{2\alpha}\right) \underset{t\to+\infty}{\sim} \frac{\sigma^2}{2\alpha}. \tag{23}$$

For now let us suppose that X_0 is deterministic. Let us find a bound for the following quantity:

$$\mathbb{P}(|X_t - 0| \geq h) \text{ und } \mathbb{P}(|X_t - X_t^{det}| \geq h),$$

where $h > 0$ and X_t^{det} is the solution of the following ODE:

$$\begin{cases} \forall t \in [0,T],\ \frac{\mathrm{d}}{\mathrm{d}t} X_t^{det} = -\alpha X_t^{det} & , \\ X_0^{det} = X_0 & . \end{cases}$$

Let us prove the following:

Result 46 — Let $h > 0$, we have the following:

$$\mathbb{P}(|X_t - X_t^{det}| \geq h) = \mathbb{P}\left(\left|\int_0^t \mathrm{e}^{\alpha s}\,\mathrm{d}W_s\right| \geq h\right) \leq \frac{2}{\varphi(h,t)\sqrt{2\pi}} \exp\left(-\frac{\varphi(h,t)^2}{2}\right),$$

where $\varphi : (h,t) \mapsto \sqrt{\frac{2\alpha}{\mathrm{e}^{2\alpha t}-1}} \cdot \frac{h\mathrm{e}^{\alpha t}}{\sigma}$, and as soon as $h > |X_0|\mathrm{e}^{-\alpha t}$

$$\mathbb{P}(|X_t| \geq h) \leq \frac{2}{\varphi(h,t)\sqrt{2\pi}} \exp\left(-\frac{\varphi(h,t)^2}{2}\right),$$

where $\varphi : (h,t) \mapsto \sqrt{\frac{2\alpha}{\mathrm{e}^{2\alpha t}-1}} \cdot \frac{h\mathrm{e}^{\alpha t} - |X_0|}{\sigma}$.

Proof. Let $h > 0$ and $t \in [0,T]$. At first we will calculate the bound for $\mathbb{P}(|X_t| \geq h)$. We know that $\mathbb{P}(|X_t| \geq h) = \mathbb{P}(X_t \geq h) + \mathbb{P}(X_t \leq -h)$. Moreover:

$$\begin{aligned}
\mathbb{P}(X_t \geq h) &= \mathbb{P}\left(\int_0^t \mathrm{e}^{\alpha s}\,\mathrm{d}W_s \geq \frac{h\mathrm{e}^{\alpha t} - X_0}{\sigma}\right) \\
&= \mathbb{P}\left(\sqrt{\frac{2\alpha}{\mathrm{e}^{2\alpha t}-1}} \cdot \int_0^t \mathrm{e}^{\alpha s}\,\mathrm{d}W_s \geq \sqrt{\frac{2\alpha}{\mathrm{e}^{2\alpha t}-1}} \cdot \frac{h\mathrm{e}^{\alpha t} - X_0}{\sigma}\right) \\
&= \overline{\Phi}\left(\sqrt{\frac{2\alpha}{\mathrm{e}^{2\alpha t}-1}} \cdot \frac{h\mathrm{e}^{\alpha t} - X_0}{\sigma}\right),
\end{aligned}$$

where $\overline{\Phi} : x \in \mathbb{R} \mapsto \frac{1}{\sqrt{2\pi}} \int_x^{+\infty} \mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{d}x \in \mathbb{R}$, because $\sqrt{\frac{2\alpha}{\mathrm{e}^{2\alpha t}-1}} \cdot \int_0^t \mathrm{e}^{\alpha s}\,\mathrm{d}W_s \sim \mathcal{N}(0,1)$, and with similar computations:

$$\begin{aligned}
\mathbb{P}(X_t \leq -h) &= \Phi\left(-\sqrt{\frac{2\alpha}{\mathrm{e}^{2\alpha t}-1}} \cdot \frac{h\mathrm{e}^{\alpha t} + X_0}{\sigma}\right) \\
&= \overline{\Phi}\left(\sqrt{\frac{2\alpha}{\mathrm{e}^{2\alpha t}-1}} \cdot \frac{h\mathrm{e}^{\alpha t} + X_0}{\sigma}\right),
\end{aligned}$$

where $\Phi = 1 - \overline{\Phi}$.

This would give us an exact value of the searched quantity. However, as we don't know the exact value of Φ and

Φ, it may be useful to find another way to bound it. Thus we will use the following inequality:

$$\forall x > 0, \quad \overline{\Phi}(x) \leq \frac{1}{x\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}}. \tag{24}$$

Thus for all $h > 0$, such that $h > X_0 \mathrm{e}^{-\alpha t}$ and $h > -X_0 \mathrm{e}^{-\alpha t}$ *ie* $h > |X_0| \mathrm{e}^{-\alpha t}$ we have:

$$\begin{aligned}
\mathbb{P}(|X_t| \geq h) &= \overline{\Phi}\left(\varphi_1(h,t)\right) + \overline{\Phi}\left(\varphi_2(h,t)\right) \\
&\overset{(24)}{\leq} \frac{1}{\varphi_1(h,t)\sqrt{2\pi}} \exp\left(-\frac{\varphi_1(h,t)^2}{2}\right) + \frac{1}{\varphi_2(h,t)\sqrt{2\pi}} \exp\left(-\frac{\varphi_2(h,t)^2}{2}\right) \\
&\leq \frac{2}{\varphi(h,t)\sqrt{2\pi}} \exp(-\frac{\varphi(h,t)^2}{2}),
\end{aligned}$$

where $\varphi_1 : (h,t) \mapsto \sqrt{\frac{2\alpha}{\mathrm{e}^{2\alpha t}-1}} \cdot \frac{h\mathrm{e}^{\alpha t} - X_0}{\sigma}$, $\varphi_2 : (h,t) \mapsto \sqrt{\frac{2\alpha}{\mathrm{e}^{2\alpha t}-1}} \cdot \frac{h\mathrm{e}^{\alpha t} + X_0}{\sigma}$ and $\varphi : (h,t) \mapsto \sqrt{\frac{2\alpha}{\mathrm{e}^{2\alpha t}-1}} \cdot \frac{h\mathrm{e}^{\alpha t} - |X_0|}{\sigma}$.

Now, let us calculate the bound for $\mathbb{P}(|X_t - X_t^{det}| \geq h)$. We have with very similar but easier computations:

$$\mathbb{P}(|X_t - X_t^{det}| \geq h) = \mathbb{P}\left(\left|\int_0^t \mathrm{e}^{\alpha s}\,\mathrm{d}W_s\right| \geq h\right) \overset{(24)}{\leq} \frac{2}{\phi(h,t)\sqrt{2\pi}} \exp\left(-\frac{\phi(h,t)^2}{2}\right),$$

where $\phi : (h,t) \mapsto \sqrt{\frac{2\alpha}{\mathrm{e}^{2\alpha t}-1}} \cdot \frac{h\mathrm{e}^{\alpha t}}{\sigma}$. □

The previous result allow us to estimate two things:

- we can estimate how far away from the deterministic solution the stochastic one is, as for small times the right-hand quantity is really small, we see that the stochastic solution stays near the deterministic one. Moreover we can see that for really small perturbation (σ small), we need greater times to be far away from the determnistic solution ;
- we can estimate where the stochastic solution goes as time goes by. We can see that, in opposite to the deterministic case, even though the stochastic solution would start from 0, the probability that it goes far away from the limit behaviour of the deterministic solution has a non zero (and potentially big as t goes to infinity) probability to be far from zero at certain times.

Let us find a bound for the following quantity:

$$\mathbb{P}\left(\sup_{t \leq T} |X_t - 0| \geq h\right) \text{ and } \mathbb{P}\left(\sup_{t \leq T} |X_t - X_t^{det}| \geq h\right).$$

We will at first need the following lemma whose proof is simple :

Lemma 47 — For $\theta \geq 0$, $N \sim \mathcal{N}(0,1)$ and $0 \leq \gamma < \frac{1}{2\theta}$, we have

$$\mathbb{E}\left[\exp(\gamma\theta N^2)\right] = \frac{1}{\sqrt{1 - 2\gamma\theta}}.$$

Let us now prove the following:

Result 48 — Let $h > 0$, we have the following:

$$\forall \gamma \in [0, \gamma_{0,\Delta}[, \quad \mathbb{P}\left(\sup_{0\leq t\leq \Delta} |X_t| \geq h\right) \leq \frac{1}{1-\gamma/\gamma_{0,\Delta}} \cdot \exp\left(-\gamma\left[\frac{h-|X_0|}{\sigma}\right]^2\right),$$

where $\gamma_{0,\Delta} = \frac{\alpha}{\mathrm{e}^{2\alpha\Delta}-1}$, it means that

$$\mathbb{P}\left(\sup_{0\leq t\leq \Delta} |X_t| \geq h\right) \leq \inf_{\gamma\in[0,\gamma_{0,\Delta}[}\left(\frac{1}{1-\gamma/\gamma_{0,\Delta}} \cdot \exp\left(-\gamma\left[\frac{h-|X_0|}{\sigma}\right]^2\right)\right).$$

Proof. Let us start with $\mathbb{P}\left(\sup(\{|X_t|\}_{t\in[0,T]}) \geq h\right)$. We know that if $\mathbb{M} = (M_t)_{t\in[0,T]}$ is a submartingale we have:

$$\forall p \in \mathbb{N}^*,\ \mathbb{P}\left(\sup_{t\in[0,T]} M_t \geq h\right) \leq \frac{1}{h^p} \sup_{t\in[0,T]} \mathbb{E}[|M_t|^p]. \tag{25}$$

So we want to find a submartingale that could help us bound these two quantities. Let us start by recalling that by (21) we know the explicit formulation of X and that we can deduce from it that for all $0 \leq a < b \leq T$, we have:

$$\sup_{a\leq s\leq b} |X_t| = \sup_{a\leq s\leq b}\left\{\mathrm{e}^{-\alpha t}\left(X_0 + \sigma\int_0^s \mathrm{e}^{\alpha s}\,\mathrm{d}W_u\right)\right\} \leq \mathrm{e}^{-\alpha a}\left(|X_0| + \sigma \sup_{a\leq s\leq b}\left|\int_0^s \mathrm{e}^{\alpha s}\,\mathrm{d}W_u\right|\right).$$

That means that for $\Delta \in]0,T]$, $h > 0$ and $0 \leq \gamma < \gamma_{0,\Delta} := \frac{\alpha}{\mathrm{e}^{2\alpha\Delta}-1}$, we have

$$\begin{aligned}
\mathbb{P}\left(\sup_{0\leq t\leq \Delta} |X_t| \geq h\right) &\leq \mathbb{P}\left(\sup_{0\leq t\leq \Delta}\left|\int_0^s \mathrm{e}^{\alpha u}\,\mathrm{d}W_u\right| \geq \frac{h-|X_0|}{\sigma}\right) \leq \mathbb{P}\left(\sup_{0\leq t\leq \Delta}\left|\int_0^s \mathrm{e}^{\alpha u}\,\mathrm{d}W_u\right|^2 \geq \left[\frac{h-|X_0|}{\sigma}\right]^2\right) \\
&\leq \mathbb{P}\left\{\sup_{0\leq t\leq \Delta}\left\{\exp\left(\gamma\left|\int_0^s \mathrm{e}^{\alpha u}\,\mathrm{d}W_u\right|^2\right)\right\} \geq \exp\left(\gamma\left[\frac{h-|X_0|}{\sigma}\right]^2\right)\right\} \\
&\overset{\text{Doob}}{\leq} \exp\left(-\gamma\left[\frac{h-|X_0|}{\sigma}\right]^2\right) \cdot \sup_{0\leq s\leq \Delta} \mathbb{E}\left[\exp\left(\gamma\left|\int_0^s \mathrm{e}^{\alpha u}\,\mathrm{d}W_u\right|^2\right)\right].
\end{aligned}$$

That's why we got, thanks to the previous lemma,

$$\begin{aligned}
\mathbb{P}\left(\sup_{0\leq t\leq \Delta} |X_t| \geq h\right) &\leq \exp\left(-\gamma\left[\frac{h-|X_0|}{\sigma}\right]^2\right) \cdot \sup_{0\leq s\leq \Delta} \frac{1}{\sqrt{1-2\gamma\frac{\mathrm{e}^{\alpha s}-1}{2\alpha}}} \\
&\leq \frac{1}{1-\gamma/\gamma_{0,\Delta}} \cdot \exp\left(-\gamma\left[\frac{h-|X_0|}{\sigma}\right]^2\right).
\end{aligned}$$ □

One last would be to an either an estimation or the true value of the infimum in order to have the best bound. An other way to obtain an estimate of the probability would be to split the interval from 0 to T and estimate the probability on the subintervals and thanks to the Markov property of the solution of the SDE, we would get an other way to estimate. Let us show the following result :

Result 49 — For all $K \in \mathbb{N}$, $\Delta = \frac{T}{K+1}$, $0 \leq \gamma^{(0)} < \gamma_{0,\Delta}, \dots, 0 \leq \gamma^{(K)} < \gamma_{K,\Delta}$,

$$\mathbb{P}\left(\sup_{0\leq s\leq T} |X_s| \geq h\right) \leq \sum_{k=0}^{K} \frac{1}{1-\gamma/\gamma_{k,\Delta}} \cdot \mathbb{E}\left[\exp\left(-\gamma\left[\frac{h-|N_{k,\Delta}|}{\sigma e^{\alpha k\Delta}}\right]^2\right)\right],$$

where for all $k \in [\![0, K]\!]$, $N_{k,\Delta} \sim \mathcal{N}(\mathrm{e}^{-\alpha t} X_0, \frac{\sigma^2}{2\alpha}(1-\mathrm{e}^{-2\alpha t})$.

Proof. For all $k \in \mathbb{N}$, $\Delta \in]0, T]$ such that $(k+1)\Delta \leq T$, $h > 0$ and $s \in [k\Delta, (k+1)\Delta]$,

$$X_s = \mathrm{e}^{-\alpha(s-k\Delta)} X_{k\Delta} + \mathrm{e}^{-\alpha s}\sigma \int_{k\Delta}^{s} \mathrm{e}^{\alpha u}\,\mathrm{d}W_u$$

and thus

$$\sup_{k\Delta\leq s\leq(k+1)\Delta} |X_s| \leq |X_{k\Delta}| + \sigma\mathrm{e}^{-\alpha k\Delta} \sup_{s\in[k\Delta,(k+1)\Delta]} \left|\int_{k\Delta}^{s} \mathrm{e}^{\alpha u}\,\mathrm{d}W_u\right|.$$

Moreover for all $0 \leq \gamma < \gamma_{k,\Delta} := \frac{\alpha}{\mathrm{e}^{2k\alpha\Delta}(\mathrm{e}^{2\alpha\Delta}-1)} = \mathrm{e}^{-2k\alpha\Delta}\gamma_{0,\Delta}$,

$$\begin{aligned}
\mathbb{P}\left(\sup_{k\Delta\leq s\leq(k+1)\Delta} |X_s| \geq h \,\middle|\, X_{k\Delta}\right) &\leq \mathbb{P}\left(\sup_{k\Delta\leq s\leq(k+1)\Delta} \left|\int_{k\Delta}^{s} \mathrm{e}^{\alpha u}\,\mathrm{d}W_u\right|^2 \geq \left[\frac{h-|X_{k\Delta}|}{\sigma\mathrm{e}^{\alpha k\Delta}}\right]^2 \,\middle|\, X_{k\Delta}\right) \\
&\leqq \mathbb{P}\left\{\sup_{0\leq s\leq\Delta}\left\{\exp\left(\gamma\left|\int_{k\Delta}^{k\Delta+s} \mathrm{e}^{\alpha u}\,\mathrm{d}W_u\right|^2\right)\right\} \geq \exp\left(\gamma\left[\frac{h-|X_{k\Delta}|}{\sigma\mathrm{e}^{\alpha k\Delta}}\right]^2\right) \,\middle|\, X_{k\Delta}\right\} \\
&\overset{\text{Doob}}{\leq} \exp\left(-\gamma\left[\frac{h-|X_{k\Delta}|}{\sigma\mathrm{e}^{\alpha k\Delta}}\right]^2\right) \cdot \sup_{0\leq s\leq\Delta} \mathbb{E}\left[\exp\left(\gamma\left|\int_{k\Delta}^{k\Delta+s} \mathrm{e}^{\alpha u}\,\mathrm{d}W_u\right|^2\right)\right] \\
&\leqq \exp\left(-\gamma\left[\frac{h-|X_{k\Delta}|}{\sigma\mathrm{e}^{\alpha k\Delta}}\right]^2\right) \cdot \sup_{0\leq s\leq\Delta} \mathbb{E}\left[\exp\left(\gamma\left|\int_{k\Delta}^{k\Delta+s} \mathrm{e}^{\alpha u}\,\mathrm{d}W_u\right|^2\right)\right] \\
&\leq \frac{1}{1-\gamma/\gamma_{k,\Delta}} \cdot \exp\left(-\gamma\left[\frac{h-|X_{k\Delta}|}{\sigma e^{\alpha k\Delta}}\right]^2\right),
\end{aligned}$$

the third line is due to a conditional doob inequality associated with the fact that $X_{k\Delta} \perp\!\!\!\perp \int_{k\Delta}^{k\Delta+s} \mathrm{e}^{\alpha u}\,\mathrm{d}W_u$, the fifth one is due to the previous lemma. Taking the expectancy, we obtain :

$$\mathbb{P}\left(\sup_{k\Delta\leq s\leq(k+1)\Delta} |X_s| \geq h\right) \leq \frac{1}{1-\gamma/\gamma_{k,\Delta}} \cdot \mathbb{E}\left[\exp\left(-\gamma\left[\frac{h-|X_{k\Delta}|}{\sigma e^{\alpha k\Delta}}\right]^2\right)\right].$$

Thus we have for all $K \in \mathbb{N}$, $\Delta = \frac{T}{K+1}$, $0 \leq \gamma^{(0)} < \gamma_{0,\Delta}, \dots, 0 \leq \gamma^{(K)} < \gamma_{K,\Delta}$,

$$\mathbb{P}\left(\sup_{0\leq s\leq T} |X_s| \geq h\right) \overset{\sigma\text{-add.}}{\leq} \sum_{k=0}^{K} \mathbb{P}\left(\sup_{k\Delta\leq s\leq(k+1)\Delta} |X_s| \geq h\right) \leq \sum_{k=0}^{K} \frac{1}{1-\gamma/\gamma_{k,\Delta}} \cdot \mathbb{E}\left[\exp\left(-\gamma\left[\frac{h-|X_{k\Delta}|}{\sigma e^{\alpha k\Delta}}\right]^2\right)\right].$$

As we know that for all $k \in \mathbb{N}$ and $\Delta > 0$ with $k\Delta \leq T$, $X_{k\Delta} \sim \mathcal{N}\left(\mathrm{e}^{-\alpha t} X_0, \frac{\sigma^2}{2\alpha}(1-\mathrm{e}^{-2\alpha t})\right)$, we have the result.□

In order to finish with the estimations, we would need to :

1. compute the expectancies ;
2. optimize on the $\gamma^{(k)}$s.

3.2 Stationary measures on a system with a potential

We want to compute (in the case it exists) the stationary measure of the following SDE

$$\mathrm{d}X_u = -\nabla U(X_u)\,\mathrm{d}u + \sigma\,\mathrm{d}W_u, \tag{26}$$

with U a $\mathcal{C}^2(\mathbb{R},\mathbb{R})$ function. In order to do so, we will use the notion of infinitesimal generetor as explined in [Hol15]. In that case, the infinitesimal generator of the SDE is the operator $\mathcal{L}$ defined as follow:

$$\mathcal{L} : \mathcal{C}^2(\mathbb{R}\times\mathbb{R}_+,\mathbb{R}) \ni f \mapsto -\nabla U(x)\nabla_x f(x,t) + \frac{\sigma^2}{2}\nabla_x^2 f(x,t).$$

Thanks to the results of [Hol15], we have that ρ is a stationary measure of (26) if and only if $\partial_t\rho = 0 \Leftrightarrow \mathcal{L}^*\rho = 0$ with for all $f,g \in \mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{R}_+,\mathbb{R})$,

$$\begin{aligned}
\langle \mathcal{L}f, g\rangle &= \int_{\mathbb{R}} (\mathcal{L}f)(x,t)\cdot g(x,t)\,\mathrm{d}x = \int_{\mathbb{R}} \left(-\nabla U(x)\cdot\nabla_x f(x,t) + \frac{\sigma^2}{2}\nabla_x^2 f(x,t)\right)\cdot g(x,t)\,\mathrm{d}x \\
&= -\int_{\mathbb{R}} \nabla U(x) g(x,t)\cdot\nabla_x f(x,t)\,\mathrm{d}x + \frac{\sigma^2}{2}\int_{\mathbb{R}} \nabla_x^2 g(x,t)\cdot f(x,t)\,\mathrm{d}x \\
&= \int_{\mathbb{R}} \big(\nabla^2 U(x) g(x,t) + \nabla U(x)\partial_x g(x,t)\big)\cdot f(x,t)\,\mathrm{d}x + \frac{\sigma^2}{2}\int_{\mathbb{R}} \nabla_x^2 g(x,t)\cdot f(x,t)\,\mathrm{d}x \\
&= \int_{\mathbb{R}} f(x,t)\cdot\left(\nabla^2 U(x) g(x,t) + \nabla U(x)\partial_x g(x,t) + \frac{\sigma^2}{2}\partial_{xx}^2 g(x,t)\right)\,\mathrm{d}x = \langle f, \mathcal{L}^* g\rangle
\end{aligned}$$

with $\mathcal{L}^* : \mathcal{C}^2(\mathbb{R}\times\mathbb{R}_+,\mathbb{R}) \ni g \mapsto \partial_x\left(\nabla U(x) g(x,t) + \frac{\sigma^2}{2}\partial_x g(x,t)\right)$.

Thanks to this result, we have the following for any measure $\rho \ll \lambda$ (with the abuse of notation that the density function of ρ is all denoted ρ)

$$\begin{aligned}
\rho \text{ is a stationary measure} &\Leftrightarrow \begin{cases} \rho \text{ is a distribution ;} \\ \forall x\in\mathbb{R},\quad \partial_x\left(\nabla U(x)\rho(x) + \frac{\sigma^2}{2}\partial_x\rho(x)\right) = 0 \end{cases} \\
&\Leftrightarrow \begin{cases} \rho \text{ is a distribution ;} \\ \exists\lambda\in\mathbb{R} \mid \forall x\in\mathbb{R},\quad \nabla U(x)\rho(x) + \frac{\sigma^2}{2}\partial_x\rho(x) = \lambda \end{cases}
\end{aligned}$$

Moreover

$$\forall x\in\mathbb{R},\quad \nabla U(x)\rho(x) + \frac{\sigma^2}{2}\partial_x\rho(x) = 0 \Leftrightarrow \exists C\in\mathbb{R} \mid \forall x\in\mathbb{R},\quad \rho(x) = C\exp\left(-\frac{2}{\sigma^2}U(x)\right)$$

and using the method of variation of the constant for all C of class $\mathcal{C}^1$,

$$\begin{cases} \forall x \in \mathbb{R}, \quad \nabla U(x)\rho(x) + \frac{\sigma^2}{2}\partial_x \rho(x) = \lambda\,; \\ \forall x \in \mathbb{R}, \quad \rho(x) = C(x)\exp\left(-\frac{2}{\sigma^2}U(x)\right). \end{cases} \Leftrightarrow \forall x \in \mathbb{R}, \quad \left(\nabla U(x)C(x) + \frac{\sigma^2}{2}\left[C'(x) - \frac{2}{\sigma^2}\nabla U(x)C(x)\right]\right)\mathrm{e}^{-\frac{2}{\sigma^2}U(x)} = \lambda$$

$$\Leftrightarrow \forall x \in \mathbb{R}, \quad \frac{\sigma^2}{2}C'(x)\exp\left(-\frac{2}{\sigma^2}U(x)\right) = \lambda \Leftrightarrow \exists \mu \in \mathbb{R} \mid \forall x \in \mathbb{R}, \quad C(x) = \mu + \lambda\frac{2}{\sigma^2}\int_0^x \exp\left(+\frac{2}{\sigma^2}U(t)\right)\,\mathrm{d}t.$$

Thus

$$\rho \text{ is a stationary measure} \Leftrightarrow \begin{cases} \exists \lambda, \mu \in \mathbb{R} \mid \forall x \in \mathbb{R}, \quad \rho(x) = \left[\mu + \lambda \int_0^x \exp\left(+\frac{2}{\sigma^2}U(t)\right)\,\mathrm{d}t\right]\exp\left(-\frac{2}{\sigma^2}U(t)\right)\,; \\ \rho \text{ is a distribution.} \end{cases}$$

Let $\mu, \lambda \in \mathbb{R}$ and denote $\rho_{\mu,\lambda}(x) = \overline{\left[\mu + \lambda \int_0^x \exp\left(+\frac{2}{\sigma^2}U(t)\right)\,\mathrm{d}t\right]\exp\left(-\frac{2}{\sigma^2}U(t)\right)}$, let us find which $\rho_{\mu,\lambda}$ are distributions. Firstly on what conditions on (μ, λ) is $\rho_{\mu,\lambda} \geq 0$?

- If $\lambda = 0$, then we need and it suffices that $\mu \geq 0$, so that $\rho_{\mu,\lambda} \geq 0$.
- If $\lambda > 0$, then in order to have $\rho_{\mu,\lambda} \geq 0$, we need and it suffices that for all $x \in \mathbb{R}$, $\int_0^x \exp\left(+\frac{2}{\sigma^2}U(t)\right)\mathrm{d}x \geq -\frac{\mu}{\lambda}$ *ie* $\frac{\mu}{\lambda} \geq \int_{-\infty}^0 \exp\left(+\frac{2}{\sigma^2}U(t)\right)\mathrm{d}x \geq 0$. Then it is equivalent to $\mu \geq \lambda \int_{-\infty}^0 \exp\left(+\frac{2}{\sigma^2}U(t)\right)\mathrm{d}x$. Then $\rho_{\mu,\lambda} \geq 0$ is equivalent to $\exp\left(+\frac{2}{\sigma^2}U(t)\right)$ is L^1 at $-\infty$ and $\mu \geq \lambda \int_{-\infty}^0 \exp\left(+\frac{2}{\sigma^2}U(t)\right)\mathrm{d}x$.
- If $\lambda < 0$, then in order to have $\rho_{\mu,\lambda} \geq 0$, we need and it suffices that for all $x \in \mathbb{R}$, $\int_0^x \exp\left(+\frac{2}{\sigma^2}U(t)\right)\mathrm{d}x \leq -\frac{\mu}{\lambda}$ *ie* $\frac{\mu}{|\lambda|} \geq \int_0^{+\infty} \exp\left(+\frac{2}{\sigma^2}U(t)\right)\mathrm{d}x \geq 0$. Then it is equivalent to $\mu \geq |\lambda| \int_0^{+\infty} \exp\left(+\frac{2}{\sigma^2}U(t)\right)\mathrm{d}x$. Then $\rho_{\mu,\lambda} \geq 0$ is equivalent to $\exp\left(+\frac{2}{\sigma^2}U(t)\right)$ is L^1 at $+\infty$ and $\mu \geq |\lambda| \int_0^{+\infty} \exp\left(+\frac{2}{\sigma^2}U(t)\right)\mathrm{d}x$.

Due to the conditions we got and the fact that, in order to be a distribution $\rho_{\mu,\lambda}$ must be L^1, the only plausible conditions have $\lambda = 0$ as we need at least $\exp\left(-\frac{2}{\sigma^2}U(t)\right)$ to be integrable on $\mathbb{R}$ and it is incompatible with $\exp\left(+\frac{2}{\sigma^2}U(t)\right)$ being integrable at either $+\infty$ or $-\infty$.
Finally, $\rho_{\mu,\lambda}$ needs to be a distribution an only possibility for the stationary measure. To resume we got the following result.

Result 50 — The SDE (26) has a stationary measure if and only if $\exp\left(-\frac{2}{\sigma^2}U(t)\right) \in L^1(\mathbb{R})$.
In this case, the stationary distribution is unique and given by

$$\forall x \in \mathbb{R}, \rho(x) = \frac{\exp\left(-\frac{2}{\sigma^2}U(x)\right)}{\int_{\mathbb{R}} \exp\left(-\frac{2}{\sigma^2}U(t)\right)\,\mathrm{d}t}.$$

Thanks to this expression, we can see that the minima of the potential will give us the maxima of probability.

3.3 The Adler equation

The Adler equation is a fundamental concept in the field of synchronization (refering to the adjustment of rhythms in self-sustained periodic oscillators due to their weak interactions). The Adler equation plays a crucial role in understanding synchronization dynamics and has applications across various disciplines, from physics to biology. The Adler equation is named after its developer. It describes the dynamics of the phase difference between two coupled oscillators, which can be used to analyze their synchronization behavior. The Adler equation takes the form:

$$\frac{d\Delta\phi}{dt} = \omega - \omega_0 + \epsilon q(\Delta\phi)$$

where, $\Delta\phi$ represents the phase difference between the oscillators, ω is the frequency of the forcing, ω_0 is the natural frequency of the oscillators, ϵ is a coupling parameter, and $q(\Delta\phi)$ is a coupling function that depends on the phase difference.

The use of the Adler equation is motivated by its ability to capture the synchronization behavior between two coupled oscillators. It reveals how the phase difference between the oscillators evolves over time due to their weak interaction. By studying the solutions of the Adler equation, one can determine under what conditions synchronization occurs, the stability of synchronized states, and how different forms of coupling impact synchronization phenomena. For more information, on the subject of synchronization and the Adler equation, the reader can see [Sch]

Changement of the potential

In our case, we would like to study the following SDE

$$\mathrm{d}X_u = -\nabla U_A(X_u)\,\mathrm{d}u + \sigma\,\mathrm{d}W_u,$$

where $U_A : x \mapsto K\cos(x) - \omega x$ with $\omega > 0$ and $K > 0$. This is a special case of the Adler equation. However, the study of this SDE can be a little problematic. First and foremost, this potential is does not give any kind of stationary measure because it does not respect the conditions given by result 50. Moreover the fact that U_A has an infinite number of wells makes the study harder thant it needs to be. In order to provide result about the behaviour of the solution of the Adler equation, we will change our potential such that in a given domain, the potential will be similar to the previous SDE and outside of it it is not.

As long as the solution stays in that given domain, we will still have a solution of the previous SDE and when it is exiting the given domain, we just need to take a greater domain such that we can continue our study.
In order to do so, we will define the following family of potentials, for all $N \in \mathbb{N}^*$, $\omega > 0$ and $K > 0$, let us define

$$\forall x \in \mathbb{R}, \quad U_N(x) = \begin{cases} P_N(x) & \text{, si } x \leq a_N; \\ K\cos(x) - \omega x & \text{, si } a_n \leq x \leq b_N; \\ Q_N(x) & \text{, si } b_N \leq x, \end{cases}$$

where $P_N = (K\cos(a_N) - \omega a_N) + (-K\sin(a_N) - \omega)(X - a_N) + (-K\cos(a_N))\frac{(X-a_N)^2}{2}$
and $Q_N = (K\cos(b_N) - \omega b_N) + (-K\sin(b_N) - \omega)(X - b_N) + (-K\cos(b_N))\frac{(X-b_N)^2}{2}$
with $a_N = -2\pi\left\lceil \frac{N}{2} \right\rceil + \frac{2}{3}\pi$ and $b_N = 2\pi\left\lfloor \frac{N}{2} \right\rfloor + \frac{4}{3}\pi$.
In that case all U_Ns follow the conditions of result 50 and provide what we needed as on $[a_N, b_N]$, U_N and U_A have the same values.

Stopping time

In the deterministic case, the solution of the ODE associated with (3.3) will either stay on a saddle of the potential or fall into the bottom of a well. We want to show that in the stochastic case, the solution will always fall into

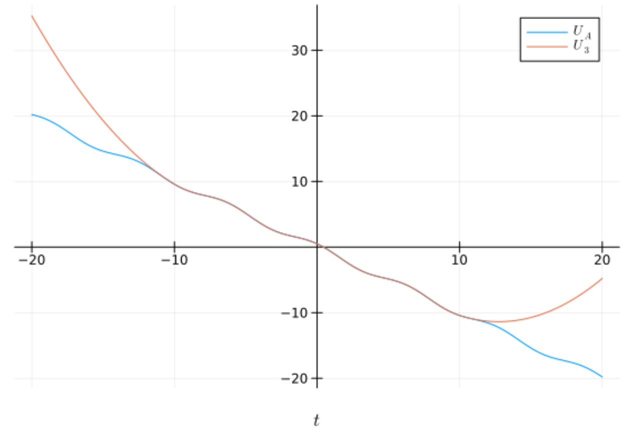


Figure 9: Representation of the Adler potential U_A and U_3 for $\omega = 1$ and $K = 0.5$

lower and lower wells.
In order to do so, we will to use theorem 5.3 of chapter 6 of [FW98]. Let us prepare everything to use it.
Firstly let us fix some notations :

- for $n \in \mathbb{N}^*$, we denote $x_1 < \cdots < x_{2n-1}$ the $2n-1$ critical points of U_n ;
- for $D \subset \mathbb{R}$, we denote $V_D(x,y) = \inf_{T>0} \left\{ \frac{1}{2} \int_0^T \|\varphi + \nabla U \circ \phi\|^2 \ : \varphi \in \mathcal{C}([0,T], \overline{D}),\, \varphi_0 = x,\, \varphi_T = y \right\}$;

Let us now fix $n \in \mathbb{N}^*$ and $\epsilon > 0$, we will study the following SDE :

$$\mathrm{d}X_u = -\nabla U_n(X_u)\,\mathrm{d}u + \epsilon\,\mathrm{d}W_u \quad \text{ and } \quad X_0 = x_1.$$

We will denote $\Delta U_1 = U(x_2) - U(x_1) = \cdots = U(x_{2n}) - U(x_{2n-1})$ and $\Delta U_2 = U(x_2) - U(x_3) = \cdots = U(x_{2n}) - U(x_{2n+1})$.

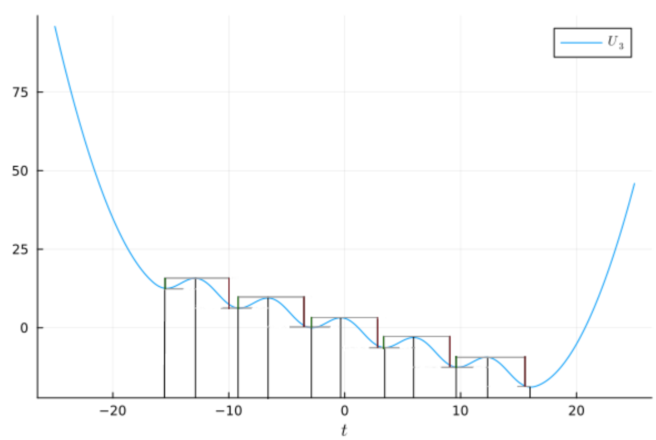


Figure 10: Illustration of the potential : the green lines illustrate ΔU_1 the red ones ΔU_2 and the black vertical ones the x_is

The domain D on which the behaviour of the stochastic process is studied must have the following properties :

- D must have a smooth boundary ;
- D must have a compact closure ;
- there is in D a finite number of compacta $K_1, \dots, K_l$ such that:
  - for all $i \in [\![1, l]\!]$, for all $x, y \in K_i$, $x \sim_D y$ (in the sense that $V_D(x, y) = V_D(y, x) = 0$);
  - for all $i \in [\![1, l]\!]$, for all $x \in K_i$ and $y \in D \setminus K_i$, $x \not\sim_D y$;
  - each ω-limitset of $\dot{x}_t = b(x_t)$ that is contained in $\overline{D}$, is in one of the K_is.

Thus we take $R > 0$ (with $U_n(-R) - U_n(x_1) > 2(n-1) \times \Delta U_1$) and $\delta > 0$ (as little as possible) and we denote $D =]-R, x_{2n-1} - \delta[$ (where x_{2n-2} is defined by the figure). Then D has the previous properties with $K_1 = \{x_1\}, \dots, K_{2n-1} = \{x_{2n-1}\}$. Thanks to theorem 5.3 of FW we can affirm:

$$\epsilon^2 \ln \mathbb{E}^{\epsilon}_{x_1} \tau^{\epsilon} \xrightarrow[\epsilon \to 0]{} W_D - M_D(x_1)$$

where $\tau^{\epsilon} = \inf\{t > 0 : X^{\epsilon}_t \notin D\}$ and W_D and $M_D(x_1)$ are defined in the book, their definitions will be recalled as we calculate them.

Before we compute these quantities, we need to compute the $V_D(K_i, K_j)$ and $V_D(K_i, \partial D)$. Thanks to theorem 5.3 we can only compute them for $i, j \in \{1, 3, \dots, 2n-3\}$ and in that case

$$V_D(K_i, K_j) = \begin{cases} 2 \cdot \Delta U_1 \cdot \frac{j-i}{2} & \text{, si } i \leq j \\ 2 \cdot \Delta U_2 \cdot \frac{i-j}{2} & \text{, si } j \leq i \end{cases} \quad \text{and} \quad V_D(K_i, \partial D) = 2 \cdot \Delta U_1 \cdot \frac{2n-1-i}{2}.$$

Let us compute W_D where

$$W_D = \max_{g \in G\{\partial D\}} \sum_{(\alpha \to \beta) \in g} V_D(\alpha, \beta),$$

where $G\{\partial D\}$ is defined on $\{K_1, K_3, \dots, K_{2n-1}, \partial D\}$. For all $g \in G\{\partial D\}$,

- if there are $i < j$ such that $K_i \to \partial D, K_j \to \partial D \in g$, we have

$$\sum_{(\alpha \to \beta) \in g'} V_D(\alpha, \beta) \leq \sum_{(\alpha \to \beta) \in g} V_D(\alpha, \beta),$$

  where g' is the same as g but for $K_i \to \partial D$ being exchanged by $K_i \to K_j$.
- if there are $i < j$ such that $K_j \to K_i \in g$, we have

$$\sum_{(\alpha \to \beta) \in g'} V_D(\alpha, \beta) \leq \sum_{(\alpha \to \beta) \in g} V_D(\alpha, \beta),$$

  where g' is the same as g but for $K_j \to K_i$ being exchanged by $K_j \to K_{j+1}$.

Thus

$$\boldsymbol{W_D} = V_D(K_1, K_3) + \cdots + V_D(K_{2n-3}, K_{2n-1}) + V_D(K_{2n-1}, \partial D) = \mathbf{2 \times \Delta U_1 \times (n-1)}.$$

Let us compute $M_D(x_1)$ where

$$M_D(x_1) = W_D \wedge \min_{i \in \{1,3,\dots,2n-3\}} (V_D(x_1, K_i) + M_D(K_i)) = W_D \wedge \min_{i \in \{1,3,\dots,2n-3\}} (V_D(K_1, K_i) + M_D(K_i)).$$

- For $i \in \{1,3,\dots,2n-3\}$, we have

$$V_D(K_1, K_i) = 2 \cdot \Delta U_1 \cdot \frac{i-1}{2}.$$

- For $i \in \{1,3,\dots,2n-3\}$, we have

$$M_D(K_i) = \min_{g \in G(K_i \not\to \{\partial D\})} \sum_{(\alpha \to \beta) \in g} V_D(\alpha, \beta) \leq 2 \cdot \Delta U_1 \cdot (n-2).$$

  For $g = (K_1 \to K_3 \to \cdots \to K_{2n-3}\ \partial D) \in G(K_i \not\to \{\partial D\})$,

$$\sum_{(\alpha \to \beta) \in g} V_D(\alpha, \beta) = 2 \cdot \Delta U_1 \cdot (n-2),$$

  that's why $M_D(K_i) = 2 \cdot \Delta U_1 \cdot (n-2)$.

Thus

$$\boldsymbol{M_D(x_1) = 2 \cdot \Delta U_1 \cdot (n-2)}.$$

Thats why we have this result

Result 51 — We have the following estimation

$$\mathbb{E}^{\epsilon}_{x_1} \tau^{\epsilon} \underset{\epsilon \to 0}{\asymp} \exp\left(\frac{2 \cdot \Delta U_1}{\epsilon^2}\right).$$

We can estimate the time needed to get out of the domain D, but we can also ask ourselves in which part of ∂D the process exit.

Intuitively we can say that if R is big enough, the probability to exit the domain in $-R$ is really littler than the probability to exit the domain in $x_{2n-1} - \delta$. Moreover the theorem 5.2 of chapter 6 of [FW98] will help us to determine the exit point. We have with the notations of theorem for $i = 1$

- $Y_1 = \arg\min_{y \in \partial D} V_D(K_1, y) = \{x_{2n-1} - \delta\}$ (when $U(-R) - U(x_1) > n\Delta U_1$) ;
- $x = x_1$;
- as $\{g \in G\{\partial D\} : W_D = \sum_{(\alpha \to \beta) \in g} V_D(\alpha, \beta)\}$ we have $M(1) = \{2n-2\}$,

then we have that final result

Result 52 — We have the following

$$\mathbb{P}\left(X^{\epsilon}_{\tau^{\epsilon}} = x_{2n-1} - \delta\right) \xrightarrow[\epsilon \to 0]{} 1.$$

(If we look precisely into theorem 5.2 $X_{\tau^{\epsilon}}$ is only in a neighbourhood of x_{2n-1} but with THEOREM 5.1 we can choose this neighbourhood as small as possible.)

Conclusion

To conclude, we have seen that the behaviour of the stochastic Adler equation tends to goes deeper and deeper in the lower potential. However, it could interesting to see what would differ if we want to study directly the stochastic Adler equation without usin another potential to do the calulations.

References

[Mey66] Paul André Meyer. *Probability and potentials*. 1966.

[McK69] Henry P McKean. *Stochastic integrals*. Vol. 353. American Mathematical Soc., 1969.

[KS91] Ioannis Karatzas and Steven Shreve. *Brownian motion and stochastic calculus*. Vol. 113. Springer Science & Business Media, 1991.

[Kal97] Olav Kallenberg. *Foundations of modern probability*. Vol. 2. Springer, 1997.

[FW98] Mark Iosifovich Freidlin and Alexander D Wentzell. *Random perturbations*. Springer, 1998.

[Kle13] Achim Klenke. *Probability theory: a comprehensive course*. Springer Science & Business Media, 2013.

[Hol15] Miranda Holmes-Cerfon. *Applied Stochastic Analysis*. `https://cims.nyu.edu/~holmes/teaching/asa19/handout_Lecture10_2019.pdf`. Accessed: 2023-06-21. 2015.

[Li17] Xue-Mei Li. "Strict local martingales: Examples". In: *Statistics & Probability Letters* 129 (2017), pp. 65–68.

[Sch] Scholarpedia. *Synchronization - Scholarpedia*. `http://www.scholarpedia.org/article/Synchronization`. Accessed: 2023-07-29.

Appendices

A Algorithms

A.1 Construction of a Brownian motion thanks to the Haar basis

```julia
using LaTeXStrings
using Plots
using Random, Distributions

const N::Int = 10
const NAME::String = "brownian_motion_haar_basis"
const NB_BROWNIAN_MOTION::Int = 8
const NB_STEPS::Int = 2^N
const STANDARD_NORMAL = Normal()

function integral_haar_basis_element(n::Int,k::Int)
  if n == 0
    return (t::Float64 -> t < 0.0 ? 0.0 : (t <= 1.0 ? t : 1.0))
  else
    function F(t::Float64)
      if (2^n * t < 2k - 2) || (2^n * t >= 2k)
        return 0.0
      elseif (2^n * t < 2k - 1)
        return (2^((n-1)/2)) * (t - (2k-2)/(2^n))
      else
        return (2 ^ ((n-1)/2)) * ((2k)/(2^n) - t)
      end
    end

    return F
  end
end

function brownian_weights(N::Int)
  return vcat([[rand(STANDARD_NORMAL)]],
               [[rand(STANDARD_NORMAL) for k in 1:(2^(n-1))] for n in 1:N])
end

function standard_brownian_motion_haar_basis(N::Int,NB_STEPS::Int,Ξ,nb_motions::Int=1)
  time = [i/NB_STEPS for i in 0:NB_STEPS]

  ξ = [brownian_weights(N::Int) for l in 1:nb_motions]

  values = [[Ξ[1][1] * integral_haar_basis_element(0,0)(t) for t in time] for Ξ in ξ]

  for l in 1:nb_motions
    for n in 1:N
      for k in 1:(2^(n-1))
        for i in 1:(NB_STEPS+1)
          values[l][i] += ξ[l][n+1][k] * integral_haar_basis_element(n,k)(time[i])
        end
      end
    end
  end

  return (time, values)
end
```

```

Ξ = brownian_weights(N)
#print(Ξ)

t, W = standard_brownian_motion_haar_basis(N,NB_STEPS,Ξ,NB_BROWNIAN_MOTION)

window_size = maximum([maximum(W[1]),-minimum(W[1])])
p = plot(t, W[1], lw=1, framestyle=:origin, legend=false)

for i in 2:NB_BROWNIAN_MOTION
  global window_size = maximum([window_size,maximum(W[i]),-minimum(W[i])])
  plot!(t, W[i], lw=1)
end

title!("Some Standard Brownian Motions Obtained\nWith Haar Basis")
xlabel!(L"$t$")
ylims!(-window_size,window_size)

Plots.png(p, NAME)
#Plots.pdf(p, NAME)
#Plots.svg(p, NAME)
```

A.2 Construction of a Brownian motion thanks to the random walk

```
using LaTeXStrings
using Plots
using Random

const NAME::String = "brownian_motion_random_walk"
const NB_BROWNIAN_MOTION::Int = 8
const NB_STEPS::Int = 2^10

function random_walk(nb_steps::Int, position::Int=0)
    walk = [position]
    for i in 1:nb_steps
        movement = rand([-1, 1])
        position += movement
        push!(walk,position)
    end
    return walk
end

function standard_brownian_motion_random_walk(nb_steps::Int, nb_brownian_motions::Int=1)
    walk = [random_walk(nb_steps) / sqrt(nb_steps) for i in 1:nb_brownian_motions]
    return ((collect(0:nb_steps) / nb_steps), walk)
end

# Exemple d'utilisation avec N pas

t, W = standard_brownian_motion_random_walk(NB_STEPS,NB_BROWNIAN_MOTION)

window_size = maximum([maximum(W[1]),-minimum(W[1])])
p = plot(t, W[1], lw=1, framestyle=:origin, legend=false)

for i in 2:NB_BROWNIAN_MOTION
  global window_size = maximum([window_size,maximum(W[i]),-minimum(W[i])])
  plot!(t, W[i], lw=1)
end
```

```

title!("Some Standard Brownian Motions Obtained\nWith Random Walks")
xlabel!(L"$t$")
ylims!(-window_size,window_size)

Plots.png(p, NAME)
#Plots.pdf(p, NAME)
#Plots.svg(p, NAME)
```

B Figures

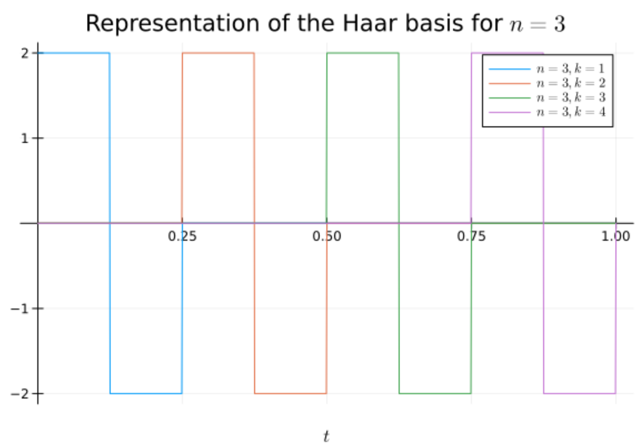


Figure 11: Representation of the Haar basis

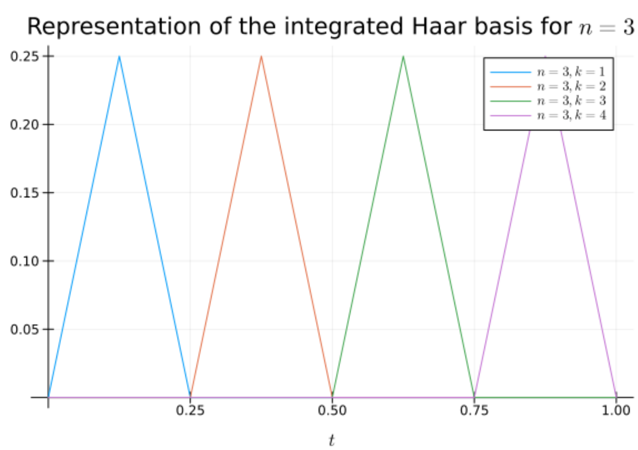


Figure 12: Representation of the integrated Haar basis

C Complements to section 1

Before defining our objects, let us fix an interval $I \subset \mathbb{R}$ starting at 0 *ie* either $I = [0, T]$ for $T > 0$ or $I = \mathbb{R}_+$.

C.1 General stochastic processes in continuous time

In this subsection, we will define the general concept of E-valued stochastic processes in continuous time. We want to generalise the notion of stochastic processes in discrete time on the interval I thanks to definition 53.

Definition 53 (Stochastic process in continuous time and sample paths) — A sequence $X = (X_t)_{t\in I}$ of E-valued random elements $X_t : \Omega \to E$ is called a (time-continuous) E-valued stochastic process.

For any time-continuous E-valued stochastic process $X = (X_t)_{t\in I}$,

(a) the maps $T \ni t \mapsto X_t(\omega) \in E$, considered for fixed $\omega \in \Omega$, are called the *sample paths of the stochastic process X*.

(b) assume that E is equipped with a topology. We say that the stochastic process X has *continuous sample paths* if and only if the sample paths $t \mapsto X_t(\omega)$ are continuous for all $\omega \in \Omega$.

We will study $\mathbb{P}$-almost sure properties of sample paths, viewing sample paths as E-valued functions, as well as statistical properties of stochastic process X, *ie*, the probability measures $\mathbb{P}\overset{-1}{X}$. However for $\mathbb{P}\overset{-1}{X}$ to be a probability measure, we need to show that X is a random element, what theorem 54 gives us.

Theorem 54 (Measurability of stochastic processes in continuous time) — Fix a subset $U \subset E^I$. Then, $X : \Omega \to U$ is $\mathcal{F}$-$\mathcal{E}^I|_U$-measurable if and only if for all $t \in I$, X_t is $\mathcal{F}$-$\mathcal{E}$-measurable.

Proof (Theorem 54). We have easily that $X : \Omega \to U$ is $\mathcal{F}$-$\mathcal{E}^I|_U$-measurable if and only if $X : \Omega \to U$ is $\mathcal{F}$-$\mathcal{E}^I$-measurable. Indeed for all $A \in \mathcal{E}^I$, $\{X \in A\} = \{X \in A \cap U\}$ and thus one measurability gives the other.
For any $n \in \mathbb{N}^*$, for all $t = (t_1, \dots, t_n) \in I^n$, for all $B_1, \dots, B_n \in \mathcal{E}$ and $A = \overset{-1}{\pi_t}(B_1 \times \cdots \times B_n)$,

$$\{X \in A\} = \{\pi_t \circ X \in B_1 \times \cdots \times B_n\} = \{(X_{t_1}, \dots, X_{t_n}) \in B_1 \times \cdots \times B_n\}.$$

And $(X_{t_1}, \dots, X_{t_n})$ is $\mathcal{F}$-$\mathcal{E}^n$-measurable if and only if $X_{t_1}, \dots, X_{t_n}$ are $\mathcal{F}$-$\mathcal{E}$-measurable.
That's why $X : \Omega \to U$ is $\mathcal{F}$-$\mathcal{E}^I|_U$-measurable if and only if for all $t \in I$, X_t is $\mathcal{F}$-$\mathcal{E}$-measurable. □

Another useful concept is the one of finite-dimensional distributions (defined in definition 55).

Definition 55 (Finite-dimensional distributions) — We define the following.

(a) Let $\mathbb{Q}$ be a probability measure on $(E^I, \mathcal{E}^I)$, then for any $n \in \mathbb{N}^*$ and any choices of $t = (t_1, \dots, t_n) \in I^n$,

$$\mathbb{Q}_t = \mathbb{Q}\overset{-1}{\pi_t}$$

defines a probability measure on $(E^n, \mathcal{E}^{\otimes n})$. The probability measures $\mathbb{Q}_t$ are called finite-dimensional distributions of $\mathbb{Q}$.

(b) The family of *finite-dimensional distributions of a (time-continuous) E-valued stochastic process X* is the family

$$\left\{\mu_t^X : n \in \mathbb{N}^*, t = (t_1, \dots, t_n) \in I^n\right\}$$

of finite-dimensional distributions of $\mathbb{P}\overset{-1}{X}$ ie μ_t^X is given by

$$\forall A \in \mathcal{E}^{\otimes n}, \quad \mu_t^X(A) = \mathbb{P}\overset{-1}{X}\overset{-1}{\pi_t}(A) = \mathbb{P}((X_{t_1}, \dots, X_{t_n}) \in A).$$

The finite-dimensional distributions determine the probability measures on $(E^I, \mathcal{E}^I)$ uniquely as can be seen thanks to theorem 56.

Theorem 56 (The finite-dimensional distributions determine a probability measure uniquely) — The family of its finite-dimensional distributions $\{\mathbb{Q}_t : n \in \mathbb{N}^*, t = (t_1, \dots, t_n) \in I^n\}$ determines a probability measure $\mathbb{Q}$ on $(E^I, \mathcal{E}^I)$ uniquely.

Proof (Theorem 56). The result is obtained thanks to the fact that $\mathcal{E}^I$ is generated by the π-system formed by the finite-dimensional cylinder set. □

Given a family of distributions on the $(E^n, \mathcal{E}^{\otimes n})$, are we able to construct a probability measure on $(E^I, \mathcal{E}^I)$ whose finite-dimensional are given by the family? To answer this question, let us first show the following result.

Lemma 57 (Finite-dimensional distributions and projections) — Let $\{\mathbb{Q}_t : n \in \mathbb{N}^*, t = (t_1, \dots, t_n) \in I^n\}$ be the family of finite-dimensional distributions of some probability measures $\mathbb{Q}$ on $(E^I, \mathcal{E}^I)$. Then, the following assertion holds:

$$\forall n \in \mathbb{N}^*, \forall t = (t_1, \dots, t_n) \in I^n, \quad \mathbb{Q}_t \overset{-1}{\widehat{\varphi_j^{(n)}}} = \mathbb{Q}_{(t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_n)}.$$

Proof (Lemma 57). With the notations of the lemma, let $n \in \mathbb{N}$ and $t = (t_1, \dots, t_n) \in I^n$. Let $B_1, \dots, B_n \in \mathcal{E}$ with $B_j = E$. We have the following

$$\begin{aligned}
\mathbb{Q}_t \overset{-1}{\widehat{\varphi_j^{(n)}}} (B_1 \times \cdots \times B_{j-1} \times B_{j+1} \times \cdots \times B_n) &= \mathbb{Q}_t (B_1 \times \cdots \times B_n) = \mathbb{Q}_t (\pi_{t_1} \in B_1, \dots, \pi_{t_n} \in B_n) \\
&= \mathbb{Q}_t \left(\pi_{t_1} \in B_1, \dots, \pi_{t_{j-1}} \in B_{j-1}, \pi_{t_{j+1}} \in B_{j+1}, \dots, \pi_{t_n} \in B_n\right) \\
&= \mathbb{Q}_{(t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_n)} (B_1 \times \cdots \times B_{j-1} \times B_{j+1} \times \cdots \times B_n)
\end{aligned}$$

Thus we have the expected result. □

In order to get a probability measure based on the projections, we will first define the Kolmogorov's consistency condition in definition 58

Definition 58 (Consistent family of probability measures) — We say that a family $\mathcal{M} = \{\mu_t : n \in \mathbb{N}^*, t = (t_1, \dots, t_n) \in I^n\}$ of probability measures, where μ_t is defined on $(E^n, \mathcal{E}^{\otimes n})$ for all $n \in \mathbb{N}$

and $t = (t_1, \ldots, t_n) \in I^n$, satisfies *Kolmogorov's consistency condition* if and only if the following holds:

$$\forall n \in \mathbb{N}^n, \forall j \in [\![1, n]\!], \forall t = (t_1, \ldots, t_n) \in I^n, \quad \mu_t \overset{-1}{\widehat{\varphi_j^{(n)}}} = \mu_{(t_1, \ldots, t_{j-1}, t_{j+1} \ldots, t_n)}.$$

We recall the definition of a Borel space.

Definition 59 (Borel space) — Let $(S, \mathcal{S})$ be a measurable space.
(a) A one-to-one map $\phi : S \to \mathbb{R}$ or to measurable subset of $\mathbb{R}$ such that ϕ and ϕ^{-1} are both measurable, is called a *Borel isomorphism.*
(b) If $(S, \mathcal{S})$ is Borel isomorphic to a measurable subset of the real numbers, then $(S, \mathcal{S})$ is called a *Borel space.*

For consistent families of probability measures, theorem 60 holds.

Theorem 60 (Kolmogorov's extension theorem, see [Kle13] theorem 14.36) — Assume that $(E, \mathcal{E})$ is a Borel space. Then, for any consistent family $\mathcal{M}$ of probability measures, there exists a uniquely determined probability measure $\mathbb{Q}$ on $(E^I, \mathcal{E}^I)$ such that the finite-dimensional distributions of $\mathbb{Q}$ are given by $\mathcal{M}$ ie

$$\forall n \in \mathbb{N}^n, \forall t = (t_1, \ldots, t_n) \in I^n, \quad \mathbb{Q}\overset{-1}{\pi_t} = \mu_t.$$

Before proceeding on the notions of Gaussian processes, let us define the concept of *canonical model* thanks to the following definition.

Definition 61 (Canonical model) — Given a consistent family $\mathcal{M}$ of probability measures, let $\mathbb{Q}$ denote the probability measure on $(E^I, \mathcal{E}^I)$ with finite-dimensional distributions $\mathcal{M}$. Then, the evaluation maps $(\pi_t)_{t \in I}$ define an E-valued stochastic process with these finite-dimensional distributions *ie* we can work on $(E^I, \mathcal{E}^I, \mathbb{Q})$ with the stochastic process $(\pi_t)_{t \in I}$ instead of working on $(\Omega, \mathcal{F}, \mathbb{P})$.

C.2 Gaussian processes

In this subsection, we will define the notion of Gaussian processes.

Definition 62 (Gaussian processes on $\mathbb{R}^n$) — We definite the following.
(a) An $\mathbb{R}$-valued stochastic process X is called a *Gaussian process* if and only if all its finite-dimensional distributions are normal distributions *ie* Gaussian vectors.
(b) A Gaussian process X is called *centred* if and only if for all $t \in I$, $\mathbb{E}X_t = 0$.

Since a n-dimensional normal distribution is uniquely determined by its expectation and its covariance matrix, this motivates theorem 63 whose proof is in the appendix on page **??**.

Theorem 63 (The finite-dimensional distributions of a Gaussian process) — We have the following results.

(a) The finite-dimensional of a Gaussian process X are uniquely determined by the functions $m : I \ni t \mapsto \mathbb{E}X_t \in \mathbb{R}$ and $\sigma : I \times I \ni (s,t) \mapsto \mathrm{Cov}(X_s, X_t) \in \mathbb{R}$.

(b) The function σ as defined in (a) has the following properties:
- for all $s, t \in I$, $\sigma(s,t) = \sigma(t,s)$;
- for any choice of $t_1, \dots, t_n \in I$, the symmetric matrix $(\sigma(t_i, t_j))_{i,j \in [\![1,n]\!]}$ is positive semi-definite.

(c) Let functions $\tilde{m} : I \to \mathbb{R}$ and $\tilde{\sigma} : I \times I \to \mathbb{R}$ be given and assume that $\tilde{\sigma}$ satisfies the properties (b). Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a Gaussian process X defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that for all $t \in I$, $\tilde{m}(t) = \mathbb{E}X_t$ and for $s, t \in I$, $\tilde{\sigma}(s,t) = \mathrm{Cov}(X_s, X_t)$.

Proof (Theorem 60). The points (a) and (b) are easy to prove. Let us prove the third point.
We will work with the canonical model *ie* we choose $(\Omega, \mathcal{F}) = (\mathbb{R}^I, \mathcal{B}^I)$ and the stochastic process will be given by the evaluation maps, so that $X = (\pi_t)_{t \in I}$. All we need to do is to verify the consistency condition from definition 58.
Let $t \in I^n$, assume that μ_t^X is the normal distribution on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ with mean $(\tilde{m}(t_1), \dots, \tilde{m}(t_n))$ and covariance matrix $(\sigma(t_i, t_j))_{i,j \in [\![1,n]\!]}$. Fix $j \in [\![1,n]\!]$. $\phi_j^{(n)}$ is a linear map and thus $\mu_t^X \overset{-1}{\pi_t}$ has a normal ditribution. It is easy to check that $\mu_t^X \overset{-1}{\pi_t}$ has mean $(\tilde{m}(t_1), \dots, \tilde{m}(t_{j-1}), \tilde{m}(t_{j+1}), \dots, \tilde{m}(t_n))$ and covariance matrix $(\sigma(t_i, t_j))_{i,j \in [\![1,n]\!] \setminus \{j\}}$. Since mean and covariance determine a normal distribution uniquely

$$\mu_t^X \overset{-1}{\pi_t} = \mu_{(t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_n)}.$$

Thus we have the consistency condition. □

C.3 Version of a stochastic process and indistiguishability

In this subsection, we will discuss about tools showing that two different stochastic processes can be considered as the same (except for one change) and how to get a stochastic process with continuous sample paths from a stochastic process.

Definition 64 (Version of a stochastic process, indistinguishability) — Let X and Y be two E-valued stochastic processes.

(a) We say that Y is a version (or a modification) of X if and only if for all $t \in I$, $\mathbb{P}(X_t = Y_t) = 1$.

(b) We say that X and Y are indistinguishable if and only if there exists a set $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that for all $t \in I$, $\omega \in \Omega_0$, $X_t(\omega) = Y_t(\omega)$.

In the general case, we don't know whether the set $\{X = Y\}$ is an event, that's why we have such a definition of indistinguishability. The lemma 65 42 gives us the possibility to write $\mathbb{P}(X = Y)$.

Lemma 65 (Continuity of sample paths and indistiguishability) — Let X and Y be two right-continuous (or left-continuous) stochastic processes such that for all $t \in T$, $\mathbb{P}(X_t = Y_t) = 1$. Then,

$$\{X = Y\} \in \mathcal{F} \text{ and } \mathbb{P}(X = Y) = 1.$$

Proof (Lemma 65). With the notations of the lemma, we want to show that $\{X = Y\} \in \mathcal{F}$ and that $\mathbb{P}(X = Y) = 1$. Whether X, Y are both right-continuous or both left-continuous, we have the following

$$\{X = Y\} = \bigcap_{t \in I} \{X_t = Y_t\} = \bigcap_{t \in I \cap \mathbb{Q}} \{X_t = Y_t\} \in \mathcal{F}.$$

Moreover, we have then

$$\mathbb{P}(X = Y) = \mathbb{P}\left(\bigcap_{t \in I \cap \mathbb{Q}} \{X_t = Y_t\}\right) = 1 - \mathbb{P}\left(\bigcup_{t \in I \cap \mathbb{Q}} \{X_t \neq Y_t\}\right) \overset{\sigma\text{-add.}}{\geq} 1 - \sum_{t \in I \cap \mathbb{Q}} \mathbb{P}\left(X_t \neq Y_t\right) = 1,$$

thus we have $\mathbb{P}(X = Y) = 1$. □

One last thing we can see is that the two defined concepts defined in definition 64 aren't the same.
Let $U \sim \mathcal{U}([0,1])$ be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. Consider $X = (\mathbb{1}_{\{t=U\}})_{t \in [0,1]}$ and $Y = 0$. Then:

- $\forall t \in [0,1], \quad \mathbb{P}(X_t = Y_t) = \mathbb{P}(X_t = 0) = \mathbb{P}(U \neq t) = 1$;
- $\{\omega \in \Omega : \forall t \in T, X_t(\omega) = Y_t(\omega)\} = \emptyset$.

Thus Y is a version of X but they are not indistinguishable. The problem lies in the fact that $[0,1]$ is uncountable.

Having a stochastic process, can we construct a new stochastic process that is like the first one but with continuous paths? The following theorem allows us to answer this question.

Theorem 66 (Kolmogorov's continuity theorem, see [Kal97] theorem 3.23) — Let X be a stochastic process taking values in a complete metric space (M, d) and assume that there exists constants $q > 0$ and $\alpha > 1$ as well as $C, r > 0$ such that

$$\forall s, t \in I, 0 < t - s \leq r \Rightarrow \mathbb{E}[d(X_s, X_t)^q] \leq C|s - t|^\alpha.$$

Then X has a version with continuous sample paths.

C.4 Filtrations

A first useful notions concerning stochastic processes in continuous time is an idea we already saw in the case of discrete time: the notion of filtrations.

Definition 67 (Filtrations and adapted stochastic processes) — Let $X = (X_t)_{t \in I}$ be a stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We define the folowing objects.

(a) For all $t \in I$, define $\mathcal{F}_t^X = \sigma(\{X_s : 0 \leq s \leq t\})$. The family of σ-fields $\mathcal{F}^X$ is called the filtration generated by X.

(b) More generally, a family $\{\mathcal{F}_t\}_{t\in I}$ of σ-fields is called a filtration of $\mathcal{F}$ if and only if $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ holds for all $s < t$ in I.

(c) An E-valued stochastic process X is called $\{\mathcal{F}_t\}_{t\in I}$-adapted if and only if X_t is $\mathcal{F}_t$-$\mathcal{E}$-measurable for all $t \in I$.

We define for $\{\mathcal{F}_t\}_{t\in I}$ a filtration

$$\forall t \in I, \mathcal{F}_{t+} = \bigcap_{u\in I\,:\,u>t} \mathcal{F}_u = \bigcap_{n\in\mathbb{N}^*\,:\,t+1/n\in I} \mathcal{F}_{t+1/n}.$$

We can think of $\mathcal{F}_{t+}$ as the events that occurs immediatly after time t.

A filtration $\{\mathcal{F}_t\}_{t\in I}$ is called right-continuous if and only if for all $t \in I$, $\mathcal{F}_{t+} = \mathcal{F}_t$ holds.
A filtration $\{\mathcal{F}_t\}_{t\in I}$ is said to satisfy the usual conditions if and only if

(i) $\{\mathcal{F}_t\}_{t\in I}$ is right-continuous ;
(ii) $\mathcal{F}_0$ contains all $N \in \mathcal{F}_\infty = \sigma(\mathcal{F}_t : t \in I)$ with $\mathbb{P}(N) = 0$.

In some cases, we will need that our stochastic processes are not only measurable with respect to the whole σ-algebra but also at each time for each element of our filtrations, thus we need the following definition.

Definition 68 (Progressively measurable stochastic processes) — An E-valued stochastic process X is called progressively measurable with respect to $\{\mathcal{F}_t\}_{t\in I}$ if and only if for every $t \in I$, the map $[0,t] \times \Omega \ni (s,\omega) \mapsto X_s(\omega) \in E$ is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-$\mathcal{E}$-measurable.

Lemma 69 shows that the assumption that a stochastic process is progressively measurable is not too restrictive as soon as the stochastic processes is right-continuous.

Lemma 69 (Right-continuity implies progressive measurability for adapted processes) — Every $\{\mathcal{F}_t\}_{t\in I}$-adapted $\mathbb{R}^d$-valued stochastic process X with right-continuous sample paths is progressively measurable with respect to $\{\mathcal{F}_t\}_{t\in I}$.

Proof (Lemma 69). Let us fix $t \in T$, for any $n \in \mathbb{N}$, we define

$$\forall s \in [0,t], \quad X_s^{(n)} = X_0 \mathbb{1}_{\{0\}}(s) + \sum_{k=1}^{n} X_{\frac{k}{n}t} \mathbb{1}_{\left]\frac{k-1}{n}t, \frac{k}{n}t\right]}(s).$$

Then $(s,\omega) \mapsto X_s^{(n)}(\omega)$ is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-$\mathcal{B}(\mathbb{R}^d)$-mesurable. Now as X has right-continuous sample paths, for any fixed $\omega \in \Omega$ and $s \in [0,t]$, $X_s^{(n)}(\omega) \xrightarrow[n\to+\infty]{} X_s(\omega)$. Thus X is progressively measurable with respect to $\{\mathcal{F}_t\}_{t\in I}$. $\square$

C.5 Stopping times

In this subsection we will define the notion of stopping times which will be as useful as it is in the case of stochastic processes in discrete time. As always, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ a filtration of $\mathcal{F}$. Let us define what is a stopping time.

Definition 70 (Stopping time) — An $\mathcal{F}$-$\overline{\mathcal{B}}$-measurable function $\tau : \Omega \to \overline{\mathbb{R}_+}$ is called an $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$-stopping time if and only if

$$\forall t \geq 0, \quad \{\tau \leq t\} \in \mathcal{F}_t.$$

The stopping times in the continuous case share with the ones in the discrete case the same basic properties as can be seen in lemma 71.

Lemma 71 (Properties of stopping times) — (a) If τ and σ are two $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$-stopping times, then $\tau \wedge \sigma$, $\tau \vee \sigma$, and $\tau + \sigma$ are $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$-stopping times.

(b) If for all $n \in \mathbb{N}$, τ_n is an $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$-stopping time and that $\tau_n \nearrow \tau$, then τ is an $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$-stopping time.

(c) If for all $n \in \mathbb{N}$, τ_n is an $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$-stopping time and that $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ is a right-continuous filtration, then $\liminf_{n\to+\infty} \tau_n$ and $\limsup_{n\to+\infty} \tau_n$ are also $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$-stopping times.

Proof (Lemma 71). (a) The property is clear for $\sigma \wedge \tau$ and $\sigma \vee \tau$. Let $t \in \mathbb{R}_+$, we have

$$\{\tau + \sigma \leq t\} = \{\tau = 0, \sigma \leq t\} \cup \{\tau \leq t, \sigma = 0\} \cup \{0 < \tau \leq t, \sigma + \tau \leq t\} \cup \{0 < \sigma \leq t, \sigma + \tau \leq t\}.$$

The first two sets are clearly in $\mathcal{F}_t$ and because

$$\{0 < \tau \leq t, \sigma + \tau \leq t\} = \bigcup_{r\in\mathbb{Q}\cap]0,t[} \{r \leq \sigma \leq t, 0 \leq \tau \leq t - r\} \in \mathcal{F}_t,$$

we clearly have $\{\tau + \sigma \leq t\} \in \mathcal{F}_t$ by the symetry of the roles played by τ and σ.

(b) We simply need to observe that for any $t \in \mathbb{R}_+$, $\{\tau \leq t\} = \bigcap_{n\in\mathbb{N}}\{\tau_n \leq t\} \in \mathcal{F}_t$.

(c) Let $t \geq 0$, we have

$$\forall t \in T, \quad \left\{\inf_{n\in\mathbb{N}} \tau_n < t\right\} = \bigcup_{n\in\mathbb{N}} \{\tau_n < t\} = \bigcup_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}^*} \left\{\tau_n \leq t - \frac{1}{k}\right\} \in \mathcal{F}_t, \quad \left\{\inf_{n\in\mathbb{N}} \tau_n \leq t\right\} = \bigcap_{k\in\mathbb{N}^*} \{\tau_n \leq t + \frac{1}{k}\} \in \mathcal{F}_{t+} = \mathcal{F}_t.$$

Moreover for all $t \in T$, $\{\sup_{n\in\mathbb{N}} \tau_n \leq t\} = \bigcap_{k\in\mathbb{N}^*}\{\tau_n \leq t\} \in \mathcal{F}_t$. Thus, $\sup_{n\in\mathbb{N}}\{\tau_n\}$ and $\inf_{n\in\mathbb{N}}\{\tau_n\}$ are $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$-stopping time and because $\liminf$ and $\limsup$ are combinaisons of supremum and infimum, we got the result. $\square$

The most important examples of stopping times are first-exit times or first-entry times of a set by a stochastic process. Let X be an $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$-adapted $\mathbb{R}^d$-valued stochastic process. We are interested in the first time this process enters a given set A. Unfortunately the first-entry time

$$\tau_A = \inf\{t \geq 0 : X_t \in A\}$$

is in general not for every $A \in \mathcal{B}(\mathbb{R}^d)$ a stopping time, even if X has continuous sample paths as can be seen in example 10.

Example 10 (A first-entry time which is not a stopping time). Let Y be a Rademacher variable and set for all $t \in \mathbb{R}_+$ $X_t = tY$. Then $X_0 = 0$ and Y decides whether X_t moves to the right-side or the left-side of 0 with speed 1. The filtration generated by X is given by

$$\mathcal{F}_0^X = \{\emptyset, \Omega\} \qquad \text{and} \qquad \forall t > 0,\ \mathcal{F}_0^X = \sigma(Y).$$

For $A = \mathbb{R}_+^*$ and $\omega \in \Omega$, if $Y(\omega) = 1$, $\tau_A(\omega) = 0$ and if $Y(\omega) = -1$, $\tau_A(\omega) = +\infty$.
Therefore, $\{\tau_A \leq 0\} = \{\tau_A = 0\} = \{Y = 1\} \notin \mathcal{F}_0^X$ and thus τ_A is no stopping time.

However we have some conditions that assure that it is a stopping times as can be seen in lemma 72 whose proof is in the appendix.

Lemma 72 (First-entry times of open and closed sets) — Suppose the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ is right-continuous and X is an $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$-adapted $\mathbb{R}^d$-valued stochastic process.
(a) If X has right-continuous sample paths and G is open, then τ_G is an $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$-stopping time.
(b) If X has continuous sample paths and F is closed, then τ_F is an $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$-stopping time.

Proof (Lemma 72). (a) Let $t \in \mathbb{R}_+$. Since X has right-continuous sample paths and G is open,

$$\{\tau_G \leq t\} = \bigcap_{\epsilon>0} \bigcup_{s<t+\epsilon} \{X_s \in G\} = \bigcap_{k\in\mathbb{N}^*} \bigcup_{s<t+\frac{1}{k}} \{X_s \in G\} \in \mathcal{F}_{t+} = \mathcal{F}_t,$$

what gives us our result.
(b) Let F be a closed set of $\mathbb{R}^d$, we denote the open $\frac{1}{n}$-neighbourhood of F by

$$F^{(1/n)} = \left\{x \in \mathbb{R}^d : \exists y \in F |\ \|x - y\| \leq \frac{1}{n}\right\}.$$

By (a), $\tau_{F^{(1/n)}}$ is a stopping time for all $n \in \mathbb{N}^*$. Since $\left(\tau_{F^{(1/n)}}\right)_{n\in\mathbb{N}^*}$ is nondecreasing, lemma 71 shows that $\tau = \lim_{n\to+\infty} \tau_{F^{(1/n)}}$ is an $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$-stopping time.
Let us whow that $\tau = \tau_F$. Because for all $n \in \mathbb{N}^*$, $\tau_{F^{(1/n)}} \leq \tau_F$, $\tau \leq \tau_F$. If $\tau(\omega) = +\infty$, then because for all $n \in \mathbb{N}^*$, $\tau_{F^{(1/n)}} \leq \tau_F$, $\tau_F(\omega) = +\infty = \tau(\omega)$. If $\tau(\omega) < +\infty$, by the continuity of sample paths of X, $X_{\tau(\omega)}(\omega) = \lim_{n\to+\infty} X_{\tau_{F^{(1/n)}}(\omega)}(\omega) \in F$, where we used that for all $n \in \mathbb{N}^*$ $X_{\tau_{F^{(1/n)}}(\omega)}(\omega) \in F$ and the fact that F is a closed set. By the definition of τ_F as the first-entry time of F, $\tau_F(\omega) \leq \tau(\omega)$ follows. Thus $\tau = \tau_F$. □

We sometime want to study the properties of a stochastic process at the moment given by a stopping time in order to do so we will need to prove that the functions $\omega \mapsto X_{\tau(\omega)}(\omega)$ is a random variable. In order to do so we will need to define another type of filtration.

Definition 73 (σ-field of the τ-past) — We call

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : \forall t \in I, A \cap \{\tau \leq t\} \in \mathcal{F}_t\}$$

the σ-field of the τ-past.

Before proceedin with the rest, the σ-field of the τ-past posseses the following properties.

Lemma 74 (Properties of $\mathcal{F}_\tau$) — (a) Let σ and τ be two $\{\mathcal{F}_t\}_{t\in T}$-stopping times. If $\sigma \leq \tau$, then $\mathcal{F}_\sigma \subset \mathcal{F}_\tau$.
(b) If $\{\mathcal{F}_t\}_{t\in I}$ is right-continuous, then $\mathcal{F}_\tau = \bigcap_{n\in N^*} \mathcal{F}_{\tau+1/n}$.

Proof (Lemma 74). (a) Let $A \in \mathcal{F}_\sigma$, for any $t \in T$,

$$A \cap \{\tau \leq t\} = A \cap \{\sigma \leq \tau\} \cap \{\tau \leq t\} \in \mathcal{F}_t.$$

Thus $A \in \mathcal{F}_\tau$.
(b) $\tau + 1/n$ is a stopping time for all $n \in \mathbb{N}^*$, hence thanks to the previous point we can write $\mathcal{F}_\tau \subset \bigcap_{n\in\mathbb{N}^*} \mathcal{F}_{\tau+\frac{1}{n}}$. Moreover for $A \in \bigcap_{n\in\mathbb{N}} \mathcal{F}_{\tau+\frac{1}{n}}$,

$$\forall n \in \mathbb{N}^*, \quad A \cap \{\tau \leq t\} = A \cap \left\{\tau + \frac{1}{n} \leq t + \frac{1}{n}\right\} \in \mathcal{F}_{t+\frac{1}{n}}.$$

Thus $A \cap \{\tau \leq t\} \in \bigcap_{n\in\mathbb{N}^*} \mathcal{F}_{t+\frac{1}{n}} = \mathcal{F}_{t+} = \mathcal{F}_t$. □

We want to prove that X_τ is a random variable thanks to this new object. That's when comes lemma 75.

Lemma 75 (X_τ is a random variable, general case) — Let $\{\mathcal{F}_t\}_{t\in I}$ be a right-continuous filtration. Suppose that X is $\mathbb{R}^d$-valued, $\{\mathcal{F}_t\}_{t\in I}$-adapted and has right-continuous sample paths. Then X_τ is $\mathcal{F}_\tau$-$\mathcal{B}(\mathbb{R}^d)$-measurable on $\{\tau < +\infty\}$ *ie*

$$\forall A \in \mathcal{B}(\mathbb{R}^d), \quad \{X_\tau \in A\} \cap \{\tau < +\infty\} \in \mathcal{F}_\tau.$$

Before proceeding with the proof, let us firstly show the following result.

Lemma 76 (X_τ is a random variable) — Suppose that X is $\mathbb{R}^d$-valued and progressively measurable with respect to $\{\mathcal{F}_t\}_{t\in I}$. Let τ be a finite $\{\mathcal{F}_t\}_{t\in I}$-stopping time. Then X_τ is an $\mathcal{F}_\tau$-$\mathcal{B}(\mathbb{R}^d)$-measurable random variable.

Proof (Lemma 76). We need to show that for any $B \in \mathcal{B}(\mathbb{R}^d)$ and $t \in T$, we have $\{X_t \in B\} \cap \{\tau \leq t\} \in \mathcal{F}_t$. We have

$$\{X_\tau \in B\} \cap \{\tau \leq t\} = \{X_{\tau\wedge t} \in B\} \cap \{\tau \leq t\}.$$

Since τ is a stopping time, $\{\tau \leq t\} \in \mathcal{F}_t$. Hence it suffices to show that $X_{\tau\wedge t}$ is $\mathcal{F}_t$-$\mathcal{B}(\mathbb{R}^d)$-measurable. The latter can be deduces as follows:

- $\omega \mapsto (\tau(\omega) \wedge t, \omega)$ is $\mathcal{F}_t$-$\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-measurable,
- since X is progressively measurable with respect to $\{\mathcal{F}_t\}_{t\in I}$, the map $(s, \omega) \mapsto X_s(\omega)$ is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-$\mathcal{B}(\mathbb{R}^d)$-

measurable.

Hence $\omega \mapsto X_{\tau\wedge t}$ is $\mathcal{F}_t$-$\mathcal{B}(\mathbb{R}^d)$-measurable. $\square$

The previous lemma is not quite satisfactory as the assumption that the stopping time is finite everywhere might be too restrictive. We will therefore show lemma 75. In this proof, we will need the following approximation result for stopping times.

Lemma 77 (Approximating stopping times) — Let τ be an $\{\mathcal{F}_t\}_{t\in I}$-stopping time. Then there exist $\{\mathcal{F}_t\}_{t\in I}$-stopping times τ_n, taking only finitely many values, such that $\tau_n \searrow \tau$ $\mathbb{P}$-almost-surely.

Proof (Lemma 77). Define

$$\tau_n = \sum_{k=1}^{n2^n} \frac{k}{2^n} \mathbb{1}_{\tau\in]\frac{k-1}{2^n},\frac{k}{2^n}]} + \infty \mathbb{1}_{\tau>n}.$$

Then for any $n\in\mathbb{N}$, τ_n takes only a finite number of values, τ_n is a stopping times and $\tau_n \searrow \tau$. $\square$

With the help of this approximation procedure, we can prove lemma 75.

Proof (Lemma 75). First note that $\{\tau < +\infty\} \in \mathcal{F}_t$ as for all $t \in T$ $\{\tau < +\infty\} \cap \{\tau \leq t\} \in \mathcal{F}_t$.

We use the approximation stopping times given by the proof of lemma 77, by the right-continuity of the sample paths of X, $\tau_n \searrow \tau$ implies that

$$X_\tau = \lim_{n\to+\infty} X_{\tau_n} \text{ on } \{\tau < +\infty\}.$$

Since τ_n only takes a finite number of values, it is easy to see that X_{τ_n} is $\mathcal{F}_{\tau_n}$-$\mathcal{B}(\mathbb{R}^d)$-measurable on $\{\tau_n < +\infty\}$. For any $\epsilon > 0$ and $n \in \mathbb{N}$ large enough to guarantee $2^{-n} \leq \epsilon$, we have $\tau_n \leq \tau + \epsilon$ on $\{\tau_n < +\infty\}$. Since $\tau+\epsilon$ is an $\{\mathcal{F}_t\}_{t\in I}$ as well, $\mathcal{F}_{\tau_n} \subset \mathcal{F}_{\tau+\epsilon}$ thanks to lemma 74. Hence $X_\tau = \lim_{n\to+\infty} X_{\tau_n}$ is $\mathcal{F}_{\tau+\epsilon}$-$\mathcal{B}(\mathbb{R}^d)$-measurable on $\{\tau < +\infty\}$. Since ϵ is arbitrary, we conclude that X_τ $\mathcal{F}_{\tau+}$-$\mathcal{B}(\mathbb{R}^d)$-measurable and thanks to lemma 74, we know that $\mathcal{F}_{\tau+} = \mathcal{F}_\tau$. $\square$

Before we continue to study Markov processes, let us remark that we will frequently consider so-called stopped stochastic process X^τ defined by

$$\forall t \in I, \quad X_t^\tau = X_{t\wedge\tau}.$$

We will show the following property.

Lemma 78 (The stopped process X^τ is adapted) — If X is progressively measurable with respect to $\{\mathcal{F}_t\}_{t\in I}$ and τ an $\{\mathcal{F}_t\}_{t\in I}$-stopping time, then the stopped process X^τ is $\{\mathcal{F}_t\}_{t\in I}$- and $\{\mathcal{F}_{t\wedge\tau}\}_{t\in I}$-adapted.

Proof (Lemma 78). Let $s \in I$ and $A \in \mathcal{B}(\mathbb{R}^d)$. We need to show that

$$\{X_s^\tau \in A\} \in \mathcal{F}_s \quad \text{and} \quad \{X_s^\tau \in A\} \in \mathcal{F}_{t\wedge\tau}.$$

Since $\tau \wedge s$ is a $\{\mathcal{F}_t\}_{t\in I}$-stopping time and $\tau\wedge s \leq s$, $\mathcal{F}_{\tau\wedge s} \subset \mathcal{F}_s$ thanks to 74 which completes the proof. $\square$

C.6 Martingales

In this section, we will focus on another class of stochastic processes called martingales, submartingales and supermartingales. In discrete-time the most obvious example are random walks which are partial sums of centred random variables and we will see that in continuous time, Brownian motions and certain functions of Brownian motion are martingales. For us martingales are important because

- we will see that a stochastic integral, considered as a function of the upper integration limit, defines a stochastic process which turns out to be a martingale,
- and martingales allow for every practical estimates of their supremum, a valuable feature when controlling sample paths.

C.6.1 Definition and first examples

For the moment, let $I = \mathbb{N}$ or $I = \mathbb{R}_+$. The definitions and results below can easily be adjusted to the more cases for I.

As usual, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\mathcal{F}_t\}_{t\in I}$ be a filtration of $\mathcal{F}$. The stochastic process we consider in this chapter are assumed to be $\mathbb{R}$-valued

Definition 79 (Martingale) — An $\{\mathcal{F}_t\}_{t\in I}$-adapted stochastic process X such that for all $t \in I$, $X_t \in \mathcal{L}^1(\mathbb{P})$ holds is called

(a) an $\{\mathcal{F}_t\}_{t\in I}$-martingale if and only if

$$\forall t \geq s, \mathbb{E}[X_t|\mathcal{F}_s] = X_s \; \mathbb{P}\text{-almost surely}\,;$$

(b) an $\{\mathcal{F}_t\}_{t\in I}$-submartingale if and only if

$$\forall t \geq s, \mathbb{E}[X_t|\mathcal{F}_s] \geq X_s \; \mathbb{P}\text{-almost surely}\,;$$

(c) an $\{\mathcal{F}_t\}_{t\in I}$-supermartingale if and only if $-X$ is a submartingale.

If the filtration is $\mathcal{F}^X$ we simply call X a martingale, submartingale or supermartingale.

We have also the following result.

Lemma 80 (Centred independent increments imply martingale property) — Let $\mathbb{X}$ be a stochastic process which has independent increments with respect to a filtration $\{\mathcal{F}_t\}_{t\in I}$ and assume that the conditional expectations $\mathbb{E}[X_t|\mathcal{F}_s]$ are defined for all $s, t \in I$ with $t \geq s$. Then X is an

- an $\{\mathcal{F}_t\}_{t\in I}$-martingale if and only if for all $t > s$, $\mathbb{E}[X_t - X_s] = 0$;
- an $\{\mathcal{F}_t\}_{t\in I}$-submartingale if and only if for all $t > s$, $\mathbb{E}[X_t - X_s] \geq 0$;
- an $\{\mathcal{F}_t\}_{t\in I}$-supermartingale if and only if for all $t > s$, $\mathbb{E}[X_t - X_s] \leq 0$.

Example 11. Let $X \in \mathcal{L}^1(\mathbb{P})$ and $\{\mathcal{F}_t\}_{t\in I}$, then

$$\forall t \in I, \quad X_t = \mathbb{E}[X|\mathcal{F}_t]$$

defines a martingale. We will see later that some but not all martingales can be represented in this form.

We will conclude this subsection by providing a useful lemma, based on Jensen's inequality for conditional expactations, we will admit the result.

Lemma 81 (Convex function of martingales) — Let X be a stochastic process taking values in an open interval $I \subset \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ a convex function such that for all $t \in T$, $X_t, \varphi(X_t) \in \mathcal{L}^1(\mathbb{P})$.
Suppose that one of the following two following conditions is satisified:

- X is a martingale ;
- X is a submartingale and φ is nondecreasing on I.

Then $\varphi(X)$ is a submartingale.

C.6.2 Continuous-time martingales

Since we are mainly interested in stochastic processes in continuous time, we will now show how to extend the results from the previous chapter to this case.

C.6.3 Examples

Let $I = \mathbb{R}_+$. Before we start: do we know examples of continuous-time martingales? So far we have only seen one, namely Brownian motion. Actually we can actually construct other examples from Brownian motion.

C.6.4 Approximation by discrete-time processes

The main idea which will allow us to extend the results from the discrete-time case is to consider only a dense subset of times. As long as martingales, sub- or supermartingales has nice sample paths (at least left- or right-continuous), it suffices to work on such a dense subset of the time interval I. In this subsection, we gather all the results which will be useful in carrying out this idea.
First we need to establish a continuous-time version of the upcrossing lemma [ADD REF]. We define the number of upcrossings in the same way as before, pretending this instead of a function defined on $\mathbb{R}$ we only have finite or countable number of values ie a sequence.

Definition 82 (Upcrossings) — For any function $\varphi : I \to \mathbb{R}$, any finite set $F \subset I$ and $a, b \in \mathbb{R}$ with $a < b$, let

$$U_F[a,b](\varphi) = \max\{k \in \mathbb{N} : \exists s_1 < t_1 < \cdots < s_k < t_k \in F \,|\, \forall i \in [\![1,k]\!], \varphi(s_i) < a \text{ and } \varphi(t_i) > b\}$$

denote the number of upcrossings of the interval $[a, b]$ that the finite sequence $(\varphi(t))_{t \in F}$ does.

For continuous-time martingales, the upcrossing lemma becomes :

Lemma 83 (Doob's upcrossing lemma for continuous-time martingales) — Let M be a supermartingale. Then for all countable G and all $a < b$, the number of upcrossings $U_G[a,b](M)$ of the interval

$[a, b]$ by $(M(t))_{t\in G}$ satisfies

$$(b-a)\mathbb{U}_G[a,b](M) \le \sup_{t\in G} \mathbb{E}[(M_t - a)^-].$$

In addition we will later need the following fact for supermartingales with right-continuous sample paths.

Lemma 84 (Supermartingales and right-continuous filtrations) — Let M be an $\{\mathcal{F}_t\}_{t\in I}$-supermartingale with right-continuous sample paths. Then M is also a $\{\mathcal{F}_{t+}\}_{t\in I}$-supermartingale.

To complete proof of lemma 84, we have to briefly return to discrete time and introduce the concept of a backwards martingale where backwards refers to the time index.

C.6.5 Martingale inequalities for continuous-time martingales

Theorem 85 (Doob's submartingale inequality) — Let M be a martingale with right-continuous sample paths and $J \subset I$ an interval. We define $M^\star = \sup_{t\in J} |M_t|$. Then for $p \ge 1$,

$$\forall \lambda > 0, \mathbb{P}(M^\star \ge \lambda) \le \frac{1}{\lambda^p} \sup_{t\in J} \mathbb{E}|M_t|^p.$$

Proof (COMPLETE).

Note that the same proof also holds if M is a nonnegative submartingale. What is needed in the proof od Doob's submartingale inequality is just that $(|M_t|)_{t\in I}$ is a submartingale.

Corollary 86 (Doob's submartingale inequality) — Let M be a nonnegative submartingale with right-continuous sample paths and $J \subset I$ an interval. We define $M^\star = \sup_{t\in J} M_t$. Then for $p \ge 1$,

$$\forall \lambda > 0, \mathbb{P}(M^\star \ge \lambda) \le \frac{1}{\lambda^p} \sup_{t\in J} \mathbb{E}[M_t^p].$$

Along the same lines we can prove Doob's $\mathcal{L}^p$-inequality.

Theorem 87 (Doob's $\mathcal{L}^p$-inequality) — Let M be a martingale with right-continuous sample paths and $I \subset T$ an interval. We again define $M^\star = \sup_{t\in I} |M_t|$. Then for $p > 1$,

$$\|M_t\|_{\mathcal{L}^p} \ge \frac{p}{p-1} \sup_{t\in I} \|M_t\|_{\mathcal{L}^p}.$$

C.7 Independent increments

Definition 88 (Independent increments) — An $\{\mathcal{F}_t\}_{t\in I}$-adapted $\mathbb{R}^d$-valued stochastic process X has independent increments with respect to $\{\mathcal{F}_t\}_{t\in I}$ if and only if for any $s \in I$, the stochastic process

$$(X_t - X_s)_{t\in I\cap[s,+\infty[}$$

and the σ-field $\mathcal{F}_s$ are independent.
We say that X has independent increments if and only if X has independent increments with respect to $\mathcal{F}^X$.

C.8 Markov chains

In this subsection, we will define the notion of Markov chain for stochastic processes in continuous time.

Definition 89 (Markov process) — Let $\{\mathcal{F}_t\}_{t\in I}$ be a filtration. An $\{\mathcal{F}_t\}_{t\in I}$-adapted $\mathbb{R}^d$-valued stochastic process X is called an $\{\mathcal{F}_t\}_{t\in I}$-Markov process if and only if

$$\forall t \in I,\ \ \forall A \in \sigma(\{X_u : u \in I, u \geq t\}), \quad \mathbb{P}(A|\mathcal{F}_t) = \mathbb{P}(A|X_t) \quad \mathbb{P}\text{-almost surely.} \tag{27}$$

If (27) holds with $\mathcal{F}^X$ we call X a Markov process.

Remark 12. (a) Since X is $\{\mathcal{F}_t\}_{t\in I}$-adapted, W_t is $\mathcal{F}_t$-$\mathbb{B}(\mathbb{R}^d)$-measurable. Thus $\sigma(X_t) \subset \mathcal{F}_t$, and we conclude that $\mathbb{P}(A|X_t)$ is $\mathcal{F}_t$-$\mathcal{B}(\mathbb{R})$-measurable. Hence (27) is equivalent to

$$\forall t \in I, \forall A \in \sigma(\{X_u : u \in I, u \geq t\}), \forall B \in \mathcal{F}_t, \quad \mathbb{P}(A \cap B) = \int_B \mathbb{P}(A|X_t)\,\mathrm{d}\mathbb{P}.$$

(b) X is a Markov process if and only if for any $\sigma(\{X_u : u \in I, u \geq t\})$-$\mathcal{B}(\mathbb{R})$-measurable, $\mathbb{P}$-integrable function $\varphi : \Omega \to \mathbb{R}$,

$$\mathbb{E}(\varphi|\mathcal{F}_t) = \mathbb{E}(\varphi|\mathcal{F}_t) \quad \mathbb{P}\text{-almost surely.} \tag{28}$$

This can be established in the usual way. On the one hand, if (28) holds, we can simply choose $\varphi = \mathbb{1}_A$ for $A \in \sigma(\{X_u : u \in I, u \geq t\})$ to obtain (27). On the other hand, if (27) holds, we can employ the usual approximation procedure to obtain (28).

The previous definition is not the most pratical to use, thus we need the following lemma which will offer us a way to handle it better.

Lemma 90 (An equivalent formulation for the Markov property) — An $\{\mathcal{F}_t\}_{t\in I}$-adapted stochastic process is an $\{\mathcal{F}_t\}_{t\in I}$-Markov process if and only if

$$\forall s, t \in I, \forall \varphi \in \mathcal{C}_b(\mathbb{R}^d, \mathbb{R}), \quad t > s \Rightarrow \mathbb{E}\left[\varphi(X_t)|\mathcal{F}_s\right] = \mathbb{E}\left[\varphi(X_t)|X_s\right]. \tag{29}$$

Proof (Lemma 90). We will assume that this property holds. □

Theorem 91 (Independent increments imply Markov property) — An $\mathbb{R}^d$-valued stochastic process X which has independent increments with respect to $\{\mathcal{F}_t\}_{t\in I}$ is an $\{\mathcal{F}_t\}_{t\in I}$-Markov process.

Proof (Lemma 91). We will assume that this property holds. □

C.9 Complements to subsection 1.1

Brownian motion exists

Proof (Corollary 2). We want to use the point (c) of theorem 63 in order to prove the result. It is clear that $\sigma : (s,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \mapsto s \wedge t$ is symmetric and for any choice of $0 = t_0 \leq t_1 \leq \cdots \leq t_n$, for any choice of $\lambda \in \mathbb{R}^n$ we have (with $\Sigma = (\sigma(t_i, t_j))_{1\leq i,j\leq n}$)

$$\begin{aligned}
\lambda^T \Sigma \lambda &= \sum_{i,j=1}^{n} \lambda_i (t_i \wedge t_j) \lambda_j = t_n \lambda_n^2 + \sum_{k=1}^{n-1} t_k \lambda_k \left[\lambda_k + 2 \sum_{i=k+1}^{n} \lambda_i \right] = \sum_{k=1}^{n} t_n \lambda_n^2 + 2 \sum_{k=1}^{n-1} \sum_{i=k+1}^{n} t_k \lambda_k \lambda_i \\
&= t_n \lambda_n^2 + \sum_{k=1}^{n-1} t_k \left[\lambda_k^2 + 2\lambda_k (\lambda_{k+1} + \cdots + \lambda_n) \right] = t_n \lambda_n^2 + \sum_{k=1}^{n-1} t_k \left[(\lambda_k + \cdots + \lambda_n)^2 - (\lambda_{k+1} + \cdots + \lambda_n)^2 \right] \\
&= \sum_{k=1}^{n} (t_k - t_{k-1}) \left(\sum_{i=k}^{n} \lambda_i \right)^2 \geq 0.
\end{aligned}$$

That's why σ is positive semi-definite and we get the existence of a centred Gaussian process X with the covariance function of Brownian motion.
Moreover we want to use theorem 66 to find a version of X with continuous sample paths. However for all $s, t \in T$ such that $0 < t - s \leq r$

$$\mathbb{E}\left[|X_t - X_s|^4 \right] = \mathbb{E}\left[\left(\frac{X_t - X_s}{\sqrt{t-s}} \right)^4 \right] (t-s)^2.$$

Thus there is a version of X with continuous sample paths, thus Brownian motion exists. □

C.10 Complements to subsection 1.2

Hilbert spaces

Here are some recalls on Hilbert spaces.

Definition 92 (Hilbert space) — We define the following.

(a) A real Hilbert space H is a vector space over $\mathbb{R}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$, which is complete with respect to the induced norm $\| \cdot \|$ given by $\|x\| = \sqrt{\langle x, x \rangle}$ for all $x \in H$.

(b) A Hilbert space H is called separable, if there exists a countable dense subset.

(c) A sequence $(h_n)_{n\in\mathbb{N}} \in H^{\mathbb{N}}$ is called a complete orthonormal basis of H if and only if the two following conditions hold:

  (i) $\forall i, j \in \mathbb{N}, \langle h_i, h_j \rangle = \delta_{i,j}$;

  (ii) $\{\sum_{k=1}^{n} a_k h_k : n \in \mathbb{N}, a_1, \ldots, a_n \in \mathbb{R}\}$ is dense in H.

Lemma 93 (Existence of complete orthonormal basis) — Every separable Hilbert space possesses a complete orthonormal basis.

Lemma 94 ($\mathcal{L}^2([0,1])$ as a Hilbert space) — $\mathcal{L}^2([0,1]) = \mathcal{L}^2([0,1], \mathcal{B}([0,1]), \lambda|_{[0,1]})$ equipped with the scalar product $\langle \cdot, \cdot \rangle$ given by

$$\forall f, g \in \mathcal{L}^2([0,1]), \langle f, g \rangle = \int_0^1 fg \,\mathrm{d}\lambda,$$

is a separable Hilbert space.

Lemma 95 (Facts for complete orthonormal bases) — We have the following results.

(a) If H is a separable Hilbert space and the sequence $(h_n)_{n\in\mathbb{N}}$ satisifes

$$\forall i, j \in \mathbb{N}, \langle h_i, h_j \rangle = \delta_{i,j},$$

then $(h_n)_{n\in\mathbb{N}}$ is a complete orthonormal basis if and only if

$$\{x \in H : \forall n \in \mathbb{N}, \langle x, h_n \rangle = 0\} = \{0\}.$$

(b) Given a complete orthonormal basis $(h_n)_{n\in\mathbb{N}}$, Parseval's identity states that

$$\forall x, y \in H, \langle x, y \rangle = \sum_{n=1}^{+\infty} \langle x, h_n \rangle \langle h_n, y \rangle$$

Haar basis in $\mathcal{L}^2([0,1])$ is a complete orthonormal basis

Proof (Lemma 7). Let $g \in \mathcal{L}^2([0,1])$ satisfy for all $f \in \mathfrak{F}$, $\langle g, f \rangle = 0$. Let us show that for all $0 \le a < b \le 1$, $\int_a^b g = 0$. Let us prove by induction that for all $n \in \mathbb{N}$, $k \in [\![1, 2^n]\!]$, $\int_{(k-1)2^{-n}}^{k2^{-n}} g = 0$.

- For $n = 0$, we simply have $\int_0^1 g = \langle g, f_0 \rangle = 0$.
- For $n = 1$, we have $0 = \langle g, f_{1,1} \rangle = \int_0^{1/2} g - \int_{1/2}^1 g$ and with $\int_0^1 g = 0$ we have what we want.
- For $n \ge 2$, let us suppose that our property holds for $(n-1)$. Then for $k \in [\![1, 2^{n-1}]\!]$,

$$0 = \langle g, f_{n,k} \rangle = 2^{(n-1)/2} \left(\int_{(2k-2)2^{-n}}^{(2k-1)2^{-n}} g - \int_{(2k-1)2^{-n}}^{2k2^{-n}} g \right),$$

  and we also have by hypothesis that $\int_{(2k-2)2^{-n}}^{2k2^{-n}} g = 0$.

We proved that for all $a, b \in \{2^{-n}k : n \in \mathbb{N}, k \in [\![1, 2^{-n}]\!]\}$ such that $a < b$, $\int_a^b g = 0$, but $\overline{\{2^{-n}k : n \in \mathbb{N}, k \in [\![1, 2^{-n}]\!]\}} = [0,1]$, so get our the lemma thanks to Lebesgue's dominated convergence theorem. $\square$

The approximations converge uniformly

Proof (Lemma 11). Consider the quantities

$$\forall N \in \mathbb{N}^*, \forall t \in [0,1], \quad D_t^{(N)} := W_t^{(N)} - W_t^{(N-1)} = \sum_{k=1}^{2^{N-1}} F_{N,k}(t)\xi_{N,k}$$

with the convention that $W_\cdot^{(0)} = 0$.
We have for $n \in \mathbb{N}^*$ and $k \in [\![1, 2^{n-1}]\!]$ the following properties,

(i) $F_{n,k} \geq 0$;
(ii) $\max_{[0,1]} F_{n,k} = 2^{-(n-1)/2}$;
(iii) $\{t \in [0,1] : F_{n,k} > 0\} = \left]\frac{k-1}{2^{n-1}}, \frac{k}{2^{n-1}}\right[$.

(iii) gives us the fact that for any $t \in [0,1]$, at most one term of $D_t^{(N)}$ is positive. Moreover with (ii) we can write

$$\sup_{t\in[0,1]} \left|D_t^{(N)}\right| \leq 2^{-(N+1)/2}\Xi_N, \quad \text{where } \Xi_N = \max_{1\leq k\leq 2^{N-1}} |\xi_{N,k}|.$$

Thus for $x > 0$,

$$\mathbb{P}\left(\sup_{t\in[0,1]} \left|D_t^{(N)}\right| \geq x\right) \leq \mathbb{P}\left(\Xi_N \geq x2^{(N+1)/2}\right) \leq 2^{N-1}\mathbb{P}\left(|\xi_0| \geq x2^{(N+1)/2}\right) \leq 2^N\mathbb{P}\left(\xi_0 \geq x2^{(N+1)/2}\right).$$

Choosing $x = \frac{N}{2}2^{-N/2}$ yields

$$\mathbb{P}\left(\sup_{t\in[0,1]} \left|D_t^{(N)}\right| \geq \frac{N}{2}2^{-N/2}\right) \leq 2^N\mathbb{P}\left(\xi_0 \geq \frac{N}{\sqrt{2}}\right) \leq 2^N \frac{1}{\sqrt{\pi}N}\mathrm{e}^{-N^2/4},$$

thanks to the the standard tail estimate for $Z \sim \mathcal{N}(0,1)$,

$$\forall z > 0, \quad \mathbb{P}(Z \geq z) \leq \frac{1}{\sqrt{2\pi}z}\mathrm{e}^{-z^2/2}.$$

Let us now use the Borel-Cantelli lemma, for $N \in \mathbb{N}^*$ let us denote

$$A_N = \{\sup_{t\in[0,1]} \left|D_t^{(N)}\right| \geq \frac{N}{2}2^{-N/2}\}.$$

Since $\sum_{N=1}^{+\infty} \mathbb{P}(A_N) \leq \sum_{N=1}^{+\infty} 2^N \frac{1}{\sqrt{\pi}N}\mathrm{e}^{-N^2/4} < +\infty$, the first Borel-Cantelli lemma shows that $\mathbb{P}(\Omega_0^c) = 0$ for

$$\Omega_0^c = \limsup_{N\to+\infty} A_n$$

ie $\mathbb{P}(\Omega_0) = 1$ and

$$\forall \omega \in \Omega_0, \exists n_0 \in \mathbb{N}^*, \forall N \geq n_0(\omega), \quad \sup_{t\in[0,1]} \left|D_t^{(N)}(\omega)\right| \leq \frac{N}{2}2^{-N/2}.$$

For all $N \in \mathbb{N}^*$, $t \in [0,1]$ and $\omega \in \Omega_0$,

$$W_t^{(N)}(\omega) = \sum_{n=1}^{N} D_t^{(n)}(\omega).$$

Since $\sum_{n=1}^{+\infty} \frac{n}{2} 2^{-n/2} < +\infty$, for any $\omega \in \Omega_0$, $W_t^{(N)}(\omega)$ converge uniformly in $t \in [0,1]$ as $N \to +\infty$. We set

$$\forall \omega \in \Omega_0, \forall t \in [0,1], \quad W_t(\omega) = \lim_{N \to +\infty} W_t^{(N)}(\omega).$$

Since the convergence is uniform on $[0,1]$, $W(\omega)$ is continuous by construction. In addition, $W_0 = 0$. □

Extension of the Brownian motion on $\mathbb{R}_+$

Proof (Theorem 13). We have to show the following properties:

- $\overline{W}$ is well-defined ;
- $\overline{W}$ is Gaussian process ;
- $\overline{W}$ is centred ;
- $\mathrm{Cov}(\overline{W}_s, \overline{W}_t) = s \wedge t$ for all $s, t \in \mathbb{R}_+$;
- $\overline{W}$ has continuous sample paths.

The three first points are easy to prove.
For $s, t \in \mathbb{R}_+$ with $s < t$, we have

$$\begin{aligned}
\mathrm{Cov}(\overline{W}_s, \overline{W}_t) &= \mathbb{E}\left[\overline{W}_s \overline{W}_t\right] \\
&= (1+s)(1+t)\mathbb{E}\left[W_{1/(1+t)} W_{1/(1+s)}\right] - (1+s)\mathbb{E}\left[W_1 W_{1/(1+s)}\right] - (1+t)\mathbb{E}\left[W_{1/(1+t)} W_1\right] + \mathbb{E}\left[W_1^2\right] \\
&= (1+s) - 1 - 1 + 1 = s.
\end{aligned}$$

The continuity of the sample paths comes from the fact that $t \mapsto (1+t)$ and $t \mapsto 1/(1+t)$ and W has continuous sample paths. □

The σ-field generated by the Borel sets in $(\mathcal{C}([0,1], \mathbb{R}^d), d_\infty)$

Proof (Lemma 17). First note that all evaluation maps π_t are continuous and therefore measurable with respect to the respective Borel σ-fields and thus $\mathcal{B}_\infty \supset \mathfrak{C}^{[0,1]}$. To prove the other inclusion, we will use the generator family $\{B_\epsilon(g) : \epsilon > 0, g \in \mathcal{C}([0,1], \mathbb{R}^d)\}$ of open balls. As the function in $\mathcal{C}([0,1], \mathbb{R})$ are continuous, we see that for $\epsilon > 0$, $g \in \mathcal{C}([0,1], \mathbb{R}^d)$,

$$B_\epsilon(g) = \bigcap_{t \in [0,1] \cap \mathbb{Q}} \{f \in \mathcal{C}([0,1], \mathbb{R}) : \|\pi_t - g(t)\| < \epsilon\} = \bigcap_{t \in [0,1] \cap \mathbb{Q}} \overset{-1}{\pi}_t(B_\epsilon(g(t))) \in \mathfrak{C}^{[0,1]},$$

and thus we have our conclusion. □

C.11 Complements to subsection 1.3

Transformations of Brownian motion

Proof (Theorem 19). (a) $-W$ is still a stochastic process with continuous sample paths and for all $s, t \in \mathbb{R}_+$,

$$\operatorname{Cov}(-W_s, -W_t) = -- \operatorname{Cov}(W_s, W_t) = s \wedge t,$$

thus is $-W$ a Brownian motion.

(b) $(\sqrt{\lambda} W_{t/\lambda})_{t \in \mathbb{R}_+}$ is still a stochastic process with continuous sample paths and for all $s, t \in \mathbb{R}_+$,

$$\operatorname{Cov}(\sqrt{\lambda} W_{s/\lambda}, \sqrt{\lambda} W_{t/\lambda}) = (\sqrt{\lambda})^2 \operatorname{Cov}(W_{s/\lambda}, W_{t/\lambda}) = (\sqrt{\lambda})^2 \left(\frac{s}{\lambda} \wedge \frac{t}{\lambda}\right) = s \wedge t,$$

thus is $(\sqrt{\lambda} W_{t/\lambda})_{t \in \mathbb{R}_+}$ a Brownian motion.

(c) $(W_{t+s} - W_s)_{t \in \mathbb{R}_+}$ is still a stochastic process with continuous sample paths and for all $t, t' \in \mathbb{R}_+$,

$$\begin{aligned}
\operatorname{Cov}(W_{t+s} - W_s, W_{t'+s} - W_s) &= \operatorname{Cov}(W_{t+s}, W_{t'+s}) - \operatorname{Cov}(W_s, W_{t'+s}) - \operatorname{Cov}(W_{t+s}, W_s) + \operatorname{Cov}(W_s, W_s) \\
&= ((t+s) \wedge (t'+s)) - (s \wedge (t'+s)) - ((t+s) \wedge s) + (s \wedge s) \\
&= (t \wedge t') + s - s - s + s = (t \wedge t'),
\end{aligned}$$

thus is $(W_{t+s} - W_s)_{t \in \mathbb{R}_+}$ a Brownian motion.

(d) We want to prove that there exists $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that on $\left(\Omega_0, \mathcal{F}|_{\Omega_0}, \mathbb{P}|_{\mathcal{F}|_{\Omega_0}}\right)$, $(tW_{1/t})_{t \in \mathbb{R}_+}$ has continuous sample paths. For all $t \in \mathbb{R}_+^*$, we know that $W_{1/t} \sim \mathcal{N}(0, 1/t)$ and so $tW_{1/t} \sim \mathcal{N}(0, t)$, then

$$tW_{1/t} \underset{t \to 0}{\Longrightarrow} 0.$$

Thus if we denote

$$\Omega_0 = \left\{\omega \in \Omega : tW_{1/t}(\omega) \underset{t \to 0}{\longrightarrow} 0\right\},$$

$\mathbb{P}(\Omega_0) = 1$ and on $\left(\Omega_0, \mathcal{F}|_{\Omega_0}, \mathbb{P}|_{\mathcal{F}|_{\Omega_0}}\right)$, $(tW_{1/t})_{t \in \mathbb{R}_+}$ has continuous sample paths. Moreover, on $\left(\Omega_0, \mathcal{F}|_{\Omega_0}, \mathbb{P}|_{\mathcal{F}|_{\Omega_0}}\right)$,

$$\operatorname{Cov}(sW_{1/s}, tW_{1/t}) = st \cdot \left(\frac{1}{s} \wedge \frac{1}{t}\right) = s \wedge t,$$

thus is $(tW_{1/t})_{t \in \mathbb{R}_+}$ a Brownian motion on $\left(\Omega_0, \mathcal{F}|_{\Omega_0}, \mathbb{P}|_{\mathcal{F}|_{\Omega_0}}\right)$. □

Brownian motion has independent increments

Before proving theorem 20, we will need the following two results (which we admit).

Lemma 96 (Characterizing independence of random vetors) — Let $X = (X_1, \ldots, X_m)$ und $Y = (Y_1, \ldots, Y_n)$ be two random vectors, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we denote by $\chi_{(X,Y)}$, χ_X and χ_Y the respective characteristic functions. Then, X and Y and independent if and only if

$$\forall (s_1, \ldots, s_m, t_1, \ldots, t_n) \in \mathbb{R}^{m+n}, \chi_{(X,Y)}(s_1, \ldots, s_m, t_1, \ldots, t_n) = \chi_X(s_1, \ldots, s_m)\chi_X(t_1, \ldots, t_m).$$

Lemma 97 (Independence of Gaussian vectors) — Let $X = (X_1, \ldots, X_m)$ and $Y = (Y_1, \ldots, Y_n)$ be two random vectors. Suppose that (X, Y) is a Gaussian random vector. Then X and Y are independant if and only if for all $i \in [\![1, m]\!]$,$j \in [\![1, n]\!]$, $\operatorname{Cov}(X_i, Y_j) = 0$.

Here goes the proof of theorem 20.

Proof (Theorem 20). Let us fix $s \in \mathbb{R}_+$, we need to show that

$$\mathcal{F}_s^W =: \mathcal{G} \perp\!\!\!\perp \mathcal{H} := \sigma(\{W_t - W_s : t \in T \cap [s, +\infty[\}).$$

As the $\bigcap_{m\in\mathbb{N}^*} \bigcap_{0\le s_1<\cdots<s_m\le s} \sigma(W_{s_1}, \ldots, W_{s_m})$s form a π-system that generates $\mathcal{G}$ and the $\bigcap_{n\in\mathbb{N}^*} \bigcap_{s\le t_1<\cdots<t_m} \sigma(W_{t_1} - W_s, \ldots, W_{t_n} - W_s)$s form a π-system that generates $\mathcal{H}$, we need to show that the generators are independent, then it suffices to show that for any $n, m \in \mathbb{N}^*$ and $0 \le s_1 < \cdots < s_m \le s \le t_1 < \cdots < t_n$, $(W_{s_1}, \ldots, W_{s_m})$ and $(W_{t_1} - W_s, \ldots, W_{t_n} - W_s)$ are independent. Because W is a Brownian motion, $(W_{s_1}, \ldots, W_{s_m}, W_s, W_{t_1}, \ldots, W_{t_n})$ is a Gaussian vector and thus $(W_{s_1}, \ldots, W_{s_m}, W_{t_1} - W_s, \ldots, W_{t_n} - W_s)$ also. Moreover for all $i \in [\![1, m]\!]$ and $j \in [\![1, n]\!]$,

$$\operatorname{Cov}(W_{s_i}, W_{t_j} - W_s) = (s_i \wedge t_j) - (s_i \wedge s) = 0,$$

and thanks to lemma 97 we know that it proves the result. □

Brownian motion changes sign infinitely often during the time interval $[0, \epsilon]$

Before proceeding with the proof of corollary 23, we will prove the follwing result.

Corollary 98 (Blumenthal's 0-1 law) — For any $A \in \mathcal{F}_{0+}$, we have $\mathbb{P}(A) \in \{0, 1\}$.

Proof (Corollary 98). Let $A \ni \mathcal{F}_{0+}^W$, by theorem 22 and Markov property,

$$\mathbb{1}_A = \mathbb{P}(A|\mathcal{F}_{0+}^W) = \mathbb{P}(A|W_0) = \mathbb{P}(A) \quad \mathbb{P}\text{-almost surely,}$$

since $W_0 = 0$. Thus $\mathbb{1}_A = \mathbb{1}_A \mathbb{1}_A = \mathbb{P}(A)^2$ and by taking expectations $\mathbb{P}(A) = \mathbb{P}(A)^2$, thus $\mathbb{P}(A) \in \{0, 1\}$. □

We can now prove corollary 23.

Proof (Corollary 23). For $\epsilon > 0$, we denote B_ϵ the set of all $\omega \in \Omega$ such that $W(\omega)$ changes finitely many times sign on $[0, \epsilon]$. We have to show that $\mathbb{P}(B_\epsilon) = 0$.

For any $\delta > 0$, define $A_\delta^\pm = \bigcap\limits_{t\in[0,\delta]} \{\pm W_t \ge 0\} \in \mathcal{F}_\delta^W$. For any $\omega \in B_\epsilon$, there exists $n_0 \in \mathbb{N}^*$ such that $\omega \in A_{1/n_0}^+ \cup A_{1/n_0}^-$. Thus $B_\epsilon \subset A^+ \cup A^-$, where $A^\pm = \bigcup_{n\in\mathbb{N}^*} A_{1/n}^\pm$. Let us show that $\mathbb{P}(A^\pm) = 0$ which will allow us to conclude.

Since for all $n \in \mathbb{N}^*$, $A_{1/n}^\pm \subset A_{1/(n+1)}^\pm$, we can write for any $n_1 \in \mathbb{N}^*$ $A^\pm = \bigcup_{n\ge n_1} A_n^\pm$. Hence, $A^\pm \in \mathcal{F}_{1/n_1}^W$ for all $n_1 \in \mathbb{N}^*$ which implies that $A^+ \cup A^- \in \mathcal{F}_{0+}^W$. By Blumenthal's 0-1 law, we find $\mathbb{P}(A^\pm) \in \{0, 1\}$.

Moreover since W is a Brownian motion, $-W$ is one too. which implies $\mathbb{P}(A^+) = \mathbb{P}(A^-)$ and therefore $\mathbb{P}(A^+ \cap A^-) = \mathbb{P}(A^\pm)$ holds, and $A^+ \cap A^- = (\bigcup_{n\ge n_0} A_{1/n}^+ \cap A_{1/n}^+) \subset \bigcap_{n\ge n_0} \{W_{1/n} = 0\}$ and thus $\mathbb{P}(A^+ \cap A^+) = 0$.

Therefore we proved what we needed to prove. □

Restarting Brownian motion in (τ, W_τ)

Proof (Theorem 25). $W^{(\tau)}$ has continuous saple paths as W has continuous sample paths and $W_0^{(\tau)} = 0$. We need to show that $W^{(\tau)}$ is a centred Gaussian process with the desired covariance function and that $W^{(\tau)}$ is independent of $\mathcal{F}_\tau$. Let us take ψ a bounded $\mathfrak{C}^{\mathbb{R}_+}$-$\mathcal{B}(\mathbb{R})$-measurable map and ξ a bounded $\mathcal{F}_\tau$-$\mathcal{B}(\mathbb{R})$-measurable map. We will denote $\mathbb{E}$ (resp. $\mathbb{E}'$) the expectancy under $\mathbb{P}$ (res. $\mathbb{P}'$). We have to prove $\mathbb{E}'[\psi(W^{(\tau)})\xi] = \mathbb{E}[\psi(W)]\mathbb{E}'[\xi]$ which is equivalent to (after multiplying by $\mathbb{P}(\tau < +\infty)$)

$$\mathbb{E}\left[\psi(W^{(\tau)})\xi\mathbb{1}_{\{\tau<+\infty\}}\right] = \mathbb{E}[\psi(W)]\mathbb{E}\left[\xi\mathbb{1}_{\{\tau<+\infty\}}\right].$$

It suffices to prove the result for all $\psi = \varphi \circ \pi_t$ for $\varphi \in \mathcal{C}_b(\mathbb{R}^d, \mathbb{R})$ and $t \in (\mathbb{R}_+)^d$, with $d \in \mathbb{N}^*$ and $\xi = \mathbb{1}_B$ with $B \in \mathcal{F}_\tau$. In that case, we need to show

$$\int_{B\cap\{\tau<+\infty\}} \varphi\left(W_{t_1}^{(\tau)}, \dots, W_{t_d}^{(\tau)}\right) \, \mathrm{d}\mathbb{P} = \mathbb{E}\left[\varphi(W_{t_1}, \dots, W_{t_d})\right] \mathbb{P}(B \cap \{\tau < +\infty\}).$$

We will now use again the approximation of τ by stopping times given by lemma 77 and then use theorem 22. We will write from now on

$$\forall t \in \mathbb{R}_+, \forall n \in \mathbb{N}^*, \quad X_t = W_t^{(\tau)} = W_{\tau+t} - W_\tau \text{ and } X_t^{(\tau_n)} = W_t^{(\tau_n)} = W_{\tau_n+t} - W_{\tau_n}.$$

For any $B \in \mathcal{F}_{\tau_n}$, using the form of the τ_n given by the proof, we find that

$$\int_{B\cap\{\tau_n<+\infty\}} \varphi(X_{t_1}^{(n)}, \dots, X_{t_1}^{(n)}) \, \mathrm{d}\mathbb{P} = \sum_{l=1}^{n2^n} \int_{B\cap\{\tau=l2^{-n}\}} \varphi(X_{t_1}^{(n)}, \dots, X_{t_1}^{(n)}) \, \mathrm{d}\mathbb{P} = \sum_{l=1}^{n2^n} \mathbb{P}(B\cap\{\tau_n = 2^{-n}l\})\mathbb{E}[\varphi(W_{t_1}, \dots, W_{t_d})],$$

where we used that $B \cap \{\tau_n = 2^{-n}l\} \in \mathcal{F}_{l2^{-n}}$ as $B \in \mathcal{F}_{\tau_n}$ znd W has independent increments. Thus we have

$$\int_{B\cap\{\tau_n<+\infty\}} \varphi(X_{t_1}^{(n)}, \dots, X_{t_d}^{(n)}) \, \mathrm{d}\mathbb{P} = \sum_{l=1}^{n2^n} \int_{B\cap\{\tau=l2^{-n}\}} \varphi(X_{t_1}^{(n)}, \dots, X_{t_d}^{(n)}) \, \mathrm{d}\mathbb{P} = \sum_{l=1}^{n2^n} \mathbb{P}(B\cap\{\tau_n < +\infty\})\mathbb{E}[\varphi(W_{t_1}, \dots, W_{t_d})].$$

We now need to take the limit as $n \to +\infty$. For $B \in \mathcal{F}_\tau \subset \mathcal{F}_{\tau_n}$ as $\tau \le \tau_n$ for all $n \in \mathbb{N}$. Moreover $\{\tau < +\infty\} = \bigcup_{n\in\mathbb{N}}\{\tau \le n\} = \bigcup_{n\in\mathbb{N}}\{\tau_n < +\infty\}$ and since $\mathbb{1}_{\{\tau_n<+\infty\}} \to \mathbb{1}_{\{\tau<+\infty\}}$, $(X_{t_1}^{(n)}, \dots, X_{t_1}^{(n)}) \to (X_{t_1}, \dots, X_{t_1})$ and φ is continuous and bounded, dominated convergence theorem gives

$$\int_{B\cap\{\tau_n<+\infty\}} \varphi(X_{t_1}^{(n)}, \dots, X_{t_d}^{(n)}) \, \mathrm{d}\mathbb{P} \xrightarrow[n\to+\infty]{} \int_{B\cap\{\tau<+\infty\}} \varphi(X_{t_1}, \dots, X_{t_d}) \, \mathrm{d}\mathbb{P}.$$

To conclude,

- if we take $\xi = 1$, we get $\mathbb{E}'[\psi(W^{(\tau)})] = \mathbb{E}[\psi(W)]$ for all ψ a bounded $\mathfrak{C}^{\mathbb{R}_+}$-$\mathcal{B}(\mathbb{R})$-measurable maps and thus is $W^{(\tau)}$ a Brownian motion on $((\Omega', \mathcal{F}', \mathbb{P}'))$ as W is one on $(\Omega, \mathcal{F}, \mathbb{P})$;
- with the previous point, we can write $\mathbb{E}'[\psi(W^{(\tau)})] = \mathbb{E}[\psi(W)]\mathbb{E}'[\xi] = \mathbb{E}[\psi(W^{(\tau)})]\mathbb{E}'[\xi]$ and choosing $\psi = \mathbb{1}_A$ with $A \in \mathfrak{C}^{\mathbb{R}_+}$ and $\xi = \mathbb{1}_B$ with $B \in \mathcal{F}_\tau$, we find that $\mathbb{P}'(\{W^{(\tau)} \in A\} \cap B) = \mathbb{P}'(\{W^{(\tau)} \in A\})\mathbb{P}'(B)$ which proves $\sigma(W^{(\tau)}) \perp\!\!\!\perp \mathcal{F}_\tau$. $\square$

D Complements to section 2

D.1 Complements to subsection 2.1

A discrete version of $\int_0^t W_s \, \mathrm{d}W_s$

Example 13 (A discrete version of $\int_0^t W_s \, \mathrm{d}W_s$). For all $t \in \mathbb{R}_+$ and $\omega \in \Omega$, set

$$\phi_1(t,\omega) = \sum_{k=0}^{2^n-1} W_{k2^{-n}}(\omega) \mathbb{1}_{]k2^{-n},(k+1)2^{-n}]}(t) \quad \text{and} \quad \phi_2(t,\omega) = \sum_{k=0}^{2^n-1} W_{(k+1)2^{-n}}(\omega) \mathbb{1}_{]k2^{-n},(k+1)2^{-n}]}(t).$$

ϕ_1 is an elementary function with the form (1) whereas ϕ_2 isn't. Even though both of them seems like reasonable approximations of W on $[0,1]$, the following assertions hold

$$\mathbb{E}\left[\phi_1(s) \, \mathrm{d}s\right] = \sum_{k=0}^{2^n-1} \mathbb{E}\left[W_{k2^{-n}}(\omega)[W_{(k+1)2^{-n}} - W_{k2^{-n}}]\right] = 0;$$

$$\mathbb{E}\left[\int_0^1 \phi_2(s) \, \mathrm{d}s\right] = \sum_{k=0}^{2^n-1} \mathbb{E}\left[W_{(k+1)2^{-n}}(\omega)[W_{(k+1)2^{-n}} - W_{k2^{-n}}]\right] = 1.$$

Depending on where we take the measurability of our elementary functions, the defined objects wil be different because the choice of the points of the partition matters.

D.2 Complements to subsection 2.2

Approximating $\mathcal{L}^2$-integrands by elementary functions

We will split the proof of theorem 32 into a series of lemmas.

Lemma 99 (Approximating bounded continuous functions in $\mathcal{V}$) — Let $h \in \mathcal{V}$ be bounded and assume that $t \mapsto h(t,\omega)$ is continuous for all $\omega \in \Omega$. Then, for any $T > 0$, there exists a sequence of elementary functions $(h^{(n)})_{n\in\mathbb{N}}$ with $h_n \in \mathcal{V}_0$ such that

$$\lim_{n\to+\infty} \mathbb{E}\left[\int_0^T [h(t) - h^{(n)}(t)]^2 \mathrm{d}t\right] = 0.$$

Proof (Lemma 99). For $n \in \mathbb{N}$, let $t_k^{(n}) = k2^{(-n)}T$ for all $k \in [\![0, 2^n]\!]$. Define for all $n \in \mathbb{N}$, $t \in [0,T]$, $\omega \in \Omega$,

$$h^{(n)}(t,\omega) = \sum_{k=1}^{2^n-1} h(t_k^{(n)},\omega) \mathbb{1}_{]t_k^{(n)},t_{k+1}^{(n)}]}(t).$$

Since $k \in \mathcal{V}$, we know that $h(t_k^{(n)},\cdot)$ is $\mathcal{F}_{t_k^{(n)}}$-$\mathcal{B}(\mathbb{R})$-measurable. Hence each $h^{(n)}$ is an elementary function and in $\mathcal{V}$. The continuity of $h(\cdot,\omega)$ for fixed $\omega \in \Omega$ implies that $h(t_k^{(n)},\omega)$ converges pointwise for each $(t,\omega) \in [0,T] \times \Omega$.

By assumption $|h|$ is bounded on $[0,T]\times\Omega$ by a $M>0$, hence $|h-h^{(n)}|\le 4M$ on $[0,T]\times\Omega$. Now

$$\mathbb{E}\left[\int_0^T [h(t)-h^{(n)}(t)]^2 \,\mathrm{d}t\right] \xrightarrow[n\to+\infty]{} 0$$

by dominated convergence theorem. □

Next we want to drop the assumption that the integrand is continuous in t.

Lemma 100 (Approximating bounded functions in $\mathcal{V}$) — Let $g\in\mathcal{V}$ be bounded. Then, for any $T>0$, there exists a sequence of bounded functions $(g^{(n)})_{n\in\mathbb{N}}$ such that for all $n\in\mathbb{N}$ $g^{(n)}\in\mathcal{V}$ and $t\mapsto g(t,\omega)$ is continuous for all $\omega\in\Omega$ and all $n\in\mathbb{N}$ and

$$\lim_{n\to+\infty}\mathbb{E}\left[\int_0^T [g(t)-g^{(n)}(t)]^2\mathrm{d}t\right]=0.$$

Proof (Lemma 100). Let $|g|\le M$ on $[0,T]\times\Omega$. We define for all $n\in\mathbb{N}$, $t\in[0,T]$, $\omega\in\Omega$,

$$g^{(n)}(t,\omega):=n\int_{(t-\frac{1}{n})\vee 0}^{t} g(s,\omega)\,\mathrm{d}s.$$

For all $\omega\in\Omega$, because g is bounded, $t\mapsto g^{(n)}(\cdot,\omega)$ is continuous and bounded and moreover, as the upper integration limit goes to t, $g^{(n)}(t,\cdot)$ is $\mathcal{F}_t$-$\mathcal{B}(\mathbb{R})$-measurable. Thanks to these facts and lemma 69, we know that $g^{(n)}(t,\cdot)$ is progressively measurable. Finally

$$\begin{aligned}
\int_0^{+\infty}\mathbb{E}[g^{(n)}(t,\cdot)]^2\,\mathrm{d}t &= \int_0^{+\infty}\mathbb{E}\left[\left(n\int_{(t-\frac{1}{n})\vee 0}^{t} g(s,\cdot)\,\mathrm{d}s\right)^2\right]\mathrm{d}t\\
&=\int_0^{+\infty}\mathbb{E}\left[\left(\int_{(t-\frac{1}{n})}^{t}\mathbb{1}_{[0,T]}(s)g(s,\cdot)\,n\,\mathrm{d}s\right)^2\right]\mathrm{d}t\\
&\le\int_0^{+\infty}\mathbb{E}\left[\int_{(t-\frac{1}{n})}^{t}\mathbb{1}_{[0,T]}(s)g(s,\cdot)^2\,n\,\mathrm{d}s\right]\mathrm{d}t\\
&=\int_0^T\mathbb{E}\left[g(s,\cdot)^2\right]\left(n\int_s^{s+\frac{1}{n}}\mathrm{d}t\right)\mathrm{d}s=\int_0^T\mathbb{E}\left[g(s,\cdot)^2\right]\mathrm{d}s<+\infty
\end{aligned}$$

Thus $g\in\mathcal{V}$.
By the result of exercise 13.1.17 of [Kle13] *ie* the fundamental theorem of calculus in the context of Lebesgue integrals, we know that $g^{(n)}(t,\Omega)$ converges to $g(t,\omega)$ for almost all $t\in[0,T]$ and all $\omega\in\Omega$. Hence

$$\mathbb{E}\left[\int_0^T[g(t)-g^{(n)}(t)]^2\,\mathrm{d}t\right]=\int_{[0,T]\times\Omega}[g(t)-g^{(n)}(t)](\lambda\otimes\mathbb{P})(\mathrm{d}(t,\omega))\xrightarrow[n\to+\infty]{}0,$$

where we used Fubini's theorem and the dominated convergence therorem. □

In the last step, we want to drop the assumption that the integrand is bounded.

Lemma 101 (Approximating functions in $\mathcal{V}$) — Let $f \in \mathcal{V}$. Then, for any $T > 0$, there exists a sequence of bounded functions $(f^{(n)})_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$ $f^{(n)} \in \mathcal{V}$ and

$$\lim_{n\to+\infty} \mathbb{E}\left[\int_0^T [f(t) - f^{(n)}(t)]^2 \mathrm{d}t\right] = 0.$$

Proof (Lemma 101). Simply set for all $t \in [0,T]$ and $\omega \in \Omega$, $f^{(n)}(t,\omega) = f(t,\omega)\mathbb{1}_{\{|f(t,\cdot)|<n\}}$. □

We can now prove theorem 32.

Proof (Theorem 32). Let $\epsilon > 0$, since $f \in \mathcal{V}$, there exists a $T > 0$ such that $\int_0^T \mathbb{E}[f(t)^2]\ \mathrm{d}t < \epsilon$. By lemma 99, lemma 100 and lemma 101, there exists a sequence of (bounded) elementary functions $(h^{(n)})_{n\in\mathbb{N}}$, $h^{(n)} \in \mathcal{V}_0$ for all $n \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that

$$\mathbb{E}\left[\int_0^T [f(t)_h^{(n)}(t)]^2\ \mathrm{d}t\right] \leq \epsilon$$

for all $n \leq n_0$ and $h^{(n)}(t) = 0$ for all $t > T$ and $n \in \mathbb{N}$. Then,

$$\mathbb{E}\left[\int_0^{+\infty} [f(t) - h^{(n)}(t)]^2\ \mathrm{d}t\right] \leq \mathbb{E}\left[\int_0^T [f(t) - h^{(n)}(t)]^2\ \mathrm{d}t\right] + \mathbb{E}\left[\int_T^{+\infty} f(t)^2\ \mathrm{d}t\right] \leq \epsilon,$$

thus we have our conclusion. □

D.3 Complements to subsection 2.3

The Itô integral for $\mathcal{L}^2$-integrands is well defined

Proposition 102 (The Itô integral for elementary integrands is well-defined) — The Itô integral is well-defined in the sense that for $h \in \mathcal{V}_0$ the integral $\int_0^{+\infty} h(t)\ \mathrm{d}W_t$ does not depend on the choice of the partition used in the definition.

Proof (Proposition 102). Assume $h \in \mathcal{V}_0$ has two representations as elementary function, say

$$h(t,\omega) = \sum_{k=0}^{K-1} h_k(\omega)\mathbb{1}_{]t_k,t_{k+1}]}(t) = \sum_{l=0}^{L-1} \tilde{h}_k(\omega)\mathbb{1}_{]s_l,s_{l+1}]}(t).$$

We can assume by merging the partitions that the second is the finer without losing any generality. Thus for all $k \in [\![0,K]\!]$ and $l \in [\![0,L]\!]$, $]s_l, s_{l+1}] \subset]t_k, t_{k+1}]$. If $]t_k, t_{k+1}] = \bigcup_{l\in I_k}]s_l, s_{l+1}]$ for $I_k \subset [\![0,L]\!]$, then

$$\sum_{l\in I_k} \tilde{h}_l[W_{s_{l+1}} - W_{s_l}] = h_k \sum_{l\in I_k} [W_{s_{l+1}} - W_{s_l}] = h_k[W_{t_{k+1}} - W_{t_k}].$$

Applying this argument to all summands in the definition of the stochastic integral of h, we see that the Itô integral for $h \in \mathcal{V}_0$ is well-defined. □

We admit the following lemma.

Lemma 103 (The Itô integral for elementary integrands is linear) — Let $f, g \in \mathcal{V}_0$ and $c \in \mathbb{R}$. Then

$$\int_0^{+\infty} [cf(t) + g(t)] \, \mathrm{d}W_t = c \int_0^{+\infty} f(t) \, \mathrm{d}W_t + \int_0^{+\infty} g(t) \, \mathrm{d}W_t$$

where every notations are defined as in the definition of the elementary functions.

Simple properties of the Itô integral

Proof (Theorem 37). As the first three properties are either clear or already proven for functions in $\mathcal{V}_0$, the approximation by elementary functions easily give the first two results for functions in $\mathcal{V}$ and concerning the third point, using the linearity, Jensen's inequality and the Itô isometry, we find that

$$\left| \mathbb{E}\left[\int_0^{+\infty} h^{(n)}(t) \, \mathrm{d}W_t\right] - \mathbb{E}\left[\int_0^{+\infty} f(t) \, \mathrm{d}W_t\right] \right|^2 \leq \int_0^{+\infty} \mathbb{E}[h^{(n)}(t) - f(t)]^2 \, \mathrm{d}t,$$

and that gives us the third result.
For $f, g \in \mathcal{V}$, using the polarizing inequality we have thanks to the linearity and the Itô isometry

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^{+\infty} f(t) \, \mathrm{d}W_t\right)\left(\int_0^{+\infty} g(t) \, \mathrm{d}W_t\right)\right] \\
&\qquad = \frac{1}{4}\left(\mathbb{E}\left[\left(\int_0^{+\infty} [f(t) + g(t)] \, \mathrm{d}W_t\right)^2\right] - \mathbb{E}\left[\left(\int_0^{+\infty} [f(t) - g(t)] \, \mathrm{d}W_t\right)^2\right]\right) \\
&\qquad = \frac{1}{4}\left(\int_0^{+\infty} \mathbb{E}\left[(f(t) + g(t))^2\right] \, \mathrm{d}t - \int_0^{+\infty} \mathbb{E}\left[(f(t) - g(t))^2\right] \, \mathrm{d}t\right) \\
&\qquad = \int_0^{+\infty} \frac{1}{4}\mathbb{E}\left[(f(t) + g(t))^2 - (f(t) - g(t))^2\right] \, \mathrm{d}t \\
&\qquad = \int_0^{+\infty} \mathbb{E}\left[f(t)g(t))\right] \, \mathrm{d}t.
\end{aligned}$$

□

D.4 Complements to subsection 2.4

The Itô integral is a martingale

Let us return to elementary integrands for a moment. Recall that we defined the Itô integral of $h \in \mathcal{V}_0$ with representation

$$\forall s \in \mathbb{R}_+, \quad h(s) = \sum_{k=0}^{K-1} h_k \mathbb{1}_{]t_k, t_{k+1}]}(s).$$

by setting

$$\forall t \in \mathbb{R}_+, \quad \int_0^t h(s) \, \mathrm{d}W_s = \sum_{k=0}^{K-1} h_k [W_{t_{k+1} \wedge t} - W_{t_k \wedge t}].$$

We have the following results.

Lemma 104 (The Itô integral of an elementary function has continuous sample paths) — Let $h \in \mathcal{V}_0$. Then $(I_t(f))_{t\in\mathbb{R}_+}$ has continuous sample paths.

Lemma 105 (The Itô integral of an elementary function is a martingale) — Let $h \in \mathcal{V}_0$ and assume that for all $k \in [\![0, K-1]\!]$ $h_k \in \mathcal{L}^2(\mathbb{P})$. Then $(I_t(f))_{t\in\mathbb{R}_+}$ is an $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$-martingale.