National Institute of Technology, Warangal

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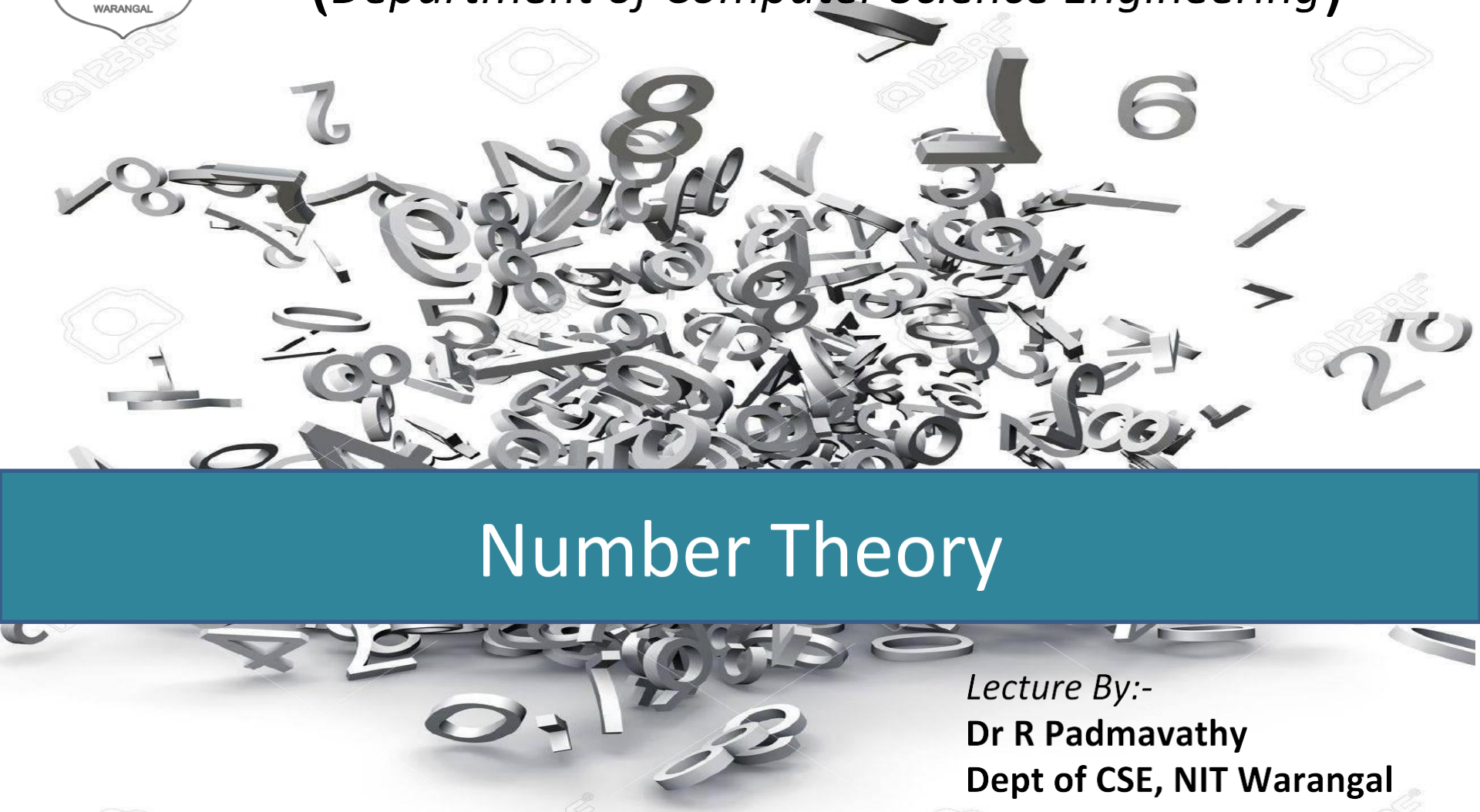

Number Theory

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Standard Notations

- $\mathbb{N}$: Set of positive Integers {1, 2, 3, . . . }
- $\mathbb{Z}$: Set of all integers {. . . , -3, -2, -1, 0, 1, 2, 3, . . .}
- $\mathbb{Q}$: Set of all rational numbers {p/q : p, q $\in \mathbb{Z}, q \neq 0$}
- $\mathbb{R}$: Set of all real numbers
- $\mathbb{C}$: Set of all Complex numbers

Basic Facts

- Every even integer is of the form $2m$ with $m \in \mathbb{Z}$
- Every odd integer is of the form $2m + 1$ or $2m-1$ with $m \in \mathbb{Z}$
- We say a *divides* b or $a \mid b$ if $b = ac$ with $c \in \mathbb{Z}$.
- When $a \mid b$, we say a is factor or a divisor of b.
- 1 and -1 are divisor of any integer.
- $a \mid 0$ for any $a \in \mathbb{Z}$
- $a \nmid b$ means a does not divide b.

Primes and positive divisors

- A positive integer $p > 1$ is prime if 1 and p are the only positive divisors of p.
- 2, 3, 5, 7, 11, 13, . . . are primes.
- A positive integer $n > 1$ is composite if it is not a prime.
- 4, 6, 8, 9, 10, . . . are composites.
- 1 is neither a prime nor a composite.

Fundamental Theorem of Arithmetic

- Every positive integer $n > 1$ can be written as product of primes and it is unique upto order of primes.
- Hence every $n \in \mathbb{N}$ can be written uniquely in the form

$$n = p_1^{a_1} \; p_2^{a_2} \; \ldots \; p_r^{a_r}$$

- where $p_1 < p_2 < \ldots < p_r$ are primes and $a_1, a_2, \ldots, a_r$ are non negative integers. e.g. $100 = 2^2 \cdot 5^2$.
- 1 is neither a prime nor a composite else this Theorem is voilated.
- Every $n > 1$ has a prime divisor.

Euclid's Theorem

Theorem 1:

The set of primes is finite.

We denote the primes by p1 =2, p2 = 3, p3 = 5, . . .

However finding large primes is a challenge. Currently the largest know prime number is $2^{43112609} - 1$ which is about 12.9 million digits and before that $2^{37156667} - 1$ (only 11.1 million digits).

Division theorem and Division algorithms

- Let $a, b \in \mathbb{Z}$ with $a > 0$. Then there exist unique integers q and r such that $b = aq + r$ with $0 \leq r < a$.
- If $b = 0$, then $q = r = 0$, i.e., $0 = a.\ 0 + 0$.
- r = 0 iff a | b
- Given a positive integer m, every other integer n is of the form Km or $km + 1$ or . . . $Km + m - 1$.

Division theorem and Division algorithms

- If $a|1$, then $a = \pm 1$.
- If $a|b$ and $b|a$, then $a = \pm b$.
- Any $b \neq 0$ divides 0.
- If $a|b$ and $b|c$, then $a|c$:

$$11|66 \text{ and } 66|198 = 11|198$$

- If $b|g$ and $b|h$, then $b|(mg + nh)$ for arbitrary integers m and n.

To see this last point, note that

- If $b|g$, then g is of the form $g = b \times g_1$ for some integer g_1.
- If $b|h$, then h is of the form $h = b \times h_1$ for some integer h_1.

So

$$mg + nh = mbg_1 + nbh_1 = b \times (mg_1 + nh_1)$$

$b = 7; g = 14; h = 63; m = 3; n = 2$

$7|14$ and $7|63$.

To show $7|(3 \times 14 + 2 \times 63)$,

we have $(3 \times 14 + 2 \times 63) = 7(3 \times 2 + 2 \times 9)$,

and it is obvious that $7|(7(3 \times 2 + 2 \times 9))$.

GCD and LCM

- a is a common divisor of b and c if $a \mid b$ and $a \mid c$.
- *Greatest Common Divisor* (*GCD*) of a and b, denoted by (a, b) is the largest positive common divisor of a and b. Also called as *HCF* of a and b.
- $(a, b) = ax + by$ for some $x, y \in \mathbb{Z}$.
- a and b are *relatively prime* or *coprime* if $(a, b) = 1$.
- *Lowest Common Multiple* or *LCM* of a and b, denoted by $[a, b]$ is the least positive integer l such that $a \mid l$ and $b \mid l$.

GCD

$$\gcd(a, b) = \max[k, \text{such that } k|a \text{ and } k|b]$$

Because we require that the greatest common divisor be positive, $\gcd(a, b) = \gcd(a, -b) = \gcd(-a, b) = \gcd(-a, -b)$. In general, $\gcd(a, b) = \gcd(|a|, |b|)$.

$$\gcd(60, 24) = \gcd(60, -24) = 12$$

8 and 15 are relatively prime because the positive divisors of 8 are 1, 2, 4, and 8, and the positive divisors of 15 are 1, 3, 5, and 15. So 1 is the only integer on both lists.

GCD Algorithm

GCD(a,b)

1. Find a/b if remainder is zero then d=gcd(a,b)=b
2. Otherwise a=b and b==remainder
3. Repeat above steps till remainder is zero

Why and how it works ?

Euclidean Algorithm

$$a = q_1 b + r_1 \qquad 0 \leq r_1 < b$$

1. If r_1 is 0 then b|a d=gcd(a,b)=b
2. If $r_1 \neq 0$ then d|r_1 ?

Divisibility theorem

What is that theorem – seen already

Since d|a and d|b shows d|(a-q_1b) this is equal to d|r1

So find gcd(b,r1)

$$b = q_2 r_1 + r_2 \qquad 0 \leq r_2 < r_1$$

Euclid's GCD Algorithm

gcd(412, 260)

	1	2	3	4	5	6	7
b	412	260	152	108	44	20	4
a	260	152	108	44	20	4	0

GCD and LCM

- If $a = p_1^{a_1} p_2^{a_2} \ldots P_r^{a_r}$ and $b = p_1^{b_1} p_2^{b_2} \ldots P_r^{b_r}$, then (a, b) = $p_1^{c_1} p_2^{c_2} \ldots P_r^{c_r}$ *and* $[a, b] = p_1^{d_1} p_2^{d_2} \ldots P_r^{a_r}$ *where* $c_i = min\{q_i, b_i\}$ and $d_i = max\{a_i, b_i\}$.
- Hence $ab = (a, b) . [a, b]$.

Stein's GCD Algorithm

Depends on following 3 observations, each of which allows one to reduce the size of at least one of the input by half:

(a) If a & b are both even and not both zero, then

$$gcd(a, b) = 2.gcd(a/2, b/2)$$

(b) If a is even & b is odd, then

$$gcd(a, b) = gcd(a/2, b)$$

(c) If a and b are both odd, then

$$gcd(a, b) = gcd((a-b)/2 , b)$$

An "extended" version of Stein's algorithm is available.

Extended Euclidean Algorithm

Now let us show how to extend the Euclidean algorithm to determine (x, y, d) given a and b. We again go through the sequence of divisions indicated in Equation (4.3), and we assume that at each step i we can find integers x_i and y_i that satisfy $r_i = ax_i + by_i$. We end up with the following sequence.

$$\begin{aligned} a &= q_1 b + r_1 & r_1 &= ax_1 + by_1 \\ b &= q_2 r_1 + r_2 & r_2 &= ax_2 + by_2 \\ r_1 &= q_3 r_2 + r_3 & r_3 &= ax_3 + by_3 \\ &\vdots & &\vdots \\ r_{n-2} &= q_n r_{n-1} + r_n & r_n &= ax_n + by_n \\ r_{n-1} &= q_{n+1} r_n + 0 \end{aligned}$$

Now, observe that we can rearrange terms to write

$$r_i = r_{i-2} - r_{i-1} q_i \tag{4.8}$$

Also, in rows $i - 1$ and $i - 2$, we find the values

$$r_{i-2} = ax_{i-2} + by_{i-2} \quad \text{and} \quad r_{i-1} = ax_{i-1} + by_{i-1}$$

Substituting into Equation (4.8), we have

$$\begin{aligned} r_i &= (ax_{i-2} + by_{i-2}) - (ax_{i-1} + by_{i-1})q_i \\ &= a(x_{i-2} - q_i x_{i-1}) + b(y_{i-2} - q_i y_{i-1}) \end{aligned}$$

But we have already assumed that $r_i = ax_i + by_i$. Therefore,

$$x_i = x_{i-2} - q_i x_{i-1} \quad \text{and} \quad y_i = y_{i-2} - q_i y_{i-1}$$

We now summarize the calculations:

Extended Euclidean Algorithm			
Calculate	**Which satisfies**	**Calculate**	**Which satisfies**
$r_{-1} = a$		$x_{-1} = 1; y_{-1} = 0$	$a = ax_{-1} + by_{-1}$
$r_0 = b$		$x_0 = 0; y_0 = 1$	$b = ax_0 + by_0$
$r_1 = a \bmod b$ $q_1 = \lfloor a/b \rfloor$	$a = q_1 b + r_1$	$x_1 = x_{-1} - q_1 x_0 = 1$ $y_1 = y_{-1} - q_1 y_0 = -q_1$	$r_1 = ax_1 + by_1$
$r_2 = b \bmod r_1$ $q_2 = \lfloor b/r_1 \rfloor$	$b = q_2 r_1 + r_2$	$x_2 = x_0 - q_2 x_1$ $y_2 = y_0 - q_2 y_1$	$r_2 = ax_2 + by_2$
$r_3 = r_1 \bmod r_2$ $q_3 = \lfloor r_1/r_2 \rfloor$	$r_1 = q_3 r_2 + r_3$	$x_3 = x_1 - q_3 x_2$ $y_3 = y_1 - q_3 y_2$	$r_3 = ax_3 + by_3$
• • •	• • •	• • •	• • •
$r_n = r_{n-2} \bmod r_{n-1}$ $q_n = \lfloor r_{n-2}/r_{n-3} \rfloor$	$r_{n-2} = q_n r_{n-1} + r_n$	$x_n = x_{n-2} - q_n x_{n-1}$ $y_n = y_{n-2} - q_n y_{n-1}$	$r_n = ax_n + by_n$
$r_{n+1} = r_{n-1} \bmod r_n = 0$ $q_{n+1} = \lfloor r_{n-1}/r_{n-2} \rfloor$	$r_{n-1} = q_{n+1} r_n + 0$		$d = \gcd(a, b) = r_n$ $x = x_n; y = y_n$

The Modulus

If a is an integer and n is a positive integer, we define $a \bmod n$ to be the remainder when a is divided by n. The integer n is called the **modulus**. Thus, for any integer a, we can rewrite Equation (4.1) as follows:

$$a = qn + r \qquad 0 \leq r < n;\ q = \lfloor a/n \rfloor$$

$$a = \lfloor a/n \rfloor \times n + (a \bmod n)$$

$$11 \bmod 7 = 4; \qquad -11 \bmod 7 = 3$$

Two integers a and b are said to be **congruent modulo** n, if $(a \bmod n) = (b \bmod n)$. This is written as $a \equiv b \pmod{n}$.[2]

Properties of Congruences

Congruences have the following properties:

1. $a \equiv b \pmod{n}$ if $n|(a - b)$.
2. $a \equiv b \pmod{n}$ implies $b \equiv a \pmod{n}$.
3. $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$ imply $a \equiv c \pmod{n}$.

1. $n|(a-b)$

$(a-b)=kn$ for some k

$a=b+kn$

Take modulo both side

$(a \bmod n) = (b \bmod n)$ so $a \equiv b \pmod n$

2. $n|(a-b)$ then $n|(b-a)$ $b \equiv a \pmod n$

2. $a=b+k1n$ $b= c+k2n$ then $a=c+(k1+k2)n$

$a=c+Kn$

Then $a \equiv c \pmod n$

Modular Arithmetic

known as modular arithmetic.

Modular arithmetic exhibits the following properties:

1. $[(a \bmod n) + (b \bmod n)] \bmod n = (a + b) \bmod n$
2. $[(a \bmod n) - (b \bmod n)] \bmod n = (a - b) \bmod n$
3. $[(a \bmod n) \times (b \bmod n)] \bmod n = (a \times b) \bmod n$

We demonstrate the first property. Define $(a \bmod n) = r_a$ and $(b \bmod n) = r_b$. Then we can write $a = r_a + jn$ for some integer j and $b = r_b + kn$ for some integer k. Then

$$\begin{aligned}(a + b) \bmod n &= (r_a + jn + r_b + kn) \bmod n \\ &= (r_a + r_b + (k + j)n) \bmod n \\ &= (r_a + r_b) \bmod n \\ &= [(a \bmod n) + (b \bmod n)] \bmod n\end{aligned}$$

Example

$$11 \bmod 8 = 3; 15 \bmod 8 = 7$$

$$[(11 \bmod 8) + (15 \bmod 8)] \bmod 8 = 10 \bmod 8 = 2$$

$$(11 + 15) \bmod 8 = 26 \bmod 8 = 2$$

$$[(11 \bmod 8) - (15 \bmod 8)] \bmod 8 = -4 \bmod 8 = 4$$

$$(11 - 15) \bmod 8 = -4 \bmod 8 = 4$$

$$[(11 \bmod 8) \times (15 \bmod 8)] \bmod 8 = 21 \bmod 8 = 5$$

$$(11 \times 15) \bmod 8 = 165 \bmod 8 = 5$$

Table 4.2 Arithmetic Modulo 8

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	0
2	2	3	4	5	6	7	0	1
3	3	4	5	6	7	0	1	2
4	4	5	6	7	0	1	2	3
5	5	6	7	0	1	2	3	4
6	6	7	0	1	2	3	4	5
7	7	0	1	2	3	4	5	6

(a) Addition modulo 8

×	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

(b) Multiplication modulo 8

w	$-w$	w^{-1}
0	0	—
1	7	1
2	6	—
3	5	3
4	4	—
5	3	5
6	2	—
7	1	7

Additive and multiplicative inverses modulo 8

Properties of Modular Arithmetic

Define the set Z_n as the set of nonnegative integers less than n:

$$Z_n = \{0, 1, \ldots, (n-1)\}$$

This is referred to as the **set of residues**, or **residue classes** (mod n).

$$[r] = \{a: a \text{ is an integer}, a = r \pmod{n}\}$$

The residue classes (mod 4) are

$$[0] = \{\ldots, -16, -12, -8, -4, 0, 4, 8, 12, 16, \ldots\}$$
$$[1] = \{\ldots, -15, -11, -7, -3, 1, 5, 9, 13, 17, \ldots\}$$
$$[2] = \{\ldots, -14, -10, -6, -2, 2, 6, 10, 14, 18, \ldots\}$$
$$[3] = \{\ldots, -13, -9, -5, -1, 3, 7, 11, 15, 19, \ldots\}$$

Residue Class

- The congruence relation $a \equiv b \pmod{n}$ on the set of integers I separates the integers into n equivalence classes, $[0]_n$, $[1]_n$, $[2]_n$, ... ,$[n-1]_n$, called ***residue classes modulo n***.

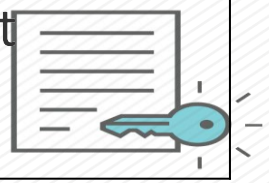
- Each equivalence class $[r]_n$ consists of all integers congruent to r where r is one of the integers 0, 1, 2, ... ,n-1.

- These n integers 0, 1, 2, ... ,n-1 are called the class representatives.

- Thus equivalence class $[3]_n$ consists of all integers congruent to 3 mod n where the integer 3 is the class representative.

Modular Arithmetic in Z_n

Property	Expression
Commutative laws	$(w + x) \bmod n = (x + w) \bmod n$ $(w \times x) \bmod n = (x \times w) \bmod n$
Associative laws	$[(w + x) + y] \bmod n = [w + (x + y)] \bmod n$ $[(w \times x) \times y] \bmod n = [w \times (x \times y)] \bmod n$
Distributive laws	$[w + (x + y)] \bmod n = [(w \times x) + (w \times y)] \bmod n$ $[w + (x \times y)] \bmod n = [(w + x) \times (w + y)] \bmod n$
Identities	$(0 + w) \bmod n = w \bmod n$ $(1 + w) \bmod n = w \bmod n$
Additive inverse ($-w$)	For each $w \in Z_n$, there exists a z such that $w + z \equiv 0 \bmod n$



Modular Symbol

- We say a is congruent to b modulo m if m| (a - b) and write a ≡ b(m).
- 3 ≡ 25(11); 5 ≡ -7(12)
- a(m) is the “remainder” of a when divided by m.
- b = aq + r is same as b ≡ r(a).
- If a ≡ b(m),then b ≡ a(m) and a ≡ bx(m) for any x ∈ $\mathbb{Z}$.
- If a ≡ b(m) and c ≡ d(m), then ax + cy ≡ bx

Extended Euclid Algorithm

Extended Euclidean algorithm also finds integer coefficients x and y such that: ax + by = gcd(a, b)

$$
\begin{aligned}
r_0 &= a & r_1 &= b \\
s_0 &= 1 & s_1 &= 0 \\
t_0 &= 0 & t_1 &= 1 \\
&\vdots & &\vdots \\
r_{i+1} &= r_{i-1} - q_i r_i & \text{and } 0 \le r_{i+1} &< |r_i| \qquad (\text{this defines } q_i) \\
s_{i+1} &= s_{i-1} - q_i s_i \\
t_{i+1} &= t_{i-1} - q_i t_i \\
&\vdots
\end{aligned}
$$

ar to Euclidean ,

Where s and t are called as Bezout's co-efficients.

Extended Euclid Algorithm

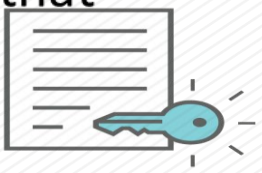
The computation also stops when $r_{k+1} = 0$ and gives

- r_k is the greatest common divisor of the input $a = r_0$ and $b = r_1$.
- The Bézout coefficients are s_k and t_k, that is $\gcd(a, b) = r_k = a*s_k + b*t_k$
- The quotients of *a* and *b* by their greatest common divisor are given by $s_{k+1} = \pm b/\gcd(a, b)$ and $t_{k+1} = \pm a/\gcd(a, b)$

Groups

A **group** G, denoted by $\{G, \cdot\}$ is a set of elements with a binary operation (.), that associates to each ordered pair (a, b) of elements in G an element $(a \cdot b)$ in G, such that the following axioms are obeyed:

- **(A1) Closure**: If a & $b \in G$, then $a \cdot b \in G$.
- **(A2) Associative**: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, $\forall\ a, b, c \in G$.
- **(A3) Identity element**: $\exists\ e \in G$, such that
$$a \cdot e = e \cdot a = a, \quad \forall a \in G.$$
- **(A4) Inverse element**: $\forall\ a \in G$ there is an element $a' \in G$ such that
$$a \cdot a' = a' \cdot a = e.$$



Abelian Groups

A group is said to be **abelian** if it satisfies the following additional condition:

- **(A5) Commutative**: $a \cdot b = b \cdot a$ for all a, b in G.

Example :

- $\mathbb{Z}$ under addition ,
- $\mathbb{R}$ (except 0) under multiplication, are abelian groups.

NOTE :

- If G has a finite no. of elements, it is referred to as a **finite group**, otherwise, **infinite group**.
- The **order** of the group is equal to the no. of elements in the group.

Cyclic Groups

A group G is **cyclic** if every element of G is a power a^k (k is an integer) of a fixed element $a \in G$.

- The element a is said to **generate** the group G, or to be a **generator** of G.
- A cyclic group is always abelian, and may be finite or infinite.

Example :

- $\mathbb{Z}$ under addition is an infinite cyclic group with 1 as generator.

Rings

A **ring** R, sometimes denoted by $\{R, +, \times\}$, is a set of elements with two binary operations, called *addition* and *multiplication*, such that for all $a, b, c \in R$ the following axioms are obeyed:

- **(A1-A5)** i.e. R is an abelian group with respect to addition.
- **(M1) Closure under multiplication**: If a & $b \in R$, then $ab \in R$.
- **(M2) Associativity of multiplication**: $a(bc) = (ab)c \; \forall \; a, b, c \in R$.
- **(M3) Distributive laws**: $a(b + c) = ab + ac \; \forall \; a, b, c \in R$.
  $(a + b)c = ac + bc \; \forall \; a, b, c \in R$.

Example :

- The set of all n-square matrices over the real numbers is a ring.

Commutative Rings

A ring is said to be **commutative** if it satisfies the following additional condition:

- **(M4) Commutativity of multiplication**: $ab = ba \; \forall \; a, b \in R$.

Example :

- Let S be the set of even integers. S is a commutative ring.

Integral Domain

An **integral domain**, which is a commutative ring that obeys the following axioms:

- (**M5) Multiplicative identity**: There is an element 1 in R such that
$$a1 = 1a = a\ ,\ \forall\ a \in R.$$

- **(M6) No zero divisors**: If $a, b \in R$ & $ab = 0$, then either $a = 0$ or $b = 0$

Example :

- $\mathbb{Z}$ is an integral domain.

Fields

A **field** F, denoted by $\{F, +, \times\}$, is a set of elements with two binary operations, called *addition* and *multiplication*, such that $\forall\ a, b, c \in F$ the following axioms are obeyed:

- **(A1-M6) i.e.** F is an integral domain.

- **(M7) Multiplicative inverse**: $\forall\ a \in F$ (except 0), $\exists\ a^{-1} \in F$ such that

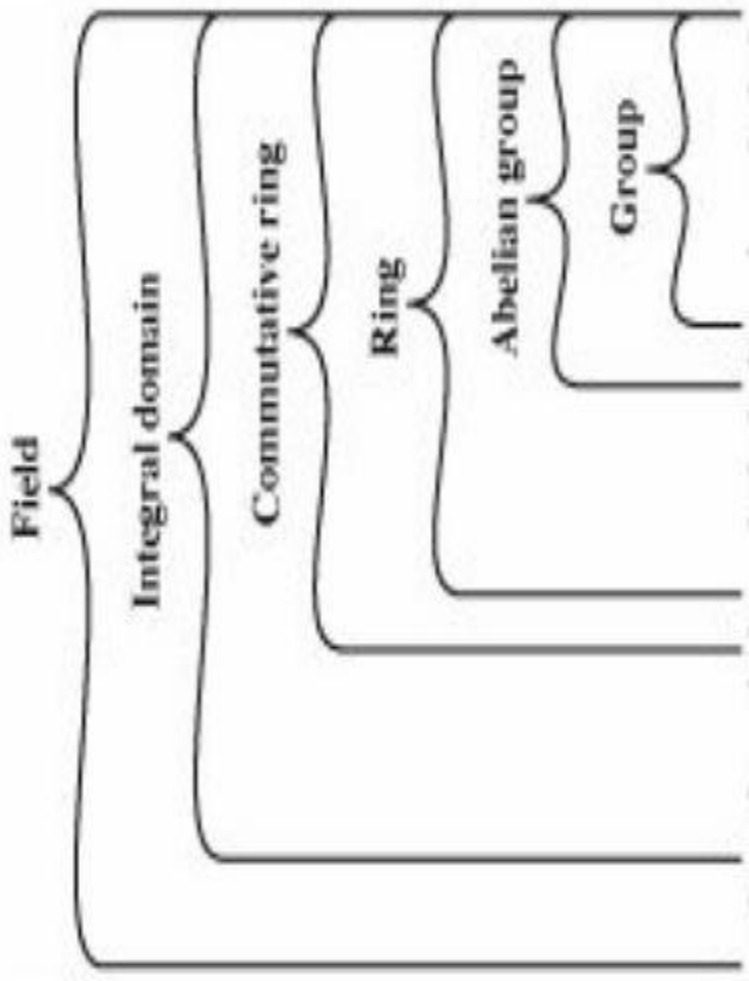
$$aa^{-1} = (a^{-1})a = 1.$$

Example :

- $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are Fields.

Note : $\mathbb{Z}$ is not a field, because not every integer has a multiplicative inverse.

Group, Ring and Field



(A1) Closure Under Addition
(A2) Associativity Under Addition
(A3) Additive Identity
(A4) Additive Inverse

(A5) Commutative under Addition

(M1) Closure Under Multiplication
(M2) Associativity Under Multiplication
(M3) Distributive Laws

(M4) Commutativity of Multiplication

(M5) Multiplicative Identity
(M6) No Zero Division

(M7) Multiplicative Inverse

Modular Arithmetic

Given any positive integer *n* and any non-negative integer *a*, if we divide *a* by *n*, we get integers *q* and *r* be quotient and remainder respectively that obey the following relationship:

$$a = qn + r \quad 0 \leq r < n;\ q = \lfloor a/n \rfloor$$

Example:

$a = 11;$	$n = 7;$	$11 = 1 \times 7 + 4;$	$r = 4$	$q = 1$
$a = -11;$	$n = 7;$	$-11 = (-2) \times 7 + 3;$	$r = 3$	$q = -2$

If *a* is an integer and *n* is a positive integer, we define *a* mod *n* to be the remainder when *a* is divided by *n*. The integer *n* is called the **modulus**. Thus, for any integer *a*, we can always write:

$$a = \lfloor a/n \rfloor \times n + (a \bmod n)$$

$11 \bmod 7 = 4;$	$-11 \bmod 7 = 3$

Modular Arithmetic

Two integers a and b are said to be **congruent modulo *n***,

if (a mod n) = (b mod n).

This is written as

$$a \equiv b \pmod{n}.$$

Example :

- $73 \equiv 4 \pmod{23}$
- $21 \equiv -9 \pmod{10}$

Modular Arithmetic

Divisors

We say that a nonzero b divides a if $a = mb$ for some m, where a, b, and m are integers. And denoted as a | b.

The following relations hold:

- If $a \mid 1$, then $a = \pm 1$.
- If $a \mid b$ and $b \mid a$, then $a = \pm b$.
- Any $b \neq 0$ divides 0.
- If $b \mid g$ and $b \mid h$, then $b \mid (mg + nh)$ for arbitrary integers m and n.

Modular Arithmetic

Congruences have the following properties:

1. $a \equiv b \bmod n$, if $n \mid (a-b)$
2. $a \equiv b \bmod n$, imples $b \equiv a \bmod n$
3. $a \equiv b \bmod n$ and $b \equiv c \bmod n$, then $a \equiv c \bmod n$

Residue Class

Example. The residue classes of integers mod 4 are:

$[0]_4$ = { ... , -16, -12, -8, -4, 0, 4, 8, 12, 16, ... }
$[1]_4$ = { ... , -15, -11, -7, -3, 1, 5, 9, 13, 17, ... }
$[2]_4$ = { ... , -14, -10, -6, -2, 2, 6, 10,14, 18, ... }
$[3]_4$ = { ... , -13, -9, -5, -1, 3, 7, 11, 15, 19, ... }

Residue Class

Complete Residue system

Any set of integers $\{a_1, a_2, ..., a_n\}$ representing all the residue classes (mod n) is called a *complete residue system (mod n).* The simplest complete residue system is 0, 1, 2, ..., n-1.

Reduced residue system

Any set of $\varphi(n)$ integers that are relatively prime to n and that are mutually incongruent modulo n, where $\varphi(n)$ denotes *Euler's Totient Function*, is called a **reduced residue system modulo *n***.

Two Types of Fields

- Prime Field , Which can be represented as , $GF(P) = F_P = Z_P$

- Polynomial Fields : represented as $GF(P^n)$

Galios Field GF(p^n)

Polynomials in which the coefficients are elements of some field F and of the degree less than “n” are said to form a polynomial ring in $GF(P)^n$, Where P is a prime number. It can be referred as **Galios field** as well.

Normally we work on field where p = 2 , which is given a $GF(2)^n$ Arithmatic over the polynomial field is similar to modular arithmatic in Zp, Where Prime number p is being replaced by an **irreducible polynomial**.

Modular operation happens with the Irreducible Polynomial that defines the field.

Modular Arithmetic in $GF(2)^n$

A polynomial f(x) over a field F is called irreducible if and only if f(x) cannot be expressed as a product of two polynomials, both over F, and both of degree lower than that of f(x). By analogy to integers, an

irreducible polynomial is also called a **prime polynomial**.

Modular Arithmatic : m(x) is the irreducible polynomial

1. f(x) mod m(x) + g(x) mod m(x) = (f(x) + g(x)) mod m(x)
2. f(x) mod m(x) * g(x) mod m(x) = (f(x) * g(x)) mod m(x)
3. f(x) mod m(x) / g(x) mod m(x) = (f(x) * g’(x)) mod m(x)

Construction of Field F_p

- Construction of Field F_p ivvolves just selecting a Prime “P”. Elements in the Field will be [0…. P-1].
- Ex : F_{11} = [0,1,2,3,4,5,6,7,8,9,10]
- We can enumerate all the elements of the field with generator g.
- F11 can be enumerated with generator 2 as, $2^0, 2^1 \ldots, 2^9$.

Construction of Field GF(P^n)

Construction of GF(P^n) can be done in two ways :

1 . Select a Prime P and degree n , Construct the field with all the polynomials of degree less than n, with co-efficients in the field F_P.

2. Select a Prime P and irreducible polynomial of degree n , Construct the field with all the polynomials of degree less than n, with co-efficients in the field F_P. The polynomial f(x) = “x” can be used as generator.

Construction of Field GF(P^n)

Eg : Construction of GF(2^3)

Constant polynomials :

000 : 0

001 : 1

Other Polynomials :

010 : x

011 : x+1

100 : x^2

101 : $x^2 + 1$

110: $x^2 + x$

111 : $x^2 + x + 1$

Enumeration of GF(2^3) with irreducible polynomial x^3+x+1

Generator = x , x3+x+1 = 0, So we have $x^3 = x+1$.

now we can enumerator all the polynomials as x^0, x, x^2,x^3 …..x^7

eg : $x^4 = x^3 . x = (x+1)x = x^2+x$

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3. Wikipedia

Any Queries ?

Thank You…