Seminar 2024-2025
Stochastic Loewner Evolution

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Introduction

The stochastic Loewner evolution (SLE) also called Schramm Loewner Evolution is a one parameter family of random fractals in the complex plane depending on one parameter introduced in 1999 by Schramm to describe the scaling limits of interfaces in two-dimensional discrete models from statistical models and, under reasonable hypotheses, they are the only possible scaling limits of such models.

The goal of this document is to develop the tools to define the Loewner evolution and to tweak it to obtain the Schramm-Loewner Evolutions. As this document wishes to be as short as possible and as it is based on them, we invite the reader to read [Mil19] and [Bef15].

In the first two sections, we review conformal mapping and the link between harmonic function are Brownian motion. In Sections 3 to 5, we develop the tools useful to define the chordal Loewner equation. From it, we derive in Section 6 the stochastic Loewner evolution. In the last section, we discuss the simulation of SLE.

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1 Conformal Mapping Review

In this section, we will define the notion of conformal transformation and recall Riemann mapping theorem.

Let us recall what a conformal transformation is.

Definition 1 (Conformal transformation)**.** Let U and V be open domains of $\mathbb{C}$ and let $f : U \to V$ be a map. We say that f is a *conformal transformation* if and only if f is holomorphic and bijective.

Remark. A conformal transformation is a function that preserves the angles and the shape (but not the size) of small figures.

We'll see at the end of the section, examples of conformal transformations.

Before recalling Riemann mapping theorem, Let us define a topological notion.

Definition 2 (Simple connectedness)**.** Let U be a domain of $\mathbb{C}$. We say that U is *simply connected* if and only if U and $\mathbb{C} \setminus U$ are connected.

Examples. Here are three example of simply connected domains.

- $\mathbb{C}$ is simply connected.
- The unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is simply connected.
- The upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$.

We may now recall the following result.

Theorem 3 (Riemann mapping theorem)**.** *Suppose that U is a simply connected domain with $U \neq \mathbb{C}$ and let $z \in U$. Then there exists a unique conformal transformation $f : \mathbb{D} \to U$ with $f(0) = z$ and $f'(0) > 0$.*

An immediate consequence of the Riemann Mapping Theorem is that any two simply connected domains which are both distinct from $\mathbb{C}$ can be mapped to each other using a conformal transformation.

Corollary 4. *If U and V are simply connected domains with $U, V \neq \mathbb{C}$, $z \in U$ and $w \in V$. Then there exists a unique conformal transformation $f : U \to V$ with $f(z) = w$ and $f'(z) > 0$.*

Sketch of proof. By Riemann mapping theorem, there exists a unique $f_U : \mathbb{D} \to U$ (resp. $f_V : \mathbb{D} \to V$) conformal transformation with $f_U(0) = z$ (resp. $f_V(0) = w$) and $f_U'(0) > 0$ (resp. $f_V'(0) > 0$). Then $f = f_V \circ {f_U}^{-1} : U \to V$ is the desired function. ☐

1.1 Examples of Conformal Transformations

Before delving deeper into the document and our desired results, let's first explore conformal transformations.

Conformal transformations of $\mathbb{D}$

Let $U = \mathbb{D}$ and $z \in \mathbb{D}$. Then $f : \mathbb{D} \to \mathbb{D}$ given by

$$f(w) = \frac{w + z}{1 + \overline{z}w}$$

is the unique conformal transformation with $f(0) = w$ and $f'(0) > 0$. More generally, every conformal transformation $f : \mathbb{D} \to \mathbb{D}$ is of the form

$$f(w) = \lambda \frac{w + z}{1 + \overline{z}w}$$

where $\lambda \in \partial\mathbb{D}$ and $z \in \mathbb{D}$. So there is a three-real-parameter family of such maps (z is two parameters and λ to one).

Conformal transformation of $\mathbb{H}$ in $\mathbb{D}$

The map $f : \mathbb{H} \to \mathbb{D}$ given by

$$f(z) = \frac{z-i}{z+i}$$

is a conformal transformation. It is the so-called Cayley transform. Its inverse $g : \mathbb{D} \to \mathbb{H}$ is given by

$$g(w) = \frac{i(1+w)}{1-w}$$

and is also a conformal transformation.

Conformal transformations of $\mathbb{H}$

The conformal transformations $\mathbb{H} \to \mathbb{H}$ consist of the maps of the form

$$f(z) = \frac{az+b}{cz+d}$$

where a, b, c, $d \in \mathbb{R}$ verify $ad - bc = 1$.

1.2 Special Example and Motivation for the SLEs

For each $t \geq 0$, let $H_t = \mathbb{H} \setminus [0, 2\sqrt{t}i]$. Let $g_t : H_t \ni z \to \sqrt{z^2 + 4t} \in \mathbb{H}$. Then g_t is a conformal transformation $H_t \to \mathbb{H}$.

We make two observations about the family of conformal transformations $(g_t)_{t\geq 0}$.

- $|g_t(z) - z| = |\sqrt{z^2+4t} - z| \to 0$ as $z \to \infty$ i.e. g_t looks like the identity at ∞.
- For each $z \in \mathbb{H}$ fixed, we have

  $$\partial_t g_t(z) = \frac{1}{2\sqrt{z^2+4t}} \cdot 4 = \frac{2}{g_t(z)}.$$

  That means that for each fixed $z \in \mathbb{H}$, we have that $g_t(z)$ solves the ODE

  $$\partial_t g_t(z) = \frac{2}{g_t(z)} \quad \text{with} \quad g_0(z) = z. \tag{1}$$

  For each $z \in \mathbb{H}$, the basic existence and uniqueness theorem for ODEs implies that (1) has a unique solution up until the denominator on the right hand side explodes, i.e.

  $$\tau(z) = \inf_{t\geq 0}\{\Im(g_t(z)) = 0\}.$$

  The family of conformal transformations $(g_t)_{t\geq 0}$ are characterized by (1). In particular, the curve $\gamma_t = 2\sqrt{t}i$ is encoded by (1). As we shall see later, this last example is a special case of Loewner's theorem.

2 Brownian Motion, Conformal Transformations and Harmonic Functions

In this section, we will give results that relate harmonic functions, Brownian motion and conformal transformations.

Definition 5 (Complex Brownian Motion)**.** We say that a process $B = B^1 + iB^2$ is a *complex Brownian motion* if and only if $\left(B^1, B^2\right)^T$ is a standard Brownian motion in $\mathbb{R}^2$.

There is a link between harmonic functions and Brownian motion as can be seen in the following theorem.

Theorem 6. *Let u be a harmonic function on a bounded domain D which is continuous on $\overline{D}$. Fix $z \in D$ and let $\mathbb{P}_z$ be the law of a complex Brownian motion B starting from z and let $\tau = \inf_{t\geq 0}\{B_t \notin D\}$. Then*

$$u(z) = \mathbb{E}_z[u(B_\tau)].$$

Sketch of proof. This is a consequence of Dynkin's formula knowing that the infinitesimal generator of B_t is $L = \Delta$ and u is harmonic. □

Moreover, we can alse prove the conformal invariance of Brownian motion, given by the following theorem.

Theorem 7. *Let $D, \widetilde{D}$ be domains and let $f : D \to \widetilde{D}$ be a conformal transformation. Let B (resp. $\widetilde{B}$) be a complex Brownian motion starting from $z \in D$ (resp. $\widetilde{z} = f(z) \in \widetilde{D}$). Let*

$$\tau = \inf_{t\geq 0}\{B_t \notin D\} \quad \text{and} \quad \widetilde{\tau} = \inf_{t\geq 0}\{\widetilde{B}_t \notin \widetilde{D}\}$$

be respectively the exit time of B from D and the exit time of $\widetilde{B}$ from $\widetilde{D}$. Set

$$\tau' = \int_0^\tau |f'(B_s)|^2 \mathrm{d}s$$

and for $t < \tau'$,

$$\sigma(t) = \inf_{s\geq 0}\left\{\int_0^s |f'(B_r)|^2 \mathrm{d}r = t\right\}.$$

With $B'_t = f(B_{\sigma(t)})$, we have

$$(\tau', B'_t : t < \tau') \sim \left(\widetilde{\tau}, \widetilde{B}_t : t < \widetilde{\tau}\right).$$

From the theorem, we can deduce the form of the exit distribution of a complex Brownian motion from a simply connected domain D. For instance, we can prove that if B is a Brownian motion starting at 0 in $\mathbb{D}$, the first exit of B is given uniformly on ∂D, as B and $\mathbb{D}$ are invariant by rotation.

These two results motivates the use of Brownian motion in the last sections.

3 Distortion estimates for conformal maps

Conformal transformations are functions that preserves the angles and the shape of small figures. The goal of this section is to estimate the distortion induced by them.

Let's denote $\mathcal{U}$ the set of all conformal transformations $f : \mathbb{D} \to D$ where D is any simple connected domain with $0 \in D$ and $D \neq \mathbb{C}$, with $f(0) = 0$ and $f'(0) = 1$.

As such, for $f \in \mathcal{U}$,

$$f(z) = z + \sum_{n=2}^\infty a_n z^n.$$

One first result we will use is the following.

Proposition 8. *If $f \in \mathcal{U}$, then $|a_2| \leq 2$.*

Proof. Admitted. □

Remark. This is a special case of the Bieberbach's conjecture made in 1916 that states that for all n, $|a_n| \leq n$, after he proved the previous proposition. The conjecture was proved by de Branges in 1985 using the Loewner equation (that was considered by Loewner in 1923 in order to prove this conjecture for $n = 3$).

Given that f is in $\mathcal{U}$, we can control $f''(0)$ by 1. Using this, we can prove the Koebe theorem.

Theorem 9 (Koebe-1/4 theorem)**.** *If $f \in \mathcal{U}$ and $0 < r \leq 1$, then $B(0, r/4) \subset f(r\mathbb{D})$.*

Sketch of proof. We only need to prove the result for $r = 1$ as otherwise, we can consider the function $f(rz)/r$. For f in $\mathcal{U}$ and $z_0 \notin D$, $\frac{z_0 f(z)}{z_0 - f(z)}$ is a conformal transformation of $\mathcal{U}$. Given the previous result, we have $|a_2|, \left|a_2 + \frac{1}{z_0}\right| \leq 2$. Hence $|z_0| \geq 1/4$. □

One consequence of Koebe's theorem is the following distortion estimate.

Proposition 10. *Suppose that $f \in \mathcal{U}$, then*

$$\mathrm{area}(f(\mathbb{D})) = \pi \sum_{n=1}^{\infty} n|a_n|^2.$$

Sketch of proof. For $r \in (0,1)$, set $\gamma_r(\theta) = F(r\mathrm{e}^{i\theta})$. We show that

$$\mathrm{area}(f(r\mathbb{D})) = \frac{1}{2i} \int_{\gamma_r} \bar{z}\mathrm{d}z = \pi \sum_{n=1}^{\infty} n|a_n|^2 r^{2n}$$

and we let $r \to 1$. □

4 Half-plane capacity

Definition 11. A set $A \subset \mathbb{H}$ is called a *compact* $\mathbb{H}$*-hull* if and only if $A = \mathbb{H} \cap \overline{A}$ and $\mathbb{H} \setminus A$ is simply connected. We let $\mathcal{Q}$ be the collection of compact $\mathbb{H}$-hulls.

In this section, we are interested on defining the "correct" conformal transformation of a compact $\mathbb{H}$-hulls and defining the notion of size of such an object.

Let's admit the following result.

Proposition 12 (Schwarz reflection principle)**.** *Let $D \subset \mathbb{H}$ be a simply connected domain and let $\phi : D \to \mathbb{H}$ be a conformal transformation which is bounded on bounded sets. Then ϕ extends by reflection to a conformal transformation on $D^* = D \cup \{\overline{z} : z \in D\} \cup \{x \in \partial\mathbb{H} : D$ is a neighbourhood of x in $\mathbb{H}\}$ by setting $\phi(\overline{z}) = \overline{\phi(z)}$.*

One of its consequences is the following result.

Proposition 13. *For each $A \in \mathcal{Q}$, there exists a unique conformal transformation $g_A : \mathbb{H} \setminus A \to \mathbb{H}$ with*

$$|g_A(z) - z| \xrightarrow[|z|\to\infty]{} 0.$$

Sketch of proof. For existence, by the Riemann mapping theorem, by post-composting by a conformal transformation $\mathbb{H} \to \mathbb{H}$ and by Schwarz reflection principle, there exists g a conformal transformation on $(\mathbb{H} \setminus A)^*$ that fixes ∞. The function g admits a Laurent expansion of the form $b_{-1}z + b_0 + \sum \frac{b_n}{z^n}$ and $g_A(z) = \frac{g(z) - b_0}{b_{-1}}$ and g_A verify the necessary conditions.

For uniqueness, if $\widetilde{g}_A$ is an other such transformation, then $\widetilde{g}_A^{-1} \circ g_A$ is a conformal transformation of $\mathbb{H} \to \mathbb{H}$ and $|\widetilde{g}_A^{-1} \circ g_A - z| \to 0$, we can deduce that $\widetilde{g}_A = g_A$. □

We now want to define some notion of size on the compact $\mathbb{H}$-hulls.

Definition 14 (Half-plane capacity)**.** Suppose that $A \in \mathcal{Q}$. The *half-plane capacity* of A is defined by:

$$\operatorname{hcap}(A) = \lim_{z \to \infty} z(g_A(z) - z).$$

Equivalently, we have

$$g_A(z) = z + \frac{\operatorname{hcap}(A)}{z} + \sum_{n=2}^{\infty} \frac{b_n}{z^n}.$$

Examples. Here are two examples of half-plane capacity.

- $\mathbb{H} \setminus [0, 2\sqrt{t}i] \ni z \mapsto \sqrt{z^2 + 4t} \in \mathbb{H}$ verifies $\left|\sqrt{z^2+4t} - z\right| \to 0$, as such $\operatorname{hcap}(\mathbb{H} \setminus [0, 2\sqrt{t}i]) = 2t$.
- $\mathbb{H} \setminus \overline{\mathbb{D}} \ni z \mapsto z + \frac{1}{z} \in \mathbb{H}$ verifies $\left|\left(z + \frac{1}{z}\right) - z\right| \to 0$, as such $\operatorname{hcap}(\mathbb{H} \setminus [0, 2\sqrt{t}i]) = 1$.

The fact that the half-plane capacity of A is some sort of "size" of A can be seen through the following properties.

Proposition 15 (Properties of the half-plane capacity)**.** *The following properties are true.*

1. ***Scaling****. For $r > 0$, $A \in \mathcal{Q}$,*
$$\operatorname{hcap}(rA) = r^2 \operatorname{hcap}(A).$$
2. ***Translation invariance****. For $x \in \mathbb{R}$, $A \in \mathcal{Q}$,*
$$\operatorname{hcap}(A + x) = \operatorname{hcap}(A).$$
3. ***Monotonicity****. For $A, \widetilde{A} \in \mathcal{Q}$,*
$$\operatorname{hcap}\left(\widetilde{A}\right) = \operatorname{hcap}(A) + \operatorname{hcap}\left(g_A\left(\widetilde{A} \setminus A\right)\right)$$

Sketch of proof. We can remark that

$$g_{rA}(z) = r g_A(z/r), \quad g_{A+x}(z) = g_A(z - x) + x, \quad g_{\widetilde{A}} = g_{g_A(\widetilde{A} \setminus A)} \circ g_A,$$

and conclude thanks to the second definition of the half-plane capacity. □

Remark. By combining the scaling and monotonicity properties of the half-plane capacity, we note that of $A \in \mathcal{Q}$ and $A \subset r\overline{\mathbb{D}} \cap \mathbb{H}$, we have

$$\operatorname{hcap}(A) \leq \operatorname{hcap}(r\overline{\mathbb{D}} \cap \mathbb{H}) = r^2 \operatorname{hcap}(\overline{\mathbb{D}} \cap \mathbb{H}) = r^2.$$

Let us denote for $A \in \mathcal{Q}$, $\operatorname{rad}(A) = \{|z| : z \in A\}$. We can now prove.

Proposition 16. *Suppose that $A \in \mathcal{Q}$, B is a complex Brownian motion and $\tau = \inf_{t \geq 0}\{B_t \notin \mathbb{H} \setminus A\}$. Then if $x > \operatorname{rad}(A)$,*

$$g_A(x) = \lim_{y \to \infty} \pi y \left(\frac{1}{2} - \mathbb{P}_{iy}\left[B_\tau \in [x, \infty)\right]\right),$$

and if $x < -\operatorname{rad}(A)$,

$$g_A(x) = \lim_{y \to \infty} \pi y \left(\mathbb{P}_{iy}\left[B_\tau \in (-\infty, x]\right] - \frac{1}{2}\right).$$

Proof. Admitted. □

We can deduce the following.

Corollary 17. *Suppose that $A \in \mathcal{Q}$ with $\operatorname{rad}(A) \leq 1$. Then*

$$\begin{cases} x \leq g_A(x) \leq x + \dfrac{1}{x} & \text{if } x > 1\ ; \\ x + \dfrac{1}{x} \leq g_A(x) \leq x & \text{if } x < -1. \end{cases}$$

Moreover, if $A \in \mathcal{Q}$, then $|g_A(z) - z| \leq 3\operatorname{rad}(A)$ for all $z \in \mathbb{H} \setminus A$.

Proof. Admitted. □

Given those results, we can control a bit more precisely the variation of g_A.

Proposition 18. *There exists $c > 0$ such that for all $A \in \mathcal{Q}$ and $|z| \geq 2\mathrm{rad}(A)$, we have that*

$$\left| g_A(z) - z - \frac{\mathrm{hcap}(A)}{z} \right| \leq c \frac{\mathrm{rad}(A)\mathrm{hcap}(A)}{|z|^2}.$$

Proof. Admitted. □

5 The chordal Loewner equation

The goal of this section is to derive the chordal Loewner equation.

We'll need the following result.

Proposition 19. *There exists a constant $c > 0$ so that the following is true. Suppose that $A, \widetilde{A} \in \mathcal{Q}$ with $A \subset \widetilde{A}$ and $\widetilde{A} \setminus A$ is connected. Then*

$$\mathrm{diam}(g_A(\widetilde{A} \setminus A)) \leq c \begin{cases} d^{1/2} r^{1/2} & \text{if} \quad d \leq r, \\ \mathrm{rad}(\widetilde{A}) & \text{if} \quad d > r, \end{cases}$$

where $d = \mathrm{diam}(\widetilde{A} \setminus A)$ and $r = \sup\{\Im(z) : z \in \widetilde{A}\}$.

Proof. Admitted □

Definition 20. Let $(A_t)_{t \geq 0}$ be a family of compact $\mathbb{H}$-hulls, we say that A is:

- *non-decreasing* if and only if $0 \leq s \leq t < \infty$, implies $A_s \subset A_t$;
- *locally growing* if and only if for every $T, \varepsilon > 0$, there exists $\delta > 0$ such that $0 \leq s \leq t \leq s + \delta \leq T$ implies that $\mathrm{diam}\left(g_s\left(A_t \setminus A_s\right)\right) \leq \varepsilon$;
- *parametrized by half-plane capacity* if and only if for all $t \geq 0$, $\mathrm{hcap}(A_t) = 2t$.

Let $\mathcal{A}$ be the collection of families of compact $\mathbb{H}$-hulls which satisfies the three previous conditions.

Example. We can prove that for any γ simple curve in $\mathbb{H}$ starting from 0 $A_t = \gamma([0, t])$ defines a family of compact $\mathbb{H}$-hulls verifying the two first conditions and that if we reparametrize γ, A will be parametrized by half-plane capacity.

Thanks to this definition we can state the following result:

Theorem 21 (Loewner equation)**.** *Suppose that $(A_t)_t$ is in $\mathcal{A}$ with A_0. For each $t \geq 0$, let $g_t = g_{A_t}$. There exists $U : [0, \infty) \to \mathbb{R}$ continuous such that*

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$

The function U is called the "Loewner driving function" for A.

Sketch of proof. Thanks to the local growth of A, we can define U_t as the only point in $\bigcap_{s>t} \overline{g_t(A_s)}$ and verify that U is continuous. By reasoning between t and $t + \varepsilon$, and using the limited development of the g_{A_t} we can prove the ODE equation. □

Thaks to this result, we can see that the whole family of compact hulls is completely determined by a simple function $U : [0, \infty) \to \mathbb{R}$! That means that having a certain fixed function U gives us a way to construct the family of compact hulls. That is exactly what we will do in the next section.

6 Derivation of the Schramm-Loewner evolution

The goal of this section is to explain the derivation and definition of SLE.

Let us start with a definition that appears naturally when we want to construct a stochastic process stable by conformal transformations.

Definition 22. Suppose that $(A_t)_t$ is a random family in $\mathcal{A}$ encoded with the Loewner driving function U. We say that $(A_t)_t$ satisfies the *conformal Markov property* if and only if for each $t \geq 0$, we denote $\mathcal{F}_t = \sigma(U_s : s \leq t)$ and

1. (Markov property)
$$((g_t(A_{t+s}) - U_t)_{s\geq 0} \,|\mathcal{F}_t\,) \sim (A_s)_{s\geq 0};$$
2. (Scale invariance)
$$\forall r > 0, \left(rA_{t/r^2}\right)_{t\geq 0} \sim (A_t)_{t\geq 0}\,.$$

As the dynamic of $((g_t(A_{t+s}) - U_t)_{s\geq 0} \,|\mathcal{F}_t\,)$ and of $(A_s)_s$ are characterized by their driving function, then the first condition of the definition is equivalent to the following:

$$(U_{t+s} - U_t)_{s\geq 0} \,|\mathcal{F}_t\,) \sim (U_s)_{s\geq 0}.$$

Thus U has stationary, independent increments and as it is continuous, there exists $\kappa \geq 0$ and $a \in \mathbb{R}$ such that $U_t = \sqrt{\kappa}B_t + at$ where B is a standard Brownian motion.

With the scale invariance, we have for $r > 0$ that

$$rU_{t/r^2} = r\sqrt{\kappa}B_{t/r^2} + at/r \sim \sqrt{\kappa}B_t + at/r = U_t.$$

Thus $a = 0$ and we can state the following result.

Theorem 23 (Schramm)**.** *If (A_t) satisfies the conformal Markov property, then there exists $\kappa \geq 0$ such that $U_t = \sqrt{\kappa}B_t$ where B is a Brownian motion.*

For $\kappa > 0$, SLE_κ is the random family of hulls obtained by solving Loewner equation with $U_t = \sqrt{\kappa}B_t$ where B is a standard Brownian motion.

7 Simulation of SLE_κ

The goal of this section is to expose methods on how to simulate SLE_κ for $\kappa \geq 0$.

The first idea one can get in order to do so would be:

1. Take a Brownian motion B on $[0, T]$.
2. Resolve the Loewner equation with driving function $U = \sqrt{\kappa}B$ and obtain the g_t for $t \in [0, T]$.
3. From g_t get the A_t back.

The first two steps are quite easy to make (the second step requiring only a discretization that the Euler scheme offers) as long as for z we do not have $g_t(z) - U_t = 0$. The third step on the other is quite hard to do as A_t is the set all points on which g_s is not defined for $s \leq t$.

An other method one can think of is that if g_t verify the ODE of the Loewner theorem, then g_t^{-1} verifies

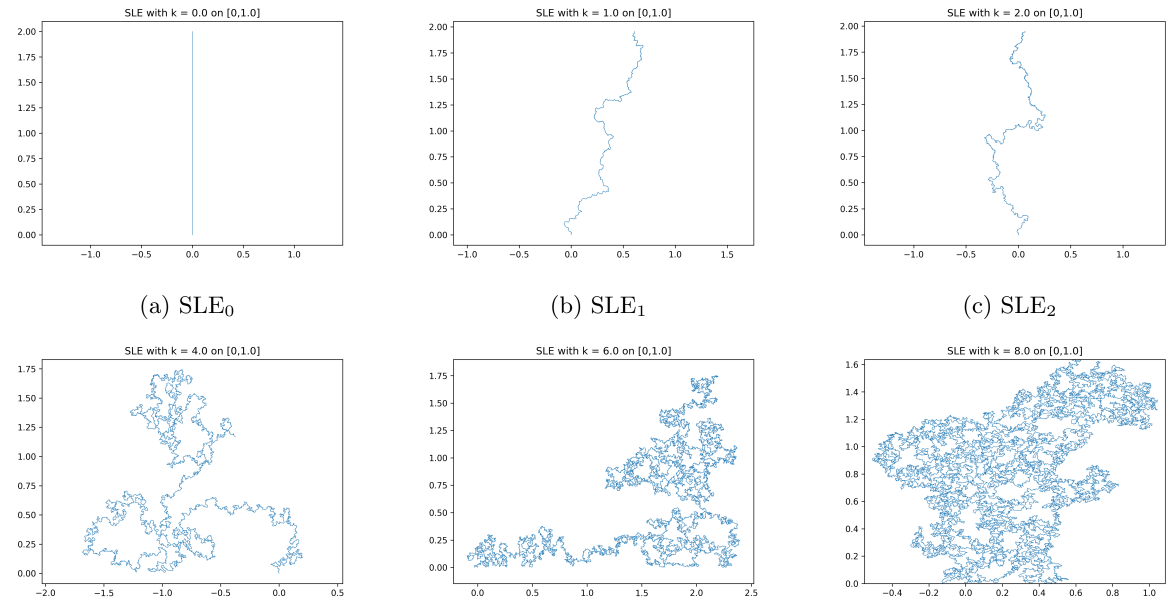
$$\partial_t g_t^{-1}(z) = \frac{2\left(g_t^{-1}\right)'(z)}{U_t - z} \quad \text{and} \quad g_0^{-1}(z) = z,$$

which is guaranteed to work for any $z \notin \{U_t : t \geq 0\}$ ie it works for all $z \in \mathbb{H}$. However, the curve $\gamma(t)$ is then given by $\gamma(t) = g_t^{-1}(U_t)$, which is not well defined. If U_t is a “nice” enough function we can define it in a asymptotic way, however *a priori* it is not clear how we should define it.

There actually are methods to simulate going around such problems that the reader may find in [KN04], [Ken09] or [FLM22].

The main idea is either to consider a backward stochastic equation based on the Schramm Loewner equation or to do sub-steps in the simulation using the Markov property of the SLE.

We have in figure 1 simulations of the SLE based on the code of [Fos24]. We can see that for $\kappa < 4$, the curve is simple, for $4 \leq \kappa < 8$, it is instersecting itself and every point will be contained in a loop but it is not space filling and for $\kappa \geq 8$ the curbe is space-filling.



(a) SLE_0 (b) SLE_1 (c) SLE_2

(d) SLE_4 (e) SLE_6 (f) SLE_8

Figure 1: Simulations of SLE with different values of κ.

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