Econ 8185 (002): Quant PS2

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Abstract

This document calculates the Optimal Ramsey policies with and without state contingent debts.

Model

Technology:

$$y_t(s^t) = \theta(s_t)n_t(s^t).$$

Resource Constraint:

$$y_t(s^t) = g_t(s^t) + c_t(s^t).$$

Utility:

$$U(c,n) = u(c) - v(n)$$

Household Budget:

$$c_t(s^t) + \sum_{s^{t+1}|s^t} P_{t+1}(s^{t+1}|s^t)b_{t+1}(s^{t+1}|s^t) = (1-\tau_t(s^t))\theta_t(s^t)n_t(s^t) + b_t(s_t|s^{t-1}).$$

Government Budget:

$$g_t(s^t) = \tau_t(s^t)\theta_t(s^t)n_t(s^t) + \sum_{s_{t+1}} P_{t+1}(s_{t+1}|s^t)b_{t+1}(s_{t+1}|s^t) - b_t(s_t|s^{t-1})$$

Shock Process:

$$\begin{aligned}
\ln(g_{t+1}) &= (1-\rho_g)\mu_g + \rho_g \ln(g_t) + N(0,\sigma_g^2) \\
\ln(\theta_{t+1}) &= (1-\rho_\theta)\mu_\theta + \rho_\theta \ln(\theta_t) + N(0,\sigma_\theta^2)
\end{aligned}$$

1 Computation Algorithm

1.1 With State Contingent Debt

1. Guess Φ.
2. Compute $\{c(s), n(s)\}$ for all s for given Φ using the following equations:

$$\begin{aligned}
&(1+\Phi)[u_c(c)\theta + u_n(n)] + \Phi[cu_{cc}(c)\theta + nu_{nn}(n)] = 0 \\
&n = \frac{g+c}{\theta}
\end{aligned}$$

   Note that given the Markov structure of shocks, the above solves for policies for all $t \geq 1$. It is understood that c and are g each functions of the Markov state s.

3. Compute $\{c_0(s_0, b_0), n_0(s_0, b_0)\}$ using:

$$\begin{aligned}
&(1+\Phi)[u_c(c_0)\theta + u_n(n_0)] + \Phi[c_0u_{cc}(c_0)\theta + n_0u_{nn}(n_0)] - \Phi u_{cc}(c_0)b_0\theta = 0 \\
&b_0 = 4\theta_0 n_0 \\
&n_0 = \frac{g_0 + c_0}{\theta_0}.
\end{aligned}$$

4. Evaluate the implementation constraint as follows:

$$u_{c,0}b_0 = u_{c,0}c_0 + u_{n,0}n_0 + \beta \sum_s \Pi(s|s_0)sum(s) \tag{1}$$

where $sum(s)$ solves the following $S \times 1$ system of equations for $s \in \{1, 2, \dots, S\}$:

$$sum(s) = u_c(s)c(s) + u_n(s)n(s) + \beta \sum_{s'} \Pi(s'|s)sum(s').$$

In matrix form the solution is given by:

$$\vec{s}um(s) = (I - \beta\Pi_s)^{-1}[\vec{u}_c(s)\vec{c}(s) + \vec{u}_n(s)\vec{n}(s)]$$

where vector products are understood as element wise product.

5. If $LHS > RHS$ in 1, increase Φ, if $LHS < RHS$ in 1, reduce Φ.

6. Repeat the procedure to find Φ such that the implementation constraint binds.

7. *Note:* Instead of updating Φ as in step $4, 5$, we can also solve for the root of 1 to arrive at Φ such that 1 binds.

1.2 Without State Contingent Debt

In this case the problem can be reduced to solving the following two part bellman equation:
for $t \geq 1$

$$V(x_-, s_-) = \max_{c(s), n(s), x(s)} \sum_s \Pi(s|s_-)\{U(c(s), n(s)) + \beta V(x(s), s)\}$$

subject to

$$\begin{aligned}
\frac{x_- U_c(s)}{\beta \sum \Pi(\tilde{s}|s_-)U_c(\tilde{s})} &= U_c(s)c(s) + U_n(s)n(s) + x(s) \\
\theta(s)n(s) &= c(s) + g(s); \quad \forall s \in S
\end{aligned}$$

for $t = 0$

$$W(b_0, s_0) = \max_{c(s_0), n(s_0), x(s_0)} U(c(s_0), n(s_0)) + \beta V(x(s_0), s_0)$$

subject to

$$\begin{aligned}
U_c(s_0)b_0 &= U_c(s_0)c(s_0) + U_n(s_0)n(s_0) + x(s_0) \\
\theta(s_0)n(s_0) &= c(s_0) + g(s_0).
\end{aligned}$$

2 Portfolio of risk free bonds

Following Angeletos(2002) the complete markets Ramsey allocation can be implemented using bonds of maturity $1, 2, \dots, N$. In this case the households budget constraint is given by:

$$c_t(s^t) + \sum_{n=1}^{N} q_t^{(n)}(s^t)b_t^{(n)}(s^t) = (1 - \tau_t(s^t))\theta_t(s^t)n_t(s^t) + \sum_{n=1}^{N} q_t^{(n-1)}(s^t)b_{t-1}^{(n)}(s^{t-1}); \; q_t^0 = 1.$$

The price of bond is given by:

$$q_t^{(n)}(s^t) = \beta \mathbb{E}_t[\frac{u_c(s^{t+1})}{u_c(s^t)} q_{t+1}^{(n-1)}(s^{t+1})]$$
$$q_t^{(n)}(s^t) = \beta^n \mathbb{E}_t[\frac{u_c(s^{t+n})}{u_c(s^t)}].$$

We can use the bond pricing equation to recursively substitute the demand for bonds. In addition we also substitute out taxes to express everything in terms of allocation. We thus get:

$$\begin{aligned}\sum_{n=1}^{N} q_t^{(n-1)}(s^t) b_{t-1}^{(n)}(s^{t-1}) u_c(s^t) &= \underbrace{u_c(s^t)c_t(s^t) - u_\ell(s^t)n_t(s^t)}_{x_t(s^t)} + \sum_{n=1}^{N} q_t^{(n)}(s^t) b_t^{(n)}(s^t) \\ &= x_t(s^t) + \sum_{n=1}^{N} q_t^{(n)}(s^t) b_t^{(n)}(s^t) u_c(s^t).\end{aligned}$$

Now we substitute the bond price

$$q_t^{(n)}(s^t) = \beta \mathbb{E}_t[\frac{u_c(s^{t+1})}{u_c(s^t)} q_{t+1}^{(n-1)}(s^{t+1})]$$

in the previous equation to get:

$$\begin{aligned}\sum_{n=1}^{N} q_t^{(n-1)}(s^t) b_{t-1}^{(n)}(s^{t-1}) u_c(s^t) &= x_t(s^t) + \beta \sum_{n=1}^{N} \mathbb{E}_t[q_{t+1}^{(n-1)}(s^{t+1}) b_t^{(n)}(s^t) u_c(s^{t+1})] \\ &= x_t(s^t) + \beta \mathbb{E}_t[x_{t+1}(s^{t+1}) + \beta \mathbb{E}_{t+1} \sum_{n=1}^{N} [q_{t+2}^{(n-1)}(s^{t+2}) b_{t+1}^{(n)}(s^{t+1}) u_c(s^{t+2})]] \\ &= \sum_{j=0}^{\infty} \sum_{s^{t+j}|s^t} \beta^j \pi_t(s^{t+j}|s^t) x_{t+j}(s^{t+j}) \\ &= \sum_{j=0}^{\infty} \sum_{s^{t+j}|s^t} \beta^j \pi_t(s^{t+j}|s^t) [u_c(s^{t+j}) c_t(s^{t+j}) - u_\ell(s^{t+j}) n_t(s^{t+j})] \\ &= sum(s^t) \\ &= u_c(s^t)c_t(s^t) - u_\ell(s^t)n_t(s^t) + \beta \sum_{s^{t+1}|s^t} \pi_t(s^{t+j}|s^t) sum(s^{t+1}).\end{aligned}$$

For Markov shock structure these simplifies to:

$$q^{(j)}(s) = \beta \sum_{s'|s} \pi(s'|s) q^{(j-1)}(s') \frac{u_c(s')}{u_c(s)}$$

$$\begin{aligned}\sum_{n=1}^{N} q^{(n-1)}(s) b_{-1}^{(n)}(s^{-1}) u_c(s) &= u_c(s)c(s) + u_n(s)n(s) + \beta \sum_{s'} \Pi(s'|s) sum(s') \\ &= \frac{sum(s)}{u_c(s)}\end{aligned}$$

For $N = S$, we have N equations in N variables which can be solved to get the portfolio of bonds of different maturity. Note that the system of equations in matrix form can be written as:

$$\mathbf{Qb} = \mathbf{Z}.$$

Result: We find that optimal portfolio of bonds involves sale and purchase of bonds in large quantities (in orders of magnitude of the gdp). This mirrors the findings in Buera-Nicolini-Pablo(2004). The interpretation is that since interest rates (bond prices) are closely correlated government has to take large asset positions in order to diversify risk.

3 Simulations

Simulating Ramsey Policies with State contingent debt

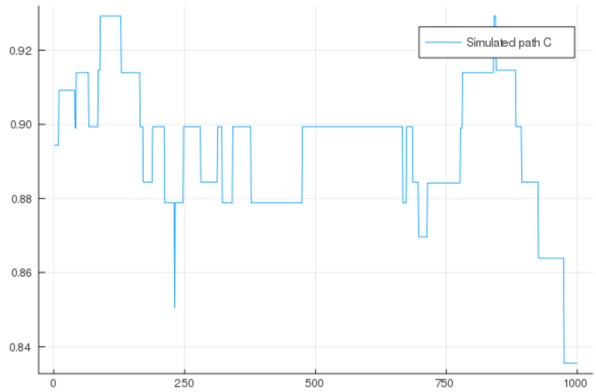

Figure 1: Consumption Policy Function

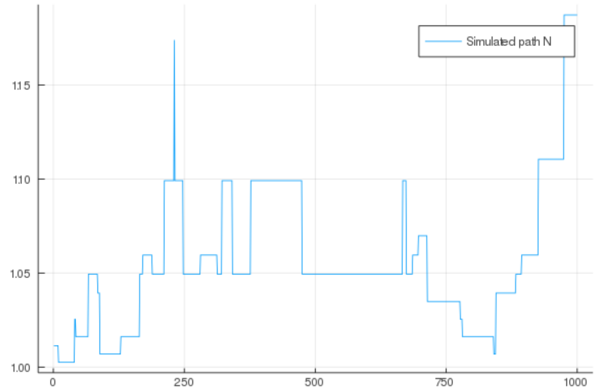

Figure 2: Labor Policy function

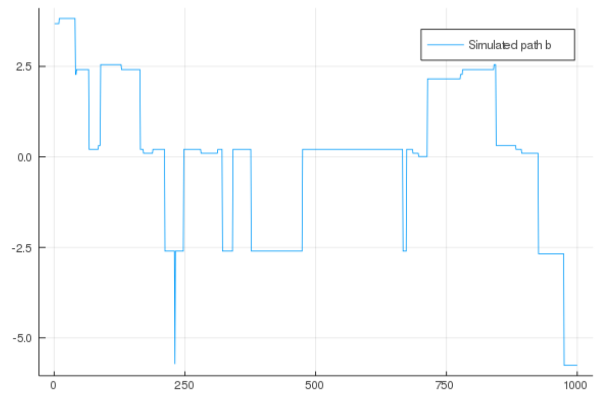

Figure 3: Bonds

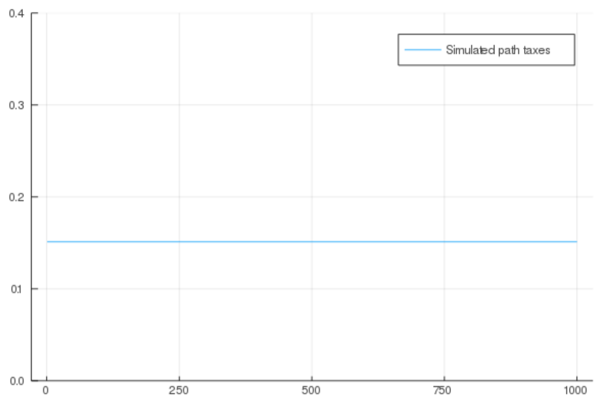

Figure 4: Taxes on labor

Simple Example

In order to verify our code we ran the model with the parameters in QuantEcon. For this case we work with the following:

$$u(c,n) = \frac{c^{1-\sigma}}{1-\sigma} - \frac{n^{1+\gamma}}{1+\gamma},$$

and set $\gamma = \sigma = 2$, and $\beta = 0.9$. We think of these 6 states as corresponding to $s = 1, 2, 3, 4, 5, 6$. The transition matrix is:

$$\Pi = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Government expenditures at each state are:

$$g = \begin{pmatrix} 0.1 \\ 0.1 \\ 0.1 \\ 0.1 \\ 0.2 \\ 0.1 \end{pmatrix}$$

The results match exactly, confirming that our code for Ramsey is correct. Below are the figures from replication :

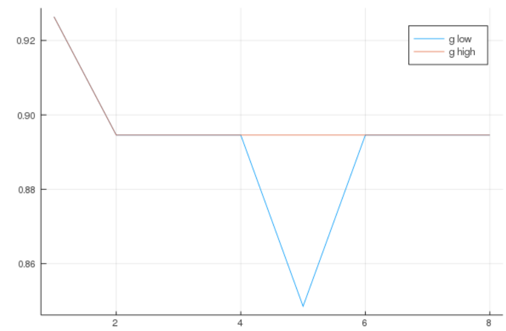

Figure 5: Consumption Policy Function

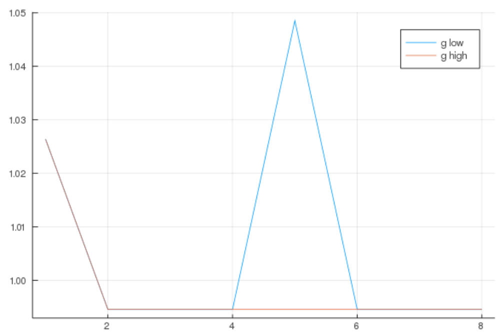

Figure 6: Labor Policy function

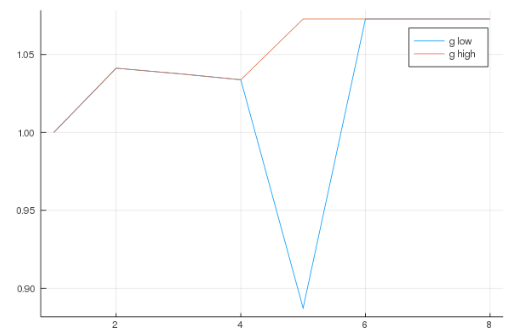

Figure 7: Bonds

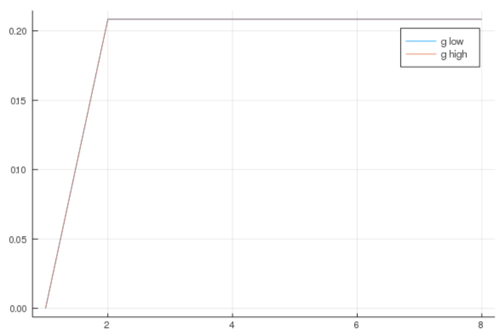

Figure 8: Taxes on labor

References

Angeletos, G.-M. (2002). Fiscal policy with noncontingent debt and the optimal maturity structure. *The Quarterly Journal of Economics*, 117(3):1105–1131.