Seminar 2024-2025
Stochastic Loewner Evolutions

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Introduction and motivation

Goal: model the scaling limits of interfaces for models of 2D statistical mechanics

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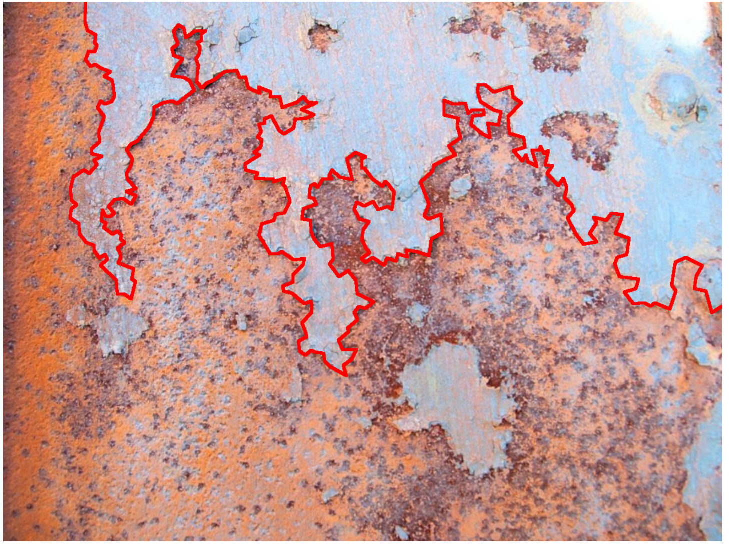

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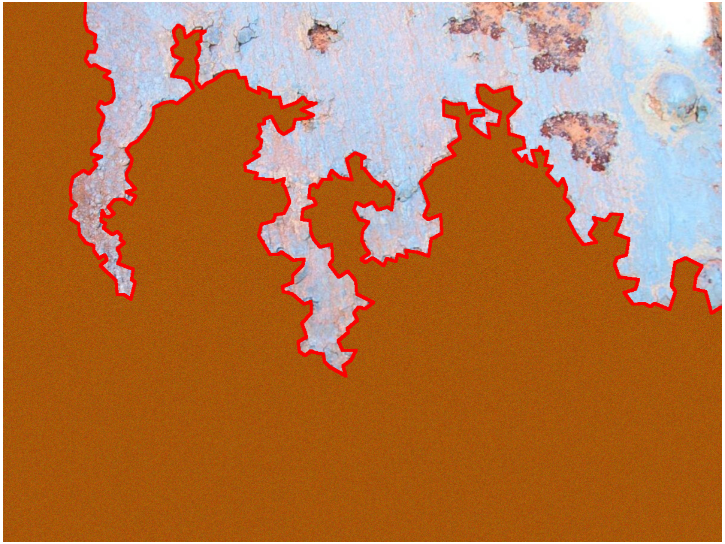

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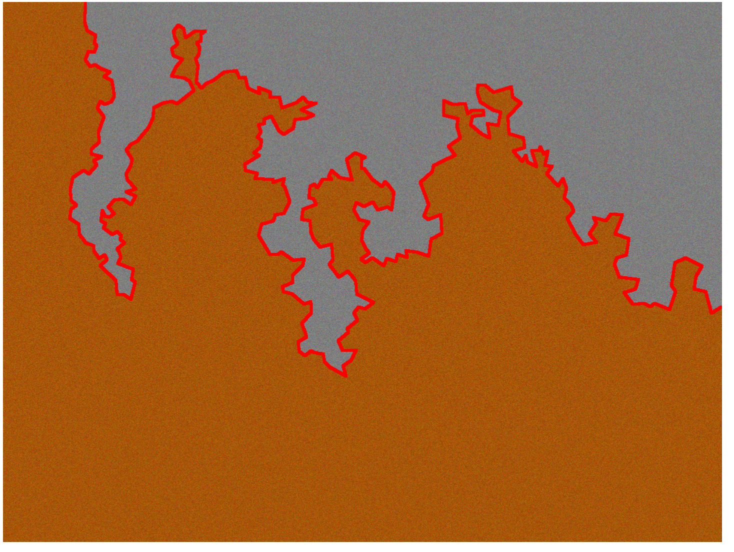

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Expectations:

- Scale invariance: $(K_t)_{t\geq 0} \sim (\lambda K_{\lambda^{-2}t})_{t\geq 0}$
- Domain Markov property

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Introduction and motivation

Create links between probability, geometry and analysis

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Conformal transformations

Definition (Conformal transformation)

Let U and V be open domains of $\mathbb{C}$ and let $f : U \to V$ be a map. We say that f is a *conformal transformation* if and only if f is holomorphic and bijective.

Examples

- $f : \mathbb{D} \ni w \mapsto \lambda \frac{w+z}{1+\bar{z}w} \in \mathbb{D}$ where $\lambda \in \mathbb{U}$ and $z \in \mathbb{D}$.
- $g : \mathbb{D} \ni z \mapsto i\frac{1+z}{1-z} \in \mathbb{H}$. Its inverse is $f : \mathbb{H} \ni z \mapsto \frac{z-i}{z+i} \in \mathbb{D}$.
- $f(z) = \frac{az+b}{cz+d}$ where $ad - bc = 1$.

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Riemann mapping theorem

Theorem (Riemann mapping theorem)

Suppose that U is a simply connected domain with $U \neq \mathbb{C}$ and let $z \in U$. Then there exists a unique conformal transformation $f : \mathbb{D} \to U$ with $f(0) = z$ and $f'(0) > 0$.

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- $f : \mathbb{D} \ni w \mapsto \frac{w+z}{1+\bar{z}w} \in \mathbb{D}$ where $z \in \mathbb{D}$

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Loewner equation

For $t \geq 0$, let $H_t = \mathbb{H} \setminus [0, 2\sqrt{t}i]$ and $g_t : H_t \ni z \to \sqrt{z^2 + 4t} \in \mathbb{H}$ is a conformal transformation $H_t \to \mathbb{H}$.

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Two observations:

- $|g_t(z) - z| = \left|\sqrt{z^2 + 4t} - z\right| \xrightarrow[z \to \infty]{} 0$;
- For each $z \in \mathbb{H}$ fixed,

$$\partial_t g_t(z) = \frac{1}{2\sqrt{z^2 + 4t}} \cdot 4 = \frac{2}{g_t(z)}.$$

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This is a special case of Loewner's theorem.

Definition

We say that $A \subset \mathbb{H}$ is a *compact* $\mathbb{H}$-*hull* iff A and $\mathbb{H} \setminus A$ are simply connected.

Proposition

For all compact $\mathbb{H}$*-hull* A*, there exists a unique conformal transformation* $g_A : \mathbb{H} \setminus A \to \mathbb{H}$ *with*

$$|g_A(z) - z| \xrightarrow[|z|\to\infty]{} 0.$$

It also verifies

$$g_A(z) = z + \frac{\operatorname{hcap}(A)}{z} + O\left(\frac{\operatorname{hcap}(A)\operatorname{rad}(A)}{|z|^2}\right).$$

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Definition

Let $(A_t)_{t\geq 0}$ be a family of compact $\mathbb{H}$-hulls, we say that A is:

- *non-decreasing* if and only if $0 \leq s \leq t < \infty$, implies $A_s \subset A_t$;
- *locally growing* if and only if for every $T, \varepsilon > 0$, there exists $\delta > 0$ such that $0 \leq s \leq t \leq s + \delta \leq T$ implies that $\operatorname{diam}\left(g_s\left(A_t \setminus A_s\right)\right) \leq \varepsilon$;
- *parametrized by half-plane capacity* if and only if for all $t \geq 0$, $\operatorname{hcap}(A_t) = 2t$.

Let $\mathcal{A}$ be the set of all families of compact $\mathbb{H}$-hulls verifying all three conditions.

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Loewner equation

Theorem (Loewner equation)

Suppose that $(A_t)_t$ *is in* $\mathcal{A}$ *with* $A_0 = \emptyset$. *For each* $\mathbf{t} \geq 0$, *let* $\mathbf{g}_t = \mathbf{g}_{A_t}$. *There exists* $U : [0, \infty) \to \mathbb{R}$ *continuous such that*

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$

The function U *is called the "Loewner driving function" for* A.

Stochastic Loewner Evolutions

Definition

Suppose that $(A_t)_t$ is a random family in $\mathcal{A}$ encoded with the Loewner driving function U. We say that $(A_t)_t$ satisfies the *conformal Markov property* if and only if for each $t \geq 0$, we denote $\mathcal{F}_t = \sigma(U_s : s \leq t)$ and

1. (Markov property)

$$\mathcal{L}\left((g_t(A_{t+s}) - U_t)_{s\geq 0} \,|\mathcal{F}_t\right) = \mathcal{L}\left((A_s)_{s\geq 0}\right)$$

$$(g_t(A_{t+s}) - U_t)_{s\geq 0} \perp\!\!\!\perp (\mathcal{F}_s)_{s\geq 0}\,;$$

2. (scale invariance)

$$\forall r > 0, \left(rA_{t/r^2}\right)_{t\geq 0} \sim (A_t)_{t\geq 0}\,.$$

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Stochastic Loewner Evolutions

Theorem (Schramm)

If (A_t) satisfies the conformal Markov property, then there exists $\kappa \geq 0$ such that $U_t = \sqrt{\kappa} B_t$ where B is a Brownian motion i.e.

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \quad g_0(z) = z.$$

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Conclusion

Thank you for your attention ! Do you have any question ?

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