Econ 8185 (002): Quant PS4

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Abstract

This document solves the Aiyagri model with Aggregate Shocks.

1 Aiyagari with Aggregate shocks

The basic Aiyagari model has only idiosyncratic labour income or productivity shocks. The Krussel-Smith model adds aggregate shocks to the Aiyagari model.

For the current case, let z denote the aggregate shocks, while ϵ denotes individual shocks. Both shocks can take two values only:

$$\begin{aligned} z &\in \{z_b = 0.99, z_g = 1.01\} \\ \epsilon &\in \{0, 1\}. \end{aligned}$$

Construction of transition matrix

We'll keep the notation same as in KS. Here $\pi_{zz'}$ denotes the probability of transition from state z today to z' tomorrow. Similarly, $\pi_{zz'\epsilon\epsilon'}$ denotes the probability of transitioning from state (z, ϵ) today to state (z', ϵ') tomorrow. The aggregate shock follows Z follows a first order markov structure with the following transition matrix:

$$\Pi_z = \begin{bmatrix} \pi_{gg} & \pi_{gb} \\ \pi_{bg} & \pi_{bb} \end{bmatrix}$$

The transition matrix for aggregate shock is calibrated such that the average duration of expansion(z_g) or recession(z_b) is 8 quarters. Given the markov process for aggregate shock, the average duration of being in good state is given by $\frac{1}{1-\pi_{gg}}$, and average duration of being in bad state is given by $\frac{1}{1-\pi_{bb}}$. This gives that $\pi_{gg} = \pi_{bb} = 1 - \frac{1}{8}$.[1]

Controlling for z, shocks (z', ϵ') are uncorrelated, however, the joint realization of (z', ϵ') next period depends on the realization of current z. Combining the aggregate shocks and individual shocks there are 4 possible state at each time period. The transition matrix Π' shows the probability of future state given the current state. The transition matrix Π' is given as:

$$\Pi' = \begin{bmatrix} \pi_{gg00} & \pi_{bg00} & \pi_{gg10} & \pi_{bg10} \\ \pi_{gb00} & \pi_{bb00} & \pi_{gb10} & \pi_{bb10} \\ \pi_{gg01} & \pi_{bg01} & \pi_{gg11} & \pi_{bg11} \\ \pi_{gb01} & \pi_{bb01} & \pi_{gb11} & \pi_{bb11} \end{bmatrix}$$

Note that this is actually the adjoint of transition matrix. Writing the matrix in form turns out to be more useful in simulation. In order to construct the transition matrix Π, we make use of the following property of conditional

[1] The average duration of being in good state is calulated using the following infinite sum : $\lim_{N\to\infty} 1(1-\pi_{gg}) + 2\pi_{gg}(1-\pi_{gg}) + \ldots N\pi_{gg}^{N-1}(1-\pi_{gg})$. A similar calculation for the average duration of being in bad state holds.

probability:[2]

$$\begin{aligned} P(A,B|C) &= P(A|B,C)P(B|C) \\ P(\epsilon',z'|z,\epsilon) &= P(\epsilon'|z',\epsilon,z)P(z'|z,\epsilon) \\ P(\epsilon',z'|z,\epsilon) &= P(\epsilon'|z',\epsilon,z)P(z'|z) \\ \pi_{zz'\epsilon\epsilon'} &= \pi_{zz'}\pi_{\epsilon'|\epsilon zz'}. \end{aligned}$$

Since ϵ can take only two values, say ϵ_1, ϵ_2, from the above formulation we see see that:

$$\pi_{zz'\epsilon\epsilon'_2} = \pi_{zz'}\pi_{\epsilon'_2|\epsilon zz'} = \pi_{zz'}(1-\pi_{\epsilon'_1|\epsilon zz'}).$$

KS mentions the above condition in the following form:

$$\pi_{zz'} = \pi_{zz'00} + \pi_{zz'01} = \pi_{zz'10} + \pi_{zz'11}.$$

We calibrate the transition matrix to match the following piece of information.

1. Average duration of unemployment($\epsilon = 0$) during expansion(z_g) is 1.5 quarters, and during recession(z_b) is 2.5 quarters.
    - Given that the average duration of unemployment, we can write:[3]

    $$\begin{aligned} \pi_{0|0bb} &= 1 - \frac{1}{2.5} \\ \pi_{0|0gg} &= 1 - \frac{1}{1.5}. \end{aligned}$$

    Thus we obtain:

    $$\begin{aligned} \pi_{gg00} &= \pi_{gg}\pi_{0|0gg} \\ \pi_{gg01} &= \pi_{gg}(1-\pi_{0|0gg}) \\ \pi_{bb00} &= \pi_{bb}\pi_{0|0bb} \\ \pi_{bb01} &= \pi_{bb}(1-\pi_{0|0bb}). \end{aligned}$$

2. KS adds the following additional restrictions to ensures that the probability of remaining unemployed is high going into a bad state as compared to the bad state, and probability of remaining unemployed is low going into a good state compared to the good state:

    $$\begin{aligned} \pi_{gb00} &= 1.25\pi_{bb00}\frac{\pi_{gb}}{\pi_{bb}} \\ \pi_{bg00} &= 0.75\pi_{gg00}\frac{\pi_{bg}}{\pi_{gg}}. \end{aligned}$$

    The above restrictions can also be stated in terms of conditional probabilities as:

    $$\begin{aligned} \pi_{0|0gb} &= 1.25\pi_{0|0bb} \\ \pi_{0|0bg} &= 0.75\pi_{0|0gg}. \end{aligned}$$

[2] Imrohoroglu(1989) takes a different approach to calibrate the transition matrix (Π). She constructs Π based on $\Pi_{\epsilon\epsilon'\_g}$, and $\Pi_{\epsilon\epsilon'\_b}$.

[3] Some authors directly set $\pi_{gg00} = 1 - \frac{1}{1.5}, \pi_{bb00} = 1 - \frac{1}{2.5}$. I think the current method to calibrate is more appropriate.

From this we can arrive at:

$$\pi_{gb01} = \pi_{gb}(1 - \pi_{0|0gb})$$
$$\pi_{bg01} = \pi_{bg}(1 - \pi_{0|0bg}).$$

3. Unemployment rate in expansion(u_g) is 4%, and in recession(u_b) is 10%.
   - Since the unemployment rate depends only on the aggregate state, we can write:[4]

     $$u_{z'} = \pi_{0|1zz'}(1 - u_z) + \pi_{0|0zz'}u_z.$$

     KS mentions the above condition as:

     $$u_{z'} = \frac{\pi_{zz'10}}{\pi_{zz'}}(1 - u_z) + \frac{\pi_{zz'00}}{\pi_{zz'}}u_z.$$

     Given the unemployment rates, we can write:

     $$\pi_{0|1gg} = (u_g - u_g\pi_{0|0gg})/(1 - u_g)$$
     $$\pi_{0|1bb} = (u_b - u_b\pi_{0|0bb})/(1 - u_b)$$
     $$\pi_{0|1bg} = (u_g - u_b\pi_{0|0bg})/(1 - u_b)$$
     $$\pi_{0|1gb} = (u_b - u_g\pi_{0|0gb})/(1 - u_g).$$

     From this we obtain:

     $$\pi_{gg10} = \pi_{gg}\pi_{0|1gg}$$
     $$\pi_{gg11} = \pi_{gg}(1 - \pi_{0|1gg})$$
     $$\pi_{bb10} = \pi_{bb}\pi_{0|1bb}$$
     $$\pi_{bb11} = \pi_{bb}(1 - \pi_{0|1bb})$$
     $$\pi_{bg10} = \pi_{bg}\pi_{0|1bg}$$
     $$\pi_{bg11} = \pi_{bg}(1 - \pi_{0|1bg})$$
     $$\pi_{gb10} = \pi_{gb}\pi_{0|1gb}$$
     $$\pi_{gb11} = \pi_{gb}(1 - \pi_{0|1gb}).$$

Thus we are able to obtain all the 16 entries of the transition matrix Π. We'll also need the four conditional transition matrices, namely, $\pi_{\epsilon'|\epsilon gg}$, $\pi_{\epsilon'|\epsilon bg}$, $\pi_{\epsilon'|\epsilon gb}\pi_{\epsilon'|\epsilon bb}$, during simulation. These matrices can be calculated based on the expressions derived above.

2 Economic Environment

Economy is populated by infinitely lived agents with CRRA utility. Each agent solves:

$$\max \sum_t \beta^t \frac{c_t^{1-\sigma} - 1}{1 - \sigma}$$

[4] This is the transition to unemployment in aggregate state z' from aggregate state z.

subject to

$$\begin{aligned} c_t + a_{t+1} &= w_t \tilde{l} \epsilon_t + (1 + r_t) a_t \\ a_{t+1} &\geq \underline{a}. \end{aligned}$$

Firms have Cobb-Douglas production function with aggregate shocks:

$$\begin{aligned} Y_t &= z_t K_t^{\theta} N_t^{1-\theta} \\ r_t + \delta &= z_t \theta (\frac{K_t}{N_t})^{\theta - 1} \\ w_t &= z_t (1 - \theta)(\frac{K_t}{N_t})^{\theta}. \end{aligned}$$

The resource constraint for this economy are as follows:

$$\begin{aligned} Y_t &= C_t + K_{t+1} - (1 - \delta) K_t \\ C_t &= \int c_t(a, \epsilon) d\mu_t(a, \epsilon) \\ K_{t+1} &= \int a_{t+1}(a, \epsilon) d\mu_t(a, \epsilon). \end{aligned}$$

2.1 Recursive Equilibrium

In order to write the recursive problem we first must think about what are the relevant state variables for the individuals problem. In Aiyagari (1994), we were solving for the steady state where the prices r, w were constant (since there was no aggregate uncertainty). In KS the prices are a function of the aggregate distribution of assets, hence they have to be included as a state variable. The individual state variable will be (a, ϵ), while the aggregate state variables includes (μ, z). Then the recursive version of the agents problem can be written as:

$$\begin{aligned} V(a, \epsilon; \mu, z) &= \max\{u(c) + \beta \mathbb{E}[V(a', \epsilon'; \mu', z')] \\ &\text{subject to} \\ c + a' &= w(\mu, z) \tilde{l} \epsilon + (1 + r(\mu, z)) a \\ \mu' &= \Psi(\mu, z, z') \end{aligned}$$

2.2 Approximate aggregation

Note that aggregate employment N_t is a function only of the aggregate state. The distribution matter only for aggregate capital(K_t). This means that in the current setup only K will influence the prices. The recursive problem then can be recast as:

$$\begin{aligned} V(a, \epsilon; z, K) &= \max\{u(c) + \beta \mathbb{E}[V(a', \epsilon'; z', K')] \\ &\text{subject to} \\ c + a' &= w(z, K) \tilde{l} \epsilon + (1 + r(z, K)) a \\ \log K' &= a_0 + a_1 \log K, \quad \text{if} \quad z = g \\ \log K' &= b_0 + b_1 \log K, \quad \text{if} \quad z = b. \end{aligned}$$

2.3 Computation Algorithm

The foremost thing is to get the policy functions, $c(a, \epsilon, K), a'(a, \epsilon, K)$ for a given initial guess of (a_0, a_1, b_0, b_1). The next step is to draw random shocks (ϵ, z) given the markov process for large N and T, and simulate the economy. Use the time series $\{K_t, z_t\}$ to update the guess for (a_0, a_1, b_0, b_1). Repeat till the convergence of the coefficients (a_0, a_1, b_0, b_1). The detailed computation algorithm is outlined below.

1. **Policy Function Computation:** For a given guess of (a_0, a_1, b_0, b_1), use the Euler:

$$U_c\Big(c(a, \epsilon; z, K)\Big) \geq \beta \mathbb{E}\Big[(1 + r(z', K'))U_c\Big(c(a'(a, \epsilon; z, K), \epsilon'; z', K')\Big)\Big]$$
$$(= \quad \text{if} \quad a'(a, \epsilon; z, K) > 0);$$
$$c + a' = w(z, K)\tilde{l}\epsilon + (1 + r(z, K))a$$
$$\log K' = a_0 + a_1 \log K, \quad \text{if} \quad z = g$$
$$\log K' = b_0 + b_1 \log K, \quad \text{if} \quad z = b.$$

to solve for the policy function using either EGM or policy function iteration. We describe the two computation algorithm in detail below.

EGM

- Guess the policy function $c^{(m)}(a, \epsilon; z, K)$ on the (a, ϵ, z) grid.
- Interpolate the policy function. We'll need to evaluate the policy function at K' which may not be on the initial K grid.
- We'll calculate the current consumption and asset level corresponding to given asset level tomorrow for each level of present aggregate capital (K).
- Compute $c^*(a', \epsilon; z, K) = U_c^{-1}\mathbb{E}_{(\epsilon',z')|z}\Big[\beta(1+r(z', K'(z, K)))U_c\Big(c(a', \epsilon'; z', K'(z, K))\Big)\Big]$.
- Compute $a^*(a', \epsilon; z, K) = (c^*(a', \epsilon; z, K)+a'-w(z, K)\tilde{l}\epsilon)/(1+r(z, K))$.
- Update the guess for the policy function as:

$$c^{(m+1)}[a, \epsilon; z, K] = \begin{cases} (1 + r(z, K))a + w(z, K)\epsilon - a_0 & ;\text{if} \quad a < a^*[a_0, \epsilon; z, K] \\ \text{Interpolate}(c^*[a_j, \epsilon; z, K], c^*[a_{j+1}, \epsilon; zK]) & ; a^*[a_j, ..] < a_i < a^*[a_{j+1}, ..] \end{cases}$$

- Repeat till convergence.
- *Note: We set $\epsilon \in \{0.25, 1\}$ such that consumption always remains non-zero.*

Policy Function Iteration

- Start with a guess for the policy function $a'(a, \epsilon; z, K)$.
- For each (z, K), get K' using the forecast equation.
- Based on the inital guess get $a''(a', \epsilon', z', K')$.
- Use the budget equation to get $c'(a', \epsilon'; z', K')$.
- Use the Euler to obtain $c(a, \epsilon; z, K)$, $a(a, \epsilon; z, K)$.
- Update the initial guess for the policy function and iterate till convergence.

- The Policy function iteration is outline in detail in Maliar, Maliar, and Valli (2010, JEDC).

Simulation: We'll use the policy function to simulate the economy for large N and T. The details of simulation is outlined below.

- Calculate the mean of initial asset holdings $K_0 = \frac{\sum_{i \in I} a_i}{I}$. KS starts with inital distribution of assets where all agents hold the same level of wealth, and hence is also the initial K_0.
- Use the transition matrix to simulate shocks for all the agents for T periods.
- Use the policy function to get the mean savings each period.
- Use the time series $\{K_t, z_t\}$ to update the coefficients (a_0, a_1, b_0, b_1).
- Repeat until convergence.

3 Results

The estimates of coefficients of aggregate law of motion of capital is:

$$\log K' = 0.12 + 0.968 \log K, \quad \text{if} \quad z = g$$
$$\log K' = 0.13 + 0.964 \log K, \quad \text{if} \quad z = b.$$

We plot the policy function for at the estimated coefficients below: