Econ 8185: Quant PS1

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Abstract

The present document calculates the value function, policy functions, of a simple economy with stochastic AR(1) shock to labor productivity. The computation is based made on two different methods- Value Function Iteration, Linear Quadratic Approximation. The LQ problem can be solved either by using iteration on Riccati Equation or directly using Vaughan Method.

1 Model

We have the following growth model:

$$\max_{\{\hat{c}_t, h_t, \hat{k}_{t+1}\}} \sum_t \hat{\beta}^t U(\hat{c}_t, h_t)$$

subject to

$$\hat{c}_t + (1+\gamma_z)(1+\gamma_n)\hat{k}_{t+1} = \hat{F}(\hat{k}_t, h_t, z_t)$$

$$log(z_{t+1}) = \rho log(z_t) + \epsilon_{t+1}$$

where $\hat{x} = \frac{x}{(1+\gamma_z)^t}$ variables denotes the de-trended variables with respect to productivity growth, $\hat{F}(k, h, z) = k^\theta (zh)^{1-\theta} + (1-\delta)k$, $U(c, h) = log(c) + \psi log(1-h)$, and $\hat{\beta} = \beta(1+\gamma_n)$.

1.1 Parameters

The parameter values for $\beta, \delta, \gamma_z, \gamma_n$ are based on Prof. Kehoe's Notes on calibrating the growth models. I have not calibrated the other parameters and tried to assume some values to begin with.

Parameters	Value
β	0.95
δ	0.05
ψ	1.6
γ_n	0.02
γ_z	0.02
θ	0.34
ρ	0.50

Table 1: Parameter Selection

1.2 Steady State

The MRS, Euler equation along with the resource constraints forms a system of 3 equations in 3 variables at the steady state. We solve the system numerically to get the steady state values of $\hat{k}, \hat{h}, \hat{c}$. For the above set of parameter values the steady state of the de-trended variables is as follows: $\hat{k}_{ss} = 1.640111, h_{ss} = 0.3543826, \hat{c}_{ss} = 0.4483689$

2 Value Function Iteration

Algorithm:

1. Create a 2 dimensional grid of capital (k) and labor supply (h).
2. Use subroutine Tauchen.jl to get an evenly spaced grid of shocks (z) along with the Transition Matrix (P) for the AR(1) process for $log(z)$.
3. Create 2 dimensional empty grids for storing value functions and policy functions.
4. Create a return function $G(k, k', h, z, V_n) = U(c, h) + \hat{\beta} E_z[V_n(k', z')]$ where $c = F(k, h, z) + (1 - \delta)k - (1 + \gamma_n)(1 + \gamma_z)k'$. The expected value is calculated using the transition matrix obtained above.
5. Create a grid search loop to look for the maximum value and the maximisers, i.e for each (k, z), $V_{n+1}(k, z) = \max_{k',h} G(k, k', h, z, V_n)$; $k'(k, z), h(k, z) = argmax_{k',h} G(k, k', h, z, V_n)$.
6. Iterate on the loop until convergence: $||V_{n+1} - V_n|| < tolerance$.

2.1 Results from VFI

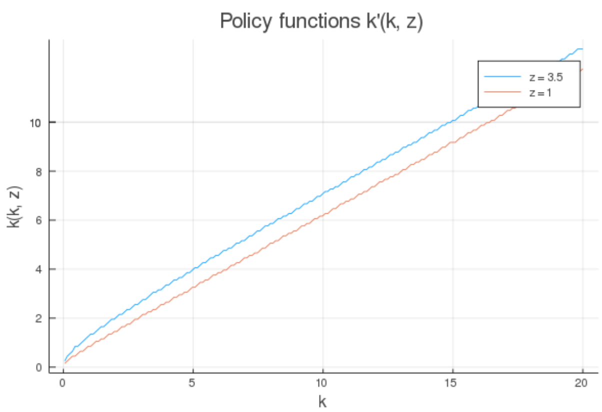

Figure 1: Capital Policy Function

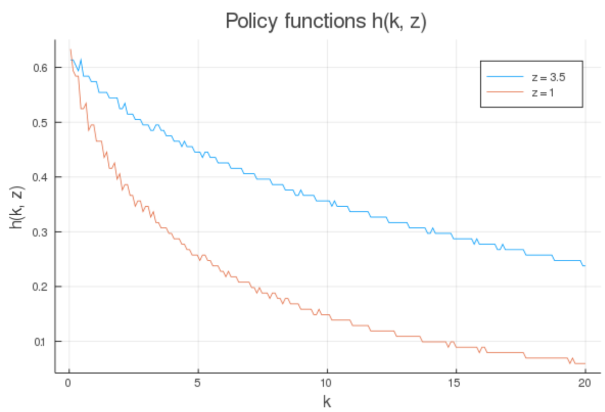

Figure 2: Labor Policy function

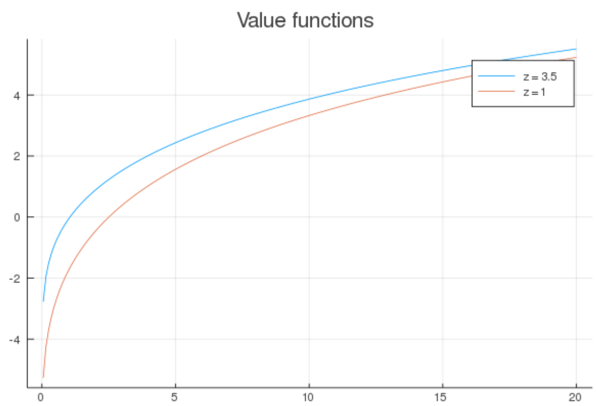

Figure 3: Value Function

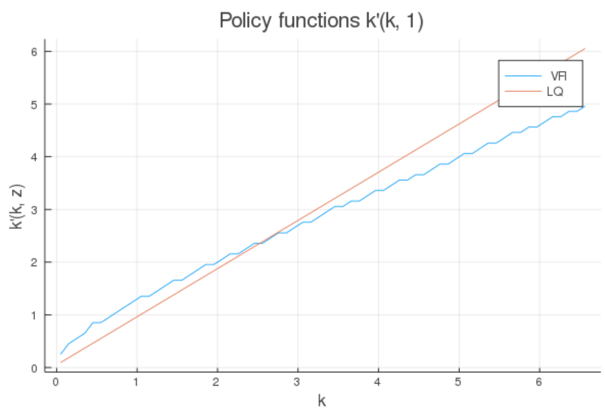

Figure 4: Comparision VFI vs LQ

3 LQ Approximation

For ease of notation define:

$$c(k,k',h,z) = \hat{F}(k,h,z) - (1+\gamma_n)(1+\gamma_z)k'$$
$$X_t = [1, k_t, log(z_t)]'$$
$$u_t = [k_{t+1}, h_t]'$$
$$r(X_t, u_t) = U(c(k_t, k_{t+1}, h_t, z_t), h_t)$$
$$Z_t = [X_t, u_t]$$

A second order Taylor approximation of return function gives -

$$r(Z) \approx Z'MZ$$

$$= \begin{bmatrix} X \\ u \end{bmatrix}' \begin{bmatrix} M11 & M12 \\ M21 & M22 \end{bmatrix} \begin{bmatrix} X \\ u \end{bmatrix}$$

3.1 (b) (ii) Return function depends on (k, k', h):

The model in this case is described as follows:

$\max_u \sum \beta^t r(X_t, u_t)$ subject to $X_{t+1} = AX_t + Bu_t + C\epsilon_{t+1}$ where Here matrices A, B, C according to state space representation are:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

3.2 (b) (iii) Return function depends on (k, k'):

We use the first order MRS condition along with the resource constraint to solve for $h(k, k'z)$ as implicit solution of the following equation:

$$\frac{U_h(c(k,k',h,z),h)}{U_c(c(k,k',h,z),h)} = -\hat{F}(k,h,z)$$

In the present case $X_t = \begin{bmatrix} 1 \\ k_t \\ log(z_t) \end{bmatrix}$, $u_t = \begin{bmatrix} k_{t+1} \end{bmatrix}$. The state space representation is as follows:

$$\underbrace{\begin{bmatrix} 1 \\ k_{t+1} \\ log(z_{t+1}) \end{bmatrix}}_{X_{t+1}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ k_t \\ log(z_t) \end{bmatrix}}_{X_t} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} k_{t+1} \end{bmatrix}}_{u_t} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{C} \begin{bmatrix} \epsilon_{t+1} \end{bmatrix}$$

3.3 (c) Vaughan Method

For Vaughan Method, we closely follow Prof. Ellen's class notes to first construct a matrix $\mathbb{H}$. We then carry out the eigenvalue decomposition of the matrix $\mathbb{H}$. The eigen-vectors are then used to construct the matrices P and F.

3.4 Results from LQ

Note that once we have a Linear Quadratic version of our initial problem we can obtain the required matrices by either solving a Riccati equation or using the Vaughan method. We note that both the approaches gives the same answer. The results are outlined below.

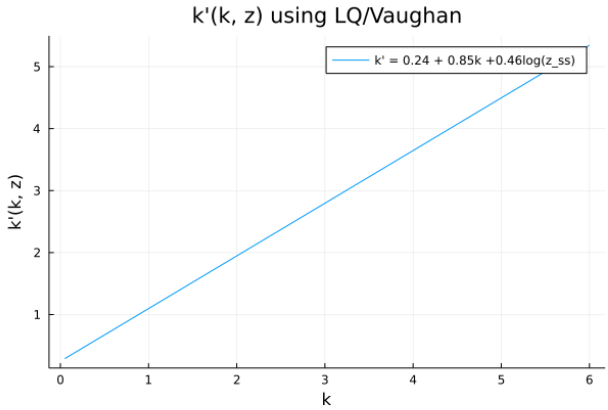

Figure 5: Capital Policy Function

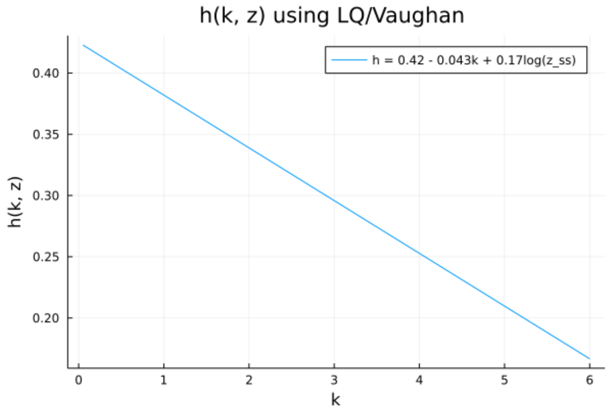

Figure 6: Labor Policy function

We note that the linearized policy function is exactly the same using all of the three formulations mentioned above. The equations for linearized policy functions are reported below:

$$k'(k,z) = 0.245832 + 0.850113k + 0.462318\ln(z)$$
$$h(k,z) = 0.425009 - 0.0430618k + 0.171364\ln(z)$$

We further note that the LQ approximation can be solved by either solving a Riccati equation or using the Vaughan method. The policy functions derived using the two methods to solve LQ are exactly the same till 6 decimal places.

4 Time series of output

Once we have the policy function with using the methods mentioned above, we can simulate the economy to study the time series properties of the output. In the plot below we graph the time series for de-trended per capita output with its steady state level for 200 time periods.

We note that de-trended per capita output moves around its steady state value. Since we have approximated the $AR(1)$ shock process using 5 productivity shocks, there are some flat portions in the graph (for some period we draw the same shock). There is slight variation in he result from VFI and LQ/Vaughan due to the fact that VFI is calculated on a limited number of grid points while LQ/Vaughan gives linear policies. However, we note that still the overall pattern of output deviation is close using the two methods.

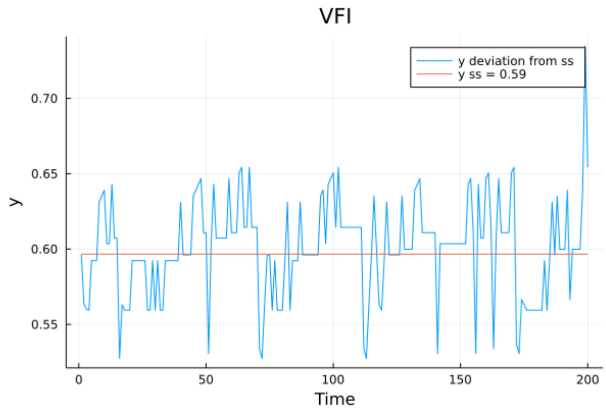

Figure 7: Output deviation VFI

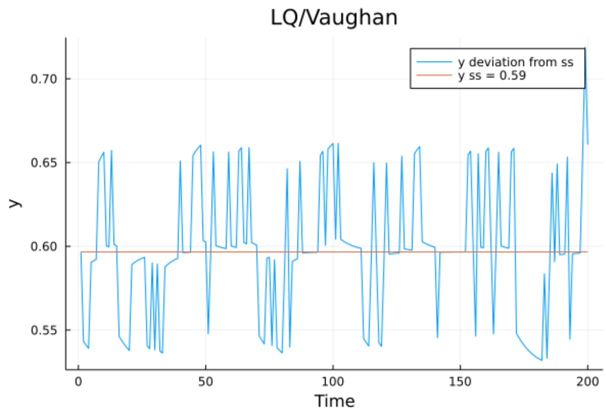

Figure 8: Output deviation LQ/Vaughan

5 Properties of Solution

Note that the code we have written is with non zero values of the parameters $\psi, \delta, \gamma_n, \gamma_z$. Thus, studying the properties of solution by changing parameter values involves changing the initial parameters in the code and re-estimating everything.

- When $\psi = 0$, $h(k, z)$ is a constant for all the cases. This is evident from the VFI, LQ as well as the Vaughan method for such parameter value.
- When $\delta \neq 1$, the steady state value of capital is higher.
- When $\gamma_z, \gamma_n = 0$, the steady state value of capital is lower.

6 Alternate Utility Function

The preference are $U(c, h) = \frac{(c(1-h)^{\psi})^{1-\gamma}}{1-\gamma}$ with $\gamma = 5$

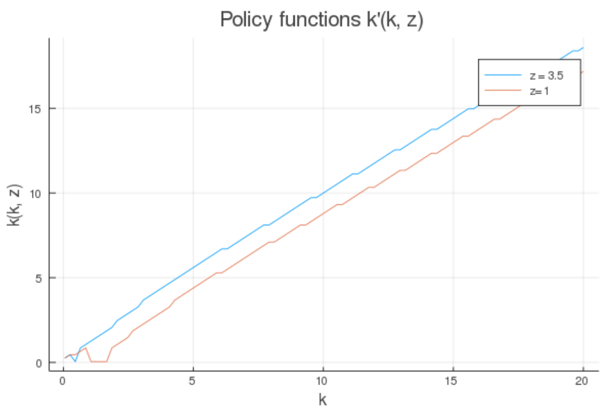

Figure 9: Capital Policy Function

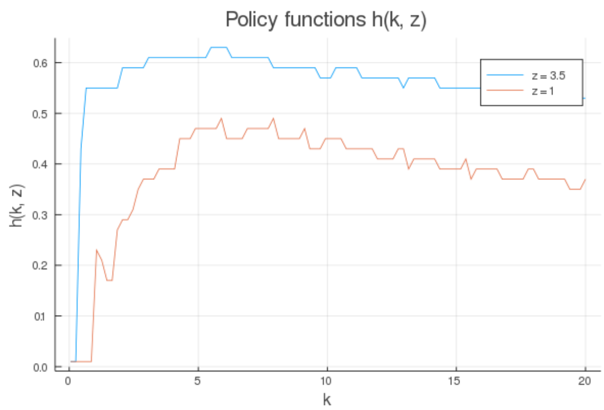

Figure 10: Labor Policy function

Note: The above VFI graphs are from a limited number of iterations and grid points and not from entire convergence since convergence was taking too much time. But the shape of policy functions are indicative of the final results.

We note that while the shape of capital policy function remains the same, the of labour policy function changes. With the separable preferences, the labour policy function is downward sloping, in the current case for low value of capital it is upward sloping. It follows the usual downward sloping graph thereafter.