10/4/21

IC-252

B20215
MOHIT VERMA

Assignment-5

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Ans 1 Total no. of possible outcomes = 16

X is no. of head in first three tosses &
Y is no. of head in last three tosses.
from previous assignment,

$P(X=0)=\frac{1}{8}$ $P(X=1)=\frac{3}{8}$ $P(X=2)=\frac{3}{8}$ $P(X=3)=\frac{1}{8}$

Also,

$P(Y=0)=\frac{1}{8}$ $P(Y=1)=\frac{3}{8}$ $P(Y=2)=\frac{3}{8}$ $P(X=3)=\frac{1}{8}$

Now, Expectation of X :

$$E(X)=\sum_{x=0}^{3} x P(X=x)$$

$$= 0.\frac{1}{8}+1.\frac{3}{8}+2.\frac{3}{8}+3.\frac{1}{8}$$

$$= \frac{12}{8} = \frac{3}{2}$$

&

$$E(Y) = \sum_{x=0}^{3} y \cdot P(Y=y)$$

$$= 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8}$$

$$= \frac{12}{8} = \frac{3}{2} = 1.5$$

& Variance is given as,

$$Var(X) = E(X^2) - ((E(X))^2)$$

for Var(X) we need $E(X^2)$,

So, $$E(X^2) = \sum_{x=0}^{3} x^2 \cdot P(X=x)$$

$$= 0^2 \cdot \frac{1}{8} + 1^2 \cdot \frac{3}{8} + 2^2 \cdot \frac{3}{8} + 3^2 \cdot \frac{1}{8}$$

$$= \frac{3}{8} + \frac{12}{8} + \frac{9}{8}$$

$$= \frac{24}{8} = 3$$

So,

$$Var(X) = 3 - \left(\frac{3}{2}\right)^2$$

$$= 3 - \frac{9}{4} = \frac{3}{4}$$

Also, since X is identically distributed as Y. So, $E(Y^2) = 3$

So, $Var(Y) = E(Y^2) - ((E(Y))^2)$

$= 3 - \left(\frac{3}{2}\right)^2 = 3 - \frac{9}{4}$

$= \frac{3}{4}$

Ans 2

From previous assignment,

X: No. of hearts drawn

Y: No. of Clubs drawn

Also,

$P(X=0) = \frac{57}{102}$, $P(X=1) = \frac{39}{102}$, $P(X=2) = \frac{6}{102}$

& $P(Y=0) = \frac{57}{102}$, $P(Y=1) = \frac{39}{102}$, $P(Y=2) = \frac{6}{102}$

Now, $E(X) = \sum_{x=0}^{2} x(P(X=x))$.

$= 0 \cdot \frac{57}{102} + 1 \cdot \frac{39}{102} + 2 \cdot \frac{6}{102}$

$= \frac{51}{102} = \frac{1}{2}$

$$E(X^2) = \sum_{x=0}^{2} x^2 \cdot P(X=x)$$

$$= 0 \cdot \frac{57}{102} + 1^2 \cdot \frac{39}{102} + 2^2 \cdot \frac{6}{102}$$

$$= \frac{63}{102}$$

$$E(Y) = \sum_{y=0}^{2} y \cdot (P(Y=y))$$

$$= 0 \cdot \frac{57}{102} + 1^2 \cdot \frac{39}{102} + 2^2 \cdot \frac{6}{102}$$

$$= \frac{51}{102} = \frac{1}{2}$$

Also, $E(Y^2) = \frac{63}{102}$

$$Var(X) = E(X^2) - (E(X))^2$$

$$= \frac{63}{102} - \left(\frac{51}{102}\right)^2 = \frac{63}{102} - \frac{1}{4} = \frac{25}{68}$$

Also,

$$Var(Y) = E(Y^2) - (E(Y))^2$$

$$= \frac{63}{102} - \frac{1}{4} = \frac{25}{68}$$

Ans 3

$X \sim Bin(n, p)$

we need to show $Var(X) = np(1-p)$

Now, PMF for X,

$$P(X=x) = ({}^{n}C_{x}) \cdot p^{x}(1-p)^{n-x}$$

To find $Var(X)$, we need $E(X)$ & $E(X^2)$

$$E(X) = \sum_{x=0}^{n} x \cdot P(X=x)$$

$$= \sum_{x=0}^{n} x \cdot \frac{n!}{x!(n-x)!} \times p^{x}(1-p)^{n-x}$$

$$= n \sum_{x=0}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} \times p \cdot p^{x-1} \cdot (1-p)^{n-x}$$

$$= np \sum_{x=0}^{n} {}^{n-1}C_{x-1} \cdot p^{x-1} (1-p)^{n-x}$$

$$= np\,(p+(1-p))^{n-1}$$

$$= np$$

Now,

$$E(X(X-1)) = \sum_{x=0}^{n} x.(x-1)\, {}^{n}C_{x}\, p^{x}\, (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1).\, \frac{n!}{x!\,(n-x)!}\, p^{x}.\, (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \frac{n!}{(x-2)!\,(n-x)!}\, p^{x}.\, (1-p)^{n-x}$$

$$= (n)(n-1) \sum_{x=0}^{n} \frac{(n-2)!}{(x-2)!\,(n-x)}\, p^{2}.\, p^{x-2}\, (1-p)^{n-x}$$

$$= (n)(n-1)(p^{2})\left(\sum_{x=0}^{n} {}^{n-2}C_{x-2}\, (p+(1-p))^{n-2}\right)$$

$$E(X(X-1)) = (n)(n-1)(p^{2})\,(1)$$

Now,

$$Var(X) = E(X^{2}) - (E(X))^{2}$$

$$= E(X(X-1)) + E(X) - (E(X))^{2}$$

from linearity,

$$Var(X) = (n)(n-1)p^2 + np - n^2p^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$Var(X) = np(1-p)$$

Hence, Proved.

<u>Ans 4</u> let X be a random varivable such that it denotes the no. of power failures in a given month at IIT Mandi.

$P(X=0) = 0.4$ $\quad$ $P(X=2) = 0.2$

$P(X=1) = 0.3$ $\quad$ $P(X=3) = 0.1$

So, $$E(X) = \sum_{x=0}^{3} x.P(X=x)$$

$$= 0.(0.4) + 1.(0.3) + 2\times(0.2) + 3.(0.1)$$

$$= \underline{\underline{1}}$$

& $$E(X^2) = 0^2(0.4) + 1^2(0.3) + 2^2\times(0.2) + 3^2(0.1)$$

$E(X^2) = 0.3 + 0.8 + 0.9 = \underline{\underline{2}}$

So, $Var(X) = E(X^2) - (E(X))^2$

$= 2 - 1 = \underline{\underline{1}}$

Ans 5 $\quad Var(X) = Var(Y) = 3$

To find, $Var(2X - 3Y + 1)$

we know, X & Y are independent,

we know that,

$$Var(cX) = c^2 Var(X)$$

$$Var(c) = 0$$

$$\& \quad Var(X+Y) = Var(X) + Var(Y)$$

So, let $(2X - 3Y) = Z$

$Var(2X - 3Y + 1) = Var(Z + 1) = Var(Z)$

$= Var(2X - 3Y) = 2^2 . Var(X) + (-3)^2 Var(Y)$

$= 4.3 + 9.3 = \underline{\underline{39}}$

Ans 6

let $X \sim Pois(\lambda)$.

Such that,

$$P(X=k) = \frac{e^{-\lambda}\lambda^k}{k!} \quad k = 0, 1, 2, \ldots$$

Now

(i) Mean or $E(X)$,

So,

$$E(X) = \sum_{k=0}^{\infty} k \cdot P(X=k)$$

$$= \sum_{k=0}^{\infty} k \cdot \frac{e^{-\lambda}\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{k \cdot \lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda \cdot \lambda^{k-1}}{(k-1)!}$$

$$= \lambda e^{-\lambda} \left(\frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots \right)$$

$$= \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

& $\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{\lambda}$ (Maclaurin Series Expansion)

$= \lambda e^{-\lambda} . e^{\lambda}$

$= \lambda$

(ii) $Var(X)$,

$$Var(X) = E(X^2) - (E(X))^2$$

$$= E(X(X-1+1)) - (E(X))^2$$

$$= E(X(X-1)) + E(X) - (E(X))^2$$

Now,

$$E(X(X-1)) = \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda} . \lambda^x}{x!}$$

$$= \lambda^2 e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!}$$

$$\left(e^{\lambda} = \frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \cdots\right)$$

$$= \lambda^2 . e^{-\lambda} . e^{\lambda}$$

$$= \lambda^2$$

So, $Var(X) = \lambda^2 + \lambda - \lambda^2$

$= \lambda$