29/4/21

IC 252

MOHIT VERMA

B20215

Assignment 7

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Ans1 $\lambda = 0.1$

We know that for an exponential distribution-

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0; \\ 0, & x < 0, \end{cases}$$

where $\lambda = 0.1$ for our case.

(a) We know that expectation $E(X) = \int_0^\infty x \cdot f_X(x)\,dx$

$$E(X) = \int_0^\infty x \cdot (0.1)\left(e^{-0.1 \times x}\right) dx$$

$$= \left[\frac{(0.1)\, x \cdot e^{-0.1 \times x}}{(-0.1)}\right]_0^\infty - \int_0^\infty \frac{(0.1)}{(-0.1)} e^{-0.1x}\, dx$$

$$= \left[-x \cdot e^{-0.1x}\right]_0^\infty + \int_0^\infty e^{-0.1x}\, dx$$

$$= 0 + \left[\frac{e^{-0.1x}}{-0.1}\right]_0^\infty$$

$= 0 - \left(-\frac{1}{0.1}\right) = 10$ minutes

(b) $P(x > 10) = 1 - P(X \le 10)$

$$= 1 - \int_0^{10} \lambda e^{-\lambda x}\, dx$$

$$= 1 + \left[e^{-\lambda x} \right]_0^{10}$$

$$= 1 + \left(e^{-10(0.1)} - e^{-0\times(0.1)}\right)$$

$$= 1 + \left(e^{-1} - 1\right)$$

$$= \frac{1}{e} = 0.37$$

(c) $P(X \le 5) = F(5)$

where $F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \ge 0 \\ 0, & x < 0 \end{cases}$

So, $P(X \le 5) = 1 - e^{-\frac{5}{10}}$

$$= 1 - \frac{1}{e^{1/2}} = 0.39$$

Ans 2 let $X \sim \text{Unif}(1.43, 1.60)$

So, $f_X(x) = \begin{cases} \frac{1}{1.60-1.43}, & 1.43 < x < 1.60; \\ 0, & \text{elsewhere:} \end{cases}$

(a) $$E(X) = \int_{1.43}^{1.6} x \cdot \frac{1}{0.17}\, dx$$

$$= \left[\frac{x^2}{2(0.17)} \right]_{1.43}^{1.6}$$

$$= \left[(1.6)^2 - (1.43)^2 \right] \times \frac{1}{0.34}$$

$$= \underline{\underline{1.515 \text{ V.}}}$$

(b) To find standard deviation,

we need var(x),

So, $\text{var}(X) = E(X^2) - (E(X))^2$

So, $$E(X^2) = \int_{1.43}^{1.60} \frac{x^2}{1.60-1.43}\, dx$$

$$E(X^2) = \left[\frac{x^3}{(0.17)\times 3}\right]_{1.43}^{1.60}$$

$$= \frac{1.1717}{0.17\times 3} = 2.2976$$

So,

$$\text{Var}(X) = 2.2976 - (E(X))^2$$
$$= 2.2976 - (1.515)^2$$
$$= 0.002408$$

$$\text{Standard deviation}(X) = \sqrt{\text{Var}(X)}$$
$$= \underline{\underline{0.0490}}$$

(c) CDF of valtage is

$$F_X(x) = \int f_X(u)\, du,$$
$$= \int \frac{1}{0.17}\, du$$
$$= \frac{x}{0.17} + c$$

at $x = 1.43$ V $\quad F_X(x) = 0$

So,

$$\underline{\underline{F_X(x)}} = \begin{cases} \dfrac{x-1.43}{0.17}, & 1.43 < x < 1.60 \\ 1, & \text{elsewhere} \end{cases}$$

(d) $P(X \leq 1.48) = F(1.48$

$$P(X \leq 1.48) = \frac{1.48-1.43}{0.17}$$

$$= \underline{\underline{0.294}}$$

(e) Box contains 50 Batteries,

& $P(X<1.5) = F(1.5)$

$$= \frac{1.5-1.43}{0.17}$$

$$= 0.4118$$

So, with 50 batteries we have binomial distribution with $n=50$ & $p=0.4118$,

Also, for $E(x) = np$ for binomial distribution, hence,

$$E(x) = 50 \times 0.4118$$

$$= \underline{\underline{20.59}}$$

$$Var(x) = np(1-p)$$

$$= 20.59(1-0.4118)$$

$$= \underline{\underline{12.11}}$$

Ans 3 $Z \sim N(0,1)$ so,

$$\Phi_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

& $$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

(a) $P(Z \leq -0.77) = \Phi_X(-0.77)$
$= 0.2206$

(b) $P(Z \geq 0.32) = 1 - P(Z \leq 0.32)$
$= 1 - \Phi(0.32)$
$= 1 - 0.6255$
$= 0.3745$

(c) $P(-0.82 \leq Z \leq 1.80) = \Phi(1.80) - \Phi(-0.82)$
$= 0.9641 - 0.2061$
$= 0.758$

(e) $P(|Z| \geq 0.91) = P(Z \geq 0.91) + P(Z \leq -0.91)$
$= 60.1814 + 0.1814$
$= 0.36282$

(f) Value of x for which $P(Z \leq x) = 0.23$

from the tables $x \approx -0.74$

(g) Value of x for which $P(Z \geq x) = 0.51$
from the tables we have $x \approx -0.03$

(h) Value of x for which $P(|Z| \geq x) = 0.42$

So, $P(|Z| \geq x) = 1 - P(|Z| \leq x)$

$= 1 - [\Phi(x) - (\Phi(-x))]$

$= 2 - 2\Phi(x)$

$0.42 = 2(1 - \Phi(x))$

$0.21 = 1 - \Phi(x)$

$\Phi(x) = 0.79$

from Tables $x \approx 0.80$

Ans 4

Since there is only one Air Conditioner

So, $P(Y=1) = 0.12 + 0.08 + 0.07 + 0.05 = 0.32$

$P(X=1 \mid Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{0.12}{0.32} = 0.375$

$$P(X=2|Y=1) = \frac{P(X=2, Y=1)}{P(Y=1)} = \frac{0.08}{0.32} = \underline{\underline{0.25}}$$

$$P(X=3|Y=1) = \frac{P(X=3, Y=1)}{P(Y=1)} = \frac{0.07}{0.32} = \underline{\underline{0.2187}}$$

$$P(X=4|Y=1) = \frac{P(X=4, Y=1)}{P(Y=1)} = \frac{0.05}{0.32} = \underline{\underline{0.1562}}$$

(b) we have service time of two hours i.e $X=2$.

$$P(X=2) = 0.08 + 0.15 + 0.01 = 0.24$$

So,

$$P(Y=1|X=2) = \frac{P(Y=1, X=2)}{P(X=2)} = \frac{0.08}{0.24} = \underline{\underline{\frac{1}{3}}}$$

$$P(Y=2|X=2) = \frac{P(Y=2\ X=2)}{P(X=2)} = \frac{0.15}{0.24} = \underline{\underline{\frac{5}{8}}}$$

$$P(Y=3|X=2) = \frac{P(Y=3, X=2)}{P(X=2)} = \frac{0.01}{0.24} = \underline{\underline{\frac{1}{24}}}$$

(c) $$\text{corr}(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}}$$

$$\text{var}(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = 1^2(P(X=1)) + 2^2(P(X=2)) + 3^2(P(X=3)) + 4^2(P(X=4))$$

$$= 0.21 + 4(0.24) + 9(0.3) + 16(0.25)$$

$$= 7.87$$

$$E(X) = 1.(0.21) + 2(0.24) + 3(0.3) + 4(0.25)$$

$$= 2.59$$

$$E(Y^2) = 1^2.(0.32) + 2^2(0.57) + 3^2(0.11)$$

$$= 3.59$$

$$E(Y) = 1.(0.32) + 2(0.57) + 3(0.11)$$

$$= 1.79$$

$$E(XY) = \sum_{x=1}^{4} \sum_{y=1}^{3} xy.P(X=x, Y=y)$$

$$= 1.1.(0.12) + 1.2.(0.08) + 1.3.(0.01) + 2.1.(0.08) + 2.2.(0.15) + 2.3.(0.01) + 3.1.(0.07) + 3.2.(0.21) + 3.3(0.13) + 4.1(0.05) + 4.2.(0.13) + 4.3(0.07)$$

$$= 4.86$$

$$\operatorname{cov}(X,Y) = E(XY) - E(X).E(Y)$$
$$= 4.86 - 2.59 \times 1.79$$
$$= 0.224$$

$$\operatorname{var}(X) = E(X^2) - (E(X))^2$$
$$= 7.87 - (2.59)^2$$
$$= \underline{\underline{1.161}}$$

$$\operatorname{var}(Y) = E(Y^2) - (E(Y))^2$$
$$= 3.59 - (3.20)$$
$$= 0.39$$

$$\operatorname{corr}(X,Y) = \frac{0.224}{\sqrt{1.161 \times 0.39}} = \underline{\underline{0.332}}$$

<u>Ans5</u> (a) $P(0.8 \leq X \leq 1,\ 25 \leq Y \leq 30) = 0.092$ (i)

$$f_{X,Y}(x,y) = \frac{39}{400} - \frac{17(x-1)^2}{50} - \frac{(y-25)^2}{10000}$$

R.H.S of eqn (i)

$$= \int_{0.8}^{1} \int_{y=25}^{30} f_{X,Y}(x,y)\,dy\,dx$$

$$= \int_{x=0.8}^{1} \int_{y=25}^{30} \frac{39}{400} - \frac{17(x-1)^2}{50} - \frac{(y-25)^2}{10000} \, dy \, dx$$

$$= \int_{x=0.8}^{1} \left[\frac{39}{400} y\right]_{25}^{30} - \left[\frac{17y}{50}(x-1)^2\right]_{25}^{30} - \left[\frac{(y-25)^3}{3 \times 10000}\right]_{25}^{30} dx$$

$$= \int_{x=0.8}^{1} \left(\frac{39 \times 5}{400} - \frac{17}{50}(x-1)^2(5) - \frac{5^3}{3 \times 10000}\right) dx$$

$$= \left[\frac{39 \times 5}{400} x\right]_{x=0.8}^{1} - \left[\frac{17}{10} \frac{(x-1)^3}{3}\right]_{0.8}^{1} - \left[\frac{5^3 \times x}{3 \times 10000}\right]_{0.8}^{1}$$

$$= \frac{39 \times 5 \times 0.2}{400} - \left[\frac{17}{10} \times \frac{(0.2)^3}{3}\right] - \left[\frac{5^3 \times 0.2}{3 \times 10000}\right]$$

$$= \frac{39}{400} - \frac{17}{30 \times 5^3} - \frac{\cancel{25}\ 1}{3 \times \cancel{10000}\ 400}$$

$$= \underline{\underline{0.092}}$$

Hence, Proved.

(b) To find the value of $E(Y)$, we need to know, $f_Y(Y)$.

So,

$$f_Y(y) = \int_{x=-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$= \int_{x=0.5}^{1.5} \left(\frac{39}{400} - \frac{17(x-1)^2}{50} - \frac{(y-25)^2}{10000}\right) dx$$

$$= \left[\frac{39}{400}x - \frac{17(x-1)^3}{3\times 50} - \frac{(y-25)^2}{10000}x\right]_{x=0.5}^{1.5}$$

$$= \frac{39}{400} - \frac{17}{3\times 50}\left((0.5)^3 + (0.5)^3\right)$$

$$- \frac{(y-25)^2}{10000}(1)$$

$$= \frac{39}{400} - \frac{17}{150}\left(\frac{1}{4}\right) - \frac{(y-25)^2}{10000}$$

$$= \frac{83}{1200} - \frac{(y-25)^2}{10000}$$

Now, $$E(Y) = \int_{20}^{35} y\left(\frac{83}{1200} - \frac{(y-25)^2}{10000}\right) dy$$

$$= \left[\frac{83y^2}{2400}\right]_{20}^{35} - \left[\frac{(y-25)^3 y}{3\times 10000}\right]_{20}^{35}$$

$$+ \int_{20}^{35} \frac{(y-25)^3}{3\times 10000}\,dy$$

$$= \left[\frac{83}{2400}\left((35)^2 - (20)^2\right)\right] - \left[\frac{(10)^3(35) - (-5)^3(20)}{3\times 10000}\right]$$

$$+ \left[\frac{(y-25)^4}{12\times 10000}\right]_{20}^{35}$$

$$= \frac{913}{32} - 1.0833 + \left[\frac{(10)^4 - (5)^4}{12\times 10000}\right]$$

$$= 28.53 - 1.25 + 0.078125$$

$$= \underline{\underline{27.36}}$$

$$\& \quad E(Y^2) = \int_{20}^{35} y^2 \left(\frac{83}{1200} - \frac{(y-25)^2}{10000}\right) dy$$

$$E(Y^2) = \left[\frac{83y^3}{3\times1200}\right]_{20}^{35} - \int_{20}^{35} \frac{y^2(y^2+625-50y)}{10000}\, dy$$

$$= 804.0625 - \left[\frac{y^5}{5} + \frac{625y^3}{3} - \frac{50y^4}{4}\right]_{20}^{35} \times \frac{1}{10000}$$

$$= 804.0625 - \left[986.4375 + 726.5625 - \frac{50(1340625)}{4\times10000}\right]$$

$$= 804.0625 - [1713 - 1675.78]$$

$$= 804.0625 - 37.21875$$

$$= \underline{\underline{766.84}}$$

standard deviation $= \sqrt{E(Y^2) - (E(Y))^2}$

$$= \sqrt{766.84 - 748.56}$$

$$= \underline{\underline{4.27}}$$

<u>Ans 6</u> $f_{X,Y}(x,y) = c(e^{x+y}\, e^{2x-y})$

$1 \le x \le 2$ & $0 \le y \le 3$ & $f_{X,Y}(x,y) = 0$

elsewhere.

(a) To find c, we know that

$$\int_{x=1}^{2}\int_{y=0}^{3} f_{X,Y}(x,y)\, dx\, dy = 1$$

$$\int_{x=1}^{2}\int_{y=0}^{3} c\left(e^{x}\cdot e^{y} + \frac{e^{2x}}{e^{y}}\right) dy\, dx = 1$$

$$\int_{x=1}^{2} c\left(e^{x}\left[e^{y}\right]_0^3 + (-1)e^{2x}\left[e^{-y}\right]_0^3\right) dx = 1$$

$$c\int_{x=1}^{2}\left(e^{x}(e^{3}-1) + (-1)\times e^{2x}(e^{-3}-1)\right) dx = 1$$

$$\left[e^{x}\right]_1^2 (e^{3}-1) + \left[\frac{e^{2x}}{2}\right]_1^2 (1-e^{-3}) = \frac{1}{c}$$

$$(e^{2}-e)(e^{3}-1) + \left[\frac{e^{4}}{2} - \frac{e^{2}}{2}\right]\left(1-\frac{1}{e^{3}}\right) = \frac{1}{c}$$

$$89.144 + 23.6\left(1-\frac{1}{e^{3}}\right) = \frac{1}{c}$$

$$c = \frac{1}{111.569} = 0.00896$$

(b) $$(0.00896)\int_{x=1.5}^{2}\int_{y=1}^{2}\left((e^{x}\cdot e^{y})+(e^{2x}\cdot e^{-y})\right)dx\,dy$$

$$= P(1.5 \le X \le 2,\ 1 \le Y \le 2)$$

$$L.H.S = (0.00896)\int_{x=1.5}^{2}\left([e^{2}-e^{1}]e^{x}+e^{2x}(-1)[e^{-y}]_{1}^{2}\right)dx$$

$$= (0.00896)\left[(e^{2}-e^{1})(e^{2}-e^{1.5})+\frac{(e^{4}-e^{3})}{2}(e^{-1}-e^{-2})\right]$$

$$= (0.00896)\left[(4.67)(2.90)+\frac{(34.51)(0.23)}{2}\right]$$

$$= \underline{\underline{0.157}}$$

(c) $$f_X(x) = \int_{0}^{3}(0.00896)(e^{x}\cdot e^{y}+e^{2x}\cdot e^{-y})\,dy$$

$$f_X(x) = (0.00896)\left(e^{x}(e^{3}-1)+e^{2x}(1-e^{-3})\right)$$

$$f_Y(y) = \int_{x=1}^{2}(0.00896)(e^{x}\cdot e^{y}+e^{2x}\cdot e^{-y})\,dx$$

$$f_Y(y) = (0.00896)\left[(e^{2}-e^{1})e^{y}+\frac{(e^{4}-e^{2})e^{-y}}{2}\right]$$

(d) $f_{X,Y}(x,y) = 0.00596\,(e^{x+y} + e^{2x-y})$

$f_X(x) = (0.00596)\,[e^{x}(e^{3}-1) + e^{2x}(1-e^{-3})]$

$f_Y(y) = (0.00596)\,\left[e^{y}(e^{2}-e^{1}) + \frac{e^{-y}(e^{4}-e^{2})}{2}\right]$

Hence,

$f_{X,Y}(x,y) \neq f_X(x) \cdot f_Y(y)$

Hence, No, the r.v. X & Y aren't independent.

(e) $f_{X|Y=0}(x) = \frac{f_{X,Y}(x,0)}{f_Y(0)} = \frac{(e^{x}+e^{2y})\cancel{(0.00596)}}{(28.28)\cancel{(0.00596)}}$

$$= \frac{e^{x}+e^{2y}}{28.28}$$