Decision Making using Fuzzy Set

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Presentation Outline

- Decision making
- Fuzzy Set and Crisp Set
- Characteristics of Fuzzy Set
- Basic Operations on Fuzzy Set
- Fuzzy Relation

Decision Making

- A decision is described as a conclusion arrived at after careful consideration
- By a decision we transfer internal action to external
- Decision making is the study of identifying and choosing alternatives based on the values and preferences of the decision maker(s)
- Decision making can be performed by a single decision maker or a group of decision makers/experts
- In group decision making (GDM), each decision maker might have their own thought but goal is common

Uncertainty in Decision Making

- Impreciseness of available data results in loss of information, leads to uncertainty
- Simplification of complex problems also results in loss of information
- Uncertainties may appear in the problem parameters or in decision situations
- In order to deal with the real life uncertain problems, a number of tools/techniques have been developed such as fuzzy set, rough set, vague set, soft set, etc.

History of Development Fuzzy Sets/Logic

- ***1937, Max Black***
  "Vagueness : an exercise in logical analysis", Philosophy and Science, 4,427-455.
- ***1962, Lotfi A. Zadeh***
  "From circuit theory to system theory", IRE Proc. 50, pp.856-865.
- ***1965, L. A. Zadeh***
  "Fuzzy Sets", Information and Control, 8, pp. 338-353.
  "Fuzzy Sets and Systems", in System Theory (ed. J. Fox.), Polytechnic Press, pp.29-37.
- ***1974, Mamdani and Assilian*** : Used fuzzy logic to regulate a steam engine.
- ***In 1985*** researchers at Bell laboratories developed the first fuzzy logic chip.
- ***2011,*** 46th anniversary celebration

Information & Complexity

simplification of complex problems (decision, management, prediction etc.)

↓

Loss of information

↓

allow some degree of **uncertainty** in its description

The information loss for reducing the complexity of the system to a manageable level is expressed in **uncertainty**

Fuzzy

Why ? & When ?

for

UNCERTAINTY Mgmt.

In the Problem

Uncertainty :

- *In the parameters which define the problem*

or

- *In the situation in which the problem occurs*

Linguistic Imprecision

***tall** man*

hot day

***long** street*

***large** number*

***sharp** corner*

very young

Fuzzy logic works with ***linguistic terms***, which are inherently imprecise and subject specific

When did you come to the seminar?

How is weather today?

How do you teach driving to your friend?

- **Fuzzy logic models the uncertainty and impreciseness observed in real world - unavoidable**
- **Handles the concept of partial truth**

- Many decision-making and problem-solving tasks are too complex to be defined precisely

- However, people succeed by using imprecise knowledge

- Fuzzy logic resembles human reasoning in its use of approximate information and uncertainty to generate decisions.

Examples:

  - Design of washing machine → Fuzzy washing machine
  - Decision regarding purchasing a car

Fuzzy logic enables us to

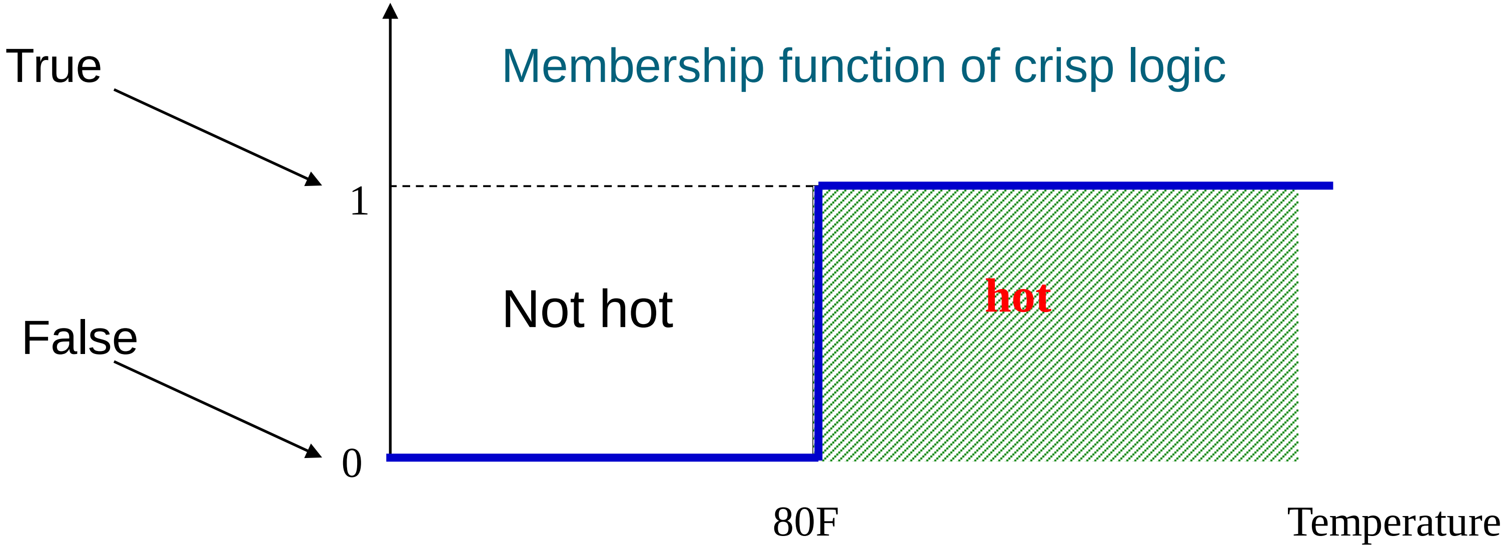
- model human reasoning process at a higher level.
- model real life situations
- use it as a tool for finding solutions to AI problems.

Fuzzy vs. Crisp

fuzzy logic

"traditional / classical / boolean logic": {true, false}

Crisp logic is concerned with absolutes-true or false, there is no in-between.



If temperature >= 80F, it is hot (1 or true);

If temperature < 80F, it is not hot (0 or false).

Drawbacks of crisp logic

The membership function of crisp logic fails

- to distinguish between the members of the same set and also
- To detect the little difference among the members of different sets

Temperature = 100F	hot
Temperature = 80.1F	**hot**
Temperature = 79.9F	**not hot**
Temperature = 50F	not hot

Fuzzy versus Probability

- ✓ Both lies in [0, 1]
- ✓ Fuzzy membership ≠ probability
- ✓ Probability theory based on frequency
- ✓ Fuzzy set theory based on similarity

Example:

X = Set of all liquids

$\tilde{A}$ = Drinkable liquids

From which glass should we drink ?

$\mu_{\tilde{A}}(P) = 0.95$ $\qquad$ $P(Q \in \tilde{A}) = 0.95$

Fuzzy versus Probability (contd.)

- ✓ Fuzzy systems and probability operate over the same numeric range
- ✓ Probabilistic approach yields there is an 95% chance that liquid is drinkable
- ✓ Fuzzy terminology corresponds to drinkability's degree of membership within the set of all liquid is 0.95
- ✓ The probability view assumes that liquid is or is not drinkable
- ✓ Fuzzy supposes that liquid is “more or less” drinkable

Fuzzy vs. Probability

Two Mathematical **tools** to handle uncertainty

Fuzzy Probability

Fuzzy :

- **partial** or **imprecise information**
- Information is not fully reliable
- Imprecision in the language

Example problem :

- understanding human speech
- recognizing handwritten characters etc.

Probability : random process, i.e., where occurrence of events are determined by chance

Examples of Crisp Set & Fuzzy Set

Crisp Sets	Fuzzy Sets
Boys passed in first div.	Good boys
All roses	Red flowers
Girls reading in IIT	Intelligent girls
Men with height $\geq$ 6 ft.	Tall men
Persons with age $\leq$ 10 yrs	Young men
Persons with age $\geq$ 18 yrs	Adults
Persons with age $\geq$ 65 yrs	Olds
Cars with price more than 6 lakhs	Expansive cars

Real Life Applications

Household Products

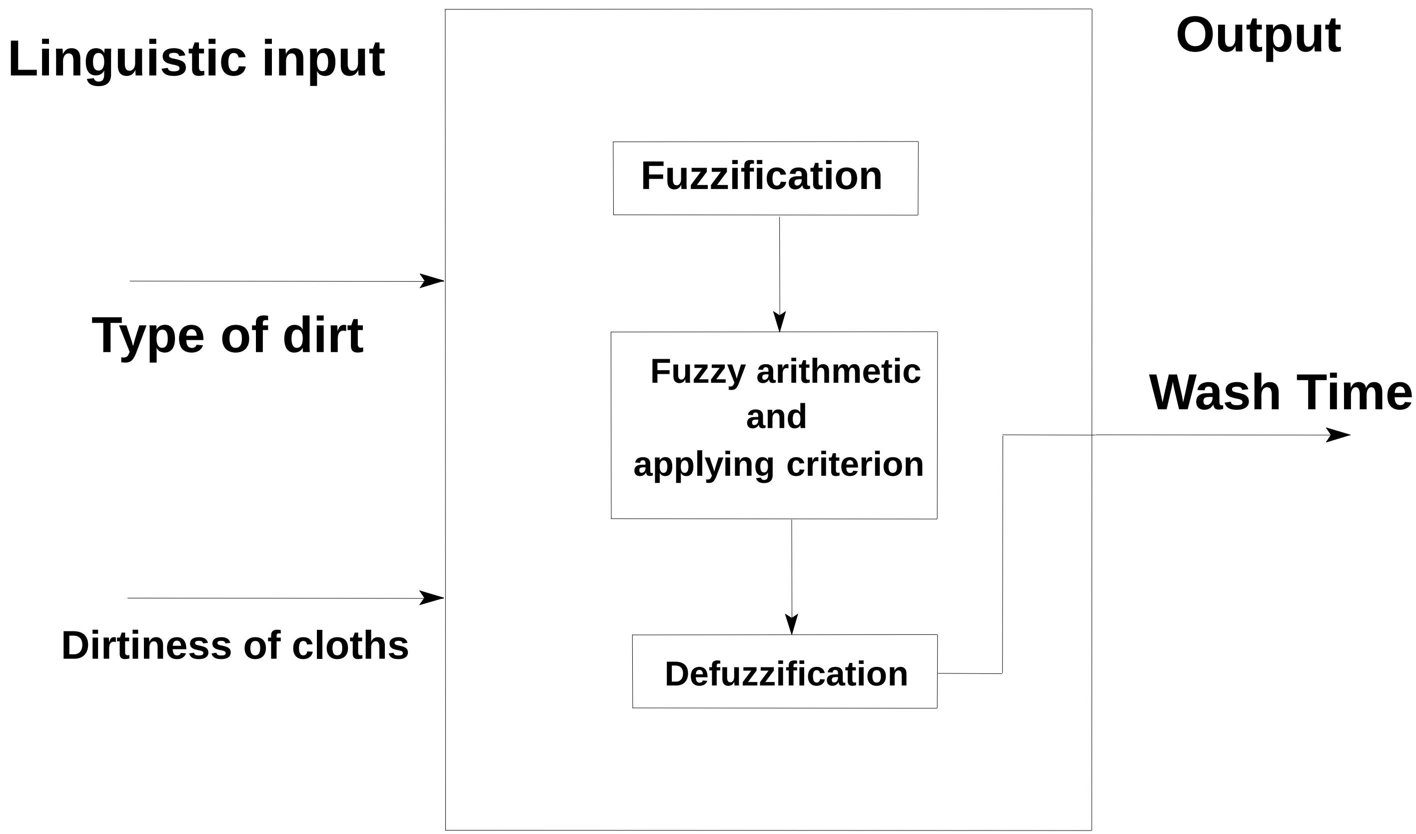
- Washing Machines
- Air-conditioners
- Microwave Ovens
- Rice Cookers
- Digital Cameras
- Toys

Commercial Systems

- Intelligent Vehicles
- Transport Systems
- Banking Systems
- Fuzzy Washing Machines

Fuzzy Controller

For Washing Machine



Example of Rule Base

Used for Washing Machine

- If dirtiness of clothes is *large* and type of dirt is *greasy* then wash time is *very long*
- If dirtiness of clothes is *medium* and type of dirt is greasy then wash time is *long*
- If dirtiness of clothes is *small* and type of dirt is greasy then wash time is *long*
- If dirtiness of clothes is *medium* and type of dirt is *medium* then wash time is *medium*

Fundamentals of “Set”

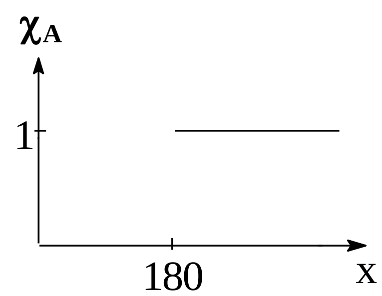
- A set is a group of entities which satisfy one or more criteria
- Sets are a mathematical way of defining concepts
- A set draws a boundary (in some universe) to distinguish between objects which belong to the set from those which do not belong to the set.
- Traditionally, the only type of boundary that could be drawn was a clear line demarcating membership versus non-membership in a set.

Crisp (traditional set)

Crisp Sets are sets defined with clear (or crisp) boundaries.

For example, a set A of real numbers greater than 180 will be defined as:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x > 180 \\ 0 & \text{otherwise} \end{cases}$$



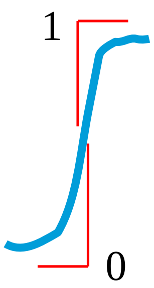
Elements are either in or out of the set.

Definition1: For a crisp set $A \subseteq X$, a characteristic function of X is a mapping from X to {0, 1} defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

and is denoted by $\chi_A(x) : X \rightarrow \{0, 1\}$ with $A = \{x : x \in A\}$.

FUZZY SET



Classical set $\mu \in \{0,1\}$ **Hard**

Fuzzy set $\mu \in [0,1]$ **Soft**

A = { ($\mu_A(x)$, x) : for all $x \in X$}

$\mu_A(x)$: degree of belonging of x to A or degree of possessing some imprecise property represented by A

Example : tall man, A = tall and x= a man

Similar Examples: long street, large number, sharp corner, very young - *Gradual transition*

Fuzzy set is a *Generalization* of classical set

Example

- A = Good football team (20 members)
  $\mu(x)$: degree goodness of a player x.

  If $\mu(x)$ = 0 or 1, no ambiguity in asserting that x is good or not.

  If $\mu(x)$= 0.5, maximum ambiguity in asserting that x is good or not.

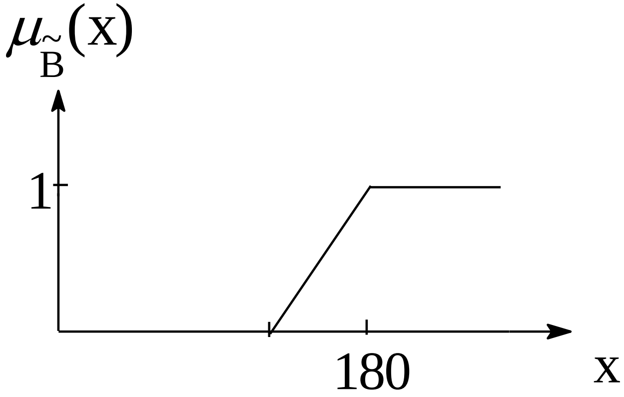
- Uncertainty $\uparrow$ in [0, 0.5] and $\downarrow$ in [0.5, 1] with maximum at μ = 0.5.

Fuzzy Sets

Fuzzy sets are a generalization of crisp sets. In fuzzy sets the boundaries are not crisp but vague.

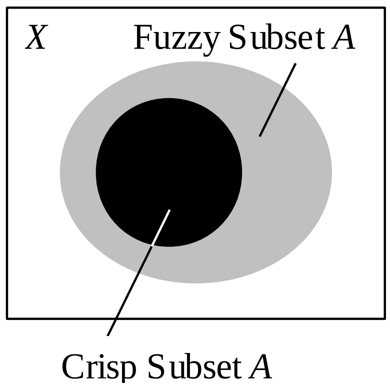
For example, a set $\tilde{A}$, of tall people could be defined as:

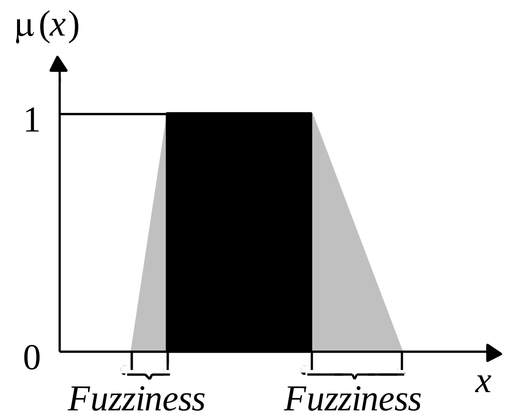
$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & \text{if } x \leq 170 \\ \dfrac{x-170}{10}, & \text{if } 170 < x \leq 180 \\ 1, & \text{if } x > 180 \end{cases}$$



Fuzzy sets help define concepts which are not (or cannot be) clearly defined. For example, congestion, spatial patterns, linguistic descriptors, etc.

Fuzzy Set Representation





Definition: A fuzzy set denoted by $\tilde{A}$ on the universe X is a set of order pairs: $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)): x \in X\}$

where $\mu_{\tilde{A}}(x)$ is termed as the grade of membership of x in $\tilde{A}$, and the function $\mu_{\tilde{A}}: X \rightarrow M$ is called the membership function from X to the membership space M. It is assumed that M is the closed interval [0, 1].

Fuzzy set with discrete order universe

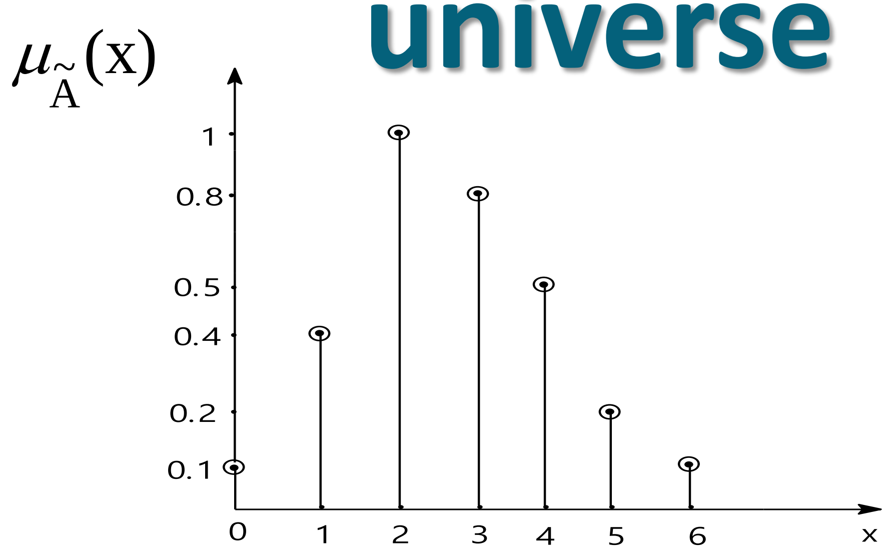


Fig.1: $\tilde{A}$ **= Number of children in a family**

Let X = {0, 1, 2, 3, 4, 5, 6} be the set of expected number of children in a family. Then the fuzzy set $\tilde{A}$ "desirable number of children in a family" may be described as follows:

$\tilde{A}$ = {(0, 0.1), (1, 0.4), (2, 1), (3, 0.8), (4, 0.5), (5, 0.2), (6, 0.1)}.

The membership function of $\tilde{B}$ is shown in figure-1 by dots; it is a discrete function.

Characteristic of a Fuzzy Set

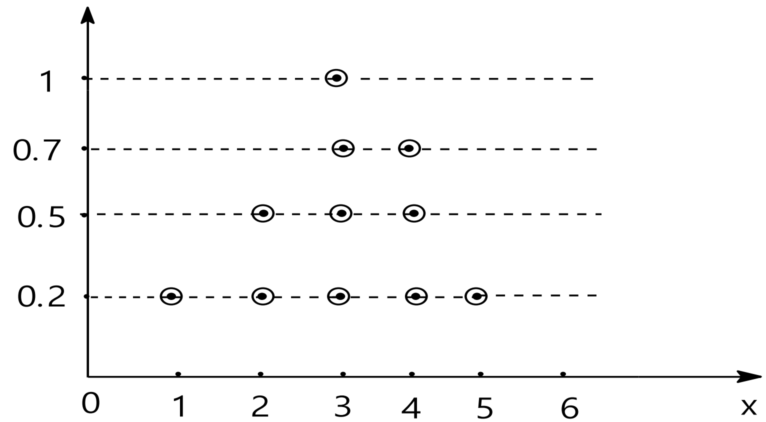
α -cuts of a fuzzy sets

The α - level set (or interval of confidence at level α or α-cut) of the fuzzy set $\tilde{A}$ of X is a crisp set A_α that contains all the elements of X that have membership values in $\tilde{A}$ greater than or equal to α i.e.,

$A_\alpha = \{x: \mu_{\tilde{A}}(x) \geq \alpha,\ x \in X, \alpha \in [0, 1]\}$.

for any fuzzy set and $\alpha_1, \alpha_2 \in [0, 1]$ of distinct values such that $\alpha_1 < \alpha_2$, we have $A_{\alpha_1} \supseteq A_{\alpha_2}$ and $A'_{\alpha_1} \supseteq A'_{\alpha_2}$

Example of α -cuts



Consider the fuzzy sets

$\tilde{A} = \{(1, 0.2), (2, 0.5), (3, 1), (4, 0.7), (5, 0.2), (6, 0)\}$.

The support $\tilde{A}$ is $S(\tilde{A}) = \{1, 2, 3, 4, 5\}$,

Since $\mu_{\tilde{A}}(6) = 0$, the number 6 is not an element of $S(\tilde{A})$.

The α - level set of $\tilde{A}$ are (c.f. figure-):

$\tilde{A}_{0.2} = \{1, 2,3,4, 5\}$, $\tilde{A}_{0.5} = \{2, 3, 4\}$, $\tilde{A}_{0.7} = \{3, 4\}$, $\tilde{A}_{1} = \{3\}$.

The strong α - level set for α = 0.5 is $\tilde{A}'_{0.5} = \{3, 4\}$.

Support, Core and Height of a Fuzzy Set

The *support* of a fuzzy set $\tilde{A}$ on X, denoted by S($\tilde{A}$) or supp($\tilde{A}$), is the set of points in X at which $\mu_{\tilde{A}}(x)$ is positive, i.e., S($\tilde{A}$) or supp ($\tilde{A}$) = $\{x \in X : \mu_{\tilde{A}}(x) > 0\}$.

It should be pointed out that the support of a fuzzy set is a crisp (classical) set

The *core* of a fuzzy set $\tilde{A}$ on X, denoted by core($\tilde{A}$), is the set of points in X at which $\mu_{\tilde{A}}(x) = 1$, i.e., core($\tilde{A}$) = $\{x \in X : \mu_{\tilde{A}}(x) = 1\}$.

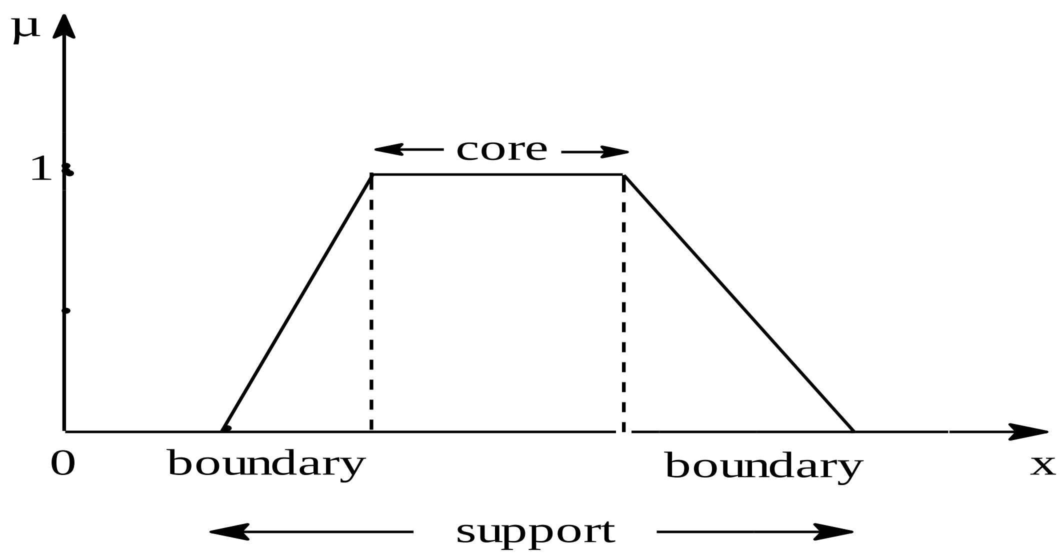
The *boundary* of a fuzzy set $\tilde{A}$ on X, denoted by boundary ($\tilde{A}$), is the set of points in X at which $0 < \mu_{\tilde{A}}(x) < 1$, i.e.,

boundary($\tilde{A}$) = $\{x \in X : 0 < \mu_{\tilde{A}}(x) < 1\}$

Contd..

The *height* of a fuzzy set $\tilde{A}$ is the largest membership grade, i.e. $h(\tilde{A}) = \sup_{x \in X} \mu_{\tilde{A}}(x)$

A fuzzy set $\tilde{A}$ is *normal* when its height is 1, i.e. $\sup_{x \in X} \mu_{\tilde{A}}(x) = 1$



Basic Operations on Fuzzy Set

Standard Fuzzy Complement

Let $\tilde{A}$ be a fuzzy set defined over X. Then its *complement*, $\tilde{A}^c$, is defined in terms of membership function as

$$\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x) \quad \text{for each } x \in X.$$

$$\tilde{A}^c = \left\{\left(x, \mu_{\tilde{A}^c}(x)\right) : x \in X, \mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x)\right\} \text{ or } = \int_{x \in X} (1 - \mu_{\tilde{A}}(x))/x$$

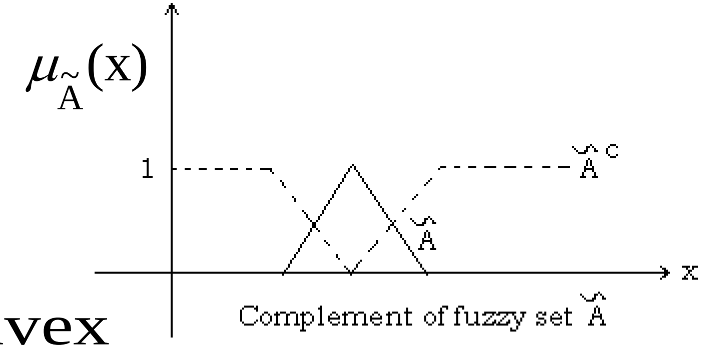
Note : The membership function $\mu_{\tilde{A}^c}(x)$ is symmetrical to $\mu_{\tilde{A}}(x)$ with respect to the line $\mu = 0.5$.

Properties:

(i) $\left(\tilde{A}^c\right)^c = \tilde{A}$

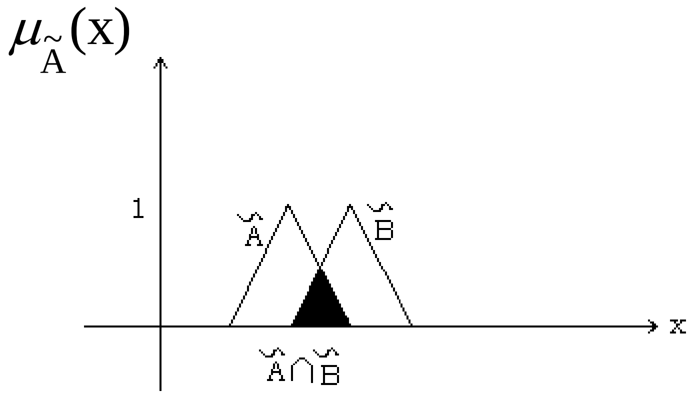
(ii) If $\tilde{A}^c = \tilde{B}$ then $\tilde{B}^c = \tilde{A}$

$\tilde{A}$ is concave if and only if $\tilde{A}^c$ is convex



Complement of fuzzy set $\tilde{A}$

Intersection of Fuzzy Sets

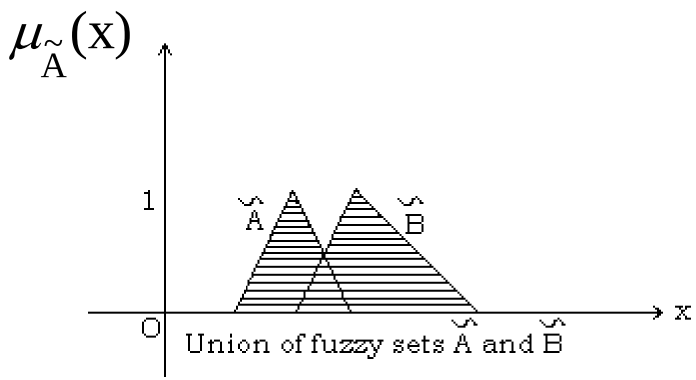


The *intersection* of $\tilde{A}$ and $\tilde{B}$ is a fuzzy set in X, denoted by $\tilde{A} \cap \tilde{B}$, whose membership function is $\mu_{\tilde{A} \cap \tilde{B}}(x) = \mu_{\tilde{A}}(x) \wedge \mu_{\tilde{B}}(x) =$ $\mathrm{Min}\left\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\right\}$ for each $x \in X$.

So $\tilde{A} \cap \tilde{B} = \left\{\left(x, \mu_{\tilde{A} \cap \tilde{B}}(x)\right):, \mu_{\tilde{A} \cap \tilde{B}}(x) = \min\left(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\right), \forall x \in X\right\}$

$$\int_{x \in X} \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))/x$$

Union of Fuzzy Sets



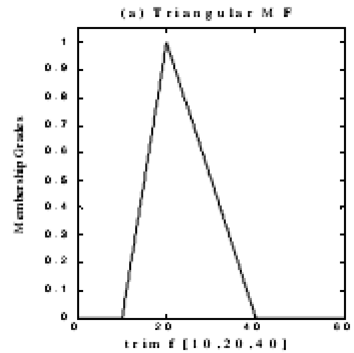
The *union* of $\tilde{A}$ and $\tilde{B}$ is a fuzzy set in X, denoted by $\tilde{A}\cup\tilde{B}$, whose membership function is

$\mu_{\tilde{A}\cup\tilde{B}}(x)=\mu_{\tilde{A}}(x)\vee\mu_{\tilde{B}}(x)=\text{Max}\{\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)\}$ for each $x\in X$.

So $\tilde{A}\cup\tilde{B} = \left\{\left(x, \mu_{\tilde{A}\cup\tilde{B}}(x)\right):, \mu_{\tilde{A}\cup\tilde{B}}(x)=\max\left(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\right), \forall x\in X\right\}$

or $\int_{x\in X} \max(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))/x$

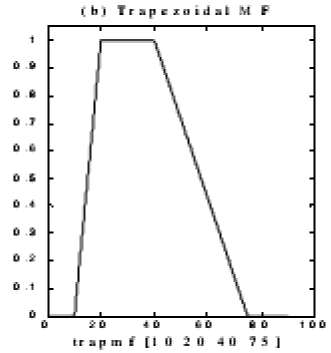
Triangular Fuzzy Number(TFN)



A "triangular MF" is specified by three parameters {a, b, c} as follows:

$$y = \mathrm{triangle}(x;a,b,c) = \begin{cases} 0, & x \le a . \\ (x-a)/(b-a), & a \le x \le b . \\ (c-x)/(c-b), & b \le x \le c . \\ 0, & c \le x . \end{cases}$$

Special Case: *Trapezoidal Fuzzy Number(TrFN)*

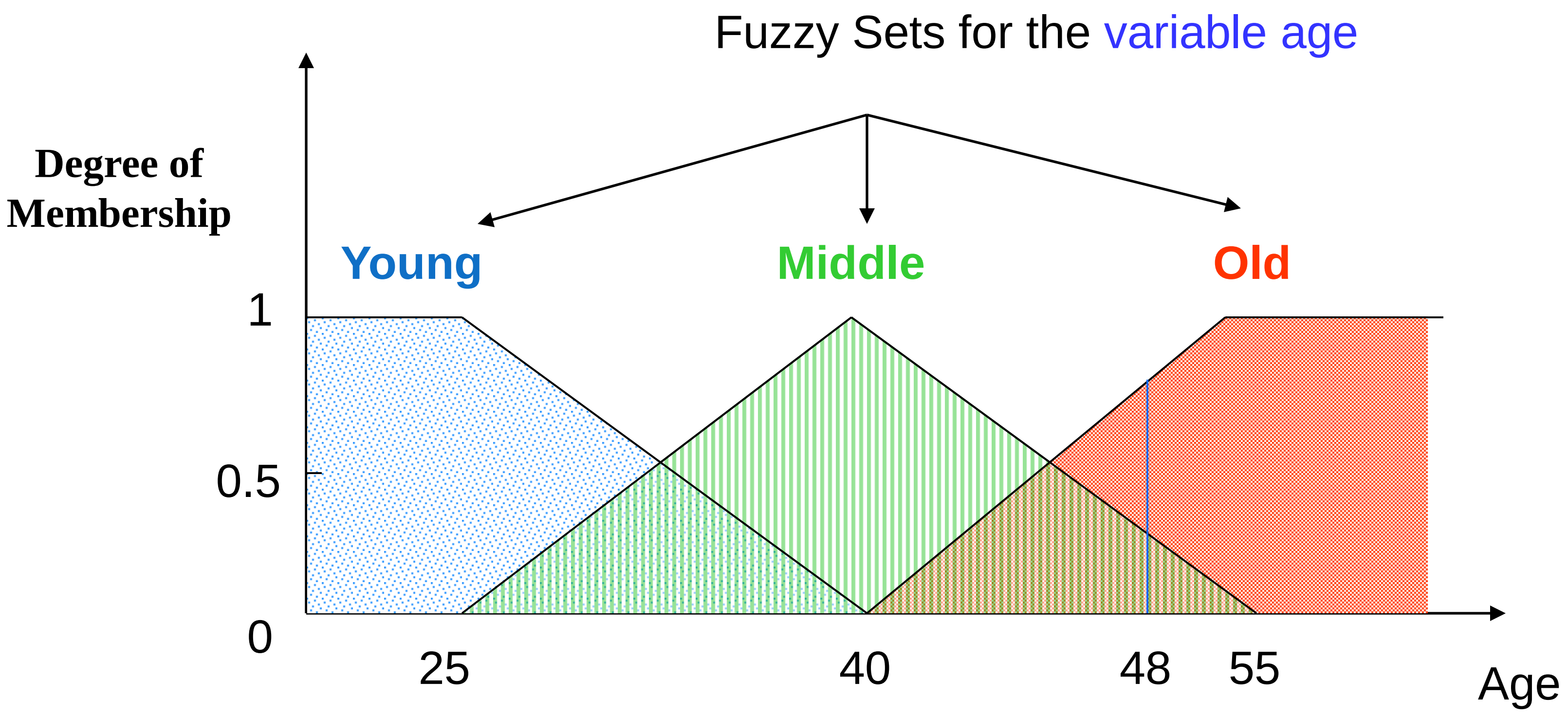


A "trapezoidal MF" is specified by four parameters {a, b, c, d} as follows:

$$\text{trapezoid}(x;a,b,c,d) = \begin{cases} 0, & x \le a. \\ (x-a)/(b-a), & a \le x \le b. \\ 1, & b \le x \le c. \\ (d-x)/(d-c), & c \le x \le d. \\ 0, & d \le x. \end{cases}$$

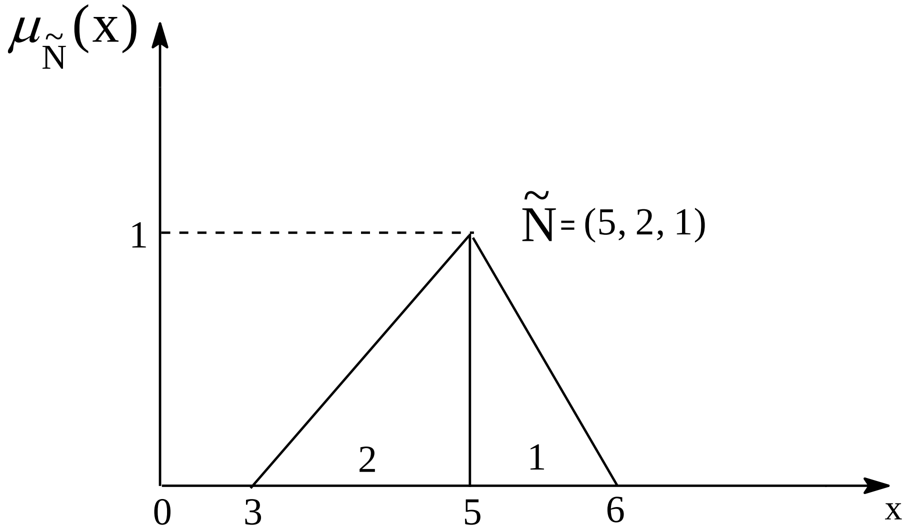
Fuzzy Variable

We want the value to switch gradually as *Young* becomes *Middle* and *Middle* becomes *Old* as it becomes really.



A person of age 48 is more compatible with old than middle

Triangular Fuzzy Number(TFN)

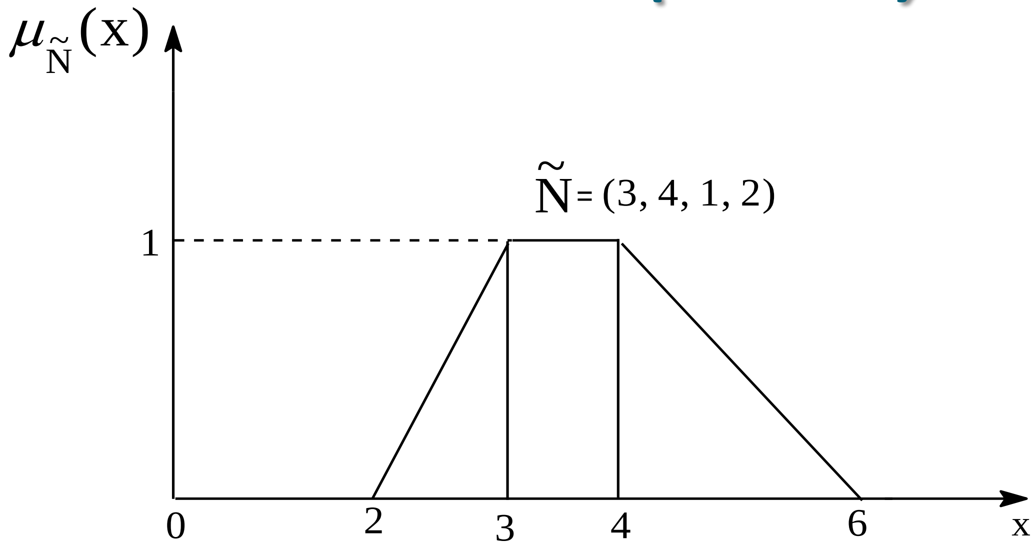


If both L and R are linear and $n_1 = n_2$, we have a triangular fuzzy number (TFN) denoted by

$$\tilde{N} = (x, \alpha, \beta)$$

where α is the left spread from x and β is the right spread from x.

Special Case: *Trapezoidal Fuzzy Number(TrFN)*



If a linear fuzzy number has $n_1 < x < n_2$, we have a trapezoidal fuzzy number (TrFN) denoted by

$$\tilde{N} = (m_1, m_2, \alpha, \beta)$$

where α is the left spread from m_1 and β is the right spread from m_2.

Fuzzy Relation

- Fuzzy relation generalizes classical relation into one that allows partial membership and describes a relationship that holds between two or more objects.
- **Example:** a fuzzy relation "Friend" describe the degree of friendship between two persons (in contrast to either being friend or not being friend in classical relation!)
- Lets consider properties of crisp relations first and then extend the mechanism to fuzzy sets.

Crisp Relations

- Ordered pairs showing connection between two sets:

  (a, b): a is related to b

  (2, 3) are related with the relation “<“

- Relations are set themselves :

  {(1, 1), (1, 2), (2, 1), (2, 2)}

<	1	2
1	×	☺
2	×	×

- Relations can be expressed as matrices.

Cartesian Product & Relation

Cartesian Product

Let X and Y are two arbitrary classical (non-fuzzy or crisp) sets. The Cartesian product of X and Y denoted by $X \times Y$ is a classical set of order pairs (x, y) such that $x \in X$ and $y \in Y$, i.e.,

$$X \times Y = \{ (x, y) : x \in X, y \in Y \}$$

In general, the Cartesian product of arbitrary n classical sets $X_1, X_2, \ldots, X_n$, denoted by $X_1 \times X_2 \times \ldots \times X_n$, is the classical set of n-tuples $(x_1, x_2, \ldots, x_n)$ such that $x_i \in X_i$ for $i \in \{1, 2, \ldots, n\}$, i.e.,

$$X_1 \times X_2 \times \ldots \times X_n, = \{(x_1, x_2, \ldots, x_n) : x_1 \in X_1, x_2 \in X_2, \ldots, x_n \in X_n\}.$$

Relation

A (classical) binary relation between two sets (crisp) X and Y, denoted by R(x, y) is a subset of the cartesian product $X \times Y$, i.e.

$$R(x, y) \subseteq X \times Y.$$

Fuzzy Relation

Fuzzy Cartesian Product

Let $\tilde{A}$ be a fuzzy set on a universe X and $\tilde{B}$ be a fuzzy set on the universe Y. The Cartesian product of two fuzzy set $\tilde{A}$ and $\tilde{B}$ is a fuzzy set, denoted by $\tilde{A}\times\tilde{B}$ defined as

$$\tilde{A}\times\tilde{B} = \left\{\left((a,b),\mu_{\tilde{A}\times\tilde{B}}(a,b)\right) : \left(a,\mu_{\tilde{A}}(a)\right)\in\tilde{A},\left(b,\mu_{\tilde{B}}(b)\right)\in\tilde{B},\mu_{\tilde{A}\times\tilde{B}}(a,b) = \min\left(\mu_{\tilde{A}}(a),\mu_{\tilde{B}}(b)\right)\right\}$$

Example : Let $\tilde{A}$ = {(3, 0.4), (5, 1), (7, 0.6)}and $\tilde{B}$ = {(5, 1), (6, 0.6)}then

$\tilde{A}\times\tilde{B}$ = {((3, 5) 0.4), ((3, 6), 0.4), ((5, 5), 1), ((5, 6), 0.6), ((7, 5), 0.6), ((7, 6), 0.6)}

The concept of relations on classical sets may be extended to fuzzy set

Fuzzy Cartesian Product: Example

Let

$\tilde{A}$ defined on a universe of three discrete temperatures, X = {x_1,x_2,x_3}, and

$\tilde{B}$ defined on a universe of two discrete pressures, Y = {y_1,y_2}

Fuzzy set $\tilde{A}$ represents the "ambient" temperature and

Fuzzy set $\tilde{B}$ the "near optimum" pressure for a certain heat exchanger, and the Cartesian product might represent the conditions (temperature-pressure pairs) of the exchanger that are associated with "efficient" operations. For example, let

$$\left.\begin{aligned} &\tilde{A} = \frac{0.2}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3} \\ &\textit{and} \\ &\tilde{B} = \frac{0.3}{y_1} + \frac{0.9}{y_2} \end{aligned}\right\} \quad \tilde{A} \times \tilde{B} = \tilde{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.5 \\ 0.3 & 0.9 \end{bmatrix} \end{array}$$

Fuzzy Relation (Contd..)

Fuzzy Relation

A fuzzy relation, defined by $\tilde{R}(x_1, x_2, \ldots, x_n)$ is a fuzzy subset of the universal product set $X_1 \times X_2 \times \ldots \times X_n$. The n-tuple $(x_1, x_2, \ldots, x_n) \in X_1 \times X_2 \times \ldots \times X_n$ may have varying degrees of membership $\mu_{\tilde{R}}(x_1, x_2, \ldots, x_n) \in [0, 1]$, i.e. the fuzzy relation is of the form

$$\tilde{R} = \left\{ \left((x_1, x_2, \ldots, x_n), \mu_{\tilde{R}}(x_1, x_2, \ldots, x_n) \right) : (x_1, x_2, \ldots, x_n) \in X_1 \times X_2 \ldots \times X_n, \mu_{\tilde{R}}(x_1, x_2, \ldots, x_n) \in \right.$$

A fuzzy binary relation, $\tilde{R}$ from the universe X to the universe Y is a fuzzy set defined on the Cartesian product of crisp sets X and Y denoted by $\tilde{R}(X, Y)$ where $(x, y) \in X \times Y$ may have varying degrees of membership function $\mu_{\tilde{R}}(x, y) \in [0, 1]$, i.e.,

$$\tilde{R}(X, Y) = \left\{ \left((x,y), \mu_{\tilde{R}}(x,y) \right) : (x, y) \in X \times Y, \mu_{\tilde{R}}(x,y) \in [0,1] \right\}$$

Example

Let A = {3, 8, 25, 30} and B = {1, 2, 5, 10} be two sets and $\tilde{R}$ denote a fuzzy relation on A×B defined by $\tilde{R}(x, y)$ = 'x is very much greater than y'. A possible realization of the fuzzy relation is concisely represented by the matrix with values of the membership functions $\mu_{\tilde{R}}(x, y)$ as entries is shown in the following table with crisp relation R for comparison.

$$M_R = \begin{array}{c|cccc|} & 1 & 2 & 5 & 10 \\ 3 & 1 & 1 & 0 & 0 \\ 8 & 1 & 1 & 1 & 0 \\ 25 & 1 & 1 & 1 & 1 \\ 30 & 1 & 1 & 1 & 1 \end{array} \qquad M_{\tilde{R}} = \begin{array}{c|cccc|} & 1 & 2 & 5 & 10 \\ 3 & 0.4 & 0.1 & 0 & 0 \\ 8 & 0.8 & 0.5 & 0.2 & 0 \\ 25 & 1 & 0.9 & 0.6 & 0.5 \\ 30 & 1 & 1 & 0.7 & 0.5 \end{array}$$

Operations on fuzzy relations

- Union relation

$$\forall (x,y) \in A \times B$$

$$\mu_{R \cup S}(x,y) = \max(\mu_R(x,y), \mu_s(x,y))$$

$$= \mu_R(x,y) \vee \mu_s(x,y)$$

- For *n* relations

$$\forall (x,y) \in A \times B$$

$$\mu_{R_1 \cup R_2 \cup \ldots \cup R_n}(x,y) = \bigvee_{R_i} \mu_{R_i}(x,y)$$

Union relation

- Example

M_R	a	b	c
1	0.3	0.2	1.0
2	0.8	1.0	1.0
3	0.0	1.0	0.0

M_S	a	b	c
1	0.3	0.0	0.1
2	0.1	0.8	1.0
3	0.6	0.9	0.3

$M_{R \cup S}$	a	b	c
1	0.3	0.2	1.0
2	0.8	1.0	1.0
3	0.6	1.0	0.3

Operations on fuzzy relations

- Intersection relation

$$\forall (x, y) \in A \times B$$

$$\mu_{R \cap S}(x, y) = \min(\mu_R(x, y), \mu_s(x, y))$$

$$= \mu_R(x, y) \wedge \mu_s(x, y)$$

- For *n* relations

$$\forall (x, y) \in A \times B$$

$$\mu_{R_1 \cap R_2 \cap \ldots \cap R_n}(x, y) = \bigwedge_{R_i} \mu_{R_i}(x, y)$$

Intersection relation

- Example

M_R	a	b	c
1	0.3	0.2	1.0
2	0.8	1.0	1.0
3	0.0	1.0	0.0

M_S	a	b	c
1	0.3	0.0	0.1
2	0.1	0.8	1.0
3	0.6	0.9	0.3

$M_{R\cap S}$	a	b	c
1	0.3	0.0	0.1
2	0.1	0.8	1.0
3	0.0	0.9	0.0

Operations on fuzzy relations

$$\forall (x, y) \in A \times B$$

$$\mu_{\overline{R}}(x, y) = 1 - \mu_R(x, y)$$

M_R	a	b	c
1	0.3	0.2	1.0
2	0.8	1.0	1.0
3	0.0	1.0	0.0

$M_{\overline{R}}$	a	b	c
1	0.7	0.8	0.0
2	0.2	0.0	0.0
3	1.0	0.0	1.0

Fuzzy Cartesian Product: Example

Let

$\tilde{A}$ defined on a universe of three discrete temperatures, X = $\{x_1,x_2,x_3\}$, and

$\tilde{B}$ defined on a universe of two discrete pressures, Y = $\{y_1,y_2\}$

Fuzzy set $\tilde{A}$ represents the "ambient" temperature and

Fuzzy set $\tilde{B}$ the "near optimum" pressure for a certain heat exchanger, and the Cartesian product might represent the conditions (temperature-pressure pairs) of the exchanger that are associated with "efficient" operations. For example, let

$$\left.\begin{aligned} &\tilde{A} = \frac{0.2}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3} \\ &\textit{and} \\ &\tilde{B} = \frac{0.3}{y_1} + \frac{0.9}{y_2} \end{aligned}\right\} \qquad \tilde{A} \times \tilde{B} = \tilde{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.5 \\ 0.3 & 0.9 \end{bmatrix} \end{array}$$

Fuzzy Composition

Suppose

$\tilde{R}$ is a fuzzy relation on the Cartesian space X x Y,

$\tilde{S}$ is a fuzzy relation on the Cartesian space Y x Z, and

$\tilde{T}$ is a fuzzy relation on the Cartesian space X x Z; then fuzzy max-min and fuzzy max-product composition are defined as

$$\tilde{T} = \tilde{R} \circ \tilde{S}$$

$$\max - \min$$

$$\mu_{\tilde{T}}(x,z) = \bigvee_{y \in Y} (\mu_{\tilde{R}}(x,y) \wedge \mu_{\tilde{S}}(y,z))$$

$$\max - product$$

$$\mu_{\tilde{T}}(x,z) = \bigvee_{y \in Y} (\mu_{\tilde{R}}(x,y) \bullet \mu_{\tilde{S}}(y,z))$$

Fuzzy Composition: Example (max-min)

$$X = \{x_1, x_2\}, \quad Y = \{y_1, y_2\}, \text{ and } \quad Z = \{z_1, z_2, z_3\}$$

Consider the following fuzzy relations:

$$\tilde{R} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \!\!\begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.7 & 0.5 \\ 0.8 & 0.4 \end{bmatrix} \end{array} \quad \text{and} \quad \tilde{S} = \begin{array}{c} \\ y_1 \\ y_2 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.9 & 0.6 & 0.5 \\ 0.1 & 0.7 & 0.5 \end{bmatrix} \end{array}$$

Using max-min composition,

$$\left.\begin{aligned} \mu_{\tilde{T}}(x_1, z_1) &= \bigvee_{y \in Y} (\mu_{\tilde{R}}(x_1, y) \wedge \mu_{\tilde{S}}(y, z_1)) \\ &= \max[\min(0.7, 0.9), \min(0.5, 0.1)] \\ &= 0.7 \end{aligned}\right\} \quad \tilde{T} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.7 & 0.6 & 0.5 \\ 0.8 & 0.6 & 0.4 \end{bmatrix} \end{array}$$

Composition of fuzzy relations

- Max-min composition

$$\forall (x, y) \in A \times B,\ \forall (y, z) \in B \times C$$

$$\mu_{S \bullet R}(x,\ z) = \max_{y}[\min(\mu_R(x,\ y),\ \mu_S(y,\ z))]$$

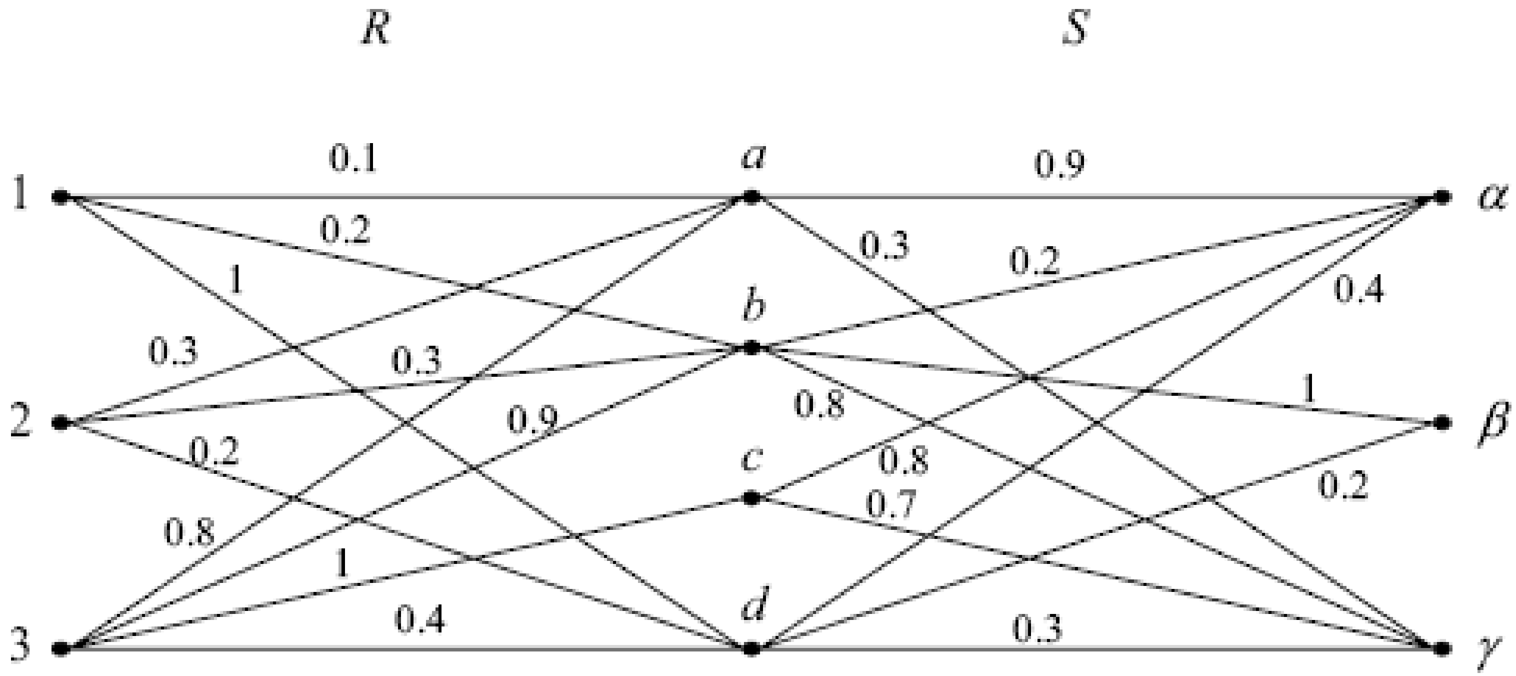
- Example

$$= \vee_{y}[\mu_R(x,\ y) \wedge \mu_S(y,\ z)]$$

R	a	b	c	d
1	0.1	0.2	0.0	1.0
2	0.3	0.3	0.0	0.2
3	0.8	0.9	1.0	0.4

S	α	β	γ
a	0.9	0.0	0.3
b	0.2	1.0	0.8
c	0.8	0.0	0.7
d	0.4	0.2	0.3

Composition of fuzzy relations



Composition of fuzzy relations

- Example

R	a	b	c	d
1	0.1	0.2	0.0	1.0
2	0.3	0.3	0.0	0.2
3	0.8	0.9	1.0	0.4

S	α	β	γ
a	0.9	0.0	0.3
b	0.2	1.0	0.8
c	0.8	0.0	0.7
d	0.4	0.2	0.3

$$\mu_{S\bullet R}(1,\ \alpha) = \max[\min(0.1, 0.9), \min(0.2, 0.2), \min(0.0, 0.8), \min(1.0, 0.4)]$$

$$= \max[0.1, 0.2, 0.0, 0.4] = 0.4$$

Composition of fuzzy relations

- Example

R	a	b	c	d
1	0.1	0.2	0.0	1.0
2	0.3	0.3	0.0	0.2
3	0.8	0.9	1.0	0.4

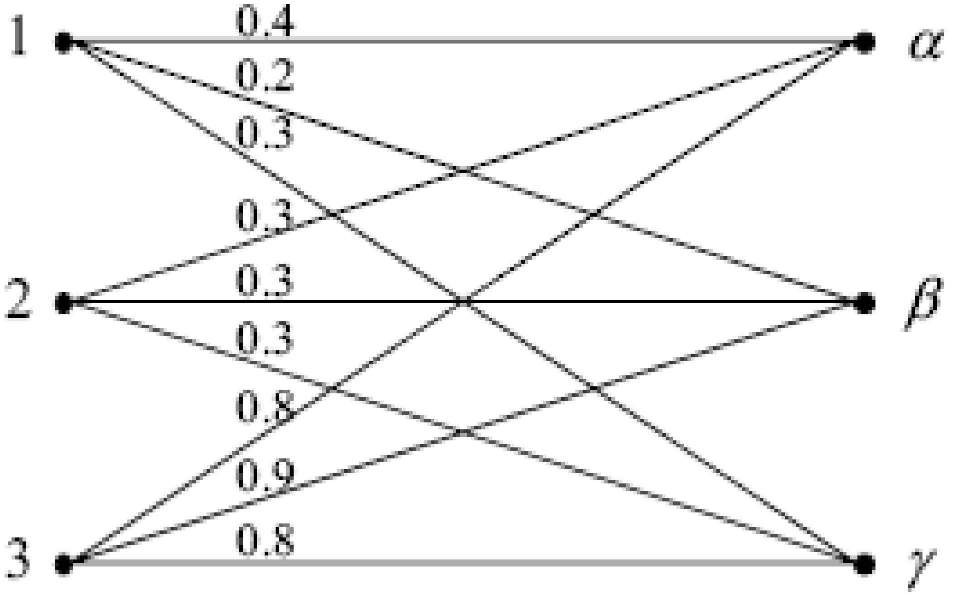
S	α	β	γ
a	0.9	0.0	0.3
b	0.2	1.0	0.8
c	0.8	0.0	0.7
d	0.4	0.2	0.3

$$\mu_{S \bullet R}(1, \beta) = \max[\min(0.1, 0.0), \min(0.2, 1.0), \min(0.0, 0.0), \min(1.0, 0.2)]$$
$$= \max[0.0, 0.2, 0.0, 0.2] = 0.2$$

Composition of fuzzy relations

S•R	α	β	γ
1	0.4	0.2	0.3
2	0.3	0.3	0.3
3	0.8	0.9	0.8

S•R



Fuzzy Composition

Suppose

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$$\max - product$$

$$\mu_{\tilde{T}}(x,z) = \bigvee_{y \in Y} (\mu_{\tilde{R}}(x,y) \bullet \mu_{\tilde{S}}(y,z))$$

Fuzzy Composition: Example (max-min)

$$X = \{x_1, x_2\}, \quad Y = \{y_1, y_2\}, \text{ and } \quad Z = \{z_1, z_2, z_3\}$$

Consider the following fuzzy relations:

$$\tilde{R} = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 0.7 & 0.5 \\ 0.8 & 0.4 \end{bmatrix} \end{array} \quad \text{and} \quad \tilde{S} = \begin{array}{c c} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} y_1 \\ y_2 \end{array} & \begin{bmatrix} 0.9 & 0.6 & 0.5 \\ 0.1 & 0.7 & 0.5 \end{bmatrix} \end{array}$$

Using max-min composition,

$$\left.\begin{aligned} \mu_{\tilde{T}}(x_1, z_1) &= \bigvee_{y \in Y} (\mu_{\tilde{R}}(x_1, y) \wedge \mu_{\tilde{S}}(y, z_1)) \\ &= \max[\min(0.7, 0.9), \min(0.5, 0.1)] \\ &= 0.7 \end{aligned}\right\} \quad \tilde{T} = \begin{array}{c c} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 0.7 & 0.6 & 0.5 \\ 0.8 & 0.6 & 0.4 \end{bmatrix} \end{array}$$

Thanks !