MATRIX METHODS

An Introduction

Second Edition

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Preface

The application of matrices to other disciplines has exploded since the publication of the first edition. It is now common to find presentations in economics, statistics, operations research, numerical analysis, probability, and differential equations—just to mention a few—peppered with matrix notation and matrix manipulations. Furthermore, advances in computing power and algorithms for using that power efficiently have drastically changed the methodologies associated with matrix operations. New techniques for locating eigenvalues, based on iterative procedures, have supplanted algebraic algorithms. Determinants have lost favor as a primary tool for matrix inversion. The factorization of matrices into more elementary products, whether triangular or orthogonal, has become popular. New tests for determining definiteness have been developed and, perhaps most important of all, elementary row operations now reign supreme as the most basic of all matrix operations.

The purpose of this new edition is to keep pace with these changes. In doing so, however, I hope to retain all the qualities of the first edition that made it useful. This edition, like its predecessor, is still a text for students rather than for teachers. It also is still a book on methodology rather than theory. Proofs of theorems are given in the main body of the text only if they are simple and revealing.

Perhaps the biggest change to this edition, besides the new material, is the number of exercises at the end of each section. These have been greatly expanded. Many are routine, allowing readers to test themselves on their mastery of the material presented in each section. Others expand upon that material, and still others explore applications of the material to other fields and disciplines.

The only prerequisite for most of the material in this book is a firm understanding of high school algebra. This is certainly the case for Chapters 1–6. A knowledge of calculus is needed for Chapter 8 and will be helpful for some parts of Chapters 7 and 9, although it is not essential there.

As in the first edition, this book is divided into two distinct parts. The first six chapters are elementary in nature and cover the matrix methodologies used most in non-scientific applications. Chapters 7–10 develop subject matter of primary importance to engineers and scientists. Chapters 1–6, augmented by the first few sections of Chapters 7 and 10, provide enough material for a full, one semester course. Other material can be added at the instructor's discretion if certain sections in the beginning chapters are either omitted or assigned as outside reading.

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Chapter 1

Matrices

1.1 Basic Concepts

Definition 1. A *matrix* is a rectangular array of elements arranged in horizontal rows and vertical columns. Thus,

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 0 & -1 \end{bmatrix}, \tag{1}$$

$$\begin{bmatrix} 4 & 1 & 1 \\ 3 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}, \tag{2}$$

and

$$\begin{bmatrix} \sqrt{2} \\ \pi \\ 19.5 \end{bmatrix} \tag{3}$$

are all examples of a matrix.

The matrix given in (1) has two rows and three columns; it is said to have *order* (or size) 2×3 (read two by three). By convention, the row index is always given first. The matrix in (2) has order 3×3, while that in (3) has order 3×1. The entries of a matrix are called *elements*.

In general, a matrix **A** (matrices will always be designated by uppercase boldface letters) of order $p \times n$ is given by

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{p1} & a_{p2} & a_{p3} & \cdots & a_{pn} \end{bmatrix}, \tag{4}$$

which is often abbreviated to $[a_{ij}]_{p \times n}$ or just $[a_{ij}]$. In this notation, a_{ij} represents the general element of the matrix and appears in the ith row and the jth column. The subscript i, which represents the row, can have any value 1 through p, while the subscript j, which represents the column, runs 1 through n. Thus, if $i = 2$ and $j = 3$, a_{ij} becomes a_{23} and designates the element in the second row and third column. If $i = 1$ and $j = 5$, a_{ij} becomes a_{15} and signifies the element in the first row, fifth column. Note again that the row index is always given before the column index.

Any element having its row index equal to its column index is a *diagonal element*. Thus, the diagonal elements of a matrix are the elements in the 1–1 position, 2–2 position, 3–3 position, and so on, for as many elements of this type that exist. Matrix (1) has 1 and 0 as its diagonal elements, while matrix (2) has 4, 2, and 2 as its diagonal elements.

If the matrix has as many rows as columns, $p = n$, it is called a *square matrix*; in general it is written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}. \tag{5}$$

In this case, the elements $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ lie on and form the *main* (or principal) *diagonal*.

It should be noted that the elements of a matrix need not be numbers; they can be, and quite often arise physically as, functions, operators or, as we shall see later, matrices themselves. Hence,

$$\begin{bmatrix} \int_0^1 (t^2+1)\,dt & t^2 & \sqrt{3t} & 2 \end{bmatrix}, \qquad \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix},$$

and

$$\begin{bmatrix} x^2 & x \\ e^x & \dfrac{d}{dx}\ln x \\ 5 & x+2 \end{bmatrix}$$

are good examples of matrices. Finally, it must be noted that a matrix is an entity unto itself; it is not a number. If the reader is familiar with determinants, he will undoubtedly recognize the similarity in form between the two. *Warning*: the similarity ends there. Whereas a determinant can be evaluated to yield a number, a matrix cannot. A matrix is a rectangular array, period.

Problems 1.1

(1) Determine the orders of the following matrices:

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & -2 & 4 & 7 \\ 2 & 5 & -6 & 5 & 7 \\ 0 & 3 & 1 & 2 & 0 \\ -3 & -5 & 2 & 2 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 3 & 2 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & -7 & 8 \\ 10 & 11 & 12 & 12 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 3 & t & t^2 & 0 \\ t-2 & t^4 & 6t & 5 \\ t+2 & 3t & 1 & 2 \\ 2t-3 & -5t^2 & 2t^5 & 3t^2 \end{bmatrix},$$

$$\mathbf{E} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{2}{3} & \frac{3}{5} & -\frac{5}{6} \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 1 \\ 5 \\ 10 \\ 0 \\ -4 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \sqrt{313} & -505 \\ 2\pi & 18 \\ 46.3 & 1.043 \\ 2\sqrt{5} & -\sqrt{5} \end{bmatrix},$$

$$\mathbf{H} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{J} = [1 \quad 5 \quad -30].$$

(2) Find, if they exist, the elements in the 1–3 and the 2–1 positions for each of the matrices defined in Problem 1.

(3) Find, if they exist, $a_{23}, a_{32}, b_{31}, b_{32}, c_{11}, d_{22}, e_{13}, g_{22}, g_{23}$, and h_{32} for the matrices defined in Problem 1.

(4) Construct the 2×2 matrix $\mathbf{A}$ having $a_{ij} = (-1)^{i+j}$.

(5) Construct the 3×3 matrix $\mathbf{A}$ having $a_{ij} = i/j$.

(6) Construct the $n \times n$ matrix $\mathbf{B}$ having $b_{ij} = n - i - j$. What will this matrix be when specialized to the 3×3 case?

(7) Construct the 2×4 matrix $\mathbf{C}$ having

$$c_{ij} = \begin{cases} i & \text{when } i = 1, \\ j & \text{when } i = 2. \end{cases}$$

(8) Construct the 3×4 matrix $\mathbf{D}$ having

$$d_{ij} = \begin{cases} i + j & \text{when } i > j, \\ 0 & \text{when } i = j, \\ i - j & \text{when } i < j. \end{cases}$$

(9) Express the following times as matrices: (a) A quarter after nine in the morning. (b) Noon. (c) One thirty in the afternoon. (d) A quarter after nine in the evening.

(10) Express the following dates as matrices:

(a) July 4, 1776
(b) December 7, 1941
(c) April 23, 1809
(d) October 31, 1688

(11) A gasoline station currently has in inventory 950 gallons of regular unleaded gasoline, 1253 gallons of premium, and 98 gallons of super. Express this inventory as a matrix.

(12) Store 1 of a three store chain has 3 refrigerators, 5 stoves, 3 washing machines, and 4 dryers in stock. Store 2 has in stock no refrigerators, 2 stoves, 9 washing machines, and 5 dryers, while store 3 has in stock 4 refrigerators, 2 stoves, and no washing machines or dryers. Present the inventory of the entire chain as a matrix.

(13) The number of damaged items delivered by the SleepTight Mattress Company from its various plants during the past year is given by the matrix

$$\begin{bmatrix} 80 & 12 & 16 \\ 50 & 40 & 16 \\ 90 & 10 & 50 \end{bmatrix}.$$

The rows pertain to its three plants in Michigan, Texas, and Utah. The columns pertain to its regular model, its firm model, and its extra-firm model, respectively. The company's goal for next year to is to reduce by 10% the number of damaged regular mattresses shipped by each plant, to reduce by 20% the number of damaged firm mattresses shipped by its Texas plant, to reduce by 30% the number of damaged extra-firm mattresses shipped by its Utah plant, and to keep all other entries the same as last year. What will next year's matrix be if all goals are realized?

(14) A person purchased 100 shares of AT&T at \$27 per share, 150 shares of Exxon at \$45 per share, 50 shares of IBM at \$116 per share, and 500 shares of PanAm at \$2 per share. The current price of each stock is \$29, \$41, \$116, and \$3, respectively. Represent in a matrix all the relevant information regarding this person's portfolio.

(15) On January 1, a person buys three certificates of deposit from different institutions, all maturing in one year. The first is for \$1000 at 7%, the second is for \$2000 at 7.5%, and the third is for \$3000 at 7.25%. All interest rates are effective on an annual basis.

(a) Represent in a matrix all the relevant information regarding this person's holdings.
(b) What will the matrix be one year later if each certificate of deposit is renewed for the current face amount and accrued interest at rates one half a percent higher than the present?

(16) **(Markov Chains)** A finite Markov chain is a set of objects, a set of consecutive time periods, and a finite set of different states such that

(i) during any given time period, each object is in only one state (although different objects can be in different states), and
(ii) the probability that an object will move from one state to another state (or remain in the same state) over a time period depends only on the beginning and ending states.

A Markov chain can be represented by a matrix $\mathbf{P} = [p_{ij}]$ where p_{ij} represents the probability of an object moving from state i to state j in one time period. Such a matrix is called a *transition matrix*.

Construct a transition matrix for the following Markov chain: Census figures show a population shift away from a large mid-western metropolitan city to its suburbs. Each year, 5% of all families living in the city move to the suburbs while during the same time period only 1% of those living in the suburbs move into the city. *Hint*: Take state 1 to represent families living in the city, state 2 to represent families living in the suburbs, and one time period to equal a year.

(17) Construct a transition matrix for the following Markov chain: Every four years, voters in a New England town elect a new mayor because a town ordinance prohibits mayors from succeeding themselves. Past data indicate that a Democratic mayor is succeeded by another Democrat 30% of the time and by a Republican 70% of the time. A Republican mayor, however, is succeeded by another Republican 60% of the time and by a Democrat 40% of the time. *Hint*: Take state 1 to represent a Republican mayor in office, state 2 to represent a Democratic mayor in office, and one time period to be four years.

(18) Construct a transition matrix for the following Markov chain: The apple harvest in New York orchards is classified as poor, average, or good. Historical data indicates that if the harvest is poor one year then there is a 40% chance of having a good harvest the next year, a 50% chance of having an average harvest, and a 10% chance of having another poor harvest. If a harvest is average one year, the chance of a poor, average, or good harvest the next year is 20%, 60%, and 20%, respectively. If a harvest is good, then the chance of a poor, average, or good harvest the next year is 25%, 65%, and 10%, respectively. *Hint*: Take state 1 to be a poor harvest, state 2 to be an average harvest, state 3 to be a good harvest, and one time period to equal one year.

(19) Construct a transition matrix for the following Markov chain. Brand X and brand Y control the majority of the soap powder market in a particular region, and each has promoted its own product extensively. As a result of past advertising campaigns, it is known that over a two year period of time 10% of brand Y customers change to brand X and 25% of all other customers change to brand X. Furthermore, 15% of brand X customers change to brand Y and 30% of all other customers change to brand Y. The major brands also lose customers to smaller competitors, with 5% of brand X customers switching to a minor brand during a two year time period and 2% of brand Y customers doing likewise. All other customers remain loyal to their past brand of soap powder. *Hint*: Take state 1 to be a brand X customer, state 2 a brand Y customer, state 3 another brand customer, and one time period to be two years.

1.2 Operations

The simplest relationship between two matrices is equality. Intuitively one feels that two matrices should be equal if their corresponding elements are equal. This is the case, providing the matrices are of the same order.

Definition 1. Two matrices $\mathbf{A} = [a_{ij}]_{p\times n}$ and $\mathbf{B} = [b_{ij}]_{p\times n}$ are equal if they have the same order and if $a_{ij} = b_{ij}$ $(i = 1, 2, 3, \ldots, p;\ j = 1, 2, 3, \ldots, n)$. Thus, the equality

$$\begin{bmatrix} 5x + 2y \\ x - 3y \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

implies that $5x + 2y = 7$ and $x - 3y = 1$.

The intuitive definition for matrix addition is also the correct one.

Definition 2. If $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ are both of order $p \times n$, then $\mathbf{A} + \mathbf{B}$ is a $p \times n$ matrix $\mathbf{C} = [c_{ij}]$ where $c_{ij} = a_{ij} + b_{ij}$ $(i = 1, 2, 3, \ldots, p;\ j = 1, 2, 3, \ldots, n)$. Thus,

$$\begin{bmatrix} 5 & 1 \\ 7 & 3 \\ -2 & -1 \end{bmatrix} + \begin{bmatrix} -6 & 3 \\ 2 & -1 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 5 + (-6) & 1 + 3 \\ 7 + 2 & 3 + (-1) \\ (-2) + 4 & (-1) + 1 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 9 & 2 \\ 2 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} t^2 & 5 \\ 3t & 0 \end{bmatrix} + \begin{bmatrix} 1 & -6 \\ t & -t \end{bmatrix} = \begin{bmatrix} t^2 + 1 & -1 \\ 4t & -t \end{bmatrix};$$

but the matrices

$$\begin{bmatrix} 5 & 0 \\ -1 & 0 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -6 & 2 \\ 1 & 1 \end{bmatrix}$$

cannot be added since they are not of the same order.

It is not difficult to show that the addition of matrices is both commutative and associative: that is, if $\mathbf{A}, \mathbf{B}, \mathbf{C}$ represent matrices of the same order, then

(A1) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$,
(A2) $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$.

We define a zero matrix $\mathbf{0}$ to be a matrix consisting of only zero elements. Zero matrices of every order exist, and when one has the same order as another matrix $\mathbf{A}$, we then have the additional property

(A3) $\mathbf{A} + \mathbf{0} = \mathbf{A}$.

Subtraction of matrices is defined in a manner analogous to addition; the orders of the matrices involved must be identical and the operation is performed elementwise.

Thus,

$$\begin{bmatrix} 5 & 1 \\ -3 & 2 \end{bmatrix} - \begin{bmatrix} 6 & -1 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -7 & 3 \end{bmatrix}.$$

Another simple operation is that of multiplying a scalar times a matrix. Intuition guides one to perform the operation elementwise, and once again intuition is correct. Thus, for example,

$$7\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ -21 & 28 \end{bmatrix} \quad \text{and} \quad t\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} t & 0 \\ 3t & 2t \end{bmatrix}.$$

Definition 3. If $\mathbf{A} = [a_{ij}]$ is a $p \times n$ matrix and if λ is a scalar, then $\lambda\mathbf{A}$ is a $p \times n$ matrix $\mathbf{B} = [b_{ij}]$ where $b_{ij} = \lambda a_{ij}$ $(i = 1, 2, 3, \ldots, p;\ j = 1, 2, 3, \ldots, n)$.

Example 1

Find $5\mathbf{A} - \frac{1}{2}\mathbf{B}$ if

$$\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 6 & -20 \\ 18 & 8 \end{bmatrix}$$

Solution.

$$\begin{aligned} 5\mathbf{A} - \tfrac{1}{2}\mathbf{B} &= 5\begin{bmatrix} 4 & 1 \\ 0 & 3 \end{bmatrix} - \tfrac{1}{2}\begin{bmatrix} 6 & -20 \\ 18 & 8 \end{bmatrix} \\ &= \begin{bmatrix} 20 & 5 \\ 0 & 15 \end{bmatrix} - \begin{bmatrix} 3 & -10 \\ 9 & 4 \end{bmatrix} = \begin{bmatrix} 17 & 15 \\ -9 & 11 \end{bmatrix}. \qquad \square \end{aligned}$$

It is not difficult to show that if λ_1 and λ_2 are scalars, and if $\mathbf{A}$ and $\mathbf{B}$ are matrices of identical order, then

(S1) $\lambda_1\mathbf{A} = \mathbf{A}\lambda_1$,
(S2) $\lambda_1(\mathbf{A} + \mathbf{B}) = \lambda_1\mathbf{A} + \lambda_1\mathbf{B}$,
(S3) $(\lambda_1 + \lambda_2)\mathbf{A} = \lambda_1\mathbf{A} + \lambda_2\mathbf{A}$,
(S4) $\lambda_1(\lambda_2\mathbf{A}) = (\lambda_1\lambda_2)\mathbf{A}$.

The reader is cautioned that there is *no* such operation as matrix division. We will, however, define a somewhat analogous operation, namely matrix inversion, in a later chapter.

Problems 1.2

In Problems 1 through 26, let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} -1 & 0 \\ 3 & -3 \end{bmatrix},$$

$$\mathbf{D} = \begin{bmatrix} 3 & 1 \\ -1 & 2 \\ 3 & -2 \\ 2 & 6 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} -2 & 2 \\ 0 & -2 \\ 5 & -3 \\ 5 & 1 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \\ 2 & 2 \end{bmatrix}.$$

(1) Find $2\mathbf{A}$.

(2) Find $-5\mathbf{A}$.

(3) Find $3\mathbf{D}$.

(4) Find $10\mathbf{E}$.

(5) Find $-\mathbf{F}$.

(6) Find $\mathbf{A} + \mathbf{B}$.

(7) Find $\mathbf{C} + \mathbf{A}$.

(8) Find $\mathbf{D} + \mathbf{E}$.

(9) Find $\mathbf{D} + \mathbf{F}$.

(10) Find $\mathbf{A} + \mathbf{D}$.

(11) Find $\mathbf{A} - \mathbf{B}$.

(12) Find $\mathbf{C} - \mathbf{A}$.

(13) Find $\mathbf{D} - \mathbf{E}$.

(14) Find $\mathbf{D} - \mathbf{F}$.

(15) Find $2\mathbf{A} + 3\mathbf{B}$.

(16) Find $3\mathbf{A} - 2\mathbf{C}$.

(17) Find $0.1\mathbf{A} + 0.2\mathbf{C}$.

(18) Find $-2\mathbf{E} + \mathbf{F}$.

(19) Find $\mathbf{X}$ if $\mathbf{A} + \mathbf{X} = \mathbf{B}$.

(20) Find $\mathbf{Y}$ if $2\mathbf{B} + \mathbf{Y} = \mathbf{C}$.

(21) Find $\mathbf{X}$ if $3\mathbf{D} - \mathbf{X} = \mathbf{E}$.

(22) Find $\mathbf{Y}$ if $\mathbf{E} - 2\mathbf{Y} = \mathbf{F}$.

(23) Find $\mathbf{R}$ if $4\mathbf{A} + 5\mathbf{R} = 10\mathbf{C}$.

(24) Find $\mathbf{S}$ if $3\mathbf{F} - 2\mathbf{S} = \mathbf{D}$.

(25) Verify directly that $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$.

(26) Verify directly that $\lambda(\mathbf{A} + \mathbf{B}) = \lambda\mathbf{A} + \lambda\mathbf{B}$.

(27) Find $6\mathbf{A} - \theta\mathbf{B}$ if

$$\mathbf{A} = \begin{bmatrix} \theta^2 & 2\theta - 1 \\ 4 & 1/\theta \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} \theta^2 - 1 & 6 \\ 3/\theta & \theta^3 + 2\theta + 1 \end{bmatrix}.$$

(28) Prove Property (A1).

(29) Prove Property (A3).

(30) Prove Property (S2).

(31) Prove Property (S3).

(32) (a) Mr. Jones owns 200 shares of IBM and 150 shares of AT&T. Determine a portfolio matrix that reflects Mr. Jones' holdings.

(b) Over the next year, Mr. Jones triples his holdings in each company. What is his new portfolio matrix?
(c) The following year Mr. Jones lists changes in his portfolio as $[-50 \quad 100]$. What is his new portfolio matrix?

(33) The inventory of an appliance store can be given by a 1×4 matrix in which the first entry represents the number of television sets, the second entry the number of air conditioners, the third entry the number of refrigerators, and the fourth entry the number of dishwashers.

(a) Determine the inventory given on January 1 by $[15 \quad 2 \quad 8 \quad 6]$.
(b) January sales are given by $[4 \quad 0 \quad 2 \quad 3]$. What is the inventory matrix on February 1?
(c) February sales are given by $[5 \quad 0 \quad 3 \quad 3]$, and new stock added in February is given by $[3 \quad 2 \quad 7 \quad 8]$. What is the inventory matrix on March 1?

(34) The daily gasoline supply of a local service station is given by a 1×3 matrix in which the first entry represents gallons of regular, the second entry gallons of premium, and the third entry gallons of super.

(a) Determine the supply of gasoline at the close of business on Monday given by $[14{,}000 \quad 8{,}000 \quad 6{,}000]$.
(b) Tuesday's sales are given by $[3{,}500 \quad 2{,}000 \quad 1{,}500]$. What is the inventory matrix at day's end?
(c) Wednesday's sales are given by $[5{,}000 \quad 1{,}500 \quad 1{,}200]$. In addition, the station received a delivery of 30,000 gallons of regular, 10,000 gallons of premium, but no super. What is the inventory at day's end?

(35) On a recent shopping trip Mary purchased 6 oranges, a dozen grapefruits, 8 apples, and 3 lemons. John purchased 9 oranges, 2 grapefruits, and 6 apples. Express each of their purchases as 1×4 matrices. What is the physical significance of the sum of these matrices?

1.3 Matrix Multiplication

Matrix multiplication is the first operation we encounter where our intuition fails. First, two matrices are *not* multiplied together elementwise. Secondly, it is not always possible to multiply matrices of the same order while it is possible to multiply certain matrices of different orders. Thirdly, if $\mathbf{A}$ and $\mathbf{B}$ are two matrices for which multiplication is defined, it is generally not the case that $\mathbf{AB} = \mathbf{BA}$; that is, *matrix multiplication is not a commutative* *operation*. There are other properties of matrix multiplication, besides the three mentioned that defy our intuition, and we shall illustrate them shortly. We begin by determining which matrices can be multiplied.

Rule 1. The product of two matrices **AB** is defined if the number of columns of **A** equals the number of rows of **B**.

Thus, if **A** and **B** are given by

$$\mathbf{A} = \begin{bmatrix} 6 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 3 & 2 & -2 & 1 \\ 4 & 1 & 1 & 0 \end{bmatrix}, \tag{6}$$

then the product **AB** is defined since **A** has three columns and **B** has three rows. The product **BA**, however, is not defined since **B** has four columns while **A** has only two rows.

When the product is written **AB**, **A** is said to *premultiply* **B** while **B** is said to *postmultiply* **A**.

Rule 2. If the product **AB** is defined, then the resultant matrix will have the same number of rows as **A** and the same number of columns as **B**.

Thus, the product **AB**, where **A** and **B** are given in (6), will have two rows and four columns since **A** has two rows and **B** has four columns.

An easy method of remembering these two rules is the following: write the orders of the matrices on paper in the sequence in which the multiplication is to be carried out; that is, if **AB** is to be found where **A** has order 2×3 and **B** has order 3×4, write

$$(2 \times 3)(3 \times 4) \tag{7}$$

If the two adjacent numbers (indicated in (7) by the curved arrow) are both equal (in the case they are both three), the multiplication is defined. The order of the product matrix is obtained by canceling the adjacent numbers and using the two remaining numbers. Thus in (7), we cancel the adjacent 3's and are left with 2×4, which in this case, is the order of **AB**.

As a further example, consider the case where **A** is 4×3 matrix while **B** is a 3×5 matrix. The product **AB** is defined since, in the notation $(4 \times 3)(3 \times 5)$, the adjacent numbers denoted by the curved arrow are equal. The product will be a 4×5 matrix. The product **BA** however is not defined since in the notation $(3 \times 5)(4 \times 3)$ the adjacent numbers are not equal. In general, one may schematically state the method as

$$(k \times n)(n \times p) = (k \times p).$$

Rule 3. If the product $\mathbf{AB} = \mathbf{C}$ is defined, where $\mathbf{C}$ is denoted by $[c_{ij}]$, then the element c_{ij} is obtained by multiplying the elements in ith row of $\mathbf{A}$ by the corresponding elements in the jth column of $\mathbf{B}$ and adding. Thus, if $\mathbf{A}$ has order $k \times n$, and $\mathbf{B}$ has order $n \times p$, and

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & & \vdots \\ c_{k1} & c_{k2} & \cdots & c_{kp} \end{bmatrix},$$

then c_{11} is obtained by multiplying the elements in the first row of $\mathbf{A}$ by the corresponding elements in the first column of $\mathbf{B}$ and adding; hence,

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1}.$$

The element c_{12} is found by multiplying the elements in the first row of $\mathbf{A}$ by the corresponding elements in the second column of $\mathbf{B}$ and adding; hence,

$$c_{12} = a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1n}b_{n2}.$$

The element c_{kp} is obtained by multiplying the elements in the kth row of $\mathbf{A}$ by the corresponding elements in the pth column of $\mathbf{B}$ and adding; hence,

$$c_{kp} = a_{k1}b_{1p} + a_{k2}b_{2p} + \cdots + a_{kn}b_{np}.$$

Example 1

Find **AB** and **BA** if

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} -7 & -8 \\ 9 & 10 \\ 0 & -11 \end{bmatrix}.$$

Solution.

$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} -7 & -8 \\ 9 & 10 \\ 0 & -11 \end{bmatrix} \\ &= \begin{bmatrix} 1(-7) + 2(9) + 3(0) & 1(-8) + 2(10) + 3(-11) \\ 4(-7) + 5(9) + 6(0) & 4(-8) + 5(10) + 6(-11) \end{bmatrix} \\ &= \begin{bmatrix} -7 + 18 + 0 & -8 + 20 - 33 \\ -28 + 45 + 0 & -32 + 50 - 66 \end{bmatrix} = \begin{bmatrix} 11 & -21 \\ 17 & -48 \end{bmatrix}, \end{aligned}$$

$$\mathbf{BA} = \begin{bmatrix} -7 & -8 \\ 9 & 10 \\ 0 & -11 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} (-7)1 + (-8)4 & (-7)2 + (-8)5 & (-7)3 + (-8)6 \\ 9(1) + 10(4) & 9(2) + 10(5) & 9(3) + 10(6) \\ 0(1) + (-11)4 & 0(2) + (-11)5 & 0(3) + (-11)6 \end{bmatrix}$$

$$= \begin{bmatrix} -7 - 32 & -14 - 40 & -21 - 48 \\ 9 + 40 & 18 + 50 & 27 + 60 \\ 0 - 44 & 0 - 55 & 0 - 66 \end{bmatrix} = \begin{bmatrix} -39 & -54 & -69 \\ 49 & 68 & 87 \\ -44 & -55 & -66 \end{bmatrix}. \qquad \square$$

The preceding three rules can be incorporated into the following formal definition:

Definition 1. If $\mathbf{A} = [a_{ij}]$ is a $k \times n$ matrix and $\mathbf{B} = [b_{ij}]$ is an $n \times p$ matrix, then the product $\mathbf{AB}$ is defined to be a $k \times p$ matrix $\mathbf{C} = [c_{ij}]$ where $c_{ij} = \sum_{l=1}^{n} a_{il}b_{lj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$ $(i = 1, 2, \ldots, k; j = 1, 2, \ldots, p)$.

Example 2

Find **AB** if

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 3 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 3 & 1 & 5 & -1 \\ 4 & -2 & 1 & 0 \end{bmatrix}.$$

Solution.

$$\mathbf{AB} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 5 & -1 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2(3) + 1(4) & 2(1) + 1(-2) & 2(5) + 1(1) & 2(-1) + 1(0) \\ -1(3) + 0(4) & -1(1) + 0(-2) & -1(5) + 0(1) & -1(-1) + 0(0) \\ 3(3) + 1(4) & 3(1) + 1(-2) & 3(5) + 1(1) & 3(-1) + 1(0) \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 0 & 11 & -2 \\ -3 & -1 & -5 & 1 \\ 13 & 1 & 16 & -3 \end{bmatrix}.$$

Note that in this example the product **BA** is not defined. $\square$

Example 3

Find **AB** and **BA** if

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}.$$

Solution.

$$\mathbf{AB} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2(4)+1(1) & 2(0)+1(2) \\ -1(4)+3(1) & -1(0)+3(2) \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ -1 & 6 \end{bmatrix};$$

$$\mathbf{BA} = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 4(2)+0(-1) & 4(1)+0(3) \\ 1(2)+2(-1) & 1(1)+2(3) \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 0 & 7 \end{bmatrix}.$$

This, therefore, is an example where both products **AB** and **BA** are defined but unequal. □

Example 4

Find **AB** and **BA** if

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}.$$

Solution.

$$\mathbf{AB} = \begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 0 & 8 \end{bmatrix},$$

$$\mathbf{BA} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 0 & 8 \end{bmatrix}.$$

This, therefore, is an example where both products **AB** and **BA** are defined and equal. □

In general, it can be shown that matrix multiplication has the following properties:

(M1) $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ (Associative Law)
(M2) $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ (Left Distributive Law)
(M3) $(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$ (Right Distributive Law)

providing that the matrices **A**, **B**, **C** have the correct order so that the above multiplications and additions are defined. The one basic property that matrix multiplication does not possess is commutativity; that is, in general, **AB** does not equal **BA** (see Example 3). We hasten to add, however, that while matrices in general do not commute, it may very well be the case that, given two particular matrices, they do commute as can be seen from Example 4.

Commutativity is not the only property that matrix multiplication lacks. We know from our experiences with real numbers that if the product $xy = 0$, then either $x = 0$ or $y = 0$ or both are zero. Matrices do not possess this property as the following example shows:

Example 5

Find **AB** if

$$\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 3 & -4 \\ -6 & 8 \end{bmatrix}.$$

Solution.

$$\mathbf{AB} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ -6 & 8 \end{bmatrix} = \begin{bmatrix} 4(3) + 2(-6) & 4(-4) + 2(8) \\ 2(3) + 1(-6) & 2(-4) + 1(8) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, even though neither **A** nor **B** is zero, their product is zero. □

One final "unfortunate" property of matrix multiplication is that the equation **AB** = **AC** does not imply **B** = **C**.

Example 6

Find **AB** and **AC** if

$$\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix}.$$

Solution.

$$\mathbf{AB} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4(1) + 2(2) & 4(1) + 2(1) \\ 2(1) + 1(2) & 2(1) + 1(1) \end{bmatrix} = \begin{bmatrix} 8 & 6 \\ 4 & 3 \end{bmatrix};$$

$$\mathbf{AC} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 4(2) + 2(0) & 4(2) + 2(-1) \\ 2(2) + 1(0) & 2(2) + 1(-1) \end{bmatrix} = \begin{bmatrix} 8 & 6 \\ 4 & 3 \end{bmatrix}.$$

Thus, cancellation is not a valid operation in the matrix algebra. □

The reader has no doubt wondered why this seemingly complicated procedure for matrix multiplication has been introduced when the more obvious methods of multiplying matrices termwise could be used. The answer lies in systems of simultaneous linear equations. Consider the set of simultaneous linear equations given by

$$\begin{aligned} 5x - 3y + 2z &= 14, \\ x + y - 4z &= -7, \\ 7x \qquad - 3z &= 1. \end{aligned} \tag{8}$$

This system can easily be solved by the method of substitution. Matrix algebra, however, will give us an entirely new method for obtaining the solution.

Consider the matrix equation

$$\mathbf{Ax} = \mathbf{b} \tag{9}$$

where

$$\mathbf{A} = \begin{bmatrix} 5 & -3 & 2 \\ 1 & 1 & -4 \\ 7 & 0 & -3 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 14 \\ -7 \\ 1 \end{bmatrix}.$$

Here $\mathbf{A}$, called the *coefficient matrix*, is simply the matrix whose elements are the coefficients of the unknowns x, y, z in (8). (Note that we have been very careful to put all the x coefficients in the first column, all the y coefficients in the second column, and all the z coefficients in the third column. The zero in the $(3, 2)$ entry appears because the y coefficient in the third equation of system (8) is zero.) $\mathbf{x}$ and $\mathbf{b}$ are obtained in the obvious manner. One note of warning: there is a basic difference between the unknown matrix $\mathbf{x}$ in (9) and the unknown variable x. The reader should be especially careful not to confuse their respective identities.

Now using our definition of matrix multiplication, we have that

$$\begin{aligned} \mathbf{Ax} &= \begin{bmatrix} 5 & -3 & 2 \\ 1 & 1 & -4 \\ 7 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (5)(x) + (-3)(y) + (2)(z) \\ (1)(x) + (1)(y) + (-4)(z) \\ (7)(x) + (0)(y) + (-3)(z) \end{bmatrix} \\ &= \begin{bmatrix} 5x - 3y + 2z \\ x + y - 4z \\ 7x \quad -3z \end{bmatrix} = \begin{bmatrix} 14 \\ -7 \\ 1 \end{bmatrix}. \end{aligned} \tag{10}$$

Using the definition of matrix equality, we see that (10) is precisely system

(8). Thus (9) is an alternate way of representing the original system. It should come as no surprise, therefore, that by redefining the matrices **A**, **x**, **b**, appropriately, we can represent any system of simultaneous linear equations by the matrix equation $\mathbf{A}\mathbf{x} = \mathbf{b}$.

Example 7

Put the following system into matrix form:

$$\begin{aligned} x - y + z + w &= 5, \\ 2x + y - z \phantom{{}+ w} &= 4, \\ 3x + 2y \phantom{{}+ z} + 2w &= 0, \\ x - 2y + 3z + 4w &= -1. \end{aligned}$$

Solution. Define

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & 1 & -1 & 0 \\ 3 & 2 & 0 & 2 \\ 1 & -2 & 3 & 4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 4 \\ 0 \\ -1 \end{bmatrix}.$$

The original system is then equivalent to the matrix system $\mathbf{A}\mathbf{x} = \mathbf{b}$. ☐

Unfortunately, we are not yet in a position to solve systems that are in matrix form $\mathbf{A}\mathbf{x} = \mathbf{b}$. One method of solution depends upon the operation of inversion, and we must postpone a discussion of it until the inverse has been defined. For the present, however, we hope that the reader will be content with the knowledge that matrix multiplication, as we have defined it, does serve some useful purpose.

Problems 1.3

(1) The order of **A** is 2×4, the order of **B** is 4×2, the order of **C** is 4×1, the order of **D** is 1×2, and the order of **E** is 4×4. Find the orders of

(a) **AB**,	(b) **BA**,	(c) **AC**,	(d) **CA**,
(e) **CD**,	(f) **AE**,	(g) **EB**,	(h) **EA**,
(i) **ABC**,	(j) **DAE**,	(k) **EBA**,	(l) **EECD**.

In Problems 2 through 19, let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} -1 & 0 & 1 \\ 3 & -2 & 1 \end{bmatrix},$$

$$\mathbf{D} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 2 & -2 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} -2 & 2 & 1 \\ 0 & -2 & -1 \\ 1 & 0 & 1 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 0 & 1 & 2 \\ -1 & -1 & 0 \\ 1 & 2 & 3 \end{bmatrix},$$

$$\mathbf{X} = [1 \quad -2], \qquad \mathbf{Y} = [1 \quad 2 \quad 1].$$

(2) Find **AB**.
(3) Find **BA**.
(4) Find **AC**.
(5) Find **BC**.
(6) Find **CB**.
(7) Find **XA**.
(8) Find **XB**.
(9) Find **XC**.
(10) Find **AX**.
(11) Find **CD**.
(12) Find **DC**.
(13) Find **YD**.
(14) Find **YC**.
(15) Find **DX**.
(16) Find **XD**.
(17) Find **EF**.
(18) Find **FE**.
(19) Find **YF**.

(20) Find **AB** if

$$\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 3 & 9 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}.$$

Note that $\mathbf{AB} = \mathbf{0}$ but neither **A** nor **B** equals the zero matrix.

(21) Find **AB** and **CB** if

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 6 \\ 3 & -4 \end{bmatrix}.$$

Thus show that $\mathbf{AB} = \mathbf{CB}$ but $\mathbf{A} \neq \mathbf{C}$

(22) Compute the product

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

(23) Compute the product

$$\begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 1 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

(24) Compute the product

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

(25) Compute the product

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

(26) Evaluate the expression $\mathbf{A}^2 - 4\mathbf{A} - 5\mathbf{I}$ for the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

(27) Evaluate the expression $(\mathbf{A} - \mathbf{I})(\mathbf{A} + 2\mathbf{I})$ for the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix}.$$

(28) Evaluate the expression $(\mathbf{I} - \mathbf{A})(\mathbf{A}^2 - \mathbf{I})$ for the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ 3 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(29) Verify property (M1) for

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 \\ -1 & 4 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix}.$$

(30) Prove Property (M2).

(31) Prove Property (M3).

(32) Put the following system of equations into matrix form:

$$\begin{aligned} 2x + 3y &= 10, \\ 4x - 5y &= 11. \end{aligned}$$

(33) Put the following system of equations into matrix form:

$$\begin{aligned} x + z + y &= 2, \\ 3z + 2x + y &= 4, \\ y + x &= 0. \end{aligned}$$

(34) Put the following system of equations into matrix form:

$$\begin{aligned} 5x + 3y + 2z + 4w &= 5, \\ x + y + w &= 0, \\ 3x + 2y + 2z &= -3, \\ x + y + 2z + 3w &= 4. \end{aligned}$$

(35) The price schedule for a Chicago to Los Angeles flight is given by $\mathbf{P} = [200 \quad 350 \quad 500]$, where the matrix elements pertain, respectively, to coach tickets, business-class tickets, and first-class tickets. The number of tickets purchased in each category for a particular flight is given by

$$\mathbf{N} = \begin{bmatrix} 130 \\ 20 \\ 10 \end{bmatrix}.$$

Compute the products (a) **PN**, and (b) **NP**, and determine their significance.

(36) The closing prices of a person's portfolio during the past week are given by the matrix

$$\mathbf{P} = \begin{bmatrix} 40 & 40\frac{1}{2} & 40\frac{7}{8} & 41 & 41 \\ 3\frac{1}{4} & 3\frac{5}{8} & 3\frac{1}{2} & 4 & 3\frac{7}{8} \\ 10 & 9\frac{3}{4} & 10\frac{1}{8} & 10 & 9\frac{5}{8} \end{bmatrix},$$

where the columns pertain to the days of the week, Monday through Friday, and the rows pertain to the prices of Orchard Fruits, Lion Airways, and Arrow Oil. The person's holdings in each of these companies are given by the matrix $\mathbf{H} = [100 \quad 500 \quad 400]$. Compute the products (a) **HP**, and (b) **PH**, and determine their significance.

(37) The time requirements for a company to produce three products is given by the matrix

$$\mathbf{T} = \begin{bmatrix} 0.2 & 0.5 & 0.4 \\ 1.2 & 2.3 & 1.7 \\ 0.8 & 3.1 & 1.2 \end{bmatrix},$$

where the rows pertain to lamp bases, cabinets, and tables, respectively. The columns pertain to the hours of labor required for cutting the wood, assembling, and painting, respectively. The hourly wages of a carpenter to cut wood, of a craftsperson to assemble a product, and of a decorator to paint is given, respectively, by the elements of the matrix

$$\mathbf{W} = \begin{bmatrix} 10.50 \\ 14.00 \\ 12.25 \end{bmatrix}.$$

Compute the product **TW** and determine its significance.

(38) Continuing with the data given in the previous problem, assume further that the number of items on order for lamp bases, cabinets, and tables, respectively, is given by the matrix $\mathbf{O} = [1000 \quad 100 \quad 200]$. Compute the product **OTW**, and determine its significance.

(39) The results of a flu epidemic at a college campus are collected in the matrix

$$\mathbf{F} = \begin{bmatrix} 0.20 & 0.20 & 0.15 & 0.15 \\ 0.10 & 0.30 & 0.30 & 0.40 \\ 0.70 & 0.50 & 0.55 & 0.45 \end{bmatrix}.$$

The elements denote percents converted to decimals. The columns pertain to freshmen, sophomores, juniors, and seniors, respectively, while the rows represent bedridden students, infected but ambulatory students, and well students, respectively. The male–female composition of each class is given by the matrix

$$\mathbf{C} = \begin{bmatrix} 1050 & 950 \\ 1100 & 1050 \\ 360 & 500 \\ 860 & 1000 \end{bmatrix}.$$

Compute the product **FC**, and determine its significance.

1.4 Special Matrices

There are certain types of matrices that occur so frequently that it becomes advisable to discuss them separately. One such type is the *transpose*. Given a matrix **A**, the transpose of **A**, denoted by $\mathbf{A}^{\mathrm{T}}$ and read *A*-transpose, is obtained by changing all the rows of **A** into the columns of $\mathbf{A}^{\mathrm{T}}$ while preserving the order; hence, the first row of **A** becomes the first column of $\mathbf{A}^{\mathrm{T}}$, while the second row of **A** becomes the second column of $\mathbf{A}^{\mathrm{T}}$, and the last row of **A** becomes the last column of $\mathbf{A}^{\mathrm{T}}$. Thus if

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \qquad \text{then } \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

and if

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}, \qquad \text{then } \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix}.$$

Definition 1. If $\mathbf{A}$, denoted by $[a_{ij}]$ is an $n \times p$ matrix, then the transpose of $\mathbf{A}$, denoted by $\mathbf{A}^{\mathsf{T}} = [a_{ij}^{\mathsf{T}}]$ is a $p \times n$ matrix where $a_{ij}^{\mathsf{T}} = a_{ji}$.

It can be shown that the transpose possesses the following properties:

(1) $(\mathbf{A}^{\mathsf{T}})^{\mathsf{T}} = \mathbf{A}$,
(2) $(\lambda\mathbf{A})^{\mathsf{T}} = \lambda\mathbf{A}^{\mathsf{T}}$ where λ represents a scalar,
(3) $(\mathbf{A} + \mathbf{B})^{\mathsf{T}} = \mathbf{A}^{\mathsf{T}} + \mathbf{B}^{\mathsf{T}}$,
(4) $(\mathbf{A} + \mathbf{B} + \mathbf{C})^{\mathsf{T}} = \mathbf{A}^{\mathsf{T}} + \mathbf{B}^{\mathsf{T}} + \mathbf{C}^{\mathsf{T}}$,
(5) $(\mathbf{AB})^{\mathsf{T}} = \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}$,
(6) $(\mathbf{ABC})^{\mathsf{T}} = \mathbf{C}^{\mathsf{T}}\mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}$

Transposes of sums and products of more than three matrices are defined in the obvious manner. We caution the reader to be alert to the ordering of properties (5) and (6). In particular, one should be aware that the transpose of a product is not the product of the transposes but rather the *commuted* product of the transposes.

Example 1

Find $(\mathbf{AB})^{\mathsf{T}}$ and $\mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}$ if

$$\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 4 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} -1 & 2 & 1 \\ 3 & -1 & 0 \end{bmatrix}.$$

Solution.

$$\mathbf{AB} = \begin{bmatrix} -3 & 6 & 3 \\ -1 & 7 & 4 \end{bmatrix}, \qquad (\mathbf{AB})^{\mathsf{T}} = \begin{bmatrix} -3 & -1 \\ 6 & 7 \\ 3 & 4 \end{bmatrix};$$

$$\mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}} = \begin{bmatrix} -1 & 3 \\ 2 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 6 & 7 \\ 3 & 4 \end{bmatrix}.$$

Note that $(\mathbf{AB})^{\mathsf{T}} = \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}$ but $\mathbf{A}^{\mathsf{T}}\mathbf{B}^{\mathsf{T}}$ is not defined. □

A zero row in a matrix is a row containing only zeros, while a nonzero row is one that contains at least one nonzero element. A matrix is in *row-reduced form* if it satisfies four conditions:

(R1) All zero rows appear below nonzero rows when both types are present in the matrix.
(R2) The first nonzero element in any nonzero row is unity.

(R3) All elements directly below (that is, in the same column but in succeeding rows from) the first nonzero element of a nonzero row are zero.

(R4) The first nonzero element of any nonzero row appears in a later column (further to the right) than the first nonzero element in any preceding row.

Such matrices are invaluable for solving sets of simultaneous linear equations and developing efficient algorithms for performing important matrix operations. We shall have much more to say on these matters in later chapters. Here we are simply interested in recognizing when a given matrix is or is not in row-reduced form.

Example 2

Determine which of the following matrices are in row-reduced form:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -2 & 4 & 7 \\ 0 & 0 & -6 & 5 & 7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 5 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} -1 & -2 & 3 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Solution. Matrix **A** is not in row-reduced form because the first nonzero element of the second row is not unity. This violates (R2). If a_{23} had been unity instead of -6, then the matrix would be in row-reduced form. Matrix **B** is not in row-reduced form because the second row is a zero row and it appears before the third row which is a nonzero row. This violates (R1). If the second and third rows had been interchanged, then the matrix would be in row-reduced form. Matrix **C** is not in row-reduced form because the first nonzero element in row two appears in a later column, column 3, than the first nonzero element of row three. This violates (R4). If the second and third rows had been interchanged, then the matrix would be in row-reduced form. Matrix **D** is not in row-reduced form because the first nonzero element in row two appears in the third column, and everything below d_{23} is not zero. This violates (R3). Had the 3–3 element been zero instead of unity, then the matrix would be in row-reduced form. □

For the remainder of this section, we concern ourselves with square matrices; that is, matrices having the same number of rows as columns. A *diagonal matrix* is a square matrix all of whose elements are zero except possibly those on the main diagonal. (Recall that the main diagonal consists of all the diagonal elements a_{11}, a_{22}, a_{33}, and so on.) Thus,

$$\begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

are both diagonal matrices of order 2×2 and 3×3 respectively. The zero matrix is the special diagonal matrix having all the elements on the main diagonal equal to zero.

An *identity* matrix is a diagonal matrix worthy of special consideration. Designated by $\mathbf{I}$, an identity is defined to be a diagonal matrix having all diagonal elements equal to one. Thus,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

are the 2×2 and 4×4 identities respectively. The identity is perhaps the most important matrix of all. If the identity is of the appropriate order so that the following multiplication can be carried out, then for any arbitrary matrix $\mathbf{A}$,

$$\mathbf{AI} = \mathbf{A} \quad \text{and} \quad \mathbf{IA} = \mathbf{A}.$$

A *symmetric* matrix is a matrix that is equal to its transpose while a *skew symmetric* matrix is a matrix that is equal to the negative of its transpose. Thus, a matrix $\mathbf{A}$ is symmetric if $\mathbf{A} = \mathbf{A}^{\mathsf{T}}$ while it is skew symmetric if $\mathbf{A} = -\mathbf{A}^{\mathsf{T}}$. Examples of each are respectively

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 1 \\ 3 & -1 & 0 \end{bmatrix}.$$

A matrix $\mathbf{A} = [a_{ij}]$ is called *lower triangular* if $a_{ij} = 0$ for $j > i$ (that is, if all the elements above the main diagonal are zero) and *upper triangular* if $a_{ij} = 0$ for $i > j$ (that is, if all the elements below the main diagonal are zero).

Examples of lower and upper triangular matrices are, respectively,

$$\begin{bmatrix} 5 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 2 & 1 & 4 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & 2 & 4 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

Theorem 1. *The product of two lower (upper) triangular matrices is also lower (upper) triangular.*

Proof. Let **A** and **B** both be $n \times n$ lower triangular matrices. Set $\mathbf{C} = \mathbf{AB}$. We need to show that **C** is lower triangular, or equivalently, that $c_{ij} = 0$ when $i < j$. Now,

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{j-1} a_{ik}b_{kj} + \sum_{k=j}^{n} a_{ik}b_{kj}.$$

We are given that $a_{ik} = 0$ when $i < k$, and $b_{kj} = 0$ when $k < j$, because both **A** and **B** are lower triangular. Thus,

$$\sum_{k=1}^{j-1} a_{ik}b_{kj} = \sum_{k=1}^{j-1} a_{ik}(0) = 0$$

because k is always less than j. Furthermore, if we restrict $i < j$, then

$$\sum_{k=j}^{n} a_{ik}b_{kj} = \sum_{k=j}^{n} (0)b_{kj} = 0$$

because $k \geq j > i$. Therefore, $c_{ij} = 0$ when $i < j$.

Finally, we define positive integral powers of a matrix in the obvious manner: $\mathbf{A}^2 = \mathbf{AA}$, $\mathbf{A}^3 = \mathbf{AAA}$ and, in general, if n is a positive integer,

$$\mathbf{A}^n = \underbrace{\mathbf{AA} \cdots \mathbf{A}.}_{n \text{ times}}$$

Thus, if

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}, \qquad \text{then } \mathbf{A}^2 = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -8 \\ 4 & 7 \end{bmatrix}.$$

It follows directly from Property 5 that

$$(\mathbf{A}^2)^{\mathsf{T}} = (\mathbf{AA})^{\mathsf{T}} = \mathbf{A}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}} = (\mathbf{A}^{\mathsf{T}})^2.$$

We can generalize this result to the following property for any integral positive power n:

(7) $(\mathbf{A}^n)^\mathsf{T} = (\mathbf{A}^\mathsf{T})^n$.

Problems 1.4

(1) Verify that $(\mathbf{A} + \mathbf{B})^\mathsf{T} = \mathbf{A}^\mathsf{T} + \mathbf{B}^\mathsf{T}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & 5 & -1 \\ 2 & 1 & 3 \\ 0 & 7 & -8 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 6 & 1 & 3 \\ 2 & 0 & -1 \\ -1 & -7 & 2 \end{bmatrix}.$$

(2) Verify that $(\mathbf{AB})^\mathsf{T} = \mathbf{B}^\mathsf{T}\mathbf{A}^\mathsf{T}$, where

$$\mathbf{A} = \begin{bmatrix} t & t^2 \\ 1 & 2t \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 3 & t & t+1 & 0 \\ t & 2t & t^2 & t^3 \end{bmatrix}.$$

(3) Simplify the following expressions:

(a) $(\mathbf{AB}^\mathsf{T})^\mathsf{T}$,
(b) $\mathbf{A}^\mathsf{T} + (\mathbf{A} + \mathbf{B}^\mathsf{T})^\mathsf{T}$,
(c) $(\mathbf{A}^\mathsf{T}(\mathbf{B} + \mathbf{C}^\mathsf{T}))^\mathsf{T}$,
(d) $((\mathbf{AB})^\mathsf{T} + \mathbf{C})^\mathsf{T}$,
(e) $((\mathbf{A} + \mathbf{A}^\mathsf{T})(\mathbf{A} - \mathbf{A}^\mathsf{T}))^\mathsf{T}$.

(4) Find $\mathbf{X}^\mathsf{T}\mathbf{X}$ and $\mathbf{XX}^\mathsf{T}$ when

$$\mathbf{X} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

(5) Find $\mathbf{X}^\mathsf{T}\mathbf{X}$ and $\mathbf{XX}^\mathsf{T}$ when $\mathbf{X} = [1 \quad -2 \quad 3 \quad -4]$.

(6) Find $\mathbf{X}^\mathsf{T}\mathbf{AX}$ when

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

(7) Determine which, if any, of the following matrices are in row-reduced form:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 4 & -7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 & 4 & -7 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 1 & 1 & 0 & 4 & -7 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 0 & 1 & 0 & 4 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{E} = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{G} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\mathbf{H} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{J} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{K} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{L} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{M} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{N} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{Q} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{R} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{S} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{T} = \begin{bmatrix} 1 & 12 \\ 0 & 1 \end{bmatrix}.$$

(8) Determine which, if any, of the matrices in Problem 7 are upper triangular.

(9) Must a square matrix in row-reduced form necessarily be upper triangular?

(10) Must an upper triangular matrix necessarily be in row-reduced form?

(11) Can a matrix be both upper and lower triangular simultaneously?

(12) Show that $\mathbf{AB} = \mathbf{BA}$, where

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(13) Prove that if $\mathbf{A}$ and $\mathbf{B}$ are diagonal matrices of the same order, then $\mathbf{AB} = \mathbf{BA}$.

(14) Does a 2×2 diagonal matrix commute with every other 2×2 matrix?

(15) Compute the products $\mathbf{AD}$ and $\mathbf{BD}$ for the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{bmatrix}.$$

What conclusions can you make about postmultiplying a square matrix by a diagonal matrix?

(16) Compute the products **DA** and **DB** for the matrices defined in Problem 15. What conclusions can you make about premultiplying a square matrix by a diagonal matrix?

(17) Prove that if a 2×2 matrix **A** commutes with every 2×2 diagonal matrix, then **A** must also be diagonal. *Hint*: Consider, in particular, the diagonal matrix

$$\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

(18) Prove that if an $n \times n$ matrix **A** commutes with every $n \times n$ diagonal matrix, then **A** must also be diagonal.

(19) Compute $\mathbf{D}^2$ and $\mathbf{D}^3$ for the matrix **D** defined in Problem 15.

(20) Find $\mathbf{A}^3$ if

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

(21) Using the results of Problems 19 and 20 as a guide, what can be said about $\mathbf{D}^n$ if **D** is a diagonal matrix and n is a positive integer?

(22) Prove that if $\mathbf{D} = [d_{ij}]$ is a diagonal matrix, then $\mathbf{D}^2 = [d_{ij}^2]$.

(23) Calculate $\mathbf{D}^{50} - 5\mathbf{D}^{35} + 4\mathbf{I}$, where

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

(24) A square matrix **A** is *nilpotent* if $\mathbf{A}^n = \mathbf{0}$ for some positive integer n. If n is the smallest positive integer for which $\mathbf{A}^n = \mathbf{0}$ then **A** is nilpotent of *index* n. Show that

$$\mathbf{A} = \begin{bmatrix} -1 & -1 & -3 \\ -5 & -2 & -6 \\ 2 & 1 & 3 \end{bmatrix}$$

is nilpotent of index 3.

(25) Show that

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is nilpotent. What is its index?

(26) Prove that if $\mathbf{A}$ is a square matrix, then $\mathbf{B} = (\mathbf{A} + \mathbf{A}^{\mathsf{T}})/2$ is a symmetric matrix.

(27) Prove that if $\mathbf{A}$ is a square matrix, then $\mathbf{C} = (\mathbf{A} - \mathbf{A}^{\mathsf{T}})/2$ is a skew-symmetric matrix.

(28) Using the results of the preceding two problems, prove that any square matrix can be written as the sum of a symmetric matrix and a skew-symmetric matrix.

(29) Write the matrix $\mathbf{A}$ in Problem 1 as the sum of a symmetric matrix and skew-symmetric matrix.

(30) Write the matrix $\mathbf{B}$ in Problem 1 as the sum of a symmetric matrix and a skew-symmetric matrix.

(31) Prove that if $\mathbf{A}$ is any matrix, then $\mathbf{A}\mathbf{A}^{\mathsf{T}}$ is symmetric.

(32) Prove that the diagonal elements of a skew-symmetric matrix must be zero.

(33) Prove that the transpose of an upper triangular matrix is lower triangular, and vice versa.

(34) If $\mathbf{P} = [p_{ij}]$ is a transition matrix for a Markov chain (see Problem 16 of Section 1.1), then it can be shown with elementary probability theory that the $i-j$ element of $\mathbf{P}^2$ denotes the probability of an object moving from state i to stage j over two time periods. More generally, the $i-j$ element of $\mathbf{P}^n$ for any positive integer n denotes the probability of an object moving from state i to state j over n time periods.

(a) Calculate $\mathbf{P}^2$ and $\mathbf{P}^3$ for the two-state transition matrix

$$\mathbf{P} = \begin{bmatrix} 0.1 & 0.9 \\ 0.4 & 0.6 \end{bmatrix}.$$

(b) Determine the probability of an object beginning in state 1 and ending in state 1 after two time periods.

(c) Determine the probability of an object beginning in state 1 and ending in state 2 after two time periods.
(d) Determine the probability of an object beginning in state 1 and ending in state 2 after three time periods.
(e) Determine the probability of an object beginning in state 2 and ending in state 2 after three time periods.

(35) Consider a two-state Markov chain. List the number of ways an object in state 1 can end in state 1 after three time periods.

(36) Consider the Markov chain described in Problem 16 of Section 1.1. Determine (a) the probability that a family living in the city will find themselves in the suburbs after two years, and (b) the probability that a family living in the suburbs will find themselves living in the city after two years.

(37) Consider the Markov chain described in Problem 17 of Section 1.1. Determine (a) the probability that there will be a Republican mayor eight years after a Republican mayor serves, and (b) the probability that there will be a Republican mayor 12 years after a Republican mayor serves.

(38) Consider the Markov chain described in Problem 18 of Section 1.1. It is known that this year the apple harvest was poor. Determine (a) the probability that next year's harvest will be poor, and (b) the probability that the harvest in two years will be poor.

(39) Consider the Markov chain described in Problem 19 of Section 1.1. Determine (a) the probability that a brand X customer will be a brand X customer after 4 years, (b) after 6 years, and (c) the probability that a brand X customer will be a brand Y customer after 4 years.

(40) A graph consists of a set of nodes, which we shall designate by positive integers, and a set of arcs that connect various pairs of nodes. An *adjacency matrix* $\mathbf{M}$ associated with a particular graph is defined by

$$m_{ij} = \text{number of distinct arcs connecting node } i \text{ to node } j$$

(a) Construct an adjacency matrix for the graph shown in Figure 1.
(b) Calculate $\mathbf{M}^2$, and note that the i–j element of $\mathbf{M}^2$ is the number of paths consisting of two arcs that connect node i to node j.

(41) (a) Construct an adjacency matrix $\mathbf{M}$ for the graph shown in Figure 2.
(b) Calculate $\mathbf{M}^2$, and use that matrix to determine the number of paths consisting of two arcs that connect node 1 to node 5.

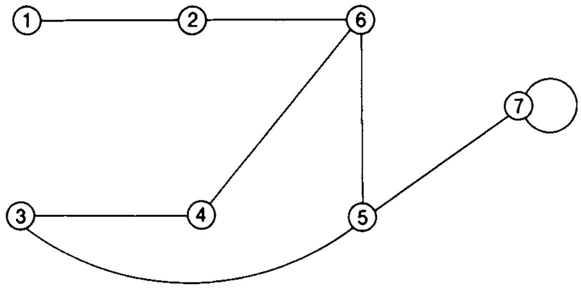


Figure 1.

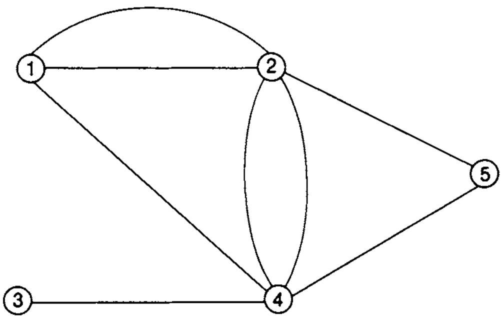


Figure 2.

(c) Calculate $\mathbf{M}^3$, and use that matrix to determine the number of paths consisting of three arcs that connect node 2 to node 4.

(42) Figure 3 depicts a road network linking various cities. A traveler in city 1 needs to drive to city 7 and would like to do so by passing through the

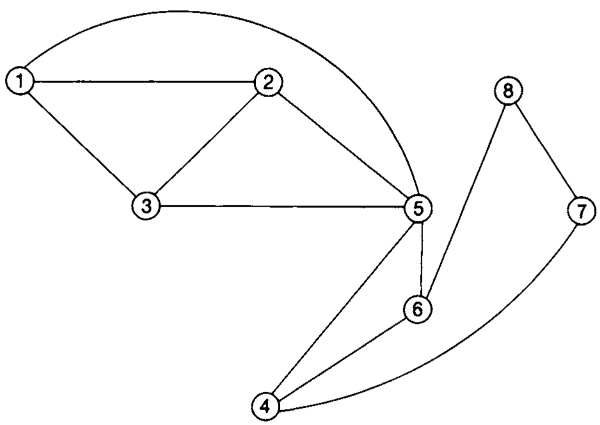


Figure 3.

least number of intermediate cities. Construct an adjacency matrix for this road network. Consider powers of this matrix to solve the traveler's problem.

1.5 Submatrices and Partitioning

Given any matrix **A**, a *submatrix* of **A** is a matrix obtained from **A** by the removal of any number of rows or columns. Thus, if

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 10 & 12 \\ 14 & 16 \end{bmatrix}, \quad \text{and} \quad \mathbf{C} = [2 \quad 3 \quad 4], \tag{11}$$

then **B** and **C** are both submatrices of **A**. Here **B** was obtained by removing from **A** the first and second rows together with the first and third columns, while **C** was obtained by removing from **A** the second, third, and fourth rows together with the first column. By removing no rows and no columns from **A**, it follows that **A** is a submatrix of itself.

A matrix is said to be partitioned if it is divided into submatrices by horizontal and vertical lines between the rows and columns. By varying the choices of where to put the horizontal and vertical lines, one can partition a matrix in many different ways. Thus,

$$\left[\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c|cc|c} 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{array}\right]$$

are examples of two different partitions of the matrix **A** given in (11).

If partitioning is carried out in a particularly judicious manner, it can be a great help in matrix multiplication. Consider the case where the two matrices **A** and **B** are to be multiplied together. If we partition both **A** and **B** into four submatrices, respectively, so that

$$\mathbf{A} = \left[\begin{array}{c|c} \mathbf{C} & \mathbf{D} \\ \hline \mathbf{E} & \mathbf{F} \end{array}\right] \quad \text{and} \quad \mathbf{B} = \left[\begin{array}{c|c} \mathbf{G} & \mathbf{H} \\ \hline \mathbf{J} & \mathbf{K} \end{array}\right]$$

where **C** through **K** represent submatrices, then the product **AB** may be obtained by simply carrying out the multiplication as if the submatrices were themselves elements. Thus,

$$\mathbf{AB} = \left[\begin{array}{c|c} \mathbf{CG} + \mathbf{DJ} & \mathbf{CH} + \mathbf{DK} \\ \hline \mathbf{EG} + \mathbf{FJ} & \mathbf{EH} + \mathbf{FK} \end{array}\right], \tag{12}$$

providing the partitioning was such that the indicated multiplications are defined.

It is not unusual to need products of matrices having thousands of rows and thousands of columns. Problem 42 of Section 1.4 dealt with a road network connecting seven cities. A similar network for a state with connections between all cities in the state would have a very large adjacency matrix associated with it, and its square is then the product of two such matrices. If we expand the network to include the entire United States, the associated matrix is huge, with one row and one column for each city and town in the country. Thus, it is not difficult to visualize large matrices that are too big to be stored in the internal memory of any modern day computer. And yet the product of such matrices must be computed.

The solution procedure is partitioning. Large matrices are stored in external memory on peripheral devices, such as disks, and then partitioned. Appropriate submatrices are fetched from the peripheral devices as needed, computed, and the results again stored on the peripheral devices. An example is the product given in (12). If **A** and **B** are too large for the internal memory of a particular computer, but **C** through **K** are not, then the partitioned product can be computed. First, **C** and **G** are fetched from external memory and multiplied; the product is then stored in external memory. Next, **D** and **J** are fetched and multiplied. Then, the product **CG** is fetched and added to the product **DJ**. The result, which is the first partition of **AB**, is then stored in external memory, and the process continues.

Example 1

Find **AB** if

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 4 & -1 \\ 3 & 1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Solution. We first partition **A** and **B** in the following manner

$$\mathbf{A} = \left[\begin{array}{cc|c} 3 & 1 & 2 \\ 1 & 4 & -1 \\ \hline 3 & 1 & 2 \end{array}\right] \quad \text{and} \quad \mathbf{B} = \left[\begin{array}{cc|c} 1 & 3 & 2 \\ -1 & 0 & 1 \\ \hline 0 & 1 & 1 \end{array}\right];$$

then,

$$\mathbf{AB} = \left[\begin{array}{c|c} \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}[0 \quad 1] & \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}[1] \\ \hline [3 \quad 1]\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} + [2][0 \quad 1] & [3 \quad 1]\begin{bmatrix} 2 \\ 1 \end{bmatrix} + [2][1] \end{array}\right]$$

$$= \left[\begin{array}{c|c} \begin{bmatrix} 2 & 9 \\ -3 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix} & \begin{bmatrix} 7 \\ 6 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ \hline [2 \quad 9] + [0 \quad 2] & [7] + [2] \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 2 & 11 & 9 \\ -3 & 2 & 5 \\ \hline 2 & 11 & 9 \end{array}\right] = \begin{bmatrix} 2 & 11 & 9 \\ -3 & 2 & 5 \\ 2 & 11 & 9 \end{bmatrix}. \quad \square$$

Example 2

Find **AB** if

$$\mathbf{A} = \left[\begin{array}{cc|c} 3 & 1 & 0 \\ 2 & 0 & 0 \\ \hline 0 & 0 & 3 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \end{array}\right] \quad \text{and} \quad \mathbf{B} = \left[\begin{array}{cc|ccc} 2 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 1 \end{array}\right].$$

Solution. From the indicated partitions, we find that

$$\mathbf{AB} = \left[\begin{array}{c|c} \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}[0 \quad 1] & \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}[0 \quad 0 \quad 1] \\ \hline \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix}[0 \quad 1] & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix}[0 \quad 0 \quad 1] \\ \hline [0 \quad 0]\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} + [0][0 \quad 1] & [0 \quad 0]\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + [0][0 \quad 0 \quad 1] \end{array}\right]$$

$$\mathbf{AB} = \left[\begin{array}{c|c} \begin{bmatrix} 5 & 4 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \\ \hline [0 \quad 0] + [0 \quad 0] & [0 \quad 0 \quad 0] + [0 \quad 0 \quad 0] \end{array}\right]$$

$$= \left[\begin{array}{cc|ccc} 5 & 4 & 0 & 0 & 0 \\ 4 & 2 & 0 & 0 & 0 \\ \hline 0 & 3 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} 5 & 4 & 0 & 0 & 0 \\ 4 & 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that we partitioned in order to make maximum use of the zero submatrices of both **A** and **B**. □

A matrix **A** that can be partitioned into the form

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & & & & \\ & \mathbf{A}_2 & & \mathbf{0} & \\ & & \mathbf{A}_3 & & \\ & \mathbf{0} & & \ddots & \\ & & & & \mathbf{A}_n \end{bmatrix}$$

is called *block diagonal*. Such matrices are particularly easy to multiply because in partitioned form they act as diagonal matrices.

Problems 1.5

(1) Which of the following are submatrices of the given **A** and why?

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

(a) $\begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$ (b) $[1]$ (c) $\begin{bmatrix} 1 & 2 \\ 8 & 9 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}$.

(2) Determine all possible submatrices of

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

(3) Given the matrices **A** and **B** (as shown), find **AB** using the partitionings indicated:

$$\mathbf{A} = \left[\begin{array}{cc|c} 1 & -1 & 2 \\ \hline 3 & 0 & 4 \\ 0 & 1 & 2 \end{array}\right], \qquad \mathbf{B} = \left[\begin{array}{ccc|c} 5 & 2 & 0 & 2 \\ 1 & -1 & 3 & 1 \\ \hline 0 & 1 & 1 & 4 \end{array}\right].$$

(4) Partition the given matrices $\mathbf{A}$ and $\mathbf{B}$ and, using the results, find $\mathbf{AB}$.

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 3 & 2 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

(5) Compute $\mathbf{A}^2$ for the matrix $\mathbf{A}$ given in Problem 4 by partitioning $\mathbf{A}$ into block diagonal form.

(6) Compute $\mathbf{B}^2$ for the matrix $\mathbf{B}$ given in Problem 4 by partitioning $\mathbf{B}$ into block diagonal form.

(7) Use partitioning to compute $\mathbf{A}^2$ and $\mathbf{A}^3$ for

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

What is $\mathbf{A}^n$ for any positive integral power of $n > 3$?

(8) Use partitioning to compute $\mathbf{A}^2$ and $\mathbf{A}^3$ for

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & -4 & 0 & 0 \\ 0 & 0 & -1 & 3 & 4 & 0 & 0 \\ 0 & 0 & 1 & -2 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

What is $\mathbf{A}^n$ for any positive integral power of n?

1.6 Vectors

Definition 1. A *vector* is a $1 \times n$ or $n \times 1$ matrix.

A $1 \times n$ matrix is called a *row vector* while an $n \times 1$ matrix is a *column vector*. The elements are called the *components* of the vector while the number of components in the vector, in this case n, is its *dimension*. Thus,

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

is an example of a 3-dimensional column vector, while

$$[t \quad 2t \quad -t \quad 0]$$

is an example of a 4-dimensional row vector.

The reader who is already familiar with vectors will notice that we have not defined vectors as directed line segments. We have done this intentionally, first because in more than three dimensions this geometric interpretation loses its significance, and second, because in the general mathematical framework, vectors are not directed line segments. However, the idea of representing a finite dimensional vector by its components and hence as a matrix is one that is acceptable to the scientist, engineer, and mathematician. Also, as a bonus, since a vector is nothing more than a special matrix, we have already defined scalar multiplication, vector addition, and vector equality.

A vector $\mathbf{y}$ (vectors will be designated by boldface lowercase letters) has associated with it a nonnegative number called its *magnitude* or length designated by $\|\mathbf{y}\|$.

Definition 2. If $\mathbf{y} = [y_1\ y_2 \cdots y_n]$ then $\|\mathbf{y}\| = \sqrt{(y_1)^2 + (y_2)^2 + \cdots + (y_n)^2}$.

Example 1

Find $\|\mathbf{y}\|$ if $\mathbf{y} = [1 \quad 2 \quad 3 \quad 4]$.

Solution.

$$\|\mathbf{y}\| = \sqrt{(1)^2 + (2)^2 + (3)^2 + (4)^2} = \sqrt{30}. \quad \square$$

If $\mathbf{z}$ is a column vector, $\|\mathbf{z}\|$ is defined in a completely analogous manner.

Example 2

Find $\|\mathbf{z}\|$ if

$$\mathbf{z} = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}.$$

Solution. $\|\mathbf{z}\| = \sqrt{(-1)^2 + (2)^2 + (-3)^2} = \sqrt{14}. \quad \square$

A vector is called a *unit vector* if its magnitude is equal to one. A nonzero vector is said to be *normalized* if it is divided by its magnitude. Thus, a normalized vector is also a unit vector.

Example 3

Normalize the vector $[1 \quad 0 \quad -3 \quad 2 \quad -1]$.

Solution. The magnitude of this vector is

$$\sqrt{(1)^2 + (0)^2 + (-3)^2 + (2)^2 + (-1)^2} = \sqrt{15}.$$

Hence, the normalized vector is

$$\begin{bmatrix} \frac{1}{\sqrt{15}} & 0 & \frac{-3}{\sqrt{15}} & \frac{2}{\sqrt{15}} & \frac{-1}{\sqrt{15}} \end{bmatrix}. \quad \square$$

In passing, we note that when a general vector is written $\mathbf{y} = [y_1 y_2 \cdots y_n]$ one of the subscripts of each element of the matrix is deleted. This is done solely for the sake of convenience. Since a row vector has only one row (a column vector has only one column), it is redundant and unnecessary to exhibit the row subscript (the column subscript).

Problems 1.6

(1) Find p if $5\mathbf{x} - 2\mathbf{y} = \mathbf{b}$, where

$$\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 2 \\ p \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 13 \\ -2 \end{bmatrix}.$$

(2) Find $\mathbf{x}$ if $3\mathbf{x} + 2\mathbf{y} = \mathbf{b}$, where

$$\mathbf{y} = \begin{bmatrix} 3 \\ 1 \\ 6 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 1 \end{bmatrix}.$$

(3) Find $\mathbf{y}$ if $2\mathbf{x} - 5\mathbf{y} = -\mathbf{b}$, where

$$\mathbf{x} = [2 \quad -1 \quad 3] \quad \text{and} \quad \mathbf{b} = [1 \quad 0 \quad -1].$$

(4) Using the vectors defined in Problem 2, calculate, if possible,

(a) $\mathbf{yb}$, (b) $\mathbf{yb}^{\mathsf{T}}$,

(c) $\mathbf{y}^{\mathsf{T}}\mathbf{b}$, (d) $\mathbf{b}^{\mathsf{T}}\mathbf{y}$.

(5) Using the vectors defined in Problem 3, calculate, if possible,

(a) $\mathbf{x} + 2\mathbf{b}$, (b) $\mathbf{xb}^{\mathsf{T}}$,

(c) $\mathbf{x}^{\mathsf{T}}\mathbf{b}$, (d) $\mathbf{b}^{\mathsf{T}}\mathbf{b}$.

(6) Determine which of the following are unit vectors:

(a) $[1 \quad 1]$, (b) $[1/2 \quad 1/2]$, (c) $[1/\sqrt{2} \quad -1/\sqrt{2}]$

(d) $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, (e) $\begin{bmatrix} 1/2 \\ 1/3 \\ 1/6 \end{bmatrix}$, (f) $\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$,

(g) $\dfrac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, (h) $\dfrac{1}{6}\begin{bmatrix} 1 \\ 5 \\ 3 \\ 1 \end{bmatrix}$, (i) $\dfrac{1}{\sqrt{3}}[-1 \quad 0 \quad 1 \quad -1]$.

(7) Find $\|\mathbf{y}\|$ if

(a) $\mathbf{y} = [1 \quad -1]$, (b) $\mathbf{y} = [3 \quad 4]$,

(c) $\mathbf{y} = [-1 \quad -1 \quad 1]$, (d) $\mathbf{y} = [\frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2}]$,

(e) $\mathbf{y} = [2 \quad 1 \quad -1 \quad 3]$, (f) $\mathbf{y} = [0 \quad -1 \quad 5 \quad 3 \quad 2]$.

(8) Find $\|\mathbf{x}\|$ if

(a) $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, (b) $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, (c) $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$,

(d) $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$, (e) $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, (f) $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

(9) Find $\|\mathbf{y}\|$ if

(a) $\mathbf{y} = [2 \quad 1 \quad -1 \quad 3]$, (b) $\mathbf{y} = [0 \quad -1 \quad 5 \quad 3 \quad 2]$.

(10) Prove that a normalized vector must be a unit vector.

(11) Show that the matrix equation

$$\begin{bmatrix} 1 & 1 & -2 \\ 2 & 5 & 3 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 11 \\ 5 \end{bmatrix}$$

is equivalent to the vector equation

$$x\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} + z\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 11 \\ 5 \end{bmatrix}.$$

(12) Convert the following system of equations into a vector equation:

$$2x + 3y = 10,$$
$$4x + 5y = 11.$$

(13) Convert the following system of equations into a vector equation:

$$\begin{aligned} 3x + 4y + 5z + 6w &= 1, \\ y - 2z + 8w &= 0, \\ -x + y + 2z - w &= 0. \end{aligned}$$

(14) Using the definition of matrix multiplication, show that the jth column of $(\mathbf{AB}) = \mathbf{A} \times (j$th column of $\mathbf{B})$.

(15) Verify the result of Problem 14 by showing that the first column of the product **AB** with

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 2 & -3 \end{bmatrix}$$

is

$$\mathbf{A}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix},$$

while the second column of the product is

$$\mathbf{A}\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}.$$

(16) A *distribution row vector* **d** for an N-state Markov chain (See Problem 16 of Section 1.1 and Problem 34 of Section 1.4) is an N-dimensional row vector having as its components, one for each state, the probabilities that an object in the system is in each of the respective states. Determine a distribution vector for a three-state Markov chain if 50% of the objects are in state 1, 30% are in state 2, and 20% are in state 3.

(17) Let $\mathbf{d}^{(k)}$ denote the distribution vector for a Markov chain after k time periods. Thus, $\mathbf{d}^{(0)}$ represents the initial distribution. It follows that

$$\mathbf{d}^{(k)} = \mathbf{d}^{(0)}\mathbf{P}^k = \mathbf{d}^{(k-1)}\mathbf{P},$$

where $\mathbf{P}$ is the transition matrix and $\mathbf{P}^k$ is its kth power.

Consider the Markov chain described in Problem 16 of Section 1.1.
(a) Explain the physical significance of saying $\mathbf{d}^{(0)} = [0.6 \quad 0.4]$.
(b) Find the distribution vectors $\mathbf{d}^{(1)}$ and $\mathbf{d}^{(2)}$.

(18) Consider the Markov chain described in Problem 19 of Section 1.1.
(a) Explain the physical significance of saying $\mathbf{d}^{(0)} = [0.4 \quad 0.5 \quad 0.1]$.
(b) Find the distribution vectors $\mathbf{d}^{(1)}$ and $\mathbf{d}^{(2)}$.

(19) Consider the Markov chain described in Problem 17 of Section 1.1. (a) Determine an initial distribution vector if the town currently has a Democratic mayor, and (b) show that the components of $\mathbf{d}^{(1)}$ are the probabilities that the next mayor will be a Republican and a Democrat, respectively.

(20) Consider the Markov chain described in Problem 18 of Section 1.1. (a) Determine an initial distribution vector if this year's crop is known to be poor. (b) Calculate $\mathbf{d}^{(2)}$ and use it to determine the probability that the harvest will be good in two years.

1.7 The Geometry of Vectors

Vector arithmetic can be described geometrically for two- and three-dimensional vectors. For simplicity, we consider two dimensions here; the extension to three-dimensional vectors is straightforward. For convenience, we restrict our examples to row vectors, but note that all constructions are equally valid for column vectors.

A two dimensional vector $\mathbf{v} = [a \quad b]$ is identified with the point (a, b) on the plane, measured from the origin a units along the horizontal axis and then b units parallel to the vertical axis. We can then draw an arrow beginning at the origin and ending at the point (a, b). This arrow or directed line

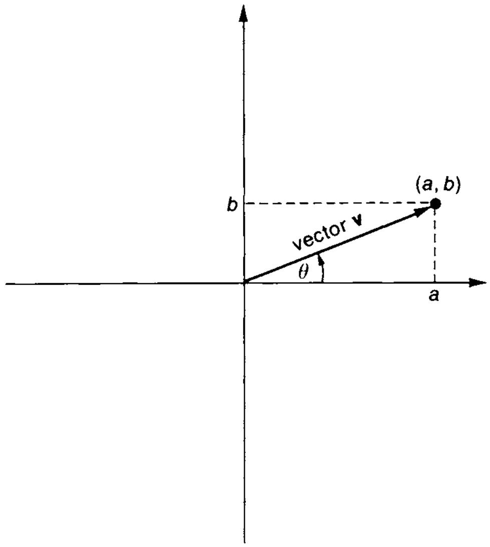


Figure 4.

segment, as shown in Figure 4, represents the vector geometrically. It follows immediately from Pythagorus's theorem and Definition 2 of Section 1.6 that the length of the directed line segment is the magnitude of the vector. The angle associated with a vector, denoted by θ in Figure 4, is the angle from the positive horizontal axis to the directed line segment measured in the counterclockwise direction.

Example 1

Graph the vectors $\mathbf{v} = [2 \quad 4]$ and $\mathbf{u} = [-1 \quad 1]$ and determine the magnitude and angle of each.

Solution. The vectors are drawn in Figure 5. Using Pythagorus's theorem and elementary trigonometry, we have, for $\mathbf{v}$,

$$\|\mathbf{v}\| = \sqrt{(2)^2 + (4)^2} = 4.47, \qquad \tan\theta = \frac{4}{2} = 2, \quad \text{and} \quad \theta = 63.4°.$$

For $\mathbf{u}$, similar computations yield

$$\|\mathbf{u}\| = \sqrt{(-1)^2 + (1)^2} = 1.14, \qquad \tan\theta = \frac{1}{-1} = -1, \quad \text{and} \quad \theta = 135°. \qquad \square$$

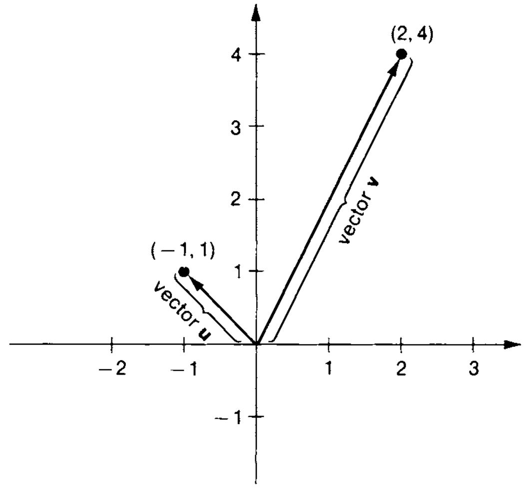


Figure 5.

To construct the sum of two vectors $\mathbf{u} + \mathbf{v}$ geometrically, graph $\mathbf{u}$ normally, translate $\mathbf{v}$ so that its initial point coincides with the terminal point of $\mathbf{u}$, *being careful to preserve both the magnitude and direction of* $\mathbf{v}$, and then draw an arrow from the origin to the terminal point of $\mathbf{v}$ after translation. This arrow geometrically represents the sum $\mathbf{u} + \mathbf{v}$. The process is depicted in Figure 6 for the two vectors defined in Example 1.

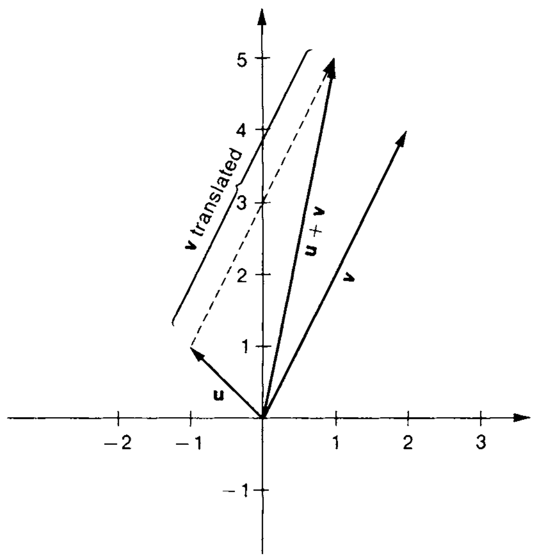


Figure 6.

To construct the difference of two vectors **u** − **v** geometrically, graph both **u** and **v** normally and construct an arrow from the terminal point of **v** to the terminal point of **u**. This arrow geometrically represents the difference **u** − **v**. The process is depicted in Figure 7 for the two vectors defined in Example 1. To measure the magnitude and direction of **u** − **v**, translate it so that its initial point is at the origin, *being careful to preserve both its magnitude and direction*, and then measure the translated vector.

Both geometrical sums and differences involve translations of vectors. This suggests that a vector is not altered by translating it to another position in the plane providing both its magnitude and direction and preserved.

Many physical phenomena such as velocity and force are completely described by their magnitudes and directions. For example, a velocity of 60 miles per hour in the northwest direction is a complete description of that velocity, and *it is independent of where that velocity occurs*. This independence is the rationale behind translating vectors geometrically. Geometrically, vectors having the same magnitude and direction are called *equivalent*, and they are regarded as being equal even though they may be located at different positions in the plane.

A scalar multiplication $k\mathbf{u}$ is defined geometrically to be a vector having length $|k|$ times the length of **u** with direction equal to **u** when k is positive, and opposite to **u** when k is negative. Effectively, $k\mathbf{u}$ is an elongation of **u** by a factor of $|k|$ when $|k|$ is greater than unity, or a contraction of **u** by a factor of

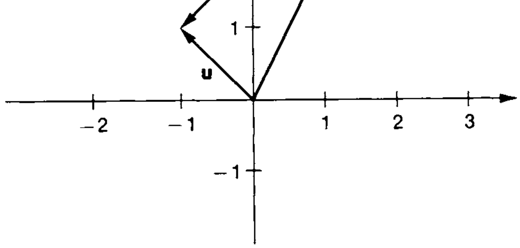


Figure 7.

$|k|$ when $|k|$ is less than unity, followed by no rotation when k is positive, or a rotation of 180 degrees when k is negative.

Example 2

Find $-2\mathbf{u}$ and $\frac{1}{2}\mathbf{v}$ geometrically for the vectors defined in Example 1.

Solution. To construct $-2\mathbf{u}$, we double the length of $\mathbf{u}$ and then rotate the resulting vector by 180°. To construct $\frac{1}{2}\mathbf{v}$ we halve the length of $\mathbf{v}$ and effect no rotation. These constructions are illustrated in Figure 8. □

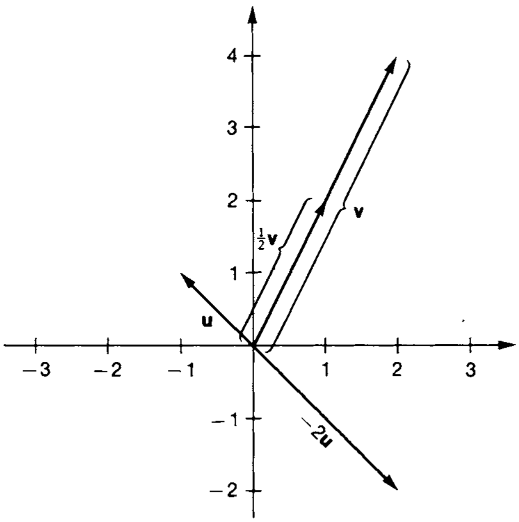


Figure 8.

Problems 1.7

In Problems 1 through 16, geometrically construct the indicated vector operations for

$$\mathbf{u} = [3 \ \ -1], \qquad \mathbf{v} = [-2 \ \ 5], \qquad \mathbf{w} = [-4 \ \ -4],$$

$$\mathbf{x} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}.$$

(1) $\mathbf{u} + \mathbf{v}$. **(2)** $\mathbf{u} + \mathbf{w}$. **(3)** $\mathbf{v} + \mathbf{w}$. **(4)** $\mathbf{x} + \mathbf{y}$.

(5) $\mathbf{x} - \mathbf{y}$. **(6)** $\mathbf{y} - \mathbf{x}$. **(7)** $\mathbf{u} - \mathbf{v}$. **(8)** $\mathbf{w} - \mathbf{u}$.

(9) $\mathbf{u} - \mathbf{w}$. **(10)** $2\mathbf{x}$. **(11)** $3\mathbf{x}$. **(12)** $-2\mathbf{x}$.

(13) $\frac{1}{2}\mathbf{u}$. **(14)** $-\frac{1}{2}\mathbf{u}$. **(15)** $\frac{1}{3}\mathbf{v}$. **(16)** $-\frac{1}{4}\mathbf{w}$.

(17) Determine the angle of $\mathbf{u}$.

(18) Determine the angle of $\mathbf{v}$.

(19) Determine the angle of $\mathbf{w}$.

(20) Determine the angle of $\mathbf{x}$.

(21) Determine the angle of $\mathbf{y}$.

(22) For arbitrary two-dimensional row vectors construct on the same graph $\mathbf{u} + \mathbf{v}$ and $\mathbf{v} + \mathbf{u}$.

(a) Show that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
(b) Show that the sum is a diagonal of a parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as two of its sides.

Chapter 2

Simultaneous Linear Equations

2.1 Linear Systems

Systems of simultaneous equations appear frequently in engineering and scientific problems. Because of their importance and because they lend themselves to matrix analysis, we devote this entire chapter to their solutions.

We are interested in systems of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned} \tag{1}$$

We assume that the coefficients a_{ij} $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ and the quantities b_i $(i = 1, 2, \ldots, m)$ are all known scalars. The quantities x_1, $x_2, \ldots, x_n$ represent unknowns.

Definition 1. A *solution* to (1) is a set of n scalars x_1, $x_2, \ldots, x_n$ that when substituted into (1) satisfies the given equations (that is, the equalities are valid).

System (1) is a generalization of systems considered earlier in that m can differ from n. If $m > n$, the system has more equations than unknowns. If $m < n$, the system has more unknowns than equations. If $m = n$, the system has as many unknowns as equations. In any case, the methods of Section 1.3 may be used to convert (1) into the matrix form

$$\mathbf{Ax} = \mathbf{b}, \tag{2}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Thus, if $m \neq n$, $\mathbf{A}$ will be rectangular and the dimensions of $\mathbf{x}$ and $\mathbf{b}$ will be different.

Example 1

Convert the following system to matrix form:

$$\begin{aligned} x + 2y - z + w &= 4, \\ x + 3y + 2z + 4w &= 9. \end{aligned}$$

Solution.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 1 & 3 & 2 & 4 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}. \qquad \square$$

Example 2

Convert the following system to matrix form:

$$\begin{aligned} x - 2y &= -9, \\ 4x + y &= 9, \\ 2x + y &= 7, \\ x - y &= -1. \end{aligned}$$

Solution.

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 4 & 1 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -9 \\ 9 \\ 7 \\ -1 \end{bmatrix}. \qquad \square$$

A system of equations given by (1) or (2) can possess no solutions, exactly one solution, or more than one solution (note that by a solution to (2) we mean a vector **x** which satisfies the matrix equality (2)). Examples of such systems are

$$\begin{aligned} x + y &= 1, \\ x + y &= 2, \end{aligned} \tag{3}$$

$$\begin{aligned} x + y &= 1, \\ x - y &= 0, \end{aligned} \tag{4}$$

$$\begin{aligned} x + y &= 0, \\ 2x + 2y &= 0. \end{aligned} \tag{5}$$

Equation (3) has no solutions, (4) admits only the solution $x = y = \frac{1}{2}$, while (5) has solutions $x = -y$ for any value of y.

Definition 2. A system of simultaneous linear equations is *consistent* if it possesses at least one solution. If no solution exists, the system is *inconsistent.*

Equation (3) is an example of an inconsistent system, while (4) and (5) represent examples of consistent systems.

Definition 3. A system given by (2) is *homogeneous* if $\mathbf{b} = \mathbf{0}$ (the zero vector). If $\mathbf{b} \neq \mathbf{0}$ (at least one component of $\mathbf{b}$ differs from zero) the system is *nonhomogeneous.*

Equation (5) is an example of a homogeneous system.

Problems 2.1

In Problems 1 and 2, determine whether or not the proposed values of x, y and z are solutions of the given systems.

(1) $x + y + 2z = 2,$ (a) $x = 1, y = -3, z = 2.$

$x - y - 2z = 0,$ (b) $x = 1, y = -1, z = 1.$

$x + 2y + 2z = 1.$

(2) $x + 2y + 3z = 6,$ (a) $x = 1, y = 1, z = 1.$

$x - 3y + 2z = 0,$ (b) $x = 2, y = 2, z = 0.$

$3x - 4y + 7z = 6.$ (c) $x = 14, y = 2, z = -4.$

(3) Find a value for k such that $x = 1$, $y = 2$, and $z = k$ is a solution of the system

$$\begin{aligned} 2x + 2y + 4z &= 1, \\ 5x + y + 2z &= 5, \\ x - 3y - 2z &= -3. \end{aligned}$$

(4) Find a value for k such that $x = 2$ and $y = k$ is a solution of the system

$$\begin{aligned} 3x + 5y &= 11, \\ 2x - 7y &= -3. \end{aligned}$$

(5) Find a value for k such that $x = 2k$, $y = -k$, and $z = 0$ is a solution of the system

$$\begin{aligned} x + 2y + z &= 0, \\ -2x - 4y + 2z &= 0, \\ 3x - 6y - 4z &= 1. \end{aligned}$$

(6) Find a value for k such that $x = 2k$, $y = -k$, and $z = 0$ is a solution of the system

$$\begin{aligned} x + 2y + 2z &= 0, \\ 2x + 4y + 2z &= 0, \\ -3x - 6y - 4z &= 0. \end{aligned}$$

(7) Find a value for k such that $x = 2k$, $y = -k$, and $z = 0$ is a solution of the system

$$\begin{aligned} x + 2y + 2z &= 0, \\ 2x + 4y + 2z &= 0, \\ -3x - 6y - 4z &= 1. \end{aligned}$$

(8) Put the system of equations given in Problem 4 into the matrix form $\mathbf{Ax} = \mathbf{b}$.

(9) Put the system of equations given in Problem 1 into the matrix form $\mathbf{Ax} = \mathbf{b}$.

(10) Put the system of equations given in Problem 2 into the matrix form $\mathbf{Ax} = \mathbf{b}$.

(11) Put the system of equations given in Problem 6 into the matrix form $\mathbf{Ax} = \mathbf{b}$.

(12) A manufacturer receives daily shipments of 70,000 springs and 45,000 pounds of stuffing for producing regular and support mattresses. Regular mattresses r require 50 springs and 30 pounds of stuffing; support mattresses s require 60 springs and 40 pounds of stuffing. The manufacturer wants to know how many mattresses of each type should be produced daily to utilize all available inventory. Show that this problem is equivalent to solving two equations in the two unknowns r and s.

(13) A manufacturer produces desks and bookcases. Desks d require 5 hours of cutting time and 10 hours of assembling time. Bookcases b require 15 minutes of cutting time and one hour of assembling time. Each day, the manufacturer has available 200 hours for cutting and 500 hours for assembling. The manufacturer wants to know how many desks and bookcases should be scheduled for completion each day to utilize all available workpower. Show that this problem is equivalent to solving two equations in the two unknowns d and b.

(14) A mining company has a contract to supply 70,000 tons of low-grade ore, 181,000 tons of medium-grade ore, and 41,000 tons of high-grade ore to a supplier. The company has three mines which it can work. Mine A produces 8000 tons of low-grade ore, 5000 tons of medium-grade ore, and 1000 tons of high-grade ore during each day of operation. Mine B produces 3000 tons of low-grade ore, 12,000 tons of medium-grade ore, and 3000 tons of high-grade ore for each day it is in operation. The figures for mine C are 1000, 10,000, and 2000, respectively. Show that the problem of determining how many days each mine must be operated to meet contractual demands without surplus is equivalent to solving a set of three equations in A, B, and C, where the unknowns denote the number of days each mine will be in operation.

(15) A pet store has determined that each rabbit in its care should receive 80 units of protein, 200 units of carbohydrates, and 50 units of fat daily. The store carries four different types of feed that are appropriate for rabbits with the following compositions:

Feed	Protein units/oz	Carbohydrates units/oz	Fat units/oz
A	5	20	3
B	4	30	3
C	8	15	10
D	12	5	7

The store wants to determine a blend of these four feeds that will meet the daily requirements of the rabbits. Show that this problem is equivalent to solving three equations in the four unknowns A, B, C, and D, where each unknown denotes the number of ounces of that feed in the blend.

(16) A small company computes its end-of-the-year bonus b as 5% of the net profit after city and state taxes have been paid. The city tax c is 2% of taxable income, while the state tax s is 3% of taxable income with credit allowed for the city tax as a pretax deduction. This year, taxable income was \$400,000. Show that b, c, and s are related by three simultaneous equations.

(17) A gasoline producer has \$800,000 in fixed annual costs and incurs an additional variable cost of \$30 per barrel B of gasoline. The total cost C is the sum of the fixed and variable costs. The net sales S is computed on a wholesale price of \$40 per barrel. (a) Show that C, B, and S are related by two simultaneous equations. (b) Show that the problem of determining how many barrels must be produced to break even, that is, for net sales to equal cost, is equivalent to solving a system of three equations.

(18) **(Leontief Closed Models)** A closed economic model involves a society in which all the goods and services produced by members of the society are consumed by those members. No goods and services are imported from without and none are exported. Such a system involves N members, each of whom produces goods or services and charges for their use. The problem is to determine the prices each member should charge for his or her labor so that everyone breaks even after one year. For simplicity, it is assumed that each member produces one unit per year.

Consider a simple closed system consisting of a farmer, a carpenter, and a weaver. The farmer produces one unit of food each year, the carpenter produces one unit of finished wood products each year, and the weaver produces one unit of clothing each year. Let p_1 denote the farmer's annual income (that is, the price she charges for her unit of food), let p_2 denote the carpenter's annual income (that is, the price he charges for his unit of finished wood products), and let p_3 denote the weaver's annual income. Assume on an annual basis that the farmer and the carpenter consume 40% each of the available food, while the weaver eats the remaining 20%. Assume that the carpenter uses 25% of the wood products he makes, while the farmer uses 30% and the weaver uses 45%. Assume further that the farmer uses 50% of the weaver's clothing while the carpenter uses 35% and the weaver consumes the remaining 15%. Show that a break-even equation for the farmer is

$$0.40p_1 + 0.30p_2 + 0.50p_3 = p_1,$$

while the break-even equation for the carpenter is

$$0.40p_1 + 0.25p_2 + 0.35p_3 = p_2.$$

What is the break-even equation for the weaver? Rewrite all three equations as a homogeneous system.

(19) Paul, Jim, and Mary decide to help each other build houses. Paul will spend half his time on his own house and a quarter of his time on each of the houses of Jim and Mary. Jim will spend one third of his time on each of the three houses under construction. Mary will spend one sixth of her time on Paul's house, one third on Jim's house, and one half of her time on her own house. For tax purposes each must place a price on his or her labor, but they want to do so in a way that each will break even. Show that the process of determining break-even wages is a Leontief closed model comprised of three homogeneous equations.

(20) Four third world countries each grow a different fruit for export and each uses the income from that fruit to pay for imports of the fruits from the other countries. Country A exports 20% of its fruit to country B, 30% to country C, 35% to country D, and uses the rest of its fruit for internal consumption. Country B exports 10% of its fruit to country A, 15% to country C, 35% to country D, and retains the rest for its own citizens. Country C does not export to country A; it divides its crop equally between countries B and D and its own people. Country D does not consume its own fruit. All of its fruit is for export with 15% going to country A, 40% to country B, and 45% to country C. Show that the problem of determining prices on the annual harvests of fruit so that each country breaks even is equivalent to solving four homogeneous equations in four unknowns.

(21) **(Leontief Input–Output Models)** Consider an economy consisting of N sectors, with each producing goods or services unique to that sector. Let x_i denote the amount produced by the ith sector, measured in dollars. Thus x_i represents the dollar value of the supply of product i available in the economy. Assume that every sector in the economy has a demand for a proportion (which may be zero) of the output of every other sector. Thus, each sector j has a demand, measured in dollars, for the item produced in sector i. Let a_{ij} denote the proportion of item j's revenues that must be committed to the purchase of items from sector i in order for sector j to produce its goods or services. Assume also that there is an external demand, denoted by d_i and measured in dollars, for each item produced in the economy.

The problem is to determine how much of each item should be produced to meet external demand without creating a surplus of any item. Show that for a two sector economy, the solution to this problem is given by the supply/demand equations

$$\begin{aligned} &\underline{\text{supply}} \qquad \underline{\text{demand}} \\ x_1 &= a_{11}x_1 + a_{12}x_2 + d_1, \\ x_2 &= a_{21}x_1 + a_{22}x_2 + d_2. \end{aligned}$$

Show that this system is equivalent to the matrix equations

$$\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{d} \quad \text{and} \quad (\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{d}.$$

In this formulation, $\mathbf{A}$ is called the *consumption matrix* and $\mathbf{d}$ the *demand vector*.

(22) Determine $\mathbf{A}$ and $\mathbf{d}$ in the previous problem if sector 1 must expend half of its revenues purchasing goods from its own sector and one third of its revenues purchasing goods from the other sector, while sector 2 must expend one quarter of its revenues purchasing items from sector 1 and requires nothing from itself. In addition, the demand for items from these two sectors are \$20,000 and \$30,000, respectively.

(23) A small town has three primary industries, coal mining (sector 1), transportation (sector 2), and electricity (sector 3). Production of one dollar of coal requires the purchase of 10 cents of electricity and 20 cents of transportation. Production of one dollar of transportation requires the purchase of 2 cents of coal and 35 cents of electricity. Production of one unit of electricity requires the purchase of 10 cents of electricity, 50 cents of coal, and 30 cents of transportation. The town has external contracts for \$50,000 of coal, \$80,000 of transportation, and \$30,000 units of electricity. Show that the problem of determining how much coal, electricity, and transportation is required to supply the external demand without a surplus is equivalent to solving a Leontief input–output model. What are **A** and **d**?

(24) An economy consists of four sectors: energy, tourism, transportation, and construction. Each dollar of income from energy requires the expenditure of 20 cents on energy costs, 10 cents on transportation, and 30 cents on construction. Each dollar of income gotten by the tourism sector requires the expenditure of 20 cents on tourism (primarily in the form of complimentary facilities for favored customers), 15 cents on energy, 5 cents on transportation, and 30 cents on construction. Each dollar of income from transportation requires the expenditure of 40 cents on energy and 10 cents on construction; while each dollar of income from construction requires the expenditure of 5 cents on construction, 25 cents on energy, and 10 cents on transportation. The only external demand is for tourism, and this amounts to \$5 million dollars a year. Show that the problem of determining how much energy, tourism, transportation, and construction is required to supply the external demand without a surplus is equivalent to solving a Leontief input–output model. What are **A** and **d**?

(25) A constraint is often imposed on each column of the consumption matrix of a Leontief input–output model, that the sum of the elements in each column be less than unity. Show that this guarantees that each sector in the economy is profitable.

2.2 Solutions by Substitution

Most readers have probably encountered simultaneous equations in high school algebra. At that time, matrices were not available; hence other methods were developed to solve these systems, in particular, the method of substitution. We review this method in this section. In the next section, we develop its matrix equivalent, which is slightly more efficient and, more importantly, better suited for computer implementations.

Consider the system given by (1):

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m.
\end{aligned}$$

The method of substitution is the following: take the first equation and solve for x_1 in terms of $x_2, x_3, \ldots, x_n$ and then substitute this value of x_1 into all the other equations, thus eliminating it from those equations. (If x_1 does not appear in the first equation, rearrange the equations so that it does. For example, one might have to interchange the order of the first and second equations.) This new set of equations is called the *first derived set*. Working with the first derived set, solve the second equation for x_2 in terms of $x_3, x_4, \ldots, x_n$ and then substitute this value of x_2 into the third, fourth, etc. equations, thus eliminating it. This new set is the second derived set. This process is kept up until the following set of equations is obtained:

$$\begin{aligned}
x_1 &= c_{12}x_2 + c_{13}x_3 + c_{14}x_4 + \cdots + c_{1n}x_n + d_1, \\
x_2 &= \qquad\quad\; c_{23}x_3 + c_{24}x_4 + \cdots + c_{2n}x_n + d_2, \\
x_3 &= \qquad\qquad\qquad\quad c_{34}x_4 + \cdots + c_{3n}x_n + d_3, \\
&\vdots \\
x_m &= \qquad\quad\; c_{m,m+1}x_{m+1} + \cdots + c_{mn}x_n + d_m,
\end{aligned} \tag{6}$$

where the c_{ij}'s and the d_i's are some combination of the original a_{ij}'s and b_i's. System (6) can be quickly solved by back substitution.

Example 1

Use the method of substitution to solve the system

$$\begin{aligned}
r + 2s + t &= 3, \\
2r + 3s - t &= -6, \\
3r - 2s - 4t &= -2.
\end{aligned}$$

Solution. By solving the first equation for r and then substituting it into the second and third equations, we obtain the first derived set

$$\begin{aligned} r &= 3 - 2s - t, \\ -s - 3t &= -12, \\ -8s - 7t &= -11. \end{aligned}$$

By solving the second equation for s and then substituting it into the third equation, we obtain the second derived set

$$\begin{aligned} r &= 3 - 2s - t, \\ s &= 12 - 3t, \\ 17t &= 85. \end{aligned}$$

By solving for t in the third equation and then substituting it into the remaining equations (of which there are none), we obtain the third derived set

$$\begin{aligned} r &= 3 - 2s - t, \\ s &= 12 - 3t, \\ t &= 5. \end{aligned}$$

Thus, the solution is $t = 5$, $s = -3$, $r = 4$. □

Example 2

Use the method of substitution to solve the system

$$\begin{aligned} x + y + 3z &= -1, \\ 2x - 2y - z &= 1, \\ 5x + y + 8z &= -2. \end{aligned}$$

Solution. The first derived set is

$$\begin{aligned} x &= -1 - y - 3z, \\ -4y - 7z &= 3, \\ -4y - 7z &= 3. \end{aligned}$$

The second derived set is

$$x = -1 - y - 3z,$$
$$y = -\tfrac{3}{4} - \tfrac{7}{4}z,$$
$$0 = 0.$$

Since the third equation can not be solved for z, this is as far as we can go. Thus, since we can not obtain a unique value for z, the first and second equations will not yield a unique value for x and y. *Caution*: The third equation does *not* imply that $z = 0$. On the contrary, this equation says nothing at all about z, consequently z is completely arbitrary. The second equation gives y in terms of z. Substituting this value into the first equation, we obtain x in terms of z. The solution therefore is $x = -\frac{1}{4} - \frac{5}{4}z$ and $y = -\frac{3}{4} - \frac{7}{4}z$, z is arbitrary. Thus there are infinitely many solutions to the above system. However, once z is chosen, x and y are determined. If z is chosen to be -1, then $x = y = 1$, while if z is chosen to be 3, then $x = -4$, $y = -6$. The solutions can be expressed in the vector form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} - \frac{5}{4}z \\ -\frac{3}{4} - \frac{7}{4}z \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} \\ -\frac{3}{4} \\ 0 \end{bmatrix} + z\begin{bmatrix} -\frac{5}{4} \\ -\frac{7}{4} \\ 1 \end{bmatrix}. \qquad \square$$

Example 3

Use the method of substitution to solve

$$a + 2b - 3c + d = 1,$$
$$2a + 6b + 4c + 2d = 8.$$

Solution. The first derived set is

$$a = 1 - 2b + 3c - d,$$
$$2b + 10c = 6.$$

The second derived set is

$$a = 1 - 2b + 3c - d$$
$$b = 3 - 5c$$

Again, since there are no more equations, this is as far as we can go, and since there are no defining equations for c and d, these two unknowns must be arbitrary. Solving for a and b in terms of c and d, we obtain the solution

$a = -5 + 13c - d$, $b = 3 - 5c$; c and d are arbitrary. The solutions can be expressed in the vector form

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -5 + 13c - d \\ 3 - 5c \\ c \\ d \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} 13 \\ -5 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Note that while c and d are arbitrary, once they are given a particular value, a and b are automatically determined. For example, if c is chosen as -1 and d as 4, a solution is $a = -22$, $b = 8$, $c = -1$, $d = 4$, while if c is chosen as 0 and d as -3, a solution is $a = -2$, $b = 3$, $c = 0$, $d = -3$. □

Example 4

Use the method of substitution to solve the following system:

$$\begin{aligned} x + 3y &= 4, \\ 2x - y &= 1, \\ 3x + 2y &= 5, \\ 5x + 15y &= 20. \end{aligned}$$

Solution. The first derived set is

$$\begin{aligned} x &= 4 - 3y, \\ -7y &= -7, \\ -7y &= -7, \\ 0 &= 0. \end{aligned}$$

The second derived set is

$$\begin{aligned} x &= 4 - 3y, \\ y &= 1, \\ 0 &= 0, \\ 0 &= 0. \end{aligned}$$

Thus, the solution is $y = 1$, $x = 1$, or in vector form

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad \square$$

Problems 2.2

Use the method of substitution to solve the following systems:

(1)
$$\begin{aligned} x + 2y - 2z &= -1, \\ 2x + y + z &= 5, \\ -x + y - z &= -2. \end{aligned}$$

(2)
$$\begin{aligned} x + y - z &= 0, \\ 3x + 2y + 4z &= 0. \end{aligned}$$

(3)
$$\begin{aligned} x + 3y &= 4, \\ 2x - y &= 1, \\ -2x - 6y &= -8, \\ 4x - 9y &= -5, \\ -6x + 3y &= -3. \end{aligned}$$

(4)
$$\begin{aligned} 4r - 3s + 2t &= 1, \\ r + s - 3t &= 4, \\ 5r - 2s - t &= 5. \end{aligned}$$

(5)
$$\begin{aligned} 2l - m + n - p &= 1, \\ l + 2m - n + 2p &= -1, \\ l - 3m + 2n - 3p &= 2. \end{aligned}$$

(6)
$$\begin{aligned} 2x + y - z &= 0, \\ x + 2y + z &= 0, \\ 3x - y + 2z &= 0. \end{aligned}$$

(7)
$$\begin{aligned} x + 2y - z &= 5, \\ 2x - y + 2z &= 1, \\ 2x + 2y - z &= 7, \\ x + 2y + z &= 3. \end{aligned}$$

(8)
$$\begin{aligned} x + 2y + z - 2w &= 1, \\ 2x + 2y - z - w &= 3, \\ 2x - 2y + 2z + 3w &= 3, \\ 3x + y - 2z - 3w &= 1. \end{aligned}$$

2.3 Gaussian Elimination

Although the method of substitution is straightforward, it is not the most efficient way to solve simultaneous equations, and it does not lend itself well to electronic computing. Computers have difficulty symbolically manipulating the unknowns in algebraic equations. A striking feature of the method of substitution, however, is that the unknowns remain unaltered throughout the process: x remains x, y remains y, z remains z. Only the coefficients of the unknowns and the numbers on the right side of the equations change from one derived set to the next. Thus, we can save a good deal of writing, and develop a useful representation for computer processing, if we direct our attention to just the numbers themselves.

Definition 1. Given the system $\mathbf{Ax} = \mathbf{b}$, the *augmented matrix*, designated by $\mathbf{A^b}$, is a matrix obtained from $\mathbf{A}$ by adding to it one extra column, namely $\mathbf{b}$.

Thus, if

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 7 \\ 8 \end{bmatrix},$$

then

$$\mathbf{A^b} = \begin{bmatrix} 1 & 2 & 3 & 7 \\ 4 & 5 & 6 & 8 \end{bmatrix},$$

while if

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix},$$

then

$$\mathbf{A^b} = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 4 & 5 & 6 & -2 \\ 7 & 8 & 9 & -3 \end{bmatrix}.$$

In particular, the system

$$\begin{aligned} x + y - 2z &= -3, \\ 2x + 5y + 3z &= 11, \\ -x + 3y + z &= 5 \end{aligned}$$

has the matrix representation

$$\begin{bmatrix} 1 & 1 & -2 \\ 2 & 5 & 3 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 11 \\ 5 \end{bmatrix},$$

with an augmented matrix of

$$\mathbf{A^b} = \begin{bmatrix} 1 & 1 & -2 & -3 \\ 2 & 5 & 3 & 11 \\ -1 & 3 & 1 & 5 \end{bmatrix}.$$

Example 1

Write the set of equations in x, y, and z associated with the augmented matrix

$$\mathbf{A}^b = \begin{bmatrix} -2 & 1 & 3 & 8 \\ 0 & 4 & 5 & -3 \end{bmatrix}.$$

Solution.

$$\begin{aligned} -2x + \quad y + 3z &= \quad 8, \\ 4y + 5z &= -3. \qquad \square \end{aligned}$$

A second striking feature to the method of substitution is that every derived set is different from the system that preceded it. The method continues creating new derived sets until it has one that is particularly easy to solve by back-substitution. Of course, there is no purpose in solving any derived set, regardless how easy it is, unless we are assured beforehand that it has the same solution as the original system. Three elementary operations that alter equations but do not change their solutions are:

(i) Interchange the positions of any two equations.
(ii) Multiply an equation by a nonzero scalar.
(iii) Add to one equation a scalar times another equation.

If we restate these operations in words appropriate to an augmented matrix, we obtain the *elementary row operations*:

(E1) Interchange any two rows in a matrix.
(E2) Multiply any row of a matrix by a nonzero scalar.
(E3) Add to one row of a matrix a scalar times another row of that same matrix.

Gaussian elimination is a matrix method for solving simultaneous linear equations. The augmented matrix for the system is created, and then it is transformed into a row-reduced matrix (see Section 1.4) using elementary row operations. This is most often accomplished by using operation (E3) with each diagonal element in a matrix to create zeros in all columns directly below it, beginning with the first column and moving successively through the matrix, column by column. The system of equations associated with a row-reduced matrix can be solved easily by back-substitution, if we solve each equation for the first unknown that appears in it. This is the unknown associated with the first nonzero element in each nonzero row of the final augmented matrix.

Example 2

Use Gaussian elimination to solve

$$\begin{aligned} x + 3y &= 4, \\ 2x - y &= 1, \\ 3x + 2y &= 5, \\ 5x + 15y &= 20. \end{aligned}$$

Solution. The augmented matrix for this system is

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & -1 & 1 \\ 3 & 2 & 5 \\ 5 & 15 & 20 \end{bmatrix}.$$

Then,

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & -1 & 1 \\ 3 & 2 & 5 \\ 5 & 15 & 20 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 4 \\ 0 & -7 & -7 \\ 3 & 2 & 5 \\ 5 & 15 & 20 \end{bmatrix} \quad \left\{\begin{array}{l}\text{by adding to the} \\ \text{second row } (-2) \text{ times} \\ \text{the first row}\end{array}\right.$$

$$\to \begin{bmatrix} 1 & 3 & 4 \\ 0 & -7 & -7 \\ 0 & -7 & -7 \\ 5 & 15 & 20 \end{bmatrix} \quad \left\{\begin{array}{l}\text{by adding to the} \\ \text{third row } (-3) \text{ times} \\ \text{the first row}\end{array}\right.$$

$$\to \begin{bmatrix} 1 & 3 & 4 \\ 0 & -7 & -7 \\ 0 & -7 & -7 \\ 0 & 0 & 0 \end{bmatrix} \quad \left\{\begin{array}{l}\text{by adding to the} \\ \text{fourth row } (-5) \text{ times} \\ \text{the first row}\end{array}\right.$$

$$\to \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & -7 & -7 \\ 0 & 0 & 0 \end{bmatrix} \quad \left\{\begin{array}{l}\text{by multiplying the} \\ \text{second row by } \dfrac{-1}{7}\end{array}\right.$$

$$\to \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad \left\{\begin{array}{l}\text{by adding to the} \\ \text{second row } (7) \text{ times} \\ \text{the first row}\end{array}\right.$$

The system of equations associated with this last augmented matrix in row-reduced form is

$$\begin{aligned} x + 3y &= 4, \\ y &= 1, \\ 0 &= 0, \\ 0 &= 0. \end{aligned}$$

Solving the second equation for y and then the first equation for x, we obtain $x = 1$ and $y = 1$, which is also the solution to the original set of equations. Compare this solution with Example 4 of the previous section. □

The notation ($\rightarrow$) should be read "is transformed into"; an equality sign is not correct because the transformed matrix is not equal to the original one.

Example 3

Use Gaussian elimination to solve

$$\begin{aligned} r + 2s + t &= 3, \\ 2r + 3s - t &= -6, \\ 3r - 2s - 4t &= -2. \end{aligned}$$

Solution. The augmented matrix for this system is

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 3 & -1 & -6 \\ 3 & -2 & -4 & -2 \end{bmatrix}.$$

Then,

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 3 & -1 & -6 \\ 3 & -2 & -4 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & -3 & -12 \\ 3 & -2 & -4 & -2 \end{bmatrix} \quad \left\{\begin{array}{l}\text{by adding to the} \\ \text{second row } (-2) \text{ times} \\ \text{the first row}\end{array}\right.$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & -3 & -12 \\ 0 & -8 & -7 & -11 \end{bmatrix} \quad \left\{\begin{array}{l}\text{by adding to the} \\ \text{third row } (-3) \text{ times} \\ \text{the first row}\end{array}\right.$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 3 & 12 \\ 0 & -8 & -7 & -11 \end{bmatrix} \quad \left\{\begin{array}{l}\text{by multiplying the} \\ \text{the second row by } (-1)\end{array}\right.$$

$$\rightarrow\begin{bmatrix}1 & 2 & 1 & 3\\0 & 1 & 3 & 12\\0 & 0 & 17 & 85\end{bmatrix}\qquad\left\{\begin{array}{l}\text{by adding to the}\\\text{third row (8) times}\\\text{the second row}\end{array}\right.$$

$$\rightarrow\begin{bmatrix}1 & 2 & 1 & 3\\0 & 1 & 3 & 12\\0 & 0 & 1 & 5\end{bmatrix}.\qquad\left\{\begin{array}{l}\text{by multiplying the}\\\text{third row by }\left(\dfrac{1}{17}\right)\end{array}\right.$$

The system of equations associated with this last augmented matrix in row-reduced form is

$$\begin{aligned}r + 2s + t &= 3,\\ s + 3t &= 12,\\ t &= 5.\end{aligned}$$

Solving the third equation for t, then the second equation for s, and, lastly, the first equation for r, we obtain $r = 4$, $s = -3$, and $t = 5$, which is also the solution to the original set of equations. Compare this solution with Example 1 of the previous section. □

Whenever one element in a matrix is used to cancel another element to zero by elementary row operation (E3), the first element is called the *pivot*. In Example 3, we first used the element in the 1–1 position to cancel the element in the 2–1 position, and then to cancel the element in the 3–1 position. In both of these operations, the unity element in the 1–1 position was the pivot. Later, we used the unity element in the 2–2 position to cancel the element -8 in the 3–2 position; here, the 2–2 element was the pivot.

While transforming a matrix into row-reduced form, it is advisable to adhere to three basic principles:

- Completely transform one column to the required form before considering another column.
- Work on columns in order, from left to right.
- Never use an operation if it will change a zero in a previously transformed column.

As a consequence of this last principle, one never involves the ith row of a matrix in an elementary row operation after the ith column has been transformed into its required form. That is, once the first column has the proper form, no pivot element should ever again come from the first row; once the second column has the proper form, no pivot element should ever again come from the second row; and so on.

When an element we want to use as a pivot is itself zero, we interchange rows using operation (E1).

Example 4

Use Gaussian elimination to solve

$$\begin{aligned} 2c + 3d &= 4,\\ a \qquad + 3c + d &= 2,\\ a + b + 2c \qquad &= 0. \end{aligned}$$

Solution. The augmented matrix is

$$\begin{bmatrix} 0 & 0 & 2 & 3 & 4 \\ 1 & 0 & 3 & 1 & 2 \\ 1 & 1 & 2 & 0 & 0 \end{bmatrix}.$$

Normally, we would use the element in the 1–1 position to cancel to zero the two elements directly below it, but we cannot because it is zero. To proceed with the reduction process, we must interchange the first row with either of the other two rows. The choice is arbitrary.

$$\begin{bmatrix} 0 & 0 & 2 & 3 & 4 \\ 1 & 0 & 3 & 1 & 2 \\ 1 & 1 & 2 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 & 1 & 2 \\ 0 & 0 & 2 & 3 & 4 \\ 1 & 1 & 2 & 0 & 0 \end{bmatrix} \quad \left\{\begin{array}{l}\text{by interchanging the}\\ \text{first row with the}\\ \text{second row}\end{array}\right.$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 3 & 1 & 2 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 1 & -1 & -1 & -2 \end{bmatrix}. \quad \left\{\begin{array}{l}\text{by adding to the}\\ \text{third row } (-1) \text{ times}\\ \text{the first row}\end{array}\right.$$

Next, we would like to use the element in the 2–2 position to cancel to zero the element in the 3–2 position, but we cannot because that prospective pivot is zero. We use elementary row operation (E1) once again. The transformation yields

$$\rightarrow \begin{bmatrix} 1 & 0 & 3 & 1 & 2 \\ 0 & 1 & -1 & -1 & -2 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix} \quad \left\{\begin{array}{l}\text{by interchanging the}\\ \text{second row with the}\\ \text{third row}\end{array}\right.$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 3 & 1 & 2 \\ 0 & 1 & -1 & -1 & -2 \\ 0 & 0 & 1 & 1.5 & 2 \end{bmatrix}. \quad \begin{cases} \text{by multiplying the} \\ \text{third row by (0.5)} \end{cases}$$

The system of equations associated with this last augmented matrix in row-reduced form is

$$\begin{aligned} a \quad\ + 3c + \quad d &= \ \ 2, \\ b - \ \ c - \quad d &= -2, \\ c + 1.5d &= \ \ 2. \end{aligned}$$

We use the third equation to solve for c, the second equation to solve for b, and the first equation to solve for a, because these are the unknowns associated with the first nonzero element of each nonzero row in the final augmented matrix. We have no defining equation for d, so this unknown remains arbitrary. The solution is, $a = -4 + 3.5d$. $b = -0.5d$, $c = 2 - 1.5d$, and d arbitrary, or in vector form

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -4 + 3.5d \\ -0.5d \\ 2 - 1.5d \\ d \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 2 \\ 0 \end{bmatrix} + \frac{d}{2}\begin{bmatrix} 7 \\ -1 \\ -3 \\ 2 \end{bmatrix}.$$

This is also the solution to the original set of equations. □

The derived set of equations associated with a row-reduced, augmented matrix may contain an absurd equation, such as $0 = 1$. In such cases, we conclude that the derived set is inconsistent, because no values of the unknowns can simultaneously satisfy all the equations. In particular, it is impossible to choose values of the unknowns that will make the absurd equation true. Since the derived set has the same solutions as the original set, it follows that the original set of equations is also inconsistent.

Example 5

Use Gaussian elimination to solve

$$\begin{aligned} 2x + 4y + 3z &= \ \ 8, \\ 3x - 4y - 4z &= \ \ 3, \\ 5x \qquad\quad - \ z &= 12. \end{aligned}$$

Solution. The augmented matrix for this system is

$$\begin{bmatrix} 2 & 4 & 3 & 8 \\ 3 & -4 & -4 & 3 \\ 5 & 0 & -1 & 12 \end{bmatrix}.$$

Then,

$$\begin{bmatrix} 2 & 4 & 3 & 8 \\ 3 & -4 & -4 & 3 \\ 5 & 0 & -1 & 12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1.5 & 4 \\ 3 & -4 & -4 & 3 \\ 5 & 0 & -1 & 12 \end{bmatrix} \quad \left\{\begin{array}{l}\text{by multiplying the} \\ \text{first row by } \left(\frac{1}{2}\right)\end{array}\right.$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 1.5 & 4 \\ 0 & -10 & -8.5 & -9 \\ 5 & 0 & -1 & 12 \end{bmatrix} \quad \left\{\begin{array}{l}\text{by adding to the} \\ \text{second row } (-3) \text{ times} \\ \text{the first row}\end{array}\right.$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 1.5 & 4 \\ 0 & -10 & -8.5 & -9 \\ 0 & -10 & -8.5 & -8 \end{bmatrix} \quad \left\{\begin{array}{l}\text{by adding to the} \\ \text{third row } (-5) \text{ times} \\ \text{the first row}\end{array}\right.$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 1.5 & 4 \\ 0 & 1 & 0.85 & 0.9 \\ 0 & -10 & -8.5 & -8 \end{bmatrix} \quad \left\{\begin{array}{l}\text{by multiplying the} \\ \text{second row by } \left(\frac{-1}{10}\right)\end{array}\right.$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 1.5 & 4 \\ 0 & 1 & 0.85 & 0.9 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad \left\{\begin{array}{l}\text{by adding to the} \\ \text{third row } (10) \text{ times} \\ \text{the second row}\end{array}\right.$$

The system of equations associated with this last augmented matrix in row-reduced form is

$$\begin{aligned} x + 2y + 1.5z &= 4, \\ y + 0.85z &= 0.9, \\ 0 &= 1. \end{aligned}$$

Since no values of x, y, and z can make this last equation true, this system, as well as the original one, has no solution. □

Finally, we note that most matrices can be transformed into a variety of row-reduced forms. If a row-reduced matrix has two nonzero rows, then a different row-reduced matrix is easily constructed by adding to the first row any nonzero constant times the second row. The equations associated with both augmented matrices, however, will have identical solutions.

Problems 2.3

In Problems 1 through 5, construct augmented matrices for the given systems of equations:

(1) $$\begin{aligned} x + 2y &= -3, \\ 3x + y &= 1. \end{aligned}$$

(2) $$\begin{aligned} x + 2y - z &= -1, \\ 2x - 3y + 2z &= 4. \end{aligned}$$

(3) $$\begin{aligned} a + 2b &= 5, \\ -3a + b &= 13, \\ 4a + 3b &= 0. \end{aligned}$$

(4) $$\begin{aligned} 2r + 4s &= 2, \\ 3r + 2s + t &= 8, \\ 5r - 3s + 7t &= 15. \end{aligned}$$

(5) $$\begin{aligned} 2r + 3s - 4t &= 12, \\ 3r - 2s &= -1, \\ 8r - s - 4t &= 10. \end{aligned}$$

In Problems 6 through 11, write the set of equations associated with the given augmented matrix and the specified variables.

(6) $\mathbf{A}^{\mathbf{b}} = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 8 \end{bmatrix}$ variables: x and y.

(7) $\mathbf{A}^{\mathbf{b}} = \begin{bmatrix} 1 & -2 & 3 & 10 \\ 0 & 1 & -5 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$ variables: x, y, and z.

(8) $\mathbf{A}^{\mathbf{b}} = \begin{bmatrix} 1 & -3 & 12 & 40 \\ 0 & 1 & -6 & -200 \\ 0 & 0 & 1 & 25 \end{bmatrix}$ variables: r, s, and t.

(9) $\mathbf{A}^{\mathbf{b}} = \begin{bmatrix} 1 & 3 & 0 & -8 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ variables: x, y, and z.

(10) $\mathbf{A}^{\mathbf{b}} = \begin{bmatrix} 1 & -7 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ variables: a, b, and c.

(11) $\mathbf{A}^b = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ variables: u, v, and w.

(12) Solve the system of equations defined in Problem 6.

(13) Solve the system of equations defined in Problem 7.

(14) Solve the system of equations defined in Problem 8.

(15) Solve the system of equations defined in Problem 9.

(16) Solve the system of equations defined in Problem 10.

(17) Solve the system of equations defined in Problem 11.

In Problems 18 through 24, use elementary row operations to transform the given matrices into row-reduced form:

(18) $\begin{bmatrix} 1 & -2 & 5 \\ -3 & 7 & 8 \end{bmatrix}$.

(19) $\begin{bmatrix} 4 & 24 & 20 \\ 2 & 11 & -8 \end{bmatrix}$.

(20) $\begin{bmatrix} 0 & -1 & 6 \\ 2 & 7 & -5 \end{bmatrix}$.

(21) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & 2 & 3 \\ -2 & 3 & 0 & 0 \end{bmatrix}$.

(22) $\begin{bmatrix} 0 & 1 & -2 & 4 \\ 1 & 3 & 2 & 1 \\ -2 & 3 & 1 & 2 \end{bmatrix}$.

(23) $\begin{bmatrix} 1 & 3 & 2 & 0 \\ -1 & -4 & 3 & -1 \\ 2 & 0 & -1 & 3 \\ 2 & -1 & 4 & 2 \end{bmatrix}$.

(24) $\begin{bmatrix} 2 & 3 & 4 & 6 & 0 & 10 \\ -5 & -8 & 15 & 1 & 3 & 40 \\ 3 & 3 & 5 & 4 & 4 & 20 \end{bmatrix}$.

(25) Solve Problem 1.

(26) Solve Problem 2.

(27) Solve Problem 3.

(28) Solve Problem 4.

(29) Solve Problem 5.

(30) Use Gaussian elimination to solve Problem 1 of Section 2.2.

(31) Use Gaussian elimination to solve Problem 2 of Section 2.2.

(32) Use Gaussian elimination to solve Problem 3 of Section 2.2.

(33) Use Gaussian elimination to solve Problem 4 of Section 2.2.

(34) Use Gaussian elimination to solve Problem 5 of Section 2.2.

(35) Determine a production schedule that satisfies the requirements of the manufacturer described in Problem 12 of Section 2.1.

(36) Determine a production schedule that satisfies the requirements of the manufacturer described in Problem 13 of Section 2.1.

(37) Determine a production schedule that satisfies the requirements of the manufacturer described in Problem 14 of Section 2.1.

(38) Determine feed blends that satisfy the nutritional requirements of the pet store described in Problem 15 of Section 2.1.

(39) Determine the bonus for the company described in Problem 16 of Section 2.1.

(40) Determine the number of barrels of gasoline that the producer described in Problem 17 of Section 2.1 must manufacture to breakeven.

(41) Determine the annual incomes of each sector of the Leontief closed model described in Problem 18 of Section 2.1.

(42) Determine the wages of each person in the Leontief closed model described in Problem 19 of Section 2.1.

(43) Determine the total sales revenue for each country of the Leontief closed model described in Problem 20 of Section 2.1.

(44) Determine the production quotas for each sector of the economy described in Problem 22 of Section 2.1.

(45) An *elementary matrix* is a square matrix $\mathbf{E}$ having the property that the product $\mathbf{EA}$ is the result of applying a single elementary row operation on the matrix $\mathbf{A}$. Form a matrix $\mathbf{H}$ from the 4×4 identity matrix $\mathbf{I}$ by interchanging any two rows of $\mathbf{I}$, and then compute the product $\mathbf{HA}$ for any 4×4 matrix $\mathbf{A}$ of your choosing. Is $\mathbf{H}$ an elementary matrix? How would one construct elementary matrices corresponding to operation (E1)?

(46) Form a matrix $\mathbf{G}$ from the 4×4 identity matrix $\mathbf{I}$ by multiplying any one row of $\mathbf{I}$ by the number 5, and then compute the product $\mathbf{HA}$ for any 4×4 matrix $\mathbf{A}$ of your choosing. Is $\mathbf{G}$ an elementary matrix? How would one construct elementary matrices corresponding to operation (E2)?

(47) Form a matrix $\mathbf{F}$ from the 4×4 identity matrix $\mathbf{I}$ by adding to one row of $\mathbf{I}$ five times another row of $\mathbf{I}$. Use any two rows of your choosing. Compute the product $\mathbf{FA}$ for any 4×4 matrix $\mathbf{A}$ of your choosing. Is $\mathbf{F}$ an elementary matrix? How would one construct elementary matrices corresponding to operation (E3)?

(48) A solution procedure uniquely suited to matrix equations of the form $\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{d}$ is iteration. A trial solution $\mathbf{x}^{(0)}$ is proposed, and then progressively better estimates $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)}, \ldots$ for the solution are obtained iteratively from the formula

$$\mathbf{x}^{(i+1)} = \mathbf{A}\mathbf{x}^{(i)} + \mathbf{d}.$$

The iterations terminate when two successive estimates differ by less than a prespecified acceptable tolerance.

If the system comes from a Leontief input–output model, then a reasonable initialization is $\mathbf{x}^{(0)} = 2\mathbf{d}$. Apply this method to the system defined by Problem 22 of Section 2.1. Stop after two iterations.

(49) Use the iteration method described in the previous problem to solve the system defined in Problem 23 of Section 2.1. In particular, find the first two iterations by hand calculations, and then use a computer to complete the iteration process.

(50) Use the iteration method described in Problem 48 to solve the system defined in Problem 24 of Section 2.1. In particular, find the first two iterations by hand calculations, and then use a computer to complete the iteration process.

2.4 Pivoting Strategies

Gaussian elimination is often programmed for computer implementation. Since all computers round or truncate numbers to a finite number of digits (e.g., the fraction 1/3 might be stored as 0.33333, but never as the *infinite* decimal 0.333333...) roundoff error can be significant. A number of strategies have been developed to minimize the effects of such errors.

The most popular strategy is *partial pivoting*, which requires that a pivot element always be larger in absolute value than any element below it in the same column. This is accomplished by interchanging rows whenever necessary.

Example 1

Use partial pivoting with Gaussian elimination to solve the system

$$\begin{aligned} x + 2y + 4z &= 18, \\ 2x + 12y - 2z &= 9, \\ 5x + 26y + 5z &= 14. \end{aligned}$$

Solution. The augmented matrix for this system is

$$\begin{bmatrix} 1 & 2 & 4 & 18 \\ 2 & 12 & -2 & 9 \\ 5 & 26 & 5 & 14 \end{bmatrix}.$$

Normally, the unity element in the 1–1 position would be the pivot. With partial pivoting, we compare this prospective pivot to all elements directly below it in the same column, and if any is larger in absolute value, as is the case here with the element 5 in the 3–1 position, we interchange rows to bring the largest element into the pivot position.

$$\begin{bmatrix} 1 & 2 & 4 & 18 \\ 2 & 12 & -2 & 9 \\ 5 & 26 & 5 & 14 \end{bmatrix} \to \begin{bmatrix} 5 & 26 & 5 & 14 \\ 2 & 12 & -2 & 9 \\ 1 & 2 & 4 & 18 \end{bmatrix}. \quad \begin{cases} \text{by interchanging the} \\ \text{the first and third rows} \end{cases}$$

Then,

$$\to \begin{bmatrix} 1 & 5.2 & 1 & 2.8 \\ 2 & 12 & -2 & 9 \\ 1 & 2 & 4 & 18 \end{bmatrix} \quad \begin{cases} \text{by multiplying the} \\ \text{first row by } \frac{1}{5} \end{cases}$$

$$\to \begin{bmatrix} 1 & 5.2 & 1 & 2.8 \\ 0 & 1.6 & -4 & 3.4 \\ 1 & 2 & 4 & 18 \end{bmatrix} \quad \begin{cases} \text{by adding to the} \\ \text{second row } (-2) \text{ times} \\ \text{the first row} \end{cases}$$

$$\to \begin{bmatrix} 1 & 5.2 & 1 & 2.8 \\ 0 & 1.6 & -4 & 3.4 \\ 0 & -3.2 & 3 & 15.2 \end{bmatrix}. \quad \begin{cases} \text{by adding to the} \\ \text{third row } (-1) \text{ times} \\ \text{the first row} \end{cases}$$

The next pivot would normally be the element 1.6 in the 2–2 position. Before accepting it, however, we compare it to all elements directly below it in the same column. The largest element in absolute value is the the element -3.2 in the 3–2 position. Therefore, we interchange rows to bring this larger element into the pivot position.

NOTE. We do not consider the element 5.2 in the 1–2 position, even though it is the largest element in its column. Comparisons are only made between a prospective pivot and all elements directly below it. Recall one of the three basic principles of row-reduction: never involve the first row of matrix in a row operation after the first column has been transformed into its required form.

$$\rightarrow \begin{bmatrix} 1 & 5.2 & 1 & 2.8 \\ 0 & -3.2 & 3 & 15.2 \\ 0 & 1.6 & -4 & 3.4 \end{bmatrix} \quad \begin{cases} \text{by interchanging the} \\ \text{second and third rows} \end{cases}$$

$$\rightarrow \begin{bmatrix} 1 & 5.2 & 1 & 2.8 \\ 0 & 1 & -0.9375 & -4.75 \\ 0 & 1.6 & -4 & 3.4 \end{bmatrix} \quad \begin{cases} \text{by multiplying the} \\ \text{second row by } \dfrac{-1}{3.2} \end{cases}$$

$$\rightarrow \begin{bmatrix} 1 & 5.2 & 1 & 2.8 \\ 0 & 1 & -0.9375 & -4.75 \\ 0 & 0 & -2.5 & 11 \end{bmatrix} \quad \begin{cases} \text{by adding to the} \\ \text{third row } (-1.6) \text{ times} \\ \text{the second row} \end{cases}$$

$$\rightarrow \begin{bmatrix} 1 & 5.2 & 1 & 2.8 \\ 0 & 1 & -0.9375 & -4.75 \\ 0 & 0 & 1 & -4.4 \end{bmatrix}. \quad \begin{cases} \text{by multiplying the} \\ \text{third row by } \dfrac{-1}{2.5} \end{cases}$$

The new derived set of equations is

$$\begin{aligned} x + 5.2y + z &= 2.8, \\ y - 0.9375z &= -4.75, \\ z &= -4.4, \end{aligned}$$

which has as its solution $x = 53.35$, $y = -8.875$, and $z = -4.4$. □

Scaled pivoting involves ratios. A prospective pivot is divided by the largest element in absolute value in its row, ignoring the last column. The result is compared to the ratios formed by dividing every element directly below the pivot by the largest element in absolute value in its respective row, again ignoring the last column. Of these, the element that yields the largest ratio in absolute value is designated as the pivot, and if that element is not already in the pivot position, then row interchanges are performed to move it there.

Example 2

Use scaled pivoting with Gaussian elimination to solve the system given in Example 1.

Solution. The augmented matrix for this system is

$$\begin{bmatrix} 1 & 2 & 4 & 18 \\ 2 & 12 & -2 & 9 \\ 5 & 26 & 5 & 14 \end{bmatrix}.$$

Normally, we would use the element in the 1–1 position as the pivot. With scaled pivoting, however, we first compare ratios between elements in the first column to the largest elements in absolute value in each row, ignoring the last column. The ratios are

$$\frac{1}{4} = 0.25, \qquad \frac{2}{12} = 0.1667, \quad \text{and} \quad \frac{5}{26} = 0.1923.$$

The largest ratio in absolute value corresponds to the unity element in the 1–1 position, so that element remains the pivot. Transforming the first column into reduced form, we obtain

$$\begin{bmatrix} 1 & 2 & 4 & 18 \\ 0 & 8 & -10 & -27 \\ 0 & 16 & -15 & -76 \end{bmatrix}.$$

Normally, the next pivot would be the element in the 2–2 position. Instead, we consider the ratios

$$\frac{8}{10} = 0.8 \quad \text{and} \quad \frac{16}{16} = 1,$$

which are obtained by dividing the pivot element and every element directly below it by the largest element in absolute value appearing in their respective rows, ignoring elements in the last column. The largest ratio in absolute value corresponds to the element 16 appearing in the 3–2 position. We move it into the pivot position by interchanging the second and third rows. The new matrix is

$$\begin{bmatrix} 1 & 2 & 4 & 18 \\ 0 & 16 & -15 & -76 \\ 0 & 8 & -10 & -27 \end{bmatrix}.$$

Completing the row-reduction transformation, we get

$$\begin{bmatrix} 1 & 2 & 4 & 18 \\ 0 & 1 & -0.9375 & -4.75 \\ 0 & 0 & 1 & -4.4 \end{bmatrix}.$$

The system of equations associated with this matrix is

$$\begin{aligned} x + 2y + \quad 4z &= \quad 18, \\ y - 0.9375z &= -4.75, \\ z &= -4.4. \end{aligned}$$

The solution is, as before, $x = 53.35$, $y = -8.875$, and $z = -4.4$. $\square$

Complete pivoting compares prospective pivots with all elements in the largest submatrix for which the prospective pivot is in the upper left position, ignoring the last column. If any element in this submatrix is larger in absolute value than the prospective pivot, both row and column interchanges are made to move this larger element into the pivot position. Because column interchanges rearrange the order of the unknowns, a bookkeeping method must be implemented to record all rearrangements. This is done by adding a new row, designated as row 0, to the matrix. The entries in the new row are initially the positive integers in ascending order, to denote that column 1 is associated with variable 1, column 2 with variable 2, and so on. This new top row is only affected by column interchanges; *none of the elementary row operations is applied to it.*

Example 3

Use complete pivoting with Gaussian elimination to solve the system given in Example 1.

Solution. The augmented matrix for this system is

$$\left[\begin{array}{cccc} 1 & 2 & 3 & \\ \hdashline 1 & 2 & 4 & 18 \\ 2 & 12 & -2 & 9 \\ 5 & 26 & 5 & 14 \end{array}\right].$$

Normally, we would use the element in the 1–1 position of the coefficient matrix **A** as the pivot. With complete pivoting, however, we first compare this prospective pivot to all elements in the submatrix shaded below. In this case, the element 26 is the largest, so we interchange rows and columns to bring it into the pivot position.

$$\left[\begin{array}{cccc} 1 & 2 & 3 & \\ \hdashline 1 & 2 & 4 & 18 \\ 2 & 12 & -2 & 9 \\ 5 & 26 & 5 & 14 \end{array}\right] \to \left[\begin{array}{cccc} 1 & 2 & 3 & \\ \hdashline 5 & 26 & 5 & 14 \\ 2 & 12 & -2 & 9 \\ 1 & 2 & 4 & 18 \end{array}\right] \quad \left\{\begin{array}{l}\text{by interchanging the}\\ \text{first and third rows}\end{array}\right.$$

$$\to \left[\begin{array}{cccc} 2 & 1 & 3 & \\ \hdashline 26 & 5 & 5 & 14 \\ 12 & 2 & -2 & 9 \\ 2 & 1 & 4 & 18 \end{array}\right]. \quad \left\{\begin{array}{l}\text{by interchanging the}\\ \text{first and second columns}\end{array}\right.$$

Applying Gaussian elimination to the first column, we obtain

$$\begin{array}{c} \begin{array}{cccc} 2 & 1 & 3 & \end{array} \\ \begin{bmatrix} 1 & 0.1923 & 0.1923 & 0.5385 \\ 0 & -0.3077 & -4.3077 & 2.5385 \\ 0 & 0.6154 & 3.6154 & 16.9231 \end{bmatrix} \end{array}.$$

Normally, the next pivot would be -0.3077. Instead, we compare this number in absolute value to all the numbers in the submatrix shaded above. The largest such element in absolute value is -4.3077, which we move into the pivot position by interchanging the second and third column. The result is

$$\begin{array}{c} \begin{array}{cccc} 2 & 3 & 1 & \end{array} \\ \begin{bmatrix} 1 & 0.1923 & 0.1923 & 0.5385 \\ 0 & -4.3077 & -0.3077 & 2.5385 \\ 0 & 3.6154 & 0.6154 & 16.9231 \end{bmatrix} \end{array}.$$

Continuing with Gaussian elimination, we obtain the row-reduced matrix

$$\begin{array}{c} \begin{array}{cccc} 2 & 3 & 1 & \end{array} \\ \begin{bmatrix} 1 & 0.1923 & 0.1923 & 0.5385 \\ 0 & 1 & 0.0714 & -0.5893 \\ 0 & 0 & 1 & 53.35 \end{bmatrix} \end{array}.$$

The system associated with this matrix is

$$\begin{aligned} y + 0.1923z + 0.1923x &= 0.5385, \\ z + 0.0714x &= -0.5893, \\ x &= 53.35. \end{aligned}$$

Its solution is, $x = 53.35$, $y = -8.8749$, and $z = -4.3985$, which is within round-off error of the answers gotten previously. □

Complete pivoting generally identifies a better pivot than scaled pivoting which, in turn, identifies a better pivot than partial pivoting. Nonetheless, partial pivoting is most often the strategy of choice. Pivoting strategies are used to avoid roundoff error. We do not need the best pivot; we only need to avoid bad pivots.

Problems 2.4

In Problems 1 through 6, determine the first pivot under (a) partial pivoting, (b) scaled pivoting, and (c) complete pivoting for given augmented matrices.

(1) $\begin{bmatrix} 1 & 3 & 35 \\ 4 & 8 & 15 \end{bmatrix}$.

(2) $\begin{bmatrix} 1 & -2 & -5 \\ 5 & 3 & 85 \end{bmatrix}$.

(3) $\begin{bmatrix} 1 & 8 & 15 \\ 3 & -4 & 11 \end{bmatrix}$.

(4) $\begin{bmatrix} -2 & 8 & -3 & 100 \\ 4 & 5 & 4 & 75 \\ -3 & -1 & 2 & 250 \end{bmatrix}$.

(5) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$.

(6) $\begin{bmatrix} 0 & 2 & 3 & 4 & 0 \\ 1 & 0.4 & 0.8 & 0.1 & 90 \\ 4 & 10 & 1 & 8 & 40 \end{bmatrix}$.

(7) Solve Problem 3 of Section 2.3 using Gaussian elimination with each of the three pivoting strategies.

(8) Solve Problem 4 of Section 2.3 using Gaussian elimination with each of the three pivoting strategies.

(9) Solve Problem 5 of Section 2.3 using Gaussian elimination with each of the three pivoting strategies.

(10) Computers internally store numbers in formats similar to the scientific notation 0.−−−−E−−, representing the number 0.−−−− multiplied by the power of 10 signified by the digits following E. Therefore, 0.1234E06 is 123,400 while 0.9935E02 is 99.35. The number of digits between the decimal point and E is finite and fixed; it is the number of significant figures. Arithmetic operations in computers are performed in registers, which have twice the number of significant figures as storage locations.

Consider the system

$$0.00001x + y = 1.00001,$$

$$x + y = 2.$$

Show that when Gaussian elimination is implemented on this system by a computer limited to four significant figures, the result is $x = 0$ and $y = 1$, which is incorrect. Show further that the difficulty is resolved when partial pivoting is employed.

2.5 Linear Independence

We momentarily digress from our discussion of simultaneous equations to develop the concepts of linearly independent vectors and rank of a matrix, both of which will prove indispensable to us in the ensuing sections.

Definition 1. A vector $\mathbf{V}_1$ is a *linear combination* of the vectors $\mathbf{V}_2, \mathbf{V}_3, \dots, \mathbf{V}_n$ if there exist scalars $d_2, d_3, \dots, d_n$ such that

$$\mathbf{V}_1 = d_2\mathbf{V}_2 + d_3\mathbf{V}_3 + \cdots + d_n\mathbf{V}_n.$$

Example 1

Show that $[1 \quad 2 \quad 3]$ is a linear combination of $[2 \quad 4 \quad 0]$ and $[0 \quad 0 \quad 1]$.

Solution. $[1 \quad 2 \quad 3] = \frac{1}{2}[2 \quad 4 \quad 0] + 3[0 \quad 0 \quad 1]$. □

Referring to Example 1, we could say that the row vector $[1 \quad 2 \quad 3]$ depends linearly on the other two vectors or, more generally, that the set of vectors $\{[1 \quad 2 \quad 3], [2 \quad 4 \quad 0], [0 \quad 0 \quad 1]\}$ is *linearly dependent.* Another way of expressing this dependence would be to say that there exist constants c_1, c_2, c_3 not all zero such that $c_1[1 \quad 2 \quad 3] + c_2[2 \quad 4 \quad 0] + c_3[0 \quad 0 \quad 1] = [0 \quad 0 \quad 0]$. Such a set would be $c_1 = -1$, $c_2 = \frac{1}{2}$, $c_3 = 3$. Note that the set $c_1 = c_2 = c_3 = 0$ is also a suitable set. The important fact about dependent sets, however, is that there exists a set of constants, *not all equal to zero*, that satisfies the equality.

Now consider the set given by $\mathbf{V}_1 = [1 \quad 0 \quad 0]$ $\mathbf{V}_2 = [0 \quad 1 \quad 0]$ $\mathbf{V}_3 = [0 \quad 0 \quad 1]$. It is easy to verify that no vector in this set is a linear combination of the other two. Thus, each vector is linearly independent of the other two or, more generally, the set of vectors is *linearly independent.* Another way of expressing this independence would be to say the only scalars that satisfy the equation $c_1[1 \quad 0 \quad 0] + c_2[0 \quad 1 \quad 0] + c_3[0 \quad 0 \quad 1] = [0 \quad 0 \quad 0]$ are $c_1 = c_2 = c_3 = 0$.

Definition 2. A set of vectors $\{\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_n\}$, of the same dimension, is *linearly dependent* if there exist scalars $c_1, c_2, \dots, c_n$, not all zero, such that

$$c_1\mathbf{V}_1 + c_2\mathbf{V}_2 + c_3\mathbf{V}_3 + \cdots + c_n\mathbf{V}_n = \mathbf{0}. \tag{7}$$

The vectors are *linearly independent* if the only set of scalars that satisfies (7) is the set $c_1 = c_2 = \cdots = c_n = 0$.

Therefore, to test whether or not a given set of vectors is linearly independent, first form the vector equation (7) and ask "What values for the c's satisfy this equation?" Clearly $c_1 = c_2 = \cdots = c_n = 0$ is a suitable set. If this is the only set of values that satisfies (7) then the vectors are linearly independent. If there exists a set of values that is not all zero, then the vectors are linearly dependent.

Note that it is not necessary for all the c's to be different from zero for a set of vectors to be linearly dependent. Consider the vectors $\mathbf{V}_1 = [1, 2]$, $\mathbf{V}_2 = [1, 4]$, $\mathbf{V}_3 = [2, 4]$. $c_1 = 2$, $c_2 = 0$, $c_3 = -1$ is a set of scalars, *not all zero*, such that $c_1\mathbf{V}_1 + c_2\mathbf{V}_2 + c_3\mathbf{V}_3 = \mathbf{0}$. Thus, this set is linearly dependent.

Example 2

Is the set $\{[1 \quad 2], [3 \quad 4]\}$ linearly independent?

Solution. The vector equation is

$$c_1[1 \quad 2] + c_2[3 \quad 4] = [0 \quad 0].$$

This equation can be rewritten as

$$[c_1 \quad 2c_1] + [3c_2 \quad 4c_2] = [0 \quad 0]$$

or as

$$[c_1 + 3c_2 \quad 2c_1 + 4c_2] = [0 \quad 0].$$

Equating components, we see that this vector equation is equivalent to the system

$$c_1 + 3c_2 = 0,$$

$$2c_1 + 4c_2 = 0.$$

Using Gaussian elimination, we find that the only solution to this system is $c_1 = c_2 = 0$, hence the original set of vectors is linearly independent. □

Although we have worked exclusively with row vectors, the above definitions are equally applicable to column vectors.

Example 3

Is the set

$$\left\{ \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 8 \\ 16 \\ -3 \end{bmatrix} \right\}$$

linearly independent?

Solution. Consider the vector equation

$$c_1\begin{bmatrix}2\\6\\-2\end{bmatrix}+c_2\begin{bmatrix}3\\1\\2\end{bmatrix}+c_3\begin{bmatrix}8\\16\\-3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}. \tag{8}$$

This equation can be rewritten as

$$\begin{bmatrix}2c_1\\6c_1\\-2c_1\end{bmatrix}+\begin{bmatrix}3c_2\\c_2\\2c_2\end{bmatrix}+\begin{bmatrix}8c_3\\16c_3\\-3c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

or as

$$\begin{bmatrix}2c_1+3c_2+8c_3\\6c_1+c_2+16c_3\\-2c_1+2c_2-3c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

By equating components, we see that this vector equation is equivalent to the system

$$\begin{aligned}2c_1+3c_2+8c_3&=0,\\6c_1+c_2+16c_3&=0,\\-2c_1+2c_2-3c_3&=0.\end{aligned}$$

By using Gaussian elimination, we find that the solution to this system is $c_1=(-\frac{5}{2})c_3$, $c_2=-c_3$, c_3 arbitrary. Thus, choosing $c_3=2$, we obtain $c_1=-5$, $c_2=-2$, $c_3=2$ as a particular nonzero set of constants that satisfies (8); hence, the original vectors are linearly dependent. □

Example 4

Is the set

$$\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}5\\7\end{bmatrix},\begin{bmatrix}-3\\1\end{bmatrix}\right\}$$

linearly independent?

Solution. Consider the vector equation

$$c_1\begin{bmatrix}1\\2\end{bmatrix}+c_2\begin{bmatrix}5\\7\end{bmatrix}+c_3\begin{bmatrix}-3\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}.$$

This is equivalent to the system

$$\begin{aligned}c_1+5c_2-3c_3&=0,\\2c_1+7c_2+c_3&=0.\end{aligned}$$

By using Gaussian elimination, we find that the solution to this system is $c_1 = (-26/3)c_3$, $c_2 = (7/3)c_3$, c_3 arbitrary. Hence a particular nonzero solution is found by choosing $c_3 = 3$; then $c_1 = -26$, $c_2 = 7$, and, therefore, the vectors are linearly dependent. □

We conclude this section with a few important theorems on linear independence and dependence.

Theorem 1. *A set of vectors is linearly dependent if and only if one of the vectors is a linear combination of the others.*

Proof. Let $\{\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_n\}$ be a linearly dependent set. Then there exist scalars $c_1, c_2, \ldots, c_n$, not all zero, such that (7) is satisfied. Assume $c_1 \neq 0$. (Since at least one of the c's must differ from zero, we lose no generality in assuming it is c_1.) Equation (7) can be rewritten as

$$c_1\mathbf{V}_1 = -c_2\mathbf{V}_2 - c_3\mathbf{V}_3 - \cdots - c_n\mathbf{V}_n,$$

or as

$$\mathbf{V}_1 = -\frac{c_2}{c_1}\mathbf{V}_2 - \frac{c_3}{c_1}\mathbf{V}_3 - \cdots - \frac{c_n}{c_1}\mathbf{V}_n.$$

Thus, $\mathbf{V}_1$ is a linear combination of $\mathbf{V}_2, \mathbf{V}_3, \ldots, \mathbf{V}_n$. To complete the proof, we must show that if one vector is a linear combination of the others, then the set is linearly dependent. We leave this as an exercise for the student (see Problem 36.)

Observation. In order for a set of vectors to be linearly dependent, it is not necessary for *every* vector to be a linear combination of the others, only that there exists *one* vector that is a linear combination of the others. For example, consider the vectors $[1 \quad 0]$, $[2 \quad 0]$, $[0 \quad 1]$. Here, $[0, \quad 1]$ cannot be written as a linear combination of the other two vectors; however, $[2 \quad 0]$ can be written as a linear combination of $[1 \quad 0]$ and $[0 \quad 1]$, namely, $[2 \quad 0] = 2[1 \quad 0] + 0[0 \quad 1]$; hence, the vectors are linearly dependent.

Theorem 2. *The set consisting of the single vector* $\mathbf{V}_1$ *is a linearly independent set if and only if* $\mathbf{V}_1 \neq 0$.

Proof. Consider the equation $c_1\mathbf{V}_1 = \mathbf{0}$. If $\mathbf{V}_1 \neq \mathbf{0}$, then the only way this equation can be valid is if $c_1 = 0$; hence, the set is linearly independent. If $\mathbf{V}_1 = \mathbf{0}$, then any $c_1 \neq 0$ will satisfy the equation; hence, the set is linearly dependent.

Theorem 3. *Any set of vectors that contains the zero vector is linearly dependent.*

Proof. Consider the set $\{\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_n, \mathbf{0}\}$. Pick $c_1 = c_2 = \cdots = c_n = 0$, $c_{n+1} = 5$ (any other number will do). Then this is a set of scalars, not all zero, such that

$$c_1\mathbf{V}_1 + c_2\mathbf{V}_2 + \cdots + c_n\mathbf{V}_n + c_{n+1}\mathbf{0} = \mathbf{0};$$

hence, the set of vectors is linearly dependent.

Theorem 4. *If a set of vectors is linearly independent, any subset of these vectors is also linearly independent.*

Proof. See Problem 37.

Theorem 5. *If a set of vectors is linearly dependent, then any larger set, containing this set, is also linearly dependent.*

Proof. See Problem 38.

Problems 2.5

In Problems 1 through 19, determine whether or not the given set is linearly independent.

(1) $\{[1 \quad 0], \quad [0 \quad 1]\}$.

(2) $\{[1 \quad 1], \quad [1 \quad -1]\}$.

(3) $\{[2 \quad -4], \quad [-3 \quad 6]\}$.

(4) $\{[1 \quad 3], \quad [2 \quad -1], \quad [1 \quad 1]\}$.

(5) $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right\}$.

(6) $\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$.

(7) $\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$.

(8) $\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right\}$.

(9) $\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$.

(10) $\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}\right\}$.

(11) $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\2\\1\end{bmatrix}, \begin{bmatrix}2\\1\\3\end{bmatrix}\right\}.$

(12) $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\2\\1\end{bmatrix}, \begin{bmatrix}2\\1\\3\end{bmatrix}, \begin{bmatrix}-1\\2\\-3\end{bmatrix}\right\}.$

(13) $\left\{\begin{bmatrix}4\\5\\1\end{bmatrix}, \begin{bmatrix}3\\0\\2\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}.$

(14) $\{[1 \quad 1 \quad 0], \ [1 \quad -1 \quad 0]\}.$

(15) $\{[1 \quad 2 \quad 3], \ [-3 \quad -6 \quad -9]\}.$

(16) $\{[10 \quad 20 \quad 20], \ [10 \quad -10 \quad 10], \ [10 \quad 20 \quad 10]\}.$

(17) $\{[10 \quad 20 \quad 20], \ [10 \quad -10 \quad 10], \ [10 \quad 20 \quad 10], \ [20 \quad 10 \quad 20]\}.$

(18) $\{[2 \quad 1 \quad 1], \ [3 \quad -1 \quad 4], \ [1 \quad 3 \quad -2]\}.$

(19) $\left\{\begin{bmatrix}2\\1\\1\\3\end{bmatrix}, \begin{bmatrix}4\\-1\\2\\-1\end{bmatrix}, \begin{bmatrix}8\\1\\4\\5\end{bmatrix}\right\}.$

(20) Express the vector

$$\begin{bmatrix}2\\1\\2\end{bmatrix}$$

as a linear combination of

$$\begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \begin{bmatrix}1\\0\\-1\end{bmatrix}, \quad \begin{bmatrix}1\\1\\1\end{bmatrix}.$$

(21) Can the vector $[2 \quad 3]$ be expressed as a linear combination of the vectors given in (a) Problem 1, (b) Problem 2, or (c) Problem 3?

(22) Can the vector $[1 \quad 1 \quad 1]^{\mathsf{T}}$ be expressed as a linear combination of the vectors given in (a) Problem 7, (b) Problem 8, or (c) Problem 9?

(23) Can the vector $[2 \quad 0 \quad 3]^{\mathsf{T}}$ be expressed as a linear combination of the vectors given in Problem 8?

(24) A set of vectors S is a *spanning set* for another set of vectors R if every vector in R can be expressed as a linear combination of the vectors in S. Show that the vectors given in Problem 1 are a spanning set for all two-dimensional row vectors. *Hint*: Show that for any arbitrary real numbers a and b, the vector $[a \quad b]$ can be expressed as a linear combination of the vectors in Problem 1

(25) Show that the vectors given in Problem 2 are a spanning set for all two-dimensional row vectors.

(26) Show that the vectors given in Problem 3 are not a spanning set for all two-dimensional row vectors.

(27) Show that the vectors given in Problem 3 are a spanning set for all vectors of the form $[a \quad -2a]$, where a designates any real number.

(28) Show that the vectors given in Problem 4 are a spanning set for all two-dimensional row vectors.

(29) Determine whether the vectors given in Problem 7 are a spanning set for all three-dimensional column vectors.

(30) Determine whether the vectors given in Problem 8 are a spanning set for all three-dimensional column vectors.

(31) Determine whether the vectors given in Problem 8 are a spanning set for vectors of the form $[a \quad 0 \quad a]^T$, whre a denotes an arbitrary real number.

(32) A set of vectors S is a *basis* for another set of vectors R if S is a spanning set for R and S is linearly independent. Determine which, if any, of the sets given in Problems 1 through 4 are a basis for the set of all two dimensional row vectors.

(33) Determine which, if any, of the sets given in Problems 7 through 12 are a basis for the set of all three dimensional column vectors.

(34) Prove that the columns of the 3×3 identity matrix form a basis for the set of all three dimensional column vectors.

(35) Prove that the rows of the 4×4 identity matrix form a basis for the set of all four dimensional row vectors.

(36) Finish the proof of Theorem 1. (*Hint*: Assume that $\mathbf{V}_1$ can be written as a linear combination of the other vectors.)

(37) Prove Theorem 4.

(38) Prove Theorem 5.

(39) Prove that the set of vectors $\{\mathbf{x}, k\mathbf{x}\}$ is linearly dependent for any choice of the scalar k.

(40) Prove that if $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, then so too are $\mathbf{x} + \mathbf{y}$ and $\mathbf{x} - \mathbf{y}$.

(41) Prove that if the set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is linearly independent then so too is the set $\{k_1\mathbf{x}_1, k_2\mathbf{x}_2, \ldots, k_n\mathbf{x}_n\}$ for any choice of the *non-zero* scalars $k_1, k_2, \ldots, k_n$.

(42) Let $\mathbf{A}$ be an $n \times n$ matrix and let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ and $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k\}$ be two sets of n-dimensional column vectors having the property that $\mathbf{A}\mathbf{x}_i = \mathbf{y}_i = 1, 2, \ldots, k$. Show that the set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ is linearly independent if the set $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k\}$ is.

2.6 Rank

If we interpret each row of a matrix as a row vector, then elementary row operations are precisely the operations used to form linear combinations; namely, multiplying vectors (rows) by scalars and adding vectors (rows) to other vectors (rows). This observation allows us to develop a straightforward matrix procedure for determining when a set of vectors is linearly independent. It rests on the concept of rank.

Definition 1. The *row rank* of a matrix is the maximum number of linearly independent vectors that can be formed from the rows of that matrix, considering each row as a separate vector. Analagously, the *column rank* of a matrix is the maximum number of linearly independent columns, considering each column as a separate vector.

Row rank is particularly easy to determine for matrices in row-reduced form.

Theorem 1. *The row rank of a row-reduced matrix is the number of nonzero rows in that matrix.*

Proof. We must prove two facts: First, that the nonzero rows, considered as vectors, form a linearly independent set, and second, that every larger set is linearly dependent. Consider the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_r\mathbf{v}_r = \mathbf{0}, \tag{9}$$

where $\mathbf{v}_1$ is the first nonzero row, $\mathbf{v}_2$ is the second nonzero row, ..., and $\mathbf{v}_r$ is the last nonzero row of a row-reduced matrix. The first nonzero element in the first nonzero row of a row-reduced matrix must be unity. Assume it appears in column j. Then, no other rows have a nonzero element in that column. Consequently, when the left side of Eq. (9) is computed, it will have c_1 as its jth component. Since the right side of Eq. (9) is the zero vector, it follows that $c_1 = 0$. A similar argument then shows iteratively that $c_2, \ldots, c_r$, are all zero. Thus, the nonzero rows are linearly independent.

If all the rows of the matrix are nonzero, then they must comprise a maximum number of linearly independent vectors, because the row rank cannot be greater than the number of rows in the matrix. If there are zero rows in the row-reduced matrix, then it follows from Theorem 3 of Section 2.5 that including them could not increase the number of linearly independent rows. Thus, the largest number of linearly independent rows comes from including just the nonzero rows.

Example 1

Determine the row rank of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -2 & 5 & 3 \\ 0 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution. **A** is in row-reduced form. Since it contains three nonzero rows, its row rank is three. □

The following two theorems, which are proved in the appendix to this chapter, are fundamental.

Theorem 2. *The row rank and column rank of a matrix are equal.*

For any matrix **A**, we call this common number the *rank* of **A** and denote it by $r(\mathbf{A})$.

Theorem 3. *If* **B** *is obtained from* **A** *by an elementary row (or column) operation, then* $r(\mathbf{B}) = r(\mathbf{A})$.

Theorems 1 through 3 suggest a useful procedure for determining the rank of any matrix: Simply use elementary row operations to transform the given matrix to row-reduced form, and then count the number of nonzero rows.

Example 2

Determine the rank of

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 4 \\ 2 & -1 & 1 \\ 3 & 2 & 5 \\ 5 & 15 & 20 \end{bmatrix}.$$

Solution. In Example 2 of Section 2.3, we transferred this matrix into the row-reduced form

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This matrix has two nonzero rows so its rank, as well as that of **A**, is two. □

Example 3

Determine the rank of

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 3 & -1 & -6 \\ 3 & -2 & -4 & -2 \end{bmatrix}.$$

Solution. In Example 3 of Section 2.3, we transferred this matrix into the row-reduced form

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 3 & 12 \\ 0 & 0 & 1 & 5 \end{bmatrix}.$$

This matrix has three nonzero rows so its rank, as well as that of **B**, is three. □

A similar procedure can be used for determining whether a set of vectors is linearly independent: Form a matrix in which each row is one of the vectors in the given set, and then determine the rank of that matrix. If the rank equals the number of vectors, the set is linearly independent; if not, the set is linearly dependent. In either case, the rank is the maximal number of linearly independent vectors that can be formed from the given set.

Example 4

Determine whether the set

$$\left\{\begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 8 \\ 16 \\ -3 \end{bmatrix}\right\}$$

is linearly independent. □

Solution. We consider the matrix

$$\begin{bmatrix} 2 & 6 & -2 \\ 3 & 1 & 2 \\ 8 & 16 & -3 \end{bmatrix}.$$

Reducing this matrix to row-reduced form, we obtain

$$\begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -\frac{5}{8} \\ 0 & 0 & 0 \end{bmatrix}.$$

This matrix has two nonzero rows, so its rank is two. Since this is less than the number of vectors in the given set, that set is linearly dependent.

We can say even more: The original set of vectors contains a subset of two linearly independent vectors, the same number as the rank. Also, since no row interchanges were involved in the transformation to row-reduced form, we can conclude that the third vector is linear combination of the first two. □

Example 5

Determine whether the set

$$\{[0 \quad 1 \quad 2 \quad 3 \quad 0], \; [1 \quad 3 \quad -1 \quad 2 \quad 1],$$
$$[2 \quad 6 \quad -1 \quad -3 \quad 1], \; [4 \quad 0 \quad 1 \quad 0 \quad 2]\}$$

is linearly independent.

Solution. We consider the matrix

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 0 \\ 1 & 3 & -1 & 2 & 1 \\ 2 & 6 & -1 & -3 & 1 \\ 4 & 0 & 1 & 0 & 2 \end{bmatrix},$$

which can be reduced (after the first two rows are interchanged) to the row-reduced form

$$\begin{bmatrix} 1 & 3 & -1 & 2 & 1 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & -7 & -1 \\ 0 & 0 & 0 & 1 & \dfrac{27}{175} \end{bmatrix}.$$

This matrix has four nonzero rows, hence its rank is four, which is equal to the number of vectors in the given set. Therefore, the set is linearly independent. □

Example 6

Can the vector

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

be written as a linear combination of the vectors

$$\begin{bmatrix} 3 \\ 6 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 4 \end{bmatrix}?$$

Solution. The matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$$

can be transformed into the row-reduced form

$$\begin{bmatrix} 3 & 6 \\ 0 & 0 \end{bmatrix},$$

which has rank one; hence **A** has just one linearly independent row vector. In contrast, the matrix

$$\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 3 & 6 \\ 2 & 4 \end{bmatrix}$$

can be transformed into the row-reduced form

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

which has rank two; hence **B** has two linearly independent row vectors. Since **B** is precisely **A** with one additional row, it follows that the additional row $[1, \quad 1]^T$ is independent of the other two and, therefore, cannot be written as a linear combination of the other two vectors. □

We did not have to transform **B** in Example 6 into row-reduced form to determine whether the three-vector set was linearly independent. There is a more direct approach. Since **B** has only two columns, its column rank must be less than or equal to two (why?). Thus, the column rank is less than three. It follows from Theorem 3 that the row rank of **B** is less than three, so the three vectors must be linearly dependent. Generalizing this reasoning, we deduce one of the more important results in linear algebra.

Theorem 4. *In an n-dimensional vector space, every set of $n + 1$ vectors is linearly dependent.*

Problems 2.6

In Problems 1–5, find the rank of the given matrix.

(1) $\begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & -5 \end{bmatrix}$.

(2) $\begin{bmatrix} 4 & 1 \\ 2 & 3 \\ 2 & 2 \end{bmatrix}$.

(3) $\begin{bmatrix} 1 & 4 & -2 \\ 2 & 8 & -4 \\ -1 & -4 & 2 \end{bmatrix}$.

(4) $\begin{bmatrix} 1 & 2 & 4 & 2 \\ 1 & 1 & 3 & 2 \\ 1 & 2 & 4 & 2 \end{bmatrix}$.

(5) $\begin{bmatrix} 1 & 7 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

In Problems 6 through 22, use rank to determine whether the given set of vectors is linearly independent.

(6) $\{[1 \quad 0], \quad [0 \quad 1]\}$.

(7) $\{[1 \quad 1], \quad [1 \quad -1]\}$.

(8) $\{[2 \quad -4], \quad [-3 \quad 6]\}$.

(9) $\{[1 \quad 3], \quad [2 \quad -1], \quad [1 \quad 1]\}$.

(10) $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right\}$.

(11) $\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$.

(12) $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$.

(13) $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}\right\}$.

(14) $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}\right\}$.

(15) $\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}3\\2\\1\end{bmatrix}, \begin{bmatrix}2\\1\\3\end{bmatrix}\right\}$.

(16) $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\2\\1\end{bmatrix}, \begin{bmatrix}2\\1\\3\end{bmatrix}\right\}$.

(17) $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\2\\1\end{bmatrix}, \begin{bmatrix}2\\1\\3\end{bmatrix}, \begin{bmatrix}-1\\2\\-3\end{bmatrix}\right\}$.

(18) $\{[1 \quad 1 \quad 0], \ [1 \quad -1 \quad 0]\}$.

(19) $\{[1 \quad 2 \quad 3], \ [-3 \quad -6 \quad -9]\}$.

(20) $\{[10 \quad 20 \quad 20], \ [10 \quad -10 \quad 10], \ [10 \quad 20 \quad 10]\}$.

(21) $\{[10 \quad 20 \quad 20], \ [10 \quad -10 \quad 10], \ [10 \quad 20 \quad 10], \ [20 \quad 10 \quad 20]\}$.

(22) $\{[2 \quad 1 \quad 1], \ [3 \quad -1 \quad 4], \ [1 \quad 3 \quad -2]\}$.

(23) Can the vector $[2 \quad 3]$ be expressed as a linear combination of the vectors given in (a) Problem 6, (b) Problem 7, or (c) Problem 8?

(24) Can the vector $[1 \quad 1 \quad 1]^{\mathsf{T}}$ be expressed as a linear combination of the vectors given in (a) Problem 12, (b) Problem 13, or (c) Problem 14?

(25) Can the vector $[2 \quad 0 \quad 3]^{\mathsf{T}}$ be expressed as a linear combination of the vectors given in Problem 13?

(26) Can $[3 \quad 7]$ be written as a linear combination of the vectors $[1 \quad 2]$ and $[3 \quad 2]$?

(27) Can $[3 \quad 7]$ be written as a linear combination of the vectors $[1 \quad 2]$ and $[4 \quad 8]$?

(28) Find a maximal linearly independent subset of the vectors given in Problem 9.

(29) Find a maximal linearly independent subset of the vectors given in Problem 13.

(30) Find a maximal linearly independent subset of the set

$$\{[1 \quad 2 \quad 4 \quad 0], \ [2 \quad 4 \quad 8 \quad 0], \ [1 \quad -1 \quad 0 \quad 1],$$
$$[4 \quad 2 \quad 8 \quad 2], \ [4 \quad -1 \quad 4 \quad 3]\}.$$

(31) What is the rank of the zero matrix?

(32) Show $r(\mathbf{A}^{\mathsf{T}}) = r(\mathbf{A})$.

2.7 Theory of Solutions

Consider once again the system $\mathbf{Ax} = \mathbf{b}$ of m equations and n unknowns given in Eq. (2). Designate the n columns of $\mathbf{A}$ by the vectors $\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_n$. Then Eq. (2) can be rewritten in the vector form

$$x_1\mathbf{V}_1 + x_2\mathbf{V}_2 + \cdots + x_n\mathbf{V}_n = \mathbf{b}. \tag{10}$$

Example 1

Rewrite the following system in the vector form (10):

$$x - 2y + 3z = 7,$$
$$4x + 5y - 6z = 8.$$

Solution.

$$x\begin{bmatrix}1\\4\end{bmatrix} + y\begin{bmatrix}-2\\5\end{bmatrix} + z\begin{bmatrix}3\\-6\end{bmatrix} = \begin{bmatrix}7\\8\end{bmatrix}. \qquad \square$$

Thus, finding solutions to (1) and (2) is equivalent to finding scalars $x_1, x_2, \ldots, x_n$ that satisfy (10). This, however, is asking precisely the question "Is the vector $\mathbf{b}$ a linear combination of $\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_n$?" If $\mathbf{b}$ is a linear combination of $\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_n$, then there will exist scalars $x_1, x_2, \ldots, x_n$ that satisfy (10) and the system is consistent. If $\mathbf{b}$ is not a linear combination of these vectors, that is, if $\mathbf{b}$ is linearly independent of the vectors $\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_n$, then no scalars $x_1, x_2, \ldots, x_n$ will exist that satisfy (10) and the system is inconsistent.

Taking a hint from Example 6 of Section 2.6, we have the following theorem.

▶ | **Theorem 1.** *The system* $\mathbf{Ax} = \mathbf{b}$ *is consistent if and only if* $r(\mathbf{A}) = r(\mathbf{A}^{\mathbf{b}})$.

Once a system is deemed consistent, the following theorem specifies the number of solutions.

Theorem 2. *If the system* $\mathbf{Ax} = \mathbf{b}$ *is consistent and* $r(\mathbf{A}) = k$ *then the solutions are expressible in terms of* $n - k$ *arbitrary unknowns (where* n *represents the number of unknowns in the system).*

Theorem 2 is almost obvious. To determine the rank of $\mathbf{A}^{\mathbf{b}}$, we must reduce it to row-reduced form. The rank is the number of nonzero rows. With Gaussian elimination, we use each nonzero row to solve for the variable associated with the first nonzero entry in it. Thus, each nonzero row defines one variable, and all other variables remain arbitrary.

Example 2

Discuss the solutions of the system

$$x + y - z = 1,$$
$$x + y - z = 0.$$

Solution.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{A}^{\mathbf{b}} = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}.$$

Here, $r(\mathbf{A}) = 1$, $r(\mathbf{A}^{\mathbf{b}}) = 2$. Thus, $r(\mathbf{A}) \neq r(\mathbf{A}^{\mathbf{b}})$ and no solution exists. □

Example 3

Discuss the solutions of the system

$$x + y + w = 3,$$
$$2x + 2y + 2w = 6,$$
$$-x - y - w = -3.$$

Solution.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}, \quad \mathbf{A}^{\mathbf{b}} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & 6 \\ -1 & -1 & -1 & -3 \end{bmatrix}.$$

Here $r(\mathbf{A}) = r(\mathbf{A}^{\mathbf{b}}) = 1$; hence, the system is consistent. In this case, $n = 3$ and $k = 1$; thus, the solutions are expressible in terms of $3 - 1 = 2$ arbitrary unknowns. Using Gaussian elimination, we find that the solution is $x = 3 - y - w$ where y and w are both arbitrary. □

Example 4

Discuss the solutions of the system

$$\begin{aligned} 2x - 3y + z &= -1, \\ x - y + 2z &= 2, \\ 2x + y - 3z &= 3. \end{aligned}$$

Solution.

$$\mathbf{A} = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \qquad \mathbf{A}^{\mathbf{b}} = \begin{bmatrix} 2 & -3 & 1 & -1 \\ 1 & -1 & 2 & 2 \\ 2 & 1 & -3 & 3 \end{bmatrix}.$$

Here $r(\mathbf{A}) = r(\mathbf{A}^{\mathbf{b}}) = 3$, hence the system is consistent. Since $n = 3$ and $k = 3$, the solution will be in $n - k = 0$ arbitrary unknowns. Thus, the solution is unique (none of the unknowns are arbitrary) and can be obtained by Gaussian elimination as $x = y = 2$, $z = 1$. □

Example 5

Discuss the solutions of the system

$$\begin{aligned} x + y - 2z &= 1, \\ 2x + y + z &= 2, \\ 3x + 2y - z &= 3, \\ 4x + 2y + 2z &= 4. \end{aligned}$$

Solution.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & 1 \\ 3 & 2 & -1 \\ 4 & 2 & 2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \qquad \mathbf{A}^{\mathbf{b}} = \begin{bmatrix} 1 & 1 & -2 & 1 \\ 2 & 1 & 1 & 2 \\ 3 & 2 & -1 & 3 \\ 4 & 2 & 2 & 4 \end{bmatrix}.$$

Here $r(\mathbf{A}) = r(\mathbf{A}^{\mathbf{b}}) = 2$. Thus, the system is consistent and the solutions will be in terms of $3 - 2 = 1$ arbitrary unknowns. Using Gaussian elimination, we find that the solution is $x = 1 - 3z$, $y = 5z$, and z is arbitrary. □

In a consistent system, the solution is unique if $k = n$. If $k \neq n$, the solution will be in terms of arbitrary unknowns. Since these arbitrary unknowns can be chosen to be any constants whatsoever, it follows that there will be an infinite number of solutions. Thus, a consistent system will possess exactly one solution or an infinite number of solutions; there is no inbetween.

A homogeneous system of simultaneous linear equations has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0, \end{aligned} \tag{11}$$

or the matrix form

$$\mathbf{Ax} = \mathbf{0}. \tag{12}$$

Since Eq. (12) is a special case of Eq. (2) with $\mathbf{b} = \mathbf{0}$, all the theory developed for the system $\mathbf{Ax} = \mathbf{b}$ remains valid. Because of the simplified structure of a homogeneous system, one can draw conclusions about it that are not valid for a nonhomogeneous system. For instance, a homogeneous system is always consistent. To verify this statement, note that $x_1 = x_2 = \cdots = x_n = 0$ is always a solution to Eq. (12). Such a solution is called the *trivial solution*. It is, in general, the *nontrivial solutions* (solutions in which one or more of the unknowns is different from zero) that are of the greatest interest.

It follows from Theorem 2, that if the rank of $\mathbf{A}$ is less than n (n being the number of unknowns), then the solution will be in terms of arbitrary unknowns. Since these arbitrary unknowns can be assigned nonzero values, it follows that nontrivial solutions exist. On the other hand, if the rank of $\mathbf{A}$ equals n, then the solution will be unique, and, hence, must be the trivial solution (why?). Thus, it follows that:

Theorem 3. *The homogeneous system (12) will admit nontrivial solutions if and only if* $r(\mathbf{A}) \neq n$.

Problems 2.7

In Problems 1–9, discuss the solutions of the given system in terms of consistency and number of solutions. Check your answers by solving the systems wherever possible.

(1) $\begin{aligned} x - 2y &= 0, \\ x + y &= 1, \\ 2x - y &= 1. \end{aligned}$

(2) $\begin{aligned} x + y &= 0, \\ 2x - 2y &= 1, \\ x - y &= 0. \end{aligned}$

(3) $\begin{aligned} x + y + z &= 1, \\ x - y + z &= 2, \\ 3x + y + 3z &= 4. \end{aligned}$

(4) $\begin{aligned} x + 3y + 2z - w &= 2, \\ 2x - y + z + w &= 3. \end{aligned}$

(5) $\begin{aligned} 2x - y + z &= 0, \\ x + 2y - z &= 4, \\ x + y + z &= 1. \end{aligned}$

(6) $\begin{aligned} 2x + 3y &= 0, \\ x - 4y &= 0. \end{aligned}$

(7) $\begin{aligned} x - y + 2z &= 0, \\ 2x + 3y - z &= 0, \\ -2x + 7y - 7z &= 0. \end{aligned}$

(8) $\begin{aligned} x - y + 2z &= 0, \\ 2x - 3y - z &= 0, \\ -2x + 7y - 9z &= 0. \end{aligned}$

(9) $\begin{aligned} x - 2y + 3z + 3w &= 0, \\ y - 2z + 2w &= 0, \\ x + y - 3z + 9w &= 0. \end{aligned}$

Appendix to Chapter 2

We would like to show that the column rank of a matrix equals its row rank, and that an elementary row operation of any kind does not alter the rank.

Lemma 1. *If* $\mathbf{B}$ *is obtained from* $\mathbf{A}$ *by interchanging two columns of* $\mathbf{A}$*, then both* $\mathbf{A}$ *and* $\mathbf{B}$ *have the same column rank.*

Proof. The set of vectors formed from the columns of $\mathbf{A}$ is identical to the set formed from the columns of $\mathbf{B}$, and, therefore, the two matrices must have the same column rank.

Lemma 2. *If* $\mathbf{Ax} = \mathbf{0}$ *and* $\mathbf{Bx} = \mathbf{0}$ *have the same set of solutions, then the column rank of* $\mathbf{A}$ *is less than or equal to the column rank of* $\mathbf{B}$.

Proof. Let the order of $\mathbf{A}$ be $m \times n$. Then, the system $\mathbf{Ax} = \mathbf{0}$ is a set of m equations in the n unknowns $x_1, x_2, \ldots, x_n$, which has the vector form

$$x_1\mathbf{A}_1 + x_2\mathbf{A}_2 + \cdots + x_n\mathbf{A}_n = \mathbf{0}, \tag{13}$$

where $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$ denote the columns of $\mathbf{A}$. Similarly, the system $\mathbf{Bx} = \mathbf{0}$

has the vector form

$$x_1\mathbf{B}_1 + x_2\mathbf{B}_2 + \cdots + x_n\mathbf{B}_n = \mathbf{0}. \tag{14}$$

We shall assume that the column rank of **A** is greater than the column rank of **B** and show that this assumption leads to a contradiction. It will then follow that the reverse must be true, which is precisely what we want to prove.

Denote the column rank of **A** as a and the column rank of **B** as b. We assume that $a > b$. Since the column rank of **A** is a, there must exist a columns of **A** that are linearly independent. If these columns are not the first a columns, rearrange the order of the columns so they are. Lemma 1 guarantees such reorderings do not alter the column rank. Thus, $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_a$ are linearly independent. Since a is assumed greater than b, we know that the first a columns of **B** are not linearly independent. Since they are linearly dependent, there must exist constants $c_1, c_2, \ldots, c_a$—not all zero—such that

$$c_1\mathbf{B}_1 + c_2\mathbf{B}_2 + \cdots + c_a\mathbf{B}_a = \mathbf{0}.$$

It then follows that

$$c_1\mathbf{B}_1 + c_2\mathbf{B}_2 + \cdots + c_a\mathbf{B}_a + 0\mathbf{B}_{a+1} + \cdots + 0\mathbf{B}_n = \mathbf{0},$$

from which we conclude that

$$x_1 = c_1, \qquad x_2 = c_2, \qquad \ldots, x_a = c_a, \qquad x_{a+1} = 0, \qquad \ldots, x_n = 0$$

is a solution of Eq. (14). Since every solution to Eq. (14) is also a solution to Eq. (12), we have

$$c_1\mathbf{A}_1 + c_2\mathbf{A}_2 + \cdots + c_a\mathbf{A}_a + 0\mathbf{A}_{a+1} + \cdots + 0\mathbf{A}_n = \mathbf{0},$$

or more simply

$$c_1\mathbf{A}_1 + c_2\mathbf{A}_2 + \cdots + c_a\mathbf{A}_a = \mathbf{0},$$

where all the c's are not all zero. But this implies that the first a columns of **A** are linearly dependent, which is a contradiction of the assumption that they were linearly independent.

Lemma 3. *If* $\mathbf{Ax} = \mathbf{0}$ *and* $\mathbf{Bx} = \mathbf{0}$ *have the same set of solutions, then* **A** *and* **B** *have the same column rank.*

Proof. If follows from Lemma 2 that the column rank of **A** is less than or equal to the column rank of **B**. By reversing the roles of **A** and **B**, we can also conclude from Lemma 2 that the column rank of **B** is less than or equal to the column rank of **A**. As a result, the two column ranks must be equal.

Theorem 1. *An elementary row operation does not alter the column rank of a matrix.*

Proof. Denote the original matrix as **A**, and let **B** denote a matrix obtained by applying an elementary row operation to **A**; and consider the two homogeneous systems $\mathbf{Ax} = \mathbf{0}$ and $\mathbf{Bx} = \mathbf{0}$. Since elementary row operations do not alter solutions, both of these systems have the same solution set. Theorem 1 follows immediately from Lemma 3.

Lemma 4. *The column rank of a matrix is less than or equal to its row rank.*

Proof. Denote rows of **A** by $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m$, the column rank of matrix **A** by c and its row rank by r. There must exist r rows of **A** which are linearly independent. If these rows are not the first r rows, rearrange the order of the rows so they are. Theorem 1 guarantees such reorderings do not alter the column rank, and they certainly do not alter the row rank. Thus, $\mathbf{A}_1$, $\mathbf{A}_2, \ldots, \mathbf{A}_r$ are linearly independent. Define partitioned matrices **R** and **S** by

$$\mathbf{R} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_r \end{bmatrix} \quad \text{and} \quad \mathbf{S} = \begin{bmatrix} \mathbf{A}_{r+1} \\ \mathbf{A}_{r+2} \\ \vdots \\ \mathbf{A}_n \end{bmatrix}.$$

Then **A** has the partitioned form

$$\mathbf{A} = \begin{bmatrix} \mathbf{R} \\ \mathbf{S} \end{bmatrix}.$$

Every row of **S** is a linear combination of the rows of **R**. Therefore, there exist constants t_{ij} such that

$$\begin{aligned}
\mathbf{A}_{r+1} &= t_{r+1,1}\mathbf{A}_1 + t_{r+1,2}\mathbf{A}_2 + \cdots + t_{r+1,r}\mathbf{A}_r, \\
\mathbf{A}_{r+2} &= t_{r+2,1}\mathbf{A}_1 + t_{r+2,2}\mathbf{A}_2 + \cdots + t_{r+2,r}\mathbf{A}_r, \\
&\vdots \\
\mathbf{A}_n &= t_{n,1}\mathbf{A}_1 \quad + t_{n,2}\mathbf{A}_2 \quad + \cdots + t_{n,r}\mathbf{A}_r,
\end{aligned}$$

which may be written in the matrix form

$$\mathbf{S} = \mathbf{TR},$$

where

$$\mathbf{T} = \begin{bmatrix} t_{r+1,1} & t_{r+1,2} & \cdots & t_{r+1,n} \\ t_{r+2,1} & t_{r+2,2} & \cdots & t_{r+2,n} \\ \vdots & \vdots & \vdots & \vdots \\ t_{n,1} & t_{n,2} & \cdots & t_{n,n} \end{bmatrix}.$$

Then, for any n-dimensional vector $\mathbf{x}$, we have

$$\mathbf{Ax} = \begin{bmatrix} \mathbf{R} \\ \mathbf{S} \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{Rx} \\ \mathbf{Sx} \end{bmatrix} = \begin{bmatrix} \mathbf{Rx} \\ \mathbf{TRx} \end{bmatrix}.$$

Thus, $\mathbf{Ax} = \mathbf{0}$ if and only if $\mathbf{Rx} = \mathbf{0}$. It follows from Lemma 3 that both $\mathbf{A}$ and $\mathbf{R}$ have the same column rank. But the columns of $\mathbf{R}$ are r-dimensional vectors, so its column rank cannot be larger than r. Thus,

$$c = \text{column rank of } \mathbf{A} = \text{column rank of } \mathbf{R} \leq r = \text{row rank of } \mathbf{A}.$$

Lemma 5. *The row rank of a matrix is less than or equal to its column rank.*

Proof. By applying Lemma 4 to $\mathbf{A}^{\mathsf{T}}$, we conclude that the column rank of $\mathbf{A}^{\mathsf{T}}$ is less than or equal to the row rank of $\mathbf{A}^{\mathsf{T}}$. But since the columns of $\mathbf{A}^{\mathsf{T}}$ are the rows of $\mathbf{A}$ and vice-versa, the result follows immediately.

Theorem 2. *The row rank of a matrix equals its column rank.*

Proof. The result is immediate from Lemmas 4 and 5.

Theorem 3. *An elementary row operation does not alter the row rank of a matrix.*

Proof. This theorem is an immediate consequence of both Theorems 1 and 2.

Chapter 3

The Inverse

3.1 Introduction

Definition 1. An *inverse* of an $n \times n$ matrix $\mathbf{A}$ is a $n \times n$ matrix $\mathbf{B}$ having the property that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}. \tag{1}$$

Here, $\mathbf{B}$ is called an inverse of $\mathbf{A}$ and is usually denoted by $\mathbf{A}^{-1}$. If a square matrix $\mathbf{A}$ has an inverse, it is said to be *invertible* or *nonsingular*. If $\mathbf{A}$ does not possess an inverse, it is said to be *singular*. Note that inverses are only defined for square matrices. In particular, the identity matrix is invertible and is its own inverse because

$$\mathbf{II} = \mathbf{I}.$$

Example 1

Determine whether

$$\mathbf{B} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix} \quad \text{or} \quad \mathbf{C} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

are inverses for

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Solution. **B** is an inverse if and only if $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$; **C** is an inverse if and only if $\mathbf{AC} = \mathbf{CA} = \mathbf{I}$. Here,

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{5}{3} & 1 \\ \frac{13}{3} & \frac{5}{2} \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

while

$$\mathbf{AC} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \mathbf{CA}.$$

Thus, **B** is not an inverse for **A**, but **C** is. We may write $\mathbf{A}^{-1} = \mathbf{C}$. □

Definition 1 is a test for checking whether a given matrix is an inverse of another given matrix. In the appendix to this chapter we prove that if $\mathbf{AB} = \mathbf{I}$ for two square matrices of the same order, then **A** and **B** commute, and $\mathbf{BA} = \mathbf{I}$. Thus, we can reduce the checking procedure by half. A matrix **B** is an inverse for a square matrix **A** if either $\mathbf{AB} = \mathbf{I}$ or $\mathbf{BA} = \mathbf{I}$; each equality automatically guarantees the other for square matrices. We will show in Section 3.4 that an inverse is unique. If a square matrix has an inverse, it has only one.

Definition 1 does not provide a method for finding inverses. We develop such a procedure in the next section. Still, inverses for some matrices can be found directly.

The inverse for a diagonal matrix **D** having only nonzero elements on its main diagonal is also a diagonal matrix whose diagonal elements are the reciprocals of the corresponding diagonal elements of **D**. That is, if

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & & & & \mathbf{0} \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ \mathbf{0} & & & & \lambda_n \end{bmatrix},$$

then

$$\mathbf{D}^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & & & & \mathbf{0} \\ & \frac{1}{\lambda_2} & & & \\ & & \frac{1}{\lambda_3} & & \\ & & & \ddots & \\ \mathbf{0} & & & & \frac{1}{\lambda_n} \end{bmatrix}.$$

It is easy to show that if any diagonal element in a diagonal matrix is zero, then that matrix is singular. (See Problem 57.)

An elementary matrix **E** is a square matrix that generates an elementary row operation on a matrix **A** (which need not be square) under the multiplication **EA**. Elementary matrices are constructed by applying the desired elementary row operation to an identity matrix of appropriate order. The appropriate order for both **I** and **E** is a square matrix having as many columns as there are rows in **A**; then, the multiplication **EA** is defined. Because identity matrices contain many zeros, the process for constructing elementary matrices can be simplified still further. After all, nothing is accomplished by interchanging the positions of zeros, multiplying zeros by nonzero constants, or adding zeros to zeros.

(i) To construct an elementary matrix that interchanges the ith row with the jth row, begin with an identity matrix of the appropriate order. First, interchange the unity element in the $i-i$ position with the zero in the $j-i$ position, and then interchange the unity element in the $j-j$ position with the zero in the $i-j$ position.

(ii) To construct an elementary matrix that multiplies the ith row of a matrix by the nonzero scalar k, replace the unity element in the $i-i$ position of the identity matrix of appropriate order with the scalar k.

(iii) To construct an elementary matrix that adds to the jth row of a matrix k times the ith row, replace the zero element in the $j-i$ position of the identity matrix of appropriate order with the scalar k.

Example 2

Find elementary matrices that when multiplied on the right by any 4×3 matrix **A** will (a) interchange the second and fourth rows of **A**, (b) multiply the third row of **A** by 3, and (c) add to the fourth row of **A** -5 times its second row.

Solution.

$$\text{(a)}\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \text{(b)}\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{(c)}\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -5 & 0 & 1 \end{bmatrix}. \qquad \square$$

Example 3

Find elementary matrices that when multiplied on the right by any 3×5 matrix **A** will (a) interchange the first and second rows of **A**, (b) multiply the third row of **A** by -0.5, and (c) add to the third row of **A** -1 times its second row.

Solution.

$$\text{(a)}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix},\quad \text{(b)}\begin{bmatrix}1&0&0\\0&1&0\\0&0&-0.5\end{bmatrix},\quad \text{(c)}\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}.$$ □

The inverse of an elementary matrix that interchanges two rows is the matrix itself; it is its own inverse. The inverse of an elementary matrix that multiplies one row by a nonzero scalar k is gotten by replacing k by $1/k$. The inverse of an elementary matrix which adds to one row a constant k times another row is obtained by replacing the scalar k by $-k$.

Example 4

Compute the inverses of the elementary matrices found in Example 2.

Solution.

$$\text{(a)}\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix},\quad \text{(b)}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&\frac{1}{3}&0\\0&0&0&1\end{bmatrix},\quad \text{(c)}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&5&0&1\end{bmatrix}.$$ □

Example 5

Compute the inverses of the elementary matrices found in Example 3.

Solution.

$$\text{(a)}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix},\quad \text{(b)}\begin{bmatrix}1&0&0\\0&1&0\\0&0&-2\end{bmatrix}\quad \text{(c)}\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}.$$

Finally, if **A** can be partitioned into the block diagonal form,

$$\mathbf{A}=\begin{bmatrix}\mathbf{A}_1 & & & & \mathbf{0}\\ & \mathbf{A}_2 & & & \\ & & \mathbf{A}_3 & & \\ & & & \ddots & \\ \mathbf{0} & & & & \mathbf{A}_n\end{bmatrix},$$

then $\mathbf{A}$ is invertible if and only if each of the diagonal blocks $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$ is invertible and

$$\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{A}_1^{-1} & & & & \mathbf{0} \\ & \mathbf{A}_2^{-1} & & & \\ & & \mathbf{A}_3^{-1} & & \\ & & & \ddots & \\ \mathbf{0} & & & & \mathbf{A}_n^{-1} \end{bmatrix}. \quad \square$$

Example 6

Find the inverse of

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Solution. Set

$$\mathbf{A}_1 = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}, \qquad \mathbf{A}_2 = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{A}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix};$$

then, $\mathbf{A}$ is in the block diagonal form

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & & \mathbf{0} \\ & \mathbf{A}_2 & \\ \mathbf{0} & & \mathbf{A}_3 \end{bmatrix}.$$

Here $\mathbf{A}_1$ is a diagonal matrix with nonzero diagonal elements, $\mathbf{A}_2$ is an elementary matrix that adds to the second row four times the first row, and $\mathbf{A}_3$ is an elementary matrix that interchanges the second and third rows; thus

$$\mathbf{A}_1^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{5} \end{bmatrix}, \qquad \mathbf{A}_2^{-1} = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{A}_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

and

$$\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \qquad \square$$

Problems 3.1

(1) Determine if any of the following matrices are inverses for

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 9 \end{bmatrix}:$$

(a) $\begin{bmatrix} 1 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{9} \end{bmatrix}$, (b) $\begin{bmatrix} -1 & -3 \\ -2 & -9 \end{bmatrix}$,

(c) $\begin{bmatrix} 3 & -1 \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$, (d) $\begin{bmatrix} 9 & -3 \\ -2 & 1 \end{bmatrix}$.

(2) Determine if any of the following matrices are inverses for

$$\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}:$$

(a) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, (b) $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$,

(c) $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$, (d) $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.

(3) Calculate directly the inverse of

$$\mathbf{A} = \begin{bmatrix} 8 & 2 \\ 5 & 3 \end{bmatrix}.$$

Hint: Define

$$\mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Calculate **AB**, set the product equal to **I**, and then solve for the elements of **B**.

(4) Use the procedure described in Problem 3 to calculate the inverse of

$$\mathbf{C} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

(5) Use the procedure described in Problem 3 to calculate the inverse of

$$\mathbf{D} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

(6) Show directly that the inverse of

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

when $ad - bc \neq 0$ is

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

(7) Use the results of Problem 6 to calculate the inverse of

$$\begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}.$$

(8) Use the results of Problem 6 to calculate the inverse of

$$\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}.$$

(9) Use the results of Problem 6 to calculate the inverse of

$$\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}.$$

(10) Use the results of Problem 6 to calculate the inverse of

$$\begin{bmatrix} 10 & 20 \\ 30 & 40 \end{bmatrix}.$$

In Problems 11 through 26, find elementary matrices that when multiplied on the right by a matrix **A** will generate the specified result.

(11) Interchange the order of the first and second row of the 2×2 matrix **A**.

(12) Multiply the first row of a 2×2 matrix **A** by three.

(13) Multiply the second row of a 2×2 matrix $\mathbf{A}$ by -5.

(14) Multiply the second row of a 3×3 matrix $\mathbf{A}$ by -5.

(15) Add to the second row of a 2×2 matrix $\mathbf{A}$ three times its first row.

(16) Add to the first row of a 2×2 matrix $\mathbf{A}$ three times its second row.

(17) Add to the second row of a 3×3 matrix $\mathbf{A}$ three times its third row.

(18) Add to the third row of a 3×4 matrix $\mathbf{A}$ five times its first row.

(19) Add to the second row of a 4×4 matrix $\mathbf{A}$ eight times its fourth row.

(20) Add to the fourth row of a 5×7 matrix $\mathbf{A}$ -2 times its first row.

(21) Interchange the second and fourth rows of a 4×6 matrix $\mathbf{A}$.

(22) Interchange the second and fourth rows of a 4×4 matrix $\mathbf{A}$.

(23) Interchange the second and fourth rows of a 6×6 matrix $\mathbf{A}$.

(24) Multiply the second row of a 2×5 matrix $\mathbf{A}$ by seven.

(25) Multiply the third row of a 5×2 matrix $\mathbf{A}$ by seven.

(26) Multiply the second row of a 3×5 matrix $\mathbf{A}$ by -0.2.

In Problems 27 through 42, find the inverses of the given elementary matrices.

(27) $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, (28) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, (29) $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$, (30) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$,

(31) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, (32) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, (33) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$,

(34) $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, (35) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$, (36) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$,

(37) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, (38) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, (39) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

(40) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, **(41)** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, **(42)** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

In Problems 43 through 55, find the inverses, if they exist, of the given diagonal or block diagonal matrices.

(43) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, **(44)** $\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$, **(45)** $\begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$, **(46)** $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{2}{3} \end{bmatrix}$,

(47) $\begin{bmatrix} 10 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, **(48)** $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, **(49)** $\begin{bmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & \frac{3}{5} \end{bmatrix}$,

(50) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$, **(51)** $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, **(52)** $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

(53) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, **(54)** $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$, **(55)** $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

(56) Prove that a square zero matrix does not have an inverse.

(57) Prove that if a diagonal matrix has at least one zero on its main diagonal, then that matrix cannot have an inverse.

(58) Prove that if $\mathbf{A}^2 = \mathbf{I}$, then $\mathbf{A}^{-1} = \mathbf{A}$.

3.2 Calculating Inverses

In Section 2.3, we developed a method for transforming any matrix into row-reduced form using elementary row operations. If we now restrict our attention to square matrices, we may say that the resulting row-reduced matrices are upper triangular matrices having either a unity or zero element in each entry on the main diagonal. This provides a simple test for determining which matrices have inverses.

Theorem 1. *A square matrix has an inverse if and only if reduction to row-reduced form by elementary row operations results in a matrix having all unity elements on the main diagonal.*

We shall prove this theorem in the appendix to this chapter as

Theorem 2. *An $n \times n$ matrix has an inverse if and only if it has rank n.*

Theorem 1 not only provides a test for determining when a matrix is invertible, but it also suggests a technique for obtaining the inverse when it exists. Once a matrix has been transformed to a row-reduced matrix with unity elements on the main diagonal, it is a simple matter to reduce it still further to the identity matrix. This is done by applying elementary row operation (E3)—adding to one row of a matrix a scalar times another row of the same matrix—to each column of the matrix, *beginning with the last column and moving sequentially towards the first column*, placing zeros in all positions above the diagonal elements.

Example 1

Use elementary row operations to transform the upper triangular matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

to the identity matrix.

Solution.

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \left\{\begin{array}{l}\text{by adding to}\\ \text{the second row } (-3)\\ \text{times the third row}\end{array}\right.$$

$$\to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \left\{\begin{array}{l}\text{by adding to}\\ \text{the first row } (-1)\\ \text{times the third row}\end{array}\right.$$

$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad \left\{\begin{array}{l}\text{by adding to}\\ \text{the first row } (-2)\\ \text{times the second row}\end{array}\right. \quad \square$$

To summarize, we now know that a square matrix $\mathbf{A}$ has an inverse if and only if it can be transformed into the identity matrix by elementary row operations. Moreover, it follows from the previous section that each elementary row operation is represented by an elementary matrix $\mathbf{E}$ that generates the row operation under the multiplication $\mathbf{EA}$. Therefore, $\mathbf{A}$ has an inverse if and only if there exist a sequence of elementary matrices $\mathbf{E}_1$, $\mathbf{E}_2, \ldots, \mathbf{E}_k$ such that

$$\mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{I}.$$

But, if we denote the product of these elementary matrices as $\mathbf{B}$, we then have $\mathbf{BA} = \mathbf{I}$, which implies that $\mathbf{B} = \mathbf{A}^{-1}$. That is, the inverse of a square matrix $\mathbf{A}$ of full rank is the product of those elementary matrices that reduce $\mathbf{A}$ to the identity matrix! Thus, to calculate the inverse of $\mathbf{A}$, we need only keep a record of the elementary row operations, or equivalently the elementary matrices, that were used to reduce $\mathbf{A}$ to $\mathbf{I}$. This is accomplished by simultaneously applying the same elementary row operations to both $\mathbf{A}$ and an identity matrix of the same order, because if

$$\mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{I},$$

then

$$(\mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1)\mathbf{I} = \mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1 = \mathbf{A}^{-1}.$$

We have, therefore, the following procedure for calculating inverses when they exist. Let $\mathbf{A}$ be the $n \times n$ matrix we wish to invert. Place next to it another $n \times n$ matrix $\mathbf{B}$ which is initially the identity. Using elementary row operations on $\mathbf{A}$, transform it into the identity. Each time an operation is performed on $\mathbf{A}$, repeat the exact same operation on $\mathbf{B}$. After $\mathbf{A}$ is transformed into the identity, the matrix obtained from transforming $\mathbf{B}$ will be $\mathbf{A}^{-1}$.

If $\mathbf{A}$ cannot be transformed into an identity matrix, which is equivalent to saying that its row-reduced form contains at least one zero row, then $\mathbf{A}$ does not have an inverse.

Example 2

Invert

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Solution.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \quad \left\{\begin{array}{l}\text{by adding to} \\ \text{the second row } (-3) \\ \text{times the first row}\end{array}\right.$$

$$\to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right]. \quad \left\{\begin{array}{l}\text{by multiplying} \\ \text{the second row by } (\frac{-1}{2})\end{array}\right.$$

$\mathbf{A}$ has been transformed into row-reduced form with a main diagonal of only unity elements; it has an inverse. Continuing with the transformation process, we get

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right]. \qquad \left\{\begin{array}{l}\text{by adding to} \\ \text{the first row } (-2) \\ \text{times the second row}\end{array}\right.$$

Thus,

$$\mathbf{A}^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}. \qquad \square$$

Example 3

Find the inverse of

$$\mathbf{A} = \begin{bmatrix} 5 & 8 & 1 \\ 0 & 2 & 1 \\ 4 & 3 & -1 \end{bmatrix}.$$

Solution.

$$\left[\begin{array}{ccc|ccc} 5 & 8 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 4 & 3 & -1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1.6 & 0.2 & 0.2 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 4 & 3 & -1 & 0 & 0 & 1 \end{array}\right] \qquad \left\{\begin{array}{l}\text{by multiplying the} \\ \text{first row by } (0.2)\end{array}\right.$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1.6 & 0.2 & 0.2 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & -3.4 & -1.8 & -0.8 & 0 & 1 \end{array}\right] \qquad \left\{\begin{array}{l}\text{by adding to the} \\ \text{third row } (-4) \\ \text{times the first row}\end{array}\right.$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1.6 & 0.2 & 0.2 & 0 & 0 \\ 0 & 1 & 0.5 & 0 & 0.5 & 0 \\ 0 & -3.4 & -1.8 & -0.8 & 0 & 1 \end{array}\right] \qquad \left\{\begin{array}{l}\text{by multiplying the} \\ \text{second row by } (0.5)\end{array}\right.$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1.6 & 0.2 & 0.2 & 0 & 0 \\ 0 & 1 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & -0.1 & -0.8 & 1.7 & 1 \end{array}\right] \qquad \left\{\begin{array}{l}\text{by adding to the} \\ \text{third row } (3.4) \\ \text{times the second row}\end{array}\right.$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1.6 & 0.2 & 0.2 & 0 & 0 \\ 0 & 1 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 1 & 8 & -17 & -10 \end{array}\right]. \qquad \left\{\begin{array}{l}\text{by multiplying the} \\ \text{third row by } (-0.1)\end{array}\right.$$

A has been transformed into row-reduced form with a main diagonal of only unity elements; it has an inverse. Continuing with the transformation process, we get

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1.6 & 0.2 & 0.2 & 0 & 0 \\ 0 & 1 & 0 & -4 & 9 & 5 \\ 0 & 0 & 1 & 8 & -17 & -10 \end{array}\right] \quad \begin{cases}\text{by adding to the} \\ \text{second row } (-0.5) \\ \text{times the third row}\end{cases}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1.6 & 0 & -1.4 & 3.4 & 2 \\ 0 & 1 & 0 & -4 & 9 & 5 \\ 0 & 0 & 1 & 8 & -17 & -10 \end{array}\right] \quad \begin{cases}\text{by adding to the} \\ \text{first row } (-0.2) \\ \text{times the third row}\end{cases}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & -11 & -6 \\ 0 & 1 & 0 & -4 & 9 & 5 \\ 0 & 0 & 1 & 8 & -17 & -10 \end{array}\right]. \quad \begin{cases}\text{by adding to the} \\ \text{first row } (-1.6) \\ \text{times the second row}\end{cases}$$

Thus,

$$\mathbf{A}^{-1} = \begin{bmatrix} 5 & -11 & -6 \\ -4 & 9 & 5 \\ 8 & -17 & -10 \end{bmatrix}. \qquad \square$$

Example 4

Find the inverse of

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}.$$

Solution.

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \quad \begin{cases}\text{by interchanging the} \\ \text{first and second rows}\end{cases}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 & 1 \end{array}\right] \quad \begin{cases}\text{by adding to the} \\ \text{the third row } (-1) \\ \text{times the first row}\end{cases}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right] \quad \begin{cases}\text{by multiplying the} \\ \text{third row by } (\frac{1}{2})\end{cases}$$

$$\to\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right] \quad \left\{\begin{array}{l}\text{by adding to the}\\ \text{second row } (-1) \text{ times}\\ \text{times the third row}\end{array}\right.$$

$$\to\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right] \quad \left\{\begin{array}{l}\text{by adding to the}\\ \text{first row } (-1) \text{ times}\\ \text{times the third row}\end{array}\right.$$

$$\to\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right]. \quad \left\{\begin{array}{l}\text{by adding to the}\\ \text{first row } (-1) \text{ times}\\ \text{times the second row}\end{array}\right.$$

Thus,

$$\mathbf{A}^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}. \qquad \square$$

Example 5

Invert

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

Solution.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]. \quad \left\{\begin{array}{l}\text{by adding to}\\ \text{the second row } (-2)\\ \text{times the first row}\end{array}\right.$$

A has been transformed into row-reduced form. Since the main diagonal contains a zero element, here in the 2–2 position, the matrix **A** does not have an inverse. It is singular. $\square$

Problems 3.2

In Problems 1–20, find the inverses of the given matrices, if they exist.

(1) $\begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$, **(2)** $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, **(3)** $\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$,

(4) $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, **(5)** $\begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix}$, **(6)** $\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}$,

(7) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, **(8)** $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, **(9)** $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix}$,

(10) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, **(11)** $\begin{bmatrix} 2 & 0 & 0 \\ 5 & 1 & 0 \\ 4 & 1 & 1 \end{bmatrix}$, **(12)** $\begin{bmatrix} 2 & 1 & 5 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{bmatrix}$,

(13) $\begin{bmatrix} 3 & 2 & 1 \\ 4 & 0 & 1 \\ 3 & 9 & 2 \end{bmatrix}$, **(14)** $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ -1 & 1 & 3 \end{bmatrix}$, **(15)** $\begin{bmatrix} 1 & 2 & 1 \\ 3 & -2 & -4 \\ 2 & 3 & -1 \end{bmatrix}$,

(16) $\begin{bmatrix} 2 & 4 & 3 \\ 3 & -4 & -4 \\ 5 & 0 & -1 \end{bmatrix}$, **(17)** $\begin{bmatrix} 5 & 0 & -1 \\ 2 & -1 & 2 \\ 2 & 3 & -1 \end{bmatrix}$, **(18)** $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 2 & 3 & -1 \end{bmatrix}$,

(19) $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & -2 \end{bmatrix}$, **(20)** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 4 & 6 & 2 & 0 \\ 3 & 2 & 4 & -1 \end{bmatrix}$.

(21) Use the results of Problems 11 and 20 to deduce a theorem involving inverses of lower triangular matrices.

(22) Use the results of Problems 12 and 19 to deduce a theorem involving the inverses of upper triangular matrices.

(23) Matrix inversion can be used to encode and decode sensitive messages for transmission. Initially, each letter in the alphabet is assigned a unique positive integer, with the simplest correspondence being

$$\begin{matrix} A & B & C & D & E & F & G & H & I & J & K & L & M & N & O & P & Q & R & S & T & U & V & W & X & Y & Z \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 \end{matrix}.$$

Zeros are used to separate words. Thus, the message

SHE IS A SEER

is encoded

19 8 5 0 9 19 0 1 0 19 5 5 18 0.

This scheme is too easy to decipher, however, so a scrambling effect is added prior to transmission. One scheme is to package the coded string as a set of 2-tuples, multiply each 2-tuple by a 2×2 invertible matrix, and then transmit the new string. For example, using the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix},$$

the coded message above would be scrambled into

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 19 \\ 8 \end{bmatrix} = \begin{bmatrix} 35 \\ 62 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 9 \\ 19 \end{bmatrix} = \begin{bmatrix} 47 \\ 75 \end{bmatrix}, \qquad \text{etc.,}$$

and the scrambled message becomes

35 62 5 10 47 75 ....

Note an immediate benefit from the scrambling: the letter S, which was originally always coded as 19 in each of its three occurrences, is now coded as a 35 the first time and as 75 the second time. Continue with the scrambling, and determine the final code for transmitting the above message.

(24) Scramble the message SHE IS A SEER using the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}.$$

(25) Scramble the message AARON IS A NAME using the matrix and steps described in Problem 23.

(26) Transmitted messages are unscrambled by again packaging the received message into 2-tuples and multiplying each vector by the inverse of **A**. To decode the scrambled message

18 31 44 72

using the encoding scheme described in Problem 23, we first calculate

$$\mathbf{A}^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix},$$

and then

$$\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 18 \\ 31 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix},$$

$$\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 44 \\ 72 \end{bmatrix} = \begin{bmatrix} 12 \\ 16 \end{bmatrix}.$$

The unscrambled message is

$$8 \quad 5 \quad 12 \quad 16$$

which, according to the letter–integer correspondence given in Problem 23, translates to HELP. Using the same procedure, decode the scrambled message

$$26 \quad 43 \quad 40 \quad 60 \quad 18 \quad 31 \quad 28 \quad 51.$$

(27) Use the decoding procedure described in Problem 26, but with the matrix **A** given in Problem 24, to decipher the transmitted message

$$16 \quad 120 \quad -39 \quad 131 \quad -27 \quad 45 \quad 38 \quad 76 \quad -51 \quad 129 \quad 28 \quad 56.$$

(28) Scramble the message SHE IS A SEER by packaging the coded letters into 3-tuples and then multiplying by the 3×3 invertible matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Add as many zeros as necessary to the end of the message to generate complete 3-tuples.

3.3 Simultaneous Equations

One use of the inverse is in the solution of systems of simultaneous linear equations. Recall, from Section 1.3 that any such system may be written in the form

$$\mathbf{Ax} = \mathbf{b}, \tag{2}$$

where **A** is the coefficient matrix, **b** is a known vector, and **x** is the unknown vector we wish to find. If **A** is invertible, then we can premultiply (2) by $\mathbf{A}^{-1}$ and obtain

$$\mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{b}.$$

But $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, therefore

$$\mathbf{Ix} = \mathbf{A}^{-1}\mathbf{b}$$

or

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}. \tag{3}$$

Hence, (3) shows that if $\mathbf{A}$ is invertible, then $\mathbf{x}$ can be obtained by premultiplying $\mathbf{b}$ by the inverse of $\mathbf{A}$.

Example 1

Solve the following system for x and y:

$$\begin{aligned} x - 2y &= -9, \\ -3x + y &= 2. \end{aligned}$$

Solution. Define

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -9 \\ 2 \end{bmatrix};$$

then the system can be written as $\mathbf{Ax} = \mathbf{b}$, hence $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. Using the method given in Section 3.2 we find that

$$\mathbf{A}^{-1} = (-\tfrac{1}{5})\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = (-\tfrac{1}{5})\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -9 \\ 2 \end{bmatrix} = (-\tfrac{1}{5})\begin{bmatrix} -5 \\ -25 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

Using the definition of matrix equality (two matrices are equal if and only if their corresponding elements are equal), we have that $x = 1$ and $y = 5$.

□

Example 2

Solve the following system for x, y and z:

$$\begin{aligned} 5x + 8y + z &= 2, \\ 2y + z &= -1, \\ 4x + 3y - z &= 3. \end{aligned}$$

Solution.

$$\mathbf{A} = \begin{bmatrix} 5 & 8 & 1 \\ 0 & 2 & 1 \\ 4 & 3 & -1 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

$\mathbf{A}^{-1}$ is found to be (see Example 3 of Section 3.2)

$$\begin{bmatrix} 5 & -11 & -6 \\ -4 & 9 & 5 \\ 8 & -17 & -10 \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \begin{bmatrix} 5 & -11 & -6 \\ -4 & 9 & 5 \\ 8 & -17 & -10 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix},$$

hence $x = 3$, $y = -2$, and $z = 3$. □

Not only does the invertibility of **A** provide us with a solution of the system $\mathbf{Ax} = \mathbf{b}$, it also provides us with a means of showing that this solution is unique (that is, there is no other solution to the system).

Theorem 1. *If* **A** *is invertible, then the system of simultaneous linear equations given by* $\mathbf{Ax} = \mathbf{b}$ *has one and only one solution.*

Proof. Define $\mathbf{w} = \mathbf{A}^{-1}\mathbf{b}$. Since we have already shown that **w** is a solution to $\mathbf{Ax} = \mathbf{b}$, it follows that

$$\mathbf{Aw} = \mathbf{b}. \tag{4}$$

Assume that there exists another solution **y**. Since **y** is a solution, we have that

$$\mathbf{Ay} = \mathbf{b}. \tag{5}$$

Equations (4) and (5) imply that

$$\mathbf{Aw} = \mathbf{Ay}. \tag{6}$$

Premultiply both sides of (6) by $\mathbf{A}^{-1}$. Then

$$\mathbf{A}^{-1}\mathbf{Aw} = \mathbf{A}^{-1}\mathbf{Ay},$$

$$\mathbf{Iw} = \mathbf{Iy},$$

or

$$\mathbf{w} = \mathbf{y}.$$

Thus, we see that if $\mathbf{y}$ is assumed to be a solution of $\mathbf{Ax} = \mathbf{b}$, it must, in fact, equal $\mathbf{w}$. Therefore, $\mathbf{w} = \mathbf{A}^{-1}\mathbf{b}$ is the only solution to the problem.

If $\mathbf{A}$ is singular, so that $\mathbf{A}^{-1}$ does not exist, then (3) is not valid and other methods, such as Gaussian elimination, must be used to solve the given system of simultaneous equations.

Problems 3.3

In Problems 1 through 12, use matrix inversion, if possible, to solve the given systems of equations:

(1) $\begin{aligned} x + 2y &= -3,\\ 3x + y &= 1. \end{aligned}$

(2) $\begin{aligned} a + 2b &= 5,\\ -3a + b &= 13. \end{aligned}$

(3) $\begin{aligned} 4x + 2y &= 6,\\ 2x - 3y &= 7. \end{aligned}$

(4) $\begin{aligned} 4l - p &= 1,\\ 5l - 2p &= -1. \end{aligned}$

(5) $\begin{aligned} 2x + 3y &= 8,\\ 6x + 9y &= 24. \end{aligned}$

(6) $\begin{aligned} x + 2y - z &= -1,\\ 2x + 3y + 2z &= 5,\\ y - z &= 2. \end{aligned}$

(7) $\begin{aligned} 2x + 3y - z &= 4,\\ -x - 2y + z &= -2,\\ 3x - y &= 2. \end{aligned}$

(8) $\begin{aligned} 60l + 30m + 20n &= 0,\\ 30l + 20m + 15n &= -10,\\ 20l + 15m + 12n &= -10. \end{aligned}$

(9) $\begin{aligned} 2r + 4s &= 2,\\ 3r + 2s + t &= 8,\\ 5r - 3s + 7t &= 15. \end{aligned}$

(10) $\begin{aligned} 2r + 4s &= 3,\\ 3r + 2s + t &= 8,\\ 5r - 3s + 7t &= 15. \end{aligned}$

(11) $\begin{aligned} 2r + 3s - 4t &= 12,\\ 3r - 2s &= -1,\\ 8r - s - 4t &= 10. \end{aligned}$

(12) $\begin{aligned} x + 2y - 2z &= -1,\\ 2x + y + z &= 5,\\ -x + y - z &= -2. \end{aligned}$

(13) Use matrix inversion to determine a production schedule that satisfies the requirements of the manufacturer described in Problem 12 of Section 2.1.

(14) Use matrix inversion to determine a production schedule that satisfies the requirements of the manufacturer described in Problem 13 of Section 2.1.

(15) Use matrix inversion to determine a production schedule that satisfies the requirements of the manufacturer described in Problem 14 of Section 2.1.

(16) Use matrix inversion to determine the bonus for the company described in Problem 16 of Section 2.1.

(17) Use matrix inversion to determine the number of barrels of gasoline that the producer described in Problem 17 of Section 2.1 must manufacture to break even.

(18) Use matrix inversion to solve the Leontief input–output model described in Problem 22 of Section 2.1.

(19) Use matrix inversion solve the Leontief input–output model described in Problem 23 of Section 2.1.

3.4 Properties of the Inverse

Theorem 1. *If* $\mathbf{A}$, $\mathbf{B}$, *and* $\mathbf{C}$ *are square matrices of the same order with* $\mathbf{AB} = \mathbf{I}$ *and* $\mathbf{CA} = \mathbf{I}$, *then* $\mathbf{B} = \mathbf{C}$.

Proof. $\mathbf{C} = \mathbf{CI} = \mathbf{C}(\mathbf{AB}) = (\mathbf{CA})\mathbf{B} = \mathbf{IB} = \mathbf{B}$.

Theorem 2. *The inverse of a matrix is unique.*

Proof. Suppose that $\mathbf{B}$ and $\mathbf{C}$ are inverses of $\mathbf{A}$. Then, by (1), we have that

$$\mathbf{AB} = \mathbf{I}, \qquad \mathbf{BA} = \mathbf{I}, \qquad \mathbf{AC} = \mathbf{I}, \quad \text{and} \quad \mathbf{CA} = \mathbf{I}.$$

It follows from Theorem 1 that $\mathbf{B} = \mathbf{C}$. Thus, if $\mathbf{B}$ and $\mathbf{C}$ are both inverses of $\mathbf{A}$, they must in fact be equal. Hence, the inverse is unique.

Using Theorem 2, we can prove some useful properties of the inverse of a matrix $\mathbf{A}$ when $\mathbf{A}$ is nonsingular.

Property 1. $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.

Proof. See Problem 1.

Property 2. $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Proof. $(\mathbf{AB})^{-1}$ denotes the inverse of $\mathbf{AB}$. However, $(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) = \mathbf{B}^{-1}(\mathbf{A}^{-1}\mathbf{A})\mathbf{B} = \mathbf{B}^{-1}\mathbf{IB} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}$. Thus, $\mathbf{B}^{-1}\mathbf{A}^{-1}$ is also an inverse for $\mathbf{AB}$, and, by uniqueness of the inverse, $\mathbf{B}^{-1}\mathbf{A}^{-1} = (\mathbf{AB})^{-1}$.

Property 3. $(\mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_n)^{-1} = \mathbf{A}_n^{-1}\mathbf{A}_{n-1}^{-1} \cdots \mathbf{A}_2^{-1}\mathbf{A}_1^{-1}$.

Proof. This is an extension of Property 2 and, as such, is proved in a similar manner.

CAUTION. Note that Property 3 states that the inverse of a product is *not* the product of the inverses but rather the product of the inverses commuted.

Property 4. $(\mathbf{A}^{\mathsf{T}})^{-1} = (\mathbf{A}^{-1})^{\mathsf{T}}$

Proof. $(\mathbf{A}^{\mathsf{T}})^{-1}$ denotes the inverse of $\mathbf{A}^{\mathsf{T}}$. However, using the property of the transpose that $(\mathbf{AB})^{\mathsf{T}} = \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}$, we have that

$$(\mathbf{A}^{\mathsf{T}})(\mathbf{A}^{-1})^{\mathsf{T}} = (\mathbf{A}^{-1}\mathbf{A})^{\mathsf{T}} = \mathbf{I}^{\mathsf{T}} = \mathbf{I}.$$

Thus, $(\mathbf{A}^{-1})^{\mathsf{T}}$ is an inverse of $\mathbf{A}^{\mathsf{T}}$, and by uniqueness of the inverse, $(\mathbf{A}^{-1})^{\mathsf{T}} = (\mathbf{A}^{\mathsf{T}})^{-1}$.

Property 5. $(\lambda\mathbf{A})^{-1} = (1/\lambda)(\mathbf{A})^{-1}$ *if λ is a nonzero scalar.*

Proof. $(\lambda\mathbf{A})^{-1}$ denotes the inverse of $\lambda\mathbf{A}$. However,

$$(\lambda\mathbf{A})(1/\lambda)\mathbf{A}^{-1} = \lambda(1/\lambda)\mathbf{AA}^{-1} = 1 \cdot \mathbf{I} = \mathbf{I}.$$

Thus, $(1/\lambda)\mathbf{A}^{-1}$ is an inverse of $\lambda\mathbf{A}$, and by uniqueness of the inverse, $(1/\lambda)\mathbf{A}^{-1} = (\lambda\mathbf{A})^{-1}$.

Property 6. *The inverse of a nonsingular symmetric matrix is symmetric.*

Proof. See Problem 18.

Property 7. *The inverse of a nonsingular upper or lower triangular matrix is again an upper or lower triangular matrix respectively.*

Proof. This is immediate from Theorem 2 and the constructive procedure described in Section 3.2 for calculating inverses.

Finally, the inverse provides us with a straightforward way of defining square matrices raised to negative integral powers. If $\mathbf{A}$ is nonsingular then we define $\mathbf{A}^{-n} = (\mathbf{A}^{-1})^n$.

Example 1

Find $\mathbf{A}^{-2}$ if

$$\mathbf{A} = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}.$$

Solution.

$$\begin{aligned} \mathbf{A}^{-2} &= (\mathbf{A}^{-1})^2 \\ &= \begin{bmatrix} 12 & -6 \\ -6 & 4 \end{bmatrix}^2 = \begin{bmatrix} 12 & -6 \\ -6 & 4 \end{bmatrix}\begin{bmatrix} 12 & -6 \\ -6 & 4 \end{bmatrix} = \begin{bmatrix} 180 & -96 \\ -96 & 52 \end{bmatrix}. \end{aligned}$$ □

Problems 3.4

(1) Prove Property 1.

(2) Verify Property 2 for

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}.$$

(3) Verify Property 2 for

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix}.$$

(4) Verify Property 2 for

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(5) Prove that $(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$.

(6) Verify the result of Problem 5 if

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} -1 & 0 \\ 2 & 2 \end{bmatrix}.$$

(7) Verify Property 4 for the matrix $\mathbf{A}$ defined in Problem 2.

(8) Verify Property 4 for the matrix $\mathbf{A}$ defined in Problem 3.

(9) Verify Property 4 for the matrix $\mathbf{A}$ defined in Problem 4.

(10) Verify Property 5 for $\lambda = 2$ and

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & -1 \\ -1 & 0 & 3 \end{bmatrix}.$$

(11) Find $\mathbf{A}^{-2}$ and $\mathbf{B}^{-2}$ for the matrices defined in Problem 2.

(12) Find $\mathbf{A}^{-3}$ and $\mathbf{B}^{-3}$ for the matrices defined in Problem 2.

(13) Find $\mathbf{A}^{-2}$ and $\mathbf{B}^{-4}$ for the matrices defined in Problem 3.

(14) Find $\mathbf{A}^{-2}$ and $\mathbf{B}^{-2}$ for the matrices defined in Problem 4.

(15) Find $\mathbf{A}^{-3}$ and $\mathbf{B}^{-3}$ for the matrices defined in Problem 4.

(16) Find $\mathbf{A}^{-3}$ if

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}.$$

(17) If $\mathbf{A}$ is symmetric, prove the identity

$$(\mathbf{BA}^{-1})^{\mathsf{T}}(\mathbf{A}^{-1}\mathbf{B}^{\mathsf{T}})^{-1} = \mathbf{I}.$$

(18) Prove Property 6.

3.5 LU Decomposition

Matrix inversion of elementary matrices (see Section 3.1) can be combined with the third elementary row operation (see Section 2.3) to generate a good numerical technique for solving simultaneous equations. It rests on being able to decompose a *nonsingular* square matrix $\mathbf{A}$ into the product of lower triangular matrix **L** with an upper triangular matrix **U**. Generally, there are many such factorizations. If, however, we add the additional condition that all diagonal elements of **L** be unity, then the decomposition, when it exists, is unique, and we may write

$$\mathbf{A} = \mathbf{LU} \tag{7}$$

with

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ l_{31} & l_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & 1 \end{bmatrix}$$

and

$$\mathbf{U} = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{bmatrix}.$$

To decompose **A** into form (7), we first reduce **A** to upper triangular form using just the third elementary row operation: namely, add to one row of a matrix a scalar times another row of that same matrix. This is completely analogous to transforming a matrix to row-reduced form, except that we no longer use the first two elementary row operations. We do not interchange rows, and we do not multiply a row by a nonzero constant. Consequently, we no longer require the first nonzero element of each nonzero row to be unity, and if any of the pivots are zero—which in the row-reduction scheme would require a row interchange operation—then the decomposition scheme we seek cannot be done.

Example 1

Use the third elementary row operation to transform the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 1 \\ -6 & -1 & 2 \end{bmatrix}$$

into upper triangular form.

Solution.

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 1 \\ -6 & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -5 \\ -6 & -1 & 2 \end{bmatrix} \quad \left\{\begin{array}{l}\text{by adding to the}\\ \text{second row } (-2) \text{ times}\\ \text{the first row}\end{array}\right.$$

$$\rightarrow \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -5 \\ 0 & -4 & 11 \end{bmatrix} \quad \left\{\begin{array}{l}\text{by adding to the}\\ \text{third row (3) times}\\ \text{the first row}\end{array}\right.$$

$$\rightarrow \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -5 \\ 0 & 0 & 6 \end{bmatrix}. \quad \left\{\begin{array}{l}\text{by adding to the}\\ \text{third row (1) times}\\ \text{the second row}\end{array}\right.$$

□

If a square matrix **A** can be reduced to upper triangular form **U** by a sequence of elementary row operations of the third type, then there exists a sequence of elementary matrices $\mathbf{E}_{21}, \mathbf{E}_{31}, \mathbf{E}_{41}, \ldots, \mathbf{E}_{n,n-1}$ such that

$$(\mathbf{E}_{n-1,n} \cdots \mathbf{E}_{41}\mathbf{E}_{31}\mathbf{E}_{21})\mathbf{A} = \mathbf{U}, \tag{8}$$

where $\mathbf{E}_{21}$ denotes the elementary matrix that places a zero in the 2–1 position, $\mathbf{E}_{31}$ denotes the elementary matrix that places a zero in the 3–1 position, $\mathbf{E}_{41}$ denotes the elementary matrix that places a zero in the 4–1 position, and so on. Since elementary matrices have inverses, we can write (8) as

$$\mathbf{A} = (\mathbf{E}_{21}^{-1}\mathbf{E}_{31}^{-1}\mathbf{E}_{41}^{-1} \cdots \mathbf{E}_{n,n-1}^{-1})\mathbf{U}. \tag{9}$$

Each elementary matrix in (8) is lower triangular. It follows from Property 7 of Section 3.4 that each of the inverses in (9) are lower triangular, and then from Theorem 1 of Section 1.4 that the product of these lower triangular matrices is itself lower triangular. Setting

$$\mathbf{L} = \mathbf{E}_{21}^{-1}\mathbf{E}_{31}^{-1}\mathbf{E}_{41}^{-1} \cdots \mathbf{E}_{n,n-1}^{-1},$$

we see that (9) is identical to (7), and we have the decomposition we seek.

Example 2

Construct an **LU** decomposition for the matrix given in Example 1.

Solution. The elementary matrices associated with the elementary row operations described in Example 1 are

$$\mathbf{E}_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{E}_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{E}_{42} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

with inverses given respectively by

$$\mathbf{E}_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{E}_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{E}_{42}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Then,

$$\begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 1 \\ -6 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -5 \\ 0 & 0 & 6 \end{bmatrix}$$

or, upon multiplying together the inverses of the elementary matrices,

$$\begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 1 \\ -6 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -5 \\ 0 & 0 & 6 \end{bmatrix}.$$

□

Example 2 suggests an important simplification of the decomposition process. Note the elements in **L** below the main diagonal are *the negatives of the scalars* used in the elementary row operations to reduce the original matrix to upper triangular form! This is no coincidence. In general,

Observation 1. If an elementary row operation is used to put a zero in the i–j position of $\mathbf{A}$ ($i > j$) by adding to row i a scalar k times row j, then the i–j element of $\mathbf{L}$ in the **LU** decomposition of $\mathbf{A}$ is $-k$.

We summarize the decomposition process as follows: Use only the third elementary row operation to transform a given square matrix **A** to upper triangular form. If this is not possible, because of a zero pivot, then stop; otherwise, the **LU** decomposition is found by defining the resulting upper triangular matrix as **U** and constructing the lower triangular matrix **L** utilizing Observation 1.

Example 3

Construct an **LU** decomposition for the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 2 & 3 \\ 6 & 2 & 4 & 8 \\ 1 & -1 & 0 & 4 \\ 0 & 1 & -3 & -4 \end{bmatrix}.$$

Solution. Transforming **A** to upper triangular form, we get

$$\begin{bmatrix} 2 & 1 & 2 & 3 \\ 6 & 2 & 4 & 8 \\ 1 & -1 & 0 & 4 \\ 0 & 1 & -3 & -4 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 2 & 3 \\ 0 & -1 & -2 & -1 \\ 1 & -1 & 0 & 4 \\ 0 & 1 & -3 & -4 \end{bmatrix} \quad \left\{\begin{array}{l}\text{by adding to the} \\ \text{second row } (-3) \text{ times} \\ \text{the first row}\end{array}\right.$$

$$\to \begin{bmatrix} 2 & 1 & 2 & 3 \\ 0 & -1 & -2 & -1 \\ 0 & -\frac{3}{2} & -1 & \frac{5}{2} \\ 0 & 1 & -3 & -4 \end{bmatrix} \quad \left\{\begin{array}{l}\text{by adding to the} \\ \text{third row } (-\frac{1}{2}) \text{ times} \\ \text{the first row}\end{array}\right.$$

$$\to \begin{bmatrix} 2 & 1 & 2 & 3 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 1 & -3 & -4 \end{bmatrix} \quad \left\{\begin{array}{l}\text{by adding to the} \\ \text{third row } (-\frac{3}{2}) \text{ times} \\ \text{the second row}\end{array}\right.$$

$$\to \begin{bmatrix} 2 & 1 & 2 & 3 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & -5 & -5 \end{bmatrix} \quad \left\{\begin{array}{l}\text{by adding to the} \\ \text{fourth row } (1) \text{ times} \\ \text{the second row}\end{array}\right.$$

$$\to \begin{bmatrix} 2 & 1 & 2 & 3 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 5 \end{bmatrix}. \quad \left\{\begin{array}{l}\text{by adding to the} \\ \text{fourth row } (\frac{5}{2}) \text{ times} \\ \text{the third row}\end{array}\right.$$

We now have an upper triangular matrix **U**. To get the lower triangular matrix **L** in the decomposition, we note that we used the scalar -3 to place a zero in the 2–1 position, so its negative $-(-3) = 3$ goes into the 2–1 position of **L**. We used the scalar $-\frac{1}{2}$ to place a zero in the 3–1 position in the second step of the above triangularization process, so its negative, $\frac{1}{2}$, becomes the 3–1 element in **L**; we used the scalar $\frac{5}{2}$ to place a zero in the 4–3 position during the last step of the triangularization process, so its negative, $-\frac{5}{2}$, becomes the 4–3 element in **L**. Continuing in this manner, we generate the decomposition

$$\begin{bmatrix} 2 & 1 & 2 & 3 \\ 6 & 2 & 4 & 8 \\ 1 & -1 & 0 & 4 \\ 0 & 1 & -3 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{3}{2} & 1 & 0 \\ 0 & -1 & -\frac{5}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 & 3 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 5 \end{bmatrix}. \qquad \square$$

LU decompositions, when they exist, can be used to solve systems of simultaneous linear equations. If a square matrix $\mathbf{A}$ can be factored into $\mathbf{A} = \mathbf{LU}$, then the system of equations $\mathbf{Ax} = \mathbf{b}$ can be written as $\mathbf{L}(\mathbf{Ux}) = \mathbf{b}$. To find $\mathbf{x}$, we first solve the system

$$\mathbf{Ly} = \mathbf{b} \tag{10}$$

for $\mathbf{y}$, and then, once $\mathbf{y}$ is determined, we solve the system

$$\mathbf{Ux} = \mathbf{y} \tag{11}$$

for $\mathbf{x}$. Both systems (10) and (11) are easy to solve, the first by forward substitution and the second by backward substitution.

Example 4

Solve the system of equations:

$$\begin{aligned} 2x - y + 3z &= 9, \\ 4x + 2y + z &= 9, \\ -6x - y + 2z &= 12. \end{aligned}$$

Solution. This system has the matrix form

$$\begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 1 \\ -6 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ 12 \end{bmatrix}.$$

The **LU** decomposition for the coefficient matrix **A** is given in Example 2. If we define the components of **y** by α, β, and γ, respectively, the matrix system $\mathbf{Ly} = \mathbf{b}$ is

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ 12 \end{bmatrix},$$

which is equivalent to the system of equations

$$\begin{aligned} \alpha &= 9, \\ 2\alpha + \beta &= 9, \\ -3\alpha - \beta + \gamma &= 12. \end{aligned}$$

Solving this system from top to bottom, we get $\alpha = 9$, $\beta = -9$, and $\gamma = 30$. Consequently, the matrix system $\mathbf{Ux} = \mathbf{y}$ is

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -9 \\ 30 \end{bmatrix}$$

which is equivalent to the system of equations

$$\begin{aligned} 2x - y + 3z &= 9, \\ 4y - 5z &= -9, \\ 6z &= 30. \end{aligned}$$

Solving this system from bottom to top, we obtain the final solution $x = -1$, $y = 4$, and $z = 5$. □

Example 5

Solve the system:

$$\begin{aligned} 2a + b + 2c + 3d &= 5, \\ 6a + 2b + 4c + 8d &= 8, \\ a - b + 4d &= -4, \\ b - 3c - 4d &= -3. \end{aligned}$$

Solution. The matrix representation for this system has as its coefficient matrix the matrix **A** of Example 3. Define

$$\mathbf{y} = [\alpha, \beta, \gamma, \delta]^{\mathrm{T}}.$$

Then, using the decomposition determined in Example 3, we can write the matrix system $\mathbf{Ly} = \mathbf{b}$ as the system of equations

$$\begin{aligned} \alpha &= 5, \\ 3\alpha + \beta &= 8, \\ \tfrac{1}{2}\alpha + \tfrac{3}{2}\beta + \gamma &= -4, \\ -\beta - \tfrac{5}{2}\gamma + \delta &= -3, \end{aligned}$$

which has as its solution $\alpha = 5$, $\beta = -7$, $\gamma = 4$, and $\delta = 0$. Thus, the matrix system $\mathbf{Ux} = \mathbf{y}$ is equivalent to the system of equations

$$\begin{aligned} 2a + b + 2c + 3d &= 5, \\ -b - 2c - d &= -7, \\ 2c + 4d &= 4, \\ 5d &= 0. \end{aligned}$$

Solving this set from bottom to top, we calculate the final solution as $a = -1$, $b = 3$, $c = 2$, and $d = 0$. □

LU decomposition and Gaussian elimination are equally efficient for solving $\mathbf{Ax} = \mathbf{b}$, when the decomposition exists. **LU** decomposition is superior when $\mathbf{Ax} = \mathbf{b}$ must be solved repeatedly for different values of $\mathbf{b}$ but the same $\mathbf{A}$, because once the factorization of $\mathbf{A}$ is determined it can be used with all $\mathbf{b}$. (See Problems 17 and 18.) A disadvantage of **LU** decomposition is that it does not exist for all nonsingular matrices, in particular whenever a pivot is zero. Fortunately, this occurs rarely, and when it does the difficulty usually is overcome by simply rearranging the order of the equations. (See Problems 19 and 20.)

Problems 3.5

In Problems 1 through 14, $\mathbf{A}$ and $\mathbf{b}$ are given. Construct an **LU** decomposition for the matrix $\mathbf{A}$ and then use it to solve the system $\mathbf{Ax} = \mathbf{b}$ for $\mathbf{x}$.

(1) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}.$

(2) $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 11 \\ -2 \end{bmatrix}.$

(3) $\mathbf{A} = \begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 625 \\ 550 \end{bmatrix}.$

(4) $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}.$

(5) $\mathbf{A} = \begin{bmatrix} -1 & 2 & 0 \\ 1 & -3 & 1 \\ 2 & -2 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}.$

(6) $\mathbf{A} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 0 \\ -2 & -1 & -2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 10 \\ -40 \\ 0 \end{bmatrix}.$

(7) $\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 0 & 1 \\ 3 & 9 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 50 \\ 80 \\ 20 \end{bmatrix}.$

(8) $\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ -1 & 1 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 80 \\ 159 \\ -75 \end{bmatrix}.$

(9) $\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 8 \\ -1 \\ 5 \end{bmatrix}.$

(10) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}.$

(11) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ -3 \\ -2 \\ -2 \end{bmatrix}.$

(12) $\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & 4 & 2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1000 \\ 200 \\ 100 \\ 100 \end{bmatrix}.$

(13) $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 30 \\ 30 \\ 10 \\ 10 \end{bmatrix}.$

(14) $\mathbf{A} = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 2 & 2 & 0 & 6 \\ -4 & 3 & 1 & 1 \\ 1 & 0 & 3 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -2 \\ 4 \\ 9 \\ 4 \end{bmatrix}.$

(15) (a) Use **LU** decomposition to solve the system

$$-x + 2y = -9,$$
$$2x + 3y = 4.$$

(b) Resolve when the right sides of each equation are replaced by 1 and -1, respectively.

(16) (a) Use **LU** decomposition to solve the system

$$\begin{aligned} x + 3y - z &= -1, \\ 2x + 5y + z &= 4, \\ 2x + 7y - 4z &= -6. \end{aligned}$$

(b) Resolve when the right sides of each equation are replaced by 10, 10, and 10, respectively.

(17) Solve the system $\mathbf{Ax} = \mathbf{b}$ for the following vectors $\mathbf{b}$ when $\mathbf{A}$ is given as in Problem 4:

(a) $\begin{bmatrix} 5 \\ 7 \\ -4 \end{bmatrix}$, (b) $\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$, (c) $\begin{bmatrix} 40 \\ 50 \\ 20 \end{bmatrix}$, (d) $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$.

(18) Solve the system $\mathbf{Ax} = \mathbf{b}$ for the following vectors $\mathbf{b}$ when $\mathbf{A}$ is given as in Problem 13:

(a) $\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, (b) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, (c) $\begin{bmatrix} 190 \\ 130 \\ 160 \\ 60 \end{bmatrix}$, (d) $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

(19) Show that **LU** decomposition cannot be used to solve the system

$$\begin{aligned} 2y + z &= -1, \\ x + y + 3z &= 8, \\ 2x - y - z &= 1, \end{aligned}$$

but that the decomposition can be used if the first two equations are interchanged.

(20) Show that **LU** decomposition cannot be used to solve the system

$$\begin{aligned} x + 2y + z &= 2, \\ 2x + 4y - z &= 7, \\ x + y + 2z &= 2, \end{aligned}$$

but that the decomposition can be used if the first and third equations are interchanged.

(21) (a) Show that the **LU** decomposition procedure given in this chapter cannot be applied to

$$\mathbf{A} = \begin{bmatrix} 0 & 2 \\ 0 & 9 \end{bmatrix}.$$

(b) Verify that $\mathbf{A} = \mathbf{LU}$, when

$$\mathbf{L} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{U} = \begin{bmatrix} 0 & 2 \\ 0 & 7 \end{bmatrix}.$$

(c) Verify that $\mathbf{A} = \mathbf{LU}$, when

$$\mathbf{L} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{U} = \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}.$$

(d) Why do you think the **LU** decomposition procedure fails for this **A**? What might explain the fact that **A** has more than one **LU** decomposition?

Appendix to Chapter 3

We now prove the answers to two questions raised earlier. First, what matrices have inverses? Second, if $\mathbf{AB} = \mathbf{I}$, is it necessarily true that $\mathbf{AB} = \mathbf{I}$ too?

Lemma 1. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be* $n \times n$ *matrices. If* $\mathbf{AB} = \mathbf{I}$, *then the system of equations* $\mathbf{Ax} = \mathbf{y}$ *has a solution for every choice of the vector* $\mathbf{y}$.

Proof. Once $\mathbf{y}$ is specified, set $\mathbf{x} = \mathbf{By}$. Then

$$\mathbf{Ax} = \mathbf{A}(\mathbf{By}) = (\mathbf{AB})\mathbf{y} = \mathbf{Iy} = \mathbf{y},$$

so $\mathbf{x} = \mathbf{By}$ is a solution of $\mathbf{Ax} = \mathbf{y}$.

Lemma 2. *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are* $n \times n$ *matrices with* $\mathbf{AB} = \mathbf{I}$, *then* $\mathbf{A}$ *has rank* n.

Proof. Designate the rows of $\mathbf{A}$ by $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$. We want to show that these n rows constitute a linearly independent set of vectors, in which case the rank of $\mathbf{A}$ is n. Designate the columns of $\mathbf{I}$ as the vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, respectively. It follows from Lemma 1 that the set of equations $\mathbf{Ax} = \mathbf{e}_j$ $(j = 1, 2, \ldots, n)$ has a solution for each j. Denote these solutions by $\mathbf{x}_1$,

$\mathbf{x}_2, \ldots, \mathbf{x}_n$, respectively. Therefore,

$$\mathbf{A}\mathbf{x}_j = \mathbf{e}_j.$$

Since $\mathbf{e}_j$ ($j = 1, 2, \ldots, n$) is an n-dimensional column vector having a unity element in row j and zeros everywhere else, it follows from the last equation that

$$\mathbf{A}_i\mathbf{x}_j = \begin{cases} 1 & \text{when } i = j, \\ 0 & \text{when } i \neq j. \end{cases}$$

This equation can be notationally simplified if we make use of the *Kronecker delta* δ_{ij} defined by

$$\delta_{ij} = \begin{cases} 1 & \text{when } i = j, \\ 0 & \text{when } i \neq j. \end{cases}$$

Then,

$$\mathbf{A}_i\mathbf{x}_j = \delta_{ij}.$$

Now consider the equation

$$\sum_{i=0}^{n} c_i\mathbf{A}_i = \mathbf{0}.$$

We wish to show that each constant c_i must be zero. Multiplying both sides of this last equation on the right by the vector $\mathbf{x}_j$, we have

$$\left(\sum_{i=0}^{n} c_i\mathbf{A}_i\right)\mathbf{x}_j = \mathbf{0}\mathbf{x}_j,$$

$$\sum_{i=0}^{n} (c_i\mathbf{A}_i)\mathbf{x}_j = 0,$$

$$\sum_{i=0}^{n} c_i(\mathbf{A}_i\mathbf{x}_j) = 0,$$

$$\sum_{i=0}^{n} c_i\delta_{ij} = 0,$$

$$c_j = 0.$$

Thus for each $\mathbf{x}_j$ ($j = 1, 2, \ldots, n$) we have $c_j = 0$, which implies that $c_1 = c_2 = \cdots = c_n = 0$ and that the rows $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$ are linearly independent.

It follows directly from Lemma 2 and the definition of an inverse that if an $n \times n$ matrix $\mathbf{A}$ has an inverse, then $\mathbf{A}$ must have rank n. This in turn implies directly that if **A** does not have rank n, then it does not have an inverse. We now want to show the converse: that is, if **A** has rank n, then **A** has an inverse.

We already have part of the result. If an $n \times n$ matrix **A** has rank n, then the procedure described in Section 3.2 is a constructive method for obtaining a matrix **C** having the property that $\mathbf{CA} = \mathbf{I}$. The procedure transforms **A** to an identity matrix by a sequence of elementary row operations $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_{k-1}, \mathbf{E}_k$. That is,

$$\mathbf{E}_k\mathbf{E}_{k-1}\ldots\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{I}.$$

Setting

$$\mathbf{C} = \mathbf{E}_k\mathbf{E}_{k-1}\ldots\mathbf{E}_2\mathbf{E}_1, \tag{12}$$

we have

$$\mathbf{CA} = \mathbf{I}. \tag{13}$$

We need only show that $\mathbf{AC} = \mathbf{I}$, too.

Theorem 1. *If* **A** *and* **B** *are* $n \times n$ *matrices such that* $\mathbf{AB} = \mathbf{I}$, *then* $\mathbf{BA} = \mathbf{I}$.

Proof. If $\mathbf{AB} = \mathbf{I}$, then from Lemma 1 **A** has rank n, and from (12) and (13) there exists a matrix **C** such that $\mathbf{CA} = \mathbf{I}$. It follows from Theorem 1 of Section 3.4 that $\mathbf{B} = \mathbf{C}$.

The major implication of Theorem 1 is that if **B** is a right inverse of **A**, then **B** is also a left inverse of **A**; and also if **A** is a left inverse of **B**, then **A** is also a right inverse of **B**. Thus, one needs only check whether a matrix is a right or left inverse; once one is verified for square matrices, the other is guaranteed. In particular, if an $n \times n$ matrix **A** has rank n, then (13) is valid. Thus, **C** is a left inverse of **A**. As a result of Theorem 1, however, **C** is also a right inverse of **A**—just replace **A** with **C** and **B** with **A** in Theorem 1—so **C** is both a left and right inverse of **A**, which means that **C** is the inverse of **A**. We have now proven:

▶ | **Theorem 2.** *An* $n \times n$ *matrix* **A** *has an inverse if and only if* **A** *has rank* n.

Chapter 4

Determinants

4.1 Introduction

Every square matrix has associated with it a scalar called its *determinant*. To be extremely rigorous we would have to define this scalar in terms of permutations on positive integers. However, since in practice it is difficult to apply a definition of this sort, other procedures have been developed which yield the determinant in a more straightforward manner. In this chapter, therefore, we concern ourselves solely with those methods that can be applied easily. We note here for reference that determinants are only defined for square matrices.

Given a square matrix $\mathbf{A}$, we use $\det(\mathbf{A})$ or $|\mathbf{A}|$ to designate its determinant. If the matrix can actually be exhibited, we then designate the determinant of $\mathbf{A}$ by replacing the brackets by vertical straight lines. For example, if

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \tag{1}$$

then

$$\det(\mathbf{A}) = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}. \tag{2}$$

We cannot overemphasize the fact that (1) and (2) represent entirely different animals. (1) represents a matrix, a rectangular array, an entity unto itself while (2) represents a scalar, a number associated with the matrix in (1). There is absolutely no similarity between the two other than form!

We are now ready to calculate determinants.

Definition 1. The determinant of a 1×1 matrix $[a]$ is the scalar a.

Thus, the determinant of the matrix [5] is 5 and the determinant of the matrix $[-3]$ is -3.

Definition 2. The determinant of a 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is the scalar $ad - bc$.

Example 1

Find det(**A**) if

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

Solution.

$$\det(\mathbf{A}) = \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = (1)(3) - (2)(4) = 3 - 8 = -5. \quad \square$$

Example 2

Find $|\mathbf{A}|$ if

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}.$$

Solution.

$$|\mathbf{A}| = \begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix} = (2)(3) - (-1)(4) = 6 + 4 = 10. \quad \square$$

We now could proceed to give separate rules which would enable one to compute determinants of 3×3, 4×4, and higher order matrices. This is unnecessary. In the next section, we will give a method that enables us to reduce all determinants of order n ($n > 2$) (if **A** has order $n \times n$ then det(**A**) is said to have order n) to a sum of determinants of order 2.

Problems 4.1

In Problems 1 through 18, find the determinants of the given matrices.

(1) $\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$, **(2)** $\begin{bmatrix} 3 & -4 \\ 5 & 6 \end{bmatrix}$, **(3)** $\begin{bmatrix} 3 & 4 \\ -5 & 6 \end{bmatrix}$,

(4) $\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$, **(5)** $\begin{bmatrix} 5 & 6 \\ -7 & 8 \end{bmatrix}$, **(6)** $\begin{bmatrix} 5 & 6 \\ 7 & -8 \end{bmatrix}$,

(7) $\begin{bmatrix} 1 & -1 \\ 2 & 7 \end{bmatrix}$, **(8)** $\begin{bmatrix} -2 & -3 \\ -4 & 4 \end{bmatrix}$, **(9)** $\begin{bmatrix} 3 & -1 \\ -3 & 8 \end{bmatrix}$,

(10) $\begin{bmatrix} 0 & 1 \\ -2 & 6 \end{bmatrix}$, **(11)** $\begin{bmatrix} -2 & 3 \\ -4 & -4 \end{bmatrix}$, **(12)** $\begin{bmatrix} 9 & 0 \\ 2 & 0 \end{bmatrix}$,

(13) $\begin{bmatrix} 12 & 20 \\ -3 & -5 \end{bmatrix}$, **(14)** $\begin{bmatrix} -36 & -3 \\ -12 & -1 \end{bmatrix}$, **(15)** $\begin{bmatrix} -8 & -3 \\ -7 & 9 \end{bmatrix}$,

(16) $\begin{bmatrix} t & 2 \\ 3 & 4 \end{bmatrix}$, **(17)** $\begin{bmatrix} 2t & 3 \\ -2 & t \end{bmatrix}$, **(18)** $\begin{bmatrix} 3t & -t^2 \\ 2 & t \end{bmatrix}$.

(19) Find t so that

$$\begin{vmatrix} t & 2t \\ 1 & t \end{vmatrix} = 0.$$

(20) Find t so that

$$\begin{vmatrix} t-2 & t \\ 3 & t+2 \end{vmatrix} = 0.$$

(21) Find λ so that

$$\begin{vmatrix} 4-\lambda & 2 \\ -1 & 1-\lambda \end{vmatrix} = 0.$$

(22) Find λ so that

$$\begin{vmatrix} 1-\lambda & 5 \\ 1 & -1-\lambda \end{vmatrix} = 0.$$

(23) Find $\det(\mathbf{A} - \lambda\mathbf{I})$ if $\mathbf{A}$ is the matrix defined in Problem 1.

(24) Find $\det(\mathbf{A} - \lambda\mathbf{I})$ if $\mathbf{A}$ is the matrix defined in Problem 2.

(25) Find $\det(\mathbf{A} - \lambda\mathbf{I})$ if $\mathbf{A}$ is the matrix defined in Problem 4.

(26) Find $\det(\mathbf{A} - \lambda\mathbf{I})$ if $\mathbf{A}$ is the matrix defined in Problem 7.

(27) Find $|\mathbf{A}|$, $|\mathbf{B}|$, and $|\mathbf{AB}|$ if

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 4 & 2 \\ -1 & 2 \end{bmatrix}.$$

What is the relationship between these three determinants?

(28) Interchange the rows for each of the matrices given in Problems 1 through 15, and calculate the new determinants. How do they compare with the determinants of the original matrices?

(29) The second elementary row operation is to multiply any row of a matrix by a nonzero constant. Apply this operation to the matrices given in Problems 1 through 15 for any constants of your choice, and calculate the new determinants. How do they compare with the determinants of the original matrix?

(30) Redo Problem 29 for the third elementary row operation.

(31) What is the determinant of a 2×2 matrix if one row or one column contains only zero entries?

(32) What is the relationship between the determinant of a 2×2 matrix and its transpose?

(33) What is the determinant of a 2×2 matrix if one row is a linear combination of the other row?

4.2 Expansion by Cofactors

Definition 1. Given a matrix $\mathbf{A}$, a *minor* is the determinant of any square submatrix of $\mathbf{A}$.

That is, given a square matrix $\mathbf{A}$, a minor is the determinant of any matrix formed from $\mathbf{A}$ by the removal of an equal number of rows and columns. As an example, if

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix},$$

then

$$\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix}$$

are both minors because

$$\begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}$$

are both submatrices of **A**, while

$$\begin{vmatrix} 1 & 2 \\ 8 & 9 \end{vmatrix} \quad \text{and} \quad |1 \quad 2|$$

are not minors because

$$\begin{bmatrix} 1 & 2 \\ 8 & 9 \end{bmatrix}$$

is not a submatrix of **A** and $[1 \quad 2]$, although a submatrix of **A**, is not square.

A more useful concept for our immediate purposes, since it will enable us to calculate determinants, is that of the cofactor of an element of a matrix.

Definition 2. Given a matrix $\mathbf{A} = [a_{ij}]$, the *cofactor of the element* a_{ij} is a scalar obtained by multiplying together the term $(-1)^{i+j}$ and the minor obtained from **A** by removing the ith row and jth column.

In other words, to compute the cofactor of the element a_{ij} we first form a submatrix of **A** by crossing out both the row and column in which the element a_{ij} appears. We then find the determinant of the submatrix and finally multiply it by the number $(-1)^{i+j}$.

Example 1

Find the cofactor of the element 4 in the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Solution. We first note that 4 appears in the (2, 1) position. The submatrix obtained by crossing out the second row and first column is

$$\begin{bmatrix} \cancel{1} & 2 & 3 \\ \cancel{4} & \cancel{5} & \cancel{6} \\ \cancel{7} & 8 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix},$$

which has a determinant equal to $(2)(9) - (3)(8) = -6$. Since 4 appears in the (2, 1) position, $i = 2$ and $j = 1$. Thus, $(-1)^{i+j} = (-1)^{2+1} = (-1)^3 = (-1)$. The cofactor of 4 is $(-1)(-6) = 6$. □

Example 2

Using the same $\mathbf{A}$ as in Example 1, find the cofactor of the element 9.

Solution. The element 9 appears in the (3, 3) position. Thus, crossing out the third row and third column, we obtain the submatrix

$$\begin{bmatrix} 1 & 2 & \cancel{3} \\ 4 & 5 & \cancel{6} \\ \cancel{7} & \cancel{8} & \cancel{9} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}.$$

which has a determinant equal to $(1)(5) - (2)(4) = -3$. Since, in this case, $i = j = 3$, the cofactor of 9 is $(-1)^{3+3}(-3) = (-1)^6(-3) = -3$. □

We now have enough tools at hand to find the determinant of any matrix.

▶ Expansion by Cofactors. To find the determinant of a matrix $\mathbf{A}$ of arbitrary order, (a) pick any one row or any one column of the matrix (dealer's choice), (b) for each element in the row or column chosen, find its cofactor, (c) multiply each element in the row or column chosen by its cofactor and sum the results. This sum is the determinant of the matrix.

Example 3

Find $\det(\mathbf{A})$ if

$$\mathbf{A} = \begin{bmatrix} 3 & 5 & 0 \\ -1 & 2 & 1 \\ 3 & -6 & 4 \end{bmatrix}.$$

Solution. In this example, we expand by the second column.

$$\begin{aligned} |\mathbf{A}| &= (5)(\text{cofactor of } 5) + (2)(\text{cofactor of } 2) + (-6)(\text{cofactor of } -6) \\ &= (5)(-1)^{1+2}\begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} + (2)(-1)^{2+2}\begin{vmatrix} 3 & 0 \\ 3 & 4 \end{vmatrix} + (-6)(-1)^{3+2}\begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix} \\ &= 5(-1)(-4-3) + (2)(1)(12-0) + (-6)(-1)(3-0) \\ &= (-5)(-7) + (2)(12) + (6)(3) = 35 + 24 + 18 = 77. \quad \square \end{aligned}$$

Example 4

Using the $\mathbf{A}$ of Example 3 and expanding by the first row, find $\det(\mathbf{A})$.

Solution.

$$\begin{aligned}
|\mathbf{A}| &= 3(\text{cofactor of } 3) + 5(\text{cofactor of } 5) + 0(\text{cofactor of } 0)\\
&= (3)(-1)^{1+1}\begin{vmatrix} 2 & 1\\ -6 & 4\end{vmatrix} + 5(-1)^{1+2}\begin{vmatrix} -1 & 1\\ 3 & 4\end{vmatrix} + 0\\
&= (3)(1)(8+6) + (5)(-1)(-4-3)\\
&= (3)(14) + (-5)(-7) = 42 + 35 = 77. \qquad \square
\end{aligned}$$

The previous examples illustrate two important properties of the method. First, the value of the determinant is the same regardless of which row or column we choose to expand by and second, expanding by a row or column that contains zeros significantly reduces the number of computations involved.

Example 5

Find det(**A**) if

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 5 & 2\\ -1 & 4 & 1 & 0\\ 3 & 0 & 4 & 1\\ -2 & 1 & 1 & 3\end{bmatrix}.$$

Solution. We first check to see which row or column contains the most zeros and expand by it. Thus, expanding by the second column gives

$$|\mathbf{A}| = 0(\text{cofactor of } 0) + 4(\text{cofactor of } 4) + 0(\text{cofactor of } 0) + 1(\text{cofactor of } 1)$$

$$= 0 + 4(-1)^{2+2}\begin{vmatrix} 1 & 5 & 2\\ 3 & 4 & 1\\ -2 & 1 & 3\end{vmatrix} + 0 + 1(-1)^{4+2}\begin{vmatrix} 1 & 5 & 2\\ -1 & 1 & 0\\ 3 & 4 & 1\end{vmatrix}$$

$$= 4\begin{vmatrix} 1 & 5 & 2\\ 3 & 4 & 1\\ -2 & 1 & 3\end{vmatrix} + \begin{vmatrix} 1 & 5 & 2\\ -1 & 1 & 0\\ 3 & 4 & 1\end{vmatrix}.$$

Using expansion by cofactors on each of the determinants of order 3 yields

$$\begin{aligned}
\begin{vmatrix} 1 & 5 & 2\\ 3 & 4 & 1\\ -2 & 1 & 3\end{vmatrix} &= 1(-1)^{1+1}\begin{vmatrix} 4 & 1\\ 1 & 3\end{vmatrix} + 5(-1)^{1+2}\begin{vmatrix} 3 & 1\\ -2 & 3\end{vmatrix} + 2(-1)^{1+3}\begin{vmatrix} 3 & 4\\ -2 & 1\end{vmatrix}\\
&= -22 \quad \text{(expanding by the first row)}
\end{aligned}$$

and

$$\begin{vmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ 3 & 4 & 1 \end{vmatrix} = 2(-1)^{1+3}\begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} + 0 + 1(-1)^{3+3}\begin{vmatrix} 1 & 5 \\ -1 & 1 \end{vmatrix}$$

$$= -8 \quad \text{(expanding by the third column).}$$

Hence,

$$|\mathbf{A}| = 4(-22) - 8 = -88 - 8 = -96. \qquad \square$$

For $n \times n$ matrices with $n > 3$, expansion by cofactors is an inefficient procedure for calculating determinants. It simply takes too long. A more elegant method, based on elementary row operations, is given in Section 4.4 for matrices whose elements are all numbers.

Problems 4.2

In Problems 1 through 22, use expansion by cofactors to evaluate the determinants of the given matrices.

(1) $\begin{bmatrix} 1 & 2 & -2 \\ 0 & 2 & 3 \\ 0 & 0 & -3 \end{bmatrix}$, **(2)** $\begin{bmatrix} 3 & 2 & -2 \\ 1 & 0 & 4 \\ 2 & 0 & -3 \end{bmatrix}$, **(3)** $\begin{bmatrix} 1 & -2 & -2 \\ 7 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$,

(4) $\begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 1 \\ 3 & 2 & -3 \end{bmatrix}$, **(5)** $\begin{bmatrix} 3 & 5 & 2 \\ -1 & 0 & 4 \\ -2 & 2 & 7 \end{bmatrix}$, **(6)** $\begin{bmatrix} 1 & -3 & -3 \\ 2 & 8 & 3 \\ 4 & 5 & 0 \end{bmatrix}$,

(7) $\begin{bmatrix} 2 & 1 & -9 \\ 3 & -1 & 1 \\ 3 & -1 & 2 \end{bmatrix}$, **(8)** $\begin{bmatrix} -1 & 3 & 3 \\ 1 & 1 & 4 \\ -1 & 1 & 2 \end{bmatrix}$, **(9)** $\begin{bmatrix} 1 & -3 & -3 \\ 2 & 8 & 4 \\ 3 & 5 & 1 \end{bmatrix}$,

(10) $\begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & 2 \\ 2 & 3 & 5 \end{bmatrix}$, **(11)** $\begin{bmatrix} -1 & 3 & 3 \\ 4 & 5 & 6 \\ -1 & 3 & 3 \end{bmatrix}$, **(12)** $\begin{bmatrix} 1 & 2 & -3 \\ 5 & 5 & 1 \\ 2 & -5 & -1 \end{bmatrix}$,

(13) $\begin{bmatrix} -4 & 0 & 0 \\ 2 & -1 & 0 \\ 3 & 1 & -2 \end{bmatrix}$, **(14)** $\begin{bmatrix} 1 & 3 & 2 \\ -1 & 4 & 1 \\ 5 & 3 & 8 \end{bmatrix}$, **(15)** $\begin{bmatrix} 3 & -2 & 0 \\ 1 & 1 & 2 \\ -3 & 4 & 1 \end{bmatrix}$,

(16) $\begin{bmatrix} -4 & 0 & 0 & 0 \\ 1 & -5 & 0 & 0 \\ 2 & 1 & -2 & 0 \\ 3 & 1 & -2 & 1 \end{bmatrix}$, **(17)** $\begin{bmatrix} -1 & 2 & 1 & 2 \\ 1 & 0 & 3 & -1 \\ 2 & 2 & -1 & 1 \\ 2 & 0 & -3 & 2 \end{bmatrix}$,

(18) $\begin{bmatrix} 1 & 1 & 2 & -2 \\ 1 & 5 & 2 & -1 \\ -2 & -2 & 1 & 3 \\ -3 & 4 & -1 & 8 \end{bmatrix}$, **(19)** $\begin{bmatrix} -1 & 3 & 2 & -2 \\ 1 & -5 & -4 & 6 \\ 3 & -6 & 1 & 1 \\ 3 & -4 & 3 & -3 \end{bmatrix}$,

(20) $\begin{bmatrix} 1 & 1 & 0 & -2 \\ 1 & 5 & 0 & -1 \\ -2 & -2 & 0 & 3 \\ -3 & 4 & 0 & 8 \end{bmatrix}$, **(21)** $\begin{bmatrix} 1 & 2 & 1 & -1 \\ 4 & 0 & 3 & 0 \\ 1 & 1 & 0 & 5 \\ 2 & -2 & 1 & 1 \end{bmatrix}$,

(22) $\begin{bmatrix} 11 & 1 & 0 & 9 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 4 & -1 & 1 & 0 & 0 \\ 3 & 2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix}$.

(23) Use the results of Problems 1, 13, and 16 to develop a theorem about the determinants of triangular matrices.

(24) Use the results of Problems 3, 20, and 22 to develop a theorem regarding determinants of matrices containing a zero row or column.

(25) Find $\det(\mathbf{A} - \lambda\mathbf{I})$ if $\mathbf{A}$ is the matrix given in Problem 2.

(26) Find $\det(\mathbf{A} - \lambda\mathbf{I})$ if $\mathbf{A}$ is the matrix given in Problem 3.

(27) Find $\det(\mathbf{A} - \lambda\mathbf{I})$ if $\mathbf{A}$ is the matrix given in Problem 4.

(28) Find $\det(\mathbf{A} - \lambda\mathbf{I})$ if $\mathbf{A}$ is the matrix given in Problem 5.

4.3 Properties of Determinants

In this section, we list some useful properties of determinants. For the sake of expediency, we only give proofs for determinants of order three, keeping in mind that these proofs may be extended in a straightforward manner to determinants of higher order.

Property 1. *If one row of a matrix consists entirely of zeros, then the determinant is zero.*

Proof. Expanding by the zero row, we immediately obtain the desired result.

Property 2. *If two rows of a matrix are interchanged, the determinant changes sign.*

Proof. Consider

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Expanding by the third row, we obtain

$$|\mathbf{A}| = a_{31}(a_{12}a_{23} - a_{13}a_{22}) - a_{32}(a_{11}a_{23} - a_{13}a_{21}) + a_{33}(a_{11}a_{22} - a_{12}a_{21}).$$

Now consider the matrix **B** obtained from **A** by interchanging the second and third rows:

$$\mathbf{B} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}.$$

Expanding by the second row, we find that

$$|\mathbf{B}| = -a_{31}(a_{12}a_{23} - a_{13}a_{22}) + a_{32}(a_{11}a_{23} - a_{13}a_{21}) - a_{33}(a_{11}a_{22} - a_{12}a_{21}).$$

Thus, $|\mathbf{B}| = -|\mathbf{A}|$. Through similar reasoning, one can demonstrate that the result is valid regardless of which two rows are interchanged.

Property 3. *If two rows of a determinant are identical, the determinant is zero.*

Proof. If we interchange the two identical rows of the matrix, the matrix remains unaltered; hence the determinant of the matrix remains constant. From Property 2, however, by interchanging two rows of a matrix, we change the sign of the determinant. Thus, the determinant must on one hand remain the same while on the other hand change sign. The only way both of these conditions can be met simultaneously is for the determinant to be zero.

Property 4. *If the matrix* **B** *is obtained from the matrix* **A** *by multiplying every element in one row of* **A** *by the scalar* λ, *then* $|\mathbf{B}| = \lambda|\mathbf{A}|$

Proof.

$$\begin{vmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \lambda a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - \lambda a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + \lambda a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= \lambda\left(a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}\right)$$

$$= \lambda\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

In essence, Property 4 shows us how to multiply a scalar times a determinant. We know from Chapter 1 that multiplying a scalar times a matrix simply multiplies every element of the matrix by that scalar. Property 4, however, implies that multiplying a scalar times a determinant simply multiplies *one* row of the determinant by the scalar. Thus, while in matrices

$$8\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 16 \\ 24 & 32 \end{bmatrix},$$

in determinants we have

$$8\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 24 & 32 \end{vmatrix},$$

or alternatively

$$8\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4(2)\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4\begin{vmatrix} 2 & 4 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 12 & 16 \end{vmatrix}.$$

Property 5. *For an* $n \times n$ *matrix* **A** *and any scalar* λ, $\det(\lambda\mathbf{A}) = \lambda^n \det(\mathbf{A})$.

Proof. This proof makes continued use of Property 4.

$$\det(\lambda\mathbf{A}) = \det\left\{\lambda\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right\} = \det\left\{\begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \\ \lambda a_{31} & \lambda a_{32} & \lambda a_{33} \end{bmatrix}\right\}$$

$$= \begin{vmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \\ \lambda a_{31} & \lambda a_{32} & \lambda a_{33} \end{vmatrix} = \lambda \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \\ \lambda a_{31} & \lambda a_{32} & \lambda a_{33} \end{vmatrix}$$

$$= (\lambda)(\lambda)\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \lambda a_{31} & \lambda a_{32} & \lambda a_{33} \end{vmatrix} = \lambda(\lambda)(\lambda)\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \lambda^3 \det(\mathbf{A}).$$

Note that for a 3×3 matrix, $n = 3$.

Property 6. *If a matrix* **B** *is obtained from a matrix* **A** *by adding to one row of* **A**, *a scalar times another row of* **A**, *then* $|\mathbf{A}| = |\mathbf{B}|$.

Proof. Let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

and

$$\mathbf{B} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} + \lambda a_{11} & a_{32} + \lambda a_{12} & a_{33} + \lambda a_{13} \end{bmatrix},$$

where **B** has been obtained from **A** by adding λ times the first row of **A** to the third row of **A**. Expanding $|\mathbf{B}|$ by its third row, we obtain

$$|\mathbf{B}| = (a_{31} + \lambda a_{11})\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} - (a_{32} + \lambda a_{12})\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$+ (a_{33} + \lambda a_{13})\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$+ \lambda\left\{a_{11}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} - a_{12}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{13}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\right\}.$$

The first three terms of this sum are exactly $|\mathbf{A}|$ (expand $|\mathbf{A}|$ by its third row), while the last three terms of the sum are

$$\lambda\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{vmatrix}$$

(expand this determinant by its third row). Thus, it follows that

$$|\mathbf{B}| = |\mathbf{A}| + \lambda\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{vmatrix}.$$

From Property 3, however, this second determinant is zero since its first and third rows are identical, hence $|\mathbf{B}| = |\mathbf{A}|$.

The same type of argument will quickly show that this result is valid regardless of the two rows chosen.

Example 1

Without expanding, show that

$$\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix} = \begin{vmatrix} a-r & b-s & c-t \\ r+2x & s+2y & t+2z \\ x & y & z \end{vmatrix}.$$

Solution. Using Property 6, we have that

$$\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix} = \begin{vmatrix} a-r & b-s & c-t \\ r & s & t \\ x & y & z \end{vmatrix}, \quad \left\{\begin{array}{l}\text{by adding to the first} \\ \text{row } (-1) \text{ times the} \\ \text{second row}\end{array}\right.$$

$$= \begin{vmatrix} a-r & b-s & c-t \\ r+2x & s+2y & t+2z \\ x & y & z \end{vmatrix}. \quad \left\{\begin{array}{l}\text{by adding to the} \\ \text{second row (2) times} \\ \text{the third row}\end{array}\right.$$

□

Property 7. $\det(\mathbf{A}) = \det(\mathbf{A}^{\mathsf{T}})$.

Proof. If

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \text{then} \quad \mathbf{A}^{\mathsf{T}} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}.$$

Expanding $\det(\mathbf{A}^T)$ by the first column, it follows that

$$|\mathbf{A}^T| = a_{11}\begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{31} \\ a_{23} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}).$$

This, however, is exactly the expression we would obtain if we expand $\det(\mathbf{A})$ by the first row. Thus $|\mathbf{A}^T| = |\mathbf{A}|$.

It follows from Property 7 that any property about determinants dealing with row operations is equally true for column operations (the analogous elementary row operation applied to columns), because a row operation on $\mathbf{A}^T$ is the same as a column operation on $\mathbf{A}$. Thus, if one column of a matrix consists entirely of zeros, then its determinant is zero; if two columns of a matrix are interchanged, the determinant changes sign; if two columns of a matrix are identical, its determinant is zero; multiplying a determinant by a scalar is equivalent to multiplying one column of the matrix by that scalar and then calculating the new determinant; and the third elementary column operation when applied to a matrix does not change its determinant.

Property 8. *The determinant of a triangular matrix, either upper or lower, is the product of the elements on the main diagonal.*

Proof. See Problem 2.

Property 9. *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are of the same order, then* $\det(\mathbf{A})\det(\mathbf{B}) = \det(\mathbf{AB})$.

Because of its difficulty, the proof of Property 9 is omitted here.

Example 2

Show that Property 9 is valid for

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 6 & -1 \\ 7 & 4 \end{bmatrix}.$$

Solution. $|\mathbf{A}| = 5$, $|\mathbf{B}| = 31$.

$$\mathbf{AB} = \begin{bmatrix} 33 & 10 \\ 34 & 15 \end{bmatrix} \quad \text{thus} \quad |\mathbf{AB}| = 155 = |\mathbf{A}||\mathbf{B}|. \qquad \square$$

Problems 4.3

(1) Prove that the determinant of a diagonal matrix is the product of the elements on the main diagonal.

(2) Prove that the determinant of an upper or lower triangular matrix is the product of the elements on the main diagonal.

(3) Without expanding, show that

$$\begin{vmatrix} a+x & r-x & x \\ b+y & s-y & y \\ c+z & t-z & z \end{vmatrix} = \begin{vmatrix} a & r & x \\ b & s & y \\ c & t & z \end{vmatrix}.$$

(4) Verify Property 5 for $\lambda = -3$ and

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 5 & -1 & 3 \\ 2 & 1 & 1 \end{bmatrix}.$$

(5) Verify Property 9 for

$$\mathbf{A} = \begin{vmatrix} 6 & 1 \\ 1 & 2 \end{vmatrix} \quad \text{and} \quad \mathbf{B} = \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix}.$$

(6) Without expanding, show that

$$\begin{vmatrix} 2a & 3r & x \\ 4b & 6s & 2y \\ -2c & -3t & -z \end{vmatrix} = -12 \begin{vmatrix} a & r & x \\ b & s & y \\ c & t & z \end{vmatrix}.$$

(7) Without expanding, show that

$$\begin{vmatrix} a-3b & r-3s & x-3y \\ b-2c & s-2t & y-2z \\ 5c & 5t & 5z \end{vmatrix} = 5 \begin{vmatrix} a & r & x \\ b & s & y \\ c & t & z \end{vmatrix}.$$

(8) Without expanding, show that

$$\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix} = - \begin{vmatrix} a & x & r \\ b & y & s \\ c & z & t \end{vmatrix}.$$

(9) Without expanding, show that

$$\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix} = -\frac{1}{4} \begin{vmatrix} 2a & 4b & 2c \\ -r & -2s & -t \\ x & 2y & z \end{vmatrix}.$$

(10) Without expanding, show that

$$\begin{vmatrix} a-3x & b-3y & c-3z \\ a+5x & b+5y & c+5z \\ x & y & z \end{vmatrix} = 0.$$

(11) Without expanding, show that

$$\begin{vmatrix} 2a & 3a & c \\ 2r & 3r & t \\ 2x & 3x & z \end{vmatrix} = 0.$$

(12) Prove that if one column of a square matrix is a linear combination of another column, then the determinant of that matrix is zero.

(13) Prove that if $\mathbf{A}$ is invertible, then $\det(\mathbf{A}^{-1}) = 1/\det(\mathbf{A})$.

4.4 Pivotal Condensation

Properties 2, 4, and 6 of the previous section describe the effects on the determinant of a matrix of applying elementary row operations to the matrix itself. They comprise part of an efficient algorithm for calculating determinants of matrices whose elements are numbers. The technique is known as *pivotal condensation*: A given matrix is transformed into row-reduced form using elementary row operations. A record is kept of the changes to the determinant as a result of Properties 2, 4, and 6. Once the transformation is complete, the row-reduced matrix is in upper triangular form, and its determinant is found easily by Property 8. In fact, since a row-reduced matrix has either unity elements or zeros on its main diagonal, its determinant will be unity if all its diagonal elements are unity, or zero if any one diagonal element is zero.

Example 1

Use pivotal condensation to evaluate

$$\begin{vmatrix} 1 & 2 & 3 \\ -2 & 3 & 2 \\ 3 & -1 & 1 \end{vmatrix}.$$

Solution.

$$\begin{vmatrix} 1 & 2 & 3 \\ -2 & 3 & 2 \\ 3 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 7 & 8 \\ 3 & -1 & 1 \end{vmatrix} \qquad \left\{\begin{array}{l}\text{Property 6: adding to} \\ \text{the second row (2)} \\ \text{times the first row}\end{array}\right.$$

$$= \begin{vmatrix} 1 & 2 & 3 \\ 0 & 7 & 8 \\ 0 & -7 & -8 \end{vmatrix} \quad \left\{ \begin{array}{l} \text{Property 6: adding to} \\ \text{the third row } (-3) \\ \text{times the first row} \end{array} \right.$$

$$= 7 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{8}{7} \\ 0 & -7 & -8 \end{vmatrix} \quad \left\{ \begin{array}{l} \text{Property 4: applied} \\ \text{to the second row} \end{array} \right.$$

$$= 7 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{8}{7} \\ 0 & 0 & 0 \end{vmatrix} \quad \left\{ \begin{array}{l} \text{Property 6: adding} \\ \text{to the third row (7)} \\ \text{times the second row} \end{array} \right.$$

$$= 7(0) = 0. \quad \{\text{Property 8} \quad \square$$

Example 2

Use pivotal condensation to evaluate

$$\begin{vmatrix} 0 & -1 & 4 \\ 1 & -5 & 1 \\ -6 & 2 & -3 \end{vmatrix}.$$

Solution.

$$\begin{vmatrix} 0 & -1 & 4 \\ 1 & -5 & 1 \\ -6 & 2 & -3 \end{vmatrix} = (-1) \begin{vmatrix} 1 & -5 & 1 \\ 0 & -1 & 4 \\ -6 & 2 & -3 \end{vmatrix} \quad \left\{ \begin{array}{l} \text{Property 2: interchanging} \\ \text{the first and second rows} \end{array} \right.$$

$$= (-1) \begin{vmatrix} 1 & -5 & 1 \\ 0 & -1 & 4 \\ 0 & -28 & 3 \end{vmatrix} \quad \left\{ \begin{array}{l} \text{Property 6: adding} \\ \text{to the third row (6)} \\ \text{times the first row} \end{array} \right.$$

$$= (-1)(-1) \begin{vmatrix} 1 & -5 & 1 \\ 0 & 1 & -4 \\ 0 & -28 & 3 \end{vmatrix} \quad \left\{ \begin{array}{l} \text{Property 4: applied} \\ \text{to the second row} \end{array} \right.$$

$$= \begin{vmatrix} 1 & -5 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & -109 \end{vmatrix} \quad \left\{ \begin{array}{l} \text{Property 6: adding} \\ \text{to the third row (28)} \\ \text{times the second row} \end{array} \right.$$

$$= (-109) \begin{vmatrix} 1 & -5 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{vmatrix} \quad \left\{ \begin{array}{l} \text{Property 4: applied} \\ \text{to the third row} \end{array} \right.$$

$$= (-109)(1) = -109. \quad \{\text{Property 8} \quad \square$$

Pivotal condensation is easily coded for implementation on a computer. Although shortcuts can be had by creative individuals evaluating determinants by hand, this rarely happens. The orders of most matrices that occur in practice are too large and, therefore, too time consuming to consider hand calculations in the evaluation of their determinants. In fact, such determinants can bring computer algorithms to their knees. As a result, calculating determinants is avoided whenever possible.

Still, when determinants are evaluated by hand, appropriate shortcuts are taken, as illustrated in the next two examples. The general approach involves operating on a matrix so that one row or one column is transformed into a new row or column containing at most one nonzero element. Expansion by cofactors is then applied to that row or column.

Example 3

Evaluate

$$\begin{vmatrix} 10 & -6 & -9 \\ 6 & -5 & -7 \\ -10 & 9 & 12 \end{vmatrix}.$$

Solution.

$$\begin{vmatrix} 10 & -6 & -9 \\ 6 & -5 & -7 \\ -10 & 9 & 12 \end{vmatrix} = \begin{vmatrix} 10 & -6 & -9 \\ 6 & -5 & -7 \\ 0 & 3 & 3 \end{vmatrix} \qquad \left\{\begin{array}{l}\text{by adding (1) times the} \\ \text{first row to the third row} \\ \text{(Property 6)}\end{array}\right.$$

$$= \begin{vmatrix} 10 & -6 & -3 \\ 6 & -5 & -2 \\ 0 & 3 & 0 \end{vmatrix} \qquad \left\{\begin{array}{l}\text{by adding } (-1) \text{ times the} \\ \text{second column to the} \\ \text{third column (Property 6)}\end{array}\right.$$

$$= -3\begin{vmatrix} 10 & -3 \\ 6 & -2 \end{vmatrix} \qquad \{\text{by expansion by cofactors}$$

$$= -3(-20 + 18) = 6. \qquad \square$$

Example 4

Evaluate

$$\begin{vmatrix} 3 & -1 & 0 & 2 \\ 0 & 1 & 4 & 1 \\ 3 & -2 & 3 & 5 \\ 9 & 7 & 0 & 2 \end{vmatrix}.$$

Solution. Since the third column already contains two zeros, it would seem advisable to work on that one.

$$\begin{vmatrix} 3 & -1 & 0 & 2 \\ 0 & 1 & 4 & 1 \\ 3 & -2 & 3 & 5 \\ 9 & 7 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 3 & -1 & 0 & 2 \\ 0 & 1 & 4 & 1 \\ 3 & -\frac{11}{4} & 0 & \frac{17}{4} \\ 9 & 7 & 0 & 2 \end{vmatrix} \quad \left\{\begin{array}{l}\text{by adding } (-\tfrac{3}{4}) \text{ times} \\ \text{the second row to} \\ \text{the third row.}\end{array}\right.$$

$$= -4\begin{vmatrix} 3 & -1 & 2 \\ 3 & -\frac{11}{4} & \frac{17}{4} \\ 9 & 7 & 2 \end{vmatrix} \quad \left\{\begin{array}{l}\text{by expansion by} \\ \text{cofactors}\end{array}\right.$$

$$= -4(\tfrac{1}{4})\begin{vmatrix} 3 & -1 & 2 \\ 12 & -11 & 17 \\ 9 & 7 & 2 \end{vmatrix} \quad \{\text{by Property 4}$$

$$= (-1)\begin{vmatrix} 3 & -1 & 2 \\ 0 & -7 & 9 \\ 9 & 7 & 2 \end{vmatrix} \quad \left\{\begin{array}{l}\text{by adding } (-4) \text{ times} \\ \text{the first row to the} \\ \text{second row}\end{array}\right.$$

$$= (-1)\begin{vmatrix} 3 & -1 & 2 \\ 0 & -7 & 9 \\ 0 & 10 & -4 \end{vmatrix} \quad \left\{\begin{array}{l}\text{by adding } (-3) \text{ times} \\ \text{the first row to the} \\ \text{third row}\end{array}\right.$$

$$= (-1)(3)\begin{vmatrix} -7 & 9 \\ 10 & -4 \end{vmatrix} \quad \left\{\begin{array}{l}\text{by expansion by} \\ \text{cofactors}\end{array}\right.$$

$$= (-3)(28 - 90) = 186. \quad \square$$

Problems 4.4

In Problems 1 through 18, evaluate the determinants of the given matrices.

(1) $\begin{bmatrix} 1 & 2 & -2 \\ 1 & 3 & 3 \\ 2 & 5 & 0 \end{bmatrix}$, **(2)** $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, **(3)** $\begin{bmatrix} 3 & -4 & 2 \\ -1 & 5 & 7 \\ 1 & 9 & -6 \end{bmatrix}$,

(4) $\begin{bmatrix} -1 & 3 & 3 \\ 1 & 1 & 4 \\ -1 & 1 & 2 \end{bmatrix}$, **(5)** $\begin{bmatrix} 1 & -3 & -3 \\ 2 & 8 & 4 \\ 3 & 5 & 1 \end{bmatrix}$, **(6)** $\begin{bmatrix} 2 & 1 & -9 \\ 3 & -1 & 1 \\ 3 & -1 & 2 \end{bmatrix}$,

(7) $\begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & 2 \\ 2 & 3 & 5 \end{bmatrix}$, (8) $\begin{bmatrix} -1 & 3 & 3 \\ 4 & 5 & 6 \\ -1 & 3 & 3 \end{bmatrix}$, (9) $\begin{bmatrix} 1 & 2 & -3 \\ 5 & 5 & 1 \\ 2 & -5 & -1 \end{bmatrix}$,

(10) $\begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 1 \\ 3 & 2 & -3 \end{bmatrix}$, (11) $\begin{bmatrix} 3 & 5 & 2 \\ -1 & 0 & 4 \\ -2 & 2 & 7 \end{bmatrix}$, (12) $\begin{bmatrix} 1 & -3 & -3 \\ 2 & 8 & 3 \\ 4 & 5 & 0 \end{bmatrix}$,

(13) $\begin{bmatrix} 3 & 5 & 4 & 6 \\ -2 & 1 & 0 & 7 \\ -5 & 4 & 7 & 2 \\ 8 & -3 & 1 & 1 \end{bmatrix}$, (14) $\begin{bmatrix} -1 & 2 & 1 & 2 \\ 1 & 0 & 3 & -1 \\ 2 & 2 & -1 & 1 \\ 2 & 0 & -3 & 2 \end{bmatrix}$,

(15) $\begin{bmatrix} 1 & 1 & 2 & -2 \\ 1 & 5 & 2 & -1 \\ -2 & -2 & 1 & 3 \\ -3 & 4 & -1 & 8 \end{bmatrix}$, (16) $\begin{bmatrix} -1 & 3 & 2 & -2 \\ 1 & -5 & -4 & 6 \\ 3 & -6 & 1 & 1 \\ 3 & -4 & 3 & -3 \end{bmatrix}$,

(17) $\begin{bmatrix} 1 & 1 & 0 & -2 \\ 1 & 5 & 0 & -1 \\ -2 & -2 & 0 & 3 \\ -3 & 4 & 0 & 8 \end{bmatrix}$, (18) $\begin{bmatrix} -2 & 0 & 1 & 3 \\ 4 & 0 & 2 & -2 \\ -3 & 1 & 0 & 1 \\ 5 & 4 & 1 & 7 \end{bmatrix}$.

(19) What can you say about the determinant of an $n \times n$ matrix that has rank less than n?

(20) What can you say about the determinant of a singular matrix?

4.5 Inversion

As an immediate consequence of Theorem 1 of Section 3.2 and the method of pivotal condensation, we have:

Theorem 1. *A square matrix has an inverse if and only if its determinant is not zero.*

In this section, we develop a method to calculate inverses of nonsingular matrices using determinants. For matrices with order greater than 3×3, this method is less efficient than the one described in Section 3.2, and is generally avoided.

Definition 1. The *cofactor matrix* associated with an $n \times n$ matrix $\mathbf{A}$ is an $n \times n$ matrix $\mathbf{A}^c$ obtained from $\mathbf{A}$ by replacing each element of $\mathbf{A}$ by its cofactor.

Example 1

Find $\mathbf{A}^c$ if

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 2 \\ -2 & 5 & 4 \\ 1 & 3 & 6 \end{bmatrix}.$$

Solution.

$$\mathbf{A}^c = \begin{bmatrix} (-1)^{1+1}\begin{vmatrix} 5 & 4 \\ 3 & 6 \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} -2 & 4 \\ 1 & 6 \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} -2 & 5 \\ 1 & 3 \end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} 3 & 2 \\ 1 & 6 \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} 1 & 2 \\ 5 & 4 \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} 3 & 2 \\ -2 & 4 \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} 3 & 1 \\ -2 & 5 \end{vmatrix} \end{bmatrix},$$

$$\mathbf{A}^c = \begin{bmatrix} 18 & 16 & -11 \\ 0 & 16 & -8 \\ -6 & -16 & 17 \end{bmatrix}. \quad \square$$

If $\mathbf{A} = [a_{ij}]$, we will use the notation $\mathbf{A}^c = [a^c_{ij}]$ to represent the cofactor matrix. Thus a^c_{ij} represents the cofactor of a_{ij}.

Definition 2. The *adjugate* of an $n \times n$ matrix $\mathbf{A}$ is the transpose of the cofactor matrix of $\mathbf{A}$.

Thus, if we designate the adjugate of $\mathbf{A}$ by $\mathbf{A}^a$, we have that $\mathbf{A}^a = (\mathbf{A}^c)^\mathsf{T}$.

Example 2

Find $\mathbf{A}^a$ for the $\mathbf{A}$ given in Example 1.

Solution.

$$\mathbf{A}^a = \begin{bmatrix} 18 & 0 & -6 \\ 16 & 16 & -16 \\ -11 & -8 & 17 \end{bmatrix}. \quad \square$$

The importance of the adjugate is given in the following theorem, which is proved in the appendix to this chapter.

Theorem 2. $\mathbf{AA}^a = \mathbf{A}^a\mathbf{A} = |\mathbf{A}|\mathbf{I}$.

If $|\mathbf{A}| \neq 0$, we may divide by it in Theorem 2 and obtain

$$\mathbf{A}\left(\frac{\mathbf{A}^a}{|\mathbf{A}|}\right) = \left(\frac{\mathbf{A}^a}{|\mathbf{A}|}\right)\mathbf{A} = \mathbf{I}.$$

Thus, using the definition of the inverse, we have that

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\mathbf{A}^a \qquad \text{if } |\mathbf{A}| \neq 0.$$

That is, if $|\mathbf{A}| \neq 0$, then $\mathbf{A}^{-1}$ may be obtained by dividing the adjugate of $\mathbf{A}$ by the determinant of $\mathbf{A}$.

Example 3

Find $\mathbf{A}^{-1}$ for the $\mathbf{A}$ given in Example 1.

Solution. The determinant of $\mathbf{A}$ is found to be 48. Using the solution to Example 2, we have

$$\mathbf{A}^{-1} = \left(\frac{\mathbf{A}^a}{|\mathbf{A}|}\right) = 1/48\begin{bmatrix} 18 & 0 & -6 \\ 16 & 16 & -16 \\ -11 & -8 & 17 \end{bmatrix} = \begin{bmatrix} 3/8 & 0 & -1/8 \\ 1/3 & 1/3 & -1/3 \\ -11/48 & -1/6 & 17/48 \end{bmatrix}. \qquad \square$$

Example 4

Find $\mathbf{A}^{-1}$ if

$$\mathbf{A} = \begin{bmatrix} 5 & 8 & 1 \\ 0 & 2 & 1 \\ 4 & 3 & -1 \end{bmatrix}.$$

Solution. $\det(\mathbf{A}) = -1 \neq 0$, therefore $\mathbf{A}^{-1}$ exists.

$$\mathbf{A}^c = \begin{bmatrix} -5 & 4 & -8 \\ 11 & -9 & 17 \\ 6 & -5 & 10 \end{bmatrix}, \qquad \mathbf{A}^a = (\mathbf{A}^c)^{\mathsf{T}} = \begin{bmatrix} -5 & 11 & 6 \\ 4 & -9 & -5 \\ -8 & 17 & 10 \end{bmatrix},$$

$$\mathbf{A}^{-1} = \frac{\mathbf{A}^a}{|\mathbf{A}|} = \begin{bmatrix} 5 & -11 & -6 \\ -4 & 9 & 5 \\ 8 & -17 & -10 \end{bmatrix}, \qquad \square$$

Example 5

Find $\mathbf{A}^{-1}$ if

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Solution. $|\mathbf{A}| = -2$, therefore $\mathbf{A}^{-1}$ exists.

$$\mathbf{A}^c = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}, \qquad \mathbf{A}^a = (\mathbf{A}^c)^{\mathsf{T}} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix},$$

$$\mathbf{A}^{-1} = \frac{\mathbf{A}^a}{|\mathbf{A}|} = (-\tfrac{1}{2})\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}. \quad \square$$

Problems 4.5

In Problems 1 through 15, find the inverses of the given matrices, if they exist.

(1) $\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$, **(2)** $\begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$, **(3)** $\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}$,

(4) $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, **(5)** $\begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix}$, **(6)** $\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$,

(7) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, **(8)** $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, **(9)** $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix}$,

(10) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, **(11)** $\begin{bmatrix} 2 & 0 & 0 \\ 5 & 1 & 0 \\ 4 & 1 & 1 \end{bmatrix}$, **(12)** $\begin{bmatrix} 1 & 2 & 1 \\ 3 & -2 & -4 \\ 2 & 3 & -1 \end{bmatrix}$,

(13) $\begin{bmatrix} 2 & 4 & 3 \\ 3 & -4 & -4 \\ 5 & 0 & -1 \end{bmatrix}$, **(14)** $\begin{bmatrix} 5 & 0 & -1 \\ 2 & -1 & 2 \\ 2 & 3 & -1 \end{bmatrix}$, **(15)** $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 2 & 3 & -1 \end{bmatrix}$.

(16) Find a formula for the inverse of

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

if its determinant is nonzero.

(17) Prove that if **A** and **B** are square matrices of the same order, then the product **AB** is nonsingular if and only if both **A** and **B** are.

(18) Prove Theorem 1.

(19) What can be said about the rank of a square matrix having a nonzero determinant?

4.6 Cramer's Rule

Cramer's rule is a method, based on determinants, for solving systems of simultaneous linear equations. In this section, we first state the rule, then illustrate its usage by an example, and finally prove its validity using the properties derived in Section 4.3. We also discuss the many limitations of the method.

Cramer's rule states that given a system of simultaneous linear equations in the matrix form $\mathbf{Ax} = \mathbf{b}$ (see Section 1.3), the ith component of $\mathbf{x}$ (or equivalently the ith unknown) is the quotient of two determinants. The determinant in the numerator is the determinant of a matrix obtained from $\mathbf{A}$ by replacing the ith column of $\mathbf{A}$ by the vector $\mathbf{b}$, while the determinant in the denominator is just $|\mathbf{A}|$. Thus, if we are considering the system

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1,\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2,\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3,
\end{aligned}$$

where x_1, x_2, and x_3 represent the unknowns, then Cramer's rule states that

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{|\mathbf{A}|}, \qquad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{|\mathbf{A}|},$$

$$x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{|\mathbf{A}|}, \quad \text{where} \quad |\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Two restrictions on the application of Cramer's rule are immediate. First, the systems under consideration must have exactly the same number of equations as unknowns to insure that all matrices involved are square and hence have determinants. Second, the determinant of the coefficient matrix must not be zero since it appears in the denominator. If $|\mathbf{A}| = 0$, then Cramer's rule can not be applied.

Example 1

Solve the system

$$\begin{aligned} x + 2y - 3z + w &= -5, \\ y + 3z + w &= 6, \\ 2x + 3y + z + w &= 4, \\ x + z + w &= 1. \end{aligned}$$

Solution.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 3 & 1 \\ 2 & 3 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} -5 \\ 6 \\ 4 \\ 1 \end{bmatrix}.$$

Since $|\mathbf{A}| = 20$, Cramer's rule can be applied, and

$$x = \frac{\begin{vmatrix} -5 & 2 & -3 & 1 \\ 6 & 1 & 3 & 1 \\ 4 & 3 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{vmatrix}}{20} = \frac{0}{20} = 0, \qquad y = \frac{\begin{vmatrix} 1 & -5 & -3 & 1 \\ 0 & 6 & 3 & 1 \\ 2 & 4 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix}}{20} = \frac{20}{20} = 1,$$

$$z = \frac{\begin{vmatrix} 1 & 2 & -5 & 1 \\ 0 & 1 & 6 & 1 \\ 2 & 3 & 4 & 1 \\ 1 & 0 & 1 & 1 \end{vmatrix}}{20} = \frac{40}{20} = 2, \qquad w = \frac{\begin{vmatrix} 1 & 2 & -3 & -5 \\ 0 & 1 & 3 & 6 \\ 2 & 3 & 1 & 4 \\ 1 & 0 & 1 & 1 \end{vmatrix}}{20} = \frac{-20}{20} = -1. \qquad \square$$

We now derive Cramer's rule using only those properties of determinants given in Section 4.3. We consider the general system $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}.$$

Then

$$x_1|\mathbf{A}| = \begin{vmatrix} a_{11}x_1 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21}x_1 & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31}x_1 & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1}x_1 & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} \quad \{\text{by Property 4 modified to columns}$$

$$= \begin{vmatrix} a_{11}x_1 + a_{12}x_2 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21}x_1 + a_{22}x_2 & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31}x_1 + a_{32}x_2 & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1}x_1 + a_{n2}x_2 & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} \quad \begin{cases} \text{by adding } (x_2) \text{ times} \\ \text{the second column to} \\ \text{the first column} \end{cases}$$

$$= \begin{vmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} \quad \begin{cases} \text{by adding } (x_3) \\ \text{times the third} \\ \text{column to the} \\ \text{first column} \end{cases}$$

$$= \begin{vmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

by making continued use of Property 6 in the obvious manner. We now note that the first column of the new determinant is nothing more than $\mathbf{Ax}$, and since, $\mathbf{Ax} = \mathbf{b}$, the first column reduces to $\mathbf{b}$.
Thus,

$$x_1|\mathbf{A}| = \begin{vmatrix} b_1 & a_{12} & a_{13} & \cdots & a_{1n} \\ b_2 & a_{22} & a_{23} & \cdots & a_{2n} \\ b_3 & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ b_n & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

or

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}}{|\mathbf{A}|}$$

providing $|\mathbf{A}| \neq 0$. This expression is Cramer's rule for obtaining x_1. A similar argument applied to the jth column, instead of the first column, quickly shows that Cramer's rule is valid for every x_j, $j = 1, 2, \ldots, n$.

Although Cramer's rule gives a systematic method for the solution of simultaneous linear equations, the number of computations involved can become awesome if the order of the determinant is large. Thus, for large systems, Cramer's rule is never used. The recommended algorithms include Gaussian elimination (Section 2.3) and **LU** decomposition (Section 3.5).

Problems 4.6

Solve the following systems of equations by Cramer's rule.

(1) $\begin{aligned} x + 2y &= -3, \\ 3x + y &= 1. \end{aligned}$

(2) $\begin{aligned} 2x + y &= 3, \\ x - y &= 6. \end{aligned}$

(3) $\begin{aligned} 4a + 2b &= 0, \\ 5a - 3b &= 10. \end{aligned}$

(4) $\begin{aligned} 3s - 4t &= 30, \\ -2s + 3t &= -10. \end{aligned}$

(5) $\begin{aligned} 2x - 8y &= 200, \\ -x + 4y &= 150. \end{aligned}$

(6) $\begin{aligned} x + y - 2z &= 3, \\ 2x - y + 3z &= 2. \end{aligned}$

(7) $\begin{aligned} x + y \phantom{{}+z} &= 15, \\ x + \phantom{y+{}} z &= 15, \\ y + z &= 10. \end{aligned}$

(8) $\begin{aligned} 3x + y + z &= 4, \\ x - y + 2z &= 15, \\ 2x - 2y - z &= 5. \end{aligned}$

(9) $\begin{aligned} x + 2y - 2z &= -1, \\ 2x + y + z &= 5, \\ -x + y - z &= -2. \end{aligned}$

(10) $\begin{aligned} 2a + 3b - c &= 4, \\ -a - 2b + c &= -2, \\ 3a - b \phantom{{}+c} &= 2. \end{aligned}$

(11) $\begin{aligned} 2x + 3y + 2z &= 3, \\ 3x + y + 5z &= 2, \\ 7y - 4z &= 5. \end{aligned}$

(12) $\begin{aligned} 5r + 8s + t &= 2, \\ 2s + t &= -1, \\ 4r + 3s - t &= 3. \end{aligned}$

(13)
$$\begin{aligned} x + 2y + z + w &= 7, \\ 3x + 4y - 2z - 4w &= 13, \\ 2x + y - z + w &= -4, \\ x - 3y + 4z + 5w &= 0. \end{aligned}$$

Appendix to Chapter 4

We shall now prove Theorem 2 of Section 4.5 dealing with the product of a matrix with its adjugate. For this proof we will need the following lemma:

Lemma 1. *If each element of one row of a matrix is multiplied by the cofactor of the corresponding element of a different row, the sum is zero.*

Proof. We prove this lemma only for an arbitrary 3×3 matrix $\mathbf{A}$ where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Consider the case in which we multiply every element of the third row by the cofactor of the corresponding element in the second row and then sum the results. Thus,

$$a_{31}(\text{cofactor of } a_{21}) + a_{32}(\text{cofactor of } a_{22}) + a_{33}(\text{cofactor of } a_{23})$$

$$= a_{31}(-1)^3\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{32}(-1)^4\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{33}(-1)^5\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0 \ \{\text{from Property 3, Section 4.3.}$$

Note that this property is equally valid if we replace the word row by the word column.

Theorem 1. $\mathbf{AA}^a = |\mathbf{A}|\mathbf{I}$.

Proof. We prove this theorem only for matrices of order 3×3. The proof easily may be extended to cover matrices of any arbitrary order. This extension is left as an exercise for the student.

$$\mathbf{AA}^a = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11}^c & a_{21}^c & a_{31}^c \\ a_{12}^c & a_{22}^c & a_{32}^c \\ a_{13}^c & a_{23}^c & a_{33}^c \end{bmatrix}.$$

If we denote this product matrix by $[b_{ij}]$, then

$$b_{11} = a_{11}a_{11}^c + a_{12}a_{12}^c + a_{13}a_{13}^c,$$

$$b_{12} = a_{11}a_{21}^c + a_{12}a_{22}^c + a_{13}a_{23}^c,$$

$$b_{23} = a_{21}a_{31}^c + a_{22}a_{32}^c + a_{23}a_{33}^c,$$

$$b_{22} = a_{21}a_{21}^c + a_{22}a_{22}^c + a_{23}a_{23}^c,$$

etc.

We now note that $b_{11} = |\mathbf{A}|$ since it is precisely the term obtained when one computes det(**A**) by cofactors, expanding by the first row. Similarly, $b_{22} = |\mathbf{A}|$ since it is precisely the term obtained by computing det(**A**) by cofactors after expanding by the second row. It follows from the above lemma that $b_{12} = 0$ and $b_{23} = 0$ since b_{12} is the term obtained by multiplying each element in the first row of **A** by the cofactor of the corresponding element in the second row and adding, while b_{23} is the term obtained by multiplying each element in the second row of **A** by the cofactor of the corresponding element in the third row and adding. Continuing this analysis for each b_{ij}, we find that

$$\mathbf{AA}^a = \begin{bmatrix} |\mathbf{A}| & 0 & 0 \\ 0 & |\mathbf{A}| & 0 \\ 0 & 0 & |\mathbf{A}| \end{bmatrix} = |\mathbf{A}| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{AA}^a = |\mathbf{A}|\mathbf{I}.$$

Theorem 2. $\mathbf{A}^a\mathbf{A} = |\mathbf{A}|\mathbf{I}$.

Proof. This proof is completely analogous to the previous one and is left as an exercise for the student.

Chapter 5

Eigenvalues and Eigenvectors

5.1 Definitions

Consider the matrix $\mathbf{A}$ and the vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ given by

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \qquad \mathbf{x}_1 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}, \qquad \mathbf{x}_3 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}.$$

Forming the products $\mathbf{Ax}_1$, $\mathbf{Ax}_2$, and $\mathbf{Ax}_3$, we obtain

$$\mathbf{Ax}_1 = \begin{bmatrix} 8 \\ 2 \\ 0 \end{bmatrix}, \qquad \mathbf{Ax}_2 = \begin{bmatrix} 9 \\ 6 \\ 6 \end{bmatrix}, \qquad \mathbf{Ax}_3 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}.$$

But

$$\begin{bmatrix} 8 \\ 2 \\ 0 \end{bmatrix} = 2\mathbf{x}_1, \qquad \begin{bmatrix} 9 \\ 6 \\ 6 \end{bmatrix} = 3\mathbf{x}_2, \quad \text{and} \quad \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = 1\mathbf{x}_3;$$

hence,

$$\mathbf{Ax}_1 = 2\mathbf{x}_1,$$

$$\mathbf{Ax}_2 = 3\mathbf{x}_2,$$

$$\mathbf{Ax}_3 = 1\mathbf{x}_3.$$

That is, multiplying $\mathbf{A}$ by any one of the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, or $\mathbf{x}_3$ is equivalent to simply multiplying the vector by a suitable scalar.

▶ ***Definition 1.*** A nonzero vector $\mathbf{x}$ is an *eigenvector* (or characteristic vector) of a square matrix $\mathbf{A}$ if there exists a scalar λ such that $\mathbf{Ax} = \lambda\mathbf{x}$. Then λ is an *eigenvalue* (or characteristic value) of $\mathbf{A}$.

Thus, in the above example, $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are eigenvectors of $\mathbf{A}$ and 2, 3, 1 are eigenvalues of $\mathbf{A}$.

Note that eigenvectors and eigenvalues are only defined for square matrices. Furthermore, note that the zero vector can *not* be an eigenvector even though $\mathbf{A} \cdot \mathbf{0} = \lambda \cdot \mathbf{0}$ for every scalar λ. An eigenvalue, however, can be zero.

Example 1

Show that

$$\mathbf{x} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$$

is an eigenvector of

$$\mathbf{A} = \begin{bmatrix} 0 & 5 & 7 \\ 0 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}.$$

Solution.

$$\mathbf{Ax} = \begin{bmatrix} 0 & 5 & 7 \\ 0 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}.$$

Thus, $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ and $\lambda = 0$ is an eigenvalue. □

Example 2

Is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

an eigenvector of

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}?$$

Solution.

$$\mathbf{Ax} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}.$$

Thus, if $\mathbf{x}$ is to be an eigenvector of $\mathbf{A}$, there must exist a scalar λ such that $\mathbf{Ax} = \lambda\mathbf{x}$, or such that

$$\begin{bmatrix} 3 \\ 7 \end{bmatrix} = \lambda\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda \end{bmatrix}.$$

It is quickly verified that no such λ exists, hence $\mathbf{x}$ is not an eigenvector of $\mathbf{A}$. □

Problems 5.1

(1) Determine which of the following vectors are eigenvectors for

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -4 & 7 \end{bmatrix}.$$

(a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, (b) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, (c) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, (d) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$,

(e) $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$, (f) $\begin{bmatrix} -4 \\ -4 \end{bmatrix}$, (g) $\begin{bmatrix} 4 \\ -4 \end{bmatrix}$, (h) $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

(2) What are the eigenvalues that correspond to the eigenvectors found in Problem 1?

(3) Determine which of the following vectors are eigenvectors for

$$\mathbf{B} = \begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix}.$$

(a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, (b) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, (c) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, (d) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

(e) $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$, (f) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, (g) $\begin{bmatrix} -4 \\ -6 \end{bmatrix}$, (h) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(4) What are the eigenvalues that correspond to the eigenvectors found in Problem 3?

(5) Determine which of the following vectors are eigenvectors for

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 2 & 1 \\ -1 & 0 & 2 \end{bmatrix}.$$

(a) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, (b) $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, (c) $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, (d) $\begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}$,

(e) $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, (f) $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, (g) $\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$, (h) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(6) What are the eigenvalues that correspond to the eigenvectors found in Problem 5?

(7) Determine which of the following vectors are eigenvectors for

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 4 & 3 \end{bmatrix}.$$

(a) $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, (b) $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$, (c) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$,

(d) $\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, (e) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, (f) $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

(8) What are the eigenvalues that correspond to the eigenvectors found in Problem 7?

5.2 Eigenvalues

Let $\mathbf{x}$ be an eigenvector of the matrix $\mathbf{A}$. Then there must exist an eigenvalue λ such that

$$\mathbf{Ax} = \lambda\mathbf{x} \tag{1}$$

or, equivalently,

$$\mathbf{Ax} - \lambda\mathbf{x} = \mathbf{0}$$

or

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}. \tag{2}$$

CAUTION. We could not have written (2) as $(\mathbf{A} - \lambda)\mathbf{x} = \mathbf{0}$ since the term $\mathbf{A} - \lambda$ would require subtracting a scalar from a matrix, an operation which is not defined. The quantity $\mathbf{A} - \lambda\mathbf{I}$, however, is defined since we are now subtracting one matrix from another.

Define a new matrix

$$\mathbf{B} = \mathbf{A} - \lambda\mathbf{I}. \tag{3}$$

Then (2) may be rewritten as

$$\mathbf{Bx} = \mathbf{0}, \tag{4}$$

which is a linear homogeneous system of equations for the unknown $\mathbf{x}$. If $\mathbf{B}$ has an inverse, then we can solve Eq. (4) for $\mathbf{x}$, obtaining $\mathbf{x} = \mathbf{B}^{-1}\mathbf{0}$, or $\mathbf{x} = \mathbf{0}$. This result, however, is absurd since $\mathbf{x}$ is an eigenvector and cannot be zero. Thus, it follows that $\mathbf{x}$ will be an eigenvector of $\mathbf{A}$ if and only if $\mathbf{B}$ does not have an inverse. But if a square matrix does not have an inverse, then its determinant must be zero (Theorem 1 of Section 4.5). Therefore, $\mathbf{x}$ will be an eigenvector of $\mathbf{A}$ if and only if

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0. \tag{5}$$

Equation (5) is called the *characteristic equation of* $\mathbf{A}$. The roots of (5) determine the eigenvalues of $\mathbf{A}$.

Example 1

Find the eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

Solution.

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix}.$$

$\det(\mathbf{A} - \lambda\mathbf{I}) = (1 - \lambda)(3 - \lambda) - 8 = \lambda^2 - 4\lambda - 5$. The characteristic equation of $\mathbf{A}$ is $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, or $\lambda^2 - 4\lambda - 5 = 0$. Solving for λ, we have that $\lambda = -1, 5$; hence the eigenvalues of $\mathbf{A}$ are $\lambda_1 = -1$, $\lambda_2 = 5$. ☐

Example 2

Find the eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}.$$

Solution.

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & -2 \\ 1 & 1-\lambda \end{bmatrix},$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (1 - \lambda)(1 - \lambda) + 2 = \lambda^2 - 2\lambda + 3.$$

The characteristic equation is $\lambda^2 - 2\lambda + 3 = 0$; hence, solving for λ by the quadratic formula, we have that $\lambda_1 = 1 + \sqrt{2}\,i$, $\lambda_2 = 1 - \sqrt{2}\,i$ which are eigenvalues of $\mathbf{A}$. ☐

NOTE. Even if the elements of a matrix are real, the eigenvalues may be complex.

Example 3

Find the eigenvalues of

$$\mathbf{A} = \begin{bmatrix} t & 2t \\ 2t & -t \end{bmatrix}.$$

Solution.

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} t & 2t \\ 2t & -t \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} t-\lambda & 2t \\ 2t & -t-\lambda \end{bmatrix}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (t - \lambda)(-t - \lambda) - 4t^2 = \lambda^2 - 5t^2.$$

The characteristic equation is $\lambda^2 - 5t^2 = 0$, hence, the eigenvalues are $\lambda_1 = \sqrt{5}t$, $\lambda_2 = -\sqrt{5}t$.

NOTE. If the matrix $\mathbf{A}$ depends on a parameter (in this case the parameter is t), then the eigenvalues may also depend on the parameter. ☐

Example 4

Find the eigenvalues for

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ 3 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution.

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 2 & -1 & 1 \\ 3 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & -1 & 1 \\ 3 & -2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}.$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (1-\lambda)[(2-\lambda)(-2-\lambda)+3] = (1-\lambda)(\lambda^2-1).$$

The characteristic equation is $(1-\lambda)(\lambda^2-1) = 0$; hence, the eigenvalues are $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = -1$. □

NOTE. The roots of the characteristic equation can be repeated. That is, $\lambda_1 = \lambda_2 = \lambda_3 = \cdots = \lambda_k$. When this happens, the eigenvalue is said to be of *multiplicity k*. Thus, in Example 4, $\lambda = 1$ is an eigenvalue of multiplicity 2 while, $\lambda = -1$ is an eigenvalue of multiplicity 1.

From the definition of the characteristic equation (5), it can be shown that if $\mathbf{A}$ is an $n \times n$ matrix then the characteristic equation of $\mathbf{A}$ is an nth degree polynomial in λ. It follows from the fundamental theorem of algebra, that the characteristic equation has n roots, counting multiplicity. Hence, $\mathbf{A}$ has exactly n eigenvalues, counting multiplicity. (See Examples 1 and 4).

In general, it is very difficult to find the eigenvalues of a matrix. First the characteristic equation must be obtained, and for matrices of high order this is a lengthy task. Then the characteristic equation must be solved for its roots. If the equation is of high order, this can be an impossibility in practice. For example, the reader is invited to find the eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{bmatrix}.$$

For these reasons, eigenvalues are rarely found by the method just given, and numerical techniques are used to obtain approximate values (see Sections 5.6 and 6.4).

Problems 5.2

In Problems 1 through 35, find the eigenvalues of the given matrices.

(1) $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$, **(2)** $\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$, **(3)** $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$,

(4) $\begin{bmatrix} 3 & 6 \\ 9 & 6 \end{bmatrix}$, **(5)** $\begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$, **(6)** $\begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}$,

(7) $\begin{bmatrix} 3 & 5 \\ 5 & -3 \end{bmatrix}$, **(8)** $\begin{bmatrix} 3 & 5 \\ -5 & -3 \end{bmatrix}$, **(9)** $\begin{bmatrix} 2 & 5 \\ -1 & -2 \end{bmatrix}$,

(10) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, **(11)** $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, **(12)** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$,

(13) $\begin{bmatrix} 2 & 2 \\ -1 & -2 \end{bmatrix}$, **(14)** $\begin{bmatrix} 4 & 10 \\ 9 & -5 \end{bmatrix}$, **(15)** $\begin{bmatrix} 5 & 10 \\ 9 & -4 \end{bmatrix}$,

(16) $\begin{bmatrix} 0 & t \\ 2t & -t \end{bmatrix}$, **(17)** $\begin{bmatrix} 0 & 2t \\ -2t & 4t \end{bmatrix}$, **(18)** $\begin{bmatrix} 4\theta & 2\theta \\ -\theta & \theta \end{bmatrix}$,

(19) $\begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 1 \\ 3 & 0 & 1 \end{bmatrix}$, **(20)** $\begin{bmatrix} 2 & 0 & -1 \\ 2 & 2 & 2 \\ -1 & 0 & 2 \end{bmatrix}$, **(21)** $\begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix}$,

(22) $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix}$, **(23)** $\begin{bmatrix} 3 & 0 & 0 \\ 2 & 6 & 4 \\ 2 & 3 & 5 \end{bmatrix}$, **(24)** $\begin{bmatrix} 5 & -7 & 7 \\ 4 & -3 & 4 \\ 4 & -1 & 2 \end{bmatrix}$,

(25) $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{bmatrix}$, **(26)** $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$, **(27)** $\begin{bmatrix} 10 & 2 & 0 \\ 2 & 4 & 6 \\ 0 & 6 & 10 \end{bmatrix}$,

(28) $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, **(29)** $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$, **(30)** $\begin{bmatrix} 4 & 2 & 1 \\ 2 & 7 & 2 \\ 1 & 2 & 4 \end{bmatrix}$,

(31) $\begin{bmatrix} 1 & 5 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$, **(32)** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$, **(33)** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 27 & -27 & 9 \end{bmatrix}$,

(34) $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 3 & 5 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & -1 & 1 \end{bmatrix}$, **(35)** $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 12 & -13 & 6 \end{bmatrix}$.

(36) Consider the matrix

$$\mathbf{C} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}.$$

Use mathematical induction to prove that

$$\det(\mathbf{C} - \lambda\mathbf{I}) = (-1)^n(\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0).$$

Deduce that the characteristic equation for this matrix is

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0 = 0.$$

The matrix **C** is called the *companion matrix* for this characteristic equation.

(37) Show that if λ is an eigenvalue of **A**, then $k\lambda$ is an eigenvalue of $k\mathbf{A}$, where k denotes an arbitrary scalar.

(38) Show that if $\lambda \neq 0$ is an eigenvalue of **A**, then $1/\lambda$ is an eigenvalue of $\mathbf{A}^{-1}$, providing the inverse exists.

(39) Show that if λ is an eigenvalue of **A**, then it is also an eigenvalue of $\mathbf{A}^{\mathrm{T}}$.

5.3 Eigenvectors

To each distinct eigenvalue of a matrix **A** there will correspond at least one eigenvector which can be found by solving the appropriate set of homogeneous equations. If an eigenvalue λ_i is substituted into (2), the corresponding eigenvector $\mathbf{x}_i$ is the solution of

$$(\mathbf{A} - \lambda_i\mathbf{I})\mathbf{x}_i = \mathbf{0}. \tag{6}$$

Example 1

Find the eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

Solution. The eigenvalues of $\mathbf{A}$ have already been found to be $\lambda_1 = -1$, $\lambda_2 = 5$ (see Example 1 of Section 5.2). We first calculate the eigenvectors corresponding to λ_1. From (6),

$$(\mathbf{A} - (-1)\mathbf{I})\mathbf{x}_1 = \mathbf{0}. \tag{7}$$

If we designate the unknown vector $\mathbf{x}_1$ by

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix},$$

Eq. (7) becomes

$$\left\{\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or, equivalently,

$$2x_1 + 2y_1 = 0,$$

$$4x_1 + 4y_1 = 0.$$

A nontrivial solution to this set of equations is $x_1 = -y_1$, y_1 arbitrary; hence, the eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -y_1 \\ y_1 \end{bmatrix} = y_1\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad y_1 \text{ arbitrary.}$$

By choosing different values of y_1, different eigenvectors for $\lambda_1 = -1$ can be obtained. Note, however, that any two such eigenvectors would be scalar multiples of each other, hence linearly dependent. Thus, there is only one linearly independent eigenvector corresponding to $\lambda_1 = -1$. For convenience we choose $y_1 = 1$, which gives us the eigenvector

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Many times, however, the scalar y_1 is chosen in such a manner that the resulting eigenvector becomes a unit vector. If we wished to achieve this result for the above vector, we would have to choose $y_1 = 1/\sqrt{2}$.

Having found an eigenvector corresponding to $\lambda_1 = -1$, we proceed to find an eigenvector $\mathbf{x}_2$ corresponding to $\lambda_2 = 5$. Designating the unknown vector $\mathbf{x}_2$ by

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

and substituting it with λ_2 into (6), we obtain

$$\left\{\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

or

$$\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

or, equivalently,

$$-4x_2 + 2y_2 = 0,$$
$$4x_2 - 2y_2 = 0.$$

A nontrivial solution to this set of equations is $x_2 = \frac{1}{2}y_2$, where y_2 is arbitrary; hence

$$\mathbf{x}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_2/2 \\ y_2 \end{bmatrix} = y_2\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}.$$

For convenience, we choose $y_2 = 2$, thus

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

In order to check whether or not $\mathbf{x}_2$ is an eigenvector corresponding to $\lambda_2 = 5$, we need only check if $\mathbf{A}\mathbf{x}_2 = \lambda_2\mathbf{x}_2$:

$$\mathbf{A}\mathbf{x}_2 = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \lambda_2\mathbf{x}_2.$$

Again note that $\mathbf{x}_2$ is *not* unique! Any scalar multiple of $\mathbf{x}_2$ is also an eigenvector corresponding to λ_2. However, in this case, there is just one *linearly independent* eigenvector corresponding to λ_2. □

Example 2

Find the eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & -1 & -2 \end{bmatrix}.$$

Solution. By using the method of the previous section, we find the eigenvalues to be $\lambda_1 = 2$, $\lambda_2 = i$, $\lambda_3 = -i$. We first calculate the eigenvectors corresponding to $\lambda_1 = 2$. Designate $\mathbf{x}_1$ by

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}.$$

Then (6) becomes

$$\left\{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

or

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & -1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

or, equivalently,

$$0 = 0,$$

$$5z_1 = 0,$$

$$-y_1 - 4z_1 = 0.$$

A nontrivial solution to this set of equations is $y_1 = z_1 = 0$, x_1 arbitrary; hence

$$\mathbf{x}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

We now find the eigenvectors corresponding to $\lambda_2 = i$. If we designate $\mathbf{x}_2$ by

$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix},$$

Eq. (6) becomes

$$\begin{bmatrix} 2-i & 0 & 0 \\ 0 & 2-i & 5 \\ 0 & -1 & -2-i \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$(2-i)x_2 = 0,$$
$$(2-i)y_2 + 5z_2 = 0,$$
$$-y_2 + (-2-i)z_2 = 0.$$

A nontrivial solution to this set of equations is $x_2 = 0$, $y_2 = (-2-i)z_2$, z_2 arbitrary; hence,

$$\mathbf{x}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ (-2-i)z_2 \\ z_2 \end{bmatrix} = z_2 \begin{bmatrix} 0 \\ -2-i \\ 1 \end{bmatrix}.$$

The eigenvectors corresponding to $\lambda_3 = -i$ are found in a similar manner to be

$$\mathbf{x}_3 = z_3 \begin{bmatrix} 0 \\ -2-i \\ 1 \end{bmatrix}, \; z_3 \text{ arbitrary.} \qquad \square$$

It should be noted that even if a mistake is made in finding the eigenvalues of a matrix, the error will become apparent when the eigenvectors corresponding to the incorrect eigenvalue are found. For instance, suppose that λ_1 in Example 2 was calculated erroneously to be 3. If we now try to find $\mathbf{x}_1$ we obtain the equations.

$$-x_1 = 0,$$
$$-y_1 + 5z_1 = 0,$$
$$-y_1 - 5z_1 = 0.$$

The only solution to this set of equations is $x_1 = y_1 = z_1 = 0$, hence

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

However, by definition, an eigenvector can not be the zero vector. Since every eigenvalue must have a corresponding eigenvector, there is a mistake. A quick check shows that all the calculations above are valid, hence the error must lie in the value of the eigenvalue.

Problems 5.3

In Problems 1 through 23, find an eigenvector corresponding to each eigenvalue of the given matrix.

(1) $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$, **(2)** $\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$, **(3)** $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$,

(4) $\begin{bmatrix} 3 & 6 \\ 9 & 6 \end{bmatrix}$, **(5)** $\begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}$, **(6)** $\begin{bmatrix} 3 & 5 \\ 5 & -3 \end{bmatrix}$,

(7) $\begin{bmatrix} 3 & 5 \\ -5 & -3 \end{bmatrix}$, **(8)** $\begin{bmatrix} 2 & 5 \\ -1 & -2 \end{bmatrix}$, **(9)** $\begin{bmatrix} 2 & 2 \\ -1 & -2 \end{bmatrix}$,

(10) $\begin{bmatrix} 4 & 10 \\ 9 & -5 \end{bmatrix}$, **(11)** $\begin{bmatrix} 0 & t \\ 2t & -t \end{bmatrix}$, **(12)** $\begin{bmatrix} 4\theta & 2\theta \\ -\theta & \theta \end{bmatrix}$,

(13) $\begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 1 \\ 3 & 0 & 1 \end{bmatrix}$, **(14)** $\begin{bmatrix} 2 & 0 & -1 \\ 2 & 2 & 2 \\ -1 & 0 & 2 \end{bmatrix}$, **(15)** $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 2 \\ -1 & 0 & 3 \end{bmatrix}$,

(16) $\begin{bmatrix} 3 & 0 & 0 \\ 2 & 6 & 4 \\ 2 & 3 & 5 \end{bmatrix}$, **(17)** $\begin{bmatrix} 5 & -7 & 7 \\ 4 & -3 & 4 \\ 4 & -1 & 2 \end{bmatrix}$, **(18)** $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{bmatrix}$,

(19) $\begin{bmatrix} 1 & 5 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$, **(20)** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$, **(21)** $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 4 & 0 \\ 0 & 1 & 5 \end{bmatrix}$,

(22) $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 3 & 5 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, **(23)** $\begin{bmatrix} 2 & 4 & 2 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 3 & -1 \\ 0 & 2 & 0 & 4 \end{bmatrix}$.

(24) Find unit eigenvectors (i.e., eigenvectors whose magnitudes equal unity) for the matrix in Problem 1.

(25) Find unit eigenvectors for the matrix in Problem 2.

(26) Find unit eigenvectors for the matrix in Problem 3.

(27) Find unit eigenvectors for the matrix in Problem 13.

(28) Find unit eigenvectors for the matrix in Problem 14.

(29) Find unit eigenvectors for the matrix in Problem 16.

(30) A nonzero vector $\mathbf{x}$ is a left eigenvector for a matrix $\mathbf{A}$ if there exists a scalar λ such that $\mathbf{xA} = \lambda\mathbf{x}$. Find a set of left eigenvectors for the matrix in Problem 1.

(31) Find a set of left eigenvectors for the matrix in Problem 2.

(32) Find a set of left eigenvectors for the matrix in Problem 3.

(33) Find a set of left eigenvectors for the matrix in Problem 4.

(34) Find a set of left eigenvectors for the matrix in Problem 13.

(35) Find a set of left eigenvectors for the matrix in Problem 14.

(36) Find a set of left eigenvectors for the matrix in Problem 16.

(37) Find a set of left eigenvectors for the matrix in Problem 18.

(38) Prove that if $\mathbf{x}$ is a right eigenvector of a symmetric matrix $\mathbf{A}$, then $\mathbf{x}^T$ is a left eigenvector of $\mathbf{A}$.

(39) A left eigenvector for a given matrix is known to be $[1 \quad 1]$. Find another left eigenvector for the same matrix satisfying the property that the sum of the vector components must equal unity.

(40) A left eigenvector for a given matrix is known to be $[2 \quad 3]$. Find another left eigenvector for the same matrix satisfying the property that the sum of the vector components must equal unity.

(41) A left eigenvector for a given matrix is known to be $[1 \quad 2 \quad 5]$. Find another left eigenvector for the same matrix satisfying the property that the sum of the vector components must equal unity.

(42) A Markov chain (see Problem 16 of Section 1.1 and Problem 16 of Section 1.6) is *regular* if some power of the transition matrix contains only positive elements. If the matrix itself contains only positive elements then the power is one, and the matrix is automatically regular. Transition matrices that are regular always have an eigenvalue of unity. They also have limiting distribution vectors denoted by $\mathbf{x}^{(\infty)}$, where the ith component of $\mathbf{x}^{(\infty)}$ represents the probability of an object being in state i after a large number of time periods have elapsed. The limiting distribution $\mathbf{x}^{(\infty)}$ is a left eigenvector of the transition matrix corresponding to the eigenvalue of unity, and having the sum of its components equal to one.

(a) Find the limiting distribution vector for the Markov chain described in Problem 16 of Section 1.1.
(b) Ultimately, what is the probability that a family will reside in the city?

(43) Find the limiting distribution vector for the Markov chain described in Problem 17 of Section 1.1. What is the probability of having a Republican mayor over the long run?

(44) Find the limiting distribution vector for the Markov chain described in Problem 18 of Section 1.1 What is the probability of having a good harvest over the long run?

(45) Find the limiting distribution vector for the Markov chain described in Problem 19 of Section 1.1. Ultimately, what is the probability that a person will use Brand Y?

5.4 Properties of Eigenvalues and Eigenvectors

Definition 1. The *trace* of a matrix $\mathbf{A}$, designated by $\operatorname{tr}(\mathbf{A})$, is the sum of the elements on the main diagonal.

Example 1

Find the $\operatorname{tr}(\mathbf{A})$ if

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 4 & 1 \\ 1 & -1 & -5 \end{bmatrix}.$$

Solution. $\operatorname{tr}(\mathbf{A}) = 3 + 4 + (-5) = 2.$ □

Property 1. *The sum of the eigenvalues of a matrix equals the trace of the matrix.*

Proof. See Problem 20.

Property 1 provides us with a quick and useful procedure for checking eigenvalues.

Example 2

Verify Property 1 for

$$\mathbf{A} = \begin{bmatrix} 11 & 3 \\ -5 & -5 \end{bmatrix}.$$

Solution. The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 10$, $\lambda_2 = -4$.

$$\operatorname{tr}(\mathbf{A}) = 11 + (-5) = 6 = \lambda_1 + \lambda_2. \quad \square$$

▶ | **Property 2.** *A matrix is singular if and only if it has a zero eigenvalue.*

Proof. A matrix $\mathbf{A}$ has a zero eigenvalue if and only if $\det(\mathbf{A} - 0\mathbf{I}) = 0$, or (since $0\mathbf{I} = \mathbf{0}$) if and only if $\det(\mathbf{A}) = 0$. But $\det(\mathbf{A}) = 0$ if and only if $\mathbf{A}$ is singular, thus, the result is immediate.

Property 3. *The eigenvalues of an upper (or lower) triangular matrix are the elements on the main diagonal.*

Proof. See Problem 15.

Example 3

Find the eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & -1 \end{bmatrix}.$$

Solution. Since $\mathbf{A}$ is lower triangular, the eigenvalues must be $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = -1$. $\square$

Property 4. *If λ is an eigenvalue of $\mathbf{A}$ and if $\mathbf{A}$ is invertible, then $1/\lambda$ is an eigenvalue of $\mathbf{A}^{-1}$.*

Proof. Since $\mathbf{A}$ is invertible, Property 2 implies that $\lambda \neq 0$; hence $1/\lambda$ exists. Since λ is an eigenvalue of $\mathbf{A}$ there must exist an eigenvector $\mathbf{x}$ such that

$\mathbf{Ax} = \lambda\mathbf{x}$. Premultiplying both sides of this equation by $\mathbf{A}^{-1}$, we obtain

$$\mathbf{x} = \lambda\mathbf{A}^{-1}\mathbf{x}$$

or, equivalently, $\mathbf{A}^{-1}\mathbf{x} = (1/\lambda)\mathbf{x}$. Thus, $1/\lambda$ is an eigenvalue of $\mathbf{A}^{-1}$.

OBSERVATION 1. If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ and if $\mathbf{A}$ is invertible, then $\mathbf{x}$ is an eigenvector of $\mathbf{A}^{-1}$ corresponding to the eigenvalue $1/\lambda$.

Property 5. *If λ is an eigenvalue of* $\mathbf{A}$, *then $\alpha\lambda$ is an eigenvalue of $\alpha\mathbf{A}$ where α is any arbitrary scalar.*

Proof. If λ is an eigenvalue of $\mathbf{A}$, then there must exist an eigenvector $\mathbf{x}$ such that $\mathbf{Ax} = \lambda\mathbf{x}$. Multiplying both sides of this equation by α, we obtain $(\alpha\mathbf{A})\mathbf{x} = (\alpha\lambda)\mathbf{x}$ which implies Property 5.

OBSERVATION 2. If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ, then $\mathbf{x}$ is an eigenvector of $\alpha\mathbf{A}$ corresponding to eigenvalue $\alpha\lambda$.

Property 6. *If λ is an eigenvalue of* $\mathbf{A}$, *then λ^k is an eigenvalue of $\mathbf{A}^k$, for any positive integer k.*

Proof. We prove the result for the special cases $k = 2$ and k equal 3. Other cases are handled by mathematical induction. (See Problem 16.) If λ is an eigenvalue of $\mathbf{A}$, there must exist an eigenvector $\mathbf{x}$ such that $\mathbf{Ax} = \lambda\mathbf{x}$. Then,

$$\mathbf{A}^2\mathbf{x} = \mathbf{A}(\mathbf{AX}) = \mathbf{A}(\lambda\mathbf{x}) = \lambda(\mathbf{Ax}) = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x},$$

which implies that λ^2 is an eigenvalue of $\mathbf{A}^2$. As a result, we also have that

$$\mathbf{A}^3\mathbf{x} = \mathbf{A}(\mathbf{A}^2\mathbf{x}) = \mathbf{A}(\lambda^2\mathbf{x}) = \lambda^2(\mathbf{Ax}) = \lambda^2(\lambda\mathbf{x}) = \lambda^3\mathbf{x},$$

which implies that λ^3 is an eigenvalue of $\mathbf{A}^3$.

OBSERVATION 3. If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ, then $\mathbf{x}$ is an eigenvector $\mathbf{A}^k$ corresponding to the eigenvalue λ^k, for any positive integer k.

Property 7. *If λ is an eigenvalue of* $\mathbf{A}$, *then for any scalar c, $\lambda - c$ is an eigenvalue of* $\mathbf{A} - c\mathbf{I}$.

Proof. If λ is an eigenvalue of $\mathbf{A}$, then there exists an eigenvector $\mathbf{x}$ such that $\mathbf{Ax} = \lambda\mathbf{x}$. Consequently,

$$\mathbf{Ax} - c\mathbf{x} = \lambda\mathbf{x} - c\mathbf{x},$$

or

$$(\mathbf{A} - c\mathbf{I})\mathbf{x} = (\lambda - c)\mathbf{x}.$$

Thus, $\lambda - c$ is an eigenvalue of $\mathbf{A} - c\mathbf{I}$.

OBSERVATION 4. If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ, then $\mathbf{x}$ is an eigenvector $\mathbf{A} - c\mathbf{I}$ corresponding to the eigenvalue $\lambda - c$.

Property 8. *If λ is an eigenvalue of $\mathbf{A}$, then λ is an eigenvalue of $\mathbf{A}^{\mathrm{T}}$.*

Proof. Since λ is an eigenvalue of $\mathbf{A}$, $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. Hence

$$\begin{aligned} 0 = |\mathbf{A} - \lambda\mathbf{I}| &= |(\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} - \lambda\mathbf{I}^{\mathrm{T}}| && \{\text{(Property 1, Section 1.4)} \\ &= |(\mathbf{A}^{\mathrm{T}} - \lambda\mathbf{I})^{\mathrm{T}}| && \{\text{(Property 3, Section 1.4)} \\ &= |\mathbf{A}^{\mathrm{T}} - \lambda\mathbf{I}| && \{\text{(Property 7, Section 4.3)} \end{aligned}$$

Thus, $\det(\mathbf{A}^{\mathrm{T}} - \lambda\mathbf{I}) = 0$, which implies that λ is an eigenvalue of $\mathbf{A}^{\mathrm{T}}$.

Property 9. *The product of the eigenvalues (counting multiplicity) of a matrix equals the determinant of the matrix.*

Proof. See Problem 21.

Example 4

Verify Property 9 for the matrix $\mathbf{A}$ given in Example 2:

Solution. For this $\mathbf{A}$, $\lambda_1 = 10$, $\lambda_2 = -4$, $\det(\mathbf{A}) = -55 + 15 = -40 = \lambda_1\lambda_2$. □

Problems 5.4

(1) One eigenvalue of the matrix

$$\mathbf{A} = \begin{bmatrix} 8 & 2 \\ 3 & 3 \end{bmatrix}$$

is known to be 2. Determine the second eigenvalue by inspection.

(2) One eigenvalue of the matrix

$$\mathbf{A} = \begin{bmatrix} 8 & 3 \\ 3 & 2 \end{bmatrix}$$

is known to be 0.7574, rounded to four decimal places. Determine the second eigenvalue by inspection.

(3) Two eigenvalues of a 3×3 matrix are known to be 5 and 8. What can be said about the remaining eigenvalue if the trace of the matrix is -4?

(4) Redo Problem 3 if the determinant of the matrix is -4 instead of its trace.

(5) The determinant of a 4×4 matrix $\mathbf{A}$ is 144 and two of its eigenvalues are known to be -3 and 2. What can be said about the remaining eigenvalues?

(6) A 2×2 matrix $\mathbf{A}$ is known to have the eigenvalues -3 and 4. What are the eigenvalues of (a) $2\mathbf{A}$, (b) $5\mathbf{A}$, (c) $\mathbf{A} - 3\mathbf{I}$, and (d) $\mathbf{A} + 4\mathbf{I}$?

(7) A 3×3 matrix $\mathbf{A}$ is known to have the eigenvalues -2, 2 and 4. What are the eigenvalues of (a) $\mathbf{A}^2$, (b) $\mathbf{A}^3$, (c) $-3\mathbf{A}$, and (d) $\mathbf{A} + 3\mathbf{I}$?

(8) A 2×2 matrix $\mathbf{A}$ is known to have the eigenvalues -1 and 1. Find a matrix in terms of $\mathbf{A}$ that has for its eigenvalues:

(a) -2 and 2, (b) -5 and 5,
(c) 1 and 1, (d) 2 and 4.

(9) A 3×3 matrix $\mathbf{A}$ is known to have the eigenvalues 2, 3 and 4. Find a matrix in terms of $\mathbf{A}$ that has for its eigenvalues:

(a) 4, 6, and 8, (b) 4, 9, and 16,
(c) 8, 27, and 64, (d) 0, 1, and 2.

(10) Verify Property 1 for

$$\mathbf{A} = \begin{bmatrix} 12 & 16 \\ -3 & -7 \end{bmatrix}.$$

(11) Verify Property 2 for

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 6 \\ -1 & 2 & -1 \\ 2 & 1 & 7 \end{bmatrix}.$$

(12) Show that if λ is an eigenvalue of $\mathbf{A}$, then it is also an eigenvalue for $\mathbf{S}^{-1}\mathbf{AS}$ for any nonsingular matrix $\mathbf{S}$.

(13) Show by example that, in general, an eigenvalue of $\mathbf{A} + \mathbf{B}$ is not the sum of an eigenvalue of $\mathbf{A}$ with an eigenvalue of $\mathbf{B}$.

(14) Show by example that, in general, an eigenvalue of $\mathbf{AB}$ is not the product of an eigenvalue of $\mathbf{A}$ with an eigenvalue of $\mathbf{B}$.

(15) Prove Property 3.

(16) Use mathematical induction to complete the proof of Property 6.

(17) The determinant of $\mathbf{A} - \lambda\mathbf{I}$ is known as the characteristic polynomial of $\mathbf{A}$. For an $n \times n$ matrix $\mathbf{A}$, it has the form

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (-1)^n(\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_2\lambda^2 + a_1\lambda + a_0),$$

where $a_{n-1}, a_{n-2}, \ldots, a_2, a_1, a_0$ are constants that depend on the elements of $\mathbf{A}$. Show that $(-1)^n a_0 = \det(\mathbf{A})$.

(18) (Problem 17 continued) Convince yourself by considering arbitrary 3×3 and 4×4 matrices that $a_{n-1} = \operatorname{tr}(\mathbf{A})$.

(19) Assume that $\mathbf{A}$ is an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, where some or all of the eigenvalues may be equal. Since each eigenvalue $\lambda_i\,(i = 1, 2, \ldots, n)$ is a root of the characteristic polynomial, $(\lambda - \lambda_i)$ must be a factor of that polynomial. Deduce that

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n).$$

(20) Use the results of Problems 18 and 19 to prove Property 1.

(21) Use the results of Problems 17 and 19 to prove Property 9.

(22) Show, by example, that an eigenvector of $\mathbf{A}$ need not be an eigenvector of $\mathbf{A}^{\mathsf{T}}$.

(23) Prove that an eigenvector of $\mathbf{A}$ is a left eigenvector of $\mathbf{A}^{\mathsf{T}}$.

5.5 Linearly Independent Eigenvectors

Since every eigenvalue has an infinite number of eigenvectors associated with it (recall that if $\mathbf{x}$ is an eigenvector, then any scalar multiple of $\mathbf{x}$ is also an eigenvector), it becomes academic to ask how many different eigenvectors can a matrix have? The answer is clearly an infinite number. A more revealing question is how many linearly independent eigenvectors can a matrix have? Theorem 4 of Section 2.6 provides us with a partial answer.

Theorem 1. *In an n-dimensional vector space, every set of $n + 1$ vectors is linearly dependent.*

Therefore, since all of the eigenvectors of an $n \times n$ matrix must be n-dimensional (why?), it follows from Theorem 1 that an $n \times n$ matrix can have *at most n* linearly independent eigenvectors. The following three examples shed more light on the subject.

Example 1

Find the eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Solution. The eigenvalues of **A** are $\lambda_1 = \lambda_2 = \lambda_3 = 2$, therefore $\lambda = 2$ is an eigenvalue of multiplicity 3. If we designate the unknown eigenvector **x** by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

then Eq. (6) gives rise to the three equations

$$y = 0,$$

$$z = 0,$$

$$0 = 0.$$

Thus, $y = z = 0$ and x is arbitrary; hence

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Setting $x = 1$, we see that $\lambda = 2$ generates only one linearly independent eigenvector,

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \qquad \square$$

Example 2

Find the eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Solution. Again, the eigenvalues are $\lambda_1 = \lambda_2 = \lambda_3 = 2$, therefore $\lambda = 2$ is an eigenvalue of multiplicity 3. Designate the unknown eigenvector $\mathbf{x}$ by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Equation (6) now gives rise to the equations

$$y = 0,$$

$$0 = 0,$$

$$0 = 0.$$

Thus, $y = 0$ and both x and z are arbitrary; hence

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Since x and z can be chosen arbitrarily, we can first choose $x = 1$ and $z = 0$ to obtain

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and then choose $x = 0$ and $z = 1$ to obtain

$$\mathbf{x}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

$\mathbf{x}_1$ and $\mathbf{x}_2$ can easily be shown to be linearly independent vectors, hence we see that $\lambda = 2$ generates the two linearly independent eigenvectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \qquad \square$$

Example 3

Find the eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Solution. Again the eigenvalues are $\lambda_1 = \lambda_2 = \lambda_3 = 2$ so again $\lambda = 2$ is an eigenvalue of multiplicity three. Designate the unknown eigenvector $\mathbf{x}$ by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Equation (6) gives rise to the equations

$$0 = 0,$$

$$0 = 0,$$

$$0 = 0,$$

Thus, x, y and z are all arbitrary; hence

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Since x, y, and z can be chosen arbitrarily, we can first choose $x = 1, y = z = 0$, then choose $x = z = 0$, $y = 1$ and finally choose $y = x = 0$, $z = 1$ to generate the three linearly independent eigenvectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

In this case we see that three linearly independent eigenvectors are generated by $\lambda = 2$. (Note that, from Theorem 1, this is the maximal number that could be generated.) □

The preceding examples are illustrations of

Theorem 2. *If λ is an eigenvalue of multiplicity k of an $n \times n$ matrix $\mathbf{A}$, then the number of linearly independent eigenvectors of $\mathbf{A}$ associated with λ is given by $\rho = n - r(\mathbf{A} - \lambda\mathbf{I})$. Furthermore, $1 \le \rho \le k$.*

Proof. Let $\mathbf{x}$ be an n-dimensional vector. If $\mathbf{x}$ is an eigenvector, then it must satisfy the vector equation $\mathbf{Ax} = \lambda\mathbf{x}$ or, equivalently, $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$. This system is homogeneous, hence consistent, so by Theorem 2 of Section 2.7, we have that the solution vector $\mathbf{x}$ will be in terms of $n - r(\mathbf{A} - \lambda\mathbf{I})$ arbitrary unknowns. Since these unknowns can be picked independently of each other, it follows that the number of linearly independent eigenvectors of $\mathbf{A}$ associated with λ is also $\rho = n - r(\mathbf{A} - \lambda\mathbf{I})$. We defer a proof that $1 \le \rho \le k$ until Chapter 9.

In Example 1, $\mathbf{A}$ is 3×3; hence $n = 3$, and $r(\mathbf{A} - 2\mathbf{I}) = 2$. Thus, there should be $3 - 2 = 1$ linearly independent eigenvector associated with $\lambda = 2$ which is indeed the case. In Example 2, once again $n = 3$ but $r(\mathbf{A} - 2\mathbf{I}) = 1$. Thus, there should be $3 - 1 = 2$ linearly independent eigenvectors associated with $\lambda = 2$ which also is the case.

The next theorem gives the relationship between eigenvectors that correspond to different eigenvalues.

Theorem 3. *Eigenvectors corresponding to distinct (that is, different) eigenvalues are linearly independent.*

Proof. For the sake of clarity, we consider the case of three distinct eigenvectors and leave the more general proof as an exercise (see Problem 17). Therefore, let λ_1, λ_2, λ_3, be distinct eigenvalues of the matrix $\mathbf{A}$ and let $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ be the associated eigenvectors. That is

$$\begin{aligned} \mathbf{Ax}_1 &= \lambda_1\mathbf{x}_1, \\ \mathbf{Ax}_2 &= \lambda_2\mathbf{x}_2, \\ \mathbf{Ax}_3 &= \lambda_3\mathbf{x}_3, \end{aligned} \tag{8}$$

and $\lambda_1 \ne \lambda_2 \ne \lambda_3 \ne \lambda_1$.

Since we want to show that $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linearly independent, we must show that the only solution to

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = \mathbf{0} \tag{9}$$

is $c_1 = c_2 = c_3 = 0$. By premultiplying (9) by $\mathbf{A}$, we obtain

$$c_1\mathbf{A}\mathbf{x}_1 + c_2\mathbf{A}\mathbf{x}_2 + c_3\mathbf{A}\mathbf{x}_3 = \mathbf{A} \cdot \mathbf{0} = \mathbf{0}.$$

It follows from (8), therefore, that

$$c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 + c_3\lambda_3\mathbf{x}_3 = \mathbf{0}. \tag{10}$$

By premultiplying (10) by $\mathbf{A}$ and again using (8), we obtain

$$c_1\lambda_1^2\mathbf{x}_1 + c_2\lambda_2^2\mathbf{x}_2 + c_3\lambda_3^2\mathbf{x}_3 = \mathbf{0}. \tag{11}$$

Equations (9)–(11) can be written in the matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix} \begin{bmatrix} c_1\mathbf{x}_1 \\ c_2\mathbf{x}_2 \\ c_3\mathbf{x}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}.$$

Define

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix}.$$

It can be shown that $\det(\mathbf{B}) = (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)$. Thus, since all the eigenvalues are distinct, $\det(\mathbf{B}) \neq 0$ and $\mathbf{B}$ is invertible. Therefore,

$$\begin{bmatrix} c_1\mathbf{x}_1 \\ c_2\mathbf{x}_2 \\ c_3\mathbf{x}_3 \end{bmatrix} = \mathbf{B}^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

or

$$\begin{aligned} c_1\mathbf{x}_1 &= \mathbf{0} \\ c_2\mathbf{x}_2 &= \mathbf{0} \\ c_3\mathbf{x}_3 &= \mathbf{0} \end{aligned} \tag{12}$$

But since $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are eigenvectors, they are nonzero, therefore, it follows from (12) that $c_1 = c_2 = c_3 = 0$. This result together with (9) implies Theorem 3.

Theorems 2 and 3 together completely determine the number of linearly independent eigenvectors of a matrix.

Example 4

Find a set of linearly independent eigenvectors for

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 3 & 2 \\ 4 & 2 & 3 \end{bmatrix}.$$

Solution. The eigenvalues of $\mathbf{A}$ are $\lambda_1 = \lambda_2 = 1$, and $\lambda_3 = 5$. For this matrix, $n = 3$ and $r(\mathbf{A} - 1\mathbf{I}) = 1$, hence $n - r(\mathbf{A} - 1\mathbf{I}) = 2$. Thus, from Theorem 2, we know that $\mathbf{A}$ has two linearly independent eigenvectors corresponding to $\lambda = 1$ and one linearly independent eigenvector corresponding to $\lambda = 5$ (why?). Furthermore, Theorem 3 guarantees that the two eigenvectors corresponding to $\lambda = 1$ will be linearly independent of the eigenvector corresponding $\lambda = 5$ and vice versa. It only remains to produce these vectors.

For $\lambda = 1$, the unknown vector

$$\mathbf{x}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

must satisfy the vector equation $(\mathbf{A} - 1\mathbf{I})\mathbf{x}_1 = \mathbf{0}$, or equivalently, the set of equations

$$\begin{aligned} 0 &= 0, \\ 4x_1 + 2y_1 + 2z_1 &= 0, \\ 4x_1 + 2y_1 + 2z_1 &= 0. \end{aligned}$$

A solution to this equation is $z_1 = -2x_1 - y_1$, x_1 and y_1 arbitrary. Thus,

$$\mathbf{x}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ -2x_1 - y_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + y_1 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

By first choosing $x_1 = 1$, $y_1 = 0$ and then $x_1 = 0$, $y_1 = 1$, we see that $\lambda = 1$ generates the two linearly independent eigenvectors

$$\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

An eigenvector corresponding to $\lambda_3 = 5$ is found to be

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Therefore, **A** possesses the three linearly independent eigenvectors,

$$\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}. \quad \square$$

Problems 5.5

In Problems 1–16 find a set of linearly independent eigenvectors for the given matrices.

(1) $\begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$, **(2)** $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$, **(3)** $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$,

(4) $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$, **(5)** $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix}$, **(6)** $\begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & -2 \\ -1 & 0 & 2 \end{bmatrix}$,

(7) $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix}$, **(8)** $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$, **(9)** $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$,

(10) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 27 & -27 & 9 \end{bmatrix}$, **(11)** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$, **(12)** $\begin{bmatrix} 4 & 2 & 1 \\ 2 & 7 & 2 \\ 1 & 2 & 4 \end{bmatrix}$,

(13) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 4 & -6 & 4 \end{bmatrix}$, **(14)** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -3 & 3 \end{bmatrix}$,

(15) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$, **(16)** $\begin{bmatrix} 3 & 1 & 1 & 2 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

(17) The Vandermonde determinant

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}$$

is known to equal the product

$$(x_2 - x_1)(x_3 - x_2)(x_3 - x_1)(x_4 - x_3)(x_4 - x_2)\cdots(x_n - x_1).$$

Using this result, prove Theorem 3 for n distinct eigenvalues.

5.6 Power Methods

The analytic methods described in Sections 5.2 and 5.3 are impractical for calculating the eigenvalues and eigenvectors of matrices of large order. Determining the characteristic equations for such matrices involves enormous effort, while finding its roots algebraically is usually impossible. Instead, iterative methods which lend themselves to computer implementation are used. Ideally, each iteration yields a new approximation, which converges to an eigenvalue and the corresponding eigenvector.

The *dominant* eigenvalue of a matrix is the one having largest absolute value. Thus, if the eigenvalues of a matrix are 2, 5, and -13, then -13 is the dominant eigenvalue because it is the largest in absolute value. The *power method* is an algorithm for locating the dominant eigenvalue and a corresponding eigenvector for a matrix of real numbers when the following two conditions exist:

Condition 1. The dominant eigenvalue of a matrix is real (not complex) and is strictly greater in absolute value than all other eigenvalues.

Condition 2. If the matrix has order $n \times n$, then it possesses n linearly independent eigenvectors.

Denote the eigenvalues of a given square matrix $\mathbf{A}$ satisfying Conditions 1 and 2 by $\lambda_1, \lambda_2, \ldots, \lambda_n$, and a set of corresponding eigenvectors by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, respectively. Assume the indexing is such that

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|.$$

Any vector $\mathbf{x}_0$ can be expressed as a linear combination of the eigenvectors of $\mathbf{A}$, so we may write

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

Multiplying this equation by $\mathbf{A}^k$, for some large, positive integer k, we get

$$\begin{aligned}\mathbf{A}^k\mathbf{x}_0 &= \mathbf{A}^k(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n)\\ &= c_1\mathbf{A}^k\mathbf{v}_1 + c_2\mathbf{A}^k\mathbf{v}_2 + \cdots + c_n\mathbf{A}^k\mathbf{v}_n.\end{aligned}$$

It follows from Property 6 and Observation 3 of Section 5.4 that

$$\begin{aligned}\mathbf{A}^k\mathbf{x}_0 &= c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 + \cdots + c_n\lambda_n^k\mathbf{v}_n\\ &= \lambda_1^k\left[c_1\mathbf{v}_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\mathbf{v}_2 + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^k\mathbf{v}_n\right]\\ &\approx \lambda_1^k c_1\mathbf{v}_1 \qquad \text{for large } k.\end{aligned}$$

This last pseudo-equality follows from noting that each quotient of eigenvalues is less than unity in absolute value, as a result of indexing the first eigenvalue as the dominant one, and therefore tends to zero as that quotient is raised to successively higher powers.

Thus, $\mathbf{A}^k\mathbf{x}_0$ approaches a scalar multiple of $\mathbf{v}_1$. But any nonzero scalar multiple of an eigenvector is itself an eigenvector, so $\mathbf{A}^k\mathbf{x}_0$ approaches an eigenvector of $\mathbf{A}$ corresponding to the dominant eigenvalue, providing c_1 is not zero. The scalar c_1 will be zero only if $\mathbf{x}_0$ is a linear combination of $\{\mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$.

The power method begins with an initial vector $\mathbf{x}_0$, usually the vector having all ones for its components, and then iteratively calculates the vectors

$$\begin{aligned}\mathbf{x}_1 &= \mathbf{A}\mathbf{x}_0,\\ \mathbf{x}_2 &= \mathbf{A}\mathbf{x}_1 = \mathbf{A}^2\mathbf{x}_0,\\ \mathbf{x}_3 &= \mathbf{A}\mathbf{x}_2 = \mathbf{A}^3\mathbf{x}_0,\\ &\vdots\\ \mathbf{x}_k &= \mathbf{A}\mathbf{x}_{k-1} = \mathbf{A}^k\mathbf{x}_0.\end{aligned}$$

As k gets larger, $\mathbf{x}_k$ approaches an eigenvector of $\mathbf{A}$ corresponding to its dominant eigenvalue.

We can even determine the dominant eigenvalue by scaling appropriately. If k is large enough so that $\mathbf{x}_k$ is a good approximation to the eigenvector, say to within acceptable roundoff error, then if follows from Eq. (1) that

$$\mathbf{A}\mathbf{x}_k = \lambda_1 \mathbf{x}_k.$$

If $\mathbf{x}_k$ is scaled so that its largest component is unity, then the component of $\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k = \lambda_1\mathbf{x}_k$ having the largest absolute value must be λ_1.

We can now formalize the power method. Begin with an initial guess $\mathbf{x}_0$ for the eigenvector, having the property that its largest component in absolute value is unity. Iteratively, calculate $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3, \ldots$ by multiplying each successive iterate by A, the matrix of interest. Each time $\mathbf{x}_k$ ($k = 1, 2, 3, \ldots$) is computed, identify its dominant component and divide each component by it. Redefine this scaled vector as the new $\mathbf{x}_k$. Each $\mathbf{x}_k$ is an estimate of an eigenvector for $\mathbf{A}$ and each dominant component is an estimate for the associated eigenvalue.

Example 1

Find the dominant eigenvalue, and a corresponding eigenvector for

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

Solution. We initialize $\mathbf{x}_0 = [1 \quad 1]^{\mathrm{T}}$. Then

First Iteration

$$\mathbf{x}_1 = \mathbf{A}\mathbf{x}_0 = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix},$$

$$\lambda \approx 7,$$

$$\mathbf{x}_1 \leftarrow \frac{1}{7}[3 \quad 7]^{\mathrm{T}} = [0.428571 \quad 1]^{\mathrm{T}}.$$

Second Iteration

$$\mathbf{x}_2 = \mathbf{A}\mathbf{x}_1 = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 0.428571 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.428571 \\ 4.714286 \end{bmatrix},$$

$$\lambda \approx 4.714286,$$

$$\mathbf{x}_2 \leftarrow \frac{1}{4.714286}[2.428571 \quad 4.714286]^{\mathrm{T}} = [0.515152 \quad 1]^{\mathrm{T}}.$$

Third Iteration

$$\mathbf{x}_3 = \mathbf{A}\mathbf{x}_2 = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 0.515152 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.515152 \\ 5.060606 \end{bmatrix},$$

$$\lambda = 5.060606,$$

$$\mathbf{x}_3 \leftarrow \frac{1}{5.060606}[2.515152 \quad 5.060606]^{\mathrm{T}} = [0.497006 \quad 1]^{\mathrm{T}}.$$

Fourth Iteration

$$\mathbf{x}_4 = \mathbf{A}\mathbf{x}_3 = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 0.497006 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.497006 \\ 4.988024 \end{bmatrix},$$

$$\lambda \approx 4.988024,$$

$$\mathbf{x}_4 \leftarrow \frac{1}{4.988024}[2.497006 \quad 4.988024]^{\mathrm{T}} = [0.500600 \quad 1]^{\mathrm{T}}.$$

The method is converging to the eigenvalue 5 and its corresponding eigenvector $[0.5 \quad 1]^{\mathrm{T}}$. □

Example 2

Find the dominant eigenvalue and a corresponding eigenvector for

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 18 & -1 & -7 \end{bmatrix}.$$

Solution. We initialize $\mathbf{x}_0 = [1 \quad 1 \quad 1]^{\mathrm{T}}$. Then

First Iteration

$$\mathbf{x}_1 = \mathbf{A}\mathbf{x}_0 = [1 \quad 1 \quad 10]^{\mathrm{T}},$$

$$\lambda \approx 10,$$

$$\mathbf{x}_1 \leftarrow \frac{1}{10}[1 \quad 1 \quad 10]^{\mathrm{T}} = [0.1 \quad 0.1 \quad 1]^{\mathrm{T}}.$$

Second Iteration

$$\mathbf{x}_2 = \mathbf{A}\mathbf{x}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 18 & -1 & -7 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 1 \\ -5.3 \end{bmatrix},$$

$$\lambda \approx -5.3,$$

$$\mathbf{x}_2 \leftarrow \frac{1}{-5.3}[0.1 \quad 1 \quad -5.3]^{\mathrm{T}} = [-0.018868 \quad -0.188679 \quad 1]^{\mathrm{T}}.$$

Third Iteration

$$\mathbf{x}_3 = \mathbf{A}\mathbf{x}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 18 & -1 & -7 \end{bmatrix} \begin{bmatrix} -0.018868 \\ -0.188679 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.188679 \\ 1 \\ -7.150943 \end{bmatrix},$$

$$\lambda \approx -7.150943,$$

$$\mathbf{x}_3 \leftarrow \frac{1}{-7.150943}[-0.188679 \quad 1 \quad -7.150943]^{\mathrm{T}}$$

$$= [0.026385 \quad -0.139842 \quad 1]^{\mathrm{T}}.$$

Continuing in this manner, we generate Table 1, where all entries are rounded to four decimal places. The algorithm is converging to the eigenvalue -6.405125 and its corresponding eigenvector

$$[0.024376 \quad -0.1561240 \quad 1]^{\mathrm{T}}. \qquad \square$$

Table 1

Iteration	Eigenvector components			Eigenvalue
0	1.0000	1.0000	1.0000	
1	0.1000	0.1000	1.0000	10.0000
2	−0.0189	−0.1887	1.0000	−5.3000
3	0.0264	−0.1398	1.0000	−7.1509
4	0.0219	−0.1566	1.0000	−6.3852
5	0.0243	−0.1551	1.0000	−6.4492
6	0.0242	−0.1561	1.0000	−6.4078
7	0.0244	−0.1560	1.0000	−6.4084
8	0.0244	−0.1561	1.0000	−6.4056

Although effective when it converges, the power method has deficiencies. It does not converge to the dominant eigenvalue when that eigenvalue is complex, and it may not converge when there are more than one equally dominant eigenvalues. (See Problem 12). Furthermore, the method, in general, cannot be used to locate all the eigenvalues.

A more powerful numerical method is the *inverse power method*, which is the power method applied to the inverse of a matrix. This, of course, adds another assumption: the inverse must exist, or equivalently, the matrix must not have any zero eigenvalues. Since a nonsingular matrix and its inverse share identical eigenvectors and reciprocal eigenvalues (see Property 4 and Observation 1 of Section 5.4), once we know the eigenvalues and eigenvectors of the inverse of a matrix, we have the analogous information about the matrix itself.

The power method applied to the inverse of a matrix $\mathbf{A}$ will generally converge to the dominant eigenvalue of $\mathbf{A}^{-1}$. Its reciprocal will be the eigenvalue of $\mathbf{A}$ having the smallest absolute value. The advantages of the inverse power method are that it converges more rapidly than the power method, and it often can be used to find all real eigenvalues of $\mathbf{A}$; a disadvantage is that it deals with $\mathbf{A}^{-1}$, which is laborious to calculate for matrices of large order. Such a calculation, however, can be avoided using **LU** decomposition.

The power method generates the sequence of vectors

$$\mathbf{x}_k = \mathbf{A}\mathbf{x}_{k-1}.$$

The inverse power method will generate the sequence

$$\mathbf{x}_k = \mathbf{A}^{-1}\mathbf{x}_{k-1},$$

which may be written as

$$\mathbf{A}\mathbf{x}_k = \mathbf{x}_{k-1}.$$

We solve for the unknown vector $\mathbf{x}_k$ using **LU**-decomposition (see Section 3.5).

Example 3

Use the inverse power method to find an eigenvalue for

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}.$$

Solution. We initialize $\mathbf{x}_0 = [1 \quad 1]^{\mathrm{T}}$. The **LU** decomposition for **A** has $\mathbf{A} = \mathbf{LU}$ with

$$\mathbf{L} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{U} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

First Iteration. We solve the system $\mathbf{LUx}_1 = \mathbf{x}_0$ by first solving the system $\mathbf{Ly} = \mathbf{x}_0$ for $\mathbf{y}$, and then solving the system $\mathbf{Ux}_1 = \mathbf{y}$ for $\mathbf{x}_1$. Set $\mathbf{y} = [y_1 \quad y_2]^{\mathrm{T}}$ and $\mathbf{x}_1 = [a \quad b]^{\mathrm{T}}$. The first system is

$$\begin{aligned} y_1 + 0y_2 &= 1, \\ y_1 + y_2 &= 1, \end{aligned}$$

which has as its solution $y_1 = 1$ and $y_2 = 0$. The system $\mathbf{Ux}_1 = \mathbf{y}$ becomes

$$\begin{aligned} 2a + b &= 1, \\ 2b &= 0, \end{aligned}$$

which admits the solution $a = 0.5$ and $b = 0$. Thus,

$$\begin{aligned} &\mathbf{x}_1 = \mathbf{A}^{-1}\mathbf{x}_0 = [0.5 \quad 0]^{\mathrm{T}}, \\ &\lambda \approx 0.5 \qquad \text{(an approximation to an eigenvalue for } \mathbf{A}^{-1}\text{)}, \\ &\mathbf{x}_1 \leftarrow \frac{1}{0.5}[0.5 \quad 0]^{\mathrm{T}} = [1 \quad 0]^{\mathrm{T}}. \end{aligned}$$

Second Iteration. We solve the system $\mathbf{LUx}_2 = \mathbf{x}_1$ by first solving the system $\mathbf{Ly} = \mathbf{x}_1$ for $\mathbf{y}$, and then solving the system $\mathbf{Ux}_2 = \mathbf{y}$ for $\mathbf{x}_2$. Set $\mathbf{y} = [y_1 \quad y_2]^{\mathrm{T}}$ and $\mathbf{x}_2 = [a \quad b]^{\mathrm{T}}$. The first system is

$$\begin{aligned} y_1 + 0y_2 &= 1, \\ y_1 + y_2 &= 0, \end{aligned}$$

which has as its solution $y_1 = 1$ and $y_2 = -1$. The system $\mathbf{Ux}_2 = \mathbf{y}$ becomes

$$\begin{aligned} 2a + b &= 1, \\ 2b &= -1, \end{aligned}$$

which admits the solution $a = 0.75$ and $b = -0.5$. Thus,

$$\mathbf{x}_2 = \mathbf{A}^{-1}\mathbf{x}_1 = [0.75 \quad -0.5]^{\mathrm{T}},$$

$$\lambda \approx 0.75,$$

$$\mathbf{x}_2 \leftarrow \frac{1}{0.75}[0.75 \quad -0.5]^{\mathrm{T}} = [1 \quad -0.666667]^{\mathrm{T}}.$$

THIRD ITERATION. We first solve $\mathbf{Ly} = \mathbf{x}_2$ to obtain $\mathbf{y} = [1 \quad -1.666667]^{\mathrm{T}}$, and then $\mathbf{Ux}_3 = \mathbf{y}$ to obtain $\mathbf{x}_3 = [0.916667 \quad -0.833333]^{\mathrm{T}}$. Then,

$$\lambda \approx 0.916667$$

$$\mathbf{x}_3 \leftarrow \frac{1}{0.916667}[0.916667 \quad -0.833333]^{\mathrm{T}} = [1 \quad -0.909091]^{\mathrm{T}}.$$

Continuing, we converge to the eigenvalue 1 for $\mathbf{A}^{-1}$ and its reciprocal $1/1 = 1$ for $\mathbf{A}$. The vector approximations are converging to $[1 \quad -1]^{\mathrm{T}}$, which is an eigenvector for both $\mathbf{A}^{-1}$ and $\mathbf{A}$. □

Example 4

Use the inverse power method to find an eigenvalue for

$$\mathbf{A} = \begin{bmatrix} 7 & 2 & 0 \\ 2 & 1 & 6 \\ 0 & 6 & 7 \end{bmatrix}.$$

Solution. We initialize $\mathbf{x}_0 = [1 \quad 1 \quad 1]^{\mathrm{T}}$. The **LU** decomposition for **A** has $\mathbf{A} = \mathbf{LU}$ with

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0.285714 & 1 & 0 \\ 0 & 14 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{U} = \begin{bmatrix} 7 & 2 & 0 \\ 0 & 0.428571 & 6 \\ 0 & 0 & -77 \end{bmatrix}.$$

FIRST ITERATION.

Set $\mathbf{y} = [y_1 \quad y_2 \quad y_3]^{\mathrm{T}}$ and $\mathbf{x}_1 = [a \quad b \quad c]^{\mathrm{T}}$. The first system is

$$\begin{aligned} y_1 + 0y_2 + 0y_3 &= 1, \\ 0.285714y_1 + y_2 + 0y_3 &= 1, \\ 0y_1 + 14y_2 + y_3 &= 1, \end{aligned}$$

which has as its solution $y_1 = 1$, and $y_2 = 0.714286$, and $y_3 = -9$. The system $\mathbf{Ux}_1 = \mathbf{y}$ becomes

$$\begin{aligned} 7a + \quad 2b \qquad\qquad &= \quad 1, \\ 0.428571b + \quad 6c &= \quad 0.714286, \\ -77c &= -9, \end{aligned}$$

which admits the solution $a = 0.134199$, $b = 0.030303$, and $c = 0.116883$. Thus,

$$\mathbf{x}_1 = \mathbf{A}^{-1}\mathbf{x}_0 = [0.134199 \quad 0.030303 \quad 0.116833]^{\mathrm{T}},$$

$$\lambda \approx 0.134199 \qquad \text{(an approximation to an eigenvalue for } \mathbf{A}^{-1}\text{)},$$

$$\mathbf{x}_1 \leftarrow \frac{1}{0.134199}[0.134199 \quad 0.030303 \quad 0.116833]^{\mathrm{T}}$$

$$= [1 \quad 0.225806 \quad 0.870968]^{\mathrm{T}}.$$

SECOND ITERATION

Solving the system $\mathbf{Ly} = \mathbf{x}_1$ for $\mathbf{y}$, we obtain

$$\mathbf{y} = [1 \quad -0.059908 \quad 1.709677]^{\mathrm{T}}.$$

Then, solving the system $\mathbf{Ux}_2 = \mathbf{y}$ for $\mathbf{x}_2$, we get

$$\mathbf{x}_2 = [0.093981 \quad 0.171065 \quad -0.022204]^{\mathrm{T}}.$$

Therefore,

$$\lambda \approx 0.171065,$$

$$\mathbf{x}_2 \leftarrow \frac{1}{0.171065}[0.093981 \quad 0.171065 \quad -0.022204]^{\mathrm{T}},$$

$$= [0.549388 \quad 1 \quad -0.129796]^{\mathrm{T}}.$$

THIRD ITERATION

Solving the system $\mathbf{Ly} = \mathbf{x}_2$ for $\mathbf{y}$, we obtain

$$\mathbf{y} = [0.549388 \quad 0.843032 \quad -11.932245]^{\mathrm{T}}.$$

Then, solving the system $\mathbf{Ux}_3 = \mathbf{y}$ for $\mathbf{x}_3$, we get

$$\mathbf{x}_3 = [0.136319 \quad -0.202424 \quad 0.154964]^{\mathrm{T}}.$$

Table 2

Iteration	Eigenvector components			Eigenvalue
0	1.0000	1.0000	1.0000	
1	1.0000	0.2258	0.8710	0.1342
2	0.5494	1.0000	−0.1298	0.1711
3	−0.6734	1.0000	−0.7655	−0.2024
4	−0.0404	1.0000	−0.5782	−0.3921
5	−0.2677	1.0000	−0.5988	−0.3197
6	−0.1723	1.0000	−0.6035	−0.3372
7	−0.2116	1.0000	−0.5977	−0.3323
8	−0.1951	1.0000	−0.6012	−0.3336
9	−0.2021	1.0000	−0.5994	−0.3333
10	−0.1991	1.0000	−0.6003	−0.3334
11	−0.2004	1.0000	−0.5999	−0.3333
12	−0.1998	1.0000	−0.6001	−0.3333

Therefore,

$$\lambda \approx -0.202424,$$

$$\mathbf{x}_3 \leftarrow \frac{1}{-0.202424}[0.136319 \quad -0.202424 \quad 0.154964]^{\mathrm{T}}$$

$$= [-0.673434 \quad 1 \quad -0.765542]^{\mathrm{T}}.$$

Continuing in this manner, we generate Table 2, where all entries are rounded to four decimal places. The algorithm is converging to the eigenvalue $-1/3$ for $\mathbf{A}^{-1}$ and its reciprocal -3 for $\mathbf{A}$. The vector approximations are converging to $[-0.2 \quad 1 \quad -0.6]^{\mathrm{T}}$, which is an eigenvector for both $\mathbf{A}^{-1}$ and $\mathbf{A}$. □

We can use Property 7 and Observation 4 of Section 5.4 in conjunction with the inverse power method to develop a procedure for finding all eigenvalues and a set of corresponding eigenvectors for a matrix, providing that the eigenvalues are real and distinct, and estimates of their locations are known. The algorithm is known as the *shifted inverse power method.*

If c is an estimate for an eigenvalue of $\mathbf{A}$, then $\mathbf{A} - c\mathbf{I}$ will have an eigenvalue near zero, and its reciprocal will be the dominant eigenvalue of $(\mathbf{A} - c\mathbf{I})^{-1}$. We use the inverse power method with an **LU** decomposition of $\mathbf{A} - c\mathbf{I}$ to calculate the dominant eigenvalue λ and its corresponding eigenvector $\mathbf{x}$ for $(\mathbf{A} - c\mathbf{I})^{-1}$. Then $1/\lambda$ and $\mathbf{x}$ are an eigenvalue and eigenvector for $\mathbf{A} - c\mathbf{I}$ while $1/\lambda + c$ and $\mathbf{x}$ are an eigenvalue and eigenvector for $\mathbf{A}$.

Example 5

Find a second eigenvalue for the matrix given in Example 4.

Solution. Since we do not have an estimate for any of the eigenvalues, we arbitrarily choose $c = 15$. Then

$$\mathbf{A} - c\mathbf{I} = \begin{bmatrix} -8 & 2 & 0 \\ 2 & -14 & 6 \\ 0 & 6 & -8 \end{bmatrix},$$

which has an **LU** decomposition with

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0.25 & 1 & 0 \\ 0 & -0.444444 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{U} = \begin{bmatrix} -8 & 2 & 0 \\ 0 & -13.5 & 6 \\ 0 & 0 & -5.333333 \end{bmatrix}.$$

Applying the inverse power method to $\mathbf{A} - 15\mathbf{I}$, we generate Table 3, which is converging to $\lambda = -0.25$ and $\mathbf{x} = [\frac{1}{3} \;\; \frac{2}{3} \;\; 1]^T$. The corresponding eigenvalue of $\mathbf{A}$ is $1/-0.25 + 15 = 11$, with the same eigenvector.

Using the results of Examples 4 and 5, we have two eigenvalues, $\lambda_1 = -3$ and $\lambda_2 = 11$, of the 3×3 matrix defined in Example 4. Since the trace of a matrix equals the sum of the eigenvalues (Property 1 of Section 5.4), we know $7 + 1 + 7 = -3 + 11 + \lambda_3$, so the last eigenvalue is $\lambda_3 = 7$. □

Table 3

Iteration	Eigenvector components			Eigenvalue
0	1.0000	1.0000	1.0000	
1	0.6190	0.7619	1.0000	−0.2917
2	0.4687	0.7018	1.0000	−0.2639
3	0.3995	0.6816	1.0000	−0.2557
4	0.3661	0.6736	1.0000	−0.2526
5	0.3496	0.6700	1.0000	−0.2513
6	0.3415	0.6683	1.0000	−0.2506
7	0.3374	0.6675	1.0000	−0.2503
8	0.3354	0.6671	1.0000	−0.2502
9	0.3343	0.6669	1.0000	−0.2501
10	0.3338	0.6668	1.0000	−0.2500
11	0.3336	0.6667	1.0000	−0.2500

Problems 5.6

In Problems 1 through 10, use the power method to locate the dominant eigenvalue and a corresponding eigenvector for the given matrices. Stop after five iterations.

(1) $\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$, **(2)** $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$, **(3)** $\begin{bmatrix} 3 & 6 \\ 9 & 6 \end{bmatrix}$,

(4) $\begin{bmatrix} 0 & 1 \\ -4 & 6 \end{bmatrix}$, **(5)** $\begin{bmatrix} 8 & 2 \\ 3 & 3 \end{bmatrix}$, **(6)** $\begin{bmatrix} 8 & 3 \\ 3 & 2 \end{bmatrix}$,

(7) $\begin{bmatrix} 3 & 0 & 0 \\ 2 & 6 & 4 \\ 2 & 3 & 5 \end{bmatrix}$, **(8)** $\begin{bmatrix} 7 & 2 & 0 \\ 2 & 1 & 6 \\ 0 & 6 & 7 \end{bmatrix}$, **(9)** $\begin{bmatrix} 3 & 2 & 3 \\ 2 & 6 & 6 \\ 3 & 6 & 11 \end{bmatrix}$,

(10) $\begin{bmatrix} 2 & -17 & 7 \\ -17 & -4 & 1 \\ 7 & 1 & -14 \end{bmatrix}$.

(11) Use the power method on

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 2 & 2 \\ -1 & 0 & 2 \end{bmatrix},$$

and explain why it does not converge to the dominant eigenvalue $\lambda = 3$.

(12) Use the power method on

$$\mathbf{A} = \begin{bmatrix} 3 & 5 \\ 5 & -3 \end{bmatrix},$$

and explain why it does not converge.

(13) Shifting can also be used with the power method to locate the next most dominant eigenvalue, if it is real and distinct, once the dominant eigenvalue has been determined. Construct $\mathbf{A} - \lambda\mathbf{I}$, where λ is the dominant eigenvalue of $\mathbf{A}$, and apply the power method to the shifted matrix. If the algorithm converges to μ and $\mathbf{x}$, then $\mu + \lambda$ is an eigenvalue of $\mathbf{A}$ with the corresponding eigenvector $\mathbf{x}$. Apply this shifted power method algorithm to the matrix in Problem 1. Use the results of Problem 1 to determine the appropriate shift.

(14) Use the shifted power method as described in Problem 13 to the matrix in Problem 9. Use the results of Problem 9 to determine the appropriate shift.

(15) Use the inverse power method on the matrix defined in Example 1. Stop after five iterations.

(16) Use the inverse power method on the matrix defined in Problem 3. Take $\mathbf{x}_0 = [1 \quad -0.5]^T$ and stop after five iterations.

(17) Use the inverse power method on the matrix defined in Problem 5. Stop after five iterations.

(18) Use the inverse power method on the matrix defined in Problem 6. Stop after five iterations.

(19) Use the inverse power method on the matrix defined in Problem 9. Stop after five iterations.

(20) Use the inverse power method on the matrix defined in Problem 10. Stop after five iterations.

(21) Use the inverse power method on the matrix defined in Problem 11. Stop after five iterations.

(22) Use the inverse power method on the matrix defined in Problem 4. Explain the difficulty, and suggest a way to avoid it.

(23) Use the inverse power method on the matrix defined in Problem 2. Explain the difficulty, and suggest a way to avoid it.

(24) Can the power method converge to a dominant eigenvalue if that eigenvalue is not distinct?

(25) Apply the shifted inverse power method to the matrix defined in Problem 9, with a shift constant of 10.

(26) Apply the shifted inverse power method to the matrix defined in Problem 10, with a shift constant of -25.

Chapter 6

Real Inner Products

6.1 Introduction

To any two vectors $\mathbf{x}$ and $\mathbf{y}$ of the same dimension having real components (as distinct from complex components), we associate a scalar called the *inner product*, denoted as $\langle \mathbf{x}, \mathbf{y} \rangle$, by multiplying together the corresponding elements of $\mathbf{x}$ and $\mathbf{y}$, and then summing the results. Students already familiar with the dot product of two- and three-dimensional vectors will undoubtedly recognize the inner product as an extension of the dot product to real vectors of all dimensions.

Example 1

Find $\langle \mathbf{x}, \mathbf{y} \rangle$ if

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 4 \\ -5 \\ 6 \end{bmatrix}.$$

Solution. $\langle \mathbf{x}, \mathbf{y} \rangle = 1(4) + 2(-5) + 3(6) = 12.$ □

Example 2

Find $\langle \mathbf{u}, \mathbf{v} \rangle$ if $\mathbf{u} = [20 \quad -4 \quad 30 \quad 10]$ and $\mathbf{v} = [10 \quad -5 \quad -8 \quad -6]$.

Solution. $\langle \mathbf{u}, \mathbf{v} \rangle = 20(10) + (-4)(-5) + 30(-8) + 10(-6) = -80.$ □

It follows immediately from the definition that the inner product of real vectors satisfies the following properties:

(I1) $\langle \mathbf{x}, \mathbf{x} \rangle$ is positive if $\mathbf{x} \neq \mathbf{0}$; $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = \mathbf{0}$.
(I2) $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$.
(I3) $\langle \lambda\mathbf{x}, \mathbf{y} \rangle = \lambda\langle \mathbf{x}, \mathbf{y} \rangle$, for any real scalar λ.
(I4) $\langle \mathbf{x} + \mathbf{z}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{z}, \mathbf{y} \rangle$.
(I5) $\langle \mathbf{0}, \mathbf{y} \rangle = 0$.

We will only prove (I1) here and leave the proofs of the other properties as exercises for the student (see Problems 29 through 32). Let $\mathbf{x} = [x_1 \quad x_2 \quad x_3 \quad \cdots \quad x_n]$ be an n-dimensional row vector whose components $x_1, x_2, x_3, \ldots, x_n$ are all real. Then,

$$\langle \mathbf{x}, \mathbf{x} \rangle = (x_1)^2 + (x_2)^2 + (x_3)^2 + \cdots + (x_n)^2.$$

This sum of squares is zero if and only if $x_1 = x_2 = x_3 = \cdots = x_n = 0$, which in turn implies $\mathbf{x} = \mathbf{0}$. If any one component is not zero, that is, if $\mathbf{x}$ is not the zero vector, then the sum of squares must be positive.

The inner product of real vectors is related to the magnitude of a vector as defined in Section 1.6. In particular,

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}.$$

Example 3

Find the magnitude of $\mathbf{x} = [2 \quad -3 \quad -4]$.

Solution. $\langle \mathbf{x}, \mathbf{x} \rangle = 2(2) + (-3)(-3) + (-4)(-4) = 29$, so the magnitude of $\mathbf{x}$ is

$$\|\mathbf{x}\| = \sqrt{29}. \quad \square$$

The concepts of a normalized vector and a unit vector are identical to the definitions given in Section 1.6. A nonzero vector is *normalized* if it is divided by its magnitude. A *unit vector* is a vector whose magnitude is unity. Thus, if $\mathbf{x}$ is any nonzero vector, then $(1/\|\mathbf{x}\|)\mathbf{x}$ is normalized. Furthermore,

$$\begin{aligned} \left\langle \frac{1}{\|\mathbf{x}\|}\mathbf{x}, \frac{1}{\|\mathbf{x}\|}\mathbf{x} \right\rangle &= \frac{1}{\|\mathbf{x}\|}\left\langle \mathbf{x}, \frac{1}{\|\mathbf{x}\|}\mathbf{x} \right\rangle && \text{(Property I3)} \\ &= \frac{1}{\|\mathbf{x}\|}\left\langle \frac{1}{\|\mathbf{x}\|}\mathbf{x}, \mathbf{x} \right\rangle && \text{(Property I2)} \end{aligned}$$

$$= \left(\frac{1}{\|\mathbf{x}\|}\right)^2 \langle \mathbf{x}, \mathbf{x} \rangle \qquad \text{(Property I3)}$$

$$= \left(\frac{1}{\|\mathbf{x}\|}\right)^2 \|\mathbf{x}\|^2 = 1,$$

so a normalized vector is always a unit vector.

Problems 6.1

In Problems 1 through 17, find (a) $\langle \mathbf{x}, \mathbf{y} \rangle$ and (b) $\langle \mathbf{x}, \mathbf{x} \rangle$ for the given vectors.

(1) $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

(2) $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$.

(3) $\mathbf{x} = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$.

(4) $\mathbf{x} = [3 \quad 14]$ and $\mathbf{y} = [7 \quad 3]$.

(5) $\mathbf{x} = [-2 \quad -8]$ and $\mathbf{y} = [-4 \quad -7]$.

(6) $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$.

(7) $\mathbf{x} = \begin{bmatrix} -2 \\ 2 \\ -4 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} -4 \\ 3 \\ -3 \end{bmatrix}$.

(8) $\mathbf{x} = \begin{bmatrix} -3 \\ -2 \\ 5 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 6 \\ -4 \\ -4 \end{bmatrix}$.

(9) $\mathbf{x} = [\frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{6}]$ and $\mathbf{y} = [\frac{1}{3} \quad \frac{3}{2} \quad 1]$.

(10) $\mathbf{x} = [1/\sqrt{2} \quad 1/\sqrt{3} \quad 1/\sqrt{6}]$ and $\mathbf{y} = [1/\sqrt{3} \quad 3/\sqrt{2} \quad 1]$.

(11) $\mathbf{x} = [\frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3}]$ and $\mathbf{y} = [\frac{1}{4} \quad \frac{1}{2} \quad \frac{1}{8}]$.

(12) $\mathbf{x} = [10 \quad 20 \quad 30]$ and $\mathbf{y} = [5 \quad -7 \quad 3]$.

(13) $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

(14) $\mathbf{x} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -4 \end{bmatrix}$.

(15) $\mathbf{x} = \begin{bmatrix} 3 \\ 5 \\ -7 \\ -8 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 4 \\ -6 \\ -9 \\ 8 \end{bmatrix}$.

(16) $\mathbf{x} = [\frac{1}{5} \quad \frac{1}{5} \quad \frac{1}{5} \quad \frac{1}{5} \quad \frac{1}{5}]$ and $\mathbf{y} = [1 \quad 2 \quad -3 \quad 4 \quad -5]$.

(17) $\mathbf{x} = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]$ and $\mathbf{y} = [-3 \quad 8 \quad 11 \quad -4 \quad 7]$.

(18) Normalize $\mathbf{y}$ as given in Problem 1.

(19) Normalize $\mathbf{y}$ as given in Problem 2.

(20) Normalize $\mathbf{y}$ as given in Problem 4.

(21) Normalize $\mathbf{y}$ as given in Problem 7.

(22) Normalize $\mathbf{y}$ as given in Problem 8.

(23) Normalize $\mathbf{y}$ as given in Problem 11.

(24) Normalize $\mathbf{y}$ as given in Problem 15.

(25) Normalize $\mathbf{y}$ as given in Problem 16.

(26) Normalize $\mathbf{y}$ as given in Problem 17.

(27) Find $\mathbf{x}$ if $\langle \mathbf{x}, \mathbf{a} \rangle \mathbf{b} = \mathbf{c}$, where

$$\mathbf{a} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}.$$

(28) Determine whether it is possible for two nonzero vectors to have an inner product that is zero.

(29) Prove Property I2.

(30) Prove Property I3.

(31) Prove Property I4.

(32) Prove Property I5.

(33) Prove that $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle + \|\mathbf{y}\|^2$.

(34) Prove the *parallelogram law*:

$$\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2.$$

(35) Prove that, for any scalar λ,

$$0 \le \|\lambda\mathbf{x} - \mathbf{y}\|^2 = \lambda^2\|\mathbf{x}\|^2 - 2\lambda\langle \mathbf{x}, \mathbf{y} \rangle + \|\mathbf{y}\|^2.$$

(36) (Problem 35 continued) Take $\lambda = \langle \mathbf{x}, \mathbf{y} \rangle / \|\mathbf{x}\|^2$ and show that

$$0 \le \frac{-\langle \mathbf{x}, \mathbf{y} \rangle^2}{\|\mathbf{x}\|^2} + \|\mathbf{y}\|^2.$$

From this, deduce that

$$\langle \mathbf{x}, \mathbf{y} \rangle^2 \le \|\mathbf{x}\|^2\|\mathbf{y}\|^2,$$

and that

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \le \|\mathbf{x}\|\,\|\mathbf{y}\|.$$

This last inequality is known as the *Cauchy–Schwarz inequality.*

(37) Using the results of Problem 33 and the Cauchy–Schwarz inequality, show that

$$\begin{aligned} \|\mathbf{x} + \mathbf{y}\|^2 &\le \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2 \\ &= (\|\mathbf{x}\| + \|\mathbf{y}\|)^2. \end{aligned}$$

From this, deduce that

$$\|\mathbf{x} + \mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|.$$

(38) Determine whether there exists a relationship between $\langle \mathbf{x}, \mathbf{y} \rangle$ and $\mathbf{x}^T\mathbf{y}$, when both $\mathbf{x}$ and $\mathbf{y}$ are column vectors of identical dimension with real components.

(39) Use the results of Problem 38 to prove that $\langle \mathbf{A}\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{A}^T\mathbf{y} \rangle$, when $\mathbf{A}$, $\mathbf{x}$, and $\mathbf{y}$ are real matrices of dimensions $n \times n$, $n \times 1$, and $n \times 1$, respectively.

(40) A generalization of the inner product for n-dimensional column vectors with real components is $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbf{A}} = \langle \mathbf{A}\mathbf{x}, \mathbf{A}\mathbf{y} \rangle$ for any real $n \times n$ nonsingular matrix $\mathbf{A}$. This definition reduces to the usual one when $\mathbf{A} = \mathbf{I}$.

Compute $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbf{A}}$ for the vectors given in Problem 1 when

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}.$$

(41) Compute $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbf{A}}$ for the vectors given in Problem 6 when

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

(42) Redo Problem 41 with

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}.$$

6.2 Orthonormal Vectors

Definition 1. Two vectors $\mathbf{x}$ and $\mathbf{y}$ are *orthogonal* (or perpendicular) if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

Thus, given the vectors

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

we see that $\mathbf{x}$ is orthogonal to $\mathbf{y}$ and $\mathbf{y}$ is orthogonal to $\mathbf{z}$ since $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{z} \rangle = 0$; but the vectors $\mathbf{x}$ and $\mathbf{z}$ are not orthogonal since $\langle \mathbf{x}, \mathbf{z} \rangle = 1 + 1 \neq 0$. In particular, as a direct consequence of Property (I5) of Section 6.1 we have that the zero vector is orthogonal to every vector.

A set of vectors is called an *orthogonal set* if each vector in the set is orthogonal to every other vector in the set. The set given above is not an orthogonal set since $\mathbf{z}$ is not orthogonal to $\mathbf{x}$ whereas the set given by $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$,

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},$$

is an orthogonal set because each vector is orthogonal to every other vector.

Definition 2. A set of vectors is *orthonormal* if it is an orthogonal set having the property that every vector is a unit vector (a vector of magnitude 1).

The set of vectors

$$\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is an example of an orthonormal set.

Definition 2 can be simplified if we make use of the Kronecker delta, δ_{ij}, defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{1}$$

A set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is an orthonormal set if and only if

$$\langle \mathbf{x}_i, \mathbf{x}_j \rangle = \delta_{ij} \qquad \text{for all } i \text{ and } j, \qquad i, j = 1, 2, \ldots, n. \tag{2}$$

The importance of orthonormal sets is that they are almost equivalent to linearly independent sets. However, since orthonormal sets have associated with them the additional structure of an inner product, they are often more convenient. We devote the remaining portion of this section to showing the equivalence of these two concepts. The utility of orthonormality will become self-evident in later sections.

Theorem 1. *An orthonormal set of vectors is linearly independent.*

Proof. Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ be an orthonormal set and consider the vector equation

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n = \mathbf{0} \tag{3}$$

where the c_j's $(j = 1, 2, \ldots, n)$ are constants. The set of vectors will be linearly independent if the only constants that satisfy (3) are $c_1 = c_2 = \cdots = c_n = 0$. Take the inner product of both sides of (3) with $\mathbf{x}_1$. Thus,

$$\langle c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n, \mathbf{x}_1 \rangle = \langle \mathbf{0}, \mathbf{x}_1 \rangle.$$

Using properties (I3), (I4), and (I5) of Section 6.1, we have

$$c_1\langle \mathbf{x}_1, \mathbf{x}_1 \rangle + c_2\langle \mathbf{x}_2, \mathbf{x}_1 \rangle + \cdots + c_n\langle \mathbf{x}_n, \mathbf{x}_1 \rangle = 0.$$

Finally, noting that $\langle \mathbf{x}_i, \mathbf{x}_1 \rangle = \delta_{i1}$, we obtain $c_1 = 0$. Now taking the inner product of both sides of (3) with $\mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_n$, successively, we obtain $c_2 = 0$, $c_3 = 0, \ldots, c_n = 0$. Combining these results, we find that $c_1 = c_2 = \cdots = c_n = 0$, which implies the theorem.

Theorem 2. *For every linearly independent set of vectors* $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$, *there exists an orthonormal set of vectors* $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n\}$ *such that each* $\mathbf{q}_j$ $(j = 1, 2, \ldots, n)$ *is a linear combination of* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_j$.

Proof. First define new vectors $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$ by

$$\mathbf{y}_1 = \mathbf{x}_1$$

$$\mathbf{y}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{y}_1 \rangle}{\langle \mathbf{y}_1, \mathbf{y}_1 \rangle} \mathbf{y}_1$$

$$\mathbf{y}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{y}_1 \rangle}{\langle \mathbf{y}_1, \mathbf{y}_1 \rangle} \mathbf{y}_1 - \frac{\langle \mathbf{x}_3, \mathbf{y}_2 \rangle}{\langle \mathbf{y}_2, \mathbf{y}_2 \rangle} \mathbf{y}_2$$

and, in general,

$$\mathbf{y}_j = \mathbf{x}_j - \sum_{k=1}^{j-1} \frac{\langle \mathbf{x}_j, \mathbf{y}_k \rangle}{\langle \mathbf{y}_k, \mathbf{y}_k \rangle} \mathbf{y}_k \qquad (j = 2, 3, \ldots, n). \tag{4}$$

Each $\mathbf{y}_j$ is a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_j$ $(j = 1, 2, \ldots, n)$. Since the $\mathbf{x}$'s are linearly independent, and the coefficient of the $\mathbf{x}_j$ term in (4) is unity, it follows that $\mathbf{y}_j$ is not the zero vector (see Problem 19). Furthermore, it can be shown that the $\mathbf{y}_j$ terms form an orthogonal set (see Problem 20), hence the only property that the $\mathbf{y}_j$ terms lack in order to be the required set is that their magnitudes may not be one. We remedy this situation by defining

$$\mathbf{q}_j = \frac{\mathbf{y}_j}{\|\mathbf{y}_j\|}. \tag{5}$$

The desired set is $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n\}$.

The process used to construct the $\mathbf{q}_j$ terms is called the *Gram–Schmidt orthonormalization process.*

Example 1

Use the Gram–Schmidt orthonormalization process to construct an orthonormal set of vectors from the linearly independent set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$, where

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Solution.

$$\mathbf{y}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Now $\langle \mathbf{x}_2, \mathbf{y}_1 \rangle = 0(1) + 1(1) + 1(0) = 1$, and $\langle \mathbf{y}_1, \mathbf{y}_1 \rangle = 1(1) + 1(1) + 0(0) = 2$; hence,

$$\mathbf{y}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{y}_1 \rangle}{\langle \mathbf{y}_1, \mathbf{y}_1 \rangle}\mathbf{y}_1 = \mathbf{x}_2 - \frac{1}{2}\mathbf{y}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix}.$$

Then,

$$\langle \mathbf{x}_3, \mathbf{y}_1 \rangle = 1(1) + 0(1) + 1(0) = 1,$$
$$\langle \mathbf{x}_3, \mathbf{y}_2 \rangle = 1(-1/2) + 0(1/2) + 1(1) = 1/2,$$
$$\langle \mathbf{y}_2, \mathbf{y}_2 \rangle = (-1/2)^2 + (1/2)^2 + (1)^2 = 3/2,$$

so

$$\mathbf{y}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{y}_1 \rangle}{\langle \mathbf{y}_1, \mathbf{y}_1 \rangle}\mathbf{y}_1 - \frac{\langle \mathbf{x}_3, \mathbf{y}_2 \rangle}{\langle \mathbf{y}_2, \mathbf{y}_2 \rangle}\mathbf{y}_2 = \mathbf{x}_3 - \frac{1}{2}\mathbf{y}_1 - \frac{1/2}{3/2}\mathbf{y}_2$$
$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -2/3 \\ 2/3 \end{bmatrix}.$$

The vectors $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ form an orthogonal set. To make this set orthonormal, we note that $\langle \mathbf{y}_1, \mathbf{y}_1 \rangle = 2$, $\langle \mathbf{y}_2, \mathbf{y}_2 \rangle = 3/2$, and $\langle \mathbf{y}_3, \mathbf{y}_3 \rangle = (2/3)(2/3) + (-2/3)(-2/3) + (2/3)(2/3) = 4/3$. Therefore,

$$\|\mathbf{y}_1\| = \sqrt{\langle \mathbf{y}_1, \mathbf{y}_1 \rangle} = \sqrt{2}, \qquad \|\mathbf{y}_2\| = \sqrt{\langle \mathbf{y}_2, \mathbf{y}_2 \rangle} = \sqrt{3/2},$$
$$\|\mathbf{y}_3\| = \sqrt{\langle \mathbf{y}_3, \mathbf{y}_3 \rangle} = 2/\sqrt{3},$$

and

$$\mathbf{q}_1 = \frac{\mathbf{y}_1}{\|\mathbf{y}_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix},$$

$$\mathbf{q}_2 = \frac{\mathbf{y}_2}{\|\mathbf{y}_2\|} = \frac{1}{\sqrt{3/2}}\begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix},$$

$$\mathbf{q}_3 = \frac{\mathbf{y}_3}{\|\mathbf{y}_3\|} = \frac{1}{2/\sqrt{3}}\begin{bmatrix} 2/3 \\ -2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}. \qquad \square$$

Example 2

Use the Gram–Schmidt orthnormalization process to construct an orthonormal set of vectors from the linearly independent set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$, where

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Solution.

$$\mathbf{y}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$

$$\langle \mathbf{y}_1, \mathbf{y}_1 \rangle = 1(1) + 1(1) + 0(0) + 1(1) = 3,$$

$$\langle \mathbf{x}_2, \mathbf{y}_1 \rangle = 1(1) + 2(1) + 1(0) + 0(1) = 3,$$

$$\mathbf{y}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{y}_1 \rangle}{\langle \mathbf{y}_1, \mathbf{y}_1 \rangle}\mathbf{y}_1 = \mathbf{x}_2 - \frac{3}{3}\mathbf{y}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix};$$

$$\langle \mathbf{y}_2, \mathbf{y}_2 \rangle = 0(0) + 1(1) + 1(1) + (-1)(-1) = 3,$$

$$\langle \mathbf{x}_3, \mathbf{y}_1 \rangle = 0(1) + 1(1) + 2(0) + 1(1) = 2,$$

$$\langle \mathbf{x}_3, \mathbf{y}_2 \rangle = 0(0) + 1(1) + 2(1) + 1(-1) = 2,$$

$$\mathbf{y}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{y}_1 \rangle}{\langle \mathbf{y}_1, \mathbf{y}_1 \rangle}\mathbf{y}_1 - \frac{\langle \mathbf{x}_3, \mathbf{y}_2 \rangle}{\langle \mathbf{y}_2, \mathbf{y}_2 \rangle}\mathbf{y}_2$$

$$= \mathbf{x}_3 - \frac{2}{3}\mathbf{y}_1 - \frac{2}{3}\mathbf{y}_2 = \begin{bmatrix} -2/3 \\ -1/3 \\ 4/3 \\ 1 \end{bmatrix};$$

$$\langle \mathbf{y}_3, \mathbf{y}_3 \rangle = \left(\frac{-2}{3}\right)^2 + \left(\frac{-1}{3}\right)^2 + \left(\frac{4}{3}\right)^2 + (1)^2 = \frac{10}{3},$$

$$\langle \mathbf{x}_4, \mathbf{y}_1 \rangle = 1(1) + 0(1) + 1(0) + 1(1) = 2,$$

$$\langle \mathbf{x}_4, \mathbf{y}_2 \rangle = 1(0) + 0(1) + 1(1) + 1(-1) = 0,$$

$$\langle \mathbf{x}_4, \mathbf{y}_3 \rangle = 1\left(\frac{-2}{3}\right) + 0\left(\frac{-1}{3}\right) + 1\left(\frac{4}{3}\right) + 1(1) = \frac{5}{3},$$

$$\mathbf{y}_4 = \mathbf{x}_4 - \frac{\langle \mathbf{x}_4, \mathbf{y}_1 \rangle}{\langle \mathbf{y}_1, \mathbf{y}_1 \rangle}\mathbf{y}_1 - \frac{\langle \mathbf{x}_4, \mathbf{y}_2 \rangle}{\langle \mathbf{y}_2, \mathbf{y}_2 \rangle}\mathbf{y}_2 - \frac{\langle \mathbf{x}_4, \mathbf{y}_3 \rangle}{\langle \mathbf{y}_3, \mathbf{y}_3 \rangle}\mathbf{y}_3$$

$$= \mathbf{x}_4 - \frac{2}{3}\mathbf{y}_1 - \frac{0}{3}\mathbf{y}_2 - \frac{5/3}{10/3}\mathbf{y}_3 = \begin{bmatrix} 2/3 \\ -1/2 \\ 1/3 \\ -1/6 \end{bmatrix}.$$

Then

$$\langle \mathbf{y}_4, \mathbf{y}_4 \rangle = (2/3)(2/3) + (-1/2)(-1/2) + (1/3)(1/3) + (-1/6)(-1/6)$$
$$= 5/6,$$

and

$$\mathbf{q}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \end{bmatrix}, \qquad \mathbf{q}_2 = \frac{1}{\sqrt{3}}\begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix},$$

$$\mathbf{q}_3 = \frac{1}{\sqrt{10/3}}\begin{bmatrix} -2/3 \\ -1/3 \\ 4/3 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/\sqrt{30} \\ -1/\sqrt{30} \\ 4/\sqrt{30} \\ 3/\sqrt{30} \end{bmatrix},$$

$$\mathbf{q}_4 = \frac{1}{\sqrt{5/6}}\begin{bmatrix} 2/3 \\ -1/2 \\ 1/3 \\ -1/6 \end{bmatrix} = \begin{bmatrix} 4/\sqrt{30} \\ -3/\sqrt{30} \\ 2/\sqrt{30} \\ -1/\sqrt{30} \end{bmatrix}. \qquad \square$$

Problems 6.2

(1) Determine which of the following vectors are orthogonal:

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

(2) Determine which of the following vectors are orthogonal:

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

(3) Find x so that

$$\begin{bmatrix} 3 \\ 5 \end{bmatrix} \quad \text{is orthogonal to} \quad \begin{bmatrix} x \\ 4 \end{bmatrix}.$$

(4) Find x so that

$$\begin{bmatrix} -1 \\ x \\ 3 \end{bmatrix} \quad \text{is orthogonal to} \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

(5) Find x so that $[x \quad x \quad 2]$ is orthogonal to $[1 \quad 3 \quad -1]$.

(6) Find x and y so that $[x \quad y]$ is orthogonal to $[1 \quad 3]$.

(7) Find x and y so that

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad \text{is orthogonal to both} \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(8) Find x, y, and z so that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{is orthogonal to both} \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

(9) Redo Problem 8 with the additional stipulation that $[x \quad y \quad z]^{\mathrm{T}}$ be a unit vector.

In Problems 10 through 18, use the Gram–Schmidt orthonormalization process to construct an orthonormal set from the given set of linearly independent vectors.

(10) $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$

(11) $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$

(12) $\mathbf{x}_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$

(13) $\mathbf{x}_1 = \begin{bmatrix}1\\2\\1\end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix}1\\0\\2\end{bmatrix}.$

(14) $\mathbf{x}_1 = \begin{bmatrix}2\\1\\0\end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix}2\\0\\2\end{bmatrix}.$

(15) $\mathbf{x}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix}2\\0\\1\end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix}2\\2\\1\end{bmatrix}.$

(16) $\mathbf{x}_1 = \begin{bmatrix}0\\3\\4\end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix}3\\5\\0\end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix}2\\5\\5\end{bmatrix}.$

(17) $\mathbf{x}_1 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix}.$

(18) $\mathbf{x}_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix}0\\1\\-1\\0\end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix}1\\0\\-1\\0\end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix}1\\0\\0\\-1\end{bmatrix}.$

(19) Prove that no $\mathbf{y}$-vector in the Gram–Schmidt orthonormalization process is zero.

(20) Prove that the $\mathbf{y}$-vectors in the Gram–Schmidt orthonormalization process form an orthogonal set. (Hint: first show that $\langle \mathbf{y}_2, \mathbf{y}_1 \rangle = 0$, hence $\mathbf{y}_2$ must be orthogonal to $\mathbf{y}_1$. Then use induction.)

(21) With $\mathbf{q}_j$ defined by Eq. (5), show that Eq. (4) can be simplified to $\mathbf{y}_j = \mathbf{x}_j - \sum_{k=1}^{j-1} \langle \mathbf{x}_j, \mathbf{q}_k \rangle \mathbf{q}_k$.

(22) The vectors

$$\mathbf{x}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix}1\\0\\-1\end{bmatrix}$$

are linearly dependent. Apply the Gram–Schmidt process to it, and use the results to deduce what occurs whenever the process is applied to a linearly dependent set of vectors.

(23) Prove that if $\mathbf{x}$ and $\mathbf{y}$ are orthogonal, then

$$\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2.$$

(24) Prove that if $\mathbf{x}$ and $\mathbf{y}$ are orthonormal, then

$$\|s\mathbf{x} + t\mathbf{y}\|^2 = s^2 + t^2$$

for any two scalars s and t.

(25) Let $\mathbf{Q}$ be any $n \times n$ matrix whose columns, when considered as n-dimensional vectors, form an orthonormal set. What can you say about the product $\mathbf{Q}^\mathsf{T}\mathbf{Q}$?

(26) Prove that if $\langle \mathbf{y}, \mathbf{x} \rangle = 0$ for every n-dimensional vector $\mathbf{y}$, then $\mathbf{x} = \mathbf{0}$.

(27) Let $\mathbf{x}$ and $\mathbf{y}$ be any two vectors of the same dimension. Prove that $\mathbf{x} + \mathbf{y}$ is orthogonal to $\mathbf{x} - \mathbf{y}$ if and only if $\|\mathbf{x}\| = \|\mathbf{y}\|$.

(28) Let $\mathbf{A}$ be an $n \times n$ real matrix and $\mathbf{p}$ be a real n-dimensional column vector. Show that if $\mathbf{p}$ is orthogonal to the columns of $\mathbf{A}$, then $\langle \mathbf{Ay}, \mathbf{p} \rangle = 0$ for any n-dimensional real column vector $\mathbf{y}$.

6.3 Projections and **QR**-Decompositions

As with other vector operations, the inner product has a geometrical interpretation in two or three dimensions. For simplicity, we consider two-dimensional vectors here; the extension to three dimensions is straightforward.

Let $\mathbf{u}$ and $\mathbf{v}$ be two nonzero vectors, considered as directed line segments (see Section 1.7), positioned so that their initial points coincide. The *angle between* $\mathbf{u}$ *and* $\mathbf{v}$ is the angle θ between the two line segments satisfying $0 \le \theta \le \pi$. See Figure 1.

Definition 1. If $\mathbf{u}$ and $\mathbf{v}$ are two-dimensional vectors and θ is the angle between them, then the *dot product* of these two vectors is $\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\| \cos\theta$.

To use Definition 1, we need the cosine of the angle between two vectors, which requires us to measure the angle. We shall take another approach.

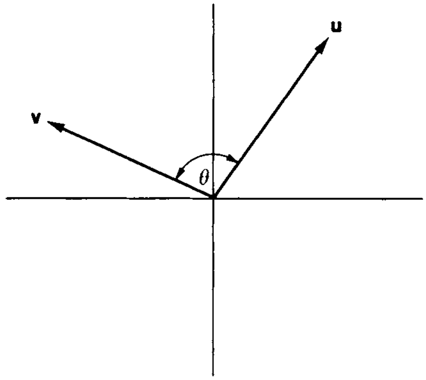


Figure 1.

The vectors **u** and **v** along with their difference **u** − **v** form a triangle (see Figure 2) having sides $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, and $\|\mathbf{u}-\mathbf{v}\|$. It follows from the law of cosines that

$$\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta,$$

whereupon

$$\begin{aligned}\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta &= \tfrac{1}{2}[\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2]\\ &= \tfrac{1}{2}[\langle\mathbf{u},\mathbf{u}\rangle + \langle\mathbf{v},\mathbf{v}\rangle - \langle\mathbf{u}-\mathbf{v},\mathbf{u}-\mathbf{v}\rangle]\\ &= \langle\mathbf{u},\mathbf{v}\rangle.\end{aligned}$$

Thus, the dot product of two-dimensional vectors is the inner product of

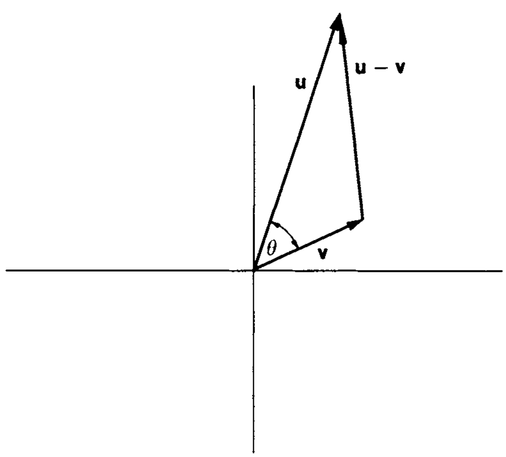


Figure 2.

those vectors. That is,

$$\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta = \langle \mathbf{u}, \mathbf{v} \rangle. \tag{6}$$

The dot product of nonzero vectors is zero if and only if $\cos\theta = 0$, or $\theta = 90°$. Consequently, the dot product of two nonzero vectors is zero if and only if the vectors are perpendicular. This, with Eq. (6), establishes the equivalence between orthogonality and perpendicularity for two-dimensional vectors. In addition, we may rewrite Eq. (6) as

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|\|\mathbf{v}\|}, \tag{7}$$

and use Eq. (7) to calculate the angle between two vectors.

Example 1

Find the angle between the vectors

$$\mathbf{u} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}.$$

Solution. $\langle \mathbf{u}, \mathbf{v} \rangle = 2(-3) + 5(4) = 14$, $\|\mathbf{u}\| = \sqrt{4 + 25} = \sqrt{29}$, $\|\mathbf{v}\| = \sqrt{9 + 16} = 5$, so $\cos\theta = 14/(5\sqrt{29}) = 0.5199$, and $\theta = 58.7°$. □

Eq. (7) is used to define the angle between any two vectors of the same, but arbitrary dimension, even though the geometrical significance of an angle becomes meaningless for dimensions greater than three. (See Problems 9 and 10.)

A problem that occurs often in the applied sciences and that has important ramifications for us in matrices involves a given nonzero vector $\mathbf{x}$ and a nonzero reference vector $\mathbf{a}$. The problem is to decompose $\mathbf{x}$ into the sum of two vectors, $\mathbf{u} + \mathbf{v}$, where $\mathbf{u}$ is parallel to $\mathbf{a}$ and $\mathbf{v}$ is perpendicular to $\mathbf{a}$. This situation is illustrated in Figure 3. In physics, $\mathbf{u}$ is called the parallel component of $\mathbf{x}$ and $\mathbf{v}$ is called the perpendicular component of $\mathbf{x}$, where parallel and perpendicular are understood to be with respect to the reference vector $\mathbf{a}$.

If $\mathbf{u}$ is to be parallel to $\mathbf{a}$, it must be a scalar multiple of $\mathbf{a}$, in particular $\mathbf{u} = \lambda\mathbf{a}$. Since we want $\mathbf{x} = \mathbf{u} + \mathbf{v}$, it follows that $\mathbf{v} = \mathbf{x} - \mathbf{u} = \mathbf{x} - \lambda\mathbf{a}$. Finally, if $\mathbf{u}$ and $\mathbf{v}$ are to be perpendicular, we require that

$$\begin{aligned} 0 = \langle \mathbf{u}, \mathbf{v} \rangle &= \langle \lambda\mathbf{a}, \mathbf{x} - \lambda\mathbf{a} \rangle \\ &= \lambda\langle \mathbf{a}, \mathbf{x} \rangle - \lambda^2\langle \mathbf{a}, \mathbf{a} \rangle \\ &= \lambda[\langle \mathbf{a}, \mathbf{x} \rangle - \lambda\langle \mathbf{a}, \mathbf{a} \rangle]. \end{aligned}$$

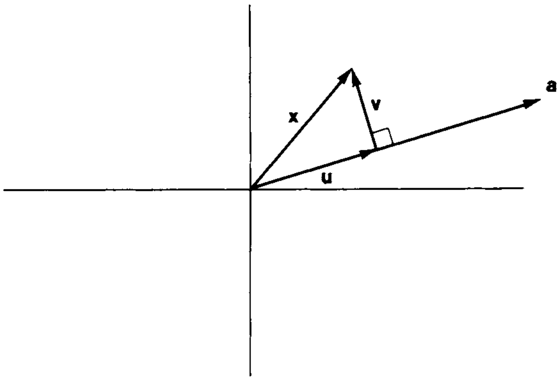


Figure 3.

Thus, either $\lambda = 0$ or $\lambda = \langle \mathbf{a}, \mathbf{x} \rangle / \langle \mathbf{a}, \mathbf{a} \rangle$. If $\lambda = 0$, then $\mathbf{u} = \lambda \mathbf{a} = \mathbf{0}$, and $\mathbf{x} = \mathbf{u} + \mathbf{v} = \mathbf{v}$, which means that $\mathbf{x}$ and $\mathbf{a}$ are perpendicular. In such a case, $\langle \mathbf{a}, \mathbf{x} \rangle = 0$. Thus, we may always infer that $\lambda = \langle \mathbf{a}, \mathbf{x} \rangle / \langle \mathbf{a}, \mathbf{a} \rangle$, with

$$\mathbf{u} = \frac{\langle \mathbf{a}, \mathbf{x} \rangle}{\langle \mathbf{a}, \mathbf{a} \rangle} \mathbf{a} \quad \text{and} \quad \mathbf{v} = \mathbf{x} - \frac{\langle \mathbf{a}, \mathbf{x} \rangle}{\langle \mathbf{a}, \mathbf{a} \rangle} \mathbf{a}.$$

In this context, $\mathbf{u}$ is the *projection of* $\mathbf{x}$ *onto* $\mathbf{a}$, and $\mathbf{v}$ is the *orthogonal complement.*

Example 2

Decompose the vector

$$\mathbf{x} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

into the sum of two vectors, one of which is parallel to

$$\mathbf{a} = \begin{bmatrix} -3 \\ 4 \end{bmatrix},$$

and one of which is perpendicular to $\mathbf{a}$.

Solution

$$\mathbf{u} = \frac{\langle \mathbf{a}, \mathbf{x} \rangle}{\langle \mathbf{a}, \mathbf{a} \rangle} \mathbf{a} = \frac{22}{25} \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} -2.64 \\ 3.52 \end{bmatrix},$$

$$\mathbf{v} = \mathbf{x} - \mathbf{u} = \begin{bmatrix} 2 \\ 7 \end{bmatrix} - \begin{bmatrix} -2.64 \\ 3.52 \end{bmatrix} = \begin{bmatrix} 4.64 \\ 3.48 \end{bmatrix}.$$

Then, $\mathbf{x} = \mathbf{u} + \mathbf{v}$, with $\mathbf{u}$ parallel to $\mathbf{a}$ and $\mathbf{v}$ perpendicular to $\mathbf{a}$. $\square$

We now extend the relationships developed in two dimensions to vectors in higher dimensions. Given a nonzero vector $\mathbf{x}$ and another nonzero reference vector $\mathbf{a}$, we define the projection of $\mathbf{x}$ onto $\mathbf{a}$ as

$$\text{proj}_{\mathbf{a}}\mathbf{x} = \frac{\langle \mathbf{a}, \mathbf{x} \rangle}{\langle \mathbf{a}, \mathbf{a} \rangle}\mathbf{a}. \tag{8}$$

As a result, we obtain the very important relationship that

$$\mathbf{x} - \frac{\langle \mathbf{a}, \mathbf{x} \rangle}{\langle \mathbf{a}, \mathbf{a} \rangle}\mathbf{a} \quad \text{is orthogonal to} \quad \mathbf{a}. \tag{9}$$

That is, if we subtract from a nonzero vector $\mathbf{x}$ its projection onto another nonzero vector $\mathbf{a}$, we are left with a vector that is orthogonal to $\mathbf{a}$. (See Problem 23.)

In this context, the Gram–Schmidt process, described in Section 6.2, is almost obvious. Consider Eq. (4) from that section:

$$\mathbf{y}_j = \mathbf{x}_j - \sum_{k=1}^{j-1} \frac{\langle \mathbf{x}_j, \mathbf{y}_k \rangle}{\langle \mathbf{y}_k, \mathbf{y}_k \rangle}\mathbf{y}_k \tag{4 repeated}$$

The quantity inside the summation sign is the projection of $\mathbf{x}_j$ onto $\mathbf{y}_k$. Thus for each k ($k = 1, 2, \ldots, j-1$), we are sequentially subtracting from $\mathbf{x}_j$ its projection onto $\mathbf{y}_k$, leaving a vector that is orthogonal to $\mathbf{y}_k$.

We now propose to alter slightly the steps of the Gram–Schmidt orthonormalization process. First, we shall normalize the orthogonal vectors as soon as they are obtained, rather than waiting until the end. This will make for messier hand calculations, but for a more efficient computer algorithm. Observe that if the $\mathbf{y}_k$ vectors in Eq. (4) are unit vectors, then the denominator is unity, and need not be calculated.

Once we have fully determined a $\mathbf{y}_k$ vector, we shall immediately subtract the various projections onto this vector from all succeeding $\mathbf{x}$ vectors. In particular, once $\mathbf{y}_1$ is determined, we shall subtract the projection of $\mathbf{x}_2$ onto $\mathbf{y}_1$ from $\mathbf{x}_2$, then we shall subtract the projection of $\mathbf{x}_3$ onto $\mathbf{y}_1$ from $\mathbf{x}_3$, and continue until we have subtracted the projection of $\mathbf{x}_n$ onto $\mathbf{y}_1$ from $\mathbf{x}_n$. Only then will we return to $\mathbf{x}_2$ and normalize it to obtain $\mathbf{y}_2$. Then, we shall subtract from $\mathbf{x}_3, \mathbf{x}_4, \ldots, \mathbf{x}_n$ the projections onto $\mathbf{y}_2$ from $\mathbf{x}_3, \mathbf{x}_4, \ldots, \mathbf{x}_n$, respectively, before returning to $\mathbf{x}_3$ and normalizing it, thus obtaining $\mathbf{y}_3$. As a result, once we have $\mathbf{y}_1$, we alter $\mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_n$ so each is orthogonal to $\mathbf{y}_1$; once we have $\mathbf{y}_2$, we alter again $\mathbf{x}_3, \mathbf{x}_4, \ldots, \mathbf{x}_n$ so each is also orthogonal to $\mathbf{y}_2$; and so on.

These changes are known as the *revised Gram–Schmidt algorithm*. Given a set of linearly independent vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$, the algorithm may be formalized as follows: Begin with $k = 1$ and, sequentially moving through $k = n$,

(i) calculate $r_{kk} = \sqrt{\langle \mathbf{x}_k, \mathbf{x}_k \rangle}$,
(ii) set $\mathbf{q}_k = (1/r_{kk})\mathbf{x}_k$,
(iii) for $j = k+1, k+2, \ldots, n$, calculate $r_{kj} = \langle \mathbf{x}_j, \mathbf{q}_k \rangle$,
(iv) for $j = k+1, k+2, \ldots, n$, replace $\mathbf{x}_j$ by $\mathbf{x}_j - r_{kj}\mathbf{q}_k$.

The first two steps normalize, the third and fourth steps subtract projections from vectors, thereby generating orthogonality.

Example 3

Use the revised Gram–Schmidt algorithm to construct an orthonormal set of vectors from the linearly independent set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$, where

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Solution.

First Iteration ($k = 1$)

$$r_{11} = \sqrt{\langle \mathbf{x}_1, \mathbf{x}_1 \rangle} = \sqrt{2},$$

$$\mathbf{q}_1 = \frac{1}{r_{11}}\mathbf{x}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix},$$

$$r_{12} = \langle \mathbf{x}_2, \mathbf{q}_1 \rangle = \frac{1}{\sqrt{2}},$$

$$r_{13} = \langle \mathbf{x}_3, \mathbf{q}_1 \rangle = \frac{1}{\sqrt{2}},$$

$$\mathbf{x}_2 \leftarrow \mathbf{x}_2 - r_{12}\mathbf{q}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix},$$

$$\mathbf{x}_3 \leftarrow \mathbf{x}_3 - r_{13}\mathbf{q}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}.$$

Note that both $\mathbf{x}_2$ and $\mathbf{x}_3$ are now orthogonal to $\mathbf{q}_1$.

SECOND ITERATION ($k = 2$)

Using vectors from the first iteration, we compute

$$r_{22} = \sqrt{\langle \mathbf{x}_2, \mathbf{x}_2 \rangle} = \sqrt{3/2},$$

$$\mathbf{q}_2 = \frac{1}{r_{22}}\mathbf{x}_2 = \frac{1}{\sqrt{3/2}}\begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix},$$

$$r_{23} = \langle \mathbf{x}_3, \mathbf{q}_2 \rangle = \frac{1}{\sqrt{6}},$$

$$\mathbf{x}_3 \leftarrow \mathbf{x}_3 - r_{23}\mathbf{q}_2 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{6}}\begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 2/3 \\ -2/3 \\ 2/3 \end{bmatrix}.$$

THIRD ITERATION ($k = 3$)

Using vectors from the second iteration, we compute

$$r_{33} = \sqrt{\langle x_3, \mathbf{x}_3 \rangle} = \frac{2}{\sqrt{3}},$$

$$\mathbf{q}_3 = \frac{1}{r_{33}}\mathbf{x}_3 = \frac{1}{2/\sqrt{3}}\begin{bmatrix} 2/3 \\ -2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}.$$

The orthonormal set is $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$. Compare with Example 1 of Section 6.2. □

The revised Gram–Schmidt algorithm has two advantages over the Gram–Schmidt process developed in the previous section. First, it is less effected by roundoff errors, and second, the inverse process—recapturing the $\mathbf{x}$-vectors from the $\mathbf{q}$-vectors—becomes trivial. To understand this second advantage, let us redo Example 3 symbolically. In the first iteration, we calculated

$$\mathbf{q}_1 = \frac{1}{r_{11}}\mathbf{x}_1,$$

so, we immediately have,

$$\mathbf{x}_1 = r_{11}\mathbf{q}_1. \tag{10}$$

We then replaced $\mathbf{x}_2$ and $\mathbf{x}_3$ with vectors that were orthogonal to $\mathbf{q}_1$. If we denote these replacement vectors as $\mathbf{x}'_2$ and $\mathbf{x}'_3$, respectively, we have

$$\mathbf{x}'_2 = \mathbf{x}_2 - r_{12}\mathbf{q}_1 \quad \text{and} \quad \mathbf{x}'_3 = \mathbf{x}_3 - r_{13}\mathbf{q}_1.$$

With the second iteration, we calculated

$$\mathbf{q}_2 = \frac{1}{r_{22}}\mathbf{x}'_2 = \frac{1}{r_{22}}(\mathbf{x}_2 - r_{12}\mathbf{q}_1).$$

Solving for $\mathbf{x}_2$, we get

$$\mathbf{x}_2 = r_{12}\mathbf{q}_1 + r_{22}\mathbf{q}_2. \tag{11}$$

We then replaced $\mathbf{x}_3$ with a vector that was orthogonal to $\mathbf{q}_2$. If we denote this replacement vector as $\mathbf{x}''_3$, we have

$$\mathbf{x}''_3 = \mathbf{x}'_3 - r_{23}\mathbf{q}_2 = (\mathbf{x}_3 - r_{13}\mathbf{q}_1) - r_{23}\mathbf{q}_2.$$

With the third iteration, we calculated

$$\mathbf{q}_3 = \frac{1}{r_{33}}\mathbf{x}''_3 = \frac{1}{r_{33}}(\mathbf{x}_3 - r_{13}\mathbf{q}_1 - r_{23}\mathbf{q}_2).$$

Solving for $\mathbf{x}_3$, we obtain

$$\mathbf{x}_3 = r_{13}\mathbf{q}_1 + r_{23}\mathbf{q}_2 + r_{33}\mathbf{q}_3. \tag{12}$$

Eqs. (10) through (12) form a pattern that is easily extended. Begin with linearly independent vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, and use the revised Gram–Schmidt algorithm to form $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$. Then, for any k ($k = 1, 2, \ldots, n$).

$$\mathbf{x}_k = r_{1k}\mathbf{q}_1 + r_{2k}\mathbf{q}_2 + r_{3k}\mathbf{q}_3 + \cdots + r_{kk}\mathbf{q}_k.$$

If we set $\mathbf{X} = [\mathbf{x}_1 \quad \mathbf{x}_2 \quad \cdots \quad \mathbf{x}_n]$,

$$\mathbf{Q} = [\mathbf{q}_1 \quad \mathbf{q}_2 \quad \cdots \quad \mathbf{q}_n] \tag{13}$$

and

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1n} \\ 0 & r_{22} & r_{23} & \cdots & r_{2n} \\ 0 & 0 & r_{33} & \cdots & r_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & r_{nn} \end{bmatrix}; \tag{14}$$

we have the matrix representation

$$\mathbf{X} = \mathbf{QR},$$

which is known as the **QR**-decomposition of the matrix **X**. The columns of **Q** form an orthonormal set of column vectors, and $\mathbf{R}$ is upper (or right) triangular.

In general, we are given a matrix $\mathbf{X}$ and are asked to generate its **QR**-decomposition. This is accomplished by applying the revised Gram–Schmidt algorithm to the columns of $\mathbf{X}$, providing those columns are linearly independent. Then Eqs. (13) and (14) yield the desired factorization.

Example 4

Construct a **QR**-decomposition for

$$\mathbf{X} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Solution. The columns of $\mathbf{X}$ are the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ of Example 3. Using the results of that problem, we generate

$$\mathbf{Q} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \quad \text{and} \quad \mathbf{R} = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3/2} & 1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix}. \qquad \square$$

Example 5

Construct a **QR**-decomposition for

$$\mathbf{X} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

Solution. The columns of $\mathbf{X}$ are the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

We apply the revised Gram–Schmidt algorithm to these vectors. Carrying eight significant figures through all computations but rounding to four decimals for presentation purposes, we get

FIRST ITERATION ($k = 1$)

$$r_{11} = \sqrt{\langle \mathbf{x}_1, \mathbf{x}_1 \rangle} = \sqrt{3} = 1.7321,$$

$$\mathbf{q}_1 = \frac{1}{r_{11}}\mathbf{x}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\0\\1\end{bmatrix} = \begin{bmatrix}0.5774\\0.5774\\0.0000\\0.5774\end{bmatrix},$$

$$r_{12} = \langle \mathbf{x}_2, \mathbf{q}_1 \rangle = 1.7321,$$

$$r_{13} = \langle \mathbf{x}_3, \mathbf{q}_1 \rangle = 1.1547,$$

$$r_{14} = \langle \mathbf{x}_4, \mathbf{q}_1 \rangle = 1.1547,$$

$$\mathbf{x}_2 \leftarrow \mathbf{x}_2 - r_{12}\mathbf{q}_1 = \begin{bmatrix}1\\2\\1\\0\end{bmatrix} - 1.7321\begin{bmatrix}0.5774\\0.5774\\0.0000\\0.5774\end{bmatrix} = \begin{bmatrix}0.0000\\1.0000\\1.0000\\-1.0000\end{bmatrix},$$

$$\mathbf{x}_3 \leftarrow \mathbf{x}_3 - r_{13}\mathbf{q}_1 = \begin{bmatrix}0\\1\\2\\1\end{bmatrix} - 1.1547\begin{bmatrix}0.5774\\0.5774\\0.0000\\0.5774\end{bmatrix} = \begin{bmatrix}-0.6667\\0.3333\\2.0000\\0.3333\end{bmatrix},$$

$$\mathbf{x}_4 \leftarrow \mathbf{x}_4 - r_{14}\mathbf{q}_1 = \begin{bmatrix}1\\0\\1\\1\end{bmatrix} - 1.1547\begin{bmatrix}0.5774\\0.5774\\0.0000\\0.5774\end{bmatrix} = \begin{bmatrix}0.3333\\-0.6667\\1.0000\\0.3333\end{bmatrix}.$$

SECOND ITERATION ($k = 2$)

Using vectors from the first iteration, we compute

$$r_{22} = \sqrt{\langle \mathbf{x}_2, \mathbf{x}_2 \rangle} = 1.7321,$$

$$\mathbf{q}_2 = \frac{1}{r_{22}}\mathbf{x}_2 = \frac{1}{1.7321}\begin{bmatrix}0.0000\\1.0000\\1.0000\\-1.0000\end{bmatrix} = \begin{bmatrix}0.0000\\0.5774\\0.5774\\-0.5774\end{bmatrix},$$

$$r_{23} = \langle \mathbf{x}_3, \mathbf{q}_2 \rangle = 1.1547,$$

$$r_{24} = \langle \mathbf{x}_4, \mathbf{q}_2 \rangle = 0.0000,$$

$$\mathbf{x}_3 \leftarrow \mathbf{x}_3 - r_{23}\mathbf{q}_2 = \begin{bmatrix} -0.6667 \\ 0.3333 \\ 2.0000 \\ 0.3333 \end{bmatrix} - 1.1547 \begin{bmatrix} 0.0000 \\ 0.5774 \\ 0.5774 \\ -0.5774 \end{bmatrix} = \begin{bmatrix} -0.6667 \\ -0.3333 \\ 1.3333 \\ 1.0000 \end{bmatrix},$$

$$\mathbf{x}_4 \leftarrow \mathbf{x}_4 - r_{24}\mathbf{q}_2 = \begin{bmatrix} 0.3333 \\ -0.6667 \\ 1.0000 \\ 0.3333 \end{bmatrix} - 0.0000 \begin{bmatrix} 0.0000 \\ 0.5774 \\ 0.5774 \\ -0.5774 \end{bmatrix} = \begin{bmatrix} 0.3333 \\ -0.6667 \\ 1.0000 \\ 0.3333 \end{bmatrix}.$$

THIRD ITERATION ($k = 3$)

Using vectors from the second iteration, we compute

$$r_{33} = \sqrt{\langle \mathbf{x}_3, \mathbf{x}_3 \rangle} = 1.8257,$$

$$\mathbf{q}_3 = \frac{1}{r_{33}}\mathbf{x}_3 = \frac{1}{1.8257} \begin{bmatrix} -0.6667 \\ -0.3333 \\ 1.3333 \\ 1.0000 \end{bmatrix} = \begin{bmatrix} -0.3651 \\ -0.1826 \\ 0.7303 \\ 0.5477 \end{bmatrix},$$

$$r_{34} = \langle \mathbf{x}_4, \mathbf{q}_3 \rangle = 0.9129,$$

$$\mathbf{x}_4 \leftarrow \mathbf{x}_4 - r_{34}\mathbf{q}_3 = \begin{bmatrix} 0.3333 \\ -0.6667 \\ 1.0000 \\ 0.3333 \end{bmatrix} - 0.9129 \begin{bmatrix} -0.3651 \\ -0.1826 \\ 0.7303 \\ 0.5477 \end{bmatrix} = \begin{bmatrix} 0.6667 \\ -0.5000 \\ 0.3333 \\ -0.1667 \end{bmatrix}.$$

FOURTH ITERATION ($k = 4$)

Using vectors from the third iteration, we compute

$$r_{44} = \sqrt{\langle \mathbf{x}_4, \mathbf{x}_4 \rangle} = 0.9129,$$

$$\mathbf{q}_4 = \frac{1}{r_{44}}\mathbf{x}_4 = \frac{1}{0.9129} \begin{bmatrix} 0.6667 \\ -0.5000 \\ 0.3333 \\ -0.1667 \end{bmatrix} = \begin{bmatrix} 0.7303 \\ -0.5477 \\ 0.3651 \\ -0.1826 \end{bmatrix}.$$

With these entries calculated (compare with Example 2 of Section 6.2), we form

$$\mathbf{Q} = \begin{bmatrix} 0.5774 & 0.0000 & -0.3651 & 0.7303 \\ 0.5774 & 0.5774 & -0.1826 & -0.5477 \\ 0.0000 & 0.5774 & 0.7303 & 0.3651 \\ 0.5774 & -0.5774 & 0.5477 & -0.1826 \end{bmatrix}$$

and

$$\mathbf{R} = \begin{bmatrix} 1.7321 & 1.7321 & 1.1547 & 1.1547 \\ 0 & 1.7321 & 1.1547 & 0.0000 \\ 0 & 0 & 1.8257 & 0.9129 \\ 0 & 0 & 0 & 0.9129 \end{bmatrix}. \qquad \square$$

Finally, we note that in contrast to **LU** decompositions, **QR**-decompositions are applicable to nonsquare matrices as well. In particular, if we consider a matrix containing just the first two columns of the matrix **X** in Example 5, and calculate r_{11}, r_{12}, r_{22}, $\mathbf{q}_1$, and $\mathbf{q}_2$ as we did there, we have the decomposition

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0.5774 & 0.0000 \\ 0.5774 & 0.5774 \\ 0.0000 & 0.5774 \\ 0.5774 & -0.5774 \end{bmatrix} \begin{bmatrix} 1.7321 & 1.7321 \\ 0 & 1.7321 \end{bmatrix}.$$

Problems 6.3

In Problems 1 through 10, determine the (a) the angle between the given vectors, (b) the projection of $\mathbf{x}_1$ onto $\mathbf{x}_2$, and (c) its orthogonal component.

(1) $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$

(2) $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$

(3) $\mathbf{x}_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$

(4) $\mathbf{x}_1 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 8 \end{bmatrix}.$

(5) $\mathbf{x}_1 = \begin{bmatrix} -7 \\ -2 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 9 \end{bmatrix}.$

(6) $\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}.$

(7) $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.$

(8) $\mathbf{x}_1 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 5 \\ 5 \end{bmatrix}.$

(9) $\mathbf{x}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$

(10) $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ -1 \end{bmatrix}.$

In Problems 11 through 21, determine **QR**-decompositions for the given matrices.

(11) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

(12) $\begin{bmatrix} 1 & 3 \\ 1 & 5 \end{bmatrix}$.

(13) $\begin{bmatrix} 3 & 3 \\ -2 & 3 \end{bmatrix}$.

(14) $\begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{bmatrix}$.

(15) $\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 3 & 5 \end{bmatrix}$.

(16) $\begin{bmatrix} 3 & 1 \\ -2 & 1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}$.

(17) $\begin{bmatrix} 2 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$.

(18) $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$.

(19) $\begin{bmatrix} 0 & 3 & 2 \\ 3 & 5 & 5 \\ 4 & 0 & 5 \end{bmatrix}$.

(20) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

(21) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$.

(22) Show that

$$\left\| \frac{\langle \mathbf{x}, \mathbf{a} \rangle}{\langle \mathbf{a}, \mathbf{a} \rangle} \mathbf{a} \right\| = \|\mathbf{x}\| |\cos\theta|,$$

where θ is the angle between $\mathbf{x}$ and $\mathbf{a}$.

(23) Prove directly that

$$\mathbf{x} - \frac{\langle \mathbf{a}, \mathbf{x} \rangle}{\langle \mathbf{a}, \mathbf{a} \rangle} \mathbf{a}$$

is orthogonal to $\mathbf{a}$.

(24) Discuss what is likely to occur in a **QR**-decomposition if the columns are not linearly independent, and all calculations are rounded.

6.4 The **QR**-Algorithm

The **QR**-algorithm is one of the more powerful numerical methods developed for computing eigenvalues of real matrices. In contrast to the power methods described in Section 5.6, which converge only to a single dominant real eigenvalue of a matrix, the **QR**-algorithm generally locates all eigenvalues, both real and complex, regardless of multiplicity.

Although a proof of the **QR**-algorithm is beyond the scope of this book, the algorithm itself is deceptively simple. As its name suggests, the algorithm is based on **QR**-decompositions. Not surprisingly then, the algorithm involves numerous arithmetic calculations, making it unattractive for hand computations but ideal for implementation on a computer.

Like many numerical methods, the **QR**-algorithm is iterative. We begin with a square real matrix $\mathbf{A}_0$. To determine its eigenvalues, we create a sequence of new matrices $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_{k-1}, \mathbf{A}_k, \ldots$, having the property that each new matrix has the same eigenvalues as $\mathbf{A}_0$, and that these eigenvalues become increasingly obvious as the sequence progresses. To calculate $\mathbf{A}_k$ $(k = 1, 2, 3, \ldots)$ once $\mathbf{A}_{k-1}$ is known, first construct a **QR**-decomposition of $\mathbf{A}_{k-1}$:

$$\mathbf{A}_{k-1} = \mathbf{Q}_{k-1}\mathbf{R}_{k-1},$$

and then reverse the order of the product to define

$$\mathbf{A}_k = \mathbf{R}_{k-1}\mathbf{Q}_{k-1}.$$

We shall show in Chapter 9 that each matrix in the sequence $\{\mathbf{A}_k\}$ $(k = 1, 2, 3, \ldots)$ has identical eigenvalues. For now, we just note that the sequence generally converges to one of the following two partitioned forms:

$$\left[\begin{array}{cccc:c} & & \mathbf{S} & & \mathbf{T} \\ \hdashline 0 & 0 & 0 \cdots 0 & & a \end{array}\right] \tag{15}$$

or

$$\left[\begin{array}{ccc:cc} & \mathbf{U} & & \mathbf{V} & \\ \hdashline 0 & 0 & 0 \cdots 0 & b & c \\ 0 & 0 & 0 \cdots 0 & d & e \end{array}\right]. \tag{16}$$

If matrix (15) occurs, then the element a is an eigenvalue, and the remaining eigenvalues are found by applying the **QR**-algorithm anew to the submatrix **S**. If, on the other hand, matrix (16) occurs, then two eigenvalues are determined by solving for the roots of the characteristic equation of the 2×2 matrix in the lower right partition, namely

$$\lambda^2 - (b + e)\lambda + (be - cd) = 0.$$

The remaining eigenvalues are found by applying the **QR**-algorithm anew to the submatrix **U**.

Convergence of the algorithm is accelerated by performing a shift at each iteration. If the orders of all matrices are $n \times n$, we denote the element in the (n, n)-position of the matrix $\mathbf{A}_{k-1}$ as w_{k-1}, and construct a $\mathbf{QR}$-decomposition for the shifted matrix $\mathbf{A}_{k-1} - w_{k-1}\mathbf{I}$. That is,

$$\mathbf{A}_{k-1} - w_{k-1}\mathbf{I} = \mathbf{Q}_{k-1}\mathbf{R}_{k-1}. \tag{17}$$

We define

$$\mathbf{A}_k = \mathbf{R}_{k-1}\mathbf{Q}_{k-1} + w_{k-1}\mathbf{I}. \tag{18}$$

Example 1

Find the eigenvalues of

$$\mathbf{A}_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 18 & -1 & -7 \end{bmatrix}.$$

Solution. Using the $\mathbf{QR}$-algorithm with shifting, carrying all calculations to eight significant figures but rounding to four decimals for presentation, we compute

$$\begin{aligned}
\mathbf{A}_0 - (-7)\mathbf{I} &= \begin{bmatrix} 7 & 1 & 0 \\ 0 & 7 & 1 \\ 18 & -1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0.3624 & 0.1695 & -0.9165 \\ 0.0000 & 0.9833 & 0.1818 \\ 0.9320 & -0.0659 & 0.3564 \end{bmatrix} \begin{bmatrix} 19.3132 & -0.5696 & 0.0000 \\ 0.0000 & 7.1187 & 0.9833 \\ 0.0000 & 0.0000 & 0.1818 \end{bmatrix} \\
&= \mathbf{Q}_0\mathbf{R}_0, \\
\mathbf{A}_1 &= \mathbf{R}_0\mathbf{Q}_0 + (-7)\mathbf{I} \\
&= \begin{bmatrix} 19.3132 & -0.5696 & 0.0000 \\ 0.0000 & 7.1187 & 0.9833 \\ 0.0000 & 0.0000 & 0.1818 \end{bmatrix} \begin{bmatrix} 0.3624 & 0.1695 & -0.9165 \\ 0.0000 & 0.9833 & 0.1818 \\ 0.9320 & -0.0659 & 0.3564 \end{bmatrix} \\
&\quad + \begin{bmatrix} -7 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{bmatrix} \\
&= \begin{bmatrix} 0.0000 & 2.7130 & -17.8035 \\ 0.9165 & -0.0648 & 1.6449 \\ 0.1695 & -0.0120 & -6.9352 \end{bmatrix},
\end{aligned}$$

$$\mathbf{A}_1 - (-6.9352)\mathbf{I} = \begin{bmatrix} 6.9352 & 2.7130 & -17.8035 \\ 0.9165 & 6.8704 & 1.6449 \\ 0.1695 & -0.0120 & 0.0000 \end{bmatrix}$$

$$= \begin{bmatrix} 0.9911 & -0.1306 & -0.0260 \\ 0.1310 & 0.9913 & 0.0120 \\ 0.0242 & -0.0153 & 0.9996 \end{bmatrix} \begin{bmatrix} 6.9975 & 3.5884 & -17.4294 \\ 0.0000 & 6.4565 & 3.9562 \\ 0.0000 & 0.0000 & 0.4829 \end{bmatrix}$$

$$= \mathbf{Q}_1\mathbf{R}_1,$$

$$\mathbf{A}_2 = \mathbf{R}_1\mathbf{Q}_1 + (-6.9352)\mathbf{I} = \begin{bmatrix} 0.0478 & 2.9101 & -17.5612 \\ 0.9414 & -0.5954 & 4.0322 \\ 0.0117 & -0.0074 & -6.4525 \end{bmatrix}.$$

Continuing in this manner, we generate sequentially

$$\mathbf{A}_3 = \begin{bmatrix} 0.5511 & 2.7835 & -16.8072 \\ 0.7826 & -1.1455 & 6.5200 \\ 0.0001 & -0.0001 & -6.4056 \end{bmatrix}$$

and

$$\mathbf{A}_4 = \begin{bmatrix} 0.9259 & 2.5510 & -15.9729 \\ 0.5497 & -1.5207 & 8.3583 \\ 0.0000 & -0.0000 & -6.4051 \end{bmatrix}.$$

$\mathbf{A}_4$ has form (15) with

$$\mathbf{S} = \begin{bmatrix} 0.9259 & 2.5510 \\ 0.5497 & -1.5207 \end{bmatrix} \quad \text{and} \quad a = -6.4051.$$

One eigenvalue is -6.4051, which is identical to the value obtained in Example 2 of Section 5.6. In addition, the characteristic equation of $\mathbf{R}$ is $\lambda^2 + 0.5948\lambda - 2.8103 = 0$, which admits both -2 and 1.4052 as roots. These are the other two eigenvalues of $\mathbf{A}_0$. $\square$

Example 2

Find the eigenvalues of

$$\mathbf{A}_0 = \begin{bmatrix} 0 & 0 & 0 & -25 \\ 1 & 0 & 0 & 30 \\ 0 & 1 & 0 & -18 \\ 0 & 0 & 1 & 6 \end{bmatrix}.$$

Solution. Using the **QR**-algorithm with shifting, carrying all calculations to eight significant figures but rounding to four decimals for presentation, we compute

$$\mathbf{A}_0 - (6)\mathbf{I} = \begin{bmatrix} -6 & 0 & 0 & -25 \\ 1 & -6 & 0 & 30 \\ 0 & 1 & -6 & -18 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -0.9864 & -0.1621 & -0.0270 & -0.0046 \\ 0.1644 & -0.9726 & -0.1620 & -0.0274 \\ 0.0000 & 0.1666 & -0.9722 & -0.1643 \\ 0.0000 & 0.0000 & 0.1667 & -0.9860 \end{bmatrix}$$

$$\times \begin{bmatrix} 6.0828 & -0.9864 & 0.0000 & 29.5918 \\ 0.0000 & 6.0023 & -0.9996 & -28.1246 \\ 0.0000 & 0.0000 & 6.0001 & 13.3142 \\ 0.0000 & 0.0000 & 0.0000 & 2.2505 \end{bmatrix}$$

$$= \mathbf{Q}_0\mathbf{R}_0,$$

$$\mathbf{A}_1 = \mathbf{R}_0\mathbf{Q}_0 + (6)\mathbf{I} = \begin{bmatrix} -0.1622 & -0.0266 & 4.9275 & -29.1787 \\ 0.9868 & -0.0044 & -4.6881 & 27.7311 \\ 0.0000 & 0.9996 & 2.3856 & -14.1140 \\ 0.0000 & 0.0000 & 0.3751 & 3.7810 \end{bmatrix},$$

$$\mathbf{A}_1 - (3.7810)\mathbf{I} = \begin{bmatrix} -3.9432 & -0.0266 & 4.9275 & -29.1787 \\ 0.9868 & -3.7854 & -4.6881 & 27.7311 \\ 0.0000 & 0.9996 & -1.3954 & -14.1140 \\ 0.0000 & 0.0000 & 0.3751 & 0.0000 \end{bmatrix}$$

$$= \begin{bmatrix} -0.9701 & -0.2343 & -0.0628 & -0.0106 \\ 0.2428 & -0.9361 & -0.2509 & -0.0423 \\ 0.0000 & 0.2622 & -0.9516 & -0.1604 \\ 0.0000 & 0.0000 & 0.1662 & -0.9861 \end{bmatrix}$$

$$\times \begin{bmatrix} 4.0647 & -0.8931 & -5.9182 & 35.0379 \\ 0.0000 & 3.8120 & 2.8684 & -22.8257 \\ 0.0000 & 0.0000 & 2.2569 & 8.3060 \\ 0.0000 & 0.0000 & 0.0000 & 1.3998 \end{bmatrix}$$

$$= \mathbf{Q}_1\mathbf{R}_1,$$

$$\mathbf{A}_2 = \mathbf{R}_1\mathbf{Q}_1 + (3.7810)\mathbf{I} = \begin{bmatrix} -0.3790 & -1.6681 & 11.4235 & -33.6068 \\ 0.9254 & 0.9646 & -7.4792 & 21.8871 \\ 0.0000 & 0.5918 & 3.0137 & -8.5524 \\ 0.0000 & 0.0000 & 0.2326 & 2.4006 \end{bmatrix}.$$

Continuing in this manner, we generate, after 25 iterations,

$$\mathbf{A}_{25} = \begin{bmatrix} 4.8641 & -4.4404 & 18.1956 & -28.7675 \\ 4.2635 & -2.8641 & 13.3357 & -21.3371 \\ 0.0000 & 0.0000 & 2.7641 & -4.1438 \\ 0.0000 & 0.0000 & 0.3822 & 1.2359 \end{bmatrix},$$

which has form (16) with

$$\mathbf{U} = \begin{bmatrix} 4.8641 & -4.4404 \\ 4.2635 & -2.8641 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} b & c \\ d & e \end{bmatrix} = \begin{bmatrix} 2.7641 & -4.1438 \\ 0.3822 & 1.2359 \end{bmatrix}.$$

The characteristic equation of $\mathbf{U}$ is $\lambda^2 - 2\lambda + 5 = 0$, which has as its roots $1 \pm 2i$; the characteristic equation of the other 2×2 matrix is $\lambda^2 - 4\lambda + 4.9999 = 0$, which has as its roots $2 \pm i$. These roots are the four eigenvalues of $\mathbf{A}_0$. $\square$

Problems 6.4

(1) Use one iteration of the **QR**-algorithm to calculate $\mathbf{A}_1$ when

$$\mathbf{A}_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 18 & -1 & 7 \end{bmatrix}.$$

Note that this matrix differs from the one in Example 1 by a single sign.

(2) Use one iteration of the **QR**-algorithm to calculate $\mathbf{A}_1$ when

$$\mathbf{A}_0 = \begin{bmatrix} 2 & -17 & 7 \\ -17 & -4 & 1 \\ 7 & 1 & -14 \end{bmatrix}.$$

(3) Use one iteration of the **QR**-algorithm to calculate $\mathbf{A}_1$ when

$$\mathbf{A}_0 = \begin{bmatrix} 0 & 0 & 0 & -13 \\ 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -14 \\ 0 & 0 & 1 & 4 \end{bmatrix}.$$

In Problems 4 through 14, use the **QR**-algorithm to calculate the eigenvalues of the given matrices:

(4) The matrix defined in Problem 1.

(5) The matrix defined in Problem 2.

(6) $\begin{bmatrix} 3 & 0 & 0 \\ 2 & 6 & 4 \\ 2 & 3 & 5 \end{bmatrix}$. **(7)** $\begin{bmatrix} 7 & 2 & 0 \\ 2 & 1 & 6 \\ 0 & 6 & 7 \end{bmatrix}$. **(8)** $\begin{bmatrix} 3 & 2 & 3 \\ 2 & 6 & 6 \\ 3 & 6 & 11 \end{bmatrix}$.

(9) $\begin{bmatrix} 2 & 0 & -1 \\ 2 & 3 & 2 \\ -1 & 0 & 2 \end{bmatrix}$. **(10)** $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 5 & -9 & 6 \end{bmatrix}$. **(11)** $\begin{bmatrix} 3 & 0 & 5 \\ 1 & 1 & 1 \\ -2 & 0 & -3 \end{bmatrix}$.

(12) The matrix in Problem 3.

(13) $\begin{bmatrix} 0 & 3 & 2 & -1 \\ 1 & 0 & 2 & -3 \\ 3 & 1 & 0 & -1 \\ 2 & -2 & 1 & 1 \end{bmatrix}$. **(14)** $\begin{bmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{bmatrix}$.

6.5 Least-Squares

Analyzing data for forecasting and predicting future events is common to business, engineering, and the sciences, both physical and social. If such data are plotted, as in Figure 4, they constitute a *scatter diagram*, which may provide insight into the underlying relationship between system variables. For example, the data in Figure 4 appears to follow a straight line relationship reasonably well. The problem then is to determine the equation of the straight line that best fits the data.

A straight line in the variables x and y having the equation

$$y = mx + c, \tag{19}$$

where m and c are constants, will have one y-value on the line for each value of x. This y-value may or may not agree with the data at the same value of x. Thus, for values of x at which data are available, we generally have two values of y, one value from the data and a second value from the straight line approximation to the data. This situation is illustrated in Figure 5. The error at each x, designated as $e(x)$, is the difference between the y-value of the data and the y-value obtained from the straight-line approximation.

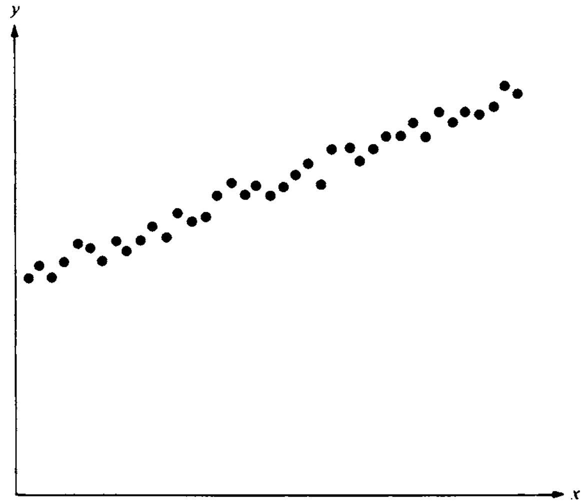


Figure 4.

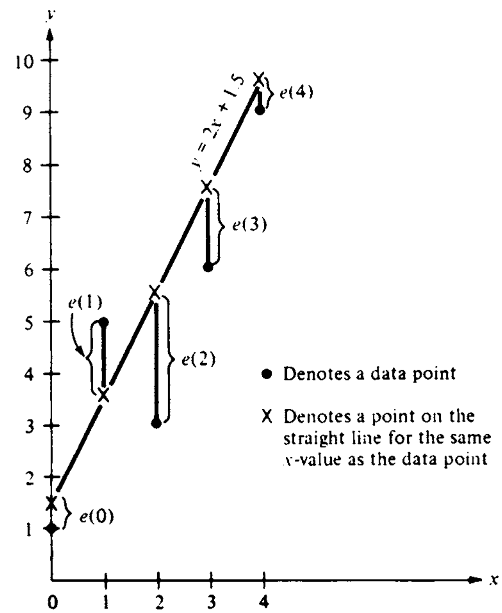


Figure 5.

Table 1.

Given data		Evaluated from $y = 2x + 1.5$
x	y	y
0	1	1.5
1	5	3.5
2	3	5.5
3	6	7.5
4	9	9.5

Example 1

Calculate the errors made in approximating the data given in Figure 5 by the line $y = 2x + 1.5$.

Solution. The line and the given data points are plotted in Figure 5. There are errors at $x = 0$, $x = 1$, $x = 2$, $x = 3$, and $x = 4$. Evaluating the equation $y = 2x + 1.5$ at these values of x, we compute Table 1.

It now follows that

$$e(0) = 1 - 1.5 = -0.5,$$

$$e(1) = 5 - 3.5 = 1.5,$$

$$e(2) = 3 - 5.5 = -2.5,$$

$$e(3) = 6 - 7.5 = -1.5,$$

and

$$e(4) = 9 - 9.5 = -0.5.$$

Note that these errors could have been read directly from the graph. □

We can extend this concept of error to the more general situation involving N data points. Let (x_1, y_1), (x_2, y_2), $(x_3, y_3), \ldots, (x_N, y_N)$ be a set of N data points for a particular situation. Any straight-line approximation to this data generates errors $e(x_1)$, $e(x_2)$, $e(x_3), \ldots, e(x_N)$ which individually can be positive, negative, or zero. The latter case occurs when the approximation agrees with the data at a particular point. We define the overall error as follows.

Definition 1. The *least-squares error* E is the sum of the squares of the individual errors. That is,

$$E = [e(x_1)]^2 + [e(x_2)]^2 + [e(x_3)]^2 + \cdots + [e(x_N)]^2.$$

The only way the total error E can be zero is for each of the individual errors to be zero. Since each term of E is squared, an equal number of positive and negative individual errors cannot sum to zero.

Example 2

Compute the least-squares error for the approximation used in Example 1.

Solution

$$\begin{aligned} E &= [e(0)]^2 + [e(1)]^2 + [e(2)]^2 + [e(3)]^2 + [e(4)]^2 \\ &= (-0.5)^2 + (1.5)^2 + (-2.5)^2 + (-1.5)^2 + (-0.5)^2 \\ &= 0.25 + 2.25 + 6.25 + 2.25 + 0.25 \\ &= 11.25. \qquad \square \end{aligned}$$

Definition 2. The *least-squares straight line* is the line that minimizes the least-squares error.

We seek values of m and c in (19) that minimize the least-squares error. For such a line,

$$e(x_i) = y_i - (mx_i + c),$$

so we want the values for m and c that minimize

$$E = \sum_{i=1}^{N} (y_i - mx_i - c)^2.$$

This occurs when

$$\frac{\partial E}{\partial m} = \sum_{i=1}^{N} 2(y_i - mx_i - c)(-x_i) = 0$$

and

$$\frac{\partial E}{\partial c} = \sum_{i=1}^{N} 2(y_i - mx_i - c)(-1) = 0,$$

or, upon simplifying, when

$$\begin{aligned}\left(\sum_{i=1}^{N} x_i^2\right)m + \left(\sum_{i=1}^{N} x_i\right)c &= \sum_{i=1}^{N} x_i y_i,\\ \left(\sum_{i=1}^{N} x_i\right)m + \qquad Nc &= \sum_{i=1}^{N} y_i.\end{aligned} \tag{20}$$

System (20) makes up the *normal equations* for a least-squares fit in two variables.

Example 3

Find the least-squares straight line for the following x–y data:

x	0	1	2	3	4
y	1	5	3	6	9

Solution. Table 2 contains the required summations.

For this data, the normal equations become

$$\begin{aligned}30m + 10c &= 65,\\ 10m + \ \ 5c &= 24,\end{aligned}$$

which has as its solution $m = 1.7$ and $c = 1.4$. The least-squares straight line is $y = 1.7x + 1.4$. □

Table 2.

	x_i	y_i	$(x_i)^2$	x_iy_i
	0	1	0	0
	1	5	1	5
	2	3	4	6
	3	6	9	18
	4	9	16	36
Sum	$\sum_{i=1}^{5} x_i = 10$	$\sum_{i=1}^{5} y_i = 24$	$\sum_{i=1}^{5} (x_i)^2 = 30$	$\sum_{i=1}^{5} x_iy_i = 65$

The normal equations have a simple matrix representation. Ideally, we would like to choose m and c for (19) so that

$$y_i = mx_i + c$$

for all data pairs (x_i, y_i), $i = 1, 2, \ldots, N$. That is, we want the constants m and c to solve the system

$$\begin{aligned} mx_1 + c &= y_1, \\ mx_2 + c &= y_2, \\ mx_3 + c &= y_3, \\ &\vdots \\ mx_N + c &= y_N, \end{aligned}$$

or, equivalently, the matrix equation

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ x_3 & 1 \\ \vdots & \vdots \\ x_N & 1 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_N \end{bmatrix}.$$

This system has the standard form $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is defined as a matrix having two columns, the first being the data vector $[x_1 \quad x_2 \quad x_3 \quad \cdots \quad x_N]^{\mathsf{T}}$, and the second containing all ones, $\mathbf{x} = [m \quad c]^{\mathsf{T}}$, and $\mathbf{b}$ is the data vector $[y_1 \quad y_2 \quad y_3 \quad \cdots \quad y_N]^{\mathsf{T}}$. In this context, $\mathbf{Ax} = \mathbf{b}$ has a solution for $\mathbf{x}$ if and only if the data falls on a straight line. If not, then the matrix system is inconsistent, and we seek the least-squares solution. That is, we seek the vector $\mathbf{x}$ that minimizes the least-squares error as stipulated in Definition 2, having the matrix form

$$E = \|\mathbf{Ax} - \mathbf{b}\|^2. \tag{21}$$

The solution is the vector $\mathbf{x}$ satisfying the normal equations, which take the matrix form

$$\mathbf{A}^{\mathsf{T}}\mathbf{Ax} = \mathbf{A}^{\mathsf{T}}\mathbf{b}. \tag{22}$$

System (22) is identical to system (20) when $\mathbf{A}$ and $\mathbf{b}$ are defined as above.

We now generalize to all linear systems of the form $\mathbf{Ax} = \mathbf{b}$. We are primarily interested in cases where the system is inconsistent (rendering the methods developed in Chapter 2 useless), and this generally occurs when $\mathbf{A}$ has more rows than columns. We shall place no restrictions on the number of columns in $\mathbf{A}$, but we will assume that *the columns are linearly independent.* We seek the vector $\mathbf{x}$ that minimizes the least-squares error defined by Eq. (21).

Theorem 1. *If* $\mathbf{x}$ *has the property that* $\mathbf{Ax} - \mathbf{b}$ *is orthogonal to the columns of* $\mathbf{A}$, *then* $\mathbf{x}$ *minimizes* $\|\mathbf{Ax} - \mathbf{b}\|^2$.

Proof. For any vector $\mathbf{x}_0$ of appropriate dimension,

$$\begin{aligned}
\|\mathbf{Ax}_0 - \mathbf{b}\|^2 &= \|(\mathbf{Ax}_0 - \mathbf{Ax}) + (\mathbf{Ax} - \mathbf{b})\|^2 \\
&= \langle(\mathbf{Ax}_0 - \mathbf{Ax}) + (\mathbf{Ax} - \mathbf{b}), (\mathbf{Ax}_0 - \mathbf{Ax}) + (\mathbf{Ax} - \mathbf{b})\rangle \\
&= \langle(\mathbf{Ax}_0 - \mathbf{Ax}), (\mathbf{Ax}_0 - \mathbf{Ax})\rangle + \langle(\mathbf{Ax} - \mathbf{b}), (\mathbf{Ax} - \mathbf{b})\rangle \\
&= +2\langle(\mathbf{Ax}_0 - \mathbf{Ax}), (\mathbf{Ax} - \mathbf{b})\rangle \\
&= \|(\mathbf{Ax}_0 - \mathbf{Ax})\|^2 + \|(\mathbf{Ax} - \mathbf{b})\|^2 \\
&= +\,2\langle(\mathbf{Ax}_0, (\mathbf{Ax} - \mathbf{b})\rangle - 2\langle\mathbf{Ax}, (\mathbf{Ax} - \mathbf{b})\rangle.
\end{aligned}$$

It follows directly from Problem 28 of Section 6.2 that the last two inner products are both zero (take $\mathbf{p} = \mathbf{Ax} - \mathbf{b}$). Therefore,

$$\begin{aligned}
\|\mathbf{Ax}_0 - \mathbf{b}\|^2 &= \|(\mathbf{Ax}_0 - \mathbf{Ax})\|^2 + \|(\mathbf{Ax} - \mathbf{b})\|^2 \\
&\geq \|(\mathbf{Ax} - \mathbf{b}\|^2,
\end{aligned}$$

and $\mathbf{x}$ minimizes Eq. (21).

As a consequence of Theorem 1, we seek a vector $\mathbf{x}$ having the property that $\mathbf{Ax} - \mathbf{b}$ is orthogonal to the columns of $\mathbf{A}$. Denoting the columns of $\mathbf{A}$ as $\mathbf{A}_1$, $\mathbf{A}_2, \ldots, \mathbf{A}_n$, respectively, we require

$$\langle\mathbf{A}_i, \mathbf{Ax} - \mathbf{b}\rangle = 0 \qquad (i = 1, 2, \ldots, n).$$

If $\mathbf{y} = [y_1 \quad y_2 \quad \cdots \quad y_n]^{\mathrm{T}}$ denotes an arbitrary vector of appropriate dimension, then

$$\mathbf{Ay} = \mathbf{A}_1 y_1 + \mathbf{A}_2 y_2 + \cdots + \mathbf{A}_n y_n,$$

and

$$\begin{aligned}
\langle\mathbf{Ay}, (\mathbf{Ax} - \mathbf{b})\rangle &= \Big\langle\sum_{i=1}^{n} \mathbf{A}_i y_i, (\mathbf{Ax} - \mathbf{b})\Big\rangle \\
&= \sum_{i=1}^{n} \langle\mathbf{A}_i y_i, (\mathbf{Ax} - \mathbf{b})\rangle \\
&= \sum_{i=1}^{n} y_i \langle\mathbf{A}_i, (\mathbf{Ax} - \mathbf{b})\rangle \\
&= 0.
\end{aligned} \tag{23}$$

It also follows from Problem 39 of Section 6.1 that

$$\langle \mathbf{Ay}, (\mathbf{Ax} - \mathbf{b}) \rangle = \langle \mathbf{y}, \mathbf{A}^{\mathsf{T}}(\mathbf{Ax} - \mathbf{b}) \rangle = \langle \mathbf{y}, (\mathbf{A}^{\mathsf{T}}\mathbf{Ax} - \mathbf{A}^{\mathsf{T}}\mathbf{b}) \rangle. \tag{24}$$

Eqs. (23) and (24) imply that $\langle \mathbf{y}, (\mathbf{A}^{\mathsf{T}}\mathbf{Ax} - \mathbf{A}^{\mathsf{T}}\mathbf{b}) \rangle = \mathbf{0}$ for any $\mathbf{y}$. We may deduce from Problem 26 of Section 6.2 that $\mathbf{A}^{\mathsf{T}}\mathbf{Ax} - \mathbf{A}^{\mathsf{T}}\mathbf{b} = \mathbf{0}$, or $\mathbf{A}^{\mathsf{T}}\mathbf{Ax} = \mathbf{A}^{\mathsf{T}}\mathbf{b}$, *which has the same form as Eq.* (22)! Therefore, a vector $\mathbf{x}$ is the least-squares solution to $\mathbf{Ax} = \mathbf{b}$ if and only if it is the solution to $\mathbf{A}^{\mathsf{T}}\mathbf{Ax} = \mathbf{A}^{\mathsf{T}}\mathbf{b}$. This set of normal equations is guaranteed to have a unique solution whenever the columns of $\mathbf{A}$ are linearly independent, *and it may be solved using any of the methods described in the previous chapters*!

Example 4

Find the least-squares solution to

$$\begin{aligned} x + 2y + z &= 1, \\ 3x - y \phantom{{}+z} &= 2, \\ 2x + y - z &= 2, \\ x + 2y + 2z &= 1. \end{aligned}$$

Solution. This system takes the matrix form $\mathbf{Ax} = \mathbf{b}$, with

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & 0 \\ 2 & 1 & -1 \\ 1 & 2 & 2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}.$$

Then,

$$\mathbf{A}^{\mathsf{T}}\mathbf{A} = \begin{bmatrix} 15 & 3 & 1 \\ 3 & 10 & 5 \\ 1 & 5 & 6 \end{bmatrix} \quad \text{and} \quad \mathbf{A}^{\mathsf{T}}\mathbf{b} = \begin{bmatrix} 12 \\ 4 \\ 1 \end{bmatrix},$$

and the normal equations become

$$\begin{bmatrix} 15 & 3 & 1 \\ 3 & 10 & 5 \\ 1 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 4 \\ 1 \end{bmatrix}.$$

Using Gaussian elimination, we obtain as the unique solution to this set of equations $x = 0.7597$, $y = 0.2607$, and $z = -0.1772$, which is also the least-squares solution to the original system. $\square$

Example 5

Find the least-squares solution to

$$\begin{aligned} 0x + 3y &= 80, \\ 2x + 5y &= 100, \\ 5x - 2y &= 60, \\ -x + 8y &= 130, \\ 10x - y &= 150. \end{aligned}$$

Solution. This system takes the matrix form $\mathbf{Ax} = \mathbf{b}$, with

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 5 & -2 \\ -1 & 8 \\ 10 & -1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 80 \\ 100 \\ 60 \\ 130 \\ 150 \end{bmatrix}.$$

Then,

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} 131 & -15 \\ -15 & 103 \end{bmatrix} \quad \text{and} \quad \mathbf{A}^{\mathrm{T}}\mathbf{b} = \begin{bmatrix} 1950 \\ 1510 \end{bmatrix},$$

and the normal equations become

$$\begin{bmatrix} 131 & -15 \\ -15 & 103 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1950 \\ 1510 \end{bmatrix}.$$

The unique solution to this set of equations is $x = 16.8450$, and $y = 17.1134$, rounded to four decimals, which is also the least-squares solution to the original system. □

Problems 6.5

In Problems 1 through 8, find the least-squares solution to the given systems of equations:

(1) $\begin{aligned} 2x + 3y &= 8, \\ 3x - y &= 5, \\ x + y &= 6. \end{aligned}$

(2) $\begin{aligned} 2x + y &= 8, \\ + y &= 4, \\ -x + y &= 0, \\ 3x + y &= 13. \end{aligned}$

(3)
$$\begin{aligned} x + 3y &= 65, \\ 2x - y &= 0, \\ 3x + y &= 50, \\ 2x + 2y &= 55. \end{aligned}$$

(4)
$$\begin{aligned} 2x + y &= 6, \\ x + y &= 8, \\ -2x + y &= 11, \\ -x + y &= 8, \\ 3x + y &= 4. \end{aligned}$$

(5)
$$\begin{aligned} 2x + 3y - 4z &= 1, \\ x - 2y + 3z &= 3, \\ x + 4y + 2z &= 6, \\ 2x + y - 3z &= 1. \end{aligned}$$

(6)
$$\begin{aligned} 2x + 3y + 2z &= 25, \\ 2x - y + 3z &= 30, \\ 3x + 4y - 2z &= 20, \\ 3x + 5y + 4z &= 55. \end{aligned}$$

(7)
$$\begin{aligned} x + y - z &= 90, \\ 2x + y + z &= 200, \\ x + 2y + 2z &= 320, \\ 3x - 2y - 4z &= 10, \\ 3x + 2y - 3z &= 220. \end{aligned}$$

(8)
$$\begin{aligned} x + 2y + 2z &= 1, \\ 2x + 3y + 2z &= 2, \\ 2x + 4y + 4z &= -2, \\ 3x + 5y + 4z &= 1, \\ x + 3y + 2z &= -1. \end{aligned}$$

(9) Which of the systems, if any, given in Problems 1 through 8 represent a least-squares, straight line fit to data?

(10) The monthly sales figures (in thousands of dollars) for a newly opened shoe store are:

month	1	2	3	4	5
sales	9	16	14	15	21

(a) Plot a scatter diagram for this data.
(b) Find the least-squares straight line that best fits this data.
(c) Use this line to predict sales revenue for month 6.

(11) The number of new cars sold at a new car dealership over the first 8 weeks of the new season are:

month	1	2	3	4	5	6	7	8
sales	51	50	45	46	43	39	35	34

(a) Plot a scatter diagram for this data.
(b) Find the least-squares straight line that best fits this data.
(c) Use this line to predict sales for weeks 9 and 10.

(12) Annual rainfall data (in inches) for a given town over the last seven years are:

year	1	2	3	4	5	6	7
rainfall	10.5	10.8	10.9	11.7	11.4	11.8	12.2

(a) Find the least-squares straight line that best fits this data.
(b) Use this line to predict next year's rainfall.

(13) Solve system (20) algebraically and explain why the solution would be susceptible to round-off error.

(14) **(Coding)** To minimize the round-off error associated with solving the normal equations for a least-squares straight line fit, the (x_i, y_i)-data are coded before using them in calculations. Each x_i-value is replaced by the difference between x_i and the average of all x_i-data. That is, if

$$\bar{X} = \frac{1}{N}\sum_{i=1}^{N} x_i, \quad \text{then set } x_i' = x_i - \bar{X},$$

and fit a straight line to the (x_i', y_i)-data instead.

Explain why this coding scheme avoids the round-off errors associated with uncoded data.

(15) (a) Code the data given in Problem 10 using the procedure described in Problem 14.
(b) Find the least-squares straight line fit for this coded data.

(16) (a) Code the data given in Problem 11 using the procedure described in Problem 14.
(b) Find the least-squares straight line fit for this coded data.

(17) Census figures for the population (in millions of people) for a particular region of the country are as follows:

year	1950	1960	1970	1980	1990
population	25.3	23.5	20.6	18.7	17.8

(a) Code this data using the procedure described in Problem 14, and then find the least-squares straight line that best fits it.
(b) Use this line to predict the population in 2000.

(18) Show that if $\mathbf{A} = \mathbf{QR}$ is a $\mathbf{QR}$-decomposition of $\mathbf{A}$, then the normal equations given by Eq. (22) can be written as $\mathbf{R}^{\mathsf{T}}\mathbf{R}\mathbf{x} = \mathbf{R}^{\mathsf{T}}\mathbf{Q}^{\mathsf{T}}\mathbf{b}$, which reduces to $\mathbf{R}\mathbf{x} = \mathbf{Q}^{\mathsf{T}}\mathbf{b}$. This is a numerically stable set of equations to solve, not subject to the same round-off errors associated with solving the normal equations directly.

(19) Use the procedure described in Problem 18 to solve Problem 1.

(20) Use the procedure described in Problem 18 to solve Problem 2.

(21) Use the procedure described in Problem 18 to solve Problem 5.

(22) Use the procedure described in Problem 18 to solve Problem 6.

(23) Determine the error vector associated with the least-squares solution of Problem 1, and then calculate the inner product of this vector with each of the columns of the coefficient matrix associated with the given set of equations.

(24) Determine the error vector associated with the least-squares solution of Problem 5, and then calculate the inner product of this vector with each of the columns of the coefficient matrix associated with the given set of equations.

Chapter 7

Matrix Calculus

7.1 Well-Defined Functions

The student should be aware of the vast importance of polynomials and exponentials to calculus and differential equations. One should not be surprised to find, therefore, that polynomials and exponentials of matrices play an equally important role in matrix calculus and matrix differential equations. Since we will be interested in using matrices to solve linear differential equations, we shall devote this entire chapter to defining matrix functions, specifically polynomials and exponentials, developing techniques for calculating these functions, and discussing some of their important properties.

Let $p_k(x)$ denote an arbitrary polynomial in x of degree k,

$$p_k(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0, \tag{1}$$

where the coefficients $a_k, a_{k-1}, \ldots, a_1, a_0$ are real numbers. We then define

$$p_k(\mathbf{A}) = a_k\mathbf{A}^k + a_{k-1}\mathbf{A}^{k-1} + \cdots + a_1\mathbf{A} + a_0\mathbf{I}. \tag{2}$$

Recall from Chapter 1, that $\mathbf{A}^2 = \mathbf{A}\cdot\mathbf{A}$, $\mathbf{A}^3 = \mathbf{A}^2\cdot\mathbf{A} = \mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}$ and, in general, $\mathbf{A}^k = \mathbf{A}^{k-1}\cdot\mathbf{A}$. Also $\mathbf{A}^0 = \mathbf{I}$.

Two observations are now immediate. Whereas a_0 in (1) is actually multiplied by $x^0 = 1$, a_0 in (2) is multiplied by $\mathbf{A}^0 = \mathbf{I}$. Also, if $\mathbf{A}$ is an $n \times n$ matrix, then $p_k(\mathbf{A})$ is an $n \times n$ matrix since the right-hand side of (2) may be summed.

Example 1

Find $p_2(\mathbf{A})$ for

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

if $p_2(x) = 2x^2 + 3x + 4$.

Solution. In this case, $p_2(\mathbf{A}) = 2\mathbf{A}^2 + 3\mathbf{A} + 4\mathbf{I}$. Thus,

$$p_2(\mathbf{A}) = 2\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^2 + 3\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= 2\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 2 \\ 0 & 4 & 3 \\ 0 & 0 & 4 \end{bmatrix}.$$

Note that had we defined $p_2(\mathbf{A}) = 2\mathbf{A}^2 + 3\mathbf{A} + 4$ (that is, without the $\mathbf{I}$ term), we could not have performed the addition since addition of a matrix and a scalar is undefined. □

Since a matrix commutes with itself, many of the properties of polynomials (addition, subtraction, multiplication, and factoring but *not* division) are still valid for polynomials of a matrix. For instance, if $f(x)$, $d(x)$, $q(x)$, and $r(x)$ represent polynomials in x and if

$$f(x) = d(x)q(x) + r(x) \tag{3}$$

then it must be the case that

$$f(\mathbf{A}) = d(\mathbf{A})q(\mathbf{A}) + r(\mathbf{A}). \tag{4}$$

Equation (4) follows from (3) only because $\mathbf{A}$ commutes with itself; thus, we multiply together two polynomials in $\mathbf{A}$ precisely in the same manner that we multiply together two polynomials in x.

If we recall from calculus that many functions can be written as a Maclaurin series, then we can define functions of matrices quite easily. For instance, the Maclaurin series for e^x is

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots. \tag{5}$$

Thus, we define the exponential of a matrix $\mathbf{A}$ as

$$e^{\mathbf{A}} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!} = \mathbf{I} + \frac{\mathbf{A}}{1!} + \frac{\mathbf{A}^2}{2!} + \frac{\mathbf{A}^3}{3!} + \cdots. \tag{6}$$

The question of convergence now arises. For an infinite series of matrices we define convergence as follows:

Definition 1. A sequence $\{\mathbf{B}_k\}$ of matrices, $\mathbf{B}_k = [b_{ij}^k]$, is said to *converge* to a matrix $\mathbf{B} = [b_{ij}]$ if the elements b_{ij}^k converge to b_{ij} for every i and j.

Definition 2. The infinite series $\sum_{n=0}^{\infty} \mathbf{B}_n$, converges to $\mathbf{B}$ if the sequence $\{\mathbf{S}_k\}$ of partial sums, where $\mathbf{S}_k = \sum_{n=0}^{k} \mathbf{B}_n$, converges to $\mathbf{B}$.

It can be shown (see Theorem 1, this section) that the infinite series given in (6) converges for any matrix $\mathbf{A}$. Thus $e^{\mathbf{A}}$ is defined for every matrix.

Example 2

Find $e^{\mathbf{A}}$ if

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Solution.

$$\begin{aligned} e^{\mathbf{A}} = e^{\left[\begin{smallmatrix} 2 & 0 \\ 0 & 0 \end{smallmatrix}\right]} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{1!}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}^2 + \frac{1}{3!}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}^3 + \cdots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2/1! & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2^2/2! & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2^3/3! & 0 \\ 0 & 0 \end{bmatrix} + \cdots \\ &= \begin{bmatrix} \sum_{k=0}^{\infty} 2^k/k! & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^2 & 0 \\ 0 & e^0 \end{bmatrix}. \qquad \square \end{aligned}$$

In general, if $\mathbf{A}$ is the diagonal matrix

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix},$$

then we can show (see Problem 12) that

$$e^{\mathbf{A}} = \begin{bmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & e^{\lambda n} \end{bmatrix}. \tag{7}$$

If $\mathbf{A}$ is not a diagonal matrix, then it is very difficult to find $e^{\mathbf{A}}$ directly from the definition given in (6). For an arbitrary $\mathbf{A}$, $e^{\mathbf{A}}$ does not have the form exhibited in (7). For example, if

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix},$$

it can be shown (however, not yet by us) that

$$e^{\mathbf{A}} = \frac{1}{6}\begin{bmatrix} 2e^5 + 4e^{-1} & 2e^5 - 2e^{-1} \\ 4e^5 - 4e^{-1} & 4e^5 + 2e^{-1} \end{bmatrix}$$

For the purposes of this book, the exponential is the only function that is needed. However, it may be of some value to know how other functions of matrices, sines, cosines, etc., are defined. The following theorem, the proof of which is beyond the scope of this book, provides this information.

Theorem 1. *Let z represent the complex variable $x + iy$. If $f(z)$ has the Taylor series $\sum_{k=0}^{\infty} a_k z^k$, which converges for $|z| < R$, and if the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of an $n \times n$ matrix $\mathbf{A}$ have the property that $|\lambda_i| < R (i = 1, 2, \ldots, n)$, then $\sum_{k=0}^{\infty} a_k \mathbf{A}^k$ will converge to an $n \times n$ matrix which is defined to be $f(\mathbf{A})$. In such a case, $f(\mathbf{A})$ is said to be well defined.*

Example 3

Define $\sin \mathbf{A}$.

Solution. A Taylor series for $\sin z$ is

$$\begin{aligned} \sin z &= \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} \\ &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots. \end{aligned}$$

This series can be shown to converge for all z (that is, $R = \infty$). Hence, since any eigenvalue λ of $\mathbf{A}$ must have the property $|\lambda| < \infty$ (that is, λ is finite) $\sin \mathbf{A}$

can be defined for every $\mathbf{A}$ as

$$\sin \mathbf{A} = \sum_{k=0}^{\infty} \frac{(-1)^k \mathbf{A}^{2k+1}}{(2k+1)!} = \mathbf{A} - \frac{\mathbf{A}^3}{3!} + \frac{\mathbf{A}^5}{5!} - \frac{\mathbf{A}^7}{7!} + \cdots \quad \square \qquad (8)$$

Problems 7.1

(1) Let $q(x) = x - 1$. Find $p_k(\mathbf{A})$ and $q(\mathbf{A})p_k(\mathbf{A})$ if

(a) $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$, $\quad k = 2$, and $\quad p_2(x) = x^2 - 2x + 1$,

(b) $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\quad k = 3$, and $\quad p_3(x) = 2x^3 - 3x^2 + 4$.

(2) If $p_k(x)$ is defined by (1), find $p_k(\mathbf{A})$ for the diagonal matrix

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}. \text{ Can you generalize?}$$

(3) By actually computing both sides of the following equation separately, verify that $(\mathbf{A} - 3\mathbf{I})(\mathbf{A} + 2\mathbf{I}) = \mathbf{A}^2 - \mathbf{A} - 6\mathbf{I}$ for

(a) $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\qquad$ (b) $\mathbf{A} = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & 1 \\ -2 & -2 & 3 \end{bmatrix}$.

The above equation is an example of matrix factoring.

(4) Although $x^2 - y^2 = (x - y)(x + y)$ whenever x and y denote real-valued variables, show by example that $\mathbf{A}^2 - \mathbf{B}^2$ need not equal the product $(\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{B})$ whenever $\mathbf{A}$ and $\mathbf{B}$ denote 2×2 real matrices. Why?

(5) It is known that $x^2 - 5x + 6$ factors into the product $(x - 2)(x - 3)$ whenever x denotes a real-valued variable. Is it necessarily true that $\mathbf{A}^2 - 5\mathbf{A} + 6\mathbf{I} = (\mathbf{A} - 2\mathbf{I})(\mathbf{A} - 3\mathbf{I})$ whenever $\mathbf{A}$ represents a square real matrix? Why?

(6) Determine $\lim_{k \to \infty} \mathbf{B}_k$ when

$$\mathbf{B}_k = \begin{bmatrix} \dfrac{1}{k} & 2 - \dfrac{2}{k^2} \\ 3 & (0.5)^k \end{bmatrix}.$$

(7) Determine $\lim_{k\to\infty} \mathbf{B}_k$ when

$$\mathbf{B}_k = \begin{bmatrix} \dfrac{2k}{k+1} \\[2ex] \dfrac{k+3}{k^2-2k+1} \\[2ex] \dfrac{3k^2+2k}{2k^2} \end{bmatrix}.$$

(8) Determine $\lim_{k\to\infty} \mathbf{D}_k$ when

$$\mathbf{D}_k = \begin{bmatrix} (0.2)^k & 1 & (0.1)^k \\ 4 & 3^k & 0 \end{bmatrix}.$$

(9) It is known that $\arctan(z) = \sum_{n=0}^{\infty}[(-1)^n/(2n+1)]z^{2n+1}$ converges for all $|z| < \pi/2$. Determine for which of the following matrices $\mathbf{A}$, $\arctan(\mathbf{A}) = \sum_{n=0}^{\infty}[(-1)^n/2n+1]\mathbf{A}^{2n+1}$ is well defined:

(a) $\begin{bmatrix} -3 & 6 \\ -2 & 4 \end{bmatrix}$. (b) $\begin{bmatrix} 5 & -4 \\ 6 & -5 \end{bmatrix}$. (c) $\begin{bmatrix} 6 & -5 \\ 2 & -1 \end{bmatrix}$.

(d) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$. (e) $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 5 \\ 0 & -1 & -3 \end{bmatrix}$. (f) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{1}{8} & \frac{3}{4} \end{bmatrix}$.

(10) It is known that $\ln(1+z) = \sum_{n=0}^{\infty}[(-1)^{n+1}/n]z^n$ converges for all $|z| < 1$. Determine for which of the matrices given in Problem 9 $\ln(\mathbf{I}+\mathbf{A}) = \sum_{n=0}^{\infty}[(-1)^{n+1}/n]\mathbf{A}^n$ is well defined.

(11) It is known that $f(z) = \sum_{n=0}^{\infty} z^n/3^n$ converges for all $|z| < 3$. Determine for which of the matrices given in Problem 9 $f(\mathbf{A}) = \sum_{n=0}^{\infty} \mathbf{A}^n/3^n$ is well defined.

(12) Derive Eq. (7).

(13) Find $e^{\mathbf{A}}$ when

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

(14) Find $e^{\mathbf{A}}$ when

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & 28 \end{bmatrix}.$$

(15) Find $e^{\mathbf{A}}$ when

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(16) Derive an expression for sin(**A**) similar to Eq. (7) when **A** is a square diagonal matrix.

(17) Find sin(**A**) for the matrix given in Problem 13.

(18) Find sin(**A**) for the matrix given in Problem 14.

(19) Using Theorem 1, give a definition for cos **A** and use this definition to find

$$\cos\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

(20) Find cos(**A**) for the matrix given in Problem 15.

7.2 Cayley–Hamilton Theorem

We now state one of the most powerful theorems of matrix theory, the proof of which is in the appendix to this chapter.

Cayley–Hamilton Theorem. *A matrix satisfies its own characteristic equation. That is, if the characteristic equation of an $n \times n$ matrix* **A** *is* $\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$, then

$$\mathbf{A}^n + a_{n-1}\mathbf{A}^{n-1} + \cdots + a_1\mathbf{A} + a_0\mathbf{I} = \mathbf{0}.$$

Note once again that when we change a scalar equation to a matrix equation, the unity element 1 is replaced by the identity matrix **I**.

Example 1

Verify the Cayley–Hamilton theorem for

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

Solution. The characteristic equation for **A** is $\lambda^2 - 4\lambda - 5 = 0$.

$$\mathbf{A}^2 - 4\mathbf{A} - 5\mathbf{I} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 \\ 16 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 8 \\ 16 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9-4-5 & 8-8-0 \\ 16-16-0 & 17-12-5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}. \qquad \square$$

Example 2

Verify the Cayley–Hamilton theorem for

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & 0 & 4 \end{bmatrix}.$$

Solution. The characteristic equation of $\mathbf{A}$ is $(3 - \lambda)(-\lambda)(4 - \lambda) = 0$.

$(3\mathbf{I} - \mathbf{A})(-\mathbf{A})(4\mathbf{I} - \mathbf{A})$

$$= \left(\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & 0 & 4 \end{bmatrix}\right)\left(-\begin{bmatrix} 3 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & 0 & 4 \end{bmatrix}\right)$$
$$\left(\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & 0 & 4 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ -2 & 3 & -1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} -3 & 0 & 1 \\ -2 & 0 & -1 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ -2 & 4 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ -2 & 3 & -1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} -3 & 0 & -3 \\ -2 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0}. \qquad \square$$

One immediate consequence of the Cayley–Hamilton theorem is a new method for finding the inverse of a nonsingular matrix. If

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$

is the characteristic equation of a matrix $\mathbf{A}$, it follows from Problem 17 of Section 5.4 that $\det(\mathbf{A}) = (-1)^n a_0$. Thus, $\mathbf{A}$ is invertible if and only if $a_0 \neq 0$.

Now assume that $a_0 \neq 0$. By the Cayley–Hamilton theorem, we have

$$\mathbf{A}^n + a_{n-1}\mathbf{A}^{n-1} + \cdots + a_1\mathbf{A} + a_0\mathbf{I} = \mathbf{0},$$

$$\mathbf{A}[\mathbf{A}^{n-1} + a_{n-1}\mathbf{A}^{n-2} + \cdots + a_1\mathbf{I}] = -a_0\mathbf{I},$$

or

$$\mathbf{A}\left[-\frac{1}{a_0}(\mathbf{A}^{n-1} + a_{n-1}\mathbf{A}^{n-2} + \cdots + a_1\mathbf{I})\right] = \mathbf{I}.$$

Thus, $(-1/a_0)(\mathbf{A}^{n-1} + a_{n-1}\mathbf{A}^{n-2} + \cdots + a_1\mathbf{I})$ is an inverse of $\mathbf{A}$. But since the inverse is unique (see Theorem 2 of Section 3.4), we have that

$$\mathbf{A}^{-1} = \frac{-1}{a_0}(\mathbf{A}^{n-1} + a_{n-1}\mathbf{A}^{n-2} + \cdots + a_1\mathbf{I}). \tag{9}$$

Example 3

Using the Cayley–Hamilton theorem, find $\mathbf{A}^{-1}$ for

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 4 \\ 0 & -1 & 2 \\ 2 & 0 & 3 \end{bmatrix}.$$

Solution. The characteristic equation for $\mathbf{A}$ is $\lambda^3 - 3\lambda^2 - 9\lambda + 3 = 0$. Thus, by the Cayley–Hamilton theorem,

$$\mathbf{A}^3 - 3\mathbf{A}^2 - 9\mathbf{A} + 3\mathbf{I} = \mathbf{0}.$$

Hence

$$\mathbf{A}^3 - 3\mathbf{A}^2 - 9\mathbf{A} = -3\mathbf{I},$$

$$\mathbf{A}(\mathbf{A}^2 - 3\mathbf{A} - 9\mathbf{I}) = -3\mathbf{I},$$

or,

$$\mathbf{A}(\tfrac{1}{3})(-\mathbf{A}^2 + 3\mathbf{A} + 9\mathbf{I}) = \mathbf{I}.$$

Thus,

$$\mathbf{A}^{-1} = (\tfrac{1}{3})(-\mathbf{A}^2 + 3\mathbf{A} + 9\mathbf{I})$$

$$= \frac{1}{3}\left(\begin{bmatrix} -9 & 0 & -12 \\ -4 & -1 & -4 \\ -8 & 4 & -17 \end{bmatrix} + \begin{bmatrix} 3 & -6 & 12 \\ 0 & -3 & 6 \\ 6 & 0 & 9 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix} 3 & -6 & 0 \\ -4 & 5 & 2 \\ -2 & 4 & 1 \end{bmatrix}.$$

□

Problems 7.2

Verify the Cayley–Hamilton theorem and use it to find $\mathbf{A}^{-1}$, where possible, for:

(1) $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, **(2)** $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$,

(3) $\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ 4 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix}$, **(4)** $\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix}$,

(5) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

7.3 Polynomials of Matrices—Distinct Eigenvalues

In general, it is very difficult to compute functions of matrices from their definition as infinite series (one exception is the diagonal matrix). The Cayley–Hamilton theorem, however, provides a starting point for the development of an alternate, straightforward method for calculating these functions. In this section, we shall develop the method for polynomials of matrices having distinct eigenvalues. In the ensuing sections, we shall extend the method to functions of matrices having arbitrary eigenvalues.

Let $\mathbf{A}$ represent an $n \times n$ matrix. Define $d(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I})$. Thus, $d(\lambda)$ is an nth degree polynomial in λ and the characteristic equation of $\mathbf{A}$ is $d(\lambda) = 0$. From Chapter 5, we know that if λ_i is an eigenvalue of $\mathbf{A}$, then λ_i is a root of the characteristic equation, hence

$$d(\lambda_i) = 0. \tag{10}$$

From the Cayley–Hamilton theorem, we know that a matrix must satisfy its own characteristic equation, hence

$$d(\mathbf{A}) = \mathbf{0}. \tag{11}$$

Let $f(\mathbf{A})$ be any matrix polynomial of arbitrary degree that we wish to compute. $f(\lambda)$ represents the corresponding polynomial of λ. A theorem of algebra states that there exist polynomials $q(\lambda)$ and $r(\lambda)$ such that

$$f(\lambda) = d(\lambda)q(\lambda) + r(\lambda), \tag{12}$$

where $r(\lambda)$ is called the remainder. The degree of $r(\lambda)$ is less than that of $d(\lambda)$, which is n, and must be less than or equal to the degree of $f(\lambda)$ (why?).

Example 1

Find $q(\lambda)$ and $r(\lambda)$ if $f(\lambda) = \lambda^4 + 2\lambda^3 - 1$ and $d(\lambda) = \lambda^2 - 1$.

Solution. For $\lambda \neq \pm 1$, $d(\lambda) \neq 0$. Dividing $f(\lambda)$ by $d(\lambda)$, we obtain

$$\frac{f(\lambda)}{d(\lambda)} = \frac{\lambda^4 + 2\lambda^3 - 1}{\lambda^2 - 1} = (\lambda^2 + 2\lambda + 1) + \frac{2\lambda}{\lambda^2 - 1},$$

$$\frac{f(\lambda)}{d(\lambda)} = (\lambda^2 + 2\lambda + 1) + \frac{2\lambda}{d(\lambda)},$$

or

$$f(\lambda) = d(\lambda)(\lambda^2 + 2\lambda + 1) + (2\lambda). \tag{13}$$

If we define $q(\lambda) = \lambda^2 + 2\lambda + 1$ and $r(\lambda) = 2\lambda$, (13) has the exact form of (12) for all λ except possibly $\lambda = \pm 1$. However, by direct substitution, we find that (13) is also valid for $\lambda = \pm 1$; hence (13) is an identity for all λ. □

From (12), (3), and (4), we have

$$f(\mathbf{A}) = d(\mathbf{A})q(\mathbf{A}) + r(\mathbf{A}). \tag{14}$$

Using (11), we obtain

$$f(\mathbf{A}) = r(\mathbf{A}). \tag{15}$$

Therefore, it follows that any polynomial in $\mathbf{A}$ may be written as a polynomial of degree $n - 1$ or less. For example, if $\mathbf{A}$ is a 4×4 matrix and if we wish to compute $f(\mathbf{A}) = \mathbf{A}^{957} - 3\mathbf{A}^{59} + 2\mathbf{A}^3 - 4\mathbf{I}$, then (15) implies that $f(\mathbf{A})$ can be written as a polynomial of degree three or less in $\mathbf{A}$, that is,

$$\mathbf{A}^{957} - 3\mathbf{A}^{59} + 2\mathbf{A}^3 - 4\mathbf{I} = \alpha_3\mathbf{A}^3 + \alpha_2\mathbf{A}^2 + \alpha_1\mathbf{A} + \alpha_0\mathbf{I} \tag{16}$$

where $\alpha_3, \alpha_2, \alpha_1, \alpha_0$ are scalars that still must be determined. Once $\alpha_3, \alpha_2, \alpha_1, \alpha_0$ are computed, the student should observe that it is much easier to calculate the right side rather than the left side of (16).

If $\mathbf{A}$ is an $n \times n$ matrix, then $r(\lambda)$ will be a polynomial having the form

$$r(\lambda) = \alpha_{n-1}\lambda^{n-1} + \alpha_{n-2}\lambda^{n-2} + \cdots + \alpha_1\lambda + \alpha_0. \tag{17}$$

If λ_i is an eigenvalue of $\mathbf{A}$, then we have, after substituting (10) into (12), that

$$f(\lambda_i) = r(\lambda_i). \tag{18}$$

Thus, using (17), Eq. (18) may be rewritten as

$$f(\lambda_i) = \alpha_{n-1}(\lambda_i)^{n-1} + \alpha_{n-2}(\lambda_i)^{n-2} + \cdots + \alpha_1(\lambda_i) + \alpha_0 \tag{19}$$

if λ_i is an eigenvalue.

If we now assume that $\mathbf{A}$ has distinct eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_n$ (note that if the eigenvalues are distinct, there must be n of them), then (19) may be used to generate n simultaneous linear equations for the n unknowns $\alpha_{n-1}, \alpha_{n-2}, \ldots, \alpha_1, \alpha_0$:

$$\begin{aligned}
f(\lambda_1) = r(\lambda_1) &= \alpha_{n-1}(\lambda_1)^{n-1} + \alpha_{n-2}(\lambda_1)^{n-2} + \cdots + \alpha_1(\lambda_1) + \alpha_0, \\
f(\lambda_2) = r(\lambda_2) &= \alpha_{n-1}(\lambda_2)^{n-1} + \alpha_{n-2}(\lambda_2)^{n-2} + \cdots + \alpha_1(\lambda_2) + \alpha_0, \\
&\vdots \\
f(\lambda_n) = r(\lambda_n) &= \alpha_{n-1}(\lambda_n)^{n-1} + \alpha_{n-2}(\lambda_n)^{n-2} + \cdots + \alpha_1(\lambda_n) + \alpha_0.
\end{aligned} \tag{20}$$

Note that $f(\lambda)$ and the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are assumed known; hence $f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n)$ are known, and the only unknowns in (20) are $\alpha_{n-1}, \alpha_{n-2}, \ldots, \alpha_1, \alpha_0$.

Example 2

Find $\mathbf{A}^{593}$ if

$$\mathbf{A} = \begin{bmatrix} -3 & -4 \\ 2 & 3 \end{bmatrix}.$$

Solution. The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 1$, $\lambda_2 = -1$. For this example, $f(\mathbf{A}) = \mathbf{A}^{593}$, thus, $f(\lambda) = \lambda^{593}$. Since $\mathbf{A}$ is a 2×2 matrix, $r(\mathbf{A})$ will be a polynomial of degree $(2 - 1) = 1$ or less, hence $r(\mathbf{A}) = \alpha_1\mathbf{A} + \alpha_0\mathbf{I}$ and $r(\lambda) = \alpha_1\lambda + \alpha_0$. From (15), we have that $f(\mathbf{A}) = r(\mathbf{A})$, thus, for this example,

$$\mathbf{A}^{593} = \alpha_1\mathbf{A} + \alpha_0\mathbf{I}. \tag{21}$$

From (18), we have that $f(\lambda_i) = r(\lambda_i)$ if λ_i is an eigenvalue of $\mathbf{A}$; thus, for this example, $(\lambda_i)^{593} = \alpha_1\lambda_i + \alpha_0$. Substituting the eigenvalues of $\mathbf{A}$ into this equation, we obtain the following system for α_1 and α_0.

$$(1)^{593} = \alpha_1(1) + \alpha_0,$$
$$(-1)^{593} = \alpha_1(-1) + \alpha_0,$$

or

$$\begin{aligned} 1 &= \alpha_1 + \alpha_0, \\ -1 &= -\alpha_1 + \alpha_0. \end{aligned} \tag{22}$$

Solving (22), we obtain $\alpha_0 = 0$, $\alpha_1 = 1$. Substituting these values into (21), we obtain $\mathbf{A}^{593} = 1 \cdot \mathbf{A} + 0 \cdot \mathbf{I}$ or

$$\begin{bmatrix} -3 & -4 \\ 2 & 3 \end{bmatrix}^{593} = \begin{bmatrix} -3 & -4 \\ 2 & 3 \end{bmatrix}. \quad \square$$

Example 3

Find $\mathbf{A}^{39}$ if

$$\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}.$$

Solution. The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 5$, $\lambda_2 = 2$. For this example, $f(\mathbf{A}) = \mathbf{A}^{39}$, thus $f(\lambda) = \lambda^{39}$. Since $\mathbf{A}$ is a 2×2 matrix, $r(\mathbf{A})$ will be a polynomial of degree 1 or less, hence $r(\mathbf{A}) = \alpha_1\mathbf{A} + \alpha_0\mathbf{I}$ and $r(\lambda) = \alpha_1\lambda + \alpha_0$. From (15), we have that $f(\mathbf{A}) = r(\mathbf{A})$, thus, for this example,

$$\mathbf{A}^{39} = \alpha_1\mathbf{A} + \alpha_0\mathbf{I}. \tag{23}$$

From (18) we have that $f(\lambda_i) = r(\lambda_i)$ if λ_i is an eigenvalue of $\mathbf{A}$, thus for this example, $(\lambda_i)^{39} = \alpha_1\lambda_i + \alpha_0$. Substituting the eigenvalues of $\mathbf{A}$ into this equation, we obtain the following system for α_1 and α_0:

$$\begin{aligned} 5^{39} &= 5\alpha_1 + \alpha_0, \\ 2^{39} &= 2\alpha_1 + \alpha_0. \end{aligned} \tag{24}$$

Solving (24), we obtain

$$\alpha_1 = \frac{5^{39} - 2^{39}}{3}, \qquad \alpha_0 = \frac{-2(5)^{39} + 5(2)^{39}}{3},$$

Substituting these values into (23), we obtain

$$\mathbf{A}^{39} = \frac{5^{39} - 2^{39}}{3}\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} + \frac{-2(5)^{39} + 5(2)^{39}}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$= \frac{1}{3}\begin{bmatrix} 2(5)^{39} + 2^{39} & 5^{39} - 2^{39} \\ 2(5)^{39} - 2(2)^{39} & 5^{39} + 2(2)^{39} \end{bmatrix}. \tag{25}$$

The number 5^{39} and 2^{39} can be determined on a calculator. For our purposes, however, the form of (25) is sufficient and no further simplification is required. □

Example 4

Find $\mathbf{A}^{602} - 3\mathbf{A}^3$ if

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & -2 \\ 0 & 0 & 0 \\ 0 & -3 & 3 \end{bmatrix}.$$

Solution. The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 3$.

$$f(\mathbf{A}) = \mathbf{A}^{602} - 3\mathbf{A}^3, \qquad r(\mathbf{A}) = \alpha_2\mathbf{A}^2 + \alpha_1\mathbf{A} + \alpha_0\mathbf{I},$$

$$f(\lambda) = \lambda^{602} - 3\lambda^3, \qquad r(\lambda) = \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0.$$

Note that since $\mathbf{A}$ is a 3×3 matrix, $r(\mathbf{A})$ must be no more than a second degree polynomial. Now

$$f(\mathbf{A}) = r(\mathbf{A});$$

thus,

$$\mathbf{A}^{602} - 3\mathbf{A}^3 = \alpha_2\mathbf{A}^2 + \alpha_1\mathbf{A} + \alpha_0\mathbf{I}. \tag{26}$$

If λ_i is an eigenvalue of $\mathbf{A}$, then $f(\lambda_i) = r(\lambda_i)$. Thus,

$$(\lambda_i)^{602} - 3(\lambda_i)^3 = \alpha_2(\lambda_i)^2 + \alpha_1\lambda_i + \alpha_0;$$

hence,

$$(0)^{602} - 3(0)^3 = \alpha_2(0)^2 + \alpha_1(0) + \alpha_0,$$

$$(1)^{602} - 3(1)^3 = \alpha_2(1)^2 + \alpha_1(1) + \alpha_0,$$

$$(3)^{602} - 3(3)^3 = \alpha_2(3)^2 + \alpha_1(3) + \alpha_0,$$

or

$$0 = \alpha_0,$$
$$-2 = \alpha_2 + \alpha_1 + \alpha_0,$$
$$3^{602} - 81 = 9\alpha_2 + 3\alpha_1 + \alpha_0.$$

Thus,

$$\alpha_2 = \frac{3^{602} - 75}{6}, \qquad \alpha_1 = \frac{-(3)^{602} + 63}{6}, \qquad \alpha_0 = 0. \tag{27}$$

Substituting (27) into (26), we obtain

$$\mathbf{A}^{602} - 3\mathbf{A}^3 = \frac{3^{602} - 75}{6}\begin{bmatrix} 1 & 10 & -8 \\ 0 & 0 & 0 \\ 0 & -9 & 9 \end{bmatrix} + \frac{-(3)^{602} + 63}{6}\begin{bmatrix} 1 & 4 & -2 \\ 0 & 0 & 0 \\ 0 & -3 & 3 \end{bmatrix}$$
$$= \frac{1}{6}\begin{bmatrix} -12 & 6(3)^{602} - 498 & -6(3)^{602} + 474 \\ 0 & 0 & 0 \\ 0 & -6(3)^{602} + 486 & 6(3)^{602} - 486 \end{bmatrix}. \qquad \square$$

Finally, the student should note that if the polynomial to be calculated is already of degree less than or equal to $n - 1$, then this method affords no simplification and the polynomial must still be computed directly.

Problems 7.3

(1) Specialize system (20) for $f(\mathbf{A}) = \mathbf{A}^7$ and

$$\mathbf{A} = \begin{bmatrix} -2 & 3 \\ -1 & 2 \end{bmatrix}.$$

Solve this system and use the results to determine $\mathbf{A}^7$. Check your answer by direct calculations.

(2) Find $\mathbf{A}^{50}$ for the matrix $\mathbf{A}$ given in Problem 1.

(3) Specialize system (20) for $f(\mathbf{A}) = \mathbf{A}^{735}$ and

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}.$$

Solve this system and use the results to determine $\mathbf{A}^{735}$.

(4) Specialize system (20) for $f(\mathbf{A}) = \mathbf{A}^{20}$ and

$$\mathbf{A} = \begin{bmatrix} -3 & 6 \\ -1 & 2 \end{bmatrix}.$$

Solve this system and use the results to determine $\mathbf{A}^{20}$.

(5) Find $\mathbf{A}^{97}$ for the matrix $\mathbf{A}$ given in Problem 4.

(6) Find $\mathbf{A}^{50}$ for the matrix $\mathbf{A}$ given in Example 3.

(7) Specialize system (20) for $f(\mathbf{A}) = \mathbf{A}^{78}$ and

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 2 & 5 \end{bmatrix}.$$

Solve this system and use the results to determine $\mathbf{A}^{78}$.

(8) Find $\mathbf{A}^{41}$ for the matrix $\mathbf{A}$ given in Problem 7.

(9) Specialize system (20) for $f(\mathbf{A}) = \mathbf{A}^{222}$ and

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$

Solve this system and use the results to determine $\mathbf{A}^{222}$.

(10) Specialize system (20) for $f(\mathbf{A}) = \mathbf{A}^{17}$, when $\mathbf{A}$ is a 3×3 matrix having 3, 5, and 10 as its eigenvalues.

(11) Specialize system (20) for $f(\mathbf{A}) = \mathbf{A}^{25}$, when $\mathbf{A}$ is a 4×4 matrix having 2, -2, 3, and 4 as its eigenvalues.

(12) Specialize system (20) for $f(\mathbf{A}) = \mathbf{A}^{25}$, when $\mathbf{A}$ is a 4×4 matrix having 1, -2, 3, and -4 as its eigenvalues.

(13) Specialize system (20) for $f(\mathbf{A}) = \mathbf{A}^{8}$, when $\mathbf{A}$ is a 5×5 matrix having 1, -1, 2, -2, and 3 as its eigenvalues.

(14) Specialize system (20) for $f(\mathbf{A}) = \mathbf{A}^8 - 3\mathbf{A}^5 + 5\mathbf{I}$, when $\mathbf{A}$ is the matrix described in Problem 10.

(15) Specialize system (20) for $f(\mathbf{A}) = \mathbf{A}^8 - 3\mathbf{A}^5 + 5\mathbf{I}$, when $\mathbf{A}$ is the matrix described in Problem 11.

(16) Specialize system (20) for $f(\mathbf{A}) = \mathbf{A}^8 - 3\mathbf{A}^5 + 5\mathbf{I}$, when $\mathbf{A}$ is the matrix described in Problem 12.

(17) Specialize system (20) for $f(\mathbf{A}) = \mathbf{A}^{10} + 6\mathbf{A}^3 + 8\mathbf{A}$, when $\mathbf{A}$ is the matrix described in Problem 12.

(18) Specialize system (20) for $f(\mathbf{A}) = \mathbf{A}^{10} + 6\mathbf{A}^3 + 8\mathbf{A}$, when $\mathbf{A}$ is the matrix described in Problem 13.

(19) Find $\mathbf{A}^{202} - 3\mathbf{A}^{147} + 2\mathbf{I}$ for the $\mathbf{A}$ of Problem 1.

(20) Find $\mathbf{A}^{1025} - 4\mathbf{A}^5$ for the $\mathbf{A}$ of Problem 1.

(21) Find $\mathbf{A}^8 - 3\mathbf{A}^5 - \mathbf{I}$ for the matrix given in Problem 7.

(22) Find $\mathbf{A}^{13} - 12\mathbf{A}^9 + 5\mathbf{I}$ for

$$\mathbf{A} = \begin{bmatrix} 3 & -5 \\ 1 & -3 \end{bmatrix}.$$

(23) Find $\mathbf{A}^{10} - 2\mathbf{A}^5 + 10\mathbf{I}$ for the matrix given in Problem 22.

(24) Find $\mathbf{A}^{593} - 2\mathbf{A}^{15}$ for

$$\mathbf{A} = \begin{bmatrix} -2 & 4 & 3 \\ 0 & 0 & 0 \\ -1 & 5 & 2 \end{bmatrix}.$$

(25) Specialize system (20) for $f(\mathbf{A}) = \mathbf{A}^{12} - 3\mathbf{A}^9 + 2\mathbf{A} + 5\mathbf{I}$ and

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 4 & 1 \end{bmatrix}.$$

Solve this system, and use the results to determine $f(\mathbf{A})$.

(26) Specialize system (20) for $f(\mathbf{A}) = \mathbf{A}^9 - 3\mathbf{A}^4 + \mathbf{I}$ and

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{16} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

Solve this system, and use the results to determine $f(\mathbf{A})$.

7.4 Polynomials of Matrices—General Case

The only restriction in the previous section was that the eigenvalues of $\mathbf{A}$ had to be distinct. The following theorem suggests how to obtain n equations for the unknown α's in (15) even if some of the eigenvalues are identical.

Theorem 1. *Let $f(\lambda)$ and $r(\lambda)$ be defined as in Eq. (12). If λ_i is an eigenvalue of multiplicity k, then*

$$\begin{aligned} f(\lambda_i) &= r(\lambda_i), \\ \frac{df(\lambda_i)}{d\lambda} &= \frac{dr(\lambda_i)}{d\lambda}, \\ \frac{d^2 f(\lambda_i)}{d\lambda^2} &= \frac{d^2 r(\lambda_i)}{d\lambda^2}, \\ &\vdots \\ \frac{d^{k-1} f(\lambda_i)}{d\lambda^{k-1}} &= \frac{d^{k-1} r(\lambda_i)}{d\lambda^{k-1}}, \end{aligned} \tag{28}$$

where the notation $d^n f(\lambda_i)/d\lambda^n$ denotes the nth derivative of $f(\lambda)$ with respect to λ evaluated at $\lambda = \lambda_i$.‡

Thus, for example, if λ_i is an eigenvalue of multiplicity 3, Theorem 1 implies that $f(\lambda)$ and its first two derivatives evaluated at $\lambda = \lambda_i$ are equal, respectively, to $r(\lambda)$ and its first two derivatives also evaluated at $\lambda = \lambda_i$. If λ_i is an eigenvalue of multiplicity 5, then $f(\lambda)$ and the first four derivatives of $f(\lambda)$ evaluated at $\lambda = \lambda_i$ are equal respectively to $r(\lambda)$ and the first four derivatives of $r(\lambda)$ evaluated at $\lambda = \lambda_i$. Note, furthermore, that if λ_i is an eigenvalue of multiplicity 1, then Theorem 1 implies that $f(\lambda_i) = r(\lambda_i)$, which is Eq. (18).

Example 1

Find $\mathbf{A}^{24} - 3\mathbf{A}^{15}$ if

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 4 \\ 0 & 1 & 0 \\ -1 & -3 & -1 \end{bmatrix}.$$

Solution. The eigenvalues of $\mathbf{A}$ are $\lambda_1 = \lambda_2 = \lambda_3 = 1$; hence, $\lambda = 1$ is an eigenvalue of multiplicity three.

‡ Theorem 1 is proved by differentiating Eq. (12) $k - 1$ times and noting that if λ_i is an eigenvalue of multiplicity k, then

$$d(\lambda_i) = \frac{d[d(\lambda_i)]}{d\lambda} = \cdots = \frac{d^{(k-1)} d(\lambda_i)}{d\lambda^{k-1}} = 0.$$

$$f(\mathbf{A}) = \mathbf{A}^{24} - 3\mathbf{A}^{15} \qquad r(\mathbf{A}) = \alpha_2\mathbf{A}^2 + \alpha_1\mathbf{A} + \alpha_0\mathbf{I}$$

$$f(\lambda) = \lambda^{24} - 3\lambda^{15} \qquad r(\lambda) = \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0$$

$$f'(\lambda) = 24\lambda^{23} - 45\lambda^{14} \qquad r'(\lambda) = 2\alpha_2\lambda + \alpha_1$$

$$f''(\lambda) = 552\lambda^{22} - 630\lambda^{13} \qquad r''(\lambda) = 2\alpha_2.$$

Now $f(\mathbf{A}) = r(\mathbf{A})$, hence

$$\mathbf{A}^{24} - 3\mathbf{A}^{15} = \alpha_2\mathbf{A}^2 + \alpha_1\mathbf{A} + \alpha_0\mathbf{I}. \tag{29}$$

Also, since $\lambda = 1$ is an eigenvalue of multiplicity 3, it follows from Theorem 1 that

$$f(1) = r(1),$$

$$f'(1) = r'(1),$$

$$f''(1) = r''(1).$$

Hence,

$$(1)^{24} - 3(1)^{15} = \alpha_2(1)^2 + \alpha_1(1) + \alpha_0,$$

$$24(1)^{23} - 45(1)^{14} = 2\alpha_2(1) + \alpha_1,$$

$$552(1)^{22} - 630(1)^{13} = 2\alpha_2,$$

or

$$-2 = \alpha_2 + \alpha_1 + \alpha_0,$$

$$-21 = 2\alpha_2 + \alpha_1,$$

$$-78 = 2\alpha_2.$$

Thus, $\alpha_2 = -39$, $\alpha_1 = 57$, $\alpha_0 = -20$, and from Eq. (29)

$$\mathbf{A}^{24} - 3\mathbf{A}^{15} = -39\mathbf{A}^2 + 57\mathbf{A} - 20\mathbf{I} = \begin{bmatrix} -44 & 270 & -84 \\ 0 & -2 & 0 \\ 21 & -93 & 40 \end{bmatrix}. \qquad \square$$

Example 2

Set up the necessary equation to find $\mathbf{A}^{15} - 6\mathbf{A}^2$ if

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 3 & 2 & 1 & -7 \\ 0 & 0 & 2 & 11 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & -1 & 17 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Solution. The eigenvalues of $\mathbf{A}$ are $\lambda_1 = \lambda_2 = \lambda_3 = 1$, $\lambda_4 = \lambda_5 = -1$, $\lambda_6 = 0$.

$$f(\mathbf{A}) = \mathbf{A}^{15} - 6\mathbf{A}^2 \qquad r(\mathbf{A}) = \alpha_5\mathbf{A}^5 + \alpha_4\mathbf{A}^4 + \alpha_3\mathbf{A}^3 + \alpha_2\mathbf{A}^2 + \alpha_1\mathbf{A} + \alpha_0\mathbf{I}$$

$$f(\lambda) = \lambda^{15} - 6\lambda^2 \qquad r(\lambda) = \alpha_5\lambda^5 + \alpha_4\lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda^1 + \alpha_0$$

$$f'(\lambda) = 15\lambda^{14} - 12\lambda \qquad r'(\lambda) = 5\alpha_5\lambda^4 + 4\alpha_4\lambda^3 + 3\alpha_3\lambda^2 + 2\alpha_2\lambda + \alpha_1$$

$$f''(\lambda) = 210\lambda^{13} - 12 \qquad r''(\lambda) = 20\alpha_5\lambda^3 + 12\alpha_4\lambda^2 + 6\alpha_3\lambda + 2\alpha_2.$$

Since $f(\mathbf{A}) = r(\mathbf{A})$,

$$\mathbf{A}^{15} - 6\mathbf{A}^2 = \alpha_5\mathbf{A}^5 + \alpha_4\mathbf{A}^4 + \alpha_3\mathbf{A}^3 + \alpha_2\mathbf{A}^2 + \alpha_1\mathbf{A} + \alpha_0\mathbf{I}. \tag{30}$$

Since $\lambda = 1$ is an eigenvalue of multiplicity 3, $\lambda = -1$ is an eigenvalue of multiplicity 2 and $\lambda = 0$ is an eigenvalue of multiplicity 1, it follows from Theorem 1 that

$$\begin{aligned} f(1) &= r(1), \\ f'(1) &= r'(1), \\ f''(1) &= r''(1), \\ f(-1) &= r(-1), \\ f'(-1) &= r'(-1), \\ f(0) &= r(0). \end{aligned} \tag{31}$$

Hence,

$$\begin{aligned} (1)^{15} - 6(1)^2 &= \alpha_5(1)^5 + \alpha_4(1)^4 + \alpha_3(1)^3 + \alpha_2(1)^2 + \alpha_1(1) + \alpha_0 \\ 15(1)^{14} - 12(1) &= 5\alpha_5(1)^4 + 4\alpha_4(1)^3 + 3\alpha_3(1)^2 + 2\alpha_2(1) + \alpha_1 \\ 210(1)^{13} - 12 &= 20\alpha_5(1)^3 + 12\alpha_4(1)^2 + 6\alpha_3(1) + 2\alpha_2 \\ (-1)^{15} - 6(-1)^2 &= \alpha_5(-1)^5 + \alpha_4(-1)^4 + \alpha_3(-1)^3 + \alpha_2(-1)^2 + \alpha_1(-1) + \alpha_0 \\ 15(-1)^{14} - 12(-1) &= 5\alpha_5(-1)^4 + 4\alpha_4(-1)^3 + 3\alpha_3(-1)^2 + 2\alpha_2(-1) + \alpha_1 \\ (0)^{15} - 12(0)^2 &= \alpha_5(0)^5 + \alpha_4(0)^4 + \alpha_3(0)^3 + \alpha_2(0)^2 + \alpha_1(0) + \alpha_0 \end{aligned}$$

or

$$\begin{aligned} -5 &= \alpha_5 + \alpha_4 + \alpha_3 + \alpha_2 + \alpha_1 + \alpha_0 \\ 3 &= 5\alpha_5 + 4\alpha_4 + 3\alpha_3 + 2\alpha_2 + \alpha_1 \\ 198 &= 20\alpha_5 + 12\alpha_4 + 6\alpha_3 + 2\alpha_2 \end{aligned}$$

$$-7 = -\alpha_5 + \alpha_4 - \alpha_3 + \alpha_2 - \alpha_1 + \alpha_0$$
$$27 = 5\alpha_5 - 4\alpha_4 + 3\alpha_3 - 2\alpha_2 + \alpha_1$$
$$0 = \alpha_0.$$

System (32) can now be solved uniquely for $\alpha_5, \alpha_4, \ldots, \alpha_0$; the results are then substituted into (30) to obtain $f(\mathbf{A})$. □

Problems 7.4

(1) Using Theorem 1, establish the equations that are needed to find $\mathbf{A}^7$ if $\mathbf{A}$ is a 2×2 matrix having 2 and 2 as multiple eigenvalues.

(2) Using Theorem 1, establish the equations that are needed to find $\mathbf{A}^7$ if $\mathbf{A}$ is a 3×3 matrix having 2 as an eigenvalue of multiplicity three.

(3) Redo Problem 2 if instead the eigenvalues are 2, 2, and 1.

(4) Using Theorem 1, establish the equations that are needed to find $\mathbf{A}^{10}$ if $\mathbf{A}$ is a 2×2 matrix having 3 as an eigenvalue of multiplicity two.

(5) Redo Problem 4 if instead the matrix has order 3×3 with 3 as an eigenvalue of multiplicity three.

(6) Redo Problem 4 if instead the matrix has order 4×4 with 3 as an eigenvalue of multiplicity four.

(7) Using Theorem 1, establish the equations that are needed to find $\mathbf{A}^9$ if $\mathbf{A}$ is a 4×4 matrix having 2 has an eigenvalue of multiplicity four.

(8) Redo Problem 7 if instead the eigenvalues are 2, 2, 2, and 1.

(9) Redo Problem 7 if instead the eigenvalues are 2 and 1, both with multiplicity two.

(10) Set up (but do not solve) the necessary equations to find $\mathbf{A}^{10} - 3\mathbf{A}^5$ if

$$\mathbf{A} = \begin{bmatrix} 5 & -2 & 1 & 1 & 5 & -7 \\ 0 & 5 & 2 & 1 & -1 & 1 \\ 0 & 0 & 5 & 0 & 1 & -3 \\ 0 & 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

(11) Find $\mathbf{A}^6$ in two different ways if

$$\mathbf{A} = \begin{bmatrix} 5 & 8 \\ -2 & -5 \end{bmatrix}.$$

(First find $\mathbf{A}^6$ using Theorem 1, and then by direct multiplication.)

(12) Find $\mathbf{A}^{521}$ if

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & -3 \\ 0 & -1 & 0 \\ 5 & 1 & -4 \end{bmatrix}.$$

(13) Find $\mathbf{A}^{14} - 3\mathbf{A}^{13}$ if

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & 2 \\ 0 & 0 & 0 \\ -8 & 1 & -4 \end{bmatrix}.$$

7.5 Functions of a Matrix

Once the student understands how to compute polynomials of a matrix, computing exponentials and other functions of a matrix is easy, because, the methods developed in the previous two sections remain valid for more general functions.

Let $f(\lambda)$ represent a *function* of λ and suppose we wish to compute $f(\mathbf{A})$. It can be shown, for a large class of problems, that there exists a function $q(\lambda)$ and an $n - 1$ degree polynomial $r(\lambda)$ (we assume $\mathbf{A}$ is of order $n \times n$) such that

$$f(\lambda) = q(\lambda)d(\lambda) + r(\lambda), \tag{33}$$

where $d(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I})$. Hence, it follows that

$$f(\mathbf{A}) = q(\mathbf{A})d(\mathbf{A}) + r(\mathbf{A}). \tag{34}$$

Since (33) and (34) are exactly Eqs. (12) and (14), where $f(\lambda)$ is now understood to be a general function and not restricted to polynomials, the analysis of Section 7.3 and 7.4 can again be applied. It then follows that

(a) $f(\mathbf{A}) = r(\mathbf{A})$, and
(b) Theorem 1 of Section 7.4 remains valid.

Thus, the methods used to compute a polynomial of a matrix can be generalized and used to compute arbitrary functions of a matrix.

Example 1

Find $e^{\mathbf{A}}$ if

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

Solution. The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 5$, $\lambda_2 = -1$; thus,

$$f(\mathbf{A}) = e^{\mathbf{A}} \qquad r(\mathbf{A}) = \alpha_1\mathbf{A} + \alpha_0\mathbf{I}$$

$$f(\lambda) = e^{\lambda} \qquad r(\lambda) = \alpha_1\lambda + \alpha_0.$$

Now $f(\mathbf{A}) = r(\mathbf{A})$; hence

$$e^{\mathbf{A}} = \alpha_1\mathbf{A} + \alpha_0\mathbf{I}. \tag{35}$$

Also, since Theorem 1 of Section 7.4 is still valid,

$$f(5) = r(5),$$

and

$$f(-1) = r(-1);$$

hence,

$$e^5 = 5\alpha_1 + \alpha_0,$$

$$e^{-1} = -\alpha_1 + \alpha_0.$$

Thus,

$$\alpha_1 = \frac{e^5 - e^{-1}}{6} \quad \text{and} \quad \alpha_0 = \frac{e^5 + 5e^{-1}}{6}.$$

Substituting these values into (35), we obtain

$$e^{\mathbf{A}} = \frac{1}{6}\begin{bmatrix} 2e^5 + 4e^{-1} & 2e^5 - 2e^{-1} \\ 4e^5 - 4e^{-1} & 4e^5 + 2e^{-1} \end{bmatrix}. \qquad \square$$

Example 2

Find $e^{\mathbf{A}}$ if

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Solution. The eigenvalues of $\mathbf{A}$ are $\lambda_1 = \lambda_2 = \lambda_3 = 2$, thus,

$$f(\mathbf{A}) = e^{\mathbf{A}} \qquad r\mathbf{A}) = \alpha_2\mathbf{A}^2 + \alpha_1\mathbf{A} + \alpha_0\mathbf{I}$$
$$f(\lambda) = e^{\lambda} \qquad r(\lambda) = \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0$$
$$f'(\lambda) = e^{\lambda} \qquad r'(\lambda) = 2\alpha_2\lambda + \alpha_1$$
$$f''(\lambda) = e^{\lambda} \qquad r''(\lambda) = 2\alpha_2.$$

since $f(\mathbf{A}) = r(\mathbf{A})$,

$$e^{\mathbf{A}} = \alpha_2\mathbf{A}^2 + \alpha_1\mathbf{A} + \alpha_0\mathbf{I}. \tag{36}$$

Since $\lambda = 2$ is an eigenvalue of multiplicity three,

$$f(2) = r(2),$$
$$f'(2) = r'(2),$$
$$f''(2) = r''(2);$$

hence,

$$e^2 = 4\alpha_2 + 2\alpha_1 + \alpha_0,$$
$$e^2 = 4\alpha_2 + \alpha_1,$$
$$e^2 = 2\alpha_2,$$

or

$$\alpha_2 = \frac{e^2}{2}, \qquad \alpha_1 = -e^2, \qquad \alpha_0 = e^2.$$

Substituting these values into (36), we obtain

$$e^{\mathbf{A}} = \frac{e^2}{2}\begin{bmatrix} 4 & 4 & 1 \\ 0 & 4 & 4 \\ 0 & 0 & 4 \end{bmatrix} - e^2\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} + e^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e^2 & e^2 & e^2/2 \\ 0 & e^2 & e^2 \\ 0 & 0 & e^2 \end{bmatrix}.$$

□

Example 3

Find sin $\mathbf{A}$ if

$$\mathbf{A} = \begin{bmatrix} \pi & 1 & 0 \\ 0 & \pi & 0 \\ 4 & 1 & \pi/2 \end{bmatrix}.$$

Solution. The eigenvalues of $\mathbf{A}$ are $\lambda_1 = \pi/2$, $\lambda_2 = \lambda_3 = \pi$; thus

$$f(\mathbf{A}) = \sin \mathbf{A} \qquad r(\mathbf{A}) = \alpha_2\mathbf{A}^2 + \alpha_1\mathbf{A} + \alpha_0\mathbf{I}$$

$$f(\lambda) = \sin \lambda \qquad r(\lambda) = \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0$$

$$f'(\lambda) = \cos \lambda \qquad r'(\lambda) = 2\alpha_2\lambda + \alpha_1.$$

But $f(\mathbf{A}) = r(\mathbf{A})$, hence

$$\sin \mathbf{A} = \alpha_2\mathbf{A}^2 + \alpha_1\mathbf{A} + \alpha_0\mathbf{I}. \tag{37}$$

Since $\lambda = \pi/2$ is an eigenvalue of multiplicity 1 and $\lambda = \pi$ is an eigenvalue of multiplicity 2, it follows that

$$f(\pi/2) = r(\pi/2),$$

$$f(\pi) = r(\pi),$$

$$f'(\pi) = r'(\pi);$$

hence,

$$\sin \pi/2 = \alpha_2(\pi/2)^2 + \alpha_1(\pi/2) + \alpha_0,$$

$$\sin \pi = \alpha_2(\pi)^2 + \alpha_1(\pi) + \alpha_0,$$

$$\cos \pi = 2\alpha_2\pi + \alpha_1,$$

or simplifying

$$4 = \alpha_2\pi^2 + 2\alpha_1\pi + 4\alpha_0,$$

$$0 = \alpha_2\pi^2 + \alpha_1\pi + \alpha_0,$$

$$-1 = 2\alpha_2\pi + \alpha_1.$$

Thus, $\alpha_2 = (1/\pi^2)(4 - 2\pi)$, $\alpha_1 = (1/\pi^2)(-8\pi + 3\pi^2)$, $\alpha_0 = (1/\pi^2)(4\pi^2 - \pi^3)$. Substituting these values into (37), we obtain

$$\sin \mathbf{A} = 1/\pi^2 \begin{bmatrix} 0 & -\pi^2 & 0 \\ 0 & 0 & 0 \\ -8\pi & 16 - 10\pi & \pi^2 \end{bmatrix}. \qquad \square$$

In closing, we point out that although exponentials of any square matrix can always be computed by the above methods, not all functions of all matrices can; $f(\mathbf{A})$ must first be "well defined" where by "well defined" (see Theorem 1 of Section 7.1) we mean that $f(z)$ has a Taylor series which converges for $|z| < R$ and all eigenvalues of $\mathbf{A}$ have the property that their absolute values are also less than R.

Problems 7.5

(1) Establish the equations necessary to find $e^{\mathbf{A}}$ if $\mathbf{A}$ is a 2×2 matrix having 1 and 2 as its eigenvalues.

(2) Establish the equations necessary to find $e^{\mathbf{A}}$ if $\mathbf{A}$ is a 2×2 matrix having 2 and 2 as multiple eigenvalues.

(3) Establish the equations necessary to find $e^{\mathbf{A}}$ if $\mathbf{A}$ is a 3×3 matrix having 2 as an eigenvalue of multiplicity three.

(4) Establish the equations necessary to find $e^{\mathbf{A}}$ if $\mathbf{A}$ is a 3×3 matrix having 1, -2, and 3 as its eigenvalues.

(5) Redo Problem 4 if instead the eigenvalues are -2, -2, and 1.

(6) Establish the equations necessary to find $\sin(\mathbf{A})$ if $\mathbf{A}$ is a 3×3 matrix having 1, 2, and 3 as its eigenvalues.

(7) Redo Problem 6 if instead the eigenvalues are -2, -2, and 1.

(8) Establish the equations necessary to find $e^{\mathbf{A}}$ if $\mathbf{A}$ is a 4×4 matrix having 2 as an eigenvalue of multiplicity four.

(9) Establish the equations necessary to find $e^{\mathbf{A}}$ if $\mathbf{A}$ is a 4×4 matrix having both 2 and -2 as eigenvalues of multiplicity two.

(10) Redo Problem 9 if instead the function of interest is $\sin(\mathbf{A})$.

(11) Establish the equations necessary to find $e^{\mathbf{A}}$ if $\mathbf{A}$ is a 4×4 matrix having 3, 3, 3, and -1 as its eigenvalues.

(12) Redo Problem 11 if instead the function of interest is $\cos(\mathbf{A})$.

(13) Find $e^{\mathbf{A}}$ for

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}.$$

(14) Find $e^{\mathbf{A}}$ for

$$\mathbf{A} = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}.$$

(15) Find $e^{\mathbf{A}}$ for

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 3 & 4 \\ 0 & 0 & 2 \end{bmatrix}.$$

(16) Find $e^{\mathbf{A}}$ for

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 3 & -1 & 4 \\ 0 & 0 & 2 \end{bmatrix}.$$

(17) Find $\cos \mathbf{A}$ if

$$\mathbf{A} = \begin{bmatrix} \pi & 3\pi \\ 2\pi & 2\pi \end{bmatrix}.$$

(18) The function $f(z) = \log(1 + z)$ has the Taylor series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^k}{k}$$

which converges for $|z| < 1$. For the following matrices, $\mathbf{A}$, determine whether or not $\log(\mathbf{A} + \mathbf{I})$ is well defined and, if so, find it.

(a) $\begin{bmatrix} \frac{1}{2} & 1 \\ 0 & -\frac{1}{2} \end{bmatrix}$ (b) $\begin{bmatrix} -6 & 9 \\ -2 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & 5 \\ -1 & -3 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

7.6 The Function $e^{\mathbf{A}t}$

A very important function in the matrix calculus is $e^{\mathbf{A}t}$, where $\mathbf{A}$ is a square constant matrix (that is, all of its entries are constants) and t is a variable. This function may be calculated by defining a new matrix $\mathbf{B} = \mathbf{A}t$ and then computing $e^{\mathbf{B}}$ by the methods of the previous section.

Example 1

Find $e^{\mathbf{A}t}$ if

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

Solution. Define

$$\mathbf{B} = \mathbf{A}t = \begin{bmatrix} t & 2t \\ 4t & 3t \end{bmatrix}.$$

The problem then reduces to finding $e^{\mathbf{B}}$. The eigenvalues of $\mathbf{B}$ are $\lambda_1 = 5t$, $\lambda_2 = -t$. Note that the eigenvalues now depend on t.

$$f(\mathbf{B}) = e^{\mathbf{B}} \qquad r(\mathbf{B}) = \alpha_1\mathbf{B} + \alpha_0\mathbf{I}$$

$$f(\lambda) = e^{\lambda} \qquad r(\lambda) = \alpha_1\lambda + \alpha_0.$$

Since $f(\mathbf{B}) = r(\mathbf{B})$,

$$e^{\mathbf{B}} = \alpha_1\mathbf{B} + \alpha_0\mathbf{I}. \tag{38}$$

Also, $f(\lambda_i) = r(\lambda_i)$; hence

$$e^{5t} = \alpha_1(5t) + \alpha_0,$$

$$e^{-t} = \alpha_1(-t) + \alpha_0.$$

Thus, $\alpha_1 = (1/6t)(e^{5t} - e^{-t})$ and $\alpha_0 = (1/6)(e^{5t} + 5e^{-t})$. Substituting these values into (38), we obtain

$$e^{\mathbf{A}t} = e^{\mathbf{B}} = \left(\frac{1}{6t}\right)(e^{5t} - e^{-t})\begin{bmatrix} t & 2t \\ 4t & 3t \end{bmatrix} + \left(\frac{1}{6}\right)(e^{5t} + 5e^{-t})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 2e^{5t} + 4e^{-t} & 2e^{5t} - 2e^{-t} \\ 4e^{5t} - 4e^{-t} & 4e^{5t} + 2e^{-t} \end{bmatrix}. \qquad \square$$

Example 2

Find $e^{\mathbf{A}t}$ if

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

Solution. Define

$$\mathbf{B} = \mathbf{A}t = \begin{bmatrix} 3t & t & 0 \\ 0 & 3t & t \\ 0 & 0 & 3t \end{bmatrix}.$$

The problem reduces to finding $e^{\mathbf{B}}$. The eigenvalues of $\mathbf{B}$ are

$$\lambda_1 = \lambda_2 = \lambda_3 = 3t$$

thus,

$$f(\mathbf{B}) = e^{\mathbf{B}} \qquad r(\mathbf{B}) = \alpha_2\mathbf{B}^2 + \alpha_1\mathbf{B} + \alpha_0\mathbf{I}$$

$$f(\lambda) = e^{\lambda} \qquad r(\lambda) = \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0 \tag{39}$$

$$f'(\lambda) = e^{\lambda} \qquad r'(\lambda) = 2\alpha_2\lambda + \alpha_1 \tag{40}$$

$$f''(\lambda) = e^{\lambda} \qquad r''(\lambda) = 2\alpha_2. \tag{41}$$

Since $f(\mathbf{B}) = r(\mathbf{B})$,

$$e^{\mathbf{B}} = \alpha_2\mathbf{B}^2 + \alpha_1\mathbf{B} + \alpha_0\mathbf{I}. \tag{42}$$

Since $\lambda = 3t$ is an eigenvalue of multiplicity 3,

$$f(3t) = r(3t), \tag{43}$$

$$f'(3t) = r'(3t), \tag{44}$$

$$f''(3t) = r''(3t). \tag{45}$$

Thus, using (39)–(41), we obtain

$$e^{3t} = (3t)^2\alpha_2 + (3t)\alpha_1 + \alpha_0,$$

$$e^{3t} = 2(3t)\alpha_2 + \alpha_1,$$

$$e^{3t} = 2\alpha_2$$

or

$$e^{3t} = 9t^2\alpha_2 + 3t\alpha_1 + \alpha_0, \tag{46}$$

$$e^{3t} = 6t\alpha_2 + \alpha_1, \tag{47}$$

$$e^{3t} = 2\alpha_2. \tag{48}$$

Solving (46)–(48) simultaneously, we obtain

$$\alpha_2 = \tfrac{1}{2}e^{3t}, \qquad \alpha_1 = (1 - 3t)e^{3t}, \qquad \alpha_0 = (1 - 3t + \tfrac{9}{2}t^2)e^{3t}.$$

From (42), it follows that

$$\begin{aligned} e^{\mathbf{A}t} = e^{\mathbf{B}} &= \tfrac{1}{2}e^{3t}\begin{bmatrix} 9t^2 & 6t^2 & t^2 \\ 0 & 9t^2 & 6t^2 \\ 0 & 0 & 9t^2 \end{bmatrix} + (1 - 3t)e^{3t}\begin{bmatrix} 3t & t & 0 \\ 0 & 3t & t \\ 0 & 0 & 3t \end{bmatrix} \\ &\quad + (1 - 3t + \tfrac{9}{2}t^2)e^{3t}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= e^{3t}\begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}. \qquad \square \end{aligned}$$

Problems 7.6

Find $e^{\mathbf{A}t}$ if $\mathbf{A}$ is given by:

(1) $\begin{bmatrix} 4 & 4 \\ 3 & 5 \end{bmatrix}$. **(2)** $\begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}$. **(3)** $\begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$.

(4) $\begin{bmatrix} 0 & 1 \\ -14 & -9 \end{bmatrix}$. **(5)** $\begin{bmatrix} -3 & 2 \\ 2 & -6 \end{bmatrix}$. **(6)** $\begin{bmatrix} -10 & 6 \\ 6 & -10 \end{bmatrix}$.

(7) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. **(8)** $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 2 \\ -1 & 4 & -1 \end{bmatrix}$.

7.7 Complex Eigenvalues

When computing $e^{\mathbf{A}t}$, it is often the case that the eigenvalues of $\mathbf{B} = \mathbf{A}t$ are complex. If this occurs the complex eigenvalues will appear in conjugate pairs, assuming the elements of $\mathbf{A}$ to be real, and these can be combined to produce real functions.

Let z represent a complex variable. Define e^z by

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \cdots \tag{49}$$

(see Eq. (5)). Setting $z = i\theta$, θ real, we obtain

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \cdots. \end{aligned}$$

Combining real and imaginary terms, we obtain

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right). \tag{50}$$

But the Maclaurin series expansions for $\sin\theta$ and $\cos\theta$ are

$$\sin\theta = \frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots;$$

hence, Eq. (50) may be rewritten as

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{51}$$

Equation (51) is referred to as DeMoivre's formula. If the same analysis is applied to $z = -i\theta$, it follows that

$$e^{-i\theta} = \cos\theta - i\sin\theta. \tag{52}$$

Adding (51) and (52), we obtain

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \tag{53}$$

while subtracting (52) from (51), we obtain

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \tag{54}$$

Equations (53) and (54) are Euler's relations and can be used to reduce complex exponentials to expressions involving real numbers.

Example 1

Find $e^{\mathbf{A}t}$ if

$$\mathbf{A} = \begin{bmatrix} -1 & 5 \\ -2 & 1 \end{bmatrix}.$$

Solution.

$$\mathbf{B} = \mathbf{A}t = \begin{bmatrix} -t & 5t \\ -2t & t \end{bmatrix}.$$

Hence the eigenvalues of $\mathbf{B}$ are $\lambda_1 = 3ti$ and $\lambda_2 = -3ti$; thus

$$f(\mathbf{B}) = e^{\mathbf{B}} \qquad r(\mathbf{B}) = \alpha_1\mathbf{B} + \alpha_0\mathbf{I}$$

$$f(\lambda) = e^{\lambda} \qquad r(\lambda) = \alpha_1\lambda + \alpha_0$$

Since $f(\mathbf{B}) = r(\mathbf{B})$,

$$e^{\mathbf{B}} = \alpha_1\mathbf{B} + \alpha_0\mathbf{I}, \tag{55}$$

and since $f(\lambda_i) = r(\lambda_i)$,

$$e^{3ti} = \alpha_1(3ti) + \alpha_0,$$

$$e^{-3ti} = \alpha_1(-3ti) + \alpha_0.$$

Thus,

$$\alpha_0 = \frac{e^{3ti} + e^{-3ti}}{2} \quad \text{and} \quad \alpha_1 = \frac{1}{3t}\left(\frac{e^{3ti} - e^{-3ti}}{2i}\right).$$

If we now use (53) and (54), where in this case $\theta = 3t$, it follows that

$$\alpha_0 = \cos 3t \quad \text{and} \quad \alpha_1 = (1/3t)\sin 3t.$$

Substituting these values into (55), we obtain

$$e^{\mathbf{A}t} = e^{\mathbf{B}} = \begin{bmatrix} -\frac{1}{3}\sin 3t + \cos 3t & \frac{5}{3}\sin 3t \\ -\frac{2}{3}\sin 3t & \frac{1}{3}\sin 3t + \cos 3t \end{bmatrix}. \quad \square$$

In Example 1, the eigenvalues of $\mathbf{B}$ are pure imaginary permitting the application of (53) and (54) in a straightforward manner. In the general case, where the eigenvalues are complex numbers, we can still use Euler's relations providing we note the following:

$$\frac{e^{\beta+i\theta} + e^{\beta-i\theta}}{2} = \frac{e^{\beta}e^{i\theta} + e^{\beta}e^{-i\theta}}{2} = \frac{e^{\beta}(e^{i\theta} + e^{-i\theta})}{2} = e^{\beta}\cos\theta, \tag{56}$$

and

$$\frac{e^{\beta+i\theta} - e^{\beta-i\theta}}{2i} = \frac{e^{\beta}e^{i\theta} - e^{\beta}e^{-i\theta}}{2i} = \frac{e^{\beta}(e^{i\theta} - e^{-i\theta})}{2i} = e^{\beta}\sin\theta. \tag{57}$$

Example 2

Find $e^{\mathbf{A}t}$ if

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 4 & 1 \end{bmatrix}.$$

Solution.

$$\mathbf{B} = \mathbf{A}t = \begin{bmatrix} 2t & -t \\ 4t & t \end{bmatrix};$$

hence, the eigenvalues of $\mathbf{B}$ are

$$\lambda_1 = \left(\frac{3}{2} + i\frac{\sqrt{15}}{2}\right)t, \qquad \lambda_2 = \left(\frac{3}{2} - i\frac{\sqrt{15}}{2}\right)t.$$

Thus,

$$f(\mathbf{B}) = e^{\mathbf{B}} \qquad r(\mathbf{B}) = \alpha_1\mathbf{B} + \alpha_0\mathbf{I}$$

$$f(\lambda) = e^{\lambda} \qquad r(\lambda) = \alpha_1\lambda + \alpha_0.$$

Since $f(\mathbf{B}) = r(\mathbf{B})$,

$$e^{\mathbf{B}} = \alpha_1\mathbf{B} + \alpha_0\mathbf{I}, \tag{58}$$

and since $f(\lambda_i) = r(\lambda_i)$,

$$e^{[3/2+i(\sqrt{15}/2)]t} = \alpha_1[\tfrac{3}{2} + i(\sqrt{15}/2)]t + \alpha_0,$$
$$e^{[3/2-i(\sqrt{15}/2)]t} = \alpha_1[\tfrac{3}{2} - i(\sqrt{15}/2)]t + \alpha_0.$$

Putting this system into matrix form, and solving for α_1 and α_0 by inversion, we obtain

$$\alpha_1 = \frac{2}{\sqrt{15}\,t}\left[\frac{e^{[(3/2)t+(\sqrt{15}/2)ti]} - e^{[(3/2)t-(\sqrt{15}/2)ti]}}{2i}\right]$$

$$\alpha_0 = \frac{-3}{\sqrt{15}}\left(\frac{e^{[(3/2)t+(\sqrt{15}/2)ti]} - e^{[(3/2)t-(\sqrt{15}/2)ti]}}{2i}\right) + \left(\frac{e^{[(3/2)t+(\sqrt{15}/2)ti]} + e^{[(3/2)t-(\sqrt{15}/2)ti]}}{2}\right).$$

Using (56) and (57) where, $\beta = \frac{3}{2}t$ and $\theta = (\sqrt{15}/2)t$, we obtain

$$\alpha_1 = \frac{2}{\sqrt{15t}}e^{3t/2}\sin\frac{\sqrt{15}t}{2}$$

$$\alpha_0 = -\frac{3}{\sqrt{15}}e^{3t/2}\sin\frac{\sqrt{15}}{2}t + e^{3t/2}\cos\frac{\sqrt{15}}{2}t.$$

Substituting these values into (58), we obtain

$$e^{\mathbf{A}t} = e^{3t/2}\begin{bmatrix} \dfrac{1}{\sqrt{15}}\sin\dfrac{\sqrt{15}}{2}t + \cos\dfrac{\sqrt{15}}{2}t & \dfrac{-2}{\sqrt{15}}\sin\dfrac{\sqrt{15}}{2}t \\ \dfrac{8}{\sqrt{15}}\sin\dfrac{\sqrt{15}}{2}t & \dfrac{-1}{\sqrt{15}}\sin\dfrac{\sqrt{15}}{2}t + \cos\dfrac{\sqrt{15}}{2}t \end{bmatrix}.$$

□

Problems 7.7

Find $e^{\mathbf{A}t}$ if $\mathbf{A}$ is given by:

(1) $\begin{bmatrix} 1 & -1 \\ 5 & -1 \end{bmatrix}$. **(2)** $\begin{bmatrix} 2 & -2 \\ 3 & -2 \end{bmatrix}$. **(3)** $\begin{bmatrix} 0 & 1 \\ -64 & 0 \end{bmatrix}$.

(4) $\begin{bmatrix} 4 & -8 \\ 10 & -4 \end{bmatrix}$. **(5)** $\begin{bmatrix} 2 & 5 \\ -1 & -2 \end{bmatrix}$. **(6)** $\begin{bmatrix} 0 & 1 \\ -25 & -8 \end{bmatrix}$.

(7) $\begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}$. **(8)** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & -5 \\ 0 & 1 & 2 \end{bmatrix}$.

7.8 Properties of $e^{\mathbf{A}}$

Since the scalar function e^x and the matrix function $e^{\mathbf{A}}$ are defined similarly (see Eqs. (5) and (6)), it should not be surprising to find that they possess some similar properties. What might be surprising, however, is that *not* all properties of e^x are common to $e^{\mathbf{A}}$. For example, while it is always true that $e^x e^y = e^{x+y} = e^y e^x$, the same cannot be said for matrices $e^{\mathbf{A}}$ and $e^{\mathbf{B}}$ *unless* **A** *and* **B** *commute.*

Example 1

Find $e^{\mathbf{A}}e^{\mathbf{B}}$, $e^{\mathbf{A}+\mathbf{B}}$, and $e^{\mathbf{B}}e^{\mathbf{A}}$ if

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Solution. Using the methods developed in Section 7.5, we find

$$e^{\mathbf{A}} = \begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix}, \qquad e^{\mathbf{B}} = \begin{bmatrix} 1 & 0 \\ 0 & e \end{bmatrix}, \qquad e^{\mathbf{A}+\mathbf{B}} = \begin{bmatrix} e & e \\ 0 & e \end{bmatrix}.$$

Therefore,

$$e^{\mathbf{A}}e^{\mathbf{B}} = \begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e \end{bmatrix} = \begin{bmatrix} e & e^2-e \\ 0 & e \end{bmatrix}$$

and

$$e^{\mathbf{B}}e^{\mathbf{A}} = \begin{bmatrix} 1 & 0 \\ 0 & e \end{bmatrix}\begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e & e-1 \\ 0 & e \end{bmatrix};$$

hence

$$e^{\mathbf{A}+\mathbf{B}} \neq e^{\mathbf{A}}e^{\mathbf{B}} \neq e^{\mathbf{B}}e^{\mathbf{A}}. \qquad \square$$

Two properties that both e^x and $e^{\mathbf{A}}$ do have in common are given by the following:

Property 1. $e^0 = \mathbf{I}$, *where* $\mathbf{0}$ *represents the zero matrix.*

Proof. From (6) we have that

$$e^{\mathbf{A}} = \sum_{k=0}^{\infty} \left(\frac{\mathbf{A}^k}{k!}\right) = \mathbf{I} + \sum_{k=1}^{\infty} \left(\frac{\mathbf{A}^k}{k!}\right).$$

Hence,

$$e^{\mathbf{0}} = \mathbf{I} + \sum_{k=1}^{\infty} \frac{\mathbf{0}^k}{k!} = \mathbf{I}.$$

Property 2. $(e^{\mathbf{A}})^{-1} = e^{-\mathbf{A}}$.

Proof

$$\begin{aligned}
&(e^{\mathbf{A}})(e^{-\mathbf{A}}) \\
&\quad = \left[\sum_{k=0}^{\infty} \left(\frac{\mathbf{A}^k}{k!}\right)\right]\left[\sum_{k=0}^{\infty} \frac{(-\mathbf{A})^k}{k!}\right] \\
&\quad = \left[\mathbf{I} + \mathbf{A} + \frac{\mathbf{A}^2}{2!} + \frac{\mathbf{A}^3}{3!} + \cdots\right]\left[\mathbf{I} - \mathbf{A} + \frac{\mathbf{A}^2}{2!} - \frac{\mathbf{A}^3}{3!} + \cdots\right] \\
&\quad = \mathbf{II} + \mathbf{A}[1 - 1] + \mathbf{A}^2[\tfrac{1}{2}! - 1 + \tfrac{1}{2}!] + \mathbf{A}^3[-\tfrac{1}{3}! + \tfrac{1}{2}! - \tfrac{1}{2}! + \tfrac{1}{3}!] + \cdots \\
&\quad = \mathbf{I}.
\end{aligned}$$

Thus, $e^{-\mathbf{A}}$ is an inverse of $e^{\mathbf{A}}$. However, by definition, an inverse of $e^{\mathbf{A}}$ is $(e^{\mathbf{A}})^{-1}$; hence, from the uniqueness of the inverse (Theorem 2 of Section 3.4), we have that $e^{-\mathbf{A}} = (e^{\mathbf{A}})^{-1}$.

Example 2

Verify Property 2 for

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Solution.

$$-\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix},$$

$$e^{\mathbf{A}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad e^{-\mathbf{A}} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Thus,

$$(e^{\mathbf{A}})^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = e^{-\mathbf{A}}. \qquad \square$$

Note that Property 2 implies that $e^{\mathbf{A}}$ *is always invertible* even if $\mathbf{A}$ itself is not.

Property 3. $(e^{\mathbf{A}})^{\mathrm{T}} = e^{\mathbf{A}^{\mathrm{T}}}$.

Proof. The proof of this property is left as an exercise for the reader (see Problem 7).

Example 3

Verify Property 3 for

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

Solution.

$$\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix},$$

$$e^{\mathbf{A}^{\mathrm{T}}} = \frac{1}{6}\begin{bmatrix} 2e^5 + 4e^{-1} & 4e^5 - 4e^{-1} \\ 2e^5 - 2e^{-1} & 4e^5 + 2e^{-1} \end{bmatrix},$$

and

$$e^{\mathbf{A}} = \frac{1}{6}\begin{bmatrix} 2e^5 + 4e^{-1} & 2e^5 - 2e^{-1} \\ 4e^5 - 4e^{-1} & 4e^5 + 2e^{-1} \end{bmatrix};$$

Hence, $(e^{\mathbf{A}})^{\mathrm{T}} = e^{\mathbf{A}^{\mathrm{T}}}$. $\square$

Problems 7.8

(1) Verify Property 2 for

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}.$$

(2) Verify Property 2 for

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -64 & 0 \end{bmatrix}.$$

(3) Verify Property 2 for

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

What is the inverse of $\mathbf{A}$?

(4) Verify Property 3 for

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}.$$

(5) Verify Property 3 for the matrix given in Problem 2.

(6) Verify Property 3 for the matrix given in Problem 3.

(7) Prove Property 3. (Hint: Using the fact that the eigenvalues of $\mathbf{A}$ are identical to eigenvalues of $\mathbf{A}^{\mathsf{T}}$, show that if $e^{\mathbf{A}} = \alpha_{n-1}\mathbf{A}^{n-1} + \cdots + \alpha_1\mathbf{A} + \alpha_0\mathbf{I}$, and if

$$e^{\mathbf{A}^{\mathsf{T}}} = \beta_{n-1}(\mathbf{A}^{\mathsf{T}})^{n-1} + \cdots + \beta_1\mathbf{A}^{\mathsf{T}} + \beta_0\mathbf{I},$$

then $\alpha_j = \beta_j$ for $j = 0, 1, \ldots, n-1$.)

(8) Find $e^{\mathbf{A}}e^{\mathbf{B}}$, $e^{\mathbf{B}}e^{\mathbf{A}}$, and $e^{\mathbf{A}+\mathbf{B}}$ if

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

and show that $e^{\mathbf{A}+\mathbf{B}} \neq e^{\mathbf{A}}e^{\mathbf{B}} \neq e^{\mathbf{B}}e^{\mathbf{A}}$.

(9) Find two matrices $\mathbf{A}$ and $\mathbf{B}$ such that

$$e^{\mathbf{A}}e^{\mathbf{B}} = e^{\mathbf{A}+\mathbf{B}}.$$

(10) By using the definition of $e^{\mathbf{A}}$, prove that if $\mathbf{A}$ and $\mathbf{B}$ commute, then $e^{\mathbf{A}}e^{\mathbf{B}} = e^{\mathbf{A}+\mathbf{B}}$.

(11) Show that if $\mathbf{A} = \mathbf{P}^{-1}\mathbf{B}\mathbf{P}$ for some invertible matrix $\mathbf{P}$, then $e^{\mathbf{A}} = \mathbf{P}^{-1}e^{\mathbf{B}}\mathbf{P}$.

7.9 Derivatives of a Matrix

Definition 1. An $n \times n$ matrix $\mathbf{A}(t) = [a_{ij}(t)]$ is *continuous* at $t = t_0$ if each of its elements $a_{ij}(t)$ $(i, j = 1, 2, \ldots, n)$ is continuous at $t = t_0$.

For example, the matrix given in (59) is continuous everywhere because each of its elements is continuous everywhere while the matrix given in (60) is not continuous at $t = 0$ because the (1, 2) element, $\sin(1/t)$, is not continuous at $t = 0$.

$$\begin{bmatrix} e^t & t^2 - 1 \\ 2 & \sin^2 t \end{bmatrix} \tag{59}$$

$$\begin{bmatrix} t^3 - 3t & \sin(1/t) \\ 2t & 45 \end{bmatrix} \tag{60}$$

We shall use the notation $\mathbf{A}(t)$ to emphasize that the matrix $\mathbf{A}$ may depend on the variable t.

Definition 2. An $n \times n$ matrix $\mathbf{A}(t) = [a_{ij}(t)]$ is *differentiable* at $t = t_0$ if each of the elements $a_{ij}(t)$ $(i, j = 1, 2, \ldots, n)$ is differentiable at $t = t_0$ and

$$\frac{d\mathbf{A}(t)}{dt} = \left[\frac{da_{ij}(t)}{dt}\right]. \tag{61}$$

Generally we will use the notation $\dot{\mathbf{A}}(t)$ to represent $d\mathbf{A}(t)/dt$.

Example 1

Find $\dot{\mathbf{A}}(t)$ if

$$\mathbf{A}(t) = \begin{bmatrix} t^2 & \sin t \\ \ln t & e^{t^2} \end{bmatrix}.$$

Solution

$$\dot{\mathbf{A}}(t) = \frac{d\mathbf{A}(t)}{dt} = \begin{bmatrix} \dfrac{d(t^2)}{dt} & \dfrac{d(\sin t)}{dt} \\ \dfrac{d(\ln t)}{dt} & \dfrac{d(e^{t^2})}{dt} \end{bmatrix} = \begin{bmatrix} 2t & \cos t \\ \dfrac{1}{t} & 2te^{t^2} \end{bmatrix}. \qquad \square$$

Example 2

Find $\dot{\mathbf{A}}(t)$ if

$$\mathbf{A}(t) = \begin{bmatrix} 3t \\ 45 \\ t^2 \end{bmatrix}.$$

Solution.

$$\dot{\mathbf{A}}(t) = \begin{bmatrix} \dfrac{d(3t)}{dt} \\ \dfrac{d(45)}{dt} \\ \dfrac{d(t^2)}{dt} \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2t \end{bmatrix}.$$

□

Example 3

Find $\dot{\mathbf{x}}(t)$ if

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}.$$

Solution.

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix}.$$

□

The following properties of the derivative can be verified:

(P1) $\dfrac{d(\mathbf{A}(t) + \mathbf{B}(t))}{dt} = \dfrac{d\mathbf{A}(t)}{dt} + \dfrac{d\mathbf{B}(t)}{dt}.$

(P2) $\dfrac{d[\alpha\mathbf{A}(t)]}{dt} = \alpha\dfrac{d\mathbf{A}(t)}{dt},$ where α is a constant.

(P3) $\dfrac{d[\beta(t)\mathbf{A}(t)}{dt} = \left(\dfrac{d\beta(t)}{dt}\right)\mathbf{A}(t) + \beta(t)\left(\dfrac{d\mathbf{A}(t)}{dt}\right),$ when $\beta(t)$ is a scalar function of t.

(P4) $$\frac{d[\mathbf{A}(t)\mathbf{B}(t)]}{dt} = \left(\frac{d\mathbf{A}(t)}{dt}\right)\mathbf{B}(t) + \mathbf{A}(t)\left(\frac{d\mathbf{B}(t)}{dt}\right).$$

We warn the student to be very careful about the order of the matrices in (P4). Any commutation of the matrices on the right side will generally yield a wrong answer. For instance, it generally is not true that

$$\frac{d}{dt}[\mathbf{A}(t)\mathbf{B}(t)] = \left(\frac{d\mathbf{A}(t)}{dt}\right)\mathbf{B}(t) + \left(\frac{d\mathbf{B}(t)}{dt}\right)\mathbf{A}(t).$$

Example 4

Verify Property (P4) for

$$\mathbf{A}(t) = \begin{bmatrix} 2t & 3t^2 \\ 1 & t \end{bmatrix} \quad \text{and} \quad \mathbf{B}(t) = \begin{bmatrix} 1 & 2t \\ 3t & 2 \end{bmatrix}.$$

Solution

$$\begin{aligned}\frac{d}{dt}[\mathbf{A}(t)\mathbf{B}(t)] &= \frac{d}{dt}\left(\begin{bmatrix} 2t & 3t^2 \\ 1 & t \end{bmatrix}\begin{bmatrix} 1 & 2t \\ 3t & 2 \end{bmatrix}\right) \\ &= \frac{d}{dt}\begin{bmatrix} 2t + 9t^3 & 10t^2 \\ 1 + 3t^2 & 4t \end{bmatrix} = \begin{bmatrix} 2 + 27t^2 & 20t \\ 6t & 4 \end{bmatrix},\end{aligned}$$

and

$$\begin{aligned}\left[\frac{d\mathbf{A}(t)}{dt}\right]\mathbf{B}(t) + \mathbf{A}(t)\left[\frac{d\mathbf{B}(t)}{dt}\right] &= \begin{bmatrix} 2 & 6t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2t \\ 3t & 2 \end{bmatrix} + \begin{bmatrix} 2t & 3t^2 \\ 1 & t \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 + 27t^2 & 20t \\ 6t & 4 \end{bmatrix} \\ &= \frac{d[\mathbf{A}(t)\mathbf{B}(t)]}{dt}. \qquad \square\end{aligned}$$

We are now in a position to establish one of the more important properties of $e^{\mathbf{A}t}$. It is this property that makes the exponential so useful in differential equations (as we shall see in Chapter 8) and hence so fundamental in analysis.

Theorem 1. *If* $\mathbf{A}$ *is a constant matrix then*

$$\frac{de^{\mathbf{A}t}}{dt} = \mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}.$$

Proof. From (6) we have that

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{(\mathbf{A}t)^k}{k!}$$

or

$$e^{\mathbf{A}t} = \mathbf{I} + t\mathbf{A} + \frac{t^2\mathbf{A}^2}{2!} + \frac{t^3\mathbf{A}^3}{3!} + \cdots + \frac{t^{n-1}\mathbf{A}^{n-1}}{(n-1)!} + \frac{t^n\mathbf{A}^n}{n!} + \frac{t^{n+1}\mathbf{A}^{n+1}}{(n+1)!} + \cdots$$

Therefore,

$$\begin{aligned}
\frac{de^{\mathbf{A}t}}{dt} &= \mathbf{0} + \frac{\mathbf{A}}{1!} + \frac{2t\mathbf{A}^2}{2!} + \frac{3t^2\mathbf{A}^3}{3!} + \cdots + \frac{nt^{n-1}\mathbf{A}^n}{n!} + \frac{(n+1)t^n\mathbf{A}^{n+1}}{(n+1)!} + \cdots \\
&= \mathbf{A} + \frac{t\mathbf{A}^2}{1!} + \frac{t^2\mathbf{A}^3}{2!} + \cdots + \frac{t^{n-1}\mathbf{A}^n}{(n-1)!} + \frac{t^n\mathbf{A}^{n+1}}{n!} + \cdots \\
&= \left[\mathbf{I} + \frac{t\mathbf{A}}{1!} + \frac{t^2\mathbf{A}^2}{2!} + \cdots + \frac{t^{n-1}\mathbf{A}^{n-1}}{(n-1)!} + \frac{t^n\mathbf{A}^n}{n!} + \cdots\right]\mathbf{A} \\
&= e^{\mathbf{A}t}\mathbf{A}.
\end{aligned}$$

If we had factored $\mathbf{A}$ on the left, instead of on the right, we would have obtained the other identity,

$$\frac{de^{\mathbf{A}t}}{dt} = \mathbf{A}e^{\mathbf{A}t}.$$

Corollary 1. *If* $\mathbf{A}$ *is a constant matrix, then*

$$\frac{de^{-\mathbf{A}t}}{dt} = -\mathbf{A}e^{-\mathbf{A}t} = -e^{-\mathbf{A}t}\mathbf{A}.$$

Proof. Define $\mathbf{C} = -\mathbf{A}$. Hence, $e^{-\mathbf{A}t} = e^{\mathbf{C}t}$. Since $\mathbf{C}$ is a constant matrix, using Theorem 1, we have

$$\frac{de^{\mathbf{C}t}}{dt} = \mathbf{C}e^{\mathbf{C}t} = e^{\mathbf{C}t}\mathbf{C}.$$

If we now substitute for $\mathbf{C}$ its value, $-\mathbf{A}$, Corollary 1 is immediate.

Definition 3. An $n \times n$ matrix $\mathbf{A}(t) = [a_{ij}(t)]$ is *integrable* if each of its elements $a_{ij}(t)(i, 1, 2, \ldots, n)$ is integrable, and if this is the case,

$$\int \mathbf{A}(t)\,dt = \left[\int a_{ij}(t)\,dt\right].$$

Example 5

Find $\int \mathbf{A}(t)\,dt$ if

$$\mathbf{A}(t) = \begin{bmatrix} 3t & 2 \\ t^2 & e^t \end{bmatrix}.$$

Solution.

$$\int \mathbf{A}(t)\,dt = \begin{bmatrix} \int 3t\,dt & \int 2\,dt \\ \int t^2\,dt & \int e^t\,dt \end{bmatrix} = \begin{bmatrix} (\frac{3}{2})t^2 + c_1 & 2t + c_2 \\ (\frac{1}{3})t^3 + c_3 & e^t + c_4 \end{bmatrix}. \qquad \square$$

Example 6

Find $\int_0^1 \mathbf{A}(t)\,dt$ if

$$\mathbf{A}(t) = \begin{bmatrix} 2t & 1 & 2 \\ e^t & 6t^2 & -1 \\ \sin \pi t & 0 & 1 \end{bmatrix}.$$

Solution.

$$\int_0^1 A(t)\,dt = \begin{bmatrix} \int_0^1 2t\,dt & \int_0^1 1\,dt & \int_0^1 2\,dt \\ \int_0^1 e^t\,dt & \int_0^1 6t^2\,dt & \int_0^1 -1\,dt \\ \int_0^1 \sin \pi t\,dt & \int_0^1 0\,dt & \int_0^1 1\,dt \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 2 \\ e-1 & 2 & -1 \\ 2/\pi & 0 & 1 \end{bmatrix}. \qquad \square$$

The following property of the integral can be verified:

(P5) $$\int [\alpha \mathbf{A}(t) + \beta \mathbf{B}(t)]\,dt = \alpha \int \mathbf{A}(t)\,dt + \beta \int \mathbf{B}(t)\,dt,$$

where α and β are constants.

Problems 7.9

(1) Find $\dot{\mathbf{A}}(t)$ if

(a) $\mathbf{A}(t) = \begin{bmatrix} \cos t & t^2 - 1 \\ 2t & e^{(t-1)} \end{bmatrix}$.

(b) $\mathbf{A}(t) = \begin{bmatrix} 2e^{t^3} & t(t-1) & 17 \\ t^2 + 3t - 1 & \sin 2t & t \\ \cos^3(3t^2) & 4 & \ln t \end{bmatrix}$.

(2) Verify Properties (P1) – (P4) for

$$\alpha = 7, \qquad \beta(t) = t^2, \qquad \mathbf{A}(t) = \begin{bmatrix} t^3 & 3t^2 \\ 1 & 2t \end{bmatrix}, \quad \text{and} \quad \mathbf{B}(t) = \begin{bmatrix} t & -2t \\ t^3 & t^5 \end{bmatrix}.$$

(3) Prove that if $d\mathbf{A}(t)/dt = \mathbf{0}$, then $\mathbf{A}(t)$ is a constant matrix. (That is, a matrix independent of t).

(4) Find $\int \mathbf{A}(t)\,dt$ for the $\mathbf{A}(t)$ given in Problem 1(a).

(5) Verify Property (P5) for

$$\alpha = 2, \qquad \beta = 10, \qquad \mathbf{A}(t) = \begin{bmatrix} 6t & t^2 \\ 2t & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{B}(t) = \begin{bmatrix} t & 4t^2 \\ 1 & 2t \end{bmatrix}.$$

(6) Using Property (P4), derive a formula for differentiating $\mathbf{A}^2(t)$. Use this formula to find $d\mathbf{A}^2(t)/dt$, where

$$\mathbf{A}(t) = \begin{bmatrix} t & 2t^2 \\ 4t^3 & e^t \end{bmatrix},$$

and, show that $d\mathbf{A}^2(t)/dt \neq 2\mathbf{A}(t)\,d\mathbf{A}(t)/dt$. Therefore, the power rule of differentiation *does not hold* for matrices unless a matrix commutes with its derivative.

Appendix to Chapter 7

We begin a proof of the Cayley–Hamilton theorem by noting that if $\mathbf{B}$ is an $n \times n$ matrix having elements which are polynomials in λ with constant coefficients, then $\mathbf{B}$ can be expressed as a matrix polynomial in λ whose coefficients are $n \times n$ constant matrices. As an example, consider the following decomposition:

$$\begin{bmatrix} \lambda^3 + 2\lambda^2 + 3\lambda + 4 & 2\lambda^3 + 3\lambda^2 + 4\lambda + 5 \\ 3\lambda^3 + 4\lambda^2 + 5\lambda & 2\lambda + 3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}\lambda^3 + \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}\lambda^2 + \begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix}\lambda + \begin{bmatrix} 4 & 5 \\ 0 & 3 \end{bmatrix}.$$

In general, if the elements of $\mathbf{B}$ are polynomials of degree k or less, then

$$\mathbf{B} = \mathbf{B}_k\lambda^k + \mathbf{B}_{k-1}\lambda^{k-1} + \cdots + \mathbf{B}_1\lambda + \mathbf{B}_0,$$

where $\mathbf{B}_j (j = 0, 1, \ldots, k)$ is an $n \times n$ constant matrix.

Now let $\mathbf{A}$ be any arbitrary $n \times n$ matrix. Define

$$\mathbf{C} = (\mathbf{A} - \lambda\mathbf{I}) \tag{62}$$

and let

$$d(\lambda) = \lambda_n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 \tag{63}$$

represent the characteristic polynomial of $\mathbf{A}$. Thus,

$$d(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \det\mathbf{C}. \tag{64}$$

Since $\mathbf{C}$ is an $n \times n$ matrix, it follows that the elements of $\mathbf{C}^a$ (see Definition 2 of Section 4.5) will be polynomials in λ of either degree $n - 1$ or $n - 2$. (Elements on the diagonal of $\mathbf{C}^a$ will be polynomials of degree $n - 1$ while all other elements will be polynomials of degree $n - 2$.) Thus, $\mathbf{C}^a$ can be written as

$$\mathbf{C}^a = \mathbf{C}_{n-1}\lambda^{n-1} + \mathbf{C}_{n-2}\lambda^{n-2} + \cdots + \mathbf{C}_1\lambda + \mathbf{C}_0, \tag{65}$$

where $\mathbf{C}_j (j = 0, 1, \ldots, n - 1)$ is an $n \times n$ constant matrix.

From Theorem 2 of Section 4.5 and (64), we have that

$$\mathbf{C}^a\mathbf{C} = [\det\mathbf{C}]\mathbf{I} = d(\lambda)\mathbf{I}. \tag{66}$$

From (62), we have that

$$\mathbf{C}^a\mathbf{C} = \mathbf{C}^a(\mathbf{A} - \lambda\mathbf{I}) = \mathbf{C}^a\mathbf{A} - \lambda\mathbf{C}^a. \tag{67}$$

Equating (66) and (67), we obtain

$$d(\lambda)\mathbf{I} = \mathbf{C}^a\mathbf{A} - \lambda\mathbf{C}^a. \tag{68}$$

Substituting (63) and (65) into (68), we find that

$$\begin{aligned}\mathbf{I}\lambda_n + a_{n-1}\mathbf{I}\lambda^{n-1} + \cdots + a_1\mathbf{I}\lambda + a_0\mathbf{I} \\ = \mathbf{C}_{n-1}\mathbf{A}\lambda^{n-1} + \mathbf{C}_{n-2}\mathbf{A}\lambda^{n-2} + \cdots + \mathbf{C}_1\mathbf{A}\lambda + \mathbf{C}_0\mathbf{A} \\ -\mathbf{C}_{n-1}\lambda^{n} - \mathbf{C}_{n-2}\lambda^{n-1} - \cdots - \mathbf{C}_1\lambda^2 - \mathbf{C}_0\lambda.\end{aligned}$$

Both sides of this equation are matrix polynomials in λ of degree n. Since two polynomials are equal if and only if their corresponding coefficients are equal we have

$$\begin{aligned}\mathbf{I} &= -\mathbf{C}_{n-1} \\ a_{n-1}\mathbf{I} &= -\mathbf{C}_{n-2} + \mathbf{C}_{n-1}\mathbf{A} \\ &\vdots \\ a_1\mathbf{I} &= -\mathbf{C}_0 + \mathbf{C}_1\mathbf{A} \\ a_0\mathbf{I} &= \mathbf{C}_0\mathbf{A}.\end{aligned} \tag{69}$$

Multiplying the first equation in (69) by $\mathbf{A}^n$, the second equation by $\mathbf{A}^{n-1}, \ldots,$ and the last equation by $\mathbf{A}^0 = \mathbf{I}$ and adding, we obtain (note that the terms on the right-hand side cancel out)

$$\mathbf{A}^n + a_{n-1}\mathbf{A}^{n-1} + \cdots + a_1\mathbf{A} + \mathbf{A}_0\mathbf{I} = \mathbf{O}. \tag{70}$$

Equation (70) is the Cayley–Hamilton theorem.

Chapter 8

Linear Differential Equations

8.1 Fundamental Form

We are now ready to solve linear differential equations. The method that we shall use involves introducing new variables $x_1(t), x_2(t), \ldots, x_n(t)$, and then reducing a given system of differential equations to the system

$$\begin{aligned}
\frac{dx_1(t)}{dt} &= a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \cdots + a_{1n}(t)x_n(t) + f_1(t) \\
\frac{dx_2(t)}{dt} &= a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \cdots + a_{2n}(t)x_n(t) + f_2(t) \\
&\vdots \\
\frac{dx_n(t)}{dt} &= a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \cdots + a_{nn}(t)x_n(t) + f_n(t).
\end{aligned} \tag{1}$$

If we define

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix},$$

$$\mathbf{A}(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}, \quad \text{and} \quad \mathbf{f}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}, \tag{2}$$

then (1) can be rewritten in the matrix form

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t). \tag{3}$$

Example 1

Put the following system into matrix form:

$$\dot{y}(t) = t^2 y(t) + 3z(t) + \sin t,$$

$$\dot{z}(t) = -e^t y(t) + tz(t) - t^2 + 1.$$

Note that we are using the standard notation $\dot{y}(t)$ and $\dot{z}(t)$ to represent

$$\frac{dy(t)}{dt} \quad \text{and} \quad \frac{dz(t)}{dt}.$$

Solution. Define $x_1(t) = y(t)$ and $x_2(t) = z(t)$. This system is then equivalent to the matrix equation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} t^2 & 3 \\ -e^t & t \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \sin t \\ -t^2 + 1 \end{bmatrix}. \tag{4}$$

If we define

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \mathbf{A}(t) = \begin{bmatrix} t^2 & 3 \\ -e^t & t \end{bmatrix}, \quad \text{and} \quad \mathbf{f}(t) = \begin{bmatrix} \sin t \\ -t^2 + 1 \end{bmatrix}$$

then (4) is in the required form, $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t)$. □

In practice, we are usually interested in solving an initial value problem; that is, we seek functions $x_1(t), x_2(t), \ldots, x_n(t)$ that satisfy not only the differential equations given by (1) but also a set of initial conditions of the form

$$x_1(t_0) = c_1, \quad x_2(t_0) = c_2, \ldots, x_n(t_0) = c_n, \tag{5}$$

where $c_1, c_2, \ldots, c_n$, and t_0 are known constants. Upon defining

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix},$$

it follows from the definition of $\mathbf{x}(t)$ (see Eqs. (2) and (5)) that

$$\mathbf{x}(t_0) = \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \\ \vdots \\ x_n(t_0) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \mathbf{c}.$$

Thus, the initial conditions can be put into the matrix form

$$\mathbf{x}(t_0) = \mathbf{c}. \tag{6}$$

Definition 1. A system of differential equations is in *fundamental form* if it is given by the matrix equations

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t) \\ \mathbf{x}(t_0) &= \mathbf{c}. \end{aligned} \tag{7}$$

Example 2

Put the following system into fundamental form:

$$\begin{aligned} \dot{x}(t) &= 2x(t) - ty(t) \\ \dot{y}(t) &= t^2x(t) + e^t \\ x(2) &= 3, \qquad y(2) = 1. \end{aligned}$$

Solution. Define $x_1(t) = x(t)$ and $x_2(t) = y(t)$. This system is then equivalent to the matrix equations

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} 2 & -t \\ t^2 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ e^t \end{bmatrix} \\ \begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} &= \begin{bmatrix} 3 \\ 1 \end{bmatrix}. \end{aligned} \tag{8}$$

Consequently, if we define

$$\mathbf{x}(t)=\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix},\qquad \mathbf{A}(t)=\begin{bmatrix}2 & -t\\t^2 & 0\end{bmatrix},$$

$$\mathbf{f}(t)=\begin{bmatrix}0\\e^t\end{bmatrix},\qquad \mathbf{c}=\begin{bmatrix}3\\1\end{bmatrix},\quad\text{and}\quad t_0=2,$$

then (8) is in fundamental form. □

Example 3

Put the following system into fundamental form:

$$\begin{aligned}\dot{l}(t) &= 2l(t) + 3m(t) - n(t)\\ \dot{m}(t) &= l(t) - m(t)\\ \dot{n}(t) &= m(t) - n(t)\end{aligned}$$

$$l(15)=0,\qquad m(15)=-170,\qquad n(15)=1007.$$

Solution. Define $x_1(t)=l(t)$, $x_2(t)=m(t)$, $x_3(t)=n(t)$. This system is then equivalent to the matrix equations

$$\begin{bmatrix}\dot{x}_1(t)\\\dot{x}_2(t)\\\dot{x}_3(t)\end{bmatrix}=\begin{bmatrix}2 & 3 & -1\\1 & -1 & 0\\0 & 1 & -1\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{bmatrix},$$

(9)

$$\begin{bmatrix}x_1(15)\\x_2(15)\\x_3(15)\end{bmatrix}=\begin{bmatrix}0\\-170\\1007\end{bmatrix}.$$

Thus, if we define

$$\mathbf{x}(t)=\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{bmatrix},\qquad \mathbf{A}(t)=\begin{bmatrix}2 & 3 & -1\\1 & -1 & 0\\0 & 1 & -1\end{bmatrix},\qquad \mathbf{f}(t)=\begin{bmatrix}0\\0\\0\end{bmatrix},$$

$$\mathbf{c}=\begin{bmatrix}0\\-170\\1007\end{bmatrix}\quad\text{and}\quad t_0=15,$$

then (9) is in fundamental form. □

Definition 2. A system in fundamental form is *homogeneous* if $\mathbf{f}(t)=\mathbf{0}$ (that is, if $f_1(t)=f_2(t)=\cdots=f_n(t)=0$) and *nonhomogeneous* if $\mathbf{f}(t)\neq\mathbf{0}$ (that is, if at least one component of $\mathbf{f}(t)$ differs from zero).

The systems given in Examples 2 and 3 are nonhomogeneous and homogeneous respectively.

Since we will be attempting to solve differential equations, it is important to know exactly what is meant by a solution.

Definition 3. $\mathbf{x}(t)$ is a *solution* of (7) if

(a) both $\mathbf{x}(t)$ and $\dot{\mathbf{x}}(t)$ are continuous in some neighborhood J of the initial time $t = t_0$,

(b) the substitution of $\mathbf{x}(t)$ into the differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t)$$

makes the equation an identity in t on the interval J; that is, the equation is valid for each t in J, and

(c) $\mathbf{x}(t_0) = \mathbf{c}$.

It would also seem advantageous, before trying to find the solutions, to know whether or not a given system has any solutions at all, and if it does, how many. The following theorem from differential equations answers both of these questions.

Theorem 1. *Consider a system given by (7). If* $\mathbf{A}(t)$ *and* $\mathbf{f}(t)$ *are continuous in some interval containing* $t = t_0$*, then this system possesses a unique continuous solution on that interval.*

Hence, to insure the applicability of this theorem, we assume for the remainder of the chapter that $\mathbf{A}(t)$ and $\mathbf{f}(t)$ are both continuous on some common interval containing $t = t_0$.

Problems 8.1

In Problems 1 through 8, put the given systems into fundamental form.

(1) $$\frac{dx(t)}{dt} = 2x(t) + 3y(t),$$

$$\frac{dy(t)}{dt} = 4x(t) + 5y(t),$$

$$x(0) = 6, \qquad y(0) = 7.$$

(2) $\dot{y}(t) = 3y(t) + 2z(t),$

$\dot{z}(t) = 4y(t) + z(t),$

$y(0) = 1, \qquad z(0) = 1.$

(3) $\dfrac{dx(t)}{dt} = -3x(t) + 3y(t) + 1,$

$\dfrac{dy(t)}{dt} = \quad 4x(t) - 4y(t) - 1,$

$x(0) = 0, \qquad y(0) = 0.$

(4) $\dfrac{dx(t)}{dt} = 3x(t) + t,$

$\dfrac{dy(t)}{dt} = 2x(t) + t + 1,$

$x(0) = 1, \qquad y(0) = -1.$

(5) $\dfrac{dx(t)}{dt} = 3t^2x(t) + 7y(t) + 2\,,$

$\dfrac{dy(t)}{dt} = \quad x(t) + \ ty(t) + 2t,$

$x(1) = 2, \qquad y(1) = -3.$

(6) $\dfrac{du(t)}{dt} = e^t u(t) + \ tv(t) + \qquad w(t),$

$\dfrac{dv(t)}{dt} = t^2u(t) - 3v(t) + (t+1)w(t),$

$\dfrac{dw(t)}{dt} = \qquad\qquad v(t) + \qquad e^{t^2}w(t),$

$u(4) = 0, \qquad v(4) = 1, \qquad z(4) = -1.$

(7) $\dfrac{dx(t)}{dt} = \quad 6y(t) + \ z(t),$

$\dfrac{dy(t)}{dt} = x(t) \qquad - 3z(t),$

$\dfrac{dz(t)}{dt} = -2y(t),$

$x(0) = 10, \qquad y(0) = 10, \qquad z(0) = 20.$

(8) $\dot{r}(t) = t^2 r(t) - 3s(t) - (\sin t)u(t) + \sin t,$

$\dot{s}(t) = r(t) - s(t) + t^2 - 1,$

$\dot{u}(t) = 2r(t) + e^t s(t) + (t^2 - 1)u(t) + \cos t,$

$r(1) = 4, \qquad s(1) = -2, \qquad u(1) = 5.$

(9) Determine which of the following are solutions to the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}:$$

(a) $\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$, (b) $\begin{bmatrix} e^t \\ 0 \end{bmatrix}$, (c) $\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$.

(10) Determine which of the following are solutions to the system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}:$$

(a) $\begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$, (b) $\begin{bmatrix} e^{-t} \\ 2e^{-t} \end{bmatrix}$, (c) $\begin{bmatrix} e^{5t} \\ 2e^{5t} \end{bmatrix}$.

(11) Determine which of the following are solutions to the system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}:$$

(a) $\begin{bmatrix} -e^{2t} + 2e^t \\ -2e^{2t} + 2e^t \end{bmatrix}$, (b) $\begin{bmatrix} -e^{2(t-1)} + 2e^{(t-1)} \\ -2e^{2(t-1)} + 2e^{(t-1)} \end{bmatrix}$,

(c) $\begin{bmatrix} e^{2(t-1)} \\ 0 \end{bmatrix}$.

8.2 Reduction of an nth Order Equation

Before seeking solutions to linear differential equations, we will first develop techniques for reducing these equations to fundamental form. In this section, we consider the initial-value problems given by

$$\begin{gathered} a_n(t)\frac{d^n x(t)}{dt^n} + a_{n-1}(t)\frac{d^{n-1}x(t)}{dt^{n-1}} + \cdots + a_1(t)\frac{dx(t)}{dt} + a_0(t)x(t) = f(t) \\ x(t_0) = c_1, \quad \frac{dx(t_0)}{dt} = c_2, \ldots, \quad \frac{d^{n-1}x(t_0)}{dt^{n-1}} = c_n. \end{gathered} \tag{10}$$

Equation (10) is an nth order differential equation for $x(t)$ where $a_0(t)$,

$a_1(t), \ldots, a_n(t)$ and $f(t)$ are assumed known and continuous on some interval containing t_0. Furthermore, we assume that $a_n(t) \neq 0$ on this interval.

A method of reduction, particularly useful for differential equations defined by system (10), is the following:

STEP 1. Rewrite (10) so that the nth derivative of $x(t)$ appears by itself;

$$\frac{d^n x(t)}{dt^n} = -\frac{a_{n-1}(t)}{a_n(t)} \frac{d^{n-1}x(t)}{dt^{n-1}} - \cdots - \frac{a_1(t)}{a_n(t)} \frac{dx(t)}{dt} - \frac{a_0(t)}{a_n(t)} x(t) + \frac{f(t)}{a_n(t)}. \tag{11}$$

STEP 2. Define n new variables (the same number as the order of the differential equation), $x_1(t), x_2(t), \ldots, x_n(t)$ by the equations

$$\begin{aligned} x_1 &= x(t), \\ x_2 &= \frac{dx_1}{dt}, \\ x_3 &= \frac{dx_2}{dt}, \\ &\vdots \\ x_{n-1} &= \frac{dx_{n-2}}{dt}, \\ x_n &= \frac{dx_{n-1}}{dt}. \end{aligned} \tag{12}$$

Generally, we will write $x_j(t)$ $(j = 1, 2, \ldots, n)$ simply as x_j when the dependence on the variable t is obvious from context. It is immediate from system (12) that we also have the following relationships between $x_1, x_2, \ldots, x_n$ and the unknown $x(t)$:

$$\begin{aligned} x_1 &= x, \\ x_2 &= \frac{dx}{dt}, \\ x_3 &= \frac{d^2x}{dt^2}, \\ &\vdots \\ x_{n-1} &= \frac{d^{n-2}x}{dt^{n-2}}, \\ x_n &= \frac{d^{n-1}x}{dt^{n-1}}. \end{aligned} \tag{13}$$

Hence, by differentiating the last equation of (13), we have

$$\frac{dx_n}{dt} = \frac{d^n x}{dt^n}. \tag{14}$$

STEP 3. Rewrite dx_n/dt in terms of the new variables $x_1, x_2, \ldots, x_n$.

Substituting (11) into (14), we have

$$\frac{dx_n}{dt} = -\frac{a_{n-1}(t)}{a_n(t)}\frac{d^{n-1}x}{dt^{n-1}} - \cdots - \frac{a_1(t)}{a_n(t)}\frac{dx}{dt} - \frac{a_0(t)}{a_n(t)}x + \frac{f(t)}{a_n(t)}.$$

Substituting (13) into this equation, we obtain

$$\frac{dx_n}{dt} = -\frac{a_{n-1}(t)}{a_n(t)}x_n - \cdots - \frac{a_1(t)}{a_n(t)}x_2 - \frac{a_0(t)}{a_n(t)}x_1 + \frac{f(t)}{a_n(t)}. \tag{15}$$

STEP 4. Form a system of n first-order differential equations for $x_1, x_2, \ldots, x_n$.

Using (12) and (15), we obtain the system

$$\begin{aligned}
\frac{dx_1}{dt} &= x_2, \\
\frac{dx_2}{dt} &= x_3, \\
&\vdots \\
\frac{dx_{n-2}}{dt} &= x_{n-1}, \\
\frac{dx_{n-1}}{dt} &= x_n, \\
\frac{dx_n}{dt} &= -\frac{a_0(t)}{a_n(t)}x_1 - \frac{a_1(t)}{a_n(t)}x_2 - \cdots - \frac{a_{n-1}(t)}{a_n(t)}x_n + \frac{f(t)}{a_n(t)}.
\end{aligned} \tag{16}$$

Note that in the last equation of (16) we have rearranged the order of (15) so that the x_1 term appears first, the x_2 term appears second, etc. This was done in order to simplify the next step.

STEP 5. Put (16) into matrix form.

Define

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix},$$

$$\mathbf{A}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -\dfrac{a_0(t)}{a_n(t)} & -\dfrac{a_1(t)}{a_n(t)} & -\dfrac{a_2(t)}{a_n(t)} & -\dfrac{a_3(t)}{a_n(t)} & \cdots & -\dfrac{a_{n-1}(t)}{a_n(t)} \end{bmatrix}, \tag{17}$$

and

$$\mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \dfrac{f(t)}{a_n(t)} \end{bmatrix}.$$

Then (16) can be written as

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t). \tag{18}$$

▶ | Step 6. Rewrite the initial conditions in matrix form.

From (17), (13), and (10), we have that

$$\mathbf{x}(t_0) = \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \\ \vdots \\ x_n(t_0) \end{bmatrix} = \begin{bmatrix} x(t_0) \\ \dfrac{dx(t_0)}{dt} \\ \vdots \\ \dfrac{d^{n-1}x(t_0)}{dt^{n-1}} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

Thus, if we define

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix},$$

the initial conditions can be put into matrix form

$$\mathbf{x}(t_0) = \mathbf{c}. \tag{19}$$

Equations (18) and (19) together represent the fundamental form for (10).

Since $\mathbf{A}(t)$ and $\mathbf{f}(t)$ are continuous (why?), Theorem 1 of the previous section guarantees that a unique solution exists to (18) and (19). Once this solution is obtained, $\mathbf{x}(t)$ will be known; hence, the components of $\mathbf{x}(t)$, $x_1(t), \ldots, x_n(t)$ will be known and, consequently, so will $x(t)$, the variable originally sought (from (12), $x_1(t) = x(t)$).

Example 1

Put the following initial-value problem into fundamental form:

$$2\ddddot{x} - 4\dddot{x} + 16t\ddot{x} - \dot{x} + 2t^2x = \sin t,$$

$$x(0) = 1, \qquad \dot{x}(0) = 2, \qquad \ddot{x}(0) = -1, \qquad \dddot{x}(0) = 0.$$

Solution. The differential equation may be rewritten as

$$\ddddot{x} = 2\dddot{x} - 8t\ddot{x} + \tfrac{1}{2}\dot{x} - t^2x + (\tfrac{1}{2})\sin t.$$

Define

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x}_1 = \dot{x}, \\ x_3 &= \dot{x}_2 = \ddot{x}, \\ x_4 &= \dot{x}_3 = \dddot{x} \end{aligned}$$

hence, $\dot{x}_4 = \ddddot{x}$. Thus,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \ddddot{x} = 2\dddot{x} - 8t\ddot{x} + \tfrac{1}{2}\dot{x} - t^2x + \tfrac{1}{2}\sin t \\ &= 2x_4 - 8tx_3 + \tfrac{1}{2}x_2 - t^2x_1 + \tfrac{1}{2}\sin t, \end{aligned}$$

or

$$\begin{aligned} \dot{x}_1 &= \quad x_2 \\ \dot{x}_2 &= \quad\quad x_3 \\ \dot{x}_3 &= \quad\quad\quad x_4 \\ \dot{x}_4 &= -t^2x_1 + \tfrac{1}{2}x_2 - 8tx_3 + 2x_4 + \tfrac{1}{2}\sin t. \end{aligned}$$

Define

$$\mathbf{x}(t) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \qquad \mathbf{A}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -t^2 & \frac{1}{2} & -8t & 2 \end{bmatrix},$$

$$\mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2}\sin t \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad t_0 = 0.$$

Thus, the initial value problem may be rewritten in the fundamental form

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t),$$
$$\mathbf{x}(t_0) = \mathbf{c}. \qquad \square$$

Example 2

Put the following initial value problem into fundamental form:

$$e^t \frac{d^5x}{dt^5} - 2e^{2t}\frac{d^4x}{dt^4} + tx = 4e^t,$$

$$x(2) = 1, \qquad \frac{dx(2)}{dt} = -1, \qquad \frac{d^2x(2)}{dt^2} = -1, \qquad \frac{d^3x(2)}{dt^3} = 2, \qquad \frac{d^4x(2)}{dt^4} = 3.$$

Solution. The differential equation may be rewritten

$$\frac{d^5x}{dt^5} = 2e^t\frac{d^4x}{dt^4} - te^{-t}x + 4.$$

Define

$$x_1 = x$$
$$x_2 = \dot{x}_1 = \dot{x}$$
$$x_3 = \dot{x}_2 = \ddot{x}$$
$$x_4 = \dot{x}_3 = \dddot{x}$$
$$x_5 = \dot{x}_4 = \frac{d^4x}{dt^4};$$

hence,

$$\dot{x}_5 = \frac{d^5x}{dt^5}.$$

Thus,

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= x_5 \\
\dot{x}_5 &= \frac{d^5x}{dt^5} = 2e^t\frac{d^4x}{dt^4} - te^{-t}x + 4 \\
&= 2e^t x_5 - te^{-t}x_1 + 4,
\end{aligned}$$

or

$$\begin{aligned}
\dot{x}_1 &= \quad x_2 \\
\dot{x}_2 &= \quad\quad x_3 \\
\dot{x}_3 &= \quad\quad\quad x_4 \\
\dot{x}_4 &= \quad\quad\quad\quad x_5 \\
\dot{x}_5 &= -te^{-t}x_1 \quad\quad + 2e^t x_5 + 4.
\end{aligned}$$

Define

$$\mathbf{x}(t) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad \mathbf{A}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -te^{-t} & 0 & 0 & 0 & 2e^t \end{bmatrix},$$

$$\mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 4 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \quad \text{and} \quad t_0 = 2.$$

Thus, the initial value problem may be rewritten in the fundamental form

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t),$$

$$\mathbf{x}(t_0) = \mathbf{c}. \quad \square$$

Problems 8.2

Put the following initial-value problems into fundamental form:

(1) $\dfrac{d^2x}{dt^2} - 2\dfrac{dx}{dt} - 3x = 0;$

$x(0) = 4, \quad \dfrac{dx(0)}{dt} = 5.$

(2) $\dfrac{d^2x}{dt^2} + e^t\dfrac{dx}{dt} - tx = 0;$

$x(1) = 2, \quad \dfrac{dx(1)}{dt} = 0.$

(3) $\dfrac{d^2x}{dt^2} - x = t^2;$

$x(0) = -3, \quad \dfrac{dx(0)}{dt} = 3.$

(4) $e^t\dfrac{d^2x}{dt^2} - 2e^{2t}\dfrac{dx}{dt} - 3e^t x = 2;$

$x(0) = 0, \quad \dfrac{dx(0)}{dt} = 0.$

(5) $\ddot{x} - 3\dot{x} + 2x = e^{-t},$

$x(1) = \dot{x}(1) = 2.$

(6) $4\dddot{x} + t\ddot{x} - x = 0,$

$x(-1) = 2, \quad \dot{x}(-1) = 1, \quad \ddot{x}(-1) = -205.$

(7) $e^t\dfrac{d^4x}{dt^4} + t\dfrac{d^2x}{dt^2} = 1 + \dfrac{dx}{dt},$

$x(0) = 1, \quad \dfrac{dx(0)}{dt} = 2, \quad \dfrac{d^2x(0)}{dt^2} = \pi, \quad \dfrac{d^3x(0)}{dt^3} = e^3.$

(8) $\dfrac{d^6x}{dt^6} + 4\dfrac{d^4x}{dt^4} = t^2 - t,$

$x(\pi) = 2, \quad \dot{x}(\pi) = 1, \quad \ddot{x}(\pi) = 0, \quad \dddot{x}(\pi) = 2,$

$\dfrac{d^4x(\pi)}{dt^4} = 1, \quad \dfrac{d^5x(\pi)}{dt^5} = 0.$

8.3 Reduction of a System

Based on our work in the preceding section, we are now able to reduce systems of higher order linear differential equations to fundamental form. The method, which is a straightforward extension of that used to reduce the

nth order differential equation to fundamental form, is best demonstrated by examples.

Example 1

Put the following system into fundamental form:

$$\begin{aligned} \dddot{x} &= 5\ddot{x} + \dot{y} - 7y + e^t, \\ \ddot{y} &= \dot{x} - 2\dot{y} + 3y + \sin t, \end{aligned} \tag{20}$$

$$x(1) = 2, \qquad \dot{x}(1) = 3, \qquad \ddot{x}(1) = -1, \qquad y(1) = 0, \qquad \dot{y}(1) = -2.$$

STEP 1. Rewrite the differential equations so that the highest derivative of *each* unknown function appears by itself. For the above system, this has already been done.

STEP 2. Define new variables $x_1(t)$, $x_2(t)$, $x_3(t)$, $y_1(t)$, and $y_2(t)$. (Since the highest derivative of $x(t)$ is of order 3, and the highest derivative of $y(t)$ is of order 2, we need 3 new variables for $x(t)$ and 2 new variables for $y(t)$. In general, for each unknown function we define a set of k new variables, where k is the order of the highest derivative of the original function appearing in the system under consideration). The new variables are defined in a manner analogous to that used in the previous section:

$$\begin{aligned} x_1 &= x, \\ x_2 &= \dot{x}_1, \\ x_3 &= \dot{x}_2, \\ y_1 &= y, \\ y_2 &= \dot{y}_1. \end{aligned} \tag{21}$$

From (21), the new variables are related to the functions $x(t)$ and $y(t)$ by the following:

$$\begin{aligned} x_1 &= x, \\ x_2 &= \dot{x}, \\ x_3 &= \ddot{x}, \\ y_1 &= y, \\ y_2 &= \dot{y}. \end{aligned} \tag{22}$$

It follows from (22), by differentiating x_3 and y_2, that

$$\begin{aligned} \dot{x}_3 &= \dddot{x}, \\ \dot{y}_2 &= \ddot{y}. \end{aligned} \tag{23}$$

STEP 3. Rewrite $\dot{x}_3$ and $\dot{y}_2$ in terms of the new variables defined in (21).

Substituting (20) into (23), we have

$$\begin{aligned} \dot{x}_3 &= 5\ddot{x} + \dot{y} - 7y + e^t, \\ \dot{y}_2 &= \dot{x} - 2\dot{y} + 3y + \sin t. \end{aligned}$$

Substituting (22) into these equations, we obtain

$$\begin{aligned} \dot{x}_3 &= 5x_3 + y_2 - 7y_1 + e^t, \\ \dot{y}_2 &= x_2 - 2y_2 + 3y_1 + \sin t. \end{aligned} \tag{24}$$

STEP 4. Set up a system of first-order differential equations for x_1, x_2, x_3, y_1, and y_2.

Using (21) and (24), we obtain the system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= 5x_3 - 7y_1 + y_2 + e^t, \\ \dot{y}_1 &= y_2, \\ \dot{y}_2 &= x_2 + 3y_1 - 2y_2 + \sin t. \end{aligned} \tag{25}$$

Note, that for convenience we have rearranged terms in some of the equations to present them in their natural order.

STEP 5. Write (25) in matrix form.

Define

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ y_1(t) \\ y_2(t) \end{bmatrix}, \quad \mathbf{A}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & -7 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 & -2 \end{bmatrix}, \quad \text{and} \quad \mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \\ e^t \\ 0 \\ \sin t \end{bmatrix}. \tag{26}$$

Thus, Eq. (25) can be rewritten in the matrix form

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t). \tag{27}$$

STEP 6. Rewrite the initial conditions in matrix form.

From Eqs. (26), (22). and (20) we have

$$\mathbf{x}(1) = \begin{bmatrix} x_1(1) \\ x_2(1) \\ x_3(1) \\ y_1(1) \\ y_2(1) \end{bmatrix} = \begin{bmatrix} x(1) \\ \dot{x}(1) \\ \ddot{x}(1) \\ y(1) \\ \dot{y}(1) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \\ 0 \\ -2 \end{bmatrix}.$$

Thus, if we define

$$\mathbf{c} = \begin{bmatrix} 2 \\ 3 \\ -1 \\ 0 \\ -2 \end{bmatrix}$$

and $t_0 = 1$, then the initial conditions can be rewritten as

$$\mathbf{x}(1) = \mathbf{c}. \qquad \square \tag{28}$$

Since $\mathbf{A}(t)$ and $\mathbf{f}(t)$ are continuous, (27) and (28) possess a unique solution. Once $\mathbf{x}(t)$ is known, we immediately have the components of $\mathbf{x}(t)$, namely $x_1(t), x_2(t), x_3(t), y_1(t)$ and $y_2(t)$. Thus, we have the functions $x(t)$ and $y(t)$ (from (21), $x_1(t) = x(t)$ and $y_1(t) = y(t)$).

All similar systems containing higher order derivatives may be put into fundamental form in exactly the same manner as that used here.

Example 2

Put the following system into fundamental form:

$$\dddot{x} = 2\dot{x} + t\dot{y} - 3z + t^2\dot{z} + t,$$

$$\dddot{y} = \dot{z} + (\sin t)y + x - t,$$

$$\ddot{z} = \ddot{x} - \ddot{y} + t^2 + 1;$$

$$x(\pi) = 15, \quad \dot{x}(\pi) = 59, \quad \ddot{x}(\pi) = -117, \quad y(\pi) = 2, \quad \dot{y}(\pi) = -19,$$

$$\ddot{y}(\pi) = 3, \quad z(\pi) = 36, \quad \dot{z}(\pi) = -212.$$

Solution. Define

$$x_1 = x$$

$$x_2 = \dot{x}_1 = \dot{x}$$

$$
\begin{aligned}
&x_3 = \dot{x}_2 = \ddot{x}; \text{ hence, } \dot{x}_3 = \dddot{x}.\\
&y_1 = y\\
&y_2 = \dot{y}_1 = \dot{y}\\
&y_3 = \dot{y}_2 = \ddot{y}; \text{ hence, } \dot{y}_3 = \dddot{y}.\\
&z_1 = z\\
&z_2 = \dot{z}_1 = \dot{z}; \text{ hence, } \dot{z}_2 = \ddot{z}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\dot{x}_1 &= x_2\\
\dot{x}_2 &= x_3\\
\dot{x}_3 &= \dddot{x} = 2\dot{x} + t\dot{y} - 3z + t^2\dot{z} + t\\
&= 2x_2 + ty_2 - 3z_1 + t^2 z_2 + t;\\
\dot{y}_1 &= y_2\\
\dot{y}_2 &= y_3\\
\dot{y}_3 &= \dddot{y} = \dot{z} + (\sin t)y + x - t\\
&= z_2 + (\sin t)y_1 + x_1 - t;\\
\dot{z}_1 &= z_2\\
\dot{z}_2 &= \ddot{z} = \ddot{x} - \ddot{y} + t^2 + 1\\
&= x_3 - y_3 + t^2 + 1;
\end{aligned}
$$

or

$$
\begin{array}{rlllllllll}
\dot{x}_1 = & & x_2 & & & & & & & \\
\dot{x}_2 = & & & x_3 & & & & & & \\
\dot{x}_3 = & & 2x_2 & & & + ty_2 & & - 3z_1 & + t^2 z_2 & + t \\
\dot{y}_1 = & & & & & y_2 & & & & \\
\dot{y}_2 = & & & & & & y_3 & & & \\
\dot{y}_3 = & x_1 & & & + (\sin t)y_1 & & & & z_2 & - t \\
\dot{z}_1 = & & & & & & & & z_2 & \\
\dot{z}_2 = & & & x^3 & & & - y_3 & & & + t^2 + 1.
\end{array}
$$

Define

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ y_1 \\ y_2 \\ y_3 \\ z_1 \\ z_2 \end{bmatrix}, \qquad \mathbf{A}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & t & 0 & -3 & t^2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & \sin t & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix},$$

$$\mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \\ t \\ 0 \\ 0 \\ -t \\ 0 \\ t^2+1 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} 15 \\ 59 \\ -117 \\ 2 \\ -19 \\ 3 \\ 36 \\ -212 \end{bmatrix}, \quad \text{and} \quad t_0 = \pi.$$

Thus, the system can now be rewritten in the fundamental form

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t),$$

$$\mathbf{x}(t_0) = \mathbf{c}. \qquad \square$$

Problems 8.3

Put the following initial-value problems into fundamental form:

(1) $$\frac{d^2x}{dt^2} = 2\frac{dx}{dt} + 3x + 4y,$$

$$\frac{dy}{dt} = 5x - 6y,$$

$$x(0) = 7,$$

$$\frac{dx(0)}{dt} = 8,$$

$$y(0) = 9.$$

(2) $$\frac{d^2x}{dt^2} = \frac{dx}{dt} + \frac{dy}{dt},$$

$$\frac{d^2y}{dt^2} = \frac{dy}{dt} - \frac{dx}{dt},$$

$$x(0) = 2, \qquad \frac{dx(0)}{dt} = 3, \qquad y(0) = 4, \qquad \frac{dy(0)}{dt} = 4.$$

(3) $$\frac{dx}{dt} = t^2\frac{dy}{dt} - 4x,$$

$$\frac{d^2y}{dt^2} = ty + t^2x,$$

$$x(2) = -1, \qquad y(2) = 0, \qquad \frac{dy(2)}{dt} = 0.$$

(4) $$\frac{dx}{dt} = 2\frac{dy}{dt} - 4x + t,$$

$$\frac{d^2y}{dt^2} = ty + 3x - 1,$$

$$x(3) = 0, \qquad y(3) = 0, \qquad \frac{dy(3)}{dt} = 0.$$

(5) $\ddot{x} = 2\dot{x} + \dddot{y} - t,$

$\dddot{y} = tx - ty + \ddot{y} - e^t;$

$x(-1) = 2, \qquad \dot{x}(-1) = 0, \qquad y(-1) = 0, \qquad \dot{y}(-1) = 3, \qquad \ddot{y}(-1) = 9,$

$\dddot{y}(-1) = 4.$

(6) $\dddot{x} = x - y + \dot{y},$

$\ddot{y} = \ddot{x} - x + 2\dot{y};$

$x(0) = 21, \qquad \dot{x}(0) = 4, \qquad \ddot{x}(0) = -5, \qquad y(0) = 5, \qquad \dot{y}(0) = 7.$

(7) $\dot{x} = y - 2,$

$\ddot{y} = z - 2,$

$\dot{z} = x + y;$

$x(\pi) = 1, \qquad y(\pi) = 2, \qquad \dot{y}(\pi) = 17, \qquad z(\pi) = 0.$

(8) $\ddot{x} = y + z + 2,$

$\ddot{y} = x + y - 1,$

$\ddot{z} = x - z + 1;$

$x(20) = 4, \quad \dot{x}(20) = -4, \quad y(20) = 5, \quad \dot{y}(20) = -5, \quad z(20) = 9,$

$\dot{z}(20) = -9.$

8.4 Solutions of Systems with Constant Coefficients

In general, when one reduces a system of differential equations to fundamental form, the matrix $\mathbf{A}(t)$ will depend explicitly on the variable t. For some systems however, $\mathbf{A}(t)$ does not vary with t (that is, every element of $\mathbf{A}(t)$ is a constant). If this is the case, the system is said to have *constant coefficients*. For instance, in Section 8.1, Example 3 illustrates a system having constant coefficients, while Example 2 illustrates a system that does not have constant coefficients.

In this section, we only consider systems having constant coefficients; hence, we shall designate the matrix $\mathbf{A}(t)$ as $\mathbf{A}$ in order to emphasize its independence of t. We seek the solution to the initial-value problem in the fundamental form

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{f}(t), \\ \mathbf{x}(t_0) &= \mathbf{c}. \end{aligned} \tag{29}$$

The differential equation in (29) can be rewritten as

$$\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) = \mathbf{f}(t). \tag{30}$$

If we premultiply each side of (30) by $e^{-\mathbf{A}t}$, we obtain

$$e^{-\mathbf{A}t}[\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t)] = e^{-\mathbf{A}t}\mathbf{f}(t). \tag{31}$$

Using matrix differentiation and Corollary 1 of Section 7.9, we find that

$$\begin{aligned} \frac{d}{dt}[e^{-\mathbf{A}t}\mathbf{x}(t)] &= e^{-\mathbf{A}t}(-\mathbf{A})\mathbf{x}(t) + e^{-\mathbf{A}t}\dot{\mathbf{x}}(t) \\ &= e^{-\mathbf{A}t}[\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t)]. \end{aligned} \tag{32}$$

Substituting (32) into (31), we obtain

$$\frac{d}{dt}[e^{-\mathbf{A}t}\mathbf{x}(t)] = e^{-\mathbf{A}t}\mathbf{f}(t). \tag{33}$$

Integrating (33) between the limits $t = t_0$ and $t = t$, we have

$$\begin{aligned}\int_{t_0}^{t} \frac{d}{dt}[e^{-\mathbf{A}t}\mathbf{x}(t)]\,dt &= \int_{t_0}^{t} e^{-\mathbf{A}t}\mathbf{f}(t)\,dt \\ &= \int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{f}(s)\,ds. \end{aligned} \tag{34}$$

Note that we have replaced the dummy variable t by the dummy variable s in the right-hand integral of (34), which *in no way* alters the definite integral (see Problem 1).

Upon evaluating the left-hand integral, it follows from (34) that

$$e^{-\mathbf{A}t}\mathbf{x}(t)\Big|_{t_0}^{t} = \int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{f}(s)\,ds$$

or that

$$e^{-\mathbf{A}t}\mathbf{x}(t) = e^{-\mathbf{A}t_0}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{f}(s)\,ds. \tag{35}$$

But $\mathbf{x}(t_0) = \mathbf{c}$, hence

$$e^{-\mathbf{A}t}\mathbf{x}(t) = e^{-\mathbf{A}t_0}\mathbf{c} + \int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{f}(s)\,ds, \tag{36}$$

Premultiplying both sides of (36) by $(e^{-\mathbf{A}t})^{-1}$, we obtain

$$\mathbf{x}(t) = (e^{-\mathbf{A}t})^{-1}e^{-\mathbf{A}t_0}\mathbf{c} + (e^{-\mathbf{A}t})^{-1}\int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{f}(s)\,ds. \tag{37}$$

Using Property 2 of Section 7.8, we have

$$(e^{-\mathbf{A}t})^{-1} = e^{\mathbf{A}t},$$

whereupon we can rewrite (37) as

$$\mathbf{x}(t) = e^{\mathbf{A}t}e^{-\mathbf{A}t_0}\mathbf{c} + e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{f}(s)\,ds. \tag{38}$$

Since $\mathbf{A}t$ and $-\mathbf{A}t_0$ commute (why?), we have from Problem 10 of Section 7.8,

$$e^{\mathbf{A}t}e^{-\mathbf{A}t_0} = e^{\mathbf{A}(t-t_0)}. \tag{39}$$

Finally using (39), we can rewrite (38) as

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{c} + e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{f}(s)\,ds. \tag{40}$$

Equation (40) is the unique solution to the initial-value problem given by (29).

A simple method for calculating the quantities $e^{\mathbf{A}(t-t_0)}$, and $e^{-\mathbf{A}s}$ is to first compute $e^{\mathbf{A}t}$ (see Section 7.6) and then replace the variable t wherever it appears by the variables $t - t_0$ and $(-s)$, respectively.

Example 1

Find $e^{\mathbf{A}(t-t_0)}$ and $e^{-\mathbf{A}s}$ for

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

Solution. Using the method of Section 7.6, we calculate $e^{\mathbf{A}t}$ as

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix}.$$

Hence,

$$e^{\mathbf{A}(t-t_0)} = \begin{bmatrix} e^{-(t-t_0)} & (t-t_0)e^{-(t-t_0)} \\ 0 & e^{-(t-t_0)} \end{bmatrix}$$

and

$$e^{-\mathbf{A}s} = \begin{bmatrix} e^{s} & -se^{s} \\ 0 & e^{s} \end{bmatrix}. \qquad \square$$

Note that when t is replaced by $(t - t_0)$ in e^{-t}, the result is $e^{-(t-t_0)} = e^{-t+t_0}$ and not e^{-t-t_0}. That is, we replaced the *quantity* t by the *quantity* $(t - t_0)$; we did not simply add $-t_0$ to the variable t wherever it appeared. Also note that the same result could have been obtained for $e^{-\mathbf{A}s}$ by first computing $e^{\mathbf{A}s}$ and then inverting by the method of cofactors (recall that $e^{-\mathbf{A}s}$ is the inverse of $e^{\mathbf{A}s}$) or by computing $e^{-\mathbf{A}s}$ directly (define $\mathbf{B} = -\mathbf{A}s$ and calculate $e^{\mathbf{B}}$). However, if $e^{\mathbf{A}t}$ is already known, the above method is by far the most expedient one for obtaining $e^{-\mathbf{A}s}$.

We can derive an alternate representation for the solution vector $\mathbf{x}(t)$ if we note that $e^{\mathbf{A}t}$ depends only on t and the integration is with respect to s. Hence, $e^{\mathbf{A}t}$ can be brought inside the integral, and (40) can be rewritten as

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{c} + \int_{t_0}^{t} e^{\mathbf{A}t}e^{-\mathbf{A}s}\mathbf{f}(s)\,ds.$$

Since $\mathbf{A}t$ and $-\mathbf{A}s$ commute, we have that

$$e^{\mathbf{A}t}e^{-\mathbf{A}s} = e^{\mathbf{A}(t-s)}$$

Thus, the solution to (29) can be written as

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{c} + \int_{t_0}^{t} e^{\mathbf{A}(t-s)}\mathbf{f}(s)\,ds. \tag{41}$$

Again the quantity $e^{\mathbf{A}(t-s)}$ can be obtained by replacing the variable t in $e^{\mathbf{A}t}$ by the variable $(t - s)$.

In general, the solution $\mathbf{x}(t)$ may be obtained quicker by using (41) than by using (40), since there is one less multiplication involved. (Note that in (40) one must premultiply the integral by $e^{\mathbf{A}t}$ while in (41) this step is eliminated.) However, since the integration in (41) is more difficult than that in (40), the reader who is not confident of his integrating abilities will probably be more comfortable using (40).

If one has a homogeneous initial-value problem with constant coefficients, that is, a system defined by

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t), \\ \mathbf{x}(t_o) &= \mathbf{c}, \end{aligned} \tag{42}$$

a great simplification of (40) is effected. In this case, $\mathbf{f}(t) \equiv \mathbf{0}$. The integral in (40), therefore, becomes the zero vector, and the solution to the system given by (42) is

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{c}. \tag{43}$$

Occasionally, we are interested in just solving a differential equation and not an entire initial-value problem. In this case, the general solution can be shown to be (see Problem 2)

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{k} + e^{\mathbf{A}t}\int e^{-\mathbf{A}t}\mathbf{f}(t)\,dt, \tag{44}$$

where $\mathbf{k}$ is an arbitrary n-dimensional constant vector. The general solution to the homogeneous differential equation by itself is given by

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{k}. \tag{45}$$

Example 2

Use matrix methods to solve

$$\dot{u}(t) = u(t) + 2v(t) + 1$$

$$\dot{v}(t) = 4u(t) + 3v(t) - 1$$

$$u(0) = 1, \qquad v(0) = 2.$$

Solution. This system can be put into fundamental form if we define $t_0 = 0$,

$$\mathbf{x}(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}, \qquad \mathbf{f}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \tag{46}$$

Since $\mathbf{A}$ is independent of t, this is a system with constant coefficients, and the solution is given by (40). For the $\mathbf{A}$ in (46), $e^{\mathbf{A}t}$ is found to be

$$e^{\mathbf{A}t} = \frac{1}{6}\begin{bmatrix} 2e^{5t} + 4e^{-t} & 2e^{5t} - 2e^{-t} \\ 4e^{5t} - 4e^{-t} & 4e^{5t} + 2e^{-t} \end{bmatrix}.$$

Hence,

$$e^{-\mathbf{A}s} = \frac{1}{6}\begin{bmatrix} 2e^{-5s} + 4e^{s} & 2e^{-5s} - 2e^{s} \\ 4e^{-5s} - 4e^{s} & 4e^{-5s} + 2e^{s} \end{bmatrix}$$

and

$$e^{\mathbf{A}(t-t_0)} = e^{\mathbf{A}t}, \quad \text{since} \quad t_0 = 0.$$

Thus,

$$\begin{aligned} e^{\mathbf{A}(t-t_0)}\mathbf{c} &= \frac{1}{6}\begin{bmatrix} 2e^{5t} + 4e^{-t} & 2e^{5t} - 2e^{-t} \\ 4e^{5t} - 4e^{-t} & 4e^{5t} + 2e^{-t} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \frac{1}{6}\begin{bmatrix} 1[2e^{5t} + 4e^{-t}] + 2[2e^{5t} - 2e^{-t}] \\ 1[4e^{5t} - 4e^{-t}] + 2[4e^{5t} + 2e^{-t}] \end{bmatrix} \\ &= \begin{bmatrix} e^{5t} \\ 2e^{5t} \end{bmatrix}, \end{aligned} \tag{47}$$

and

$$\begin{aligned} e^{-\mathbf{A}s}\mathbf{f}(s) &= \frac{1}{6}\begin{bmatrix} 2e^{-5s} + 4e^{s} & 2e^{-5s} - 2e^{s} \\ 4e^{-5s} - 4e^{s} & 4e^{-5s} + 2e^{s} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \frac{1}{6}\begin{bmatrix} 1[2e^{-5s} + 4e^{s}] - 1[2e^{-5s} - 2e^{s}] \\ 1[4e^{-5s} - 4e^{s}] - 1[4e^{-5s} + 2e^{s}] \end{bmatrix} = \begin{bmatrix} e^{s} \\ -e^{s} \end{bmatrix}. \end{aligned}$$

Hence,

$$\int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{f}(s)\,ds = \begin{bmatrix} \displaystyle\int_0^t e^s\,ds \\ \displaystyle\int_0^t -e^s\,ds \end{bmatrix} = \begin{bmatrix} e^s|_0^t \\ -e^s|_0^t \end{bmatrix} = \begin{bmatrix} e^t - 1 \\ -e^t + 1 \end{bmatrix}$$

and

$$\begin{aligned} e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{f}(s)\,ds &= \frac{1}{6}\begin{bmatrix} 2e^{5t} + 4e^{-t} & 2e^{5t} - 2e^{-t} \\ 4e^{5t} - 4e^{-t} & 4e^{5t} + 2e^{-t} \end{bmatrix}\begin{bmatrix} (e^t - 1) \\ (1 - e^t) \end{bmatrix} \\ &= \frac{1}{6}\begin{bmatrix} [2e^{5t} + 4e^{-t}][e^t - 1] + [2e^{5t} - 2e^{-t}][1 - e^t] \\ [4e^{5t} - 4e^{-t}][e^t - 1] + [4e^{5t} + 2e^{-t}][1 - e^t] \end{bmatrix} \\ &= \begin{bmatrix} (1 - e^{-t}) \\ (-1 + e^{-t}) \end{bmatrix}. \end{aligned} \tag{48}$$

Substituting (47) and (48) into (40), we have

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \mathbf{x}(t) = \begin{bmatrix} e^{5t} \\ 2e^{5t} \end{bmatrix} + \begin{bmatrix} 1 - e^{-t} \\ -1 + e^{-t} \end{bmatrix} = \begin{bmatrix} e^{5t} + 1 - e^{-t} \\ 2e^{5t} - 1 + e^{-t} \end{bmatrix},$$

or

$$u(t) = e^{5t} - e^{-t} + 1,$$

$$v(t) = 2e^{5t} + e^{-t} - 1. \qquad \square$$

Example 3

Use matrix methods to solve

$$\ddot{y} - 3\dot{y} + 2y = e^{-3t},$$

$$y(1) = 1, \qquad \dot{y}(1) = 0.$$

Solution. This system can be put into fundamental form if we define $t_0 = 1$;

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} 0 \\ e^{-3t} \end{bmatrix}, \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The solution to this system is given by (40). For this $\mathbf{A}$,

$$e^{\mathbf{A}t} = \begin{bmatrix} -e^{2t} + 2e^t & e^{2t} - e^t \\ -2e^{2t} + 2e^t & 2e^{2t} - e^t \end{bmatrix}.$$

Thus,

$$e^{\mathbf{A}(t-t_0)}\mathbf{c} = \begin{bmatrix} -e^{2(t-1)} + 2e^{(t-1)} & e^{2(t-1)} - e^{(t-1)} \\ -2e^{2(t-1)} + 2e^{(t-1)} & 2e^{2(t-1)} - e^{(t-1)} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -e^{2(t-1)} + 2e^{(t-1)} \\ -2e^{2(t-1)} + 2e^{(t-1)} \end{bmatrix}. \tag{49}$$

Now

$$\mathbf{f}(t) = \begin{bmatrix} 0 \\ e^{-3t} \end{bmatrix}, \qquad \mathbf{f}(s) = \begin{bmatrix} 0 \\ e^{-3s} \end{bmatrix},$$

and

$$e^{-\mathbf{A}s}\mathbf{f}(s) = \begin{bmatrix} -e^{-2s} + 2e^{-s} & e^{-2s} - e^{-s} \\ -2e^{-2s} + 2e^{-s} & 2e^{-2s} - e^{-s} \end{bmatrix} \begin{bmatrix} 0 \\ e^{-3s} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-5s} - e^{-4s} \\ 2e^{-5s} - e^{-4s} \end{bmatrix}.$$

Hence,

$$\int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{f}(s)\,ds = \begin{bmatrix} \displaystyle\int_1^t (e^{-5s} - e^{-4s})\,ds \\ \displaystyle\int_1^t (2e^{-5s} - e^{-4s})\,ds \end{bmatrix}$$

$$= \begin{bmatrix} (-\frac{1}{5})e^{-5t} + (\frac{1}{4})e^{-4t} + (\frac{1}{5})e^{-5} - (\frac{1}{4})e^{-4} \\ (-\frac{2}{5})e^{-5t} + (\frac{1}{4})e^{-4t} + (\frac{2}{5})e^{-5} - (\frac{1}{4})e^{-4} \end{bmatrix},$$

and

$$e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{f}(s)\,ds$$

$$= \begin{bmatrix} (-e^{2t} + 2e^{t}) & (e^{2t} - e^{t}) \\ (-2e^{2t} + 2e^{t}) & (2e^{2t} - e^{t}) \end{bmatrix} \begin{bmatrix} (-\frac{1}{5}e^{-5t} + \frac{1}{4}e^{-4t} + \frac{1}{5}e^{-5} - \frac{1}{4}e^{-4}) \\ (-\frac{2}{5}e^{-5t} + \frac{1}{4}e^{-4t} + \frac{2}{5}e^{-5} - \frac{1}{4}e^{-4}) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{20}e^{-3t} + \frac{1}{5}e^{(2t-5)} - \frac{1}{4}e^{t-4} \\ -\frac{3}{20}e^{-3t} + \frac{2}{5}e^{(2t-5)} - \frac{1}{4}e^{t-4} \end{bmatrix}. \tag{50}$$

Substituting (49) and (50) into (40), we have that

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -e^{2(t-1)} + 2e^{t-1} \\ -2e^{2(t-1)} + 2e^{t-1} \end{bmatrix} + \begin{bmatrix} \frac{1}{20}e^{-3t} + \frac{1}{5}e^{(2t-5)} - \frac{1}{4}e^{t-4} \\ -\frac{3}{20}e^{-3t} + \frac{2}{5}e^{(2t-5)} - \frac{1}{4}e^{t-4} \end{bmatrix}$$

$$= \begin{bmatrix} -e^{2(t-1)} + 2e^{t-1} + \frac{1}{20}e^{-3t} + \frac{1}{5}e^{(2t-5)} - \frac{1}{4}e^{t-4} \\ -2e^{2(t-1)} + 2e^{t-1} - \frac{3}{20}e^{-3t} + \frac{2}{5}e^{(2t-5)} - \frac{1}{4}e^{t-4} \end{bmatrix}.$$

Thus, it follows that the solution to the initial-value problem is given by

$$\begin{aligned} y(t) &= x_1(t) \\ &= -e^{2(t-1)} + 2e^{t-1} + \tfrac{1}{20}e^{-3t} + \tfrac{1}{5}e^{(2t-5)} - \tfrac{1}{4}e^{t-4}. \quad \square \end{aligned}$$

The most tedious step in Example 3 was multiplying the matrix $e^{\mathbf{A}t}$ by the vector $\int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{f}(s)\,ds$. We could have eliminated this multiplication had we used (41) for the solution rather than (40). Of course, in using (41), we would have had to handle an integral rather more complicated than the one we encountered.

If $\mathbf{A}$ and $\mathbf{f}(t)$ are relatively simple (for instance, if $\mathbf{f}(t)$ is a constant vector), then the integral obtained in (41) may not be too tedious to evaluate, and its use can be a real savings in time and effort over the use of (40). We illustrate this point in the next example.

Example 4

Use matrix methods to solve

$$\ddot{x}(t) + x(t) = 2,$$

$$x(\pi) = 0, \qquad \dot{x}(\pi) = -1.$$

Solution. This initial-valued problem can be put into fundamental form if we define $t_0 = \pi$,

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}. \tag{51}$$

Here, $\mathbf{A}$ is again independent of the variable t, hence, the solution is given by either (40) or (41). This time we elect to use (41). For the $\mathbf{A}$ given in (51), $e^{\mathbf{A}t}$ is found to be

$$e^{\mathbf{A}t} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$

Thus,

$$\begin{aligned} e^{\mathbf{A}(t-t_0)}\mathbf{c} &= \begin{bmatrix} \cos(t-\pi) & \sin(t-\pi) \\ -\sin(t-\pi) & \cos(t-\pi) \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} -\sin(t-\pi) \\ -\cos(t-\pi) \end{bmatrix}, \end{aligned} \tag{52}$$

and

$$e^{\mathbf{A}(t-s)}\mathbf{f}(s) = \begin{bmatrix} \cos(t-s) & \sin(t-s) \\ -\sin(t-s) & \cos(t-s) \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 2\sin(t-s) \\ 2\cos(t-s) \end{bmatrix}.$$

Hence,

$$\int_{t_0}^{t} e^{\mathbf{A}(t-s)}\mathbf{f}(s)\,ds = \begin{bmatrix} \displaystyle\int_{\pi}^{t} 2\sin(t-s)\,ds \\ \displaystyle\int_{\pi}^{t} 2\cos(t-s)\,ds \end{bmatrix}$$
$$= \begin{bmatrix} 2 - 2\cos(t-\pi) \\ 2\sin(t-\pi) \end{bmatrix}. \tag{53}$$

Substituting (52) and (53) into (41) and using the trigonometric identities $\sin(t-\pi) = -\sin t$ and $\cos(t-\pi) = -\cos t$, we have

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \mathbf{x}(t) = \begin{bmatrix} -\sin(t-\pi) \\ -\cos(t-\pi) \end{bmatrix} + \begin{bmatrix} 2 - 2\cos(t-\pi) \\ 2\sin(t-\pi) \end{bmatrix}$$
$$= \begin{bmatrix} \sin t + 2\cos t + 2 \\ \cos t - 2\sin t \end{bmatrix}.$$

Thus, since $x(t) = x_1(t)$, it follows that the solution to the initial-value problem is given by

$$x(t) = \sin t + 2\cos t + 2. \qquad \square$$

Example 5

Solve by matrix methods

$$\dot{u}(t) = u(t) + 2v(t),$$
$$\dot{v}(t) = 4u(t) + 3v(t).$$

Solution. This system can be put into fundamental form if we define

$$\mathbf{x}(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}, \quad \text{and} \quad \mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This is a homogeneous system with constant coefficients and no initial conditions specified; hence, the general solution is given by (45).

As in Example 2, for this **A**, we have

$$e^{\mathbf{A}t} = \frac{1}{6}\begin{bmatrix} 2e^{5t} + 4e^{-t} & 2e^{5t} - 2e^{-t} \\ 4e^{5t} - 4e^{-t} & 4e^{5t} + 2e^{-t} \end{bmatrix}.$$

Thus,

$$\begin{aligned} e^{\mathbf{A}t}\mathbf{k} &= \frac{1}{6}\begin{bmatrix} 2e^{5t} + 4e^{-t} & 2e^{5t} - 2e^{-t} \\ 4e^{5t} - 4e^{-t} & 4e^{5t} + 2e^{-t} \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \\ &= \frac{1}{6}\begin{bmatrix} k_1[2e^{5t} + 4e^{-t}] + k_2[2e^{5t} - 2e^{-t}] \\ k_1[4e^{5t} - 4e^{-t}] + k_2[4e^{5t} + 2e^{-t}] \end{bmatrix} \\ &= \frac{1}{6}\begin{bmatrix} e^{5t}(2k_1 + 2k_2) + e^{-t}(4k_1 - 2k_2) \\ e^{5t}(4k_1 + 4k_2) + e^{-t}(-4k_1 + 2k_2) \end{bmatrix}. \end{aligned} \tag{54}$$

Substituting (54) into (45), we have that

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \mathbf{x}(t) = \frac{1}{6}\begin{bmatrix} e^{5t}(2k_1 + 2k_2) + e^{-t}(4k_1 - 2k_2) \\ e^{5t}(4k_1 + 4k_2) + e^{-t}(-4k_1 + 2k_2) \end{bmatrix}$$

or

$$\begin{aligned} u(t) &= \left(\frac{2k_1 + 2k_2}{6}\right)e^{5t} + \left(\frac{4k_1 - 2k_2}{6}\right)e^{-t} \\ v(t) &= 2\left(\frac{2k_1 + 2k_2}{6}\right)e^{5t} + \left(\frac{-4k_1 + 2k_2}{6}\right)e^{-t}. \end{aligned} \tag{55}$$

We can simplify the expressions for $u(t)$ and $v(t)$ if we introduce two new arbitrary constants k_3 and k_4 defined by

$$k_3 = \frac{2k_1 + 2k_2}{6}, \qquad k_4 = \frac{4k_1 - 2k_2}{6}. \tag{56}$$

Substituting these values into (55), we obtain

$$\begin{aligned} u(t) &= k_3e^{5t} + k_4e^{-t} \\ v(t) &= 2k_3e^{5t} - k_4e^{-t}. \qquad \square \end{aligned} \tag{57}$$

Problems 8.4

(1) Show by direct integration that

$$\int_{t_0}^{t} t^2\,dt = \int_{t_0}^{t} s^2\,ds = \int_{t_0}^{t} p^2\,dp.$$

In general, show that if $f(t)$ is integrable on $[a, b]$, then

$$\int_a^b f(t)\,dt = \int_a^b f(s)\,ds.$$

(Hint: Assume $\int f(t)\,dt = F(t) + c$. Hence, $\int f(s)\,ds = F(s) + c$. Use the fundamental theorem of integral calculus to obtain result.)

(2) Derive Eq. (44). (Hint: Follow steps (30)–(33). For step (34) use indefinite integration and note that

$$\int \frac{d}{dt}[e^{-\mathbf{A}t}\mathbf{x}(t)]\,dt = e^{-\mathbf{A}t}\mathbf{x}(t) + \mathbf{k},$$

where $\mathbf{k}$ is an arbitrary constant vector of integration.)

(3) Find (a) $e^{-\mathbf{A}t}$ (b) $e^{\mathbf{A}(t-2)}$ (c) $e^{\mathbf{A}(t-s)}$ (d) $e^{-\mathbf{A}(t-2)}$, if

$$e^{\mathbf{A}t} = e^{3t}\begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}.$$

(4) For $e^{\mathbf{A}t}$ as given in Problem 3, invert by the method of cofactors to obtain $e^{-\mathbf{A}t}$ and hence verify part (a) of that problem.

(5) Find (a) $e^{-\mathbf{A}t}$, (b) $e^{-\mathbf{A}s}$, (c) $e^{\mathbf{A}(t-3)}$ if

$$e^{\mathbf{A}t} = \frac{1}{6}\begin{bmatrix} 2e^{5t} + 4e^{-t} & 2e^{5t} - 2e^{-t} \\ 4e^{5t} - 4e^{-t} & 4e^{5t} + 2e^{-t} \end{bmatrix}.$$

(6) Find (a) $e^{-\mathbf{A}t}$, (b) $e^{-\mathbf{A}s}$, (c) $e^{-\mathbf{A}(t-s)}$ if

$$e^{\mathbf{A}t} = \frac{1}{3}\begin{bmatrix} -\sin 3t + 3\cos 3t & 5\sin 3t \\ -2\sin 3t & \sin 3t + 3\cos 3t \end{bmatrix}.$$

Solve the systems given in Problems 7 through 14 by matrix methods. Note that Problems 7 through 10 have the same coefficient matrix.

(7) $\dot{x}(t) = -2x(t) + 3y(t),$

$\dot{y}(t) = -x(t) + 2y(t);$

$x(2) = 2, \quad y(2) = 4.$

(8) $\dot{x}(t) = -2x(t) + 3y(t) + 1,$

$\dot{y}(t) = -x(t) + 2y(t) + 1;$

$x(1) = 1, \quad y(1) = 1.$

(9) $\dot{x}(t) = -2x(t) + 3y(t),$

$\dot{y}(t) = -x(t) + 2y(t).$

(10) $\dot{x}(t) = -2x(t) + 3y(t) + 1,$

$\dot{y}(t) = -x(t) + 2y(t) + 1.$

(11) $\ddot{x}(t) = -4x(t) + \sin t;$

$x(0) = 1, \quad \dot{x}(0) = 0.$

(12) $\dddot{x}(t) = t;$

$x(1) = 1, \quad \dot{x}(1) = 2, \quad \ddot{x}(1) = 3$

(13) $\ddot{x} - \dot{x} - 2x = e^{-t};$

$x(0) = 1, \quad \dot{x}(0) = 0.$

(14) $\ddot{x} = 2\dot{x} + 5y + 3,$

$\dot{y} = -\dot{x} - 2y;$

$x(0) = 0, \quad \dot{x}(0) = 0, \quad y(0) = 1.$

8.5 Solutions of Systems—General Case

Having completely solved systems of linear differential equations with constant coefficients, we now turn our attention to the solutions of systems of the form

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t), \\ \mathbf{x}(t_0) &= \mathbf{c}. \end{aligned} \tag{58}$$

Note that $\mathbf{A}(t)$ may now depend on t, hence the analysis of Section 8.4 does not apply. However, since we still require both $\mathbf{A}(t)$ and $\mathbf{f}(t)$ to be continuous in some interval about $t = t_0$, Theorem 1 of Section 8.1 still guarantees that (58) has a unique solution. Our aim in this section is to obtain a representation for this solution.

Definition 1. A *transition* (or fundamental) *matrix* of the homogeneous equation $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$ is an $n \times n$ matrix $\mathbf{\Phi}(t, t_0)$ having the properties that

(a) $$\frac{d}{dt}\mathbf{\Phi}(t, t_0) = \mathbf{A}(t)\mathbf{\Phi}(t, t_0), \tag{59}$$

(b) $$\mathbf{\Phi}(t_0, t_0) = \mathbf{I}. \tag{60}$$

Here t_0 is the initial time given in (58). In the appendix to this chapter, we show that $\mathbf{\Phi}(t, t_0)$ exists and is unique.

Example 1

Find $\mathbf{\Phi}(t, t_0)$ if $\mathbf{A}(t)$ is a constant matrix.

Solution. Consider the matrix $e^{\mathbf{A}(t-t_0)}$. From Property 1 of Section 7.8, we have that $e^{\mathbf{A}(t_0-t_0)} = e^{\mathbf{0}} = \mathbf{I}$, while from Theorem 1 of Section 7.9, we have that

$$\begin{aligned}\frac{d}{dt}e^{\mathbf{A}(t-t_0)} &= \frac{d}{dt}(e^{\mathbf{A}t}e^{-\mathbf{A}t_0})\\ &= \mathbf{A}e^{\mathbf{A}t}e^{-\mathbf{A}t_0} = \mathbf{A}e^{\mathbf{A}(t-t_0)}.\end{aligned}$$

Thus, $e^{\mathbf{A}(t-t_0)}$ satisfies (59) and (60). Since $\mathbf{\Phi}(t, t_0)$ is unique, it follows for the case where $\mathbf{A}$ is a *constant* matrix that

$$\mathbf{\Phi}(t, t_0) = e^{\mathbf{A}(t-t_0)}. \tag{61}$$

CAUTION. Although $\mathbf{\Phi}(t, t_0) = e^{\mathbf{A}(t-t_0)}$ if $\mathbf{A}$ is a constant matrix, this equality is not valid if $\mathbf{A}$ actually depends on t. In fact, it is usually impossible to explicitly find $\mathbf{\Phi}(t, t_0)$ in the general time varying case. Usually, the best we can say about the transition matrix is that it exists, it is unique, and, of course, it satisfies (59) and (60). □

One immediate use of $\mathbf{\Phi}(t, t_0)$ is that it enables us to theoretically solve the general homogeneous initial-value problem

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t)\\ \mathbf{x}(t_0) &= \mathbf{c}.\end{aligned} \tag{62}$$

Theorem 1. *The unique solution to (62) is*

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{c}. \tag{63}$$

Proof. If $\mathbf{A}(t)$ is a constant matrix, (63) reduces to (43) (see (61)), hence Theorem 1 is valid. In general, however, we have that

$$\begin{aligned}\frac{d\mathbf{x}(t)}{dt} &= \frac{d}{dt}[\mathbf{\Phi}(t, t_0)\mathbf{c}] = \frac{d}{dt}[\mathbf{\Phi}(t, t_0)]\mathbf{c},\\ &= \mathbf{A}(t)\mathbf{\Phi}(t, t_0)\mathbf{c} \qquad \{\text{from } (59),\\ &= \mathbf{A}(t)\mathbf{x}(t) \qquad \{\text{from } (63),\end{aligned}$$

and

$$\begin{aligned}\mathbf{x}(t_0) &= \mathbf{\Phi}(t_0, t_0)\mathbf{c},\\ &= \mathbf{I}\mathbf{c} \qquad \{\text{from } (60),\\ &= \mathbf{c}.\end{aligned}$$

Example 2

Find $x(t)$ and $y(t)$ if

$$\dot{x} = ty$$
$$\dot{y} = -tx$$
$$x(1) = 0, \qquad y(1) = 1,$$

Solution. Putting this system into fundamental form, we obtain

$$t_0 = 1, \qquad \mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \qquad \mathbf{A}(t) = \begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix}, \qquad \mathbf{f}(t) = \mathbf{0}, \qquad \mathbf{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t),$$
$$\mathbf{x}(t_0) = \mathbf{c}.$$

The transition matrix for this system can be shown to be (see Problem 1)

$$\mathbf{\Phi}(t, t_0) = \begin{bmatrix} \cos\left(\dfrac{t^2 - t_0^2}{2}\right) & \sin\left(\dfrac{t^2 - t_0^2}{2}\right) \\ -\sin\left(\dfrac{t^2 - t_0^2}{2}\right) & \cos\left(\dfrac{t^2 - t_0^2}{2}\right) \end{bmatrix}.$$

Thus, from (63), we have

$$\mathbf{x}(t) = \begin{bmatrix} \cos\left(\dfrac{t^2 - 1}{2}\right) & \sin\left(\dfrac{t^2 - 1}{2}\right) \\ -\sin\left(\dfrac{t^2 - 1}{2}\right) & \cos\left(\dfrac{t^2 - 1}{2}\right) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} \sin\left(\dfrac{t^2 - 1}{2}\right) \\ \cos\left(\dfrac{t^2 - 1}{2}\right) \end{bmatrix}.$$

Consequently, the solution is

$$x(t) = \sin\left(\frac{t^2 - 1}{2}\right), \qquad y(t) = \cos\left(\frac{t^2 - 1}{2}\right). \qquad \square$$

The transition matrix also enables us to give a representation for the solution of the general time-varying initial-value problem

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t), \\ \mathbf{x}(t_0) &= \mathbf{c}. \end{aligned} \tag{58}$$

Theorem 2. *The unique solution to (58) is*

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{c} + \int_{t_0}^{t} \mathbf{\Phi}(t, s)\mathbf{f}(s)\, ds. \tag{64}$$

Proof. If $\mathbf{A}$ is a constant matrix, $\mathbf{\Phi}(t, t_0) = e^{\mathbf{A}(t-t_0)}$; hence $\mathbf{\Phi}(t, s) = e^{\mathbf{A}(t-s)}$ and (64) reduces to (41). We defer the proof of the general case, where $\mathbf{A}(t)$ depends on t, until later in this section.

Equation (64) is the solution to the general initial-value problem given by (58). It should be noted, however, that since $\mathbf{\Phi}(t, t_0)$ is not explicitly known, $\mathbf{x}(t)$ will not be explicitly known either, hence, (64) is not as useful a formula as it might first appear. Unfortunately, (64) is the best solution that we can obtain for the general time varying problem. The student should not despair, though. It is often the case that by knowing enough properties of $\mathbf{\Phi}(t, t_0)$, we can extract a fair amount of information about the solution from (64). In fact, we can sometimes even obtain the exact solution!

Before considering some important properties of the transition matrix, we state one lemma that we ask the student to prove (see Problem 3).

Lemma 1. *If* $\mathbf{B}(t)$ *is an* $n \times n$ *matrix having the property that* $\mathbf{B}(t)\mathbf{c} = \mathbf{0}$ *for every n-dimensional constant vector* $\mathbf{c}$, then $\mathbf{B}(t)$ *is the zero matrix.*

For the remainder of this section we assume that $\mathbf{\Phi}(t, t_0)$ is the transition matrix for $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$.

Property 1. *(The transition property)*

$$\mathbf{\Phi}(t, \tau)\mathbf{\Phi}(\tau, t_0) = \mathbf{\Phi}(t, t_0). \tag{65}$$

Proof. If $\mathbf{A}(t)$ is a constant matrix, $\mathbf{\Phi}(t, t_0) = e^{\mathbf{A}(t-t_0)}$ hence,

$$\begin{aligned} \mathbf{\Phi}(t, \tau)\mathbf{\Phi}(\tau, t_0) &= e^{\mathbf{A}(t-\tau)}e^{\mathbf{A}(\tau-t_0)} \\ &= e^{\mathbf{A}(t-\tau+\tau-t_0)} \\ &= e^{\mathbf{A}(t-t_0)} = \mathbf{\Phi}(t, t_0). \end{aligned}$$

Thus, Property 1 is immediate. For the more general case, that in which $\mathbf{A}(t)$ depends on t, the argument runs as follows: Consider the initial-value problem

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t) \\ \mathbf{x}(t_0) &= \mathbf{c}. \end{aligned} \tag{66}$$

The unique solution of (66) is

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{c}. \tag{67}$$

Hence,

$$\mathbf{x}(t_1) = \mathbf{\Phi}(t_1, t_0)\mathbf{c} \tag{68}$$

and

$$\mathbf{x}(\tau) = \mathbf{\Phi}(\tau, t_0)\mathbf{c}, \tag{69}$$

where t_1 is any arbitrary time greater than τ. If we designate the vector $\mathbf{x}(t_1)$ by $\mathbf{d}$ and the vector $\mathbf{x}(\tau)$ by $\mathbf{b}$, then we can give the solution graphically by Fig. 1.

Consider an associated system governed by

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t), \\ \mathbf{x}(\tau) &= \mathbf{b}. \end{aligned} \tag{70}$$

We seek a solution to the above differential equation that has an initial value $\mathbf{b}$ at the initial time $t = \tau$. If we designate the solution by $\mathbf{y}(t)$, it follows from Theorem 1 that

$$\mathbf{y}(t) = \mathbf{\Phi}(t, \tau)\mathbf{b}, \tag{71}$$

hence

$$\mathbf{y}(t_1) = \mathbf{\Phi}(t_1, \tau)\mathbf{b}. \tag{72}$$

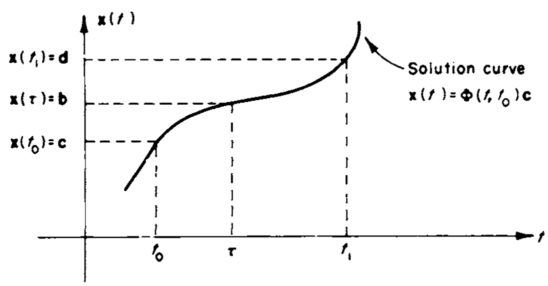


Figure 1.

But now we note that both $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are governed by the same equation of motion, namely $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$, and both $\mathbf{x}(t)$ and $\mathbf{y}(t)$ go through the same point (τ, b). Thus, $\mathbf{x}(t)$ and $\mathbf{y}(t)$ must be the same solution. That is, the solution curve for $\mathbf{y}(t)$ looks exactly like that of $\mathbf{x}(t)$, shown in Fig. 1 except that it starts at $t = \tau$, while that of $\mathbf{x}(t)$ starts at $t = t_0$. Hence,

$$\mathbf{x}(t) = \mathbf{y}(t), \qquad t \geq \tau,$$

and, in particular,

$$\mathbf{x}(t_1) = \mathbf{y}(t_1). \tag{73}$$

Thus, substituting (68) and (72) into (73) we obtain

$$\mathbf{\Phi}(t_1, t_0)\mathbf{c} = \mathbf{\Phi}(t_1, \tau)\mathbf{b}. \tag{74}$$

However, $\mathbf{x}(\tau) = \mathbf{b}$, thus (74) may be rewritten as

$$\mathbf{\Phi}(t_1, t_0)\mathbf{c} = \mathbf{\Phi}(t_1, \tau)\mathbf{x}(\tau). \tag{75}$$

Substituting (69) into (75), we have

$$\mathbf{\Phi}(t_1, t_0)\mathbf{c} = \mathbf{\Phi}(t_1, \tau)\mathbf{\Phi}(\tau, t_0)\mathbf{c}$$

or

$$[\mathbf{\Phi}(t_1, t_0) - \mathbf{\Phi}(t_1, \tau)\mathbf{\Phi}(\tau, t_0)]\mathbf{c} = \mathbf{0}. \tag{76}$$

Since $\mathbf{c}$ may represent any n-dimensional initial state, it follows from Lemma 1 that

$$\mathbf{\Phi}(t_1, t_0) - \mathbf{\Phi}(t_1, \tau)\mathbf{\Phi}(\tau, t_0) = \mathbf{0}$$

or

$$\mathbf{\Phi}(t_1, t_0) = \mathbf{\Phi}(t_1, \tau)\mathbf{\Phi}(\tau, t_0). \tag{77}$$

Since t_1 is arbitrary, it can be replaced by t; Eq. (77) therefore implies Eq. (65).

Property 2. $\mathbf{\Phi}(t, t_0)$ is invertible and

$$[\mathbf{\Phi}(t, t_0)]^{-1} = \mathbf{\Phi}(t_0, t). \tag{78}$$

Proof. This result is obvious if $\mathbf{A}(t)$ is a constant matrix. We know from Section 7.8 that the inverse of $e^{\mathbf{A}t}$ is $e^{-\mathbf{A}t}$, hence,

$$\begin{aligned}[\mathbf{\Phi}(t, t_0)]^{-1} &= [e^{\mathbf{A}(t-t_0)}]^{-1} = e^{-\mathbf{A}(t-t_0)} \\ &= e^{\mathbf{A}(t_0-t)} = \mathbf{\Phi}(t_0, t).\end{aligned}$$

In order to prove Property 2 for any $\mathbf{A}(t)$, we note that (65) is valid for any t, hence it must be valid for $t = t_0$. Thus

$$\mathbf{\Phi}(t_0, \tau)\mathbf{\Phi}(\tau, t_0) = \mathbf{\Phi}(t_0, t_0).$$

It follows from (60) that

$$\mathbf{\Phi}(t_0, \tau)\mathbf{\Phi}(\tau, t_0) = \mathbf{I}.$$

Thus, from the definition of the inverse, we have

$$[\mathbf{\Phi}(\tau, t_0)]^{-1} = \mathbf{\Phi}(t_0, \tau)$$

which implies (78).

Example 3

Find the inverse of

$$\begin{bmatrix} \cos\left(\dfrac{t^2 - t_0^2}{2}\right) & \sin\left(\dfrac{t^2 - t_0^2}{2}\right) \\ -\sin\left(\dfrac{t^2 - t_0^2}{2}\right) & \cos\left(\dfrac{t^2 - t_0^2}{2}\right) \end{bmatrix}.$$

Solution. This matrix is a transition matrix. (See Problem 1.) Hence using (78) we find the inverse to be

$$\begin{bmatrix} \cos\left(\dfrac{t_0^2 - t^2}{2}\right) & \sin\left(\dfrac{t_0^2 - t^2}{2}\right) \\ -\sin\left(\dfrac{t_0^2 - t^2}{2}\right) & \cos\left(\dfrac{t_0^2 - t^2}{2}\right) \end{bmatrix} = \begin{bmatrix} \cos\left(\dfrac{t^2 - t_0^2}{2}\right) & -\sin\left(\dfrac{t^2 - t_0^2}{2}\right) \\ \sin\left(\dfrac{t^2 - t_0^2}{2}\right) & \cos\left(\dfrac{t^2 - t_0^2}{2}\right) \end{bmatrix}.$$

Here we have used the identities $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$. □

Properties 1 and 2 enable us to prove Theorem 2, namely, that the solution of

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t),$$
$$\mathbf{x}(t_0) = \mathbf{c}$$

is

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{c} + \int_{t_0}^{t} \mathbf{\Phi}(t, s)\mathbf{f}(s)\,ds. \tag{79}$$

Using Property 1, we have that $\mathbf{\Phi}(t, s) = \mathbf{\Phi}(t, t_0)\mathbf{\Phi}(t_0, s)$; hence, (79) may be rewritten as

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{c} + \mathbf{\Phi}(t, t_0)\int_{t_0}^{t} \mathbf{\Phi}(t_0, s)\mathbf{f}(s)\, ds. \tag{80}$$

Now

$$\begin{aligned}\mathbf{x}(t_0) &= \mathbf{\Phi}(t_0, t_0)\mathbf{c} + \mathbf{\Phi}(t_0, t_0)\int_{t_0}^{t_0} \mathbf{\Phi}(t_0, s)\mathbf{f}(s)\, ds \\ &= \mathbf{Ic} + \mathbf{I0} = \mathbf{c}.\end{aligned}$$

Thus, the initial condition is satisfied by (80). To show that the differential equation is also satisfied, we differentiate (80) and obtain

$$\begin{aligned}\frac{d\mathbf{x}(t)}{dt} &= \frac{d}{dt}\left[\mathbf{\Phi}(t, t_0)\mathbf{c} + \mathbf{\Phi}(t, t_0)\int_{t_0}^{t} \mathbf{\Phi}(t_0, s)\mathbf{f}(s)\, ds\right] \\ &= \left[\frac{d}{dt}\mathbf{\Phi}(t, t_0)\right]\mathbf{c} + \left[\frac{d}{dt}\mathbf{\Phi}(t, t_0)\right]\int_{t_0}^{t} \mathbf{\Phi}(t_0, s)\mathbf{f}(s)\, ds \\ &\quad + \mathbf{\Phi}(t, t_0)\left[\frac{d}{dt}\int_{t_0}^{t} \mathbf{\Phi}(t_0, s)\mathbf{f}(s)\, ds\right] \\ &= \mathbf{A}(t)\mathbf{\Phi}(t, t_0)\mathbf{c} + \mathbf{A}(t)\mathbf{\Phi}(t, t_0)\int_{t_0}^{t} \mathbf{\Phi}(t_0, s)\mathbf{f}(s)\, ds \\ &\quad + \mathbf{\Phi}(t, t_0)\mathbf{\Phi}(t_0, t)\mathbf{f}(t) \\ &= \mathbf{A}(t)\left[\mathbf{\Phi}(t, t_0)\mathbf{c} + \mathbf{\Phi}(t, t_0)\int_{t_0}^{t} \mathbf{\Phi}(t_0, s)\mathbf{f}(s)\, ds\right] \\ &\quad + \mathbf{\Phi}(t, t_0)\mathbf{\Phi}^{-1}(t, t_0)\mathbf{f}(t).\end{aligned}$$

The quantity inside the bracket is given by (80) to be $\mathbf{x}(t)$; hence

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t). \qquad \square$$

We conclude this section with one final property of the transition matrix, the proof of which is beyond the scope of this book.

Property 3.

$$\det \mathbf{\Phi}(t, t_0) = \exp\left\{\int_{t_0}^{t} \operatorname{tr}[\mathbf{A}(t)]\, dt\right\}. \tag{81}$$

Since the exponential is never zero, (81) establishes that $\det \mathbf{\Phi}(t, t_0) \neq 0$, hence, we have an alternate proof that $\mathbf{\Phi}(t, t_0)$ is invertible.

Problems 8.5

(1) Use (59) and (60) to show that

$$\Phi(t,t_0)=\begin{bmatrix}\cos\left(\dfrac{t^2-t_0^2}{2}\right) & \sin\left(\dfrac{t^2-t_0^2}{2}\right)\\ -\sin\left(\dfrac{t^2-t_0^2}{2}\right) & \cos\left(\dfrac{t^2-t_0^2}{2}\right)\end{bmatrix}$$

is a transition matrix for

$$\dot{x}=ty,$$
$$\dot{y}=-tx.$$

(2) As a generalization of Problem 1, use (59) and (60) to show that

$$\Phi(t,t_0)=\begin{bmatrix}\cos\displaystyle\int_{t_0}^{t}g(s)\,ds & \sin\displaystyle\int_{t_0}^{t}g(s)\,ds\\ -\sin\displaystyle\int_{t_0}^{t}g(s)\,ds & \cos\displaystyle\int_{t_0}^{t}g(s)\,ds\end{bmatrix}$$

is a transition matrix for

$$\dot{x}=g(t)y,$$
$$\dot{y}=-g(t)x.$$

(3) Prove Lemma 1. (Hint: Consider the product $\mathbf{B}(t)\mathbf{c}$ where

$$\text{first, } \mathbf{c}=\begin{bmatrix}1\\0\\0\\ \vdots\\0\end{bmatrix},\qquad \text{second, } \mathbf{c}=\begin{bmatrix}0\\1\\0\\ \vdots\\0\end{bmatrix},\qquad \text{etc.)}$$

(4) If $\Phi(t,t_0)$ is a transition matrix, prove that

$$\Phi^{\mathsf{T}}(t_1,t_0)\left[\int_{t_0}^{t_1}\Phi(t_1,s)\Phi^{\mathsf{T}}(t_1,s)\,ds\right]^{-1}\Phi(t_1,t_0)$$
$$=\left[\int_{t_0}^{t_1}\Phi(t_0,s)\Phi^{\mathsf{T}}(t_0,s)\,ds\right]^{-1}.$$

Appendix to Chapter 8

We now prove that there exists a unique matrix $\mathbf{\Phi}(t, t_0)$ having properties (59) and (60).

Define n-dimensional unit vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ by

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}. \tag{82}$$

Thus,

$$[\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3 \quad \cdots \quad \mathbf{e}_n] = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 0 & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = \mathbf{I}. \tag{83}$$

Consider the homogeneous systems given by

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t) \\ \mathbf{x}(t_0) &= \mathbf{e}_j \qquad (j = 1, 2, \ldots, n), \end{aligned} \tag{84}$$

where $\mathbf{A}(t)$ and t_0 are taken from (58). For each $j (j = 1, 2, \ldots, n)$, Theorem 1 of Section 8.1 guarantees the existence of a unique solution of (84); denote this solution by $\mathbf{x}_j(t)$. Thus, $\mathbf{x}_1(t)$ solves the system

$$\begin{aligned} \dot{\mathbf{x}}_1(t) &= \mathbf{A}(t)\mathbf{x}_1(t) \\ \mathbf{x}_1(t_0) &= \mathbf{e}_1, \end{aligned} \tag{85}$$

$\mathbf{x}_2(t)$ satisfies the system

$$\begin{aligned} \dot{\mathbf{x}}_2(t) &= \mathbf{A}(t)\mathbf{x}_2(t) \\ \mathbf{x}_2(t_0) &= \mathbf{e}_2, \end{aligned} \tag{86}$$

and $\mathbf{x}_n(t)$ satisfies the system

$$\begin{aligned} \dot{\mathbf{x}}_n(t) &= \mathbf{A}(t)\mathbf{x}_n(t) \\ \mathbf{x}_n(t_0) &= \mathbf{e}_n. \end{aligned} \tag{87}$$

Define the matrix $\mathbf{\Phi}(t, t_0) = [\mathbf{x}_1(t) \quad \mathbf{x}_2(t) \quad \cdots \quad \mathbf{x}_n(t)]$. Then

$$\begin{aligned}
\mathbf{\Phi}(t_0, t_0) &= [\mathbf{x}_1(t_0) \quad \mathbf{x}_2(t_0) \quad \cdots \quad \mathbf{x}_n(t_0)] \\
&= [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \cdots \quad \mathbf{e}_n] && \{\text{from (85)–(87)} \\
&= \mathbf{I} && \{\text{from (83)}
\end{aligned}$$

and

$$\begin{aligned}
\frac{d\mathbf{\Phi}(t, t_0)}{dt} &= \frac{d}{dt}[\mathbf{x}_1(t) \quad \mathbf{x}_2(t) \quad \cdots \quad \mathbf{x}_n(t)] \\
&= [\dot{\mathbf{x}}_1(t) \quad \dot{\mathbf{x}}_2(t) \quad \cdots \quad \dot{\mathbf{x}}_n(t)] \\
&= [\mathbf{A}(t)\mathbf{x}_1(t) \quad \mathbf{A}(t)\mathbf{x}_2(t) \quad \cdots \quad \mathbf{A}(t)\mathbf{x}_n(t)] && \{\text{from (85)–(87)} \\
&= \mathbf{A}(t)[\mathbf{x}_1(t) \quad \mathbf{x}_2(t) \quad \cdots \quad \mathbf{x}_n(t)] \\
&= \mathbf{A}(t)\mathbf{\Phi}(t, t_0).
\end{aligned}$$

Thus, $\mathbf{\Phi}(t, t_0)$, as defined above, is a matrix that satisfies (59) and (60). Since this $\mathbf{\Phi}(t, t_0)$ always exists, it follows that there will always exist a matrix that satisfies these equations.

It only remains to be shown that $\mathbf{\Phi}(t, t_0)$ is unique. Let $\mathbf{\Psi}(t, t_0)$ be any matrix satisfying (59) and (60). Then the jth column of $\mathbf{\Psi}(t, t_0)$ must satisfy the initial-valued problem given by (84). However, the solution to (84) is unique by Theorem 1 of Section 8.1, hence, the jth column of $\mathbf{\Psi}(t, t_0)$ must be $\mathbf{x}_j(t)$. Thus,

$$\mathbf{\Psi}(t, t_0) = [\mathbf{x}_1(t) \quad \mathbf{x}_2(t) \quad \cdots \quad \mathbf{x}_n(t)] = \mathbf{\Phi}(t, t_0).$$

From this equation, it follows that the transition matrix is unique.

Chapter 9

Jordan Canonical Forms

9.1 Similar Matrices

▶ ***Definition 1.*** A matrix $\mathbf{A}$ is *similar* to a matrix $\mathbf{B}$ if there exists an invertible matrix $\mathbf{P}$ such that

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{B}\mathbf{P}. \tag{1}$$

If we premultiply (1) by $\mathbf{P}$, it follows that $\mathbf{A}$ is similar to $\mathbf{B}$ if and only if there exists a nonsingular matrix $\mathbf{P}$ such that

$$\mathbf{PA} = \mathbf{BP}. \tag{2}$$

Furthermore, if we postmultiply (2) by $\mathbf{P}^{-1}$, we see that $\mathbf{A}$ is similar to $\mathbf{B}$ if and only if $\mathbf{B}$ is similar to $\mathbf{A}$.

Example 1

Determine whether

$$\mathbf{A} = \begin{bmatrix} 4 & 3 \\ -2 & -1 \end{bmatrix} \quad \text{is similar to } \mathbf{B} = \begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix}.$$

Solution. $\mathbf{A}$ will be similar to $\mathbf{B}$ if and only if there exists a nonsingular matrix $\mathbf{P}$ such that (2) is satisfied. Designate $\mathbf{P}$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then $\mathbf{PA} = \mathbf{BP}$ implies that

$$\begin{bmatrix} (4a - 2b) & (3a - b) \\ (4c - 2d) & (3c - d) \end{bmatrix} = \begin{bmatrix} (5a - 4c) & (5b - 4d) \\ (3a - 2c) & (3b - 2d) \end{bmatrix}.$$

Equating corresponding elements, we find that the elements of $\mathbf{P}$ must satisfy the four equations:

$$-a - 2b + 4c = 0,$$

$$3a - 6b + 4d = 0,$$

$$-3a + 6c - 2d = 0,$$

$$-3b + 3c + d = 0.$$

A solution to this set of equations is $a = -\frac{2}{3}d$, $b = \frac{1}{3}d$, with $c = 0$, d arbitrary. Thus,

$$\mathbf{P} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{d}{3}\begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix}.$$

$\mathbf{P}$ is invertible if $d \neq 0$. Thus, by choosing $d \neq 0$, we obtain an invertible matrix $\mathbf{P}$ that satisfies (2), which implies that $\mathbf{A}$ is similar to $\mathbf{B}$. □

The following theorem reveals the basic relationship between similar matrices.

Theorem 1. *Similar matrices have the same characteristic equation (and, therefore, the same eigenvalues).*

Proof. Let $\mathbf{A}$ and $\mathbf{B}$ be similar matrices. The characteristic equation of $\mathbf{A}$ is $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, while the characteristic equation of $\mathbf{B}$ is $\det(\mathbf{B} - \lambda\mathbf{I}) = 0$. Thus, it is sufficient to show that $\det(\mathbf{A} - \lambda\mathbf{I}) = \det(\mathbf{B} - \lambda\mathbf{I})$. We first note that since $\mathbf{A}$ is similar to $\mathbf{B}$, there must exist a nonsingular matrix $\mathbf{P}$ such that (1) is satisfied, and since $\mathbf{P}$ is invertible, we can write

$$\lambda\mathbf{I} = \lambda\mathbf{P}^{-1}\mathbf{P} = \mathbf{P}^{-1}\lambda\mathbf{P} = \mathbf{P}^{-1}\lambda\mathbf{I}\mathbf{P}$$

Then,

$$\begin{aligned} |\mathbf{A} - \lambda\mathbf{I}| = |\mathbf{P}^{-1}\mathbf{B}\mathbf{P} - \lambda\mathbf{I}| &= |\mathbf{P}^{-1}\mathbf{B}\mathbf{P} - \mathbf{P}^{-1}\lambda\mathbf{I}\mathbf{P}| \\ &= |\mathbf{P}^{-1}(\mathbf{B} - \lambda\mathbf{I})\mathbf{P}| \\ &= |\mathbf{P}^{-1}||\mathbf{B} - \lambda\mathbf{I}||\mathbf{P}| \quad \{\text{by Property 9 of Section 4.3} \end{aligned}$$

$$= \frac{1}{|\mathbf{P}|}|\mathbf{B} - \lambda\mathbf{I}||\mathbf{P}| \qquad \{\text{by Problem 13 of Section 4.3}$$

$$= |\mathbf{B} - \lambda\mathbf{I}|.$$

Example 2

Determine whether

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \quad \text{is similar to } \mathbf{B} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}.$$

Solution. The characteristic equation of **A** is $\lambda^2 - 4\lambda - 5 = 0$, while that of **B** is $\lambda^2 - 3\lambda - 10 = 0$. Since these equations are not identical, it follows from Theorem 1 that **A** is not similar to **B**. □

WARNING. Theorem I does *not* imply that two matrices are similar if their characteristic equations are the same. This statement is, in fact, false. Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

Although both matrices have the same characteristic equation, namely $(\lambda - 2)^2 = 0$, they are not similar (see Problem 2). This example serves to emphasize that Theorem 1 can only be used to prove that two matrices are not similar; it *cannot* be used to prove that two matrices are similar. In other words, if two matrices do not have the same characteristic equation, then we can state, categorically, that they are not similar. However, if two matrices have the same characteristic equations, they may or may not be similar, and for the present, an analysis similar to that employed in Example 1 must be used to reach a conclusion. In later sections, we will develop more sophisticated methods to determine when two matrices are similar.

Problems 9.1

(1) Prove that **A** is similar to **B** if and only if **B** is similar to **A**.

(2) Prove that

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{is not similar to } \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

(3) For the following pairs, determine whether or not $\mathbf{A}$ is similar to $\mathbf{B}$:

(a) $\mathbf{A} = \begin{bmatrix} 3 & 5 \\ 3 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix};$

(b) $\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 6 & 3 \\ 1 & 4 \end{bmatrix};$

(c) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix};$

(d) $\mathbf{A} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 3 & 1 & 0 \\ 4 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}.$

(4) Let

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

Show that $\mathbf{PA} = \mathbf{DP}$, and then, without further calculations, determine the eigenvalues of $\mathbf{A}$.

(5) Prove that if $\mathbf{A}$ is similar to $\mathbf{B}$, then both $\mathbf{A}$ and $\mathbf{B}$ have the same trace.

(6) Determine the traces of

$$\mathbf{A} = \begin{bmatrix} 2 & 5 & 4 \\ 2 & 3 & 7 \\ 8 & 1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 4 \\ 2 & 3 & 5 \end{bmatrix},$$

and use the results to determine whether the two matrices are similar.

(7) Prove that if $\mathbf{A}$ is similar to $\mathbf{B}$ and $\mathbf{B}$ is similar to $\mathbf{C}$, then $\mathbf{A}$ is similar to $\mathbf{C}$.

(8) Prove that if $\mathbf{A}$ is similar to $\mathbf{B}$, then $\mathbf{A}^2$ is similar to $\mathbf{B}^2$.

(9) Prove that if $\mathbf{A}$ is similar to $\mathbf{B}$, then $\mathbf{A}^3$ is similar to $\mathbf{B}^3$.

(10) Prove that if $\mathbf{A}$ is similar to $\mathbf{B}$, then $\mathbf{A}^\mathsf{T}$ is similar to $\mathbf{B}^\mathsf{T}$.

(11) Prove that $\mathbf{A}$ is similar to itself whenever $\mathbf{A}$ is a square matrix.

(12) Prove that if $\mathbf{A}$ is similar to $\mathbf{B}$ then $k\mathbf{A}$ is similar to $k\mathbf{B}$ for any constant k.

(13) Prove that if $\mathbf{A}$ is similar to $\mathbf{B}$ and if $\mathbf{A}$ is invertible, then $\mathbf{B}$ is also invertible and $\mathbf{A}^{-1}$ is similar to $\mathbf{B}^{-1}$.

(14) Show that if $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ associated with the eigenvalue λ, and if Eq. (1) is valid, then $\mathbf{y} = \mathbf{P}\mathbf{x}$ is an eigenvector of $\mathbf{B}$ associated with the same eigenvalue λ.

(15) The $\mathbf{QR}$-algorithm generates a sequence of matrices $\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2, \ldots$, where for each k ($k = 1, 2, 3, \ldots$), a $\mathbf{QR}$-decomposition is obtained for $\mathbf{A}_{k-1}$, namely

$$\mathbf{A}_{k-1} = \mathbf{Q}_{k-1}\mathbf{R}_{k-1},$$

and then the order of the product is reversed to obtain

$$\mathbf{A}_k = \mathbf{R}_{k-1}\mathbf{Q}_{k-1}.$$

Show that $\mathbf{A}_k$ is similar to $\mathbf{A}_{k-1}$, and then deduce that $\mathbf{A}_k$ and $\mathbf{A}_{k-1}$ have the same eigenvalues.

9.2 Diagonalizable Matrices

Definition 1. A matrix is *diagonalizable* if it is similar to a diagonal matrix.

Diagonalizable matrices are of particular interest since matrix functions of them can be computed easily. As such we devote this section to determining which matrices are diagonalizable and to finding those matrices $\mathbf{P}$ which will perform the similarity transformations. We note by Theorem 1 of the previous section that if a matrix is similar to a diagonal matrix $\mathbf{D}$, then the form of $\mathbf{D}$ is known. Since the eigenvalues of $\mathbf{D}$ are precisely the elements on the main diagonal of $\mathbf{D}$, it follows that the main diagonal of $\mathbf{D}$ must consist of the eigenvalues of $\mathbf{A}$. For example, if

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix},$$

having eigenvalues -1 and 5, is diagonalizable, then it must be similar to either

$$\begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}.$$

Before continuing our discussion of diagonalizable matrices, we first review two important properties of matrix factoring. The verification of these properties is left as an exercise for the student.

Property 1. *Let* $\mathbf{B} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n]$ *be an* $n \times n$ *matrix, where* $\mathbf{b}_j$ $(j = 1, 2, \ldots, n)$ *is the jth column of* $\mathbf{B}$ *considered as a vector. Then for any* $n \times n$ *matrix* $\mathbf{A}$,

$$\mathbf{AB} = \mathbf{A}[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n] = [\mathbf{Ab}_1 \quad \mathbf{Ab}_2 \quad \cdots \quad \mathbf{Ab}_n].$$

Example 1

Verify Property 1 for

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & -2 & 3 \\ -2 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Solution. In this case,

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \qquad \mathbf{b}_2 = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \qquad \mathbf{b}_3 = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}.$$

Thus

$$\mathbf{Ab}_1 = \begin{bmatrix} 1 & 2 & -1 \\ 1 & -2 & 3 \\ -2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 14 \\ 9 \end{bmatrix},$$

$$\mathbf{Ab}_2 = \begin{bmatrix} 1 & 2 & -1 \\ 1 & -2 & 3 \\ -2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 \\ 16 \\ 9 \end{bmatrix},$$

$$\mathbf{Ab}_3 = \begin{bmatrix} 1 & 2 & -1 \\ 1 & -2 & 3 \\ -2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \\ 9 \end{bmatrix},$$

and

$$[\mathbf{Ab}_1 \quad \mathbf{Ab}_2 \quad \mathbf{Ab}_3] = \begin{bmatrix} 2 & 4 & 6 \\ 14 & 16 & 18 \\ 9 & 9 & 9 \end{bmatrix},$$

which is exactly **AB**. □

Property 2[‡]**.** *Designate the* $n \times n$ *matrix* $\mathbf{B}$ *by* $[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n]$ *as in Property 1 and let* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *represent scalars. Then:*

$$[\lambda_1\mathbf{b}_1 \quad \lambda_2\mathbf{b}_2 \quad \cdots \quad \lambda_n\mathbf{b}_n] = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$= \mathbf{B}\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

Now let $\mathbf{A}$ be an $n \times n$ matrix that has *n linearly independent eigenvectors* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ which correspond to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. (Recall from Section 5.5 that a matrix will have n linearly independent eigenvectors if all the eigenvalues are distinct or, depending upon the matrix, even if some or all of the eigenvalues are equal. *A priori*, therefore, we place no restrictions on the multiplicities of the eigenvalues.) Define

$$\mathbf{M} = [\mathbf{x}_1 \quad \mathbf{x}_2 \quad \cdots \quad \mathbf{x}_n] \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

Here $\mathbf{M}$ is called a *modal matrix* for $\mathbf{A}$, and $\mathbf{D}$ is called a *spectral matrix* for $\mathbf{A}$. Note that since eigenvectors themselves are not unique, and since the columns of both $\mathbf{M}$ and $\mathbf{D}$ may be interchanged (although the jth column of $\mathbf{M}$ must still correspond to the jth column of $\mathbf{D}$; that is, the jth column of $\mathbf{M}$ must be the eigenvector of $\mathbf{A}$ associated with the eigenvalue located in the (j, j) position of $\mathbf{D}$), it follows that both $\mathbf{M}$ and $\mathbf{D}$ are not unique. Using Properties 1 and 2 and the fact that $\mathbf{x}_j$ $(j = 1, 2, \ldots, n)$ is an eigenvector of $\mathbf{A}$, we have that

$$\begin{aligned} \mathbf{AM} &= \mathbf{A}[\mathbf{x}_1 \quad \mathbf{x}_2 \quad \cdots \quad \mathbf{x}_n] \\ &= [\mathbf{Ax}_1 \quad \mathbf{Ax}_2 \quad \cdots \quad \mathbf{Ax}_n] \\ &= [\lambda_1\mathbf{x}_1 \quad \lambda_2\mathbf{x}_2 \quad \cdots \quad \lambda_n\mathbf{x}_n] \\ &= [\mathbf{x}_1 \quad \mathbf{x}_2 \quad \cdots \quad \mathbf{x}_n]\mathbf{D} = \mathbf{MD} \end{aligned} \tag{3}$$

[‡] See Problem 15 of Section 1.4.

(Since the columns of **M** are linearly independent, it follows that the column rank of **M** is n, the rank of **M** is n, the determinant of **M** is nonzero, and $\mathbf{M}^{-1}$ exists.) Premultiplying (3) by $\mathbf{M}^{-1}$ we obtain,

$$\mathbf{D} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M}, \tag{4}$$

which implies that **D** is similar to **A**. Furthermore, by defining $\mathbf{P} = \mathbf{M}^{-1}$, it follows that

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{D}\mathbf{P} = \mathbf{M}\mathbf{D}\mathbf{M}^{-1}, \tag{5}$$

which implies that **A** is similar to **D**. Since we can retrace our steps and show that if (5) is satisfied, then **P** must be $\mathbf{M}^{-1}$, we have proved the following theorem.

Theorem 1. *An $n \times n$ matrix* **A** *is diagonalizable if and only if it possesses n linearly independent eigenvectors. The inverse of the matrix* **P** *is a modal matrix of* **A**.

Example 2

Determine whether

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

is diagonalizable.

Solution. The eigenvalues of **A** are -1 and 5. Since the eigenvalues are distinct, their respective eigenvectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

are linearly independent, hence the matrix is diagonalizable. We can choose either

$$\mathbf{M} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \quad \text{or} \quad \mathbf{M} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}.$$

Making the first choice, we find

$$\mathbf{D} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}.$$

Making the second choice, we obtain

$$\mathbf{D} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}. \qquad \square$$

Example 2 illustrates a point we made previously that neither **M** nor **D** is unique. However, the columns of **M** must still correspond to the columns of **D**; that is, once **M** is chosen, then **D** is uniquely determined. For example, if we choose $\mathbf{M} = [\mathbf{x}_2 \quad \mathbf{x}_1 \quad \mathbf{x}_3 \quad \cdots \quad \mathbf{x}_n]$, then **D** must be

$$\begin{bmatrix} \lambda_2 & & & & \mathbf{0} \\ & \lambda_1 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ \mathbf{0} & & & & \lambda_n \end{bmatrix}$$

while if we choose $\mathbf{M} = [\mathbf{x}_n \quad \mathbf{x}_{n-1} \quad \cdots \quad \mathbf{x}_1]$, then **D** must be

$$\begin{bmatrix} \lambda_n & & & \mathbf{0} \\ & \lambda_{n-1} & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_1 \end{bmatrix}.$$

Example 3

Is

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

diagonalizable?

Solution. The eigenvalues of **A** are 2, 2, and 4. Even though the eigenvalues of **A** are not all distinct, **A** still possesses three linearly independent eigenvectors, namely

$$\mathbf{x}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

hence it is diagonalizable. If we choose

$$\mathbf{M} = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

we find that

$$\mathbf{M}^{-1}\mathbf{AM} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}. \qquad \square$$

Example 4

Is

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

diagonalizable?

Solution. The eigenvalues of **A** are 2 and 2. **A** has only one linearly independent eigenvector associated with it, namely

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

hence it is *not* diagonalizable. (See Problem 2 of the Section 9.1). $\square$

Problems 9.2

Determine whether or not the following matrices are diagonalizable. If they are, determine **M** and compute $\mathbf{M}^{-1}\mathbf{AM}$.

(1) $\mathbf{A} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$.

(2) $\mathbf{A} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$.

(3) $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix}$.

(4) $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(5) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 3 \\ 1 & 2 & 2 \end{bmatrix}$.

(6) $\mathbf{A} = \begin{bmatrix} 5 & 1 & 2 \\ 0 & 3 & 0 \\ 2 & 1 & 5 \end{bmatrix}$.

(7) $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$.

(8) $\mathbf{A} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$.

(9) $\mathbf{A} = \begin{bmatrix} 7 & 3 & 3 \\ 0 & 1 & 0 \\ -3 & -3 & 1 \end{bmatrix}$.

(10) $\mathbf{A} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.

(11) $\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.

9.3 Functions of Matrices—Diagonalizable Matrices

By utilizing (5), we can develop a simple procedure for computing functions of a diagonalizable matrix. We begin with those matrices that are already in diagonal form. In particular, we have from Section 7.1 (Eq. (7) and Problem 2) that if

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & & & \mathbf{0} \\ & \lambda_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n \end{bmatrix} \tag{6}$$

then

$$\mathbf{D}^m = \begin{bmatrix} \lambda_1^m & & & \mathbf{0} \\ & \lambda_2^m & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n^m \end{bmatrix}, \tag{7}$$

$$p_k(\mathbf{D}) = \begin{bmatrix} p_k(\lambda_1) & & & \mathbf{0} \\ & p_k(\lambda_2) & & \\ & & \ddots & \\ \mathbf{0} & & & p_k(\lambda_n) \end{bmatrix}, \tag{8}$$

and

$$e^{\mathbf{D}} = \begin{bmatrix} e^{\lambda_1} & & & \mathbf{0} \\ & e^{\lambda_2} & & \\ & & \ddots & \\ \mathbf{0} & & & e^{\lambda_n} \end{bmatrix}. \tag{9}$$

Example 1

Find $\mathbf{D}^5$, $\mathbf{D}^3 + 2\mathbf{D} - 3\mathbf{I}$, and $e^{\mathbf{D}}$ for

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Solution. From (7), (8), and (9) we obtain

$$\mathbf{D}^5 = \begin{bmatrix} (1)^5 & 0 & 0 \\ 0 & (2)^5 & 0 \\ 0 & 0 & (2)^5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{bmatrix}.$$

$$\mathbf{D}^3 + 2\mathbf{D} - 3\mathbf{I} = \begin{bmatrix} (1)^3 + 2(1) - 3 & 0 & 0 \\ 0 & (2)^3 + 2(2) - 3 & 0 \\ 0 & 0 & (2)^3 + 2(2) - 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}.$$

$$e^{\mathbf{D}} = \begin{bmatrix} e^1 & 0 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e^2 \end{bmatrix}. \qquad \square$$

Now assume that a matrix $\mathbf{A}$ is diagonalizable. Then it follows from (5) that

$$\mathbf{A} = \mathbf{MDM}^{-1}.$$

Thus,

$$\begin{aligned} \mathbf{A}^2 = \mathbf{A} \cdot \mathbf{A} &= (\mathbf{MDM}^{-1})(\mathbf{MDM}^{-1}) \\ &= (\mathbf{MD})(\mathbf{M}^{-1}\mathbf{M})(\mathbf{DM}^{-1}) \\ &= \mathbf{MD}(\mathbf{I})\mathbf{DM}^{-1} = \mathbf{MD}^2\mathbf{M}^{-1}, \\ \mathbf{A}^3 = \mathbf{A}^2 \cdot \mathbf{A} &= (\mathbf{MD}^2\mathbf{M}^{-1})(\mathbf{MDM}^{-1}) \\ &= (\mathbf{MD}^2)(\mathbf{M}^{-1}\mathbf{M})(\mathbf{DM}^{-1}) = \mathbf{MD}^3\mathbf{M}^{-1}, \end{aligned}$$

and, in general,

$$\mathbf{A}^n = \mathbf{MD}^n\mathbf{M}^{-1}. \tag{10}$$

Therefore, to obtain any power of a diagonalizable matrix $\mathbf{A}$, we need only compute $\mathbf{D}$ to that power (this, in itself, is easily done using (7)), premultiply $\mathbf{D}^n$ by $\mathbf{M}$, and postmultiply the result by $\mathbf{M}^{-1}$.

Example 2

Find $\mathbf{A}^{915}$ for

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

Solution. The eigenvalues for $\mathbf{A}$ are $-1, 5$ and a set of linearly independent eigenvectors corresponding to these eigenvalues are

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Hence

$$\mathbf{M} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}, \qquad \mathbf{M}^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}.$$

It follows from (10) that

$$\mathbf{A}^{915} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}^{915} = \mathbf{M}\mathbf{D}^{915}\mathbf{M}^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} (-1)^{915} & 0 \\ 0 & 5^{915} \end{bmatrix}\frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} -2 + 5^{915} & 1 + 5^{915} \\ 2 + 2(5)^{915} & -1 + 2(5)^{915} \end{bmatrix}. \quad \square$$

We may also use (10) to find a simplified expression for $p_k(\mathbf{A})$ where $p_k(x)$ is a kth degree polynomial in x. For instance, suppose that $p_5(x) = 5x^5 - 3x^3 + 2x^2 + 4$ and $p_5(\mathbf{A})$ is to be calculated. Making repeated use of (10), we have

$$\begin{aligned} p_5(\mathbf{A}) &= 5\mathbf{A}^5 - 3\mathbf{A}^3 + 2\mathbf{A}^2 + 4\mathbf{I} \\ &= 5\mathbf{M}\mathbf{D}^5\mathbf{M}^{-1} - 3\mathbf{M}\mathbf{D}^3\mathbf{M}^{-1} + 2\mathbf{M}\mathbf{D}^2\mathbf{M}^{-1} + 4\mathbf{M}\mathbf{I}\mathbf{M}^{-1} \\ &= \mathbf{M}[5\mathbf{D}^5 - 3\mathbf{D}^3 + 2\mathbf{D}^2 + 4\mathbf{I}]\mathbf{M}^{-1} \\ &= \mathbf{M}p_5(\mathbf{D})\mathbf{M}^{-1}. \end{aligned}$$

Thus, to calculate $p_5(\mathbf{A})$, we need compute only $p_5(\mathbf{D})$, which can easily be done by using (8), premultiply this result by $\mathbf{M}$, and postmultiply by $\mathbf{M}^{-1}$. We can extend this reasoning in a straightforward manner (see Problem 7) to prove that if $p_k(\mathbf{A})$ is any kth degree polynomial of $\mathbf{A}$, then

$$p_k(\mathbf{A}) = \mathbf{M}p_k(\mathbf{D})\mathbf{M}^{-1}. \tag{11}$$

Example 3

Find $4\mathbf{A}^{15} - 2\mathbf{A}^{7} + \mathbf{I}$ for

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

Solution. From Example 2, we have $\mathbf{M}$, $\mathbf{M}^{-1}$, and $\mathbf{D}$. Therefore, it follows from (11) and (8) that

$$\begin{aligned} 4\mathbf{A}^{15} - 2\mathbf{A}^{7} + \mathbf{I} &= \mathbf{M}(4\mathbf{D}^{15} - 2\mathbf{D}^{7} + \mathbf{I})\mathbf{M}^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4(-1)^{15} - 2(-1)^{7} + 1 & 0 \\ 0 & 4(5)^{15} - 2(5)^{7} + 1 \end{bmatrix} \\ &\quad \times \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 4(5)^{15} - 2(5)^{7} - 1 & 4(5)^{15} - 2(5)^{7} + 2 \\ 8(5)^{15} - 4(5)^{7} + 4 & 8(5)^{15} - 4(5)^{7} + 1 \end{bmatrix}. \qquad \square \end{aligned}$$

The function of greatest interest is the exponential. If a matrix $\mathbf{A}$ is diagonalizable, then we can use (10) to obtain a useful representation for $e^{\mathbf{A}}$. In particular

$$e^{\mathbf{A}} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!} = \sum_{k=0}^{\infty} \frac{\mathbf{M}\mathbf{D}^k\mathbf{M}^{-1}}{k!} = \mathbf{M}\left(\sum_{k=0}^{\infty} \frac{\mathbf{D}^k}{k!}\right)\mathbf{M}^{-1} = \mathbf{M}e^{\mathbf{D}}\mathbf{M}^{-1}. \tag{12}$$

Thus, to calculate $e^{\mathbf{A}}$, we need only compute $e^{\mathbf{D}}$, which can be done easily by using (9), and then premultiply this result by $\mathbf{M}$ and postmultiply by $\mathbf{M}^{-1}$.

Example 4

Find $e^{\mathbf{A}}$ for

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

Solution. Once again $\mathbf{M}$, $\mathbf{M}^{-1}$, and $\mathbf{D}$ are known from Example 2. It follows from (12) and (9) that

$$\begin{aligned} e^{\mathbf{A}} = \mathbf{M}e^{\mathbf{D}}\mathbf{M}^{-1} &= \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} e^{-1} & 0 \\ 0 & e^{5} \end{bmatrix} \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 2e^{-1} + e^{5} & -e^{-1} + e^{5} \\ -2e^{-1} + 2e^{5} & e^{-1} + 2e^{5} \end{bmatrix}. \qquad \square \end{aligned}$$

(Check this result against Example 1 of Section 7.5.)

Example 5

Find $e^{\mathbf{A}}$ for

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}.$$

Solution. Using the results of Example 3 of Section 9.2, we have

$$\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \qquad \mathbf{M} = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \qquad \mathbf{M}^{-1} = \frac{1}{2}\begin{bmatrix} 0 & 2 & 0 \\ -1 & -2 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Thus,

$$\begin{aligned} e^{\mathbf{A}} = \mathbf{M}e^{\mathbf{D}}\mathbf{M}^{-1} &= \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^2 & 0 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e^4 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 0 & 2 & 0 \\ -1 & -2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} e^4 + e^2 & 2e^4 - 2e^2 & e^4 - e^2 \\ 0 & 2e^2 & 0 \\ e^4 - e^2 & 2e^4 - 2e^2 & e^4 + e^2 \end{bmatrix}. \end{aligned}$$ □

By employing the same reasoning as that used in obtaining (12), we can prove the following theorem (see Problem 8):

Theorem 1. *If $f(\mathbf{A})$ is well-defined for a square matrix $\mathbf{A}$ and if $\mathbf{A}$ is diagonalizable, then*

$$f(\mathbf{A}) = \mathbf{M}f(\mathbf{D})\mathbf{M}^{-1}. \tag{13}$$

Example 6

Find $\cos \mathbf{A}$ for

$$\mathbf{A} = \begin{bmatrix} 4\pi & 2\pi & 0 \\ -\pi & \pi & 0 \\ 3\pi & -2\pi & \pi \end{bmatrix}.$$

Solution. The eigenvalues of $\mathbf{A}$ are π, 2π, 3π, hence

$$\mathbf{D} = \begin{bmatrix} \pi & 0 & 0 \\ 0 & 2\pi & 0 \\ 0 & 0 & 3\pi \end{bmatrix}.$$

An appropriate $\mathbf{M}$ is found to be

$$\mathbf{M} = \begin{bmatrix} 0 & 1 & -2 \\ 0 & -1 & 1 \\ 1 & 5 & -4 \end{bmatrix}, \quad \text{hence} \quad \mathbf{M}^{-1} = \begin{bmatrix} 1 & 6 & 1 \\ -1 & -2 & 0 \\ -1 & -1 & 0 \end{bmatrix}.$$

Thus, it follows from (13) that

$$\begin{aligned} \cos(\mathbf{A}) &= \mathbf{M}\cos(\mathbf{D})\mathbf{M}^{-1} \\ &= \begin{bmatrix} 0 & 1 & -2 \\ 0 & -1 & 1 \\ 1 & 5 & -4 \end{bmatrix} \begin{bmatrix} \cos\pi & 0 & 0 \\ 0 & \cos 2\pi & 0 \\ 0 & 0 & \cos 3\pi \end{bmatrix} \begin{bmatrix} 1 & 6 & 1 \\ -1 & -2 & 0 \\ -1 & -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -3 & -4 & 0 \\ 2 & 3 & 0 \\ -10 & -20 & -1 \end{bmatrix}. \quad \square \end{aligned}$$

Problems 9.3

(1) Find $\mathbf{A}^{27}$ for

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}.$$

(2) Find $\mathbf{A}^{17} - 3\mathbf{A}^5 + 2\mathbf{A}^2 + \mathbf{I}$ for the $\mathbf{A}$ of Problem 1.

(3) Find $e^{\mathbf{A}}$ for the $\mathbf{A}$ of Problem 1.

(4) Find $e^{\mathbf{A}}$ for

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ -5 & 2 \end{bmatrix}.$$

(5) Find $e^{\mathbf{A}}$ for

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 3 \\ 1 & 2 & 2 \end{bmatrix}.$$

(6) Find $e^{\mathbf{A}}$ for

$$\mathbf{A} = \begin{bmatrix} 5 & 1 & 2 \\ 0 & 3 & 0 \\ 2 & 1 & 5 \end{bmatrix}.$$

(7) Prove that (11) is valid for any $p_k(\mathbf{A})$.

(8) Prove Theorem 1 for $f(\mathbf{A}) = \sin \mathbf{A}$.

(9) Find sin **A** for the **A** of Example 6.

9.4 Generalized Eigenvectors

In the previous section, we showed that if a matrix **A** has n linearly independent eigenvectors associated with it, and hence is diagonalizable, then well-defined matrix functions of **A** can be computed. We now generalize our analysis and obtain similar results for matrices which are not diagonalizable. We begin by generalizing the concept of the eigenvector.

▶ ***Definition 1.*** A vector $\mathbf{x}_m$ is a *generalized eigenvector of type m* corresponding to the matrix **A** and the eigenvalue λ if $(\mathbf{A} - \lambda\mathbf{I})^m\mathbf{x}_m = \mathbf{0}$ but $(\mathbf{A} - \lambda\mathbf{I})^{m-1}\mathbf{x}_m \neq \mathbf{0}$.

For example, if

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad \text{then} \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

is a generalized eigenvector of type 3 corresponding to $\lambda = 2$ since

$$(\mathbf{A} - 2\mathbf{I})^3\mathbf{x}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

but

$$(\mathbf{A} - 2\mathbf{I})^2\mathbf{x}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \neq \mathbf{0}.$$

Also,

$$\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

is a generalized eigenvector of type 2 corresponding to $\lambda = 2$ since

$$(\mathbf{A} - 2\mathbf{I})^2\mathbf{x}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

but

$$(\mathbf{A} - 2\mathbf{I})^1\mathbf{x}_2 = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \neq \mathbf{0}.$$

Furthermore,

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

is a generalized eigenvector of type 1 corresponding to $\lambda = 2$ since $(\mathbf{A} - \lambda\mathbf{I})^1\mathbf{x}_1 = \mathbf{0}$ but $(\mathbf{A} - \lambda\mathbf{I})^0\mathbf{x}_1 = \mathbf{I}\mathbf{x}_1 = \mathbf{x}_1 \neq \mathbf{0}$.

We note for reference that a generalized eigenvector of type 1 is, in fact, an eigenvector (see Problem 8).

Example 1

It is known, and we shall see why in Section 9.6, that the matrix

$$\mathbf{A} = \begin{bmatrix} 5 & 1 & -2 & 4 \\ 0 & 5 & 2 & 2 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

has a generalized eigenvector of type 3 corresponding to $\lambda = 5$. Find it.

Solution. We seek a vector $\mathbf{x}_3$ such that

$$(\mathbf{A} - 5\mathbf{I})^3\mathbf{x}_3 = \mathbf{0} \quad \text{and} \quad (\mathbf{A} - 5\mathbf{I})^2\mathbf{x}_3 \neq \mathbf{0}.$$

Designate $\mathbf{x}_3$ by

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}.$$

Then

$$(\mathbf{A}-5\mathbf{I})^3\mathbf{x}_3 = \begin{bmatrix} 0 & 0 & 0 & 14 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14z \\ -4z \\ 3z \\ -z \end{bmatrix}$$

and

$$(\mathbf{A}-5\mathbf{I})^2\mathbf{x}_3 = \begin{bmatrix} 0 & 0 & 2 & -8 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2y-8z \\ 4z \\ -3z \\ z \end{bmatrix}.$$

Thus, in order to satisfy the conditions that

$$(\mathbf{A}-5\mathbf{I})^3\mathbf{x}_3 = \mathbf{0}, \quad \text{and} \quad (\mathbf{A}-5\mathbf{I})^2\mathbf{x}_3 \neq \mathbf{0},$$

we must have $z = 0$ and $y \neq 0$. No restrictions are placed on w and x. By choosing $w = x = z = 0$, $y = 1$, we obtain

$$x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

as a generalized eigenvector of type 3 corresponding to $\lambda = 5$. Note that it is possible to obtain infinitely many other generalized eigenvectors of type 3 by choosing different values of w, x, and y ($y \neq 0$). For instance, if we had picked $w = -1$, $x = 2$ and $y = 15$ (z must be zero), we would have found

$$\mathbf{x}_3 = \begin{bmatrix} -1 \\ 2 \\ 15 \\ 0 \end{bmatrix}.$$

Our first choice, however, is the simplest. □

Example 2

It is known that the matrix

$$\mathbf{A} = \begin{bmatrix} 5 & 0 & 2 \\ 2 & 1 & 1 \\ -5 & 1 & -1 \end{bmatrix}$$

has a generalized eigenvector of type 2 corresponding to $\lambda = 2$. Find it.

Solution. We seek a vector $\mathbf{x}_2$ such that

$$(\mathbf{A} - 2\mathbf{I})^2\mathbf{x}_2 = \mathbf{0} \quad \text{and} \quad (\mathbf{A} - 2\mathbf{I})\mathbf{x}_2 \neq \mathbf{0}.$$

Designate $\mathbf{x}_2$ by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Then

$$(\mathbf{A} - 2\mathbf{I})^2\mathbf{x}_2 = \begin{bmatrix} -1 & 2 & 0 \\ -1 & 2 & 0 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -x + 2y \\ -x + 2y \\ 2x - 4y \end{bmatrix}$$

and

$$(\mathbf{A} - 2\mathbf{I})\mathbf{x}_2 = \begin{bmatrix} 3 & 0 & 2 \\ 2 & -1 & 1 \\ -5 & 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x + 2z \\ 2x - y + z \\ -5x + y - 3z \end{bmatrix}.$$

For $(\mathbf{A} - 2\mathbf{I})^2\mathbf{x}_2 = \mathbf{0}$, it follows that $x = 2y$. Using this result, we obtain

$$(\mathbf{A} - 2\mathbf{I})\mathbf{x}_2 = \begin{bmatrix} 6y + 2z \\ 3y + z \\ -9y - 3z \end{bmatrix}.$$

Since this vector must not be zero, it follows that $z \neq -3y$. There are infinitely many values of x, y, z that simultaneously satisfy the requirements $x = 2y$ and $z \neq -3y$ (for instance, $x = 2$, $y = 1$, $z = 4$); the simplest choice is $x = y = 0$, $z = 1$. Thus,

$$\mathbf{x}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

is a generalized eigenvector of type two corresponding to $\lambda = 2$. □

Example 3

It is known that the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$$

has a generalized eigenvector of type 2 corresponding to $\lambda = 4$. Find it.

Solution. We seek a vector $\mathbf{x}_2$ such that

$$(\mathbf{A} - 4\mathbf{I})^2\mathbf{x}_2 = \mathbf{0} \quad \text{and} \quad (\mathbf{A} - 4\mathbf{I})\mathbf{x}_2 \neq \mathbf{0}.$$

Designate $\mathbf{x}_2$ by

$$\begin{bmatrix} x \\ y \end{bmatrix}.$$

Then

$$(\mathbf{A} - 4\mathbf{I})^2\mathbf{x}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus, we see that every vector has the property that $(\mathbf{A} - 4\mathbf{I})^2\mathbf{x}_2 = \mathbf{0}$; hence, we need place no restrictions on either x or y to achieve this result. However, since

$$(\mathbf{A} - 4\mathbf{I})\mathbf{x}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix}$$

cannot be the zero vector, it must be the case that $y \neq 0$. Thus, by choosing $x = 0$ and $y = 1$ (once again there are infinitely many other choices), we obtain

$$\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

as a generalized eigenvector of type 2 corresponding to $\lambda = 4$. □

Problems 9.4

(1) Determine whether the following vectors are generalized eigenvectors of type 3 corresponding to $\lambda = 2$ for the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, (b) $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, (c) $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, (d) $\begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix}$,

(e) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, (f) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

For the matrices in Problems 2 through 6 find a generalized eigenvector of type 2 corresponding to the eigenvalue $\lambda = -1$:

(2) $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$. **(3)** $\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. **(4)** $\begin{bmatrix} 0 & 4 & 2 \\ -1 & 4 & 1 \\ -1 & -7 & -4 \end{bmatrix}$.

(5) $\begin{bmatrix} 3 & -2 & 2 \\ 2 & -2 & 1 \\ -9 & 9 & -4 \end{bmatrix}$. **(6)** $\begin{bmatrix} 2 & 0 & 3 \\ 2 & -1 & 1 \\ -1 & 0 & -2 \end{bmatrix}$.

(7) Find a generalized eigenvector of type 3 corresponding to $\lambda = 3$ and a generalized eigenvector of type 2 corresponding to $\lambda = 4$ for

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & 0 & 0 & 1 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

(8) Prove that a generalized eigenvector of type 1 is an eigenvector.

9.5 Chains

Definition 1. Let x_m be a generalized eigenvector of type m corresponding to the matrix $\mathbf{A}$ and the eigenvalue λ. The *chain generated by* $\mathbf{x}_m$ is a set of vectors $\{\mathbf{x}_m \quad \mathbf{x}_{m-1} \quad \cdots \quad \mathbf{x}_1\}$ given by

$$\begin{aligned} \mathbf{x}_{m-1} &= (\mathbf{A} - \lambda\mathbf{I})\mathbf{x}_m \\ \mathbf{x}_{m-2} &= (\mathbf{A} - \lambda\mathbf{I})^2\mathbf{x}_m = (\mathbf{A} - \lambda\mathbf{I})\mathbf{x}_{m-1} \\ \mathbf{x}_{m-3} &= (\mathbf{A} - \lambda\mathbf{I})^3\mathbf{x}_m = (\mathbf{A} - \lambda\mathbf{I})\mathbf{x}_{m-2} \\ &\vdots \\ \mathbf{x}_1 &= (\mathbf{A} - \lambda\mathbf{I})^{m-1}\mathbf{x}_m = (\mathbf{A} - \lambda\mathbf{I})\mathbf{x}_2. \end{aligned} \tag{14}$$

Thus, in general,

$$\mathbf{x}_j = (\mathbf{A} - \lambda\mathbf{I})^{m-j}\mathbf{x}_m = (\mathbf{A} - \lambda\mathbf{I})\mathbf{x}_{j+1} \quad (j = 1, 2, \ldots, m-1). \tag{15}$$

Theorem 1. $\mathbf{x}_j$ *(given by (15)) is a generalized eigenvector of type j corresponding to the eigenvalue* λ.

Proof. Since $\mathbf{x}_m$ is a generalized eigenvector of type m, $(\mathbf{A} - \lambda\mathbf{I})^m\mathbf{x}_m = \mathbf{0}$ and $(\mathbf{A} - \lambda\mathbf{I})^{m-1}\mathbf{x}_m \neq \mathbf{0}$. Thus, using (15), we find that

$$(\mathbf{A} - \lambda\mathbf{I})^j\mathbf{x}_j = (\mathbf{A} - \lambda\mathbf{I})^j(\mathbf{A} - \lambda\mathbf{I})^{m-j}\mathbf{x}_m = (\mathbf{A} - \lambda\mathbf{I})^m\mathbf{x}_m = \mathbf{0}$$

and

$$(\mathbf{A} - \lambda\mathbf{I})^{j-1}\mathbf{x}_j = (\mathbf{A} - \lambda\mathbf{I})^{j-1}(\mathbf{A} - \lambda\mathbf{I})^{m-j}\mathbf{x}_m = (\mathbf{A} - \lambda\mathbf{I})^{m-1}\mathbf{x}_m \neq \mathbf{0}$$

which together imply Theorem 1.

It follows from (14) and Theorem 1 that once we have found a generalized eigenvector of type m, it is simple to obtain a generalized eigenvector of any type less than m. For example, we found in the previous section that

$$\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

is a generalized eigenvector of type three for

$$\mathbf{A} = \begin{bmatrix} 5 & 1 & -2 & 4 \\ 0 & 5 & 2 & 2 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

corresponding to $\lambda = 5$ (see Example 1). Using Theorem 1, we now can state that

$$\mathbf{x}_2 = (\mathbf{A} - 5\mathbf{I})\mathbf{x}_3 = \begin{bmatrix} 0 & 1 & -2 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

is a generalized eigenvector of type 2 corresponding to $\lambda = 5$, while

$$\mathbf{x}_1 = (\mathbf{A} - 5\mathbf{I})^2\mathbf{x}_3 = (\mathbf{A} - 5\mathbf{I})\mathbf{x}_2 = \begin{bmatrix} 0 & 1 & -2 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} -2 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

is a generalized eigenvector of type 1, hence an eigenvector, corresponding to $\lambda = 5$. The set

$$\{\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

is the chain generated by $\mathbf{x}_3$.

The value of chains is hinted at by the following theorem.

Theorem 2. *A chain is a linearly independent set of vectors.*

Proof. Let $\{\mathbf{x}_m, \mathbf{x}_{m-1}, \ldots, \mathbf{x}_1\}$ be a chain generated from $\mathbf{x}_m$, a generalized eigenvector of type m corresponding to the eigenvalue λ of a matrix $\mathbf{A}$, and consider the vector equation

$$c_m\mathbf{x}_m + c_{m-1}\mathbf{x}_{m-1} + \cdots + c_1\mathbf{x}_1 = \mathbf{0}. \tag{16}$$

In order to prove that this chain is a linearly independent set, we must show that the only constants satisfying (16) are $c_m = c_{m-1} = \cdots = c_1 = 0$. Multiply (16) by $(\mathbf{A} - \lambda\mathbf{I})^{m-1}$, and note that for $j = 1, 2, \ldots, (m-1)$

$$\begin{aligned} (\mathbf{A} - \lambda\mathbf{I})^{m-1}c_j\mathbf{x}_j &= c_j(\mathbf{A} - \lambda\mathbf{I})^{m-j-1}(\mathbf{A} - \lambda\mathbf{I})^j\mathbf{x}_j \\ &= c_j(\mathbf{A} - \lambda\mathbf{I})^{m-j-1}\mathbf{0} \qquad \left\{\begin{array}{l}\text{since } \mathbf{x}_j \text{ is a generalized} \\ \text{eigenvector of type } j\end{array}\right. \\ &= \mathbf{0}. \end{aligned}$$

Thus, (16) becomes $c_m(\mathbf{A} - \lambda\mathbf{I})^{m-1}\mathbf{x}_m = \mathbf{0}$. However, since $\mathbf{x}_m$ is a generalized eigenvector of type m, $(\mathbf{A} - \lambda\mathbf{I})^{m-1}\mathbf{x}_m \neq \mathbf{0}$, from which it follows that $c_m = 0$. Substituting $c_m = 0$ into (16) and then multiplying (16) by $(\mathbf{A} - \lambda\mathbf{I})^{m-2}$, we find by similar reasoning that $c_{m-1} = 0$. Continuing this process, we finally obtain $c_m = c_{m-1} = \cdots = c_1 = 0$, which implies that the chain is linearly independent.

Problems 9.5

(1) The vector $[1 \quad 1 \quad 1 \quad 0]^{\mathrm{T}}$ is known to be a generalized eigenvector of type 3 corresponding to the eigenvalue 2 for

$$\mathbf{A} = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Construct a chain from this vector.

(2) Redo Problem 1 for the generalized eigenvector $[0 \quad 0 \quad 1 \quad 0]^T$, which is also of type 3 corresponding to the same eigenvalue and matrix.

(3) The vector $[0 \quad 0 \quad 0 \quad 0 \quad 1]^T$ is known to be a generalized eigenvector of type 4 corresponding to the eigenvalue 1 for

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Construct a chain from this vector.

(4) Redo Problem 3 for the generalized eigenvector $[0 \quad 0 \quad 0 \quad 1 \quad 0]^T$, which is of type 3 corresponding to the same eigenvalue and matrix.

(5) The vector $[1 \quad 0 \quad 0 \quad 0 \quad -1]^T$ is known to be a generalized eigenvector of type 3 corresponding to the eigenvalue 3 for

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & 0 & 0 & 1 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

Construct a chain from this vector.

(6) Redo Problem 5 for the generalized eigenvector $[0 \quad 1 \quad 0 \quad 0 \quad 0]^T$, which is of type 2 corresponding to the eigenvalue 4 for the same matrix.

(7) Find a generalized eigenvector of type 2 corresponding to the eigenvalue -1 and the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix},$$

and construct a chain from this vector.

(8) Find a generalized eigenvector of type 2 corresponding to the eigenvalue -1 and the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

and construct a chain from this vector.

(9) Find a generalized eigenvector of type 2 corresponding to the eigenvalue -1 and the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 4 & 2 \\ -1 & 4 & 1 \\ -1 & -7 & -4 \end{bmatrix},$$

and construct a chain from this vector.

(10) Find a generalized eigenvector of type 4 corresponding to $\lambda = 2$ for

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 3 & -1 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix},$$

and construct a chain from this vector.

(11) Find a generalized eigenvector of type 3 corresponding to the eigenvalue 3 for

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & 1 & 2 & 2 \\ -1 & 2 & 1 & 3 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix},$$

and construct a chain from this vector.

9.6 Canonical Basis

As the reader might suspect from our work with diagonalizable matrices, we are interested only in sets of linearly independent generalized eigenvectors. The following theorem, the proof of which is beyond the scope of this book, answers many of the questions regarding the number of such vectors.

Theorem 1. *Every $n \times n$ matrix $\mathbf{A}$ possesses n linearly independent generalized eigenvectors, henceforth abbreviated liges. Generalized eigenvectors corresponding to distinct eigenvalues are linearly independent. If λ is an eigenvalue of $\mathbf{A}$ of multiplicity v, then $\mathbf{A}$ will have v liges corresponding to λ.*

For any given matrix $\mathbf{A}$, there are infintely many ways to pick the n liges. If they are chosen in a particularly judicious manner, we can use these vectors to show that $\mathbf{A}$ is similar to an "almost diagonal matrix." In particular,

Definition 1. A set of n liges (linearly independent generalized eigenvectors) is a *canonical basis* for an $n \times n$ matrix if the set is composed entirely of chains.

Thus, once we have determined that a generalized eigenvector of type m is in a canonical basis, it follows that the $m - 1$ vectors $\mathbf{x}_{m-1}, \mathbf{x}_{m-2}, \ldots, \mathbf{x}_1$ that are in the chain generated by $\mathbf{x}_m$ given by (14) are also in the canonical basis.

For the remainder of this section we concern ourselves with determining a canonical basis for an arbitrary $n \times n$ matrix $\mathbf{A}$.

Let λ_i be an eigenvalue of $\mathbf{A}$ of multiplicity ν. First, find the ranks of the matrices $(\mathbf{A} - \lambda_i\mathbf{I})$, $(\mathbf{A} - \lambda_i\mathbf{I})^2$, $(\mathbf{A} - \lambda_i\mathbf{I})^3, \ldots, (\mathbf{A} - \lambda_i\mathbf{I})^m$. The integer m is determined to be the first integer for which $(\mathbf{A} - \lambda_i\mathbf{I})^m$ has rank $n - \nu$ (n being the number of rows or columns of $\mathbf{A}$, that is, $\mathbf{A}$ is $n \times n$).

Example 1

Determine m corresponding to $\lambda_i = 2$ for

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

Solution. $n = 6$ and the eigenvalue $\lambda_i = 2$ has multiplicity $\nu = 5$, hence $n - \nu = 1$.

$$(\mathbf{A} - 2\mathbf{I}) = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \tag{17}$$

has rank 4.

$$(\mathbf{A} - 2\mathbf{I})^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} \tag{18}$$

has rank 2.

$$(\mathbf{A} - 2\mathbf{I})^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 8 \end{bmatrix} \tag{19}$$

has rank $1 = n - \nu$. Therefore, corresponding to $\lambda_i = 2$, we have $m = 3$. □

Now define

$$\rho_k = r((\mathbf{A} - \lambda_i\mathbf{I})^{k-1}) - r((\mathbf{A} - \lambda_i\mathbf{I})^k) \qquad (k = 1, 2, \ldots, m). \tag{20}$$

ρ_k designates the number of liges of type k corresponding to the eigenvalue λ_i that will appear in a canonical basis for **A**. Note that $r(\mathbf{A} - \lambda_i\mathbf{I})^0 = r(\mathbf{I}) = n$.

Example 2

Determine how many generalized eigenvectors of each type corresponding to $\lambda_1 = 2$ will appear in a canonical basis for the **A** of Example 1.

Solution. Using the results of Example 1, we have that

$$\rho_3 = r(\mathbf{A} - 2\mathbf{I})^2 - r(\mathbf{A} - 2\mathbf{I})^3 = 2 - 1 = 1$$

$$\rho_2 = r(\mathbf{A} - 2\mathbf{I})^1 - r(\mathbf{A} - 2\mathbf{I})^2 = 4 - 2 = 2$$

$$\rho_1 = r(\mathbf{A} - 2\mathbf{I})^0 - r(\mathbf{A} - 2\mathbf{I})^1 = 6 - 4 = 2.$$

Thus, a canonical basis for the matrix given in Example 1 will have, corresponding to $\lambda_1 = 2$, one generalized eigenvector of type 3, two liges of type 2, and two liges of type 1. □

If in the previous example the question had been how many liges of each type corresponding to $\lambda_2 = 4$ will appear in a canonical basis for **A**, we would have found $m = 1$ and $\rho_1 = 1$; hence, a canonical basis would have contained one generalized eigenvector of rank 1 corresponding to $\lambda_2 = 4$. This is, of course, the eigenvector corresponding to $\lambda_2 = 4$ (see Problem 8 of Section 9.4).

Once we have determined the number of generalized eigenvectors of each type in a canonical basis, we can use the techniques of Section 9.4 (see Examples 1 and 2 of that section) together with (14) to obtain the vectors explicitly.

Example 3

Find a canonical basis for the **A** given in Example 1.

Solution. We first find the liges corresponding to $\lambda_1 = 2$. From Example 2, we know that there is one generalized eigenvector of type 3; using the methods of Section 9.4, we find this vector to be

$$\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then using (14), we obtain $\mathbf{x}_2$ and $\mathbf{x}_1$ as generalized eigenvectors of type 2 and 1 respectively, where

$$\mathbf{x}_2 = (\mathbf{A} - 2\mathbf{I})\mathbf{x}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{x}_1 = (\mathbf{A} - 2\mathbf{I})\mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

From Example 2, we know that a canonical basis for **A** also has two liges of type 2 corresponding to $\lambda_1 = 2$. We already found one of these vectors to be $\mathbf{x}_2$; therefore, we seek a generalized eigenvector $\mathbf{y}_2$ of type 2 that is linearly independent of $\{\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1\}$. Designate

$$\mathbf{y}_2 = \begin{bmatrix} u_2 \\ v_2 \\ w_2 \\ x_2 \\ y_2 \\ z_2 \end{bmatrix}.$$

Using the methods of Section 9.4, we find that in order for $\mathbf{y}_2$ to be a generalized eigenvector of type 2, $w_2 = z_2 = 0$, v_2 or y_2 must be nonzero, and u_2 and x_2 are arbitrary. If we pick $u_2 = w_2 = x_2 = y_2 = z_2 = 0$, $v_2 = 1$, we obtain

$$\mathbf{y}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

as a generalized eigenvector of type 2. This vector, however, is not linearly independent of $\{\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1\}$ since $\mathbf{y}_2 = \mathbf{x}_2 + \mathbf{x}_1$. If instead we choose $u_2 = v_2 = w_2 = x_2 = z_2 = 0, y_2 = 1$, we obtain

$$\mathbf{y}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

which satisfies all the necessary requirements. (Note that there are many other adequate choices for $\mathbf{y}_2$. In particular, we could have chosen $u_2 = w_2 = x_2 = z_2 = 0$, $v_2 = y_2 = 1$.) Using (14) again, we find that

$$\mathbf{y}_1 = (\mathbf{A} - 2\mathbf{I})\mathbf{y}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

is a generalized eigenvector of type 1.

From Example 2, we know that a canonical basis for $\mathbf{A}$ has two liges of type 1 corresponding to $\lambda_1 = 2$. We have determined these vectors already to be $\mathbf{x}_1$ and $\mathbf{y}_1$.

Having found all the liges corresponding to $\lambda_1 = 2$, we direct our attention to the liges corresponding to $\lambda_2 = 4$. From our previous discussion, we know that the only generalized eigenvector corresponding to $\lambda_2 = 4$ is the eigenvector itself, which we determine to be

$$\mathbf{z}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \\ 4 \end{bmatrix}.$$

Thus, a canonical basis for $\mathbf{A}$ is $\{\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1, \mathbf{y}_2, \mathbf{y}_1, \mathbf{z}_1\}$. Note that due to Theorem 1, we do not have to check whether $\mathbf{z}_1$ is linearly independent of $\{\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1, \mathbf{y}_2, \mathbf{y}_1\}$. Since $\mathbf{z}_1$ corresponds to λ_2 and all the other vectors correspond to λ_1 where $\lambda_1 \neq \lambda_2$, linear independence is guaranteed.

For future reference, we note that this canonical basis consists of one chain containing three vectors $\{\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1\}$, one chain containing two vectors $\{\mathbf{y}_2, \mathbf{y}_1\}$, and one chain containing one vector $\{\mathbf{z}_1\}$. □

Example 4

Find a canonical basis for

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Solution. $\mathbf{A}$ is a 4×4 and $\lambda_1 = 1$ is an eigenvalue of multiplicity 4; hence, $n = 4$, $v = 4$ and $n - v = 0$.

$$(\mathbf{A} - 1\mathbf{I}) = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

has type 2, and

$$(\mathbf{A} - 1\mathbf{I})^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

has type $0 = n - \nu$. Thus, $m = 2$, $\rho_2 = r(\mathbf{A} - 1\mathbf{I}) - r(\mathbf{A} - 1\mathbf{I})^2 = 2 - 0 = 2$ and $\rho_1 = r(\mathbf{A} - 1\mathbf{I})^0 - r(\mathbf{A} - 1\mathbf{I})^1 = 4 - 2 = 2$; hence, a canonical basis for $\mathbf{A}$ will have two liges of type 2 and two liges of type 1. In order for a vector

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

to be a generalized eigenvector of type 2, either x or z must be nonzero and w and y arbitrary (see Section 9.4). If we first choose $x = 1$, $w = y = z = 0$, and then choose $z = 1$, $w = x = y = 0$, we obtain two liges of type 2 to be

$$\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{y}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Note that we could have chosen w, x, y, z in such a manner as to generate *four* linearly independent generalized eigenvectors of type 2. The vectors

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

together with $\mathbf{x}_2$ and $\mathbf{y}_2$ form such a set. Thus, we immediately have found a set of four liges corresponding to $\lambda_1 = 1$. This set, however is *not* a canonical basis for $\mathbf{A}$, since it is not composed of chains. In order to obtain a canonical basis for $\mathbf{A}$, we use only two of these vectors (in particular $\mathbf{x}_2$ and $\mathbf{y}_2$) and form chains from them.

Using (14), we obtain the two liges of type 1 to be

$$\mathbf{x}_1 = (\mathbf{A} - \mathbf{I})\mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{y}_1 = (\mathbf{A} - \mathbf{I})\mathbf{y}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Thus, a canonical basis for $\mathbf{A}$ is $\{\mathbf{x}_2, \mathbf{x}_1, \mathbf{y}_2, \mathbf{y}_1\}$, which consists of the two chains $\{\mathbf{x}_2, \mathbf{x}_1\}$ and $\{\mathbf{y}_2, \mathbf{y}_1\}$ each containing two vectors. □

Example 5

Find a canonical basis for

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 \\ -1 & 0 & 2 & 0 \\ 4 & 0 & 1 & 2 \end{bmatrix}.$$

Solution. The characteristic equation for $\mathbf{A}$ is $(\lambda - 3)^2(\lambda - 2)^2 = 0$; hence, $\lambda_1 = 3$ and $\lambda_2 = 2$ are both eigenvalues of multiplicity 2. For $\lambda_1 = 3$, we find that $n - v = 2$, $m = 2$, $\rho_2 = 1$, and $\rho_1 = 1$, so that a canonical basis for $\mathbf{A}$ has one generalized eigenvector of type 2 and one generalized eigenvector of type 1 corresponding to $\lambda_1 = 3$. Using the methods of Section 9.4, we find that a generalized eigenvector of type 2 is

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}.$$

By (14), therefore, we have that

$$\mathbf{x}_1 = (\mathbf{A} - 3\mathbf{I})\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 3 \end{bmatrix}$$

is a generalized eigenvector of type 1.

For $\lambda_2 = 2$, we find that $n - v = 2$, $m = 1$, and $\rho_1 = 2$; hence, there are two generalized eigenvectors of type 1 corresponding to $\lambda_2 = 2$. Using the methods of Section 9.4 (or, equivalently, the methods of Section 5.5 since generalized eigenvectors of type 1 are themselves eigenvectors), we obtain

$$\mathbf{y}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{z}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

as the required vectors. Thus, a canonical basis for $\mathbf{A}$ is $\{\mathbf{x}_2, \mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1\}$ which consists of one chain containing two vectors $\{\mathbf{x}_2, \mathbf{x}_1\}$ and two chains containing one vector apiece $\{\mathbf{y}_1\}$ and $\{\mathbf{z}_1\}$. Note that once again, due to Theorem 1, we are guaranteed that $\{\mathbf{x}_2, \mathbf{x}_1\}$ are linearly independent of $\{\mathbf{y}_1, \mathbf{z}_1\}$ since they correspond to different eigenvalues. □

Problems 9.6

(1) Determine the length of the chains in a canonical basis if each is associated with the same eigenvalue λ and if (see Eq. (20)):

(a) $\rho_3 = \rho_2 = \rho_1 = 1$, (b) $\rho_3 = \rho_2 = \rho_1 = 2$,
(c) $\rho_3 = 1, \rho_2 = \rho_1 = 2$, (d) $\rho_3 = 1, \rho_2 = 2, \rho_1 = 3$,
(e) $\rho_3 = \rho_2 = 1, \rho_1 = 3$, (f) $\rho_3 = 3, \rho_2 = 4, \rho_1 = 3$,
(g) $\rho_2 = 2, \rho_1 = 4$, (h) $\rho_2 = 4, \rho_1 = 2$,
(i) $\rho_2 = 2, \rho_1 = 3$, (j) $\rho_2 = \rho_1 = 2$.

Find a canonical basis for the following matrices:

(2) $\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$. **(3)** $\begin{bmatrix} 7 & 3 & 3 \\ 0 & 1 & 0 \\ -3 & -3 & 1 \end{bmatrix}$. **(4)** $\begin{bmatrix} 5 & 1 & -1 \\ 0 & 5 & 2 \\ 0 & 0 & 5 \end{bmatrix}$.

(5) $\begin{bmatrix} 5 & 1 & 2 \\ 0 & 3 & 0 \\ 2 & 1 & 5 \end{bmatrix}$. **(6)** $\begin{bmatrix} 2 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$. **(7)** $\begin{bmatrix} 3 & 1 & 0 & -1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$.

(8) $\begin{bmatrix} 4 & 1 & 1 & 0 & 0 & -1 \\ 0 & 4 & 2 & 0 & 0 & 1 \\ 0 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$.

9.7 Jordan Canonical Forms

In this section, we will show that every matrix is similar to an "almost diagonal" matrix, or in more precise terminology, a matrix in Jordan canonical form. We start by defining a square matrix $\mathbf{S}_k$ (k represents some positive integer and has *no* direct bearing on the order of $\mathbf{S}_k$) given by

$$\mathbf{S}_k = \begin{bmatrix} \lambda_k & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_k & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_k & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda_k & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda_k \end{bmatrix}. \tag{21}$$

Thus, $\mathbf{S}_k$ is a matrix that has all of its diagonal elements equal to λ_k, all of its superdiagonal elements (i.e., all elements directly above the diagonal elements) equal to 1, and all of its other elements equal to zero.

Definition 1. A square matrix $\mathbf{A}$ is in *Jordan canonical form* if it is a diagonal matrix or can be expressed in either one of the following two partitioned diagonal forms:

$$\begin{bmatrix} \mathbf{D} & & & \mathbf{0} \\ & \mathbf{S}_1 & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{S}_r \end{bmatrix} \tag{22}$$

or

$$\begin{bmatrix} \mathbf{S}_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{S}_r \end{bmatrix} \tag{23}$$

Here $\mathbf{D}$ is a diagonal matrix and $\mathbf{S}_k$ $(k = 1, 2, \ldots, r)$ is defined by (21).

Consider the following matrices:

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \tag{24}$$

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \tag{25}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \tag{26}$$

$$\begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \tag{27}$$

$$\begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}. \tag{28}$$

Matrix (24) is in Jordan canonical form since it can be written

$$\begin{bmatrix} \mathbf{S}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_2 \end{bmatrix}, \quad \text{where} \quad \mathbf{S}_1 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{S}_2 = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}.$$

Matrix (25) is in Jordan canonical form since it can be expressed as

$$\begin{bmatrix} \mathbf{S}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_2 \end{bmatrix}, \quad \text{where} \quad \mathbf{S}_1 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{S}_2 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix},$$

while (26) is in Jordan canonical form since it can be written

$$\begin{bmatrix} \mathbf{D} & 0 & 0 \\ 0 & \mathbf{S}_1 & 0 \\ 0 & 0 & \mathbf{S}_2 \end{bmatrix}, \quad \text{where} \quad \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{S}_1 = \mathbf{S}_2 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

Matrices (27) and (28) are not in Jordan canonical form; the first because of the nonzero term in the $(1, 4)$ position and the second due to the 2's on the superdiagonal.

Note that a matrix in Jordan canonical form has nonzero elements only on the main diagonal and superdiagonal, and that the elements on the superdiagonal are restricted to be either zero or one. In particular, a diagonal matrix is a matrix in Jordan canonical form (by definition) that has all its superdiagonal elements equal to zero.

In order to prove that every matrix $\mathbf{A}$ is similar to a matrix in Jordan canonical form, we must first generalize the concept of the modal matrix (see Section 9.2).

Definition 2. Let $\mathbf{A}$ be an $n \times n$ matrix. A *generalized modal matrix* $\mathbf{M}$ for $\mathbf{A}$ is an $n \times n$ matrix whose columns, considered as vectors, form a canonical basis for $\mathbf{A}$ and appear in $\mathbf{M}$ according to the following rules:

(M1) All chains consisting of one vector (that is, one vector in length) appear in the first columns of $\mathbf{M}$.
(M2) All vectors of the same chain appear together in adjacent columns of $\mathbf{M}$.
(M3) Each chain appears in $\mathbf{M}$ in order of increasing type (that is, the generalized eigenvector of type 1 appears before the generalized eigenvector of type 2 of the same chain, which appears before the generalized eigenvector of type 3 of the same chain, etc.).

Example 1

Find a generalized modal matrix $\mathbf{M}$ corresponding to the $\mathbf{A}$ given in Example 5 of Section 9.6.

Solution. In that example, we found that a canonical basis for $\mathbf{A}$ has one chain of two vectors $\{\mathbf{x}_2, \mathbf{x}_1\}$ and two chains of one vector each $\{\mathbf{y}_1\}$ and $\{\mathbf{z}_1\}$. Thus, the first two columns of $\mathbf{M}$ must be $\mathbf{y}_1$ and $\mathbf{z}_1$ (however, in any order) due to (M1) while the third and fourth columns must be $\mathbf{x}_1$ and $\mathbf{x}_2$ respectively due to (M3). Hence,

$$\mathbf{M} = [\mathbf{y}_1 \quad \mathbf{z}_1 \quad \mathbf{x}_1 \quad \mathbf{x}_2] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & -1 & 3 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix}$$

or

$$\mathbf{M} = [\mathbf{z}_1 \quad \mathbf{y}_1 \quad \mathbf{x}_1 \quad \mathbf{x}_2] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 3 & 1 \end{bmatrix}. \qquad \square$$

Example 2

Find a generalized modal matrix $\mathbf{M}$ corresponding to the $\mathbf{A}$ given in Example 4 of Section 9.6.

Solution. In that example, we found that a canonical basis for $\mathbf{A}$ has two chains consisting of two vectors apiece $\{\mathbf{x}_2, \mathbf{x}_1\}$ and $\{\mathbf{y}_2, \mathbf{y}_1\}$. Since this canonical basis has no chain consisting of one vector, (M1) does not apply. From (M2), we assign either $\mathbf{x}_2$ and $\mathbf{x}_1$ to the first two columns of $\mathbf{M}$ and $\mathbf{y}_2$ and $\mathbf{y}_1$ to the last two columns of $\mathbf{M}$ or, alternatively, $\mathbf{y}_2$ and $\mathbf{y}_1$ to the first two columns of $\mathbf{M}$ and $\mathbf{x}_2$ and $\mathbf{x}_1$ to the last two columns of $\mathbf{M}$. We can not, however, define $\mathbf{M} = [\mathbf{x}_1 \quad \mathbf{y}_1 \quad \mathbf{y}_2 \quad \mathbf{x}_2]$ since this alignment would split the $\{\mathbf{x}_2, \mathbf{x}_1\}$ chain and violate (M2). Due to (M3), $\mathbf{x}_1$ must precede $\mathbf{x}_2$ and $\mathbf{y}_1$ must precede $\mathbf{y}_2$; hence

$$\mathbf{M} = [\mathbf{x}_1 \quad \mathbf{x}_2 \quad \mathbf{y}_1 \quad \mathbf{y}_2] = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

or

$$\mathbf{M} = [\mathbf{y}_1 \quad \mathbf{y}_2 \quad \mathbf{x}_1 \quad \mathbf{x}_2] = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \qquad \square$$

Examples 1 and 2 show that $\mathbf{M}$ is not unique. The important fact, however, is that for any arbitrary $n \times n$ matrix $\mathbf{A}$, there does exist at least one generalized modal matrix $\mathbf{M}$ corresponding to it. Furthermore, since the columns of $\mathbf{M}$ considered as vectors form a linearly independent set, it follows that the column rank of $\mathbf{M}$ is n, the rank of $\mathbf{M}$ is n, the determinant of $\mathbf{M}$ is nonzero, and $\mathbf{M}$ is invertible (that is, $\mathbf{M}^{-1}$ exists).

Now let $\mathbf{A}$ represent any $n \times n$ matrix and let $\mathbf{M}$ be a generalized modal matrix for $\mathbf{A}$. Then, one can show (see the appendix to this chapter for a proof) that

$$\mathbf{AM} = \mathbf{MJ}, \tag{29}$$

where $\mathbf{J}$ is a matrix in Jordan canonical form. By either premultiplying or postmultiplying (29) by $\mathbf{M}^{-1}$, we obtain either

$$\mathbf{J} = \mathbf{M}^{-1}\mathbf{AM} \quad (30$$

or

▶| $$\mathbf{A} = \mathbf{MJM}^{-1}. \tag{31}$$

Equation (31) provides us with a proof of

Theorem 1. *Every $n \times n$ matrix $\mathbf{A}$ is similar to a matrix in Jordan canonical form.*

Note the resemblance of Theorem 1, (30) and (31) above to Theorem 1, (4) and (5) of Section 9.2.

Example 3

Verify (30) for the **A** of Example 1.

Solution. In that example, we found a generalized modal matrix for **A** to be

$$\mathbf{M} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & -1 & 3 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix}.$$

We compute

$$\mathbf{M}^{-1} = \begin{bmatrix} -3 & 1 & -4 & 0 \\ -1 & 0 & 2 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Thus,

$$\mathbf{M}^{-1}\mathbf{AM} = \begin{bmatrix} -3 & 1 & -4 & 0 \\ -1 & 0 & 2 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 \\ -1 & 0 & 2 & 0 \\ 4 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & -1 & 3 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} = \mathbf{J},$$

a matrix in Jordan canonical form. □

Example 4

Find a matrix in Jordan canonical form that is similar to

$$\mathbf{A} = \begin{bmatrix} 0 & 4 & 2 \\ -3 & 8 & 3 \\ 4 & -8 & -2 \end{bmatrix}.$$

Solution. The characteristic equation of **A** is $(\lambda - 2)^3 = 0$, hence, $\lambda = 2$ is an eigenvalue of multiplicity three. Following the procedures of the previous sections, we find that $r(\mathbf{A} - 2\mathbf{I}) = 1$ and $r(\mathbf{A} - 2\mathbf{I}) = 0 = n - v$. Thus, $\rho_2 = 1$ and $\rho_1 = 2$, which implies that a canonical basis for $\mathbf{A}$ will contain one lige of type 2 and two liges of type 1, or equivalently, one chain of two vectors $\{\mathbf{x}_2, \mathbf{x}_1\}$ and one chain of one vector $\{\mathbf{y}_1\}$. Designating $\mathbf{M} = [\mathbf{y}_1 \quad \mathbf{x}_1 \quad \mathbf{x}_2]$, we find that

$$\mathbf{M} = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 3 & 0 \\ 0 & -4 & 1 \end{bmatrix}.$$

Thus,

$$\mathbf{M}^{-1} = \frac{1}{4}\begin{bmatrix} 3 & -2 & 0 \\ -1 & 2 & 0 \\ -4 & 8 & 4 \end{bmatrix}$$

and

$$\mathbf{J} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \frac{1}{4}\begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 4 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}. \qquad \square$$

Example 5

Find a matrix in Jordan canonical form that is similar to

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & -1 & 1 & 1 & 3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & -1 & -1 & -6 & 0 \\ -2 & 0 & -1 & 2 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 & 2 & 4 & 1 \end{bmatrix}.$$

Solution. The characteristic equation $\mathbf{A}$ is $(\lambda - 1)^7 = 0$, hence, $\lambda = 1$ is an eigenvalue of multiplicity 7. Following the procedures of the previous sections, we find that $r(\mathbf{A} - 1\mathbf{I}) = 3$, $r(\mathbf{A} - 1\mathbf{I})^2 = 1$, $r(\mathbf{A} - 1\mathbf{I})^3 = 0 = n - v$. Thus $\rho_3 = 1$, $\rho_2 = 2$, and $\rho_1 = 4$ which implies that a canonical basis for $\mathbf{A}$ will consist of one lige of type 3, two liges of type 2 and four liges of type 1, or equivalently, one chain of three vectors $\{\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1\}$ one chain of two vectors $\{\mathbf{y}_2, \mathbf{y}_1\}$, and two chains of one vector $\{\mathbf{z}_1\}$, $\{\mathbf{w}_1\}$. Designating

$$\mathbf{M} = [\mathbf{z}_1 \quad \mathbf{w}_1 \quad \mathbf{x}_1 \quad \mathbf{x}_2 \quad \mathbf{x}_3 \quad \mathbf{y}_1 \quad \mathbf{y}_2],$$

we find that

$$\mathbf{M} = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 & -2 & 1 \\ 0 & 3 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 1 & 0 & 2 & 0 \\ -2 & 0 & -1 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 \end{bmatrix}.$$

Thus,

$$\mathbf{M}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 4 & -2 & 2 \\ 0 & 0 & 1 & 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & -1 & -1 & -3 & 1 & -1 \\ 1 & 0 & 0 & -1 & -2 & -1 & 0 \end{bmatrix}$$

and

$$\mathbf{J} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \qquad \square$$

It is possible to construct a **J** matrix in Jordan canonical form associated with a given $n \times n$ matrix **A** just from a knowledge of the composition of **M**.

Each complete chain of more than one vector in length that goes into composing **M** will give rise to an $\mathbf{S}_k$ submatrix in **J**. Thus, for example, if a canonical basis for **A** contains a chain of three elements corresponding to the eigenvalue λ, the matrix **J** that is similar to **A** must contain a submatrix

$$\mathbf{S}_k = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}.$$

The order of $\mathbf{S}_k$ is identical to the length of the chain.

The chains consisting of only one vector give rise collectively to the **D** submatrix in **J**. Thus, if there were no chains of one vector in a canonical basis for **A**, then **J** would contain no **D** submatrix, while if a canonical basis for **A** contained four one vector chains, then **J** would contain a **D** submatrix of order 4×4. In this latter case, the elements on the main diagonal of **D** would be the eigenvalues corresponding to the one element chains.

Finally, by rearranging the order in which whole chains are placed into **M**, we merely rearrange the order in which the corresponding $\mathbf{S}_k$ submatrices appear in **J**. For example, suppose that the characteristic equation of **A** is $(\lambda - 1)(\lambda - 2)(\lambda - 3)^2(\lambda - 4)^2$. Furthermore, assume that $\lambda = 3$ gives rise to the chain $\{\mathbf{z}_2, \mathbf{z}_1\}$, $\lambda = 4$ gives rise to the chain $\{\mathbf{w}_2, \mathbf{w}_1\}$, and the eigenvalues $\lambda = 1$ and $\lambda = 2$ correspond respectively to the eigenvectors $\mathbf{x}_1$ and $\mathbf{y}_1$. Then, if we choose $\mathbf{M} = [\mathbf{x}_1 \quad \mathbf{y}_1 \quad \mathbf{z}_1 \quad \mathbf{z}_2 \quad \mathbf{w}_1 \quad \mathbf{w}_2]$, it will follow that

$$\mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

while if we pick $\mathbf{M} = [\mathbf{y}_1 \quad \mathbf{x}_1 \quad \mathbf{w}_1 \quad \mathbf{w}_2 \quad \mathbf{z}_1 \quad \mathbf{z}_2]$, it will follow that

$$\mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

Problems 9.7

In Problems 1 through 5, assume that $\lambda = 2$ is the only eigenvalue for a 5×5 matrix **A**. Find a matrix in Jordan canonical form similar to **A** if the complete set of ρ_k numbers (see Eq. (20)) associated with this eigenvalue are as follows:

(1) $\rho_3 = 1, \rho_2 = 1, \rho_1 = 3.$

(2) $\rho_3 = 1, \rho_2 = 2, \rho_1 = 2.$

(3) $\rho_2 = 2, \rho_1 = 3.$

(4) $\rho_4 = \rho_3 = \rho_2 = 1, \rho_1 = 2.$

(5) $\rho_5 = \rho_4 = \rho_3 = \rho_2 = \rho_1 = 1.$

In Problems 6 through 11, assume that $\lambda = 3$ is the only eigenvalue for a 6×6 matrix **A**. Find a matrix in Jordan canonical form similar to **A** if the complete set of ρ_k numbers (see Eq. (20)) associated with this eigenvalue are as follows:

(6) $\rho_3 = \rho_2 = \rho_1 = 2$.

(7) $\rho_3 = 1, \rho_2 = 2, \rho_1 = 3$.

(8) $\rho_3 = \rho_2 = 1, \rho_1 = 4$.

(9) $\rho_2 = \rho_1 = 3$.

(10) $\rho_2 = 2, \rho_1 = 4$.

(11) $\rho_2 = 1, \rho_1 = 5$.

In Problems 12 through 16 find a matrix in Jordan canonical form that is similar to the given matrix in

(12) Problem 2 of Section 9.6.

(13) Problem 3 of Section 9.6.

(14) Problem 4 of Section 9.6.

(15) Problem 6 of Section 9.6.

(16) Problem 8 of Section 9.6.

9.8 Functions of Matrices—General Case

By using (31), we can generalize the results of Section 9.3 and develop a method for computing functions of nondiagonalizable matrices. We begin by directing our attention to those matrices that are already in Jordan canonical form.

Consider any arbitrary $n \times n$ matrix **J** in the Jordan canonical form‡

$$\mathbf{J} = \begin{bmatrix} \mathbf{D} & & & & \mathbf{0} \\ & \mathbf{S}_1 & & & \\ & & \mathbf{S}_2 & & \\ & & & \ddots & \\ \mathbf{0} & & & & \mathbf{S}_r \end{bmatrix}$$

Using the methods of Section 1.5 for multiplying together partitioned matrices, it follows that

$$\mathbf{J}^2 = \begin{bmatrix} \mathbf{D} & & & \mathbf{0} \\ & \mathbf{S}_1 & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{S}_r \end{bmatrix} \begin{bmatrix} \mathbf{D} & & & \mathbf{0} \\ & \mathbf{S}_1 & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{S}_r \end{bmatrix} = \begin{bmatrix} \mathbf{D}^2 & & & \mathbf{0} \\ & \mathbf{S}_1^2 & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{S}_r^2 \end{bmatrix},$$

‡ If instead the matrix has a form given by (23), the same analysis may be carried over in total by suppressing the **D** term.

$$\mathbf{J}^3 = \mathbf{J}\cdot\mathbf{J}^2 = \begin{bmatrix} \mathbf{D}^3 & & & 0 \\ & \mathbf{S}_1^3 & & \\ & & \ddots & \\ 0 & & & \mathbf{S}_r^3 \end{bmatrix},$$

and, in general,

$$\mathbf{J}^n = \begin{bmatrix} \mathbf{D}^n & & & 0 \\ & \mathbf{S}_1^n & & \\ & & \ddots & \\ 0 & & & \mathbf{S}_r^n \end{bmatrix}. \qquad n = 0, 1, 2, 3, \ldots$$

Furthermore, if $f(z)$ is a well-defined function for $\mathbf{J}$, or equivalently, $\mathbf{D}$, $\mathbf{S}_1, \ldots, \mathbf{S}_r$, then we can use the procedures developed in Section 7.1 to show that

$$f(\mathbf{J}) = \begin{bmatrix} f(\mathbf{D}) & & & 0 \\ & f(\mathbf{S}_1) & & \\ & & \ddots & \\ 0 & & & f(\mathbf{S}_r) \end{bmatrix}. \tag{32}$$

Since $f(\mathbf{D})$ has already been determined in Section 9.3, we only need develop a method for calculating $f(\mathbf{S}_k)$ in order to have $f(\mathbf{J})$ determined completely.

From (21), we have the $(p + 1) \times (p + 1)$ matrix $\mathbf{S}_k$ defined by

$$\mathbf{S}_k = \begin{bmatrix} \lambda_k & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_k & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_k & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_k & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_k \end{bmatrix}.$$

It can be shown that

$$f(\mathbf{S}_k) = \begin{bmatrix} f(\lambda_k) & \dfrac{f'(\lambda_k)}{1!} & \dfrac{f''(\lambda_k)}{2!} & \cdots & \dfrac{f^{(p)}(\lambda_k)}{p!} \\ 0 & f(\lambda_k) & \dfrac{f'(\lambda_k)}{1!} & \cdots & \dfrac{f^{(p-1)}(\lambda_k)}{(p-1)!} \\ 0 & 0 & f(\lambda_k) & \cdots & \dfrac{f^{(p-2)}(\lambda_k)}{(p-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & f(\lambda_k) \end{bmatrix}. \tag{33}$$

CAUTION. The derivatives in (33) are taken with respect to λ_k. For instance, if $\lambda_k = 3t$ and $f(\lambda_k) = e^{\lambda_k} = e^{3t}$, then $f''(\lambda_k)$ is not equal to $9e^{3t}$ but rather e^{3t}. That is, the derivative must be taken with respect to $\lambda_k = 3t$ and not with respect to t. Perhaps the safest way to make the necessary computations in (33) without incurring an error is to first keep $f(\lambda_k)$ in terms of λ_k (do not substitute a numerical value for λ_k such as $3t$), then take the derivative of $f(\lambda_k)$ with respect to λ_k (the second derivative of e^{λ_k} with respect to λ_k is e^{λ_k}), and finally, as the last step, substitute in the correct value for λ_k where needed.

Example 1

Find $e^{\mathbf{S}_k}$ if

$$\mathbf{S}_k = \begin{bmatrix} 2t & 1 & 0 \\ 0 & 2t & 1 \\ 0 & 0 & 2t \end{bmatrix}.$$

Solution. In this case, $\lambda_k = 2t$, $f(\mathbf{S}_k) = e^{\mathbf{S}_k}$, and $f(\lambda_k) = e^{\lambda_k}$.

$$e^{\mathbf{S}_k} = f(\mathbf{S}_k) = \begin{bmatrix} f(\lambda_k) & f'(\lambda_k) & \dfrac{f''(\lambda_k)}{2} \\ 0 & f(\lambda_k) & f'(\lambda_k) \\ 0 & 0 & f(\lambda_k) \end{bmatrix} = \begin{bmatrix} e^{\lambda_k} & e^{\lambda_k} & \dfrac{e^{\lambda_k}}{2} \\ 0 & e^{\lambda_k} & e^{\lambda_k} \\ 0 & 0 & e^{\lambda_k} \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} & e^{2t} & \dfrac{e^{2t}}{2} \\ 0 & e^{2t} & e^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix} = e^{2t}\begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \qquad \square$$

Example 2

Find $\mathbf{J}^6$ if

$$\mathbf{J} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Solution. $\mathbf{J}$ is in the Jordan canonical form

$$\mathbf{J} = \begin{bmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_1 \end{bmatrix}.$$

In this case, $f(\mathbf{J}) = \mathbf{J}^6$. It follows, therefore, from (32) that

$$\mathbf{J}^6 = \begin{bmatrix} \mathbf{D}^6 & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_1^6 \end{bmatrix}. \tag{34}$$

Using (7) and (33), we find that

$$\mathbf{D}^6 = \begin{bmatrix} 2^6 & 0 \\ 0 & 3^6 \end{bmatrix} = \begin{bmatrix} 64 & 0 \\ 0 & 729 \end{bmatrix} \tag{35}$$

and with $\lambda_1 = 1$

$$\mathbf{S}_1^6 = \begin{bmatrix} f(\lambda_1) & \dfrac{f'(\lambda_1)}{1} & \dfrac{f''(\lambda_1)}{2} & \dfrac{f'''(\lambda_1)}{6} \\ 0 & f(\lambda_1) & f'(\lambda_1) & \dfrac{f''(\lambda_1)}{2} \\ 0 & 0 & f(\lambda_1) & f'(\lambda_1) \\ 0 & 0 & 0 & f(\lambda_1) \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1^6 & 6\lambda_1^5 & 15\lambda_1^4 & 20\lambda_1^3 \\ 0 & \lambda_1^6 & 6\lambda_1^5 & 15\lambda_1^4 \\ 0 & 0 & \lambda_1^6 & 6\lambda_1^5 \\ 0 & 0 & 0 & \lambda_1^6 \end{bmatrix}$$

$$= \begin{bmatrix} (1)^6 & 6(1)^5 & 15(1)^4 & 20(1)^3 \\ 0 & (1)^6 & 6(1)^5 & 15(1)^4 \\ 0 & 0 & (1)^6 & 6(1)^5 \\ 0 & 0 & 0 & (1)^6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 6 & 15 & 20 \\ 0 & 1 & 6 & 15 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{36}$$

Substituting (35) and (36) to (34), we obtain

$$\mathbf{J}^6 = \begin{bmatrix} 64 & 0 & 0 & 0 & 0 & 0 \\ 0 & 729 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 6 & 15 & 20 \\ 0 & 0 & 0 & 1 & 6 & 15 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \qquad \square$$

Now let $\mathbf{A}$ be an $n \times n$ matrix. We know from the previous section that there exists a matrix $\mathbf{J}$ in Jordan canonical form and an invertible generalized modal matrix $\mathbf{M}$ such that

$$\mathbf{A} = \mathbf{M}\mathbf{J}\mathbf{M}^{-1}. \tag{37}$$

By an analysis identical to that used in Section 9.3 to obtain (10), (11), and (13), we have

$$f(\mathbf{A}) = \mathbf{M}f(\mathbf{J})\mathbf{M}^{-1},$$

providing, of course, that $f(\mathbf{A})$ is well-defined. Thus, $f(\mathbf{A})$ is obtained simply by first calculating $f(\mathbf{J})$, which in view of (32) can be done quite easily, then premultiplying $f(\mathbf{J})$ by $\mathbf{M}$, and finally postmultiplying this result by $\mathbf{M}^{-1}$.

Example 3

Find $e^{\mathbf{A}}$ if

$$\mathbf{A} = \begin{bmatrix} 0 & 4 & 2 \\ -3 & 8 & 3 \\ 4 & -8 & -2 \end{bmatrix}.$$

Solution. From Example 4 of Section 9.7 we have that a modal matrix for $\mathbf{A}$ is

$$\mathbf{M} = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 3 & 0 \\ 0 & -4 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{J} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Thus, $e^{\mathbf{A}} = \mathbf{M}e^{\mathbf{J}}\mathbf{M}^{-1}$. In order to calculate $e^{\mathbf{J}}$, we note that

$$\mathbf{J} = \begin{bmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_1 \end{bmatrix},$$

where $\mathbf{D}$ is the 1×1 matrix $[2]$ and $\mathbf{S}_1$ is the 2×2 matrix

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

Using (7) and (33), we find that

$$e^{\mathbf{D}} = [e^2], \qquad e^{\mathbf{S}_1} = \begin{bmatrix} e^2 & e^2 \\ 0 & e^2 \end{bmatrix};$$

and

$$e^{\mathbf{J}} = \begin{bmatrix} e^{\mathbf{D}} & \mathbf{0} \\ \mathbf{0} & e^{\mathbf{S}_1} \end{bmatrix} = \begin{bmatrix} e^2 & 0 & 0 \\ 0 & e^2 & e^2 \\ 0 & 0 & e^2 \end{bmatrix}.$$

Thus,

$$e^{\mathbf{A}} = \mathbf{M}e^{\mathbf{J}}\mathbf{M}^{-1} = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 3 & 0 \\ 0 & -4 & 1 \end{bmatrix} \begin{bmatrix} e^2 & 0 & 0 \\ 0 & e^2 & e^2 \\ 0 & 0 & e^2 \end{bmatrix} \begin{bmatrix} 3 & -2 & 0 \\ -1 & 2 & 0 \\ -4 & 8 & 4 \end{bmatrix} \frac{1}{4}$$

$$= e^2 \begin{bmatrix} -1 & 4 & 2 \\ -3 & 7 & 3 \\ 4 & -8 & -3 \end{bmatrix}. \quad \square$$

Example 4

Find sin **A** if

$$\mathbf{A} = \begin{bmatrix} \pi & \pi/3 & -\pi \\ 0 & \pi & \pi/2 \\ 0 & 0 & \pi \end{bmatrix}$$

Solution. By the methods of the previous section, we find that **A** is similar to

$$\mathbf{J} = \begin{bmatrix} \pi & 1 & 0 \\ 0 & \pi & 1 \\ 0 & 0 & \pi \end{bmatrix}$$

with a corresponding generalized modal matrix given by

$$\mathbf{M} = \begin{bmatrix} \pi^2/6 & -\pi & 0 \\ 0 & \pi/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now $\sin \mathbf{A} = \mathbf{M}(\sin \mathbf{J})\mathbf{M}^{-1}$. If we note that $\mathbf{J} = \mathbf{S}_1$, then $\sin \mathbf{J}$ can be calculated from (33). Here $f(\lambda_1) = \sin \lambda_1$, $f'(\lambda_1) = \cos \lambda_1$, $f''(\lambda_1) = -\sin \lambda_1$ and $\lambda_1 = \pi$, hence

$$\sin \mathbf{J} = \begin{bmatrix} \sin \pi & \cos \pi & \dfrac{-\sin \pi}{2!} \\ 0 & \sin \pi & \cos \pi \\ 0 & 0 & \sin \pi \end{bmatrix},$$

and

$$\sin \mathbf{A} = \begin{bmatrix} \pi^2/6 & -\pi & 0 \\ 0 & \pi/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin \pi & \cos \pi & -\frac{1}{2}\sin \pi \\ 0 & \sin \pi & \cos \pi \\ 0 & 0 & \sin \pi \end{bmatrix} \begin{bmatrix} 6/\pi^2 & 12/\pi^2 & 0 \\ 0 & 2/\pi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\pi/3 & \pi \\ 0 & 0 & -\pi/2 \\ 0 & 0 & 0 \end{bmatrix}. \qquad \square$$

Problems 9.8

(1) Find $\mathbf{A}^4$ if

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

(2) Find $\mathbf{A}^{10}$ if

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

(3) Find $e^{\mathbf{A}}$ if

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix}.$$

(4) Find $e^{\mathbf{A}}$ if

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$

(5) Find $\cos \mathbf{A}$ for the $\mathbf{A}$ of Example 4.

(6) Find $e^{\mathbf{A}}$ for the $\mathbf{A}$ of Example 4.

(7) Find $e^{\mathbf{A}}$ for

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix}.$$

(8) Find $e^{\mathbf{A}}$ for

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

9.9 The Function $e^{\mathbf{A}t}$

As was the case in Chapter 7, once we have a procedure to find functions of arbitrary matrices, no new difficulties arise in the calculation of $e^{\mathbf{A}t}$. Again we simply define $\mathbf{B} = \mathbf{A}t$ and use the methods of the previous sections to compute $e^{\mathbf{B}}$.

Example 1

Find $e^{\mathbf{A}t}$ if

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

Solution. Define

$$\mathbf{B} = \mathbf{A}t = \begin{bmatrix} 3t & t & 0 \\ 0 & 3t & t \\ 0 & 0 & 3t \end{bmatrix}.$$

Note that although $\mathbf{A}$ is in Jordan canonical form, $\mathbf{B}$, due to the t terms on the superdiagonal, is not. We find a generalized modal matrix $\mathbf{M}$ to be

$$\mathbf{M} = \begin{bmatrix} t^2 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \text{hence} \quad \mathbf{J} = \mathbf{M}^{-1}\mathbf{B}\mathbf{M} = \begin{bmatrix} 3t & 1 & 0 \\ 0 & 3t & 1 \\ 0 & 0 & 3t \end{bmatrix}.$$

Now

$$e^{\mathbf{A}t} = e^{\mathbf{B}} = \mathbf{M}e^{\mathbf{J}}\mathbf{M}^{-1}.$$

Using the techniques of Section 9.8, we find that

$$e^{\mathbf{J}} = \begin{bmatrix} e^{3t} & e^{3t} & e^{3t}/2 \\ 0 & e^{3t} & e^{3t} \\ 0 & 0 & e^{3t} \end{bmatrix}.$$

Hence,

$$\begin{aligned} e^{\mathbf{A}t} &= \begin{bmatrix} t^2 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{bmatrix} e^{3t} \begin{bmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/t^2 & 0 & 0 \\ 0 & 1/t & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= e^{3t} \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned} \tag{38}$$

Note that this derivation of $e^{\mathbf{A}t}$ is not valid for $t = 0$ since $\mathbf{M}^{-1}$ is undefined there. Considering the case $t = 0$ separately, we find that $e^{\mathbf{A}0} = e^0 = \mathbf{I}$. However, since (38) also reduces to the identity matrix at $t = 0$, we are justified in using (38) for all t.

The student is urged to compare (38) with the result obtained in Example 2 of Section 7.6. □

Example 2

Find $e^{\mathbf{A}t}$ if

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 4 \\ 1 & 2 & 1 \\ -1 & 0 & -2 \end{bmatrix}.$$

Solution. Define

$$\mathbf{B} = \mathbf{A}t = \begin{bmatrix} 3t & 0 & 4t \\ t & 2t & t \\ -t & 0 & -2t \end{bmatrix}.$$

A generalized modal matrix for $\mathbf{B}$ is found to be

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3t & 0 \\ -1 & 0 & -1 \end{bmatrix}$$

Thus,

$$\mathbf{J} = \mathbf{M}^{-1}\mathbf{B}\mathbf{M} = \begin{bmatrix} -t & 0 & 0 \\ 0 & 2t & 1 \\ 0 & 0 & 2t \end{bmatrix}.$$

$$e^{\mathbf{A}t} = e^{\mathbf{B}} = \mathbf{M}e^{\mathbf{J}}\mathbf{M}^{-1} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3t & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{2t} & e^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix} \frac{1}{3} \begin{bmatrix} -1 & 0 & -4 \\ 0 & 1/t & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} -e^{-t} + 4e^{2t} & 0 & -4e^{-t} + 4e^{2t} \\ 3te^{2t} & 3e^{2t} & 3te^{2t} \\ e^{-t} - e^{2t} & 0 & 4e^{-t} - e^{2t} \end{bmatrix}. \tag{39}$$

Note that once again this derivation is not valid at $t = 0$ due to the $1/t$ term in $\mathbf{M}^{-1}$. However, if we consider the case $t = 0$ separately, we find that we are justified in using (39) for all t. □

We now have two methods available to calculate $e^{\mathbf{A}t}$ and other functions of a matrix. The question naturally arises as to which method is best. In general, the method developed in Chapter 7 is by far the quicker and should be the one most often employed. With this method, we only need the eigenvalues, while the method given in this chapter requires the knowledge of both the eigenvalues and a canonical basis. On the other hand, if a canonical basis is available, or if such a set of vectors is easy to obtain, then the method given in this chapter yields $e^{\mathbf{A}t}$ very quickly and should be the method employed.

Problems 9.9

(1) Find a canonical basis for the $\mathbf{B} = \mathbf{A}t$ given in Example 2 and verify that the matrix $\mathbf{M}$ is a generalized modal matrix for $\mathbf{B}$.

Find $e^{\mathbf{A}t}$ for the following matrices $\mathbf{A}$:

(2) $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$. **(3)** $\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$. **(4)** $\begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

(5) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. **(6)** $\begin{bmatrix} 2 & 3 & 0 \\ -1 & -2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. **(7)** $\begin{bmatrix} 3 & 1 & 0 \\ -1 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix}$.

(8) $\begin{bmatrix} 5 & -2 & 2 \\ 2 & 0 & 1 \\ -7 & 5 & -2 \end{bmatrix}$.

Appendix to Chapter 9

We now prove the validity of Eq. (29) which states that if $\mathbf{A}$ is an arbitrary $n \times n$ matrix and $\mathbf{M}$ is a generalized modal matrix for $\mathbf{A}$, then $\mathbf{AM} = \mathbf{MJ}$ where $\mathbf{J}$ is a matrix in Jordan canonical form. In actuality, we only will prove this result for a special 4×4 matrix, but the reasoning used in this particular case can be extended easily to cover any arbitrary case.

We assume that the canonical basis used to form $\mathbf{M}$ consisted of one chain containing three vectors $\{\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1\}$ and one chain containing one vector $\{\mathbf{y}_1\}$. The three-element chain corresponds to the eigenvalue λ_1 while $\mathbf{y}_1$ corresponds to the eigenvalue λ_2; λ_1 and λ_2 can be equal or distinct.

Since $\mathbf{x}_1$ and $\mathbf{y}_1$ are both generalized eigenvectors of type 1, they are themselves eigenvectors, hence $\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1$ and $\mathbf{Ay}_1 = \lambda_2\mathbf{y}_1$. Furthermore, since $\mathbf{x}_2$ and $\mathbf{x}_1$ belong to the chain generated by $\mathbf{x}_3$, it follows that

$$\mathbf{x}_2 = (\mathbf{A} - \lambda_1\mathbf{I})\mathbf{x}_3$$

$$\mathbf{x}_1 = (\mathbf{A} - \lambda_1\mathbf{I})\mathbf{x}_2.$$

Hence,

$$\mathbf{x}_2 = \mathbf{Ax}_3 - \lambda_1\mathbf{x}_3$$

$$\mathbf{x}_1 = \mathbf{Ax}_2 - \lambda_1\mathbf{x}_2$$

or, by a rearrangement of terms,

$$\mathbf{Ax}_3 = \lambda_1\mathbf{x}_3 + \mathbf{x}_2$$

$$\mathbf{Ax}_2 = \lambda_1\mathbf{x}_2 + \mathbf{x}_1.$$

Thus,

$$\begin{aligned}\mathbf{AM} = \mathbf{A}[\mathbf{y}_1 \quad \mathbf{x}_1 \quad \mathbf{x}_2 \quad \mathbf{x}_3] &= [\mathbf{Ay}_1 \quad \mathbf{Ax}_1 \quad \mathbf{Ax}_2 \quad \mathbf{Ax}_3] \\ &= [\lambda_2\mathbf{y}_1 \quad \lambda_1\mathbf{x}_1 \quad \lambda_1\mathbf{x}_2 + \mathbf{x}_1 \quad \lambda_1\mathbf{x}_3 + \mathbf{x}_2] \\ &= [\mathbf{y}_1 \quad \mathbf{x}_1 \quad \mathbf{x}_2 \quad \mathbf{x}_3]\begin{bmatrix} \lambda_2 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix}.\end{aligned}$$

Defining

$$\mathbf{J} = \begin{bmatrix} \lambda_2 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix},$$

which is a matrix in Jordan canonical form, we obtain $\mathbf{AM} = \mathbf{MJ}$, the desired result.

Chapter 10

Special Matrices

10.1 Complex Inner Product

By now it should be apparent that the eigenvalues and the eigenvectors (including the generalized eigenvectors) of a matrix serve to characterize the matrix almost completely. With them, we can calculate the determinant of the matrix, determine its invertibility, find its Jordan canonical form, and compute functions of the matrix. On the other hand, it is, practically speaking, impossible to obtain information of this sort and numerical methods must be employed for most matrices.

Fortunately, however, there are certain classes of matrices, for example real symmetric matrices, whose structures are such that their eigenvalues and eigenvectors are particularly simple. We devote this entire chapter to the study of such matrices.

Before beginning, we note that all matrices under consideration have been real; that is, they have not contained complex elements. (In contrast, even though the matrices themselves have been real, the eigenvalues and eigenvectors often were complex. Example 2 of Section 5.3 is a case in point.) This restriction has been for convenience only and we now remedy the situation by henceforth allowing matrices to contain both real and complex elements. Most of the analysis of the previous nine chapters still remains valid; that is, eigenvalues, eigenvectors, inverses, functions of matrices, and Jordan canonical forms for complex matrices are found by exactly the same methods that were developed for real matrices.

Example 1

Find $e^{\mathbf{A}t}$ for

$$\mathbf{A} = \begin{bmatrix} i & 1-i \\ 0 & i \end{bmatrix}.$$

Solution. Define

$$\mathbf{B} = \mathbf{A}t = \begin{bmatrix} it & t-it \\ 0 & it \end{bmatrix}.$$

The eigenvalues of $\mathbf{B}$ are $\lambda_1 = \lambda_2 = it$. We could use the methods derived in Chapter 7 to compute $e^{\mathbf{A}t}$; we choose instead to use the methods of Chapter 9. A generalized modal matrix for $\mathbf{B}$ is found to be

$$\mathbf{M} = \begin{bmatrix} t-it & 0 \\ 0 & 1 \end{bmatrix}.$$

Then,

$$\mathbf{J} = \mathbf{M}^{-1}\mathbf{B}\mathbf{M} = \begin{bmatrix} it & 1 \\ 0 & it \end{bmatrix}$$

and

$$e^{\mathbf{A}t} = e^{\mathbf{B}} = \mathbf{M}e^{\mathbf{J}}\mathbf{M}^{-1} = \mathbf{M}\begin{bmatrix} e^{it} & e^{it} \\ 0 & e^{it} \end{bmatrix}\mathbf{M}^{-1} = e^{it}\begin{bmatrix} 1 & t-it \\ 0 & 1 \end{bmatrix}. \quad \square$$

In working with complex numbers, we will frequently make use of the concept of conjugation.

Definition 1. The *complex conjugate* of a number $x = a + ib$ (where a and b are real) is the number $\bar{x} = a - ib$.

Thus, for example, the complex conjugates of $4 + 5i$, $-i$, $t - 2i$, 3, and e^{it}, t real, are $4 - 5i$, i, $t + 2i$, 3, and e^{-it}. Furthermore, it follows from the definition that the complex conjugate of a real number is itself while the complex conjugate of a pure imaginary number is its negative.

The complex conjugate of a matrix $\mathbf{A}$ is defined to be the matrix $\bar{\mathbf{A}}$ obtained by conjugating every element in $\mathbf{A}$. Thus, for the $\mathbf{A}$ given in Example 1, we have that

$$\bar{\mathbf{A}} = \begin{bmatrix} -i & 1+i \\ 0 & -i \end{bmatrix}.$$

In the following sections, we will have occasion to use certain properties of complex numbers and complex matrices. We list these properties here, and leave their verification as an exercise for the reader. Note that all properties pertaining to matrices are equally valid for vectors since vectors are a special case of matrices. Thus, for example, if

$$\mathbf{x} = \begin{bmatrix} -2i \\ 4 \\ -1+i \end{bmatrix}, \quad \text{then} \quad \bar{\mathbf{x}} = \begin{bmatrix} 2i \\ 4 \\ -1-i \end{bmatrix}.$$

(C1) $x\bar{x}$ is always real and positive, except when $x = 0$, whereupon $x\bar{x} = 0$.
(C2) x is real if and only if $x = \bar{x}$.
(C3) $\mathbf{A}$ is a real matrix if and only if $\mathbf{A} = \bar{\mathbf{A}}$.
(C4) $x + \bar{x}$ is real; $\mathbf{A} + \bar{\mathbf{A}}$ is a real matrix.
(C5) $\overline{xy} = \bar{x}\bar{y}$.
(C6) $\overline{\mathbf{Ax}} = \mathbf{A}\bar{\mathbf{x}}$ if $\mathbf{A}$ is real.
(C7) $\bar{\bar{x}} = x$; $\bar{\bar{\mathbf{A}}} = \mathbf{A}$.
(C8) $\overline{x+y} = \bar{x} + \bar{y}$

In contrast to most of the analysis developed in the previous chapters, the inner product, as defined in Section 6.1, must be generalized before it can be used with complex-valued vectors. If not, some of the properties we want all inner products to possess are no longer valid. In particular, we do not want the inner product of a nonzero vector with itself to be zero. Yet, if we use the definition of a real inner product on the nonzero complex-valued vector $\mathbf{x} = [1 \quad i]$, we find that $\langle \mathbf{x}, \mathbf{x} \rangle = 1(1) + i(i) = 0$.

For any two vectors $\mathbf{x}$ and $\mathbf{y}$, we redefine the inner product $\langle \mathbf{x}, \mathbf{y} \rangle$ to be a scalar obtained by first conjugating $\mathbf{y}$, then multiplying together the corresponding elements of $\mathbf{x}$ and $\bar{\mathbf{y}}$, and finally summing the results. Note, that when $\mathbf{x}$ and $\mathbf{y}$ are real vectors, this new definition *reduces to the real inner product* defined in Section 6.1, because $\bar{\mathbf{y}} = \mathbf{y}$.

Example 2

Find $\langle \mathbf{x}, \mathbf{y} \rangle$ if

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2i \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} -4i \\ 5 \\ 6+i \end{bmatrix}.$$

Solution.

$$\bar{\mathbf{y}} = \begin{bmatrix} 4i \\ 5 \\ 6-i \end{bmatrix},$$

hence

$$\begin{aligned}\langle \mathbf{x}, \mathbf{y} \rangle &= 1(4i) + 2i(5) + 3(6 - i) \\ &= 18 + 11i. \quad \square\end{aligned}$$

Example 3

Find $\langle \mathbf{x}, \mathbf{y} \rangle$ if

$$\mathbf{x} = \begin{bmatrix} 1 + i \\ 2 - 3i \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}.$$

Solution. Here $\mathbf{y}$ is a real vector, hence $\bar{\mathbf{y}} = \mathbf{y}$ and

$$\langle \mathbf{x}, \mathbf{y} \rangle = (1 + i)(5) + (2 - 3i)(-4) = -3 + 17i.$$

By relying on our knowledge of matrix multiplication, transposition, and determinants, we can give a formal definition for the inner product of two column vectors solely in terms of matrix notation. $\square$

Definition 2. If $\mathbf{x}$ and $\mathbf{y}$ are column vectors of the same dimension, then $\langle \mathbf{x}, \mathbf{y} \rangle = \det(\mathbf{x}^{\mathsf{T}}\bar{\mathbf{y}})$.

Thus, if

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix},$$

then

$$\mathbf{x}^{\mathsf{T}}\bar{\mathbf{y}} = [x_1 \quad x_2 \quad \cdots \quad x_n] \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \vdots \\ \bar{y}_n \end{bmatrix} = [x_1\bar{y}_1 + x_2\bar{y}_2 + \cdots + x_n\bar{y}_n]$$

and

$$\langle \mathbf{x}, \mathbf{y} \rangle = \det(\mathbf{x}^{\mathsf{T}}\bar{\mathbf{y}}) = x_1\bar{y}_1 + x_2\bar{y}_2 + \cdots + x_n\bar{y}_n.$$

From the definition of the inner product and properties of complex numbers, the inner product can be shown to possess the following properties[‡]:

(I1) $\langle \mathbf{x}, \mathbf{x} \rangle$ is real and positive if $\mathbf{x} \neq \mathbf{0}$; $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = \mathbf{0}$.
(I2) $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$
(I3) $\langle \lambda\mathbf{x}, \mathbf{y} \rangle = \lambda\langle \mathbf{x}, \mathbf{y} \rangle$ where λ is any scalar, real or complex.
(I4) $\langle \mathbf{x} + \mathbf{z}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{z}, \mathbf{y} \rangle$.

We will only prove (I2) here and leave the proofs of the other properties as exercises for the student (see Problems 16 and 17). Let

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

Then

$$\langle \mathbf{y}, \mathbf{x} \rangle = y_1\bar{x}_1 + y_2\bar{x}_2 + \cdots + y_n\bar{x}_n;$$

hence,

$$\begin{aligned} \overline{\langle \mathbf{y}, \mathbf{x} \rangle} &= \overline{y_1\bar{x}_1 + y_2\bar{x}_2 + \cdots + y_n\bar{x}_n} \\ &= \bar{y}_1\bar{\bar{x}}_1 + \bar{y}_2\bar{\bar{x}}_2 + \cdots + \bar{y}_n\bar{\bar{x}}_n \qquad \begin{cases}\text{(Properties (C5) and} \\ \text{(C8)}\end{cases} \\ &= \bar{y}_1 x_1 + \bar{y}_2 x_2 + \cdots + \bar{y}_n x_n \\ &= x_1\bar{y}_1 + x_2\bar{y}_2 + \cdots + x_n\bar{y}_n = \langle \mathbf{x}, \mathbf{y} \rangle. \end{aligned}$$

Properties (I1)–(I4) can be used to generate other properties of the inner product. For example, (I1) and (I3) can be used to show that

(I5) $\langle \mathbf{0}, \mathbf{y} \rangle = 0$,

while (I2) and (I3) can be combined to prove that

(I6) $\langle \mathbf{x}, \lambda\mathbf{y} \rangle = \bar{\lambda}\langle \mathbf{x}, \mathbf{y} \rangle$.

[‡] In other abstract treatments, these properties are used to define an inner product. The function given by Definition 1 is then shown to satisfy these properties.

Example 4

Verify (I6) for $\mathbf{x} = \begin{bmatrix} i \\ 2+3i \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, and $\lambda = 1 - i$.

Solution.

$$\lambda\mathbf{y} = \begin{bmatrix} 4-4i \\ 5-5i \end{bmatrix},$$

hence

$$\langle \mathbf{x}, \lambda\mathbf{y} \rangle = i(4+4i) + (2+3i)(5+5i) = -9 + 29i.$$

But

$$\langle \mathbf{x}, \mathbf{y} \rangle = i(4) + (2+3i)5 = 10 + 19i,$$

hence

$$\bar{\lambda}\langle \mathbf{x}, \mathbf{y} \rangle = (1+i)(10+19i) = -9 + 29i = \langle \mathbf{x}, \lambda\mathbf{y} \rangle. \qquad \square$$

Perhaps surprisingly, even with this new, generalized definition of the inner product, we retain the old definitions for the magnitude of a vector, for orthogonal vectors, and for orthonormal sets. The magnitude of a vector $\mathbf{x}$, either real or complex valued, continues to be defined as $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$.

Example 5

Find $\|\mathbf{x}\|$ if

$$\mathbf{x} = \begin{bmatrix} 1+2i \\ -3i \end{bmatrix}.$$

Solution.

$$\bar{\mathbf{x}} = \begin{bmatrix} 1-2i \\ 3i \end{bmatrix},$$

hence $\langle \mathbf{x}, \mathbf{x} \rangle = (1+2i)(1-2i) + (-3i)(3i) = 14$

and

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{14}. \qquad \square$$

As before, a vector $\mathbf{x}$ is *normalized* if its magnitude is unity; that is, if and only if $\langle \mathbf{x}, \mathbf{x} \rangle = 1$. A vector can always be normalized by dividing it by its magnitude. Also, two vectors are *orthogonal* if their inner product is zero.

Thus, given the vectors

$$\mathbf{x} = \begin{bmatrix} 1 \\ i \\ 1-i \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} -i \\ i \\ i \end{bmatrix}, \qquad \mathbf{z} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

we see that $\mathbf{x}$ is orthogonal to $\mathbf{y}$ and $\mathbf{y}$ is orthogonal to $\mathbf{z}$ since $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{z} \rangle = 0$; but the vectors $\mathbf{x}$ and $\mathbf{z}$ are not orthogonal since $\langle \mathbf{x}, \mathbf{z} \rangle = 1 + i \neq 0$.

A set of vectors in *orthonormal* if each vector is orthogonal to every other vector in the set and if every vector is a unit vector (a vector of magnitude 1). The Gram–Schmidt orthonormalization process can be used to transform a linearly independent set of vectors into an orthonormal set. As in Section 6.2, if $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is a linearly independent set, then we set

$$\mathbf{y}_1 = \mathbf{x}_1$$

$$\mathbf{y}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{y}_1 \rangle}{\langle \mathbf{y}_1, \mathbf{y}_1 \rangle} \mathbf{y}_1$$

$$\mathbf{y}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{y}_1 \rangle}{\langle \mathbf{y}_1, \mathbf{y}_1 \rangle} \mathbf{y}_1 - \frac{\langle \mathbf{x}_3, \mathbf{y}_2 \rangle}{\langle \mathbf{y}_2, \mathbf{y}_2 \rangle} \mathbf{y}_2$$

and, in general,

$$\mathbf{y}_j = \mathbf{x}_j - \sum_{k=1}^{j-1} \frac{\langle \mathbf{x}_j, \mathbf{y}_k \rangle}{\langle \mathbf{y}_k, \mathbf{y}_k \rangle} \mathbf{y}_k \qquad j = 2, 3, \ldots, n).$$

The $\mathbf{y}$-vectors are orthogonal. If we then divide each of these new vectors by its magnitude to obtain

$$\mathbf{q}_j = \frac{\mathbf{y}_j}{\|\mathbf{y}_j\|},$$

we obtain the set $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n\}$, which is orthonormal.

Example 6

Use the Gram–Schmidt orthonormalization process to construct an orthonormal set of vectors from the linearly independent set $\{\mathbf{x}_1, \mathbf{x}_2\}$, where

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}.$$

Solution.

$$\mathbf{y}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

Now $\langle \mathbf{x}_2, \mathbf{y}_1 \rangle = (1-i)(1) + 1(-i) = 1 - 2i$ and $\langle \mathbf{y}_1, \mathbf{y}_1 \rangle = 1(1) + i(-i) = 2$; hence,

$$\mathbf{y}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{y}_1 \rangle}{\langle \mathbf{y}_1, \mathbf{y}_1 \rangle}\mathbf{y}_1 = \mathbf{x}_2 - \frac{(1-2i)}{2}\mathbf{y}_1$$

$$= \begin{bmatrix} 1-i \\ 1 \end{bmatrix} - \begin{bmatrix} (1-2i)/2 \\ (2+i)/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -i/2 \end{bmatrix}.$$

To obtain $\mathbf{q}_1$ and $\mathbf{q}_2$, we first note that $\langle \mathbf{y}_1, \mathbf{y}_1 \rangle = 2$ and

$$\langle \mathbf{y}_2, \mathbf{y}_2 \rangle = (1/2)(1/2) + (-i/2)(i/2) = 1/2;$$

hence, $\|\mathbf{y}_1\| = \sqrt{\langle \mathbf{y}_1, \mathbf{y}_1 \rangle} = \sqrt{2}$ and $\|\mathbf{y}_2\| = \sqrt{\langle \mathbf{y}_2, \mathbf{y}_2 \rangle} = \sqrt{1/2}$. Thus,

$$\mathbf{q}_1 = \frac{\mathbf{y}_1}{\|\mathbf{y}_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$$

and

$$\mathbf{q}_2 = \frac{\mathbf{y}_2}{\|\mathbf{y}_2\|} = \sqrt{2}\begin{bmatrix} 1/2 \\ -i/2 \end{bmatrix}. \qquad \square$$

Example 7

Use the Gram–Schmidt orthonormalization process to construct an orthonormal set of vectors from the linearly independent set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ where

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} i \\ 0 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{x}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ i \end{bmatrix}.$$

Solution.

$$\mathbf{y}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix}.$$

$$\mathbf{y}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{y}_1 \rangle}{\langle \mathbf{y}_1, \mathbf{y}_1 \rangle}\mathbf{y}_1 = \mathbf{x}_2 - \frac{i}{2}\mathbf{y}_1 = \begin{bmatrix} i/2 \\ 1/2 \\ 2 \\ 0 \end{bmatrix}.$$

$$\mathbf{y}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{y}_1 \rangle}{\langle \mathbf{y}_1, \mathbf{y}_1 \rangle}\mathbf{y}_1 - \frac{\langle \mathbf{x}_3, \mathbf{y}_2 \rangle}{\langle \mathbf{y}_2, \mathbf{y}_2 \rangle}\mathbf{y}_2$$

$$= \mathbf{x}_3 - \frac{(-i)}{2}\mathbf{y}_1 - \frac{1/2}{9/2}\mathbf{y}_2 = \begin{bmatrix} (4/9)i \\ 4/9 \\ -2/9 \\ 1 \end{bmatrix}.$$

$$\mathbf{y}_4 = \mathbf{x}_4 - \frac{\langle \mathbf{x}_4, \mathbf{y}_1 \rangle}{\langle \mathbf{y}_1, \mathbf{y}_1 \rangle}\mathbf{y}_1 - \frac{\langle \mathbf{x}_4, \mathbf{y}_2 \rangle}{\langle \mathbf{y}_2, \mathbf{y}_2 \rangle}\mathbf{y}_2 - \frac{\langle \mathbf{x}_4, \mathbf{y}_3 \rangle}{\langle \mathbf{y}_3, \mathbf{y}_3 \rangle}\mathbf{y}_3$$

$$= \mathbf{x}_4 - \frac{0}{2}\mathbf{y}_1 - \frac{0}{9/2}\mathbf{y}_2 - \frac{i}{13/9}\mathbf{y}_3$$

$$= \begin{bmatrix} 4/13 \\ (-4/13)i \\ (2/13)i \\ (4/13)i \end{bmatrix}.$$

Thus,

$$\mathbf{q}_1 = \sqrt{\frac{1}{2}}\begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{q}_2 = \sqrt{\frac{2}{9}}\begin{bmatrix} i/2 \\ 1/2 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{q}_3 = \sqrt{\frac{9}{13}}\begin{bmatrix} (4/9)i \\ 4/9 \\ -2/9 \\ 1 \end{bmatrix},$$

$$\mathbf{q}_4 = \sqrt{\frac{13}{4}}\begin{bmatrix} 4/13 \\ -(4/13)i \\ (2/13)i \\ (4/13)i \end{bmatrix}. \qquad \square$$

Problems 10.1

In Problems 1 through 14, let

$$\mathbf{A} = \begin{bmatrix} 2+3i & 1 \\ 0 & 2+i \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2-i & 1+3i \\ 2i & 4 \end{bmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 3 & 5+6i \\ 5-6i & 4 \end{bmatrix}.$$

(1) Find $\bar{\mathbf{A}}$.

(2) Find $\bar{\mathbf{B}}$.

(3) Find $\mathbf{C}^{\mathsf{T}}$.

(4) Find $\bar{\mathbf{C}}^{\mathsf{T}}$.

(5) Find $i\mathbf{B}$.

(6) Find $(1-2i)\mathbf{A}$.

(7) Find $\mathbf{A} + \mathbf{B}$.

(8) Find $\mathbf{A} - \mathbf{C}$.

(9) Find $2\mathbf{B} + 3\mathbf{C}$.

(10) Find $3\mathbf{A} + i\mathbf{C}$.

(11) Find $\mathbf{A}\mathbf{A}^{\mathsf{T}}$.

(12) Find $\bar{\mathbf{B}}\mathbf{B}^{\mathsf{T}}$.

(13) Find $\mathbf{B}\bar{\mathbf{B}}^{\mathsf{T}}$.

(14) Find $e^{\mathbf{A}t}$.

(15) Find $\langle \mathbf{x}, \mathbf{y} \rangle$ and $\langle \mathbf{y}, \mathbf{x} \rangle$ if

(a) $\mathbf{x} = \begin{bmatrix} i \\ -i \\ 1 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 3 \\ 1-i \\ -1 \end{bmatrix},$ (b) $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix},$

(c) $\mathbf{x} = \begin{bmatrix} i \\ 1-i \\ 2 \\ 1+i \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} -i \\ 1+i \\ -3i \\ 2 \end{bmatrix}.$

(16) Use Property (C1) to prove (I1).

(17) Prove properties (I3)–(I6).

(18) Verify property (I6) for

$$\mathbf{x} = \mathbf{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \lambda = i.$$

(19) Normalize the following vectors:

(a) $\begin{bmatrix} 1 \\ i \end{bmatrix},$ (b) $\begin{bmatrix} 1 \\ 1-i \\ 2i \end{bmatrix},$ (c) $\begin{bmatrix} -i \\ 1+i \\ 1-i \end{bmatrix},$ (d) $\begin{bmatrix} 1 \\ 2 \\ i \\ 1-i \end{bmatrix}.$

(20) Prove that $\overline{\langle \mathbf{x}, \mathbf{y} \rangle} = \langle \bar{\mathbf{x}}, \bar{\mathbf{y}} \rangle$.

(21) Find x so that

$$\begin{bmatrix} 3 \\ x \\ i \end{bmatrix} \quad \text{is orthogonal to} \quad \begin{bmatrix} 1-i \\ 2 \\ -i \end{bmatrix}.$$

(22) Find x and y so that

$$\begin{bmatrix} x \\ y \\ 3 \end{bmatrix}$$

is orthogonal to both

$$\begin{bmatrix} 0 \\ i \\ 1-i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} i \\ 1 \\ 4 \end{bmatrix}.$$

(23) Find a three-dimensional unit vector that is orthogonal to both

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} i \\ 1 \\ 1-i \end{bmatrix}.$$

(24) Use the Gram–Schmidt orthonormalization process to construct an orthonormal set from

$$\mathbf{x}_1 = \begin{bmatrix} i \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ i \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} 0 \\ 0 \\ 1-i \\ 0 \end{bmatrix}.$$

(25) Use the Gram–Schmidt orthonormalization process to construct an orthonormal set from

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1+i2 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} i \\ 1+i \end{bmatrix}.$$

(26) Use the Gram–Schmidt orthonormalization process to construct an orthonormal set from

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1+i \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1-i \\ -i \end{bmatrix}.$$

(27) Redo Problem 25 using the modified Gram–Schmidt process.

(28) Redo Problem 26 using the modified Gram–Schmidt process.

(29) Let $\mathbf{x} = \sum_{i=1}^{n} c_i \mathbf{x}_i$ and let $\mathbf{y} = \sum_{j=1}^{m} d_j \mathbf{y}_j$ where $\mathbf{x}$, $\mathbf{y}$, $\mathbf{x}_i$, and $\mathbf{y}_j$ represent n-dimensional vectors and c_i and d_j represent scalars ($i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$). Use Properties (I3), (I4), and (I6) to prove that

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} c_i \bar{d}_j \langle \mathbf{x}_i, \mathbf{y}_j \rangle.$$

(30) Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ be an orthonormal set of vectors. Prove that

$$\left\langle \sum_{i=1}^{n} c_i \mathbf{x}_i, \quad \sum_{j=1}^{n} c_j \mathbf{x}_j \right\rangle = \sum_{i=1}^{n} c_i \bar{c}_i.$$

10.2 Self-Adjoint Matrices

Definition 1. Let $\mathbf{A}$ be an $m \times n$ matrix. An *adjoint of* $\mathbf{A}$, designated by $\mathbf{A}^*$, is an $n \times m$ matrix satisfying the property that

$$\langle \mathbf{x}, \mathbf{A}\mathbf{y} \rangle = \langle \mathbf{A}^*\mathbf{x}, \mathbf{y} \rangle \tag{1}$$

for all m-dimensional vectors $\mathbf{x}$ and n-dimensional vectors $\mathbf{y}$.

In other words, the inner product of the vector $\mathbf{x}$ with the vector $\mathbf{Ay}$ is equal to the inner product of the vector $\mathbf{A}^*\mathbf{x}$ with the vector $\mathbf{y}$ for every m-dimensional vector $\mathbf{x}$ and n-dimensional vector $\mathbf{y}$. Note that although $\mathbf{A}$ need not be square, its order must be such that the dimensions of $\mathbf{Ax}$ and $\mathbf{A}^*\mathbf{y}$ are identical, respectively, to $\mathbf{y}$ and $\mathbf{x}$, so that the inner products in (1) are defined.

Example 1

Verify that the adjoint of

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{is} \quad \mathbf{A}^* = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

Solution. Since the order of $\mathbf{A}$ is 2×3, $\mathbf{x}$ must be 2-dimensional and $\mathbf{y}$ must be 3-dimensional. Designate the arbitrary vectors $\mathbf{x}$ and $\mathbf{y}$ by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

Then,

$$\mathbf{Ay} = \begin{bmatrix} (y_1 + 2y_2 + 3y_3) \\ (4y_1 + 5y_2 + 6y_3) \end{bmatrix} \quad \text{and} \quad \mathbf{A}^*\mathbf{x} = \begin{bmatrix} x_1 + 4x_2 \\ 2x_1 + 5x_2 \\ 3x_1 + 6x_2 \end{bmatrix}.$$

Hence,

$$\begin{aligned} \langle \mathbf{x}, \mathbf{Ay} \rangle &= x_1(\overline{y_1 + 2y_2 + 3y_3}) + x_2(\overline{4y_1 + 5y_2 + 6y_3}) \\ &= x_1\bar{y}_1 + 2x_1\bar{y}_2 + 3x_1\bar{y}_3 + 4x_2\bar{y}_1 + 5x_2\bar{y}_2 + 6x_2\bar{y}_3. \end{aligned}$$

and

$$\begin{aligned} \langle \mathbf{A}^*\mathbf{x}, \mathbf{y} \rangle &= (x_1 + 4x_2)\bar{y}_1 + (2x_1 + 5x_2)\bar{y}_2 + (3x_1 + 6x_2)\bar{y}_3 \\ &= x_1\bar{y}_1 + 4x_2\bar{y}_1 + 2x_1\bar{y}_2 + 5x_2\bar{y}_2 + 3x_1\bar{y}_3 + 6x_2\bar{y}_3. \end{aligned}$$

After rearranging terms, we find that $\langle \mathbf{x}, \mathbf{Ay} \rangle = \langle \mathbf{A}^*\mathbf{x}, \mathbf{y} \rangle$ where $\mathbf{x}$ and $\mathbf{y}$ are arbitrary. Thus, (1) is satisfied and $\mathbf{A}^*$, as given, is the adjoint of $\mathbf{A}$. □

Example 2

Determine whether

$$\mathbf{B} = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \quad \text{is the adjoint of} \quad \mathbf{A} = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}.$$

Solution. Since the order of $\mathbf{A}$ is 2×2, both $\mathbf{x}$ and $\mathbf{y}$ must be 2-dimensional vectors. Designate $\mathbf{x}$ and $\mathbf{y}$ by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Then,

$$\begin{aligned} \langle \mathbf{x}, \mathbf{Ay} \rangle &= x_1(\overline{y_1 + iy_2}) + x_2(\overline{iy_1 + y_2}) \\ &= x_1\bar{y}_1 - ix_1\bar{y}_2 - ix_2\bar{y}_1 + x_2\bar{y}_2 \end{aligned}$$

and

$$\begin{aligned} \langle \mathbf{Bx}, \mathbf{y} \rangle &= (x_1 + ix_2)\bar{y}_1 + (ix_1 + x_2)\bar{y}_2 \\ &= x_1\bar{y}_1 + ix_2\bar{y}_1 + ix_1\bar{y}_2 + x_2\bar{y}_2. \end{aligned}$$

Thus, we see that $\langle \mathbf{x}, \mathbf{Ay} \rangle$ is not always equal to $\langle \mathbf{Bx}, \mathbf{y} \rangle$ (for example, choose

$$\mathbf{x} = \mathbf{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix});$$

hence, $\mathbf{B}$ is not the adjoint of $\mathbf{A}$. □

Theorem 1. *The adjoint matrix has the following properties:*

(A1) $(\mathbf{A}^*)^* = \mathbf{A}$
(A2) $(\mathbf{A} + \mathbf{B})^* = \mathbf{A}^* + \mathbf{B}^*$
(A3) $(\mathbf{AB})^* = \mathbf{B}^*\mathbf{A}^*$
(A4) $(\lambda\mathbf{A})^* = \bar{\lambda}\mathbf{A}^*$

Proof. We shall prove only (A3) and leave the others as an exercise for the student (see Problem 7). Let $\mathbf{x}$ and $\mathbf{y}$ be arbitrary. Then

$$\langle (\mathbf{AB})^*\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{ABy} \rangle = \langle \mathbf{A}^*\mathbf{x}, \mathbf{By} \rangle = \langle \mathbf{B}^*\mathbf{A}^*\mathbf{x}, \mathbf{y} \rangle,$$

hence

$$\langle (\mathbf{AB})^*\mathbf{x}, \mathbf{y} \rangle - \langle \mathbf{B}^*\mathbf{A}^*\mathbf{x}, \mathbf{y} \rangle = 0,$$

or

$$\langle [(\mathbf{AB})^* - \mathbf{B}^*\mathbf{A}^*]\mathbf{x}, \mathbf{y} \rangle = 0 \qquad \{\text{(Property (I4) of Section 10.1)}.$$

But since this is true for every $\mathbf{y}$, we note that in particular it must be true for $\mathbf{y} = [(\mathbf{AB})^* - \mathbf{B}^*\mathbf{A}^*]\mathbf{x}$. Thus,

$$\langle [(\mathbf{AB})^* - \mathbf{B}^*\mathbf{A}^*]\mathbf{x}, [(\mathbf{AB})^* - \mathbf{B}^*\mathbf{A}^*]\mathbf{x} \rangle = 0.$$

Now since both arguments of the inner product are equal, it follows from Property (I1) of Section 10.1 that $[(\mathbf{AB})^* - \mathbf{B}^*\mathbf{A}^*]\mathbf{x} = \mathbf{0}$. However, since $\mathbf{x}$ is also arbitrary, it follows (see Lemma 1 of Section 8.5) that $(\mathbf{AB})^* - \mathbf{B}^*\mathbf{A}^* = \mathbf{0}$, from which (A3) immediately follows.

It was not accidental that the adjoint of the matrix given in Example 1 was the transpose of $\mathbf{A}$. In fact, for real matrices, that is matrices having no complex elements, this will always be the case.

Theorem 2. *The adjoint of a real matrix* $\mathbf{A}$ *exists, is unique, and equals the transpose of* $\mathbf{A}$*; that is,* $\mathbf{A}^* = \mathbf{A}^\mathsf{T}$.

We defer discussion of $\mathbf{A}^*$ for the case where $\mathbf{A}$ has complex elements until Section 10.5. Suffice to say here that for complex matrices the adjoint exists, is unique, but does not equal the transpose (see Example 2).

Definition 2. A matrix $\mathbf{A}$ is *self-adjoint* if it equals its own adjoint, that is, $\mathbf{A} = \mathbf{A}^*$.

An immediate consequence of Definition 2 is that a self-adjoint matrix must be square (for an $m \times n$ matrix to equal an $n \times m$ (see Definition 1), n must equal m). It follows from (5) that an $n \times n$ matrix is self-adjoint if and only if

$$\langle \mathbf{x}, \mathbf{Ay} \rangle = \langle \mathbf{Ax}, \mathbf{y} \rangle \tag{2}$$

for all n-dimensional vectors $\mathbf{x}$ and $\mathbf{y}$.

Combining Definition 2 with Theorem 2, we have

Theorem 3. *A real matrix is self-adjoint if and only if it is symmetric, that is,* $\mathbf{A} = \mathbf{A}^\mathsf{T}$.

Again we defer discussion of self-adjoint complex matrices until Section 10.5.

Example 3

Verify that

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

is self-adjoint.

Solution. Proceeding exactly as we did in Example 1, we find that

$$\langle \mathbf{x}, \mathbf{Ay} \rangle = \langle \mathbf{Ax}, \mathbf{y} \rangle = x_1\bar{y}_1 + 2x_1\bar{y}_2 + 2x_2\bar{y}_1 + 3x_2\bar{y}_2;$$

hence, (2) is satisfied which implies that **A** is self-adjoint. □

Example 4

Verify that

$$\mathbf{A} = \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix}$$

is self-adjoint.

Solution. By direct calculation, we find that

$$\langle \mathbf{Ax}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{Ay} \rangle = x_1\bar{y}_1 - i(x_1\bar{y}_2) + i(x_2\bar{y}_1) + 2x_2\bar{y}_2;$$

hence (2) is satisfied so **A** is self-adjoint. □

Note in Example 4 that although **A** is self-adjoint, it is not symmetric. Why does this example not violate Theorem 3?

Problems 10.2

(1) Find the adjoints of the following matrices:

(a) $\begin{bmatrix} 2 & 4 \\ 0 & 5 \end{bmatrix}$, (b) $\begin{bmatrix} -3 & 1 \\ 10 & -6 \end{bmatrix}$,

(c) $\begin{bmatrix} -3 & 1 \\ 1 & 5 \end{bmatrix}$, (d) $\begin{bmatrix} 44 & 56 \\ 38 & 81 \\ 23 & 75 \end{bmatrix}$,

(e) $\begin{bmatrix} -3 & 1 & 9 & 7 \\ 10 & -6 & -2 & 26 \end{bmatrix}$, (f) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 7 \end{bmatrix}$,

(g) $\begin{bmatrix} 15 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & -3 \end{bmatrix}$, (h) $\begin{bmatrix} 5 & 32 & 17 \\ 0 & -3 & -4 \\ 0 & 8 & 0 \\ 1 & 18 & 45 \end{bmatrix}$.

(2) Which of the matrices in the previous problem are self-adjoint?

(3) Determine whether

$$\begin{bmatrix} i & 1 \\ 1 & 2 \end{bmatrix}$$

is self-adjoint.

(4) Determine whether

$$\begin{bmatrix} 2 & 3i \\ 3i & 4 \end{bmatrix}$$

is self-adjoint.

(5) Determine whether

$$\begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix}$$

is self-adjoint.

(6) Prove that $\langle \mathbf{Ax}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{A}^*\mathbf{y} \rangle$ (Hint: first use Property (I2) of Section 10.1. and then the definition of the adjoint.)

(7) Prove the remainder of Theorem 1.

(8) Prove Theorem 2 for the case of 2×2 matrices.

10.3 Real Symmetric Matrices

In this section and the next we restrict our attention to real matrices. We return to the general case of complex matrices in Section 10.5. Note, however, that although we require the matrices themselves to be real *a priori*, we place no such restrictions on either the eigenvalues or eigenvectors.

We begin our discussion with those real matrices that are self-adjoint, or equivalently (see Theorem 3 of Section 10.2), those real matrices that are symmetric.

Theorem 1. *The eigenvalues of a real symmetric matrix are real.*

Proof. Let $\mathbf{A}$ be a real symmetric matrix, λ an eigenvalue of $\mathbf{A}$, and $\mathbf{x}$ an eigenvector of $\mathbf{A}$ corresponding to λ. Thus, $\mathbf{Ax} = \lambda\mathbf{x}$. Combining this result with (2) and the properties of the inner product, we find that

$$\lambda\langle \mathbf{x}, \mathbf{x}\rangle = \langle \lambda\mathbf{x}, \mathbf{x}\rangle = \langle \mathbf{Ax}, \mathbf{x}\rangle = \langle \mathbf{x}, \mathbf{Ax}\rangle = \langle \mathbf{x}, \lambda\mathbf{x}\rangle = \bar{\lambda}\langle \mathbf{x}, \mathbf{x}\rangle,$$

hence

$$\lambda\langle \mathbf{x}, \mathbf{x}\rangle - \bar{\lambda}\langle \mathbf{x}, \mathbf{x}\rangle = 0,$$

or

$$(\lambda - \bar{\lambda})\langle \mathbf{x}, \mathbf{x}\rangle = 0.$$

Since $\mathbf{x}$ is an eigenvector, $\mathbf{x} \neq \mathbf{0}$, hence, $\langle \mathbf{x}, \mathbf{x}\rangle \neq 0$ (Property (I1) of Section 10.1). It follows that $\lambda - \bar{\lambda} = 0$ or $\lambda = \bar{\lambda}$, which implies that λ is real (Property (C2) of Section 10.1).

Theorem 2. *The eigenvectors of a real symmetric matrix can always be chosen to be real.*

Proof. Let $\mathbf{A}$, $\mathbf{x}$, and λ be the same as in the previous proof. We first assume that $\mathbf{x}$ is not pure imaginary (that is, $\mathbf{x} = \mathbf{a} + i\mathbf{b}$, $\mathbf{a}$ and $\mathbf{b}$ real, $\mathbf{a} \neq \mathbf{0}$) and define a new vector $\mathbf{y} = \mathbf{x} + \bar{\mathbf{x}}$. Then $\mathbf{y}$ is real (Property (C4) of Section 10.1), nonzero, and an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ. To substantiate this last claim, we note that $\mathbf{Ax} = \lambda\mathbf{x}$, hence by conjugating both sides, we obtain $\overline{\mathbf{Ax}} = \overline{\lambda\mathbf{x}}$. However, since λ is a real number (Theorem 1) and $\mathbf{A}$ is a real matrix, it follows that $\overline{\lambda\mathbf{x}} = \bar{\lambda}\bar{\mathbf{x}} = \lambda\bar{\mathbf{x}}$ and $\overline{\mathbf{Ax}} = \mathbf{A}\bar{\mathbf{x}}$, hence $\mathbf{A}\bar{\mathbf{x}} = \lambda\bar{\mathbf{x}}$. Thus, $\mathbf{Ay} = \mathbf{A}(\mathbf{x} + \bar{\mathbf{x}}) = \mathbf{Ax} + \mathbf{A}\bar{\mathbf{x}} = \lambda\mathbf{x} + \lambda\bar{\mathbf{x}} = \lambda(\mathbf{x} + \bar{\mathbf{x}}) = \lambda\mathbf{y}$ which implies that $\mathbf{y}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ. If $\mathbf{x}$ is pure imaginary, then $\mathbf{y} = i\mathbf{x}$ is the desired eigenvector (see Problem 4).

As an example of Theorem 2, consider the real symmetric matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \quad \text{and the vector} \quad \mathbf{x} = \begin{bmatrix} 2 + 4i \\ 1 + 2i \end{bmatrix}$$

which is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue $\lambda = 5$. By following the procedure given in the proof of Theorem 2, we obtain

$$\mathbf{y} = \mathbf{x} + \bar{\mathbf{x}} = \begin{bmatrix} 2+4i \\ 1+2i \end{bmatrix} + \begin{bmatrix} 2-4i \\ 1-2i \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

as a real eigenvector of $\mathbf{A}$ corresponding to the eigenvalue $\lambda = 5$.

For the remainder of this book, we will always assume that the procedures outlined in the proof of Theorem 2 have been performed so that eigenvectors corresponding to a real symmetric matrix are presumed real. Furthermore, we note that if $\mathbf{A}$ is real symmetric, then $\mathbf{A} - \lambda\mathbf{I}$, where λ is an eigenvalue of $\mathbf{A}$, is also real symmetric, hence it follows that $(\mathbf{A} - \lambda\mathbf{I})$ is self-adjoint. This, in turn, implies that

$$\langle \mathbf{x}, (\mathbf{A} - \lambda\mathbf{I})\mathbf{y} \rangle = \langle (\mathbf{A} - \lambda\mathbf{I})\mathbf{x}, \mathbf{y} \rangle \tag{3}$$

for all vectors $\mathbf{x}$ and $\mathbf{y}$ of appropriate dimension.

▶ | **Theorem 3.** *A real symmetric matrix is diagonalizable.*

Proof. Let $\mathbf{A}$ be a real symmetric matrix. Then, from our work in Chapter 9, we know that $\mathbf{A}$ is diagonalizable if it possesses n linearly independent eigenvectors, or equivalently, if no generalized eigenvector of $\mathbf{A}$ has type greater than 1.

Assume that $\mathbf{x}$ is a generalized eigenvector of type 2 corresponding to the eigenvalue λ. (By identical reasoning as that used to prove Theorem 2, we can assume that $\mathbf{x}$ is real (see Problem 5).) Then, it must be the case (Definition 1 of Section 9.4) that

$$(\mathbf{A} - \lambda\mathbf{I})^2\mathbf{x} = \mathbf{0}$$

and (4)

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} \neq \mathbf{0}.$$

But,

$$\begin{aligned} 0 = \langle \mathbf{x}, \mathbf{0} \rangle &= \langle \mathbf{x}, (\mathbf{A} - \lambda\mathbf{I})^2\mathbf{x} \rangle \\ &= \langle \mathbf{x}, (\mathbf{A} - \lambda\mathbf{I})(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} \rangle \\ &= \langle (\mathbf{A} - \lambda\mathbf{I})\mathbf{x}, (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} \rangle \qquad \{\text{from (3)}\}. \end{aligned}$$

Using Property (I1) of Section 10.1, we conclude that $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$, which contradicts (4). Something is wrong! Since all our steps are valid, it must be our assumption which is incorrect, hence, $\mathbf{A}$ cannot possess a generalized eigenvector of type 2.

This conclusion, in turn, implies that $\mathbf{A}$ has no generalized eigenvector $\mathbf{x}_m$ of type m, $m > 1$. For if it did, we then could form a chain from $\mathbf{x}_m$ and obtain a generalized eigenvector of type 2. But we have just shown that $\mathbf{A}$ cannot possess such a vector; hence, we are led to the conclusion that $\mathbf{A}$ cannot possess a generalized eigenvector of type m, $m > 1$. This in turn implies that all generalized eigenvectors of $\mathbf{A}$ are of type 1.

Theorem 4. *Eigenvectors of a real symmetric matrix corresponding to distinct eigenvalues are orthogonal.*

Proof. Let $\mathbf{A}$ be a real symmetric matrix with eigenvectors $\mathbf{x}$ and $\mathbf{y}$ which correspond respectively to the different eigenvalues λ and μ. Thus, $\mathbf{Ax} = \lambda\mathbf{x}$ and $\mathbf{A}y = \mu\mathbf{y}$. Recalling that $\mathbf{x}$, $\mathbf{y}$, λ, and μ are all real (see Theorems 1 and 2) and $\mathbf{A}$ is self-adjoint, we have

$$\begin{aligned}(\lambda - \mu)\langle \mathbf{x}, \mathbf{y}\rangle &= \lambda\langle \mathbf{x}, \mathbf{y}\rangle - \mu\langle \mathbf{x}, \mathbf{y}\rangle \\ &= \langle \lambda\mathbf{x}, \mathbf{y}\rangle - \langle \mathbf{x}, \bar{\mu}\mathbf{y}\rangle \\ &= \langle \mathbf{Ax}, \mathbf{y}\rangle - \langle \mathbf{x}, \mu\mathbf{y}\rangle \\ &= \langle \mathbf{x}, \mathbf{Ay}\rangle - \langle \mathbf{x}, \mathbf{Ay}\rangle = 0.\end{aligned}$$

But $\lambda \neq \mu$; hence it follows that $\langle \mathbf{x}, \mathbf{y}\rangle = 0$, which implies that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

Example 1

Verify Theorem 4 for

$$\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}.$$

Solution. The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 5$ and $\lambda_2 = -5$. An eigenvector corresponding to $\lambda_1 = 5$ is found to be

$$\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

while an eigenvector corresponding to $\lambda_2 = -5$ is found to be

$$\mathbf{y} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Then, $\langle \mathbf{x}, \mathbf{y}\rangle = 2(1) + (1)(-2) = 0$, which implies that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal and verifies Theorem 4. □

From Theorem 3, we know that a real symmetric matrix is diagonalizable. It follows, therefore, that every real symmetric matrix possesses a set of n

linearly independent eigenvectors. The question now arises whether such a set of eigenvectors can be chosen to be orthonormal. Note that Theorem 4 does not answer this question since it gives no information about eigenvectors corresponding to the same eigenvalue.

Definition 1. A set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p\}$ is a *complete orthonormal set of eigenvectors* for the $n \times n$ matrix $\mathbf{A}$ (real or complex) if (1) the set is an orthonormal set, (2) each vector in the set is an eigenvector of $\mathbf{A}$, and (3) the set contains exactly n vectors, that is, $p = n$.

Note that Theorem 1 of Section 6.2 guarantees that a complete orthonormal set is also a linearly independent set.

Theorem 5. *Every real symmetric $n \times n$ matrix possesses a complete orthonormal set of eigenvectors.*

Proof. Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ be a set of r linearly independent eigenvectors corresponding to the same eigenvalue λ_i. Use the Gram–Schmidt orthonormalization process on this set to construct the orthonormal set

$$\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_r\}.$$

Since each $\mathbf{q}_j$ $(j = 1, 2, \ldots, r)$ is simply a linear combination of the vectors $\mathbf{x}_1$, $\mathbf{x}_2, \ldots, \mathbf{x}_r$, it must also be an eigenvector of $\mathbf{A}$ corresponding to λ_i (see Problem 6). Theorem 4 guarantees that each $\mathbf{q}_j$ will be orthogonal to every eigenvector that corresponds to an eigenvalue distinct from λ_i; hence, taking into consideration Theorem 3, which guarantees the existence of n-linearly independent eigenvectors, the result immediately follows.

In essence, Theorem 5 states that the columns of a modal matrix of a real symmetric matrix are not only linearly independent eigenvectors, but also can be chosen to form an orthonormal set.

Example 2

Find an orthonormal set of eigenvectors for

$$\mathbf{A} = \begin{bmatrix} 9 & -2 & -2 & -4 & 0 \\ -2 & 11 & 0 & 2 & 0 \\ -2 & 0 & 7 & -2 & 0 \\ -4 & 2 & -2 & 9 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

Solution. **A** is real and symmetric. Its eigenvalues are $\lambda_1 = \lambda_2 = 9$, $\lambda_3 = \lambda_4 = 3$, $\lambda_5 = 15$. Linearly independent eigenvectors corresponding to $\lambda = 9$ are found to be

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}.$$

Using the Gram–Schmidt orthonormalization process on these vectors, we obtain

$$\mathbf{q}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{q}_2 = \begin{bmatrix} 0 \\ -1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \end{bmatrix}.$$

Linearly independent eigenvectors corresponding $\lambda = 3$ are found to be

$$\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_4 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Using the Gram–Schmidt orthonormalization process on these vectors, we obtain

$$\mathbf{q}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{q}_4 = \begin{bmatrix} 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \end{bmatrix}.$$

An eigenvector corresponding to $\lambda = 15$, is

$$\mathbf{x}_5 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Normalizing this vector, we obtain

$$\mathbf{q}_5 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \\ 0 \end{bmatrix}.$$

The set $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4, \mathbf{q}_5\}$ is the required set. □

Problems 10.3

(1) Find an orthonormal set of eigenvectors and verify Theorems 2 and 4 for the following matrices (the eigenvalues are given below each matrix):

(a) $\begin{bmatrix} 2 & -2 & 1 \\ -2 & -1 & 2 \\ 1 & 2 & 2 \end{bmatrix}$, $(3, 3, -3)$

(b) $\begin{bmatrix} 1 & 2 & 0 & 2 \\ 2 & 0 & -1 & -1 \\ 0 & -1 & 2 & 1 \\ 2 & -1 & 1 & 0 \end{bmatrix}$, $(3, 3, -3, 0)$

(c) $\begin{bmatrix} -1 & 0 & 2 & 2 \\ 0 & -1 & 2 & -2 \\ 2 & 2 & 1 & 0 \\ 2 & -2 & 0 & 1 \end{bmatrix}$, $(3, 3, -3, -3)$

(d) $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & -2 & 1 & 2 \\ 0 & 2 & 2 & 1 \end{bmatrix}$. $(3, 3, 3, -3)$

(2) Show that

$$\begin{bmatrix} 3 & i \\ i & 1 \end{bmatrix}$$

has an eigenvector of type 2. Why does this result not violate Theorem 3?

(3) Prove that the eigenvalues of a self-adjoint matrix are real.

(4) Let $\mathbf{x}$ be an eigenvector of a real symmetric matrix $\mathbf{A}$ such that every component of $\mathbf{x}$ is pure imaginary. Prove that $\mathbf{y} = i\mathbf{x}$ is a real eigenvector of $\mathbf{A}$ corresponding to the same eigenvalue to which $\mathbf{x}$ corresponds.

(5) Prove that if a generalized eigenvector of type 2 exists for a real symmetric matrix, then it can be chosen to be real. (Note that Theorem 3 implies that such a vector does not exist.)

(6) Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r$ be eigenvectors of a matrix $\mathbf{A}$ corresponding to the same eigenvalue. Prove that $\mathbf{y} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_r\mathbf{x}_r$ (the c's are scalars) is also an eigenvector of $\mathbf{A}$ corresponding to λ providing that the c's are such that $\mathbf{y}$ is not the zero vector.

(7) Let $\mathbf{A}$ be an $n \times n$ real symmetric matrix. Prove that every n-dimensional vector $\mathbf{x}$ can be written as $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$ where $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is a complete orthonormal set of eigenvectors for $\mathbf{A}$ and c_j $(j = 1, 2, \ldots, n)$ is a scalar. (Hint: use Theorem 5 of this Section and Theorem 4 of Section 2.6.)

(8) Let $\mathbf{A}$, $\mathbf{x}$, and $\mathbf{x}_j$ $(j = 1, 2, \ldots, n)$ be the same as in Problem 7, and suppose that $\mathbf{x}_j$ corresponds to the eigenvalue λ_j.

(a) Show that

$$\mathbf{x} = \sum_{i=1}^{n} c_i\mathbf{x}_i = \sum_{j=1}^{n} c_j\mathbf{x}_j.$$

(b) Show that

$$\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} c_i\bar{c}_j \langle \mathbf{A}\mathbf{x}_i, \mathbf{x}_j \rangle = \sum_{i=1}^{n} c_i\bar{c}_i\lambda_i$$

(Hint: use Problem 30 of Section 10.1)

(9) Prove that the sum of two real symmetric matrices of the same order is itself real symmetric.

(10) Prove that if $\mathbf{A}$ is real symmetric, then $\mathbf{A} - k\mathbf{I}$ is also real symmetric, for any real scalar k.

(11) Determine when the product of two real symmetric matrices is real symmetric.

(12) Prove that any integral power of a real symmetric matrix is also real symmetric.

(13) Prove that any well-defined function of a real symmetric matrix is also real symmetric.

(14) Prove, for any real square matrix $\mathbf{A}$, that the matrix $(\mathbf{A} + \mathbf{A}^{\mathrm{T}})$ is real symmetric.

(15) A matrix $\mathbf{A}$ is *skew-symmetric* if $\mathbf{A} = -\mathbf{A}^{\mathsf{T}}$. Show that any real square matrix can be written as the sum of a real symmetric matrix with a skew-symmetric matrix.

(16) What can you say about the diagonal elements of a skew-symmetric matrix?

10.4 Orthogonal Matrices

Definition 1. A real matrix $\mathbf{P}$ is *orthogonal* if $\mathbf{P}^{-1} = \mathbf{P}^{\mathsf{T}}$ (that is, if the inverse of $\mathbf{P}$ equals the transpose of $\mathbf{P}$).

Example 1

Verify that

$$\mathbf{P} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

is orthogonal.

Solution. By direct calculation, we find that $\mathbf{P}^{\mathsf{T}}\mathbf{P} = \mathbf{P}\mathbf{P}^{\mathsf{T}} = \mathbf{I}$, hence, it follows from the definition of the inverse that $\mathbf{P}^{\mathsf{T}} = \mathbf{P}^{-1}$. Thus, $\mathbf{P}$ is orthogonal. □

Theorem 1. *The determinant of an orthogonal matrix is* ± 1.

Proof. Let $\mathbf{P}$ be orthogonal, thus $\mathbf{P}^{-1} = \mathbf{P}^{\mathsf{T}}$ or $\mathbf{P}\mathbf{P}^{\mathsf{T}} = \mathbf{I}$. Taking determinants of both sides of this equation and using Property 9 of Section 4.3, we obtain $\det(\mathbf{P})\det(\mathbf{P}^{\mathsf{T}}) = \det(\mathbf{I})$. But $\det(\mathbf{P}^{\mathsf{T}}) = \det(\mathbf{P})$ (Property 7 of Section 4.3) and $\det(\mathbf{I}) = 1$, hence, it follows that $[\det(\mathbf{P})]^2 = 1$, which in turn implies the desired result.

Theorem 2. *A real matrix is orthogonal if and only if its columns (and rows), considered as vectors, form an orthonormal set.*

Proof. We must first prove that a matrix is orthogonal if its columns form an orthonormal set and then prove the converse, that the columns of a matrix form an orthonormal set if the matrix is orthogonal. We will prove the first part here and leave the converse as an exercise for the student.

Assume that the columns of a matrix **P** form an orthonormal set. Then **P** may be represented by $\mathbf{P} = [\mathbf{x}_1 \quad \mathbf{x}_2 \quad \cdots \quad \mathbf{x}_n]$ where $\mathbf{x}_j$ is the jth column of **P** having the property that $\langle \mathbf{x}_i, \mathbf{x}_j \rangle = \delta_{ij}$. It now follows from the definition of the inner product (note that $\mathbf{x}_j$ is real, hence $\langle \mathbf{x}_i, \mathbf{x}_j \rangle = \langle \mathbf{x}_j, \mathbf{x}_i \rangle$) that $\mathbf{P}^\mathsf{T}\mathbf{P}$ can be written as

$$\mathbf{P}^\mathsf{T}\mathbf{P} = \mathbf{P}\mathbf{P}^\mathsf{T} = \begin{bmatrix} \langle \mathbf{x}_1, \mathbf{x}_1 \rangle & \langle \mathbf{x}_1, \mathbf{x}_2 \rangle & \cdots & \langle \mathbf{x}_1, \mathbf{x}_n \rangle \\ \langle \mathbf{x}_2, \mathbf{x}_1 \rangle & \langle \mathbf{x}_2, \mathbf{x}_2 \rangle & \cdots & \langle \mathbf{x}_2, \mathbf{x}_n \rangle \\ \vdots & \vdots & & \vdots \\ \langle \mathbf{x}_n, \mathbf{x}_1 \rangle & \langle \mathbf{x}_n, \mathbf{x}_2 \rangle & \cdots & \langle \mathbf{x}_n, \mathbf{x}_n \rangle \end{bmatrix}.$$

Since $\langle \mathbf{x}_1, \mathbf{x}_j \rangle = \delta_{ij}$, we obtain $\mathbf{P}^\mathsf{T}\mathbf{P} = \mathbf{P}\mathbf{P}^\mathsf{T} = \mathbf{I}$ from which it follows that $\mathbf{P}^\mathsf{T} = \mathbf{P}^{-1}$. Thus, **P** is orthogonal.

If we now apply Theorem 2 to the results of the previous section (in particular to Theorems 3 and 5) then we obtain the following important conclusion.

Theorem 3. *For every $n \times n$ real symmetric matrix* **A** *there exists an $n \times n$ real orthogonal matrix* **P** *such that* $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$, *or, equivalently, such that* $\mathbf{P}^\mathsf{T}\mathbf{A}\mathbf{P} = \mathbf{D}$, *where* **D** *is a diagonal matrix.*

Example 2

Verify Theorem 3 for the matrix given in Example 1 of the previous section.

Solution. In that example, we found a complete orthonormal set of a eigenvectors given by $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4, \mathbf{q}_5$. Define

$$\mathbf{P} = [\mathbf{q}_1 \quad \mathbf{q}_2 \quad \mathbf{q}_3 \quad \mathbf{q}_4 \quad \mathbf{q}_5] = \begin{bmatrix} 1/\sqrt{3} & 0 & 0 & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{3} & -1/\sqrt{3} & 0 & 0 & 1/\sqrt{3} \\ -1/\sqrt{3} & -1/\sqrt{3} & 0 & 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{3} & 0 & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Since the columns of **P** form an orthonormal set, it follows from Theorem 2 that **P** is orthogonal. By direct calculation, we find that

$$\mathbf{P}^\mathsf{T}\mathbf{A}\mathbf{P} = \begin{bmatrix} 9 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 15 \end{bmatrix}$$

which verifies Theorem 3. □

We conclude this section with one final note: orthogonal matrices leave inner products invariant. That is,

$$\langle \mathbf{Px}, \mathbf{Py} \rangle = \langle \mathbf{x}, \mathbf{P}^*\mathbf{Py} \rangle = \langle \mathbf{x}, \mathbf{P}^{\mathrm{T}}\mathbf{Py} \rangle$$
$$= \langle \mathbf{x}, \mathbf{P}^{-1}\mathbf{Py} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$$

or

$$\langle \mathbf{Px}, \mathbf{Py} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle \tag{5}$$

Example 3

Verify (5) for the matrix

$$\mathbf{P} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

Solution. Since the columns of **P** form an orthonormal set of vectors, **P** is orthogonal. Designate **x** and **y** by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Note that **x** and **y** can both be complex. Then, $\langle \mathbf{x}, \mathbf{y} \rangle = x_1\bar{y}_1 + x_2\bar{y}_2$,

$$\mathbf{Px} = \begin{bmatrix} (x_1/\sqrt{2} + x_2/\sqrt{2}) \\ (-x_1/\sqrt{2} + x_2/\sqrt{2}) \end{bmatrix}, \quad \text{and} \quad \mathbf{Py} = \begin{bmatrix} (y_1/\sqrt{2} + y_2/\sqrt{2}) \\ (-y_1/\sqrt{2} + y_2/\sqrt{2}) \end{bmatrix}.$$

Thus,

$$\begin{aligned} \langle \mathbf{Px}, \mathbf{Py} \rangle &= (x_1/\sqrt{2} + x_2/\sqrt{2})\overline{(y_1/\sqrt{2} + y_2/\sqrt{2})} \\ &\quad + (-x_1/\sqrt{2} + x_2/\sqrt{2})\overline{(-y_1/\sqrt{2} + y_2/\sqrt{2})} \\ &= x_1\bar{y}_1 + x_2\bar{y}_2 = \langle \mathbf{x}, \mathbf{y} \rangle. \quad \square \end{aligned}$$

Problems 10.4

(1) Determine which of the following matrices are orthogonal:

(a) $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, (b) $\begin{bmatrix} 3/\sqrt{8} & i/\sqrt{8} \\ i/\sqrt{8} & -3/\sqrt{8} \end{bmatrix}$,

(c) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$,

(d) $\begin{bmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$.

(2) Verify Theorem 3 for

$$\mathbf{A} = \begin{bmatrix} 2 & -2 & 1 \\ -2 & -1 & 2 \\ 1 & 2 & 2 \end{bmatrix}.$$

(3) Let **A** be a real matrix and **P** be a real orthogonal matrix. Prove that $\mathbf{P}^{-1}\mathbf{AP}$ is symmetric if and only if **A** is symmetric.

(4) Prove that a real matrix **P** is orthogonal if and only if its inverse equals its adjoint.

(5) Show by example that it is possible for a real matrix to be both symmetric and orthogonal.

(6) Show that if **P** is orthogonal, then $\|\mathbf{Px}\| = \|\mathbf{x}\|$ for any real vector **x** of suitable dimension.

(7) Show that if **P** is orthogonal, then the angle between **Px** and **Py** is the same as that between **x** and **y**.

(8) Prove that the product of two orthogonal matrices of the same order is also an orthogonal matrix.

(9) Define an *elementary reflector* for a real column vector **x** as

$$\mathbf{R} = \mathbf{I} - 2\left(\frac{\mathbf{x}\mathbf{x}^{\mathsf{T}}}{\|\mathbf{x}\|^2}\right).$$

Find the elementary reflector associated with

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

(10) Find the elementary reflector associated with

$$\mathbf{x} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}.$$

(11) Find the elementary reflector associated with

$$\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}.$$

(12) Find the elementary reflector associated with

$$\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}.$$

(13) Prove that an elementary reflector is a real symmetric matrix.

(14) Show that the square of an elementary reflector must be identity matrix.

(15) Prove that an elementary reflector is an orthogonal matrix.

10.5 Hermitian Matrices

We now return to the general case of complex matrices. From our previous work, it would seem advisable to first determine the adjoint of a complex matrix (from Example 2 of Section 10.2, we conclude that the adjoint of a complex matrix is not its transpose as was the case with real matrices), and then ascertain whether or not self-adjoint complex matrices have properties similar to their real counterparts. We begin our discussion by defining the complex conjugate transpose matrix.

Definition 1. If $\mathbf{A} = [a_{ij}]$ is $n \times p$ matrix, then the *complex conjugate transpose* of $\mathbf{A}$ is the $p \times n$ matrix $\mathbf{A}^{\mathrm{H}} = [\bar{a}_{ij}]^{\mathrm{T}}$, obtained by first taking the complex conjugate of each element of $\mathbf{A}$, and then transposing the resulting matrix.

Thus, if

$$\mathbf{A} = \begin{bmatrix} 2+i & 3 & -i \\ 1 & 1-i & 2i \end{bmatrix}, \qquad \text{then} \quad \mathbf{A}^{\mathrm{H}} = \begin{bmatrix} 2-i & 1 \\ 3 & 1+i \\ i & -2i \end{bmatrix}.$$

Definition 2. A matrix is *Hermitian* if it is equal to its own complex conjugate transpose; that is, if $\mathbf{A} = \mathbf{A}^{\mathrm{H}}$.

An example of a Hermitian matrix is

$$\mathbf{A} = \begin{bmatrix} 1 & 2-i & 4i \\ 2+i & 3 & -1-i \\ -4i & -1+i & 4 \end{bmatrix}.$$

It follows immediately from Definition 2 that Hermitian matrices must be square and that the main diagonal elements must be real (see Problem 2). Furthermore, we note that neither Definition 1 nor Definition 2 requires the matrix under consideration to be complex; that is, the concepts of the complex conjugate transpose and the Hermitian matrix are equally applicable to real matrices. We leave it as an exercise for the student, however, to show that for real matrices these concepts reduce to those of the transpose and real symmetric matrix respectively. Thus, we may think of the Hermitian matrix as a generalization of the real symmetric matrix and the complex conjugate transpose as the generalization of the transpose. This analogy is strengthened further by the following theorem, the proof of which is beyond the scope of this book.

Theorem 1. *The adjoint* $\mathbf{A}^*$ *of a matrix* $\mathbf{A}$ *(real or complex) exists, is unique and equals the complex conjugate transpose of* $\mathbf{A}$. *That is,* $\mathbf{A}^* = \mathbf{A}^{\mathsf{H}}$.

Combining this theorem with Definition 2, we have a proof of

Theorem 2. *A matrix (real or complex) is self-adjoint if and only if it is Hermitian.*

Example 1

Verify that

$$\mathbf{A} = \begin{bmatrix} 1 & -2-i \\ -2+i & 3 \end{bmatrix}$$

is self-adjoint.

Solution. Designate the arbitrary 2-dimensional vectors $\mathbf{x}$ and $\mathbf{y}$ by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Then by direct calculation we find that

$$\langle \mathbf{Ax}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{Ay} \rangle = x_1\bar{y}_1 + (-2-i)x_2\bar{y}_1 + (-2+i)x_1\bar{y}_2 + 3x_2\bar{y}_2.$$

Thus (6) is satisfied, which implies $\mathbf{A}$ is self-adjoint. $\square$

Note that if **A** is real, Theorems 1 and 2 reduce to Theorems 2 and 3 of Section 10.2. Furthermore, by directing our attention to the theorems in Section 10.3 we note that the proofs of all those theorems, with the exception of Theorem 2, did not depend on the fact that the matrix involved was real symmetric, but rather on the fact that the matrix was self-adjoint. Hence, those theorems remain equally valid for self-adjoint complex matrices or, equivalently, Hermitian matrices. We incorporate those results, as they pertain to Hermitian matrices, into one master theorem.

Theorem 3. *Let* **A** *be a Hermitian matrix. The eigenvalues of* **A** *are real,* **A** *is diagonalizable, eigenvectors corresponding to distinct eigenvalues are orthogonal, and* **A** *possesses a complete orthonormal set of eigenvectors.*

Example 2

Find a complete orthonormal set of eigenvectors for

$$\mathbf{A} = \begin{bmatrix} 1 & -2i & 0 & 2i \\ 2i & -1 & -2 & 0 \\ 0 & -2 & 1 & -2 \\ -2i & 0 & -2 & -1 \end{bmatrix}.$$

Solution. The eigenvalues of this Hermitian matrix are $\lambda_1 = \lambda_2 = 3$, $\lambda_3 = \lambda_4 = -3$. Two linearly independent eigenvectors corresponding to $\lambda = 3$ are found to be

$$\mathbf{x}_1 = \begin{bmatrix} i \\ -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} 2i \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

Using the Gram–Schmidt orthonormalization process on these vectors we obtain

$$\mathbf{q}_1 = \begin{bmatrix} i/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{q}_2 = \begin{bmatrix} i/\sqrt{3} \\ 0 \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}.$$

Two linearly independent eigenvectors corresponding to $\lambda = -3$ are found to be

$$\mathbf{x}_3 = \begin{bmatrix} -i \\ -1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_4 = \begin{bmatrix} i \\ 2 \\ 1 \\ 0 \end{bmatrix}.$$

Using the Gram–Schmidt orthonormalization process on these two vectors, we obtain

$$\mathbf{q}_3 = \begin{bmatrix} -i/\sqrt{3} \\ -1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \end{bmatrix} \quad \text{and} \quad \mathbf{q}_4 = \begin{bmatrix} 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}.$$

The set $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4\}$ is the required set. □

Theorem 4. *A matrix* $\mathbf{A}$ *is Hermitian if and only if* $\langle \mathbf{Ax}, \mathbf{x} \rangle$ *is real for all (real and complex) vectors* $\mathbf{x}$.

Proof. If $\mathbf{A}$ is Hermitian, hence self-adjoint, it follows (see Property (I2) of Section 10.1) that $\langle \mathbf{Ax}, \mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{A}^*\mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{Ax} \rangle = \overline{\langle \mathbf{Ax}, \mathbf{x} \rangle}$. Hence, from Property (C2) of Section 10.1, we conclude that $\langle \mathbf{Ax}, \mathbf{x} \rangle$ is real. (Recall that the inner product of two vectors is itself a number.) We leave the converse, $\mathbf{A}$ is Hermitian if $\langle \mathbf{Ax}, \mathbf{x} \rangle$ is real, as an exercise for the student (See Problem 5).

In conclusion, we note that since a real symmetric matrix is just a special case of a Hermitian matrix, Theorem 4 implies the following:

Corollary 1. *A real matrix is symmetric if and only if* $\langle \mathbf{Ax}, \mathbf{x} \rangle$ *is real for all (real and complex) vectors* $\mathbf{x}$.

Example 3

Verify Theorem 4 for the Hermitian matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1-i \\ 1+i & -1 \end{bmatrix}.$$

Solution. Designate $\mathbf{x}$ by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Then,

$$\mathbf{Ax} = \begin{bmatrix} 2x_1 + (1-i)x_2 \\ (1+i)x_1 - x_2 \end{bmatrix},$$

and

$$\begin{aligned}\langle \mathbf{Ax}, \mathbf{x} \rangle &= [2x_1 + (1-i)x_2]\bar{x}_1 + [(1+i)x_1 - x_2]\bar{x}_2 \\ &= 2x_1\bar{x}_1 + (1-i)\bar{x}_1x_2 + (1+i)x_1\bar{x}_2 - x_2\bar{x}_2 \\ &= 2x_1\bar{x}_1 + [(1-i)\bar{x}_1x_2 + \overline{(1-i)\bar{x}_1x_2}] - x_2\bar{x}_2.\end{aligned}$$

Since the quantity inside the brackets is of the form $a + \bar{a}$ and all other terms are of the form $a\bar{a}$, we have, from Properties (C1) and (C4) of Section 10.1, that $\langle \mathbf{Ax}, \mathbf{x} \rangle$ is the sum of real numbers, hence, is itself real. □

Problems 10.5

(1) Determine which of the following matrices are Hermitian and find a complete orthonormal set of eigenvectors for those matrices:

(a) $\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$,

(c) $\begin{bmatrix} 1 & 2i & -i \\ -2i & i & 1 \\ i & 1 & 2 \end{bmatrix}$, (d) $\begin{bmatrix} 4 & 0 & 3i \\ 0 & 5 & 0 \\ -3i & 0 & -4 \end{bmatrix}$,

(e) $\begin{bmatrix} 1 & i & 0 & -i \\ -i & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ i & 0 & 1 & 2 \end{bmatrix}$, (f) $\begin{bmatrix} 1 & i & 0 & -i \\ -i & 1 & 0 & -i \\ 0 & 0 & 0 & 0 \\ i & -1 & 0 & 1 \end{bmatrix}$.

(2) Prove that the main diagonal elements of a Hermitian matrix must be real.

(3) Prove that $\mathbf{A}^* = \overline{\mathbf{A}^\mathrm{T}}$ for a general 2×2 complex matrix.

(4) Prove the equality

$$\begin{aligned}\langle \mathbf{Ax}, \mathbf{y} \rangle = {}& \tfrac{1}{4}[\langle \mathbf{A}(\mathbf{x}+\mathbf{y}), (\mathbf{x}+\mathbf{y}) \rangle - \langle \mathbf{A}(\mathbf{x}-\mathbf{y}), (\mathbf{x}-\mathbf{y}) \rangle] \\ & + \frac{i}{4}[\langle \mathbf{A}(\mathbf{x}+i\mathbf{y}), (\mathbf{x}+i\mathbf{y}) \rangle - \langle \mathbf{A}(\mathbf{x}-i\mathbf{y}), (\mathbf{x}-i\mathbf{y}) \rangle].\end{aligned}$$

(5) Use Problem 4 to prove that $\mathbf{A}$ is Hermitian if $\langle \mathbf{Ax}, \mathbf{x} \rangle$ is real for all vectors $\mathbf{x}$.

(6) Prove that the sum of two Hermitian matrices of the same order is itself Hermitian.

(7) Prove that if $\mathbf{A}$ is Hermitian, then $\mathbf{A} - k\mathbf{I}$ is also Hermitian, for any real scalar k.

(8) Determine when the product of two Hermitian matrices is Hermitian.

(9) Prove that any integral power of a Hermitian matrix is also Hermitian.

(10) Prove that if $\mathbf{A}$ is Hermitian, then so too is $e^{\mathbf{A}}$.

(11) Prove, for any square matrix $\mathbf{A}$, that $(\mathbf{A} + \mathbf{A}^{\mathsf{H}})$ is Hermitian.

(12) A matrix $\mathbf{A}$ is *skew-Hermitian* if $\mathbf{A} = -\mathbf{A}^{\mathsf{H}}$. Show that any square matrix can be written as the sum of a Hermitian matrix with a skew-Hermitian matrix.

(13) What can you say about the diagonal elements of a skew-Hermitian matrix?

(14) Prove that if $\mathbf{A}$ is skew-Hermitian, then $i\mathbf{A}$ is Hermitian, where $i = \sqrt{-1}$.

10.6 Unitary Matrices

Definition 1. A matrix $\mathbf{U}$ is *unitary* if $\mathbf{U}^{-1} = \mathbf{U}^{\mathsf{H}}$ (that is, if the inverse of $\mathbf{U}$ equals the complex conjugate transpose of $\mathbf{U}$).

Example 1

Verify that

$$\mathbf{U} = \begin{bmatrix} i/2 & -i/\sqrt{3} & 5/\sqrt{60} \\ 1/2 & -i/\sqrt{3} & (-4+3i)/\sqrt{60} \\ (1-i)/2 & 1/\sqrt{3} & (3-i)/\sqrt{60} \end{bmatrix}$$

is unitary.

Solution. By direct calculation, we find that $\mathbf{U}^{\mathsf{H}}\mathbf{U} = \mathbf{U}\mathbf{U}^{\mathsf{H}} = \mathbf{I}$, hence, it follows from the definition of the inverse that $\mathbf{U}^{\mathsf{H}} = \mathbf{U}^{-1}$. Thus, $\mathbf{U}$ is unitary. □

We note that if $\mathbf{U}$ is a real matrix, then $\mathbf{U}^{\mathrm{H}} = \mathbf{U}^{\mathrm{T}}$, and the concept of a unitary matrix reduces to that of an orthogonal matrix. Thus, we can think of a unitary matrix as a generalization of an orthogonal matrix. Furthermore, many of the properties of orthogonal matrices discussed in Section 10.4 remain valid in this more general case. As such, we just state the results here, and refer the student to their appropriate counterparts in Section 10.4 for the proofs.

Theorem 1. *A matrix is unitary if and only if its columns (and rows), considered as vectors, form an orthonormal set.*

Theorem 2. *For every $n \times n$ Hermitian matrix $\mathbf{A}$, there exists an $n \times n$ unitary matrix $\mathbf{U}$ such that $\mathbf{U}^{-1}\mathbf{A}\mathbf{U} = \mathbf{D}$, or equivalently, such that $\mathbf{U}^{\mathrm{H}}\mathbf{A}\mathbf{U} = \mathbf{D}$, where $\mathbf{D}$ is a real diagonal matrix.*

Example 2

Verify Theorem 2 for the matrix given in Example 2 of the previous section.

Solution. In that example, we found a complete orthonormal set of eigenvectors to be $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4\}$. Define $\mathbf{U} = [\mathbf{q}_1 \quad \mathbf{q}_2 \quad \mathbf{q}_3 \quad \mathbf{q}_4]$. Since the columns of $\mathbf{U}$ form an orthonormal set, it follows from Theorem 1 that $\mathbf{U}$ is unitary. By direct calculation, we find that

$$\mathbf{U}^{\mathrm{H}}\mathbf{A}\mathbf{U} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

which verifies Theorem 2. □

Once again, as was the case with orthogonal matrices, we find that unitary matrices leave the inner product invariant. That is,

$$\langle \mathbf{Ux}, \mathbf{Uy} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle. \tag{6}$$

Example 3

Verify (6) for the matrix

$$\mathbf{U} = \begin{bmatrix} -1/\sqrt{2} & i/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

Solution. Since the columns of **U** form an orthonormal set of vectors, **U** is unitary. Designate **x** and **y** by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Then

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1\bar{y}_1 + x_2\bar{y}_2,$$

$$\mathbf{Ux} = \begin{bmatrix} ((-1/\sqrt{2})x_1 + (i/\sqrt{2})x_2) \\ ((-i/\sqrt{2})x_1 + (1/\sqrt{2})x_2) \end{bmatrix} \quad \text{and} \quad \mathbf{Uy} = \begin{bmatrix} ((-1/\sqrt{2})y_1 + i/\sqrt{2})y_2) \\ ((-i/\sqrt{2})y_1 + (1/\sqrt{2})y_2) \end{bmatrix}.$$

Thus,

$$\begin{aligned} \langle \mathbf{Ux}, \mathbf{Uy} \rangle &= ((-1/\sqrt{2})x_1 + (i/\sqrt{2})x_2)\overline{((-1/\sqrt{2})y_1 + (i/\sqrt{2})y_2)} \\ &\quad + ((-i/\sqrt{2})x_1 + (1/\sqrt{2})x_2)\overline{((-i/\sqrt{2})y_1 + (1/\sqrt{2})y_2)} \\ &= ((-1/\sqrt{2})x_1 + (i/\sqrt{2})x_2)((-1/\sqrt{2})\bar{y}_1 + (i/\sqrt{2})\bar{y}_2) \\ &\quad + ((-i/\sqrt{2})x_1 + (1/\sqrt{2})x_2)((i/\sqrt{2})\bar{y}_1 + (1/\sqrt{2})\bar{y}_2) \\ &= x_1\bar{y}_1 + x_2\bar{y}_2 = \langle \mathbf{x}, \mathbf{y} \rangle. \quad \square \end{aligned}$$

Problems 10.6

(1) Determine which of the following matrices are unitary:

(a) $\begin{bmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ i/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$, (b) $\begin{bmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$,

(c) $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, (d) $1/\sqrt{3}\begin{bmatrix} 1 & i & -i & 0 \\ 0 & 1 & 1 & 1 \\ i & 0 & -1 & 1 \\ -i & -1 & 0 & 1 \end{bmatrix}$.

(2) Verify Theorem 2 for

$$\mathbf{A} = \begin{bmatrix} 1 & i & 0 & -i \\ -i & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ i & -1 & 0 & 1 \end{bmatrix}.$$

(3) Prove that a matrix $\mathbf{U}$ is unitary if and only if its inverse equals its adjoint.

(4) Let $\mathbf{U}$ be a unitary matrix. Prove that $\mathbf{U}^{-1}\mathbf{AU}$ is Hermitian if and only if $\mathbf{A}$ is Hermitian.

(5) Prove Eq. (6).

(6) Show by example that it is possible for a matrix to be both Hermitian and unitary.

(7) What is the relationship between the eigenvalues of $\mathbf{U}^{\mathrm{H}}\mathbf{AU}$ and those of $\mathbf{A}$ when $\mathbf{U}$ is unitary?

(8) Show that if $\mathbf{U}$ is unitary, then $\|\mathbf{Ux}\| = \|\mathbf{x}\|$ for any real vector $\mathbf{x}$ of suitable dimension.

(9) Show that if $\mathbf{U}$ is unitary, then the angle between $\mathbf{Ux}$ and $\mathbf{Uy}$ is the same as that between $\mathbf{x}$ and $\mathbf{y}$.

(10) Prove that the product of two unitary matrices of the same order is also an unitary matrix.

(11) Prove that the absolute values of the eigenvalues of a unitary matrix must be unity.

(12) Prove that the absolute value of the determinant of a unitary matrix is unity.

(13) Prove that if $\mathbf{A}$ is an orthogonal matrix, then

$$\mathbf{U} = \frac{1+i}{\sqrt{2}}\mathbf{A}$$

is a unitary matrix.

10.7 Summary

We now summarize some of the more important results of the previous five sections. In what follows, the student will note that we generally differentiate between real symmetric matrices and Hermitian matrices, and between orthogonal matrices and unitary matrices. We do this for convenience only, because such a classification is actually superfluous. Recall that real symmetric matrices are special cases of Hermitian matrices (that is, those

Hermitian matrices that are real), while orthogonal matrices are special cases of unitary matrices (for the same reason).

Given a matrix $\mathbf{A}$, its adjoint $\mathbf{A}^*$ is defined by the relation $\langle \mathbf{x}, \mathbf{Ay} \rangle = \langle \mathbf{A}^*\mathbf{x}, \mathbf{y} \rangle$ for all vectors $\mathbf{x}$ and $\mathbf{y}$. The adjoint of a real matrix is its transpose while the adjoint of a complex matrix is its complex conjugate transpose. A matrix is called self-adjoint if it equals its own adjoint. Thus, a real matrix is self-adjoint if and only if it is symmetric while a complex matrix is self-adjoint if and only if it is Hermitian.

Self-adjoint matrices (real and complex) have real eigenvalues and possess a complete orthonormal set of eigenvectors. Furthermore, self-adjoint matrices are diagonalizable. For a real matrix, the modal matrix can be chosen to be an orthogonal matrix while for a complex matrix, the modal matrix can be chosen to be a unitary matrix.

Both orthogonal and unitary matrices are defined by the property that their inverses must equal their adjoints. This property requires that the inverse of a real orthogonal matrix be equal to its transpose while the inverse of a unitary matrix be equal to its complex conjugate transpose. Both orthogonal and unitary matrices have the property that they leave the inner product invariant.

10.8 Positive Definite Matrices

We know from Theorem 4 of Section 10.5 that if $\mathbf{A}$ is Hermitian then $\langle \mathbf{Ax}, \mathbf{x} \rangle$ must be real. If in particular the quantity $\langle \mathbf{Ax}, \mathbf{x} \rangle$ is nonnegative for all $\mathbf{x}$, then the Hermitian matrix $\mathbf{A}$ is called *nonnegative definite*. If the quantity $\langle \mathbf{Ax}, \mathbf{x} \rangle$ is always positive for all nonzero vectors $\mathbf{x}$ (note that if $\mathbf{x} = \mathbf{0}$, then $\langle \mathbf{Ax}, \mathbf{x} \rangle = 0$), then the Hermitian matrix is called positive definite.

Definition 1. An $n \times n$ Hermitian matrix is *positive definite*, designated by $\mathbf{A} > 0$, if $\langle \mathbf{Ax}, \mathbf{x} \rangle$ is positive for all nonzero n-dimensional (real or complex) vectors $\mathbf{x}$.

Example 1

Verify that

$$\mathbf{A} = \begin{bmatrix} 2 & i \\ -i & 2 \end{bmatrix}$$

is positive definite.

Solution. Designate $\mathbf{x}$ by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Then,

$$\begin{aligned}\langle \mathbf{Ax}, \mathbf{x} \rangle &= 2x_1\bar{x}_1 + i\bar{x}_1x_2 - ix_1\bar{x}_2 + 2x_2\bar{x}_2 \\ &= x_1\bar{x}_1 + (x_1 + ix_2)(\bar{x}_1 - i\bar{x}_2) + x_2\bar{x}_2 \\ &= x_1\bar{x}_1 + (x_1 + ix_2)\overline{(x_1 + ix_2)} + x_2\bar{x}_2.\end{aligned}$$

But each of these three terms is in the form of a complex number times its conjugate and therefore is nonnegative (see Property (C1) of Section 10.1). Thus, the sum of these terms is positive (unless $x_1 = x_2 = 0$) which implies that $\mathbf{A}$ is positive definite. □

Theorem 1. *If* $\mathbf{A}$ *is a positive definite matrix, then its diagonal elements are all positive.*

Proof. Assume that the positive definite matrix $\mathbf{A}$ has order $n \times n$. Then $\langle \mathbf{Ax}, \mathbf{x} \rangle > 0$ for any n-dimensional column vector $\mathbf{x}$. In particular, designate as $\mathbf{e}_k$ $(k = 1, 2, \ldots, n)$ the n-dimensional column vector having its kth component equal to unity, and all other components equal to zero. For each k,

$$0 < \langle \mathbf{Ae}_k, \mathbf{e}_k \rangle = a_{kk}.$$

Theorem 2. *If* $\mathbf{A} = [a_{ij}]$ *is an* $n \times n$ *positive definite matrix, then for any distinct* i *and* j $(i, j = 1, 2, \ldots, n)$, $\quad a_{ii}a_{jj} > |a_{ij}|^2$.

Proof. Since $\mathbf{A}$ is Hermitian, $\mathbf{A} = \mathbf{A}^{\mathsf{H}}$, and $\bar{a}_{ji} = a_{ij}$. Define $\mathbf{x}$ to be an n-dimensional column vector having its ith component denoted by x_i, its jth component equal to unity, and all other components equal to zero. By direct calculation, we have

$$0 < \langle \mathbf{Ax}, \mathbf{x} \rangle = a_{ii}x_i\bar{x}_i + a_{ij}\bar{x}_i + a_{ji}x_i + a_{jj}.$$

Setting $x_i = -a_{ij}/a_{ii}$, we note that the first two terms on the right cancel, leaving

$$0 < \; -\frac{a_{ji}a_{ij}}{a_{ii}} + a_{jj} = \frac{1}{a_{ii}}(-\bar{a}_{ij}a_{ij} + a_{ii}a_{jj}).$$

But a_{ii} is positive (Theorem 1) and $\bar{a}_{ij}a_{ij} = |a_{ij}|^2$, so the desired inequality is immediate.

Theorem 3. *If* **A** *is a positive definite matrix, then its largest element in absolute value must lie on its main diagonal.*

Proof. Assume that the largest element in absolute value is a_{pq} $(p \neq q)$, so that it is not on the main diagonal. If it is the largest, then $|a_{pq}| \geq a_{pp}$ and $|a_{pq}| \geq a_{qq}$. Therefore, $|a_{pq}|^2 \geq a_{pp}a_{qq}$, which contradicts Theorem 2. Thus, the assumption is incorrect.

Theorems 1 through 3 are tests to filter out Hermitian matrices that are *not* positive definite. In particular, consider the Hermitian matrices

$$\mathbf{A} = \begin{bmatrix} 15 & -2 & 1 \\ -2 & 3 & i \\ 1 & -i & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & -1 & 3+2i \\ -1 & 2 & -i \\ 3-2i & i & 3 \end{bmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 5 & -4 & 1 \\ -4 & 2 & 3 \\ 1 & 3 & 8 \end{bmatrix}.$$

A is not positive definite, because not all of its diagonal elements are positive, contradicting Theorem 1; **B** is not positive definite, because the largest element in absolute value, $3 + 2i$ with absolute value $\sqrt{13}$, is not on the main diagonal, contradicting Theorem 3; **C** is not positive definite because $|a_{12}|^2 = 16 > 10 = 5(2) = a_{11}a_{22}$, contradicting Theorem 2. In contrast,

$$\mathbf{D} = \begin{bmatrix} 10 & 2 & 0 \\ 2 & 4 & 6 \\ 0 & 6 & 10 \end{bmatrix}$$

satisfies all the conditions of Theorems 1 through 3, yet we are unable to say it is positive definite. In fact, we shall show later that it is not. It is important to realize that Theorems 1 through 3 represent necessary conditions for a Hermitian matrix to be positive definite; they are not sufficient conditions. If a matrix fails to satisfy any one of these three theorems, it is *not* positive definite; if it satisfies all three, no conclusions can be drawn. Two necessary and sufficient conditions are given by the following theorems.

Theorem 4. *A Hermitian matrix is positive definite if and only if its eigenvalues are positive.*

Proof. We will prove only that the eigenvalues of a positive definite matrix are positive and leave the converse, if the eigenvalues of a Hermitian matrix are positive then the matrix is positive definite, as an exercise for the student (see Problem 4). Let **A** be a positive definite Hermitian matrix and **x** an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ. Then

$$\langle \mathbf{Ax}, \mathbf{x} \rangle = \langle \lambda \mathbf{x}, \mathbf{x} \rangle = \lambda \langle \mathbf{x}, \mathbf{x} \rangle$$

or

$$\lambda = \frac{\langle \mathbf{Ax}, \mathbf{x} \rangle}{\langle \mathbf{x}, \mathbf{x} \rangle}. \tag{7}$$

Since $\mathbf{x}$ is an eigenvector, it cannot be zero, hence $\langle \mathbf{x}, \mathbf{x} \rangle$ is positive. Furthermore, since $\mathbf{A}$ is positive definite and $\mathbf{x} \neq \mathbf{0}$, it follows that $\langle \mathbf{Ax}, \mathbf{x} \rangle$ is also positive. Combining these results with (7), we find that λ must be positive.

Example 2

Is

$$\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}$$

positive definite?

Solution. $\mathbf{A}$ is Hermitian with eigenvalues

$$\lambda_1 = \frac{5 + \sqrt{13}}{2} \quad \text{and} \quad \lambda_2 = \frac{5 - \sqrt{13}}{2}$$

which are both positive. Thus, by Theorem 4, $\mathbf{A}$ is positive definite. □

Example 3

Is the matrix

$$\mathbf{A} = \begin{bmatrix} 6 & 2i \\ 6i & -1 \end{bmatrix}$$

positive definite?

Solution. Although the eigenvalues of $\mathbf{A}$ are both positive $\lambda_1 = 2$ and $\lambda_2 = 3$, the matrix is not Hermitian and hence cannot be positive definite. In particular, if we choose

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

we find that $\langle \mathbf{Ax}, \mathbf{x} \rangle = 5 + 8i$, which is not even real. □

Theorem 5. *A Hermitian matrix is positive definite if and only if it can be reduced to upper triangular form using only the third elementary row operation (i.e., add to one row a scalar times another row), and the resulting matrix has only positive elements on its main diagonal.*

The proof of this result is beyond the scope of this book, but it is, perhaps, the simplest and most straightforward procedure for determining whether or not a Hermitian matrix is positive definite.

Example 4

Determine whether the matrix

$$\mathbf{D} = \begin{bmatrix} 10 & 2 & 0 \\ 2 & 4 & 6 \\ 0 & 6 & 10 \end{bmatrix}$$

is positive definite.

Solution.

$$\begin{bmatrix} 10 & 2 & 0 \\ 2 & 4 & 6 \\ 0 & 6 & 10 \end{bmatrix} \to \begin{bmatrix} 10 & 2 & 0 \\ 0 & 3.6 & 6 \\ 0 & 6 & 10 \end{bmatrix} \quad \left\{\begin{array}{l}\text{by adding to the} \\ \text{second row } -0.2 \\ \text{times the first row}\end{array}\right.$$

$$\to \begin{bmatrix} 10 & 2 & 0 \\ 0 & 3.6 & 6 \\ 0 & 0 & 0 \end{bmatrix}. \quad \left\{\begin{array}{l}\text{by adding to the} \\ \text{third row } -5/3 \\ \text{times the second row}\end{array}\right.$$

This last matrix is in upper triangular form having as its diagonal elements 10, 3.6, and 0. These elements are not all positive, so the matrix **D** is *not* positive definite. Alternatively, if we sought the eigenvalues of **D** (involving a much more tedious set of calculations), we would have found them to be 0, 10, and 14. These eigenvalues are not all positive, so it follows from Theorem 4 that **D** is not positive definite. □

Theorems 1 through 5 are equally applicable to nonnegative definite matrices when the word *positive* is replaced by *nonnegative* and the strict inequality $>$ is replaced by $\geq$. Consequently, **D** in Example 4 is a nonnegative definite matrix. Once it is reduced to upper triangular form using only the third elementary row operation, the resulting matrix has diagonal elements that are all are nonnegative. In addition, the eigenvalues of **D** are all nonnegative.

Problems 10.8

(1) Determine which of the following matrices are positive definite. For those matrices that are not positive definite, produce a vector which will verify that conclusion.

(a) $\begin{bmatrix} 5 & 3i \\ 3i & -5 \end{bmatrix}$, (b) $\begin{bmatrix} 3 & 2i \\ -2i & 2 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 0 \\ 2i & 0 \end{bmatrix}$,

(d) $\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$, (e) $\begin{bmatrix} 1 & 2i & i \\ 0 & 2 & -i \\ 0 & i & 3 \end{bmatrix}$, (f) $\begin{bmatrix} 1 & 0 & i \\ 0 & 2 & 0 \\ i & 0 & 5 \end{bmatrix}$.

(2) Determine whether the following matrices are positive definite or nonnegative definite.

(a) $\begin{bmatrix} 8 & 2 & -2 \\ 2 & 8 & -2 \\ -2 & -2 & 11 \end{bmatrix}$, (b) $\begin{bmatrix} 15 & 12 & 16 \\ 12 & 13 & 14 \\ 16 & 14 & 13 \end{bmatrix}$,

(c) $\begin{bmatrix} 3 & 10 & -2 \\ 10 & 6 & 8 \\ -2 & 8 & 12 \end{bmatrix}$, (d) $\begin{bmatrix} 11 & -3 & 5 & -8 \\ -3 & 11 & -5 & -8 \\ 5 & -5 & 19 & 0 \\ -8 & -8 & 0 & 16 \end{bmatrix}$.

(3) A Hermitian matrix is *negative definite* if and only if $\langle \mathbf{Ax}, \mathbf{x} \rangle$ is negative for all nonzero vectors $\mathbf{x}$. Prove that $\mathbf{A}$ is negative definite if and only if $-\mathbf{A}$ is positive definite.

(4) Using Problem 8 of Section 10.3, complete the proof of Theorem 4.

(5) Prove that a Hermitian matrix is negative definite if and only if its eigenvalues are negative.

(6) Let $\mathbf{A}$ be a positive definite matrix. Define $\langle \mathbf{x}, \mathbf{y} \rangle_1 = \langle \mathbf{Ax}, \mathbf{y} \rangle$, where $\langle \mathbf{Ax}, \mathbf{y} \rangle$ designates our old inner product. Show that $\langle \mathbf{x}, \mathbf{y} \rangle_1$ satisfies Properties (I1)–(I4) of Section 10.1, thereby also defining an inner product. Thus, we see that positive definite matrices can be used to generate different inner products.

(7) Prove that the sum of two positive definite matrices of the same order is also positive definite.

(8) Prove that if $\mathbf{A}$ is positive definite, then so too is $\mathbf{A}^{\mathrm{H}}$.

(9) Prove that the determinant of a positive definite matrix must be positive.

(10) What can you say about the inverse of a positive definite matrix? Does it exist? If it does, must it also be positive definite?

(11) Prove that if $\mathbf{A}$ is positive definite and $\mathbf{T}$ is invertible, then $\mathbf{B} = \mathbf{T}^H\mathbf{A}\mathbf{T}$ is also positive definite.

(12) Prove that $e^{\mathbf{A}t}$ is positive definite whenever $\mathbf{A}$ is Hermitian.

(13) Define the *square root* of a positive definite matrix $\mathbf{A}$ to be another positive definite matrix $\mathbf{A}^{1/2}$ having the property that $\mathbf{A}^{1/2}\mathbf{A}^{1/2} = \mathbf{A}$. This is a well-defined function in the sense of Theorem 1 of Section 7.1. Find the square root of the matrix given in part (b) of Problem 1.

(14) Find the square root of the matrix given in part (d) of Problem 1.

Answers and Hints to Selected Problems

Chapter 1

Section 1.1

(1) $\mathbf{A}$ is 4×5, $\mathbf{B}$ is 3×3, $\mathbf{C}$ is 3×4,
$\mathbf{D}$ is 4×4, $\mathbf{E}$ is 2×3, $\mathbf{F}$ is 5×1,
$\mathbf{G}$ is 4×2, $\mathbf{H}$ is 2×2, $\mathbf{J}$ is 1×3.

(2) $a_{13} = -2$, $a_{21} = 2$,
$b_{13} = 3$, $b_{21} = 0$,
$c_{13} = 3$, $c_{21} = 5$,
$d_{13} = t^2$, $d_{21} = t - 2$,
$e_{13} = \frac{1}{4}$, $e_{21} = \frac{2}{3}$,
f_{13} does not exist, $f_{21} = 5$,
g_{13} does not exist, $g_{21} = 2\pi$,
h_{13} does not exist, $h_{21} = 0$,
$j_{13} = -30$, j_{21} does not exist.

(3) $a_{23} = -6$, $a_{32} = 3$, $b_{31} = 4$,
$b_{32} = 3$, $c_{11} = 1$, $d_{22} = t^4$, $e_{13} = \frac{1}{4}$,
$g_{22} = 18$, g_{23} and h_{32} do not exist.

(4) $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.
(5) $\mathbf{A} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 2 & 1 & \frac{2}{3} \\ 3 & \frac{3}{2} & 1 \end{bmatrix}$.
(6) $\mathbf{B} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & -2 \\ -1 & -2 & -3 \end{bmatrix}$.

(7) $\mathbf{C} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$. **(8)** $\mathbf{D} = \begin{bmatrix} 0 & -1 & -2 & -3 \\ 3 & 0 & -1 & -2 \\ 4 & 5 & 0 & -1 \end{bmatrix}$.

(9) (a) $[9 \quad 15]$, (b) $[12 \quad 0]$, (c) $[13 \quad 30]$, (d) $[21 \quad 15]$.

(10) (a) $[7 \quad 4 \quad 1776]$, (b) $[12 \quad 7 \quad 1941]$, (c) $[4 \quad 23 \quad 1809]$, (d) $[10 \quad 31 \quad 1688]$.

(11) $[950 \quad 1253 \quad 98]$. **(12)** $\begin{bmatrix} 3 & 5 & 3 & 4 \\ 0 & 2 & 9 & 5 \\ 4 & 2 & 0 & 0 \end{bmatrix}$. **(13)** $\begin{bmatrix} 72 & 12 & 16 \\ 45 & 32 & 16 \\ 81 & 10 & 35 \end{bmatrix}$.

(14) $\begin{bmatrix} 100 & 150 & 50 & 500 \\ 27 & 45 & 116 & 2 \\ 29 & 41 & 116 & 3 \end{bmatrix}$

(15) (a) $\begin{bmatrix} 1000 & 2000 & 3000 \\ 0.07 & 0.075 & 0.0725 \end{bmatrix}$. (b) $\begin{bmatrix} 1070.00 & 2150.00 & 3217.50 \\ 0.075 & 0.08 & 0.0775 \end{bmatrix}$.

(16) $\begin{bmatrix} 0.95 & 0.05 \\ 0.01 & 0.99 \end{bmatrix}$. **(17)** $\begin{bmatrix} 0.6 & 0.4 \\ 0.7 & 0.3 \end{bmatrix}$. **(18)** $\begin{bmatrix} 0.10 & 0.50 & 0.40 \\ 0.20 & 0.60 & 0.20 \\ 0.25 & 0.65 & 0.10 \end{bmatrix}$.

(19) $\begin{bmatrix} 0.80 & 0.15 & 0.05 \\ 0.10 & 0.88 & 0.02 \\ 0.25 & 0.30 & 0.45 \end{bmatrix}$.

Section 1.2

(1) $\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$. **(2)** $\begin{bmatrix} -5 & -10 \\ -15 & -20 \end{bmatrix}$. **(3)** $\begin{bmatrix} 9 & 3 \\ -3 & 6 \\ 9 & -6 \\ 6 & 18 \end{bmatrix}$.

(4) $\begin{bmatrix} -20 & 20 \\ 0 & -20 \\ 50 & -30 \\ 50 & 10 \end{bmatrix}$. **(5)** $\begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \\ -2 & -2 \end{bmatrix}$. **(6)** $\begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$. **(7)** $\begin{bmatrix} 0 & 2 \\ 6 & 1 \end{bmatrix}$.

(8) $\begin{bmatrix} 1 & 3 \\ -1 & 0 \\ 8 & -5 \\ 7 & 7 \end{bmatrix}$. **(9)** $\begin{bmatrix} 3 & 2 \\ -2 & 2 \\ 3 & -2 \\ 4 & 8 \end{bmatrix}$. **(10)** Does not exist.

(11) $\begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}$. **(12)** $\begin{bmatrix} -2 & -2 \\ 0 & -7 \end{bmatrix}$. **(13)** $\begin{bmatrix} 5 & -1 \\ -1 & 4 \\ -2 & 1 \\ -3 & 5 \end{bmatrix}$.

(14) $\begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 3 & -2 \\ 0 & 4 \end{bmatrix}$. **(15)** $\begin{bmatrix} 17 & 22 \\ 27 & 32 \end{bmatrix}$. **(16)** $\begin{bmatrix} 5 & 6 \\ 3 & 18 \end{bmatrix}$.

(17) $\begin{bmatrix} -0.1 & 0.2 \\ 0.9 & -0.2 \end{bmatrix}$. **(18)** $\begin{bmatrix} 4 & -3 \\ -1 & 4 \\ -10 & 6 \\ -8 & 0 \end{bmatrix}$. **(19)** $\mathbf{X} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$.

(20) $\mathbf{Y} = \begin{bmatrix} -11 & -12 \\ -11 & -19 \end{bmatrix}$. **(21)** $\mathbf{X} = \begin{bmatrix} 11 & 1 \\ -3 & 8 \\ 4 & -3 \\ 1 & 17 \end{bmatrix}$.

(22) $\mathbf{Y} = \begin{bmatrix} -1.0 & 0.5 \\ 0.5 & -1.0 \\ 2.5 & -1.5 \\ 1.5 & -0.5 \end{bmatrix}$. **(23)** $\mathbf{R} = \begin{bmatrix} -2.8 & -1.6 \\ 3.6 & -9.2 \end{bmatrix}$.

(24) $\mathbf{S} = \begin{bmatrix} -1.5 & 1.0 \\ -1.0 & -1.0 \\ -1.5 & 1.0 \\ 2.0 & 0 \end{bmatrix}$. **(25)** $\begin{bmatrix} 5 & 8 \\ 13 & 9 \end{bmatrix}$.

(27) $\begin{bmatrix} -\theta^3 + 6\theta^2 + \theta & 6\theta - 6 \\ 21 & -\theta^4 - 2\theta^2 - \theta + 6/\theta \end{bmatrix}$.

(32) (a) [200 150], (b) [600 450], (c) [550 550].

(33) (b) [11 2 6 3], (c) [9 4 10 8]

(34) (b) [10,500 6,000 4,500], (c) [35,500 14,500 3,300].

Section 1.3

(1) (a) 2×2, (b) 4×4, (c) 2×1, (d) Not defined, (e) 4×2, (f) 2×4, (g) 4×2, (h) Not defined, (i) Not defined, (j) 1×4, (k) 4×4, (l) 4×2.

(2) $\begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$. **(3)** $\begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$. **(4)** $\begin{bmatrix} 5 & -4 & 3 \\ 9 & -8 & 7 \end{bmatrix}$.

(5) $\mathbf{A} = \begin{bmatrix} 13 & -12 & 11 \\ 17 & -16 & 15 \end{bmatrix}$. **(6)** Not defined. **(7)** $[-5 \quad -6]$.

(8) $[-9 \quad -10]$. **(9)** $[-7 \quad 4 \quad -1]$. **(10)** Not defined.

(11) $\begin{bmatrix} 1 & -3 \\ 7 & -3 \end{bmatrix}$. **(12)** $\begin{bmatrix} 2 & -2 & 2 \\ 7 & -4 & 1 \\ -8 & 4 & 0 \end{bmatrix}$. **(13)** $[1 \quad 3]$.

(14) Not defined. **(15)** Not defined. **(16)** Not defined.

(17) $\begin{bmatrix} -1 & -2 & -1 \\ 1 & 0 & -3 \\ 1 & 3 & 5 \end{bmatrix}$. **(18)** $\begin{bmatrix} 2 & -2 & 1 \\ -2 & 0 & 0 \\ 1 & -2 & 2 \end{bmatrix}$. **(19)** $[-1 \quad 1 \quad 5]$.

(22) $\begin{bmatrix} x + 2y \\ 3x + 4y \end{bmatrix}$. **(23)** $\begin{bmatrix} x - z \\ 3x + y + z \\ x + 3y \end{bmatrix}$. **(24)** $\begin{bmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{bmatrix}$.

(25) $\begin{bmatrix} 2b_{11} - b_{12} + 3b_{13} \\ 2b_{21} - b_{22} + 3b_{23} \end{bmatrix}$. **(26)** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. **(27)** $\begin{bmatrix} 0 & 40 \\ -16 & 8 \end{bmatrix}$.

(28) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. **(29)** $\begin{bmatrix} 7 & 5 \\ 11 & 10 \end{bmatrix}$. **(32)** $\begin{bmatrix} 2 & 3 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \end{bmatrix}$.

(33) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$. **(34)** $\begin{bmatrix} 5 & 3 & 2 & 4 \\ 1 & 1 & 0 & 1 \\ 3 & 2 & 2 & 0 \\ 1 & 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -3 \\ 4 \end{bmatrix}$.

(35) (a) **PN** = [38,000], which represents the total revenue for that flight.

(b) $\mathbf{NP} = \begin{bmatrix} 26{,}000 & 45{,}5000 & 65{,}000 \\ 4{,}000 & 7{,}000 & 10{,}000 \\ 2{,}000 & 3{,}500 & 5{,}000 \end{bmatrix}$,

which has no physical significance.

(36) (a) **HP** = [9,625 9,762.50 9,887.50 10,100 9,887.50], which represents the portfolio value each day.

(b) **PH** does not exist.

(37) **TW** = $[14.00 \quad 65.625 \quad 66.50]^{\mathrm{T}}$, which denotes the cost of producing each product.

(38) **OTW** = [33,862.50], which denotes the cost of producing all items on order.

(39) $\mathbf{FC} = \begin{bmatrix} 613 & 625 \\ 887 & 960 \\ 1870 & 1915 \end{bmatrix}$,
which represents the number of each sex in each state of sickness.

Section 1.4

(1) $\begin{bmatrix} 7 & 4 & -1 \\ 6 & 1 & 0 \\ 2 & 2 & -6 \end{bmatrix}$. **(2)** $\begin{bmatrix} t^3+3t & 2t^2+3 & 3 \\ 2t^3+t^2 & 4t^2+t & t \\ t^4+t^2+t & 2t^3+t+1 & t+1 \\ t^5 & 2t^4 & 0 \end{bmatrix}$.

(3) (a) $\mathbf{BA}^{\mathsf{T}}$, (b) $2\mathbf{A}^{\mathsf{T}}$, (c) $(\mathbf{B}^{\mathsf{T}} + \mathbf{C})\mathbf{A}$, (d) $\mathbf{AB} + \mathbf{C}^{\mathsf{T}}$,
(e) $\mathbf{A}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}} + \mathbf{A}^{\mathsf{T}}\mathbf{A} - \mathbf{AA}^{\mathsf{T}} - \mathbf{AA}$.

(4) $\mathbf{X}^{\mathsf{T}}\mathbf{X} = [29]$, and $\mathbf{XX}^{\mathsf{T}} = \begin{bmatrix} 4 & 6 & 8 \\ 6 & 9 & 12 \\ 8 & 12 & 16 \end{bmatrix}$.

(5) $\mathbf{X}^{\mathsf{T}}\mathbf{X} = \begin{bmatrix} 1 & -2 & 3 & -4 \\ -2 & 4 & -6 & 8 \\ 3 & -6 & 9 & -12 \\ -4 & 8 & -12 & 16 \end{bmatrix}$, and $\mathbf{XX}^{\mathsf{T}} = [30]$.

(6) $[2x^2 + 6xy + 4y^2]$. **(7)** **A**, **B**, **D**, **F**, **M**, **N**, **R**, and **T**.

(8) **E**, **F**, **H**, **K**, **L**, **M**, **N**, **R**, and **T**. **(9)** Yes.

(10) No, see **H** and **L** in Problem 7. **(11)** Yes, see **L** in Problem 7.

(12) $\begin{bmatrix} -5 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. **(14)** No.

(19) $\mathbf{D}^2$ is a diagonal matrix with diagonal elements 4, 9, and 25; $\mathbf{D}^3$ is a diagonal matrix with diagonal elements 8, 27, and -125.

(20) A diagonal matrix with diagonal elements 1, 8, 27.

(23) A diagonal matrix with diagonal elements 4, 0, 10. **(25)** 4.

(28) $\mathbf{A} = \mathbf{B} + \mathbf{C}$ **(29)** $\begin{bmatrix} 1 & \frac{7}{2} & -\frac{1}{2} \\ \frac{7}{2} & 1 & 5 \\ -\frac{1}{2} & 5 & -8 \end{bmatrix} + \begin{bmatrix} 0 & \frac{3}{2} & -\frac{1}{2} \\ -\frac{3}{2} & 0 & -2 \\ \frac{1}{2} & 2 & 0 \end{bmatrix}$.

(30) $\begin{bmatrix} 6 & \frac{3}{2} & 1 \\ \frac{3}{2} & 0 & -4 \\ 1 & -4 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{2} & 2 \\ \frac{1}{2} & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$.

(34) (a) $\mathbf{P}^2 = \begin{bmatrix} 0.37 & 0.63 \\ 0.28 & 0.72 \end{bmatrix}$ and $\mathbf{P}^3 = \begin{bmatrix} 0.289 & 0.711 \\ 0.316 & 0.684 \end{bmatrix}$,
(b) 0.37, (c) 0.63, (d) 0.711, (e) 0.684.

(35) $1 \to 1 \to 1 \to 1$, $1 \to 1 \to 2 \to 1$, $1 \to 2 \to 1 \to 1$, $1 \to 2 \to 2 \to 1$.

(36) (a) 0.097, (b) 0.0194. **(37)** (a) 0.64, (b) 0.636.

(38) (a) 0.1, (b) 0.21. **(39)** (a) 0.6675, (b) 0.577075, (c) 0.267.

(40) $\mathbf{M} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$.

(41) (a) $\mathbf{M} = \begin{bmatrix} 0 & 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$, (b) three arcs.

(42) (a) $\mathbf{M} = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$,

(b) $\mathbf{M}^3$ has a path from node 1 to node 7; it is the first integral power of $\mathbf{M}$ having m_{17} positive. The minimum number of *intermediate* cities is two.

Section 1.5

(1) (a), (b), and (d) are submatrices. **(3)** $\left[\begin{array}{ccc|c} 4 & 5 & -1 & 9 \\ \hline 15 & 10 & 4 & 22 \\ 1 & 1 & 5 & 9 \end{array}\right]$.

(4) Partition **A** and **B** into four 2×2 submatrices each. Then,

$$\mathbf{AB} = \left[\begin{array}{cc|cc} 11 & 9 & 0 & 0 \\ 4 & 6 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 \\ 0 & 0 & 4 & -1 \end{array}\right].$$

(5) $\left[\begin{array}{cc|cc} 18 & 6 & 0 & 0 \\ 12 & 6 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 4 \end{array}\right].$ **(6)** $\left[\begin{array}{cc|cc} 7 & 8 & 0 & 0 \\ -4 & -1 & 0 & 0 \\ \hline 0 & 0 & 5 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right].$

(7) $\mathbf{A}^2 = \left[\begin{array}{cc|cccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$ $\mathbf{A}^3 = \left[\begin{array}{cc|cccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$

(8) $\mathbf{A}^n = \mathbf{A}$ when n is odd.

Section 1.6

(1) $p = 1$. **(2)** $\begin{bmatrix} -4/3 \\ -1 \\ -8/3 \\ 1/3 \end{bmatrix}$. **(3)** $[1 \;\; -0.4 \;\; 1]$.

(4) (a) Not defined, (b) $\begin{bmatrix} 6 & -3 & 12 & 3 \\ 2 & -1 & 4 & 1 \\ 12 & -6 & 24 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, (c) [29], (d) [29].

(5) (a) $[4 \;\; -1 \;\; 1]$, (b) $[-1]$, (c) $\begin{bmatrix} 2 & 0 & -2 \\ -1 & 0 & 1 \\ 3 & 0 & -3 \end{bmatrix}$, (d) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

(6) (c), (d), (f), (g), (h), and (i).

(7) (a) $\sqrt{2}$, (b) 5, (c) $\sqrt{3}$, (d) $\frac{1}{2}\sqrt{3}$, (e) $\sqrt{15}$, (f) $\sqrt{39}$.

(8) (a) $\sqrt{2}$, (b) $\sqrt{5}$, (c) $\sqrt{3}$, (d) 2, (e) $\sqrt{30}$, (f) $\sqrt{2}$.

(9) (a) $\sqrt{15}$, (b) $\sqrt{39}$. **(12)** $x\begin{bmatrix}2\\4\end{bmatrix}+y\begin{bmatrix}3\\5\end{bmatrix}=\begin{bmatrix}10\\11\end{bmatrix}$.

(13) $x\begin{bmatrix}3\\0\\-1\end{bmatrix}+y\begin{bmatrix}4\\1\\1\end{bmatrix}+z\begin{bmatrix}5\\-2\\2\end{bmatrix}+w\begin{bmatrix}6\\8\\-1\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}$. **(16)** [0.5 0.3 0.2].

(17) (a) There is a 0.6 probability that an individual chosen at random initially will live in the city; thus, 60% of the population initially lives in the city, while 40% lives in the suburbs.
(b) $\mathbf{d}^{(1)} = [0.574 \quad 0.426]$. (c) $\mathbf{d}^{(2)} = [0.54956 \quad 0.45044]$.

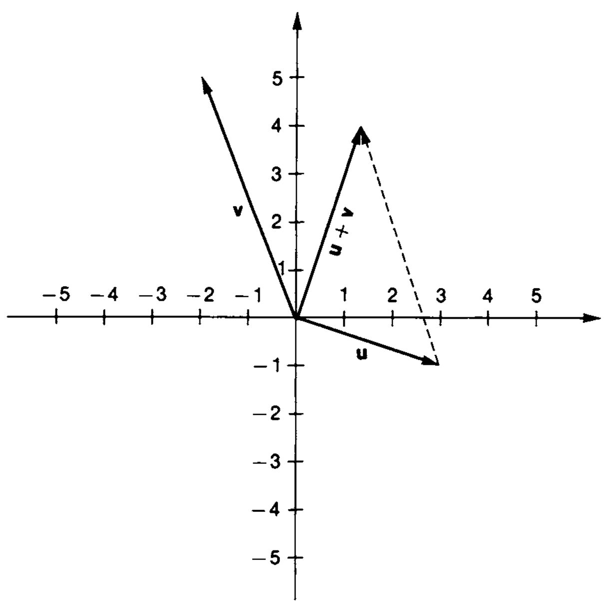
(18) (a) 40% of customers now use brand X, 50% use brand Y, and 10% use other brands.
(b) $\mathbf{d}_1 = [0.395 \quad 0.530 \quad 0.075]$, (c) $\mathbf{d}_2 = [0.38775 \quad 0.54815 \quad 0.06410]$.

(19) (a) $\mathbf{d}^{(0)} = [0 \quad 1]$. (b) $\mathbf{d}^{(1)} = [0.7 \quad 0.3]$.

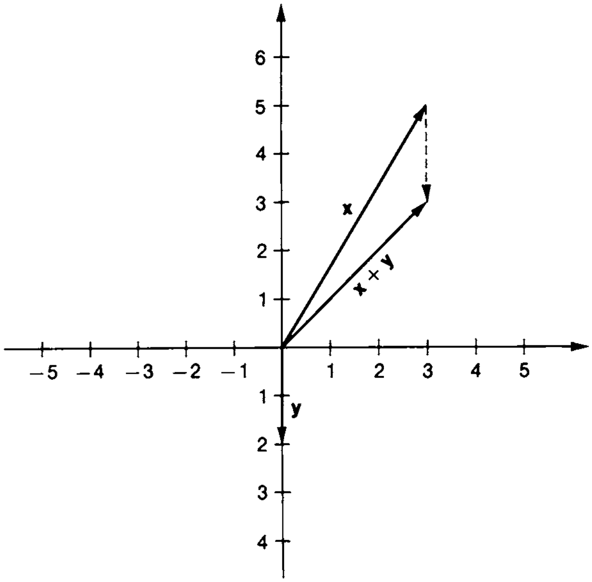
(20) (a) $\mathbf{d}^{(0)} = [1 \quad 0 \quad 0]$.
(b) $\mathbf{d}^{(2)} = [0.21 \quad 0.61 \quad 0.18]$. A probability of 0.18 that the harvest will be good in two years.

Section 1.7

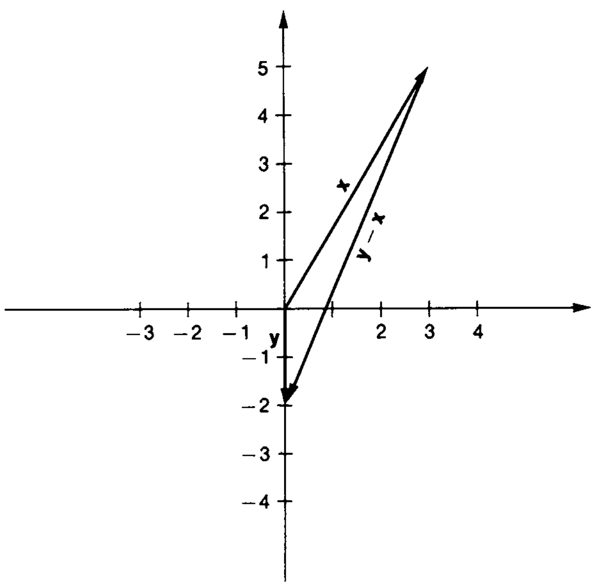
(1)



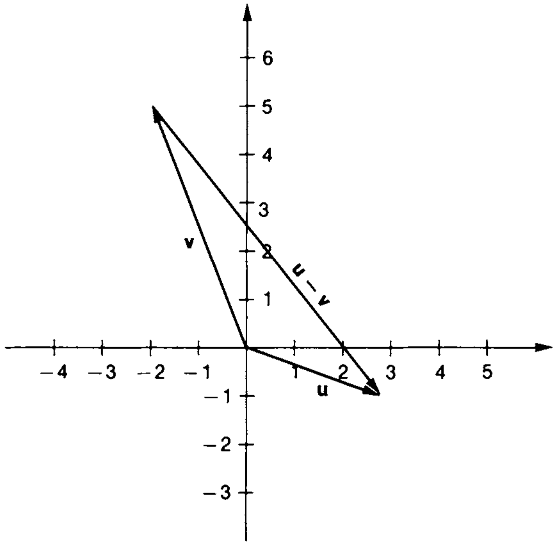
(4)



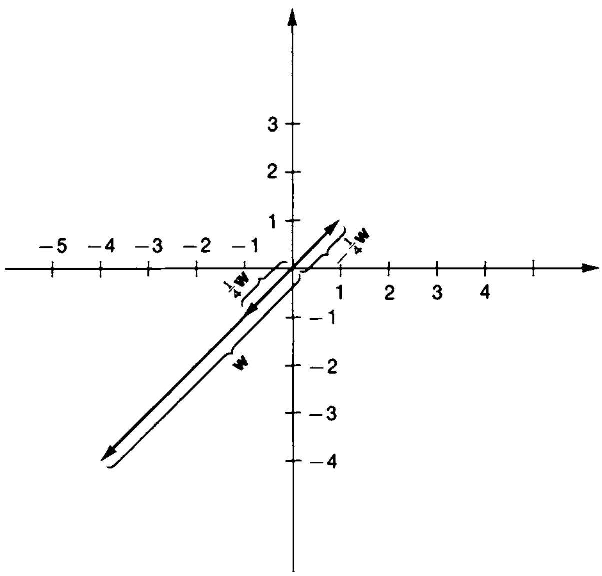
(6)



(7)



(16)



(17) 341.57°. **(18)** 111.80°. **(19)** 225°. **(20)** 59.04°.

(21) 270°.

Chapter 2

Section 2.1

(1) (a) No. (b) Yes. **(2)** (a) Yes. (b) No. (c) No.
(3) No value of k will work. **(4)** $k = 1$ **(5)** $k = 1/12$.
(6) k is arbitrary; any value will work. **(7)** No value of k will work.

(8) $\begin{bmatrix} 3 & 5 \\ 2 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ -3 \end{bmatrix}$. **(9)** $\begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & -2 \\ 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

(10) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & -3 & 2 \\ 3 & -4 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix}$. **(11)** $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 2 \\ -3 & -6 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(12) $50r + 60s = 70{,}000,$
$30r + 40s = 45{,}000.$

(13) $5d + 0.25b = 200,$
$10d + b = 500.$

(14) $8{,}000A + 3{,}000B + 1{,}000C = 70{,}000,$
$5{,}000A + 12{,}000B + 10{,}000C = 181{,}000,$
$1{,}000A + 3{,}000B + 2{,}000C = 41{,}000.$

(15) $5A + 4B + 8C + 12D = 80,$
$20A + 30B + 15C + 5D = 200,$
$3A + 3B + 10C + 7D = 50.$

(16) $b + 0.05c + 0.05s = 20{,}000,$
$c = 8{,}000,$
$0.03c + s = 12{,}000.$

(17) (a) $C = 800{,}000 + 30B,$ $S = 40B.$ (b) Add the additional equation $S = C$.

(18) $-0.60p_1 + 0.30p_2 + 0.50p_3 = 0,$
$0.40p_1 - 0.75p_2 + 0.35p_3 = 0,$
$0.20p_1 + 0.45p_2 - 0.85p_3 = 0.$

(19) $-\frac{1}{2}p_1 + \frac{1}{3}p_2 + \frac{1}{6}p_3 = 0,$
$\frac{1}{4}p_1 - \frac{2}{3}p_2 + \frac{1}{3}p_3 = 0,$
$\frac{1}{4}p_1 + \frac{1}{3}p_2 - \frac{1}{2}p_3 = 0.$

(20) $-0.85p_1 + 0.10p_2 + 0.15p_4 = 0,$
$0.20p_1 - 0.60p_2 + \frac{1}{3}p_3 + 0.40p_4 = 0,$
$0.30p_1 + 0.15p_2 - \frac{2}{3}p_3 + 0.45p_4 = 0,$
$0.35p_1 + 0.35p_2 + \frac{1}{3}p_3 - p_4 = 0.$

(22) $\mathbf{A} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & 0 \end{bmatrix}$ and $\mathbf{d} = \begin{bmatrix} 20{,}000 \\ 30{,}000 \end{bmatrix}$.

(23) $\mathbf{A} = \begin{bmatrix} 0 & 0.02 & 0.50 \\ 0.20 & 0 & 0.30 \\ 0.10 & 0.35 & 0.10 \end{bmatrix}$ and $\mathbf{d} = \begin{bmatrix} 50{,}000 \\ 80{,}000 \\ 30{,}000 \end{bmatrix}$.

(24) $\mathbf{A} = \begin{bmatrix} 0.20 & 0.15 & 0.40 & 0.25 \\ 0 & 0.20 & 0 & 0 \\ 0.10 & 0.05 & 0 & 0.10 \\ 0.30 & 0.30 & 0.10 & 0.05 \end{bmatrix}$ and $\mathbf{d} = \begin{bmatrix} 0 \\ 5{,}000{,}000 \\ 0 \\ 0 \end{bmatrix}$.

Section 2.2

(1) $x = 1, y = 1, z = 2.$ **(2)** $x = -6z, y = 7z, z$ is arbitrary.
(3) $x = y = 1.$ **(4)** $r = t + 13/7, s = 2t + 15/7, t$ is arbitrary.
(5) $l = \frac{1}{5}(-n + 1), m = \frac{1}{5}(3n - 5p - 3), n$ and p are arbitrary.
(6) $x = 0, y = 0, z = 0.$ **(7)** $x = 2, y = 1, z = -1.$
(8) $x = 1, y = 1, z = 0, w = 1.$

Section 2.3

(1) $\mathbf{A}^b = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 1 & 1 \end{bmatrix}.$ **(2)** $\mathbf{A}^b = \begin{bmatrix} 1 & 2 & -1 & -1 \\ 2 & -3 & 2 & 4 \end{bmatrix}.$

(3) $\mathbf{A}^b = \begin{bmatrix} 1 & 2 & 5 \\ -3 & 1 & 13 \\ 4 & 3 & 0 \end{bmatrix}.$ **(4)** $\mathbf{A}^b = \begin{bmatrix} 2 & 4 & 0 & 2 \\ 3 & 2 & 1 & 8 \\ 5 & -3 & 7 & 15 \end{bmatrix}.$

(5) $\mathbf{A}^b = \begin{bmatrix} 2 & 3 & -4 & 12 \\ 3 & -2 & 0 & -1 \\ 8 & -1 & -4 & 10 \end{bmatrix}.$ **(6)** $\begin{aligned} x + 2y &= 5, \\ y &= 8. \end{aligned}$

(7) $\begin{aligned} x - 2y + 3z &= 10, \\ y - 5z &= -3, \\ z &= 4. \end{aligned}$ **(8)** $\begin{aligned} r - 3s + 12t &= 40, \\ s - 6t &= -200, \\ t &= 25. \end{aligned}$

(9) $\begin{aligned} x + 3y \qquad &= -8, \\ y + 4z &= 2, \\ 0 &= 0. \end{aligned}$ **(10)** $\begin{aligned} a - 7b + 2c &= 0, \\ b - c &= 0, \\ 0 &= 0. \end{aligned}$ **(11)** $\begin{aligned} u - v \qquad &= 1, \\ v - 2w &= 2, \\ w &= -3, \\ 0 &= 1. \end{aligned}$

(12) $x = -11, y = 8.$ **(13)** $x = 32, y = 17, z = 4.$
(14) $r = -410, s = -50, t = 25.$
(15) $x = -14 + 12z, y = 2 - 4z, z$ is arbitrary.
(16) $a = 5c, b = c, c$ is arbitrary. **(17)** No solution.

(18) $\begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & 23 \end{bmatrix}$. **(19)** $\begin{bmatrix} 1 & 6 & 5 \\ 0 & 1 & 18 \end{bmatrix}$. **(20)** $\begin{bmatrix} 1 & 3.5 & -2.5 \\ 0 & 1 & -6 \end{bmatrix}$.

(21) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & 1 & 41/29 \end{bmatrix}$. **(22)** $\begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & -32/23 \end{bmatrix}$.

(23) $\begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & 1 & -9/35 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. **(24)** $\begin{bmatrix} 1 & 3/2 & 2 & 3 & 0 & 5 \\ 0 & 1 & -50 & -32 & -6 & -130 \\ 0 & 0 & 1 & 53/76 & 5/76 & 190/76 \end{bmatrix}$.

(25) $x = 1, y = -2$.
(26) $x = 5/7 - (1/7)z$, $y = -6/7 + (4/7)z$, z is arbitrary.
(27) $a = -3$, $b = 4$. **(28)** $r = 13/3$, $s = t = -5/3$.
(29) $r = \frac{1}{13}(21 + 8t)$, $s = \frac{1}{13}(38 + 12t)$, t is arbitrary.
(30) $x = 1, y = 1, z = 2$. **(31)** $x = -6z$, $y = 7z$, z is arbitrary.
(32) $x = y = 1$. **(33)** $r = t + 13/7$, $s = 2t + 15/7$, t is arbitrary.
(34) $l = \frac{1}{5}(-n + 1)$, $m = \frac{1}{5}(3n - 5p - 3)$, n and p are arbitrary.
(35) $r = 500$, $s = 750$. **(36)** $d = 30$, $b = 200$. **(37)** $A = 5$, $B = 8$, $C = 6$.
(38) $A = 19.759 - 4.145D$, $B = -7.108 + 2.735D$,
$C = 1.205 - 0.277D$, D is arbitrary. **(39)** $b = \$19{,}012$.
(40) 80,000 barrels. **(41)** $p_1 = (48/33)p_3$, $p_2 = (41/33)p_3$, p_3 is arbitrary.
(42) $p_1 = (8/9)p_3$, $p_2 = (5/6)p_3$, p_3 is arbitrary.
(43) $p_1 = 0.3435p_4$, $p_2 = 1.4195p_4$, $p_3 = 1.1489p_4$, p_4 is arbitrary.
(44) $x_1 = \$66{,}000$; $x_2 = \$52{,}000$.
(45) To construct an elementary matrix that will interchange the ith and jth rows, simply interchange those rows in the identity matrix of appropriate order.
(46) To construct an elementary matrix that will multiply the ith row of a matrix by the scalar r, simply replace the unity element in the i–i position of an identity matrix of appropriate order by r.
(47) To construct an elementary matrix that will add r times the ith row to the jth row, simply do the identical process to an identity matrix of appropriate order.

(48) $\mathbf{x}^{(0)} = \begin{bmatrix} 40{,}000 \\ 60{,}000 \end{bmatrix}$, $\mathbf{x}^{(1)} = \begin{bmatrix} 55{,}000 \\ 43{,}333 \end{bmatrix}$, $\mathbf{x}^{(2)} = \begin{bmatrix} 58{,}333 \\ 48{,}333 \end{bmatrix}$.

(49) $\mathbf{x}^{(0)} = \begin{bmatrix} 100{,}000 \\ 160{,}000 \\ 60{,}000 \end{bmatrix}$, $\mathbf{x}^{(1)} = \begin{bmatrix} 83{,}200 \\ 118{,}000 \\ 102{,}000 \end{bmatrix}$, $\mathbf{x}^{(2)} = \begin{bmatrix} 103{,}360 \\ 127{,}240 \\ 89{,}820 \end{bmatrix}$.

The solution is $x_1 = \$99{,}702$; $x_2 = \$128{,}223$; and $x_3 = \$94{,}276$, rounded to the nearest dollar.

(50) $\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 10{,}000{,}000 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{x}^{(1)} = \begin{bmatrix} 1{,}500{,}000 \\ 7{,}000{,}000 \\ 500{,}000 \\ 3{,}000{,}000 \end{bmatrix}$, $\mathbf{x}^{(2)} = \begin{bmatrix} 2{,}300{,}000 \\ 6{,}400{,}000 \\ 800{,}000 \\ 2{,}750{,}000 \end{bmatrix}$.

The solution is: energy = \$2,484,488; tourism = \$6,250,000; transportation = \$845,677; and construction = \$2,847,278, all rounded to the nearest dollar.

Section 2.4

(1) (a) 4, (b) 4, (c) 8. **(2)** (a) 5, (b) 5, (c) 5.
(3) (a) 3, (b) 3, (c) 8. **(4)** (a) 4, (b) -3, (c) 8.
(5) (a) 9, (b) 9, (c) 11. **(6)** (a) 4, (b) 1, (c) 10.
(7) $a = -3, b = 4$. **(8)** $r = 13/3, s = t = -5/3$.
(9) Depending on the roundoff procedure used, the last equation may not be $0 = 0$, but rather numbers very close to zero. Then only one answer is obtained.

Section 2.5

(1) Independent. **(2)** Independent. **(3)** Dependent.
(4) Dependent. **(5)** Independent. **(6)** Dependent.
(7) Independent. **(8)** Dependent. **(9)** Dependent.
(10) Dependent. **(11)** Independent. **(12)** Dependent.
(13) Independent **(14)** Independent. **(15)** Dependent.
(16) Independent. **(17)** Dependent. **(18)** Dependent.
(19) Dependent. **(20)** $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = (-2)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (1)\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + (3)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(21) (a) $[2 \quad 3] = 2[1 \quad 0] + 3[0 \quad 1]$, (b) $[2 \quad 3] = \frac{5}{2}[1 \quad 1] + (-\frac{1}{2})[1 \quad -1]$, (c) No.

(22) (a) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \left(\frac{1}{2}\right)\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \left(\frac{1}{2}\right)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \left(\frac{1}{2}\right)\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, (b) No,

(c) $\begin{bmatrix}1\\1\\1\end{bmatrix} = (0)\begin{bmatrix}1\\0\\1\end{bmatrix} + (1)\begin{bmatrix}1\\1\\1\end{bmatrix} + (0)\begin{bmatrix}1\\-1\\1\end{bmatrix}.$

(23) $\begin{bmatrix}2\\0\\3\end{bmatrix} = (1)\begin{bmatrix}1\\0\\1\end{bmatrix} + (1)\begin{bmatrix}1\\0\\2\end{bmatrix} + (0)\begin{bmatrix}2\\0\\1\end{bmatrix}.$

(24) $[a \quad b] = (a)[1 \quad 0] + (b)[0 \quad 1].$

(25) $[a \quad b] = \left(\dfrac{a+b}{2}\right)[1 \quad 1] + \left(\dfrac{a-b}{2}\right)[1 \quad -1].$

(26) $[1 \quad 0]$ can not be written as a linear combination of these vectors.

(27) $[a \quad -2a] = (a/2)[2 \quad -4] + (0)[-3 \quad 6].$

(28) $[a \quad b] = \left(\dfrac{a+2b}{7}\right)[1 \quad 3] + \left(\dfrac{3a-b}{7}\right)[2 \quad -1] + (0)[1 \quad 1].$

(29) $\begin{bmatrix}a\\b\\c\end{bmatrix} = \left(\dfrac{a-b+c}{2}\right)\begin{bmatrix}1\\0\\1\end{bmatrix} + \left(\dfrac{a+b-c}{2}\right)\begin{bmatrix}1\\1\\0\end{bmatrix} + \left(\dfrac{-a+b+c}{2}\right)\begin{bmatrix}0\\1\\1\end{bmatrix}.$

(30) No, impossible to write any vector with a nonzero second component as a linear combination of these vectors.

(31) $\begin{bmatrix}a\\0\\a\end{bmatrix} = (a)\begin{bmatrix}1\\0\\1\end{bmatrix} + (0)\begin{bmatrix}1\\0\\2\end{bmatrix} + (0)\begin{bmatrix}2\\0\\1\end{bmatrix}.$ **(32)** 1 and 2 are bases.

(33) 7 and 11 are bases. **(39)** $(-k)x + (1)k\mathbf{x} = \mathbf{0}.$

(42) $\mathbf{0} = \mathbf{A0} = \mathbf{A}(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k) = c_1\mathbf{A}\mathbf{x}_1 + c_2\mathbf{A}\mathbf{x}_2 + \cdots + c_k\mathbf{A}\mathbf{x}_k = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_k\mathbf{y}_k).$

Section 2.6

(1) 2. **(2)** 2. **(3)** 1. **(4)** 2. **(5)** 3.
(6) Independent. **(7)** Independent. **(8)** Dependent.
(9) Dependent. **(10)** Independent. **(11)** Dependent.
(12) Independent. **(13)** Dependent. **(14)** Dependent.
(15) Dependent. **(16)** Independent. **(17)** Dependent.
(18) Independent. **(19)** Dependent. **(20)** Independent.
(21) Dependent. **(22)** Dependent.
(23) (a) Yes, (b) Yes, (c) No. **(24)** (a) Yes, (b) No, (c) Yes.
(25) Yes. **(26)** Yes. **(27)** No. **(28)** First two.
(29) First two. **(30)** First and third. **(31)** 0.

Section 2.7

(1) Consistent with no arbitrary unknowns; $x = 2/3, y = 1/3$.
(2) Inconsistent.
(3) Consistent with one arbitrary unknown; $x = (1/2)(3 - 2z), y = -1/2$.
(4) Consistent with two arbitrary unknowns; $x = (1/7)(11 - 5z - 2w)$, $y = (1/7)(1 - 3z + 3w)$.
(5) Consistent with no arbitrary unknowns; $x = y = 1, z = -1$.
(6) Consistent with no arbitrary unknowns; $x = y = 0$.
(7) Consistent with no arbitrary unknowns; $x = y = z = 0$.
(8) Consistent with one arbitrary unknown; $x = -z, y = z$.
(9) Consistent with two arbitrary unknowns; $x = z - 7w, y = 2z - 2w$.

Chapter 3

Section 3.1

(1) (c). **(2)** None. **(3)** $\begin{bmatrix} \frac{3}{14} & \frac{-2}{14} \\ \frac{-5}{14} & \frac{8}{14} \end{bmatrix}$. **(4)** $\begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$.

(5) **D** has no inverse. **(7)** $\begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$. **(8)** $\begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{bmatrix}$.

(9) $\begin{bmatrix} 4 & -6 \\ -6 & 12 \end{bmatrix}$. **(10)** $\begin{bmatrix} \frac{-1}{5} & \frac{1}{10} \\ \frac{3}{20} & \frac{-1}{20} \end{bmatrix}$. **(11)** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. **(12)** $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$.

(13) $\begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix}$. **(14)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. **(15)** $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$. **(16)** $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$.

(17) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$. **(18)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}$. **(19)** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(20) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. **(21)** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$. **(22)** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

(23) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. **(24)** $\begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}$.

(25) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. **(26)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(27) $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$. **(28)** $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$. **(29)** $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$. **(30)** $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$.

(31) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. **(32)** $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. **(33)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$.

(34) $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. **(35)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. **(36)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{4} \end{bmatrix}$

(37) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. **(38)** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. **(39)** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(40) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. **(41)** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. **(42)** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(43) $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$. **(44)** No inverse. **(45)** $\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}$. **(46)** $\begin{bmatrix} 2 & 0 \\ 0 & -\frac{3}{2} \end{bmatrix}$.

(47) $\begin{bmatrix} \frac{1}{10} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$.
(48) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.
(49) $\begin{bmatrix} -\frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{5}{3} \end{bmatrix}$.

(50) $\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix}$.
(51) $\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.
(52) $\begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(53) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.
(54) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{7} \end{bmatrix}$.
(55) $\begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Section 3.2

(1) $\begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$.
(2) $\frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.
(3) Does not exist.

(4) $\frac{1}{11}\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$.
(5) $\begin{bmatrix} 2 & -3 \\ -5 & 8 \end{bmatrix}$.
(6) $\begin{bmatrix} 4 & -6 \\ -6 & 12 \end{bmatrix}$.

(7) $\frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$.
(8) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.
(9) $\begin{bmatrix} -1 & -1 & 1 \\ 6 & 5 & -4 \\ -3 & -2 & 2 \end{bmatrix}$.

(10) Does not exist.
(11) $\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ -5 & 2 & 0 \\ 1 & -2 & 2 \end{bmatrix}$.
(12) $\frac{1}{6}\begin{bmatrix} 3 & -1 & -8 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.

(13) $\begin{bmatrix} 9 & -5 & -2 \\ 5 & -3 & -1 \\ -36 & 21 & 8 \end{bmatrix}$.
(14) $\frac{1}{17}\begin{bmatrix} 1 & 7 & -2 \\ 7 & -2 & 3 \\ -2 & 3 & 4 \end{bmatrix}$.

(15) $\frac{1}{17}\begin{bmatrix} 14 & 5 & -6 \\ -5 & -3 & 7 \\ 13 & 1 & -8 \end{bmatrix}$.
(16) Does not exist.

(17) $\frac{1}{33}\begin{bmatrix} 5 & 3 & 1 \\ -6 & 3 & 12 \\ -8 & 15 & 5 \end{bmatrix}$.
(18) $\frac{1}{4}\begin{bmatrix} 0 & -4 & 4 \\ 1 & 5 & -4 \\ 3 & 7 & -8 \end{bmatrix}$.

(19) $\dfrac{1}{4}\begin{bmatrix} 4 & -4 & -4 & -4 \\ 0 & 4 & 2 & 5 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & -2 \end{bmatrix}$.   **(20)** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ -8 & 3 & \frac{1}{2} & 0 \\ -25 & 10 & 2 & -1 \end{bmatrix}$.

(21) Inverse of a nonsingular lower triangular matrix is lower triangular.
(22) Inverse of a nonsingular upper triangular matrix is upper triangular.
(23) 35 62 5 10 47 75 2 3 38 57 15 25 18 36.
(24) 14 116 10 20 −39 131 −3 5 −57 95 −5 45 36 72.
(25) 3 5 48 81 14 28 47 75 2 3 16 31 23 41
(26) HI THERE.   **(27)** THIS IS FUN.
(28) 24 13 27 19 28 9 0 1 1 24 10 24 18 0 18

Section 3.3

(1) $x = 1, y = -2$.   **(2)** $a = -3, b = 4$.   **(3)** $x = 2, y = -1$.
(4) $l = 1, p = 3$.   **(5)** Not possible; **A** is singular.
(6) $x = -8, y = 5, z = 3$.   **(7)** $x = y = z = 1$.
(8) $l = 1, m = -2, n = 0$.   **(9)** $r = 4.333, s = t = -1.667$.
(10) $r = 3.767, s = -1.133, t = -1.033$.   **(11)** Not possible; **A** is singular.
(12) $x = y = 1, z = 2$.   **(13)** $r = 500, s = 750$.   **(14)** $d = 30, b = 200$.
(15) $A = 5, B = 8, C = 6$.   **(16)** $B = \$19{,}012$.
(17) 80,000 barrels.   **(18)** $x_1 = 66{,}000$; $x_2 = 52{,}000$.
(19) $x_1 = 99{,}702$; $x_2 = 128{,}223$; $x_3 = 94{,}276$.

Section 3.4

(11) $\mathbf{A}^{-2} = \begin{bmatrix} 11 & -4 \\ -8 & 3 \end{bmatrix}$, $\mathbf{B}^{-2} = \begin{bmatrix} 9 & -20 \\ -4 & 9 \end{bmatrix}$.

(12) $\mathbf{A}^{-3} = \begin{bmatrix} 41 & -15 \\ -30 & 11 \end{bmatrix}$, $\mathbf{B}^{-3} = \begin{bmatrix} -38 & 85 \\ 17 & -38 \end{bmatrix}$.

(13) $\mathbf{A}^{-2} = \dfrac{1}{4}\begin{bmatrix} 22 & -10 \\ -15 & 7 \end{bmatrix}$, $\mathbf{B}^{-4} = \dfrac{1}{512}\begin{bmatrix} 47 & 15 \\ -45 & -13 \end{bmatrix}$.

(14) $\mathbf{A}^{-2} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$, $\mathbf{B}^{-2} = \begin{bmatrix} 1 & -4 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

(15) $\mathbf{A}^{-3} = \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{B}^{-3} = \begin{bmatrix} 1 & -6 & -9 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$

(16) $\dfrac{1}{125}\begin{bmatrix} -11 & -2 \\ 2 & -11 \end{bmatrix}.$

(17) First show that $(\mathbf{BA}^{-1})^{\mathrm{T}} = \mathbf{A}^{-1}\mathbf{B}^{\mathrm{T}}$ and that $(\mathbf{A}^{-1}\mathbf{B}^{\mathrm{T}})^{-1} = (\mathbf{B}^{\mathrm{T}})^{-1}\mathbf{A}$.

Section 3.5

(1) $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 10 \\ -9 \end{bmatrix}.$

(2) $\begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 1.5 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 8 \\ -5 \end{bmatrix}.$

(3) $\begin{bmatrix} 1 & 0 \\ 0.625 & 1 \end{bmatrix}\begin{bmatrix} 8 & 3 \\ 0 & 0.125 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} -400 \\ 1275 \end{bmatrix}.$

(4) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}.$

(5) $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 5 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}.$

(6) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & -6 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} -10 \\ 0 \\ 10 \end{bmatrix}.$

(7) $\begin{bmatrix} 1 & 0 & 0 \\ \frac{4}{3} & 1 & 0 \\ 1 & -\frac{21}{8} & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 & 1 \\ 0 & -\frac{8}{3} & -\frac{1}{3} \\ 0 & 0 & \frac{1}{8} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} 10 \\ -10 \\ 40 \end{bmatrix}.$

(8) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -0.75 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 0 & -4 & 3 \\ 0 & 0 & 4.25 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 79 \\ 1 \\ 1 \end{bmatrix}.$

(9) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 19 \\ -3 \\ 5 \end{bmatrix}$.

(10) $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & \frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ \frac{1}{2} \end{bmatrix}$.

(11) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 1 \\ -5 \\ 2 \\ 1 \end{bmatrix}$.

(12) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{2}{7} & \frac{5}{7} & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & \frac{7}{2} & \frac{5}{2} & -\frac{1}{2} \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & \frac{3}{7} \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 266.67 \\ -166.67 \\ 166.67 \\ 266.67 \end{bmatrix}$.

(13) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 10 \\ 10 \\ 10 \\ -10 \end{bmatrix}$.

(14) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & 1.5 & 1 & 0 \\ 0.5 & 0 & 0.25 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 8 & -8 \\ 0 & 0 & 0 & 3 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} -2.5 \\ -1.5 \\ 1.5 \\ 2.0 \end{bmatrix}$.

(15) (a) $x = 5, y = -2$; (b) $x = -5/7, y = 1/7$.

(16) (a) $x = 1, y = 0, z = 2$; (b) $x = 140, y = -50, z = -20$.

(17) (a) $\begin{bmatrix} 8 \\ -3 \\ -1 \end{bmatrix}$, (b) $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, (c) $\begin{bmatrix} 35 \\ 5 \\ 15 \end{bmatrix}$, (d) $\begin{bmatrix} -0.5 \\ 1.5 \\ 1.5 \end{bmatrix}$.

(18) (a) $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$, (b) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, (c) $\begin{bmatrix} 80 \\ 50 \\ -10 \\ 20 \end{bmatrix}$, (d) $\begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$.

(21) (d) $\mathbf{A}$ is singular.

Chapter 4

Section 4.1

(1) -2. **(2)** 38. **(3)** 38. **(4)** -2. **(5)** 82. **(6)** -82.
(7) 9. **(8)** -20. **(9)** 21. **(10)** 2. **(11)** 20. **(12)** 0.
(13) 0. **(14)** 0. **(15)** -93. **(16)** $4t - 6$. **(17)** $2t^2 + 6$.
(18) $5t^2$. **(19)** 0 and 2. **(20)** -1 and 4. **(21)** 2 and 3.
(22) $\pm\sqrt{6}$. **(23)** $\lambda^2 - 9\lambda - 2$. **(24)** $\lambda^2 - 9\lambda + 38$.
(25) $\lambda^2 - 13\lambda - 2$. **(26)** $\lambda^2 - 8\lambda + 9$. **(27)** $|\mathbf{A}||\mathbf{B}| = |\mathbf{AB}|$.
(28) They differ by a sign.
(29) The new determinants are the chosen constant times the old determinants, respectively.
(30) No change. **(31)** zero. **(32)** Identical. **(33)** Zero.

Section 4.2

(1) -6. **(2)** 22. **(3)** 0. **(4)** -9. **(5)** -33.
(6) 15. **(7)** -5. **(8)** -10. **(9)** 0. **(10)** 0.
(11) 0. **(12)** 119. **(13)** -8. **(14)** 22. **(15)** -7.
(16) -40. **(17)** 52. **(18)** 25. **(19)** 0. **(20)** 0.
(21) -11. **(22)** 0. **(23)** Product of diagonal elements.
(24) Always zero. **(25)** $-\lambda^3 + 7\lambda + 22$.
(26) $-\lambda^3 + 4\lambda^2 - 17\lambda$. **(27)** $-\lambda^3 + 6\lambda - 9$.
(28) $-\lambda^3 + 10\lambda^2 - 22\lambda - 33$.

Section 4.3

(2) For an upper triangular matrix, expand by the first column at each step.
(3) Use the third column to simply both the first and second columns.
(6) Factor the numbers -1, 2, 2, and 3 from the third row, second row, first column and second column respectively.
(7) Factor a five from the third row. Then use this new third row to simplify the second row and the new second row to simplify the first row.
(8) Interchange the second and third rows, and then transpose.

(9) Multiply the first row by 2, the second row by -1, and the second column by 2.

(10) Apply the third elementary row operation with the third row to make the first two rows identical.

(11) Multiply the first column by 1/2, the second column by 1/3, to obtain identical columns.

(13) $1 = \det(\mathbf{I}) = \det(\mathbf{A}\mathbf{A}^{-1}) = \det(\mathbf{A})\det(\mathbf{A}^{-1})$.

Section 4.4

(1) -1.	**(2)** 0.	**(3)** -311.	**(4)** -10.	**(5)** 0.
(6) -5.	**(7)** 0.	**(8)** 0.	**(9)** 119.	**(10)** -9.
(11) -33.	**(12)** 15.	**(13)** 2187.	**(14)** 52.	**(15)** 25.
(16) 0.	**(17)** 0.	**(18)** 152.	**(19)** 0.	**(20)** 0.

Section 4.5

(1) Does not exist.

(2) $\begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$.

(3) $\begin{bmatrix} 4 & -6 \\ -6 & 12 \end{bmatrix}$.

(4) $\dfrac{1}{11}\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$.

(5) $\begin{bmatrix} 2 & -3 \\ -5 & 8 \end{bmatrix}$.

(6) Does not exist.

(7) $\dfrac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$.

(8) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

(9) $\begin{bmatrix} -1 & -1 & 1 \\ 6 & 5 & -4 \\ -3 & -2 & 2 \end{bmatrix}$.

(10) Does not exist.

(11) $\dfrac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ -5 & 2 & 0 \\ 1 & -2 & 2 \end{bmatrix}$.

(12) $\dfrac{1}{17}\begin{bmatrix} 14 & 5 & -6 \\ -5 & -3 & 7 \\ 13 & 1 & -8 \end{bmatrix}$.

(13) Does not exist.

(14) $\dfrac{1}{33}\begin{bmatrix} 5 & 3 & 1 \\ -6 & 3 & 12 \\ -8 & 15 & 5 \end{bmatrix}$.

(15) $\dfrac{1}{4}\begin{bmatrix} 0 & -4 & 4 \\ 1 & 5 & -4 \\ 3 & 7 & -8 \end{bmatrix}$.

(16) $\dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

(17) $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.

(19) Equals the number of rows in the matrix.

Section 4.6

(1) $x = 1, y = -2.$ **(2)** $x = 3, y = -3.$ **(3)** $a = 10/11, b = -20/11.$
(4) $s = 50, t = 30.$ **(5)** Determinant of coefficient matrix is zero.
(6) System is not square. **(7)** $x = 10, y = z = 5.$
(8) $x = 1, y = -4, z = 5.$ **(9)** $x = y = 1, z = 2.$ **(10)** $a = b = c = 1.$
(11) Determinant of coefficient matrix is zero. **(12)** $r = 3, s = -2, t = 3.$
(13) $x = 1, y = 2, z = 5, w = -3.$

Chapter 5

Section 5.1

(1) (a), (d), (e), (f), and (h). **(2)** (a) 3, (d) 5, (e) 3, (f) 3, (h) 5.
(3) (c), (e), (f), and (g). **(4)** (c) 0, (e) 0, (f) -4, (g) -4.
(5) (b), (c), (d), (e), and (g). **(6)** (b) 2, (c) 1, (d) 1, (e) 3, (g) 3.
(7) (a), (b), and (d). **(8)** (a) -2, (b) -1, (d) 2.

Section 5.2

(1) 2, 3.	**(2)** 1, 4.	**(3)** 0, 8.	**(4)** -3, 12.
(5) 3, 3.	**(6)** 3, -3.	**(7)** $\pm\sqrt{34}$.	**(8)** $\pm 4i$.
(9) $\pm i$.	**(10)** 1, 1.	**(11)** 0, 0.	**(12)** 0, 0.
(13) $\pm\sqrt{2}$.	**(14)** 10, -11.	**(15)** -10, 11.	**(16)** $t, -2t$.
(17) $2t, 2t$.	**(18)** $2\theta, 3\theta$.	**(19)** 2, 4, -2.	**(20)** 1, 2, 3.
(21) 1, 1, 3.	**(22)** 0, 2, 2.	**(23)** 2, 3, 9.	**(24)** 1, -2, 5.
(25) 2, 3, 6.	**(26)** 0, 0, 14.	**(27)** 0, 10, 14.	**(28)** 2, 2, 5.
(29) 0, 0, 6.	**(30)** 3, 3, 9.	**(31)** 3, $\pm 2i$.	**(32)** 0, $\pm i$.
(33) 3, 3, 3.	**(34)** 2, 4, 1, $\pm i\sqrt{5}$.		**(35)** 1, 1, 2, 2.

Section 5.3

(1) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$ **(2)** $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$ **(3)** $\begin{bmatrix} 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$

(4) $\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}$. **(5)** $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-2\end{bmatrix}$. **(6)** $\begin{bmatrix}-5\\3-\sqrt{34}\end{bmatrix}, \begin{bmatrix}-5\\3+\sqrt{34}\end{bmatrix}$.

(7) $\begin{bmatrix}-5\\3-4i\end{bmatrix}, \begin{bmatrix}-5\\3+4i\end{bmatrix}$. **(8)** $\begin{bmatrix}-5\\2-i\end{bmatrix}, \begin{bmatrix}-5\\2+i\end{bmatrix}$.

(9) $\begin{bmatrix}-2-\sqrt{2}\\1\end{bmatrix}, \begin{bmatrix}-2+\sqrt{2}\\1\end{bmatrix}$. **(10)** $\begin{bmatrix}5\\3\end{bmatrix}, \begin{bmatrix}2\\-3\end{bmatrix}$. **(11)** $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix}$.

(12) $\begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}-2\\1\end{bmatrix}$. **(13)** $\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}$. **(14)** $\begin{bmatrix}1\\-4\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}$.

(15) $\begin{bmatrix}1\\-4\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}$, **(16)** $\begin{bmatrix}0\\-1\\1\end{bmatrix}, \begin{bmatrix}3\\-2\\0\end{bmatrix}, \begin{bmatrix}0\\4\\3\end{bmatrix}$. **(17)** $\begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$.

(18) $\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-2\end{bmatrix}$. **(19)** $\begin{bmatrix}9\\1\\13\end{bmatrix}, \begin{bmatrix}5\\-1+2i\\0\end{bmatrix}, \begin{bmatrix}5\\-1-2i\\0\end{bmatrix}$

(20) $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\-i\\1\end{bmatrix}, \begin{bmatrix}-1\\i\\1\end{bmatrix}$. **(21)** $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}$.

(22) $\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\3\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\2\\1\end{bmatrix}, \begin{bmatrix}0\\0\\-2\\1\end{bmatrix}$. **(23)** $\begin{bmatrix}10\\-6\\11\\4\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}2\\0\\1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\1\\-1\end{bmatrix}$.

(24) $\begin{bmatrix}2/\sqrt{5}\\1/\sqrt{5}\end{bmatrix}, \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$. **(25)** $\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}, \begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\end{bmatrix}$.

(26) $\begin{bmatrix}3/\sqrt{13}\\-2/\sqrt{13}\end{bmatrix}, \begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\end{bmatrix}$. **(27)** $\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}, \begin{bmatrix}-1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix}$.

(28) $\begin{bmatrix}1/\sqrt{18}\\-4/\sqrt{18}\\1/\sqrt{18}\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}-1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix}$. **(29)** $\begin{bmatrix}0\\-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}, \begin{bmatrix}3/\sqrt{13}\\-2/\sqrt{13}\\0\end{bmatrix}, \begin{bmatrix}0\\4/5\\3/5\end{bmatrix}$.

(30) $[1 \quad -1], [-1 \quad 2]$. **(31)** $[-2 \quad 1], [1 \quad 1]$. **(32)** $[-2 \quad 1], [2 \quad 3]$.
(33) $[-3 \quad 2], [1 \quad 1]$. **(34)** $[1 \quad -2 \quad 1], [1 \quad 0 \quad 1], [-1 \quad 0 \quad 1]$.
(35) $[1 \quad 0 \quad 1], [2 \quad 1 \quad 2], [-1 \quad 0 \quad 1]$.

(36) $[-2 \quad -3 \quad 4], [1 \quad 0 \quad 0], [2 \quad 3 \quad 3]$.
(37) $[1 \quad -1 \quad 0], [1 \quad 1 \quad 1], [1 \quad 1 \quad -2]$.
(38) $\mathbf{Ax} = \lambda\mathbf{x}$, so $(\mathbf{Ax})^{\mathrm{T}} = (\lambda\mathbf{x})^{\mathrm{T}}$, and $\mathbf{x}^{\mathrm{T}}\mathbf{A} = \lambda\mathbf{x}^{\mathrm{T}}$. **(39)** $[\frac{1}{2} \quad \frac{1}{2}]$.
(40) $[\frac{2}{5} \quad \frac{3}{5}]$. **(41)** $[\frac{1}{8} \quad \frac{2}{8} \quad \frac{5}{8}]$. **(42)** (a) $[\frac{1}{6} \quad \frac{5}{6}]$. (b) $\frac{1}{6}$.
(43) $[7/11 \quad 4/11]$; probability of having a Republican is $7/11 = 0.636$.
(44) $[23/120 \quad 71/120 \quad 26/120]$; probability of a good harvest is $26/120 = 0.217$.
(45) $[40/111 \quad 65/111 \quad 6/111]$; probability of a person using brand Y is $65/111 = 0.586$

Section 5.4

(1) 9. **(2)** 9.2426. **(3)** $5 + 8 + \lambda = -4$, $\lambda = -17$.

(4) $(5)(8)\lambda = -4$, $\lambda = -0.1$ **(5)** Their product is -24.

(6) (a) $-6, 8$; (b) $-15, 20$; (c) $-6, 1$; (d) $1, 8$.

(7) (a) $4, 4, 16$; (b) $-8, 8, 64$; (c) $6, -6, -12$; (d) $1, 5, 7$.

(8) (a) $2\mathbf{A}$, (b) $5\mathbf{A}$, (c) $\mathbf{A}^2$, (d) $\mathbf{A} + 3\mathbf{I}$.

(9) (a) $2\mathbf{A}$, (b) $\mathbf{A}^2$, (c) $\mathbf{A}^3$, (d) $\mathbf{A} - 2\mathbf{I}$.

Section 5.5

(1) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. **(2)** $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. **(3)** $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. **(4)** $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

(5) $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. **(6)** $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$. **(7)** $\begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

(8) $\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. **(9)** $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. **(10)** $\begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$.

(11) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. **(12)** $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. **(13)** $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. **(14)** $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

(15) $\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. **(16)** $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$.

Section 5.6

(1)

Iteration	Eigenvector components		Eigenvalue
0	1.0000	1.0000	
1	0.6000	1.0000	5.0000
2	0.5238	1.0000	4.2000
3	0.5059	1.0000	4.0476
4	0.5015	1.0000	4.0118
5	0.5004	1.0000	4.0029

(2)

Iteration	Eigenvector components		Eigenvalue
0	1.0000	1.0000	
1	0.5000	1.0000	10.0000
2	0.5000	1.0000	8.0000
3	0.5000	1.0000	8.0000

(3)

Iteration	Eigenvector components		Eigenvalue
0	1.0000	1.0000	
1	0.6000	1.0000	15.0000
2	0.6842	1.0000	11.4000
3	0.6623	1.0000	12.1579
4	0.6678	1.0000	11.9610
5	0.6664	1.0000	12.0098

(4)

Iteration	Eigenvector components		Eigenvalue
0	1.0000	1.0000	
1	0.5000	1.0000	2.0000
2	0.2500	1.0000	4.0000
3	0.2000	1.0000	5.0000
4	0.1923	1.0000	5.2000
5	0.1912	1.0000	5.2308

(5)

Iteration	Eigenvector components		Eigenvalue
0	1.0000	1.0000	
1	1.0000	0.6000	10.0000
2	1.0000	0.5217	9.2000
3	1.0000	0.5048	9.0435
4	1.0000	0.5011	9.0096
5	1.0000	0.5002	9.0021

(6)

Iteration	Eigenvector components		Eigenvalue
0	1.0000	1.0000	
1	1.0000	0.4545	11.0000
2	1.0000	0.4175	9.3636
3	1.0000	0.4145	9.2524
4	1.0000	0.4142	9.2434
5	1.0000	0.4142	9.2427

(7)

Iteration	Eigenvector components			Eigenvalue
0	1.0000	1.0000	1.0000	
1	0.2500	1.0000	0.8333	12.0000
2	0.0763	1.0000	0.7797	9.8333
3	0.0247	1.0000	0.7605	9.2712
4	0.0081	1.0000	0.7537	9.0914
5	0.0027	1.0000	0.7513	9.0310

(8)

Iteration	Eigenvector components			Eigenvalue
0	1.0000	1.0000	1.0000	
1	0.6923	0.6923	1.0000	13.0000
2	0.5586	0.7241	1.0000	11.1538
3	0.4723	0.6912	1.0000	11.3448
4	0.4206	0.6850	1.0000	11.1471
5	0.3883	0.6774	1.0000	11.1101

(9)

Iteration	Eigenvector components			Eigenvalue
0	1.0000	1.0000	1.0000	
1	0.4000	0.7000	1.0000	20.0000
2	0.3415	0.6707	1.0000	16.4000
3	0.3343	0.6672	1.0000	16.0488
4	0.3335	0.6667	1.0000	16.0061
5	0.3333	0.6667	1.0000	16.0008

(10)

Iteration	Eigenvector components			Eigenvalue
0	1.0000	1.0000	1.0000	
1	0.4000	1.0000	0.3000	−20.0000
2	1.0000	0.7447	0.0284	−14.1000
3	0.5244	1.0000	−0.3683	−19.9504
4	1.0000	0.7168	−0.5303	−18.5293
5	0.6814	1.0000	−0.7423	−20.3976

(11) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a linear combinations of $\begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, which are eigenvectors corresponding to $\lambda = 1$ and $\lambda = 2$, not $\lambda = 3$. Thus, the power method converges to $\lambda = 2$.

(12) There is no single dominant eigenvalue. Here, $|\lambda_1| = |\lambda_2| = \sqrt{34}$.

(13) Shift by $\lambda = 4$. Power method on $\mathbf{A} = \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix}$ converges after three iterations to $\mu = -3$. $\lambda + \mu = 1$.

(14) Shift by $\lambda = 16$. Power method on $\mathbf{A} = \begin{bmatrix} -13 & 2 & 3 \\ 2 & -10 & 6 \\ 3 & 6 & -5 \end{bmatrix}$ converges after three iterations to $\mu = -14$. $\lambda + \mu = 2$.

(15)

Iteration	Eigenvector components		Eigenvalue
0	1.0000	1.0000	
1	−0.3333	1.0000	0.6000
2	1.0000	−0.7778	0.6000
3	−0.9535	1.0000	0.9556
4	1.0000	−0.9904	0.9721
5	−0.9981	1.0000	0.9981

(16)

Iteration	Eigenvector components		Eigenvalue
0	1.0000	−0.5000	
1	−0.8571	1.0000	0.2917
2	1.0000	−0.9615	0.3095
3	−0.9903	1.0000	0.3301
4	1.0000	−0.9976	0.3317
5	−0.9994	1.0000	0.3331

(17)

Iteration	Eigenvector components		Eigenvalue
0	1.0000	1.0000	
1	0.2000	1.0000	0.2778
2	−0.1892	1.0000	0.4111
3	−0.2997	1.0000	0.4760
4	−0.3258	1.0000	0.4944
5	−0.3316	1.0000	0.4987

(18)

Iteration	Eigenvector components		Eigenvalue
0	1.0000	1.0000	
1	−0.2000	1.0000	0.7143
2	−0.3953	1.0000	1.2286
3	−0.4127	1.0000	1.3123
4	−0.4141	1.0000	1.3197
5	−0.4142	1.0000	1.3203

(19)

Iteration	Eigenvector components			Eigenvalue
0	1.0000	1.0000	1.0000	
1	1.0000	0.4000	−0.2000	0.3125
2	1.0000	0.2703	−0.4595	0.4625
3	1.0000	0.2526	−0.4949	0.4949
4	1.0000	0.2503	−0.4994	0.4994
5	1.0000	0.2500	−0.4999	0.4999

(20)

Iteration	Eigenvector components			Eigenvalue
0	1.0000	1.0000	1.0000	
1	0.3846	1.0000	0.9487	−0.1043
2	0.5004	0.7042	1.0000	−0.0969
3	0.3296	0.7720	1.0000	−0.0916
4	0.3857	0.6633	1.0000	−0.0940
5	0.3244	0.7002	1.0000	−0.0907

(21)

Iteration	Eigenvector components			Eigenvalue
0	1.0000	1.0000	1.0000	
1	−0.6667	1.0000	−0.6667	−1.5000
2	−0.3636	1.0000	−0.3636	1.8333
3	−0.2963	1.0000	−0.2963	1.2273
4	−0.2712	1.0000	−0.2712	1.0926
5	−0.2602	1.0000	−0.2602	1.0424

(22) Cannot construct an **LU** decomposition. Shift as explained in Problem 13.

(23) Cannot solve $\mathbf{L}\mathbf{x}_1 = \mathbf{y}$ uniquely for $\mathbf{x}_1$ because one eigenvalue is zero. Shift as explained in Problem 13.

(24) Yes, on occasion.

(25) Inverse power method applied to $\mathbf{A} = \begin{bmatrix} -7 & 2 & 3 \\ 2 & -4 & 6 \\ 3 & 6 & 1 \end{bmatrix}$ converges to $\mu = 1/6$. $\lambda + 1/\mu = 10 + 6 = 16$.

(26) Inverse power method applied to $\mathbf{A} = \begin{bmatrix} 27 & -17 & 7 \\ -17 & 21 & 1 \\ 7 & 1 & 11 \end{bmatrix}$ converges to $\mu = 1/3$. $\lambda + 1/\mu = -25 + 3 = -22$.

Chapter 6

Section 6.1

(1) 11, 5. **(2)** 8, 4. **(3)** −50, 74. **(4)** 63, 205. **(5)** 64, 68.
(6) 6, 5. **(7)** 26, 24. **(8)** −30, 38. **(9)** 5/6, 7/18.
(10) $5/\sqrt{6}$, 1. **(11)** 7/24, 1/3. **(12)** 0, 1400. **(13)** 2, 3.
(14) 1, 1. **(15)** −19, 147. **(16)** −1/5, 1/5. **(17)** undefined, 6.

(18) $\begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$. **(19)** $\begin{bmatrix} 4/\sqrt{41} \\ -5/\sqrt{41} \end{bmatrix}$. **(20)** $[7/\sqrt{58} \quad 3/\sqrt{58}]$.

(21) $\begin{bmatrix} -4/\sqrt{34} \\ 3/\sqrt{34} \\ -3/\sqrt{34} \end{bmatrix}$. **(22)** $\begin{bmatrix} 3/\sqrt{17} \\ -2/\sqrt{17} \\ -2/\sqrt{17} \end{bmatrix}$. **(23)** $\begin{bmatrix} 2/\sqrt{21} \\ 4/\sqrt{21} \\ 1/\sqrt{21} \end{bmatrix}$.

(24) $\begin{bmatrix} 4/\sqrt{197} \\ -6/\sqrt{197} \\ -9/\sqrt{197} \\ 8/\sqrt{197} \end{bmatrix}$. **(25)** $[1/\sqrt{55} \quad 2/\sqrt{55} \quad -3/\sqrt{55} \quad 4/\sqrt{55} \quad -5/\sqrt{55}]$.

(26) $[-3/\sqrt{259} \quad 8/\sqrt{259} \quad 11/\sqrt{259} \quad -4/\sqrt{259} \quad 7/\sqrt{259}]$.

(27) No vector $\mathbf{x}$ exists. **(28)** Yes, see Problem 12.

(33) $\|\mathbf{x}+\mathbf{y}\|^2 = \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}\rangle = \langle \mathbf{x},\mathbf{x}\rangle + 2\langle \mathbf{x},\mathbf{y}\rangle + \langle \mathbf{y},\mathbf{y}\rangle$
$= \|\mathbf{x}\|^2 + 2\langle \mathbf{x},\mathbf{y}\rangle + \|\mathbf{y}\|^2.$

(34) Show that $\|\mathbf{x}-\mathbf{y}\|^2 = \|\mathbf{x}\|^2 - 2\langle \mathbf{x},\mathbf{y}\rangle + \|\mathbf{y}\|^2$, and then use Problem 33.

(37) Note that $\langle \mathbf{x},\mathbf{y}\rangle \le |\langle \mathbf{x},\mathbf{y}\rangle|$. **(38)** $\langle \mathbf{x},\mathbf{y}\rangle = \det(\mathbf{x}^{\mathrm{T}}\mathbf{y})$.

(40) 145. **(41)** 27. **(42)** 32.

Section 6.2

(1) $\mathbf{x}$ and $\mathbf{y}$, $\mathbf{x}$ and $\mathbf{u}$, $\mathbf{y}$ and $\mathbf{v}$, $\mathbf{u}$ and $\mathbf{v}$.

(2) $\mathbf{x}$ and $\mathbf{z}$, $\mathbf{x}$ and $\mathbf{u}$, $\mathbf{y}$ and $\mathbf{u}$, $\mathbf{z}$ and $\mathbf{u}$, $\mathbf{y}$ and $\mathbf{v}$. **(3)** $-20/3$.

(4) -4. **(5)** 0.5. **(6)** $x = -3y$. **(7)** $x = 1, y = -2$.

(8) $x = y = -z$. **(9)** $x = y = -z;\ z = \pm 1/\sqrt{3}$.

(10) $\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$. **(11)** $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

(12) $\begin{bmatrix} 3/\sqrt{13} \\ -2/\sqrt{13} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{13} \\ 3/\sqrt{13} \end{bmatrix}$. **(13)** $\begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$.

(14) $\begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}, \begin{bmatrix} -2/\sqrt{45} \\ 4/\sqrt{45} \\ 5/\sqrt{45} \end{bmatrix}, \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$. **(15)** $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$.

(16) $\begin{bmatrix} 0 \\ 3/5 \\ 4/5 \end{bmatrix}, \begin{bmatrix} 3/5 \\ 16/25 \\ -12/25 \end{bmatrix}, \begin{bmatrix} 4/5 \\ -12/25 \\ 9/25 \end{bmatrix}$.

(17) $\begin{bmatrix} 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 3/\sqrt{15} \\ -2/\sqrt{15} \\ 1/\sqrt{15} \\ 1/\sqrt{15} \end{bmatrix}, \begin{bmatrix} 3/\sqrt{35} \\ 3/\sqrt{35} \\ -4/\sqrt{35} \\ 1/\sqrt{35} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{7} \\ 1/\sqrt{7} \\ 1/\sqrt{7} \\ -2/\sqrt{7} \end{bmatrix}$.

(18) $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ -1/\sqrt{3} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$

(23) $\|\mathbf{x}-\mathbf{y}\|^2 = \langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y}\rangle = \|\mathbf{x}\|^2 - 2\langle \mathbf{x},\mathbf{y}\rangle + \|\mathbf{y}\|^2$.

(24) $\|s\mathbf{x}+t\mathbf{y}\|^2 = \langle s\mathbf{x}-t\mathbf{y}, s\mathbf{x}-t\mathbf{y}\rangle = \|s\mathbf{x}\|^2 - 2st\langle \mathbf{x},\mathbf{y}\rangle + \|t\mathbf{y}\|^2$.

(25) $\mathbf{I}$. **(26)** Set $\mathbf{y} = \mathbf{x}$ and use Property (I1) of Section 6.1.

(28) Denote the columns of $\mathbf{A}$ as $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$, and the elements of $\mathbf{y}$ as $y_1, y_2, \ldots, y_n$, respectively. Then, $\mathbf{Ay} = \mathbf{A}_1 y_1 + \mathbf{A}_2 y_2 + \cdots + \mathbf{A}_n y_n$ and $\langle \mathbf{Ay}, \mathbf{p}\rangle = y_1\langle \mathbf{A}_1, \mathbf{p}\rangle + y_2\langle \mathbf{A}_2, \mathbf{p}\rangle + \cdots + y_n\langle \mathbf{A}_n, \mathbf{p}\rangle$.

Section 6.3

(1) (a) $\theta = 36.9^\circ$, (b) $\begin{bmatrix} 1.6 \\ 0.8 \end{bmatrix}$, (c) $\begin{bmatrix} -0.6 \\ 1.2 \end{bmatrix}$.

(2) (a) $\theta = 14.0^\circ$, (b) $\begin{bmatrix} 0.7059 \\ 1.1765 \end{bmatrix}$, (c) $\begin{bmatrix} 0.2941 \\ -0.1765 \end{bmatrix}$.

(3) (a) $\theta = 78.7^\circ$, (b) $\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$, (c) $\begin{bmatrix} 2.5 \\ -2.5 \end{bmatrix}$.

(4) (a) $\theta = 90^\circ$, (b) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, (c) $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$.

(5) (a) $\theta = 118.5^\circ$, (b) $\begin{bmatrix} -0.7529 \\ -3.3882 \end{bmatrix}$, (c) $\begin{bmatrix} -6.2471 \\ 1.3882 \end{bmatrix}$.

(6) (a) $\theta = 50.8^\circ$, (b) $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, (c) $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

(7) (a) $\theta = 19.5^\circ$, (b) $\begin{bmatrix} 8/9 \\ 8/9 \\ 4/9 \end{bmatrix}$, (c) $\begin{bmatrix} 1/9 \\ 1/9 \\ -4/9 \end{bmatrix}$.

(8) (a) $\theta = 17.7^\circ$, (b) $\begin{bmatrix} 1.2963 \\ 3.2407 \\ 3.2407 \end{bmatrix}$, (c) $\begin{bmatrix} -1.2963 \\ -0.2407 \\ 0.7593 \end{bmatrix}$.

(9) (a) $\theta = 48.2°$, (b) $\begin{bmatrix} 2/3 \\ 2/3 \\ 2/3 \\ 0 \end{bmatrix}$, (c) $\begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \\ 1 \end{bmatrix}$.

(10) (a) $\theta = 121.4°$, (b) $\begin{bmatrix} -7/6 \\ 7/3 \\ 0 \\ 7/6 \end{bmatrix}$, (c) $\begin{bmatrix} 13/6 \\ -1/3 \\ 3 \\ 17/6 \end{bmatrix}$.

(11) $\begin{bmatrix} 0.4472 & 0.8944 \\ 0.8944 & -0.4472 \end{bmatrix}\begin{bmatrix} 2.2361 & 1.7889 \\ 0.0000 & 1.3416 \end{bmatrix}$.

(12) $\begin{bmatrix} 0.7071 & -0.7071 \\ 0.7071 & 0.7071 \end{bmatrix}\begin{bmatrix} 1.4142 & 5.6569 \\ 0.0000 & 1.4142 \end{bmatrix}$.

(13) $\begin{bmatrix} 0.8321 & 0.5547 \\ -0.5547 & 0.8321 \end{bmatrix}\begin{bmatrix} 3.6056 & 0.8321 \\ 0.0000 & 4.1603 \end{bmatrix}$.

(14) $\begin{bmatrix} 0.3333 & 0.8085 \\ 0.6667 & 0.1617 \\ 0.6667 & -0.5659 \end{bmatrix}\begin{bmatrix} 3.0000 & 2.6667 \\ 0.0000 & 1.3744 \end{bmatrix}$.

(15) $\begin{bmatrix} 0.3015 & -0.2752 \\ 0.3015 & -0.8808 \\ 0.9045 & 0.3853 \end{bmatrix}\begin{bmatrix} 3.3166 & 4.8242 \\ 0.0000 & 1.6514 \end{bmatrix}$.

(16) $\begin{bmatrix} 0.7746 & 0.4034 \\ -0.5164 & 0.5714 \\ 0.2582 & 0.4706 \\ -0.2582 & 0.5378 \end{bmatrix}\begin{bmatrix} 3.8730 & 0.2582 \\ 0.0000 & 1.9833 \end{bmatrix}$.

(17) $\begin{bmatrix} 0.8944 & -0.2981 & 0.3333 \\ 0.4472 & 0.5963 & -0.6667 \\ 0.0000 & 0.7454 & 0.6667 \end{bmatrix}\begin{bmatrix} 2.2361 & 0.4472 & 1.7889 \\ 0.0000 & 1.3416 & 0.8944 \\ 0.0000 & 0.0000 & 2.0000 \end{bmatrix}$.

(18) $\begin{bmatrix} 0.7071 & 0.5774 & -0.4082 \\ 0.7071 & -0.5774 & 0.4082 \\ 0.0000 & 0.5774 & 0.8165 \end{bmatrix}\begin{bmatrix} 1.4142 & 1.4142 & 2.8284 \\ 0.0000 & 1.7321 & 0.5774 \\ 0.0000 & 0.0000 & 0.8165 \end{bmatrix}$.

(19) $\begin{bmatrix} 0.00 & 0.60 & 0.80 \\ 0.60 & 0.64 & -0.48 \\ 0.80 & -0.48 & 0.36 \end{bmatrix}\begin{bmatrix} 5 & 3 & 7 \\ 0 & 5 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

(20) $\begin{bmatrix} 0.0000 & 0.7746 & 0.5071 \\ 0.5774 & -0.5164 & 0.5071 \\ 0.5774 & 0.2582 & -0.6761 \\ 0.5774 & 0.2582 & 0.1690 \end{bmatrix} \begin{bmatrix} 1.7321 & 1.1547 & 1.1547 \\ 0.0000 & 1.2910 & 0.5164 \\ 0.0000 & 0.0000 & 1.1832 \end{bmatrix}.$

(21) $\begin{bmatrix} 0.7071 & -0.4082 & 0.5774 \\ 0.7071 & 0.4082 & -0.5774 \\ 0.0000 & -0.8165 & -0.5774 \\ 0.0000 & 0.0000 & 0.0000 \end{bmatrix} \begin{bmatrix} 1.4142 & 0.7071 & 0.7071 \\ 0.0000 & 1.2247 & 0.4082 \\ 0.0000 & 0.0000 & 1.1547 \end{bmatrix}.$

(24) $\mathbf{QR} \neq \mathbf{A}$.

Section 6.4

(1) $\mathbf{A}_1 = \mathbf{R}_0\mathbf{Q}_0 + 7\mathbf{I}$

$$= \begin{bmatrix} 19.3132 & -1.2945 & 0.0000 \\ 0.0000 & 7.0231 & -0.9967 \\ 0.0000 & 0.0000 & 0.0811 \end{bmatrix} \begin{bmatrix} -0.3624 & 0.0756 & 0.9289 \\ 0.0000 & -0.9967 & 0.0811 \\ 0.9320 & 0.0294 & 0.3613 \end{bmatrix}$$

$$+ 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.0000 & 2.7499 & 17.8357 \\ -0.9289 & -0.0293 & 0.2095 \\ 0.0756 & 0.0024 & 7.0293 \end{bmatrix}.$$

(2) $\mathbf{A}_1 = \mathbf{R}_0\mathbf{Q}_0 - 14\mathbf{I}$

$$= \begin{bmatrix} 24.3721 & -17.8483 & 3.8979 \\ 0.0000 & 8.4522 & -4.6650 \\ 0.0000 & 0.0000 & 3.6117 \end{bmatrix} \begin{bmatrix} 0.6565 & -0.6250 & 0.4223 \\ -0.6975 & -0.2898 & 0.6553 \\ 0.2872 & 0.7248 & 0.6262 \end{bmatrix}$$

$$-14\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 15.5690 & -7.2354 & 1.0373 \\ -7.2354 & -19.8307 & 2.6178 \\ 1.0373 & 2.6178 & -11.7383 \end{bmatrix}.$$

(3) Shift by 4.

$$\mathbf{R}_0 = \begin{bmatrix} 4.1231 & -0.9701 & 0.0000 & 13.5820 \\ 0.0000 & 4.0073 & -0.9982 & -4.1982 \\ 0.0000 & 0.0000 & 4.0005 & 12.9509 \\ 0.0000 & 0.0000 & 0.0000 & 3.3435 \end{bmatrix},$$

$$\mathbf{Q}_0 = \begin{bmatrix} -0.9701 & -0.2349 & -0.0586 & -0.0151 \\ 0.2425 & -0.9395 & -0.2344 & -0.0605 \\ 0.0000 & 0.2495 & -0.9376 & -0.2421 \\ 0.0000 & 0.0000 & 0.2500 & -0.9683 \end{bmatrix}.$$

$$\mathbf{A}_1 = \mathbf{R}_0\mathbf{Q}_0 + 4\mathbf{I} = \begin{bmatrix} -0.2353 & -0.0570 & 3.3809 & -13.1545 \\ 0.9719 & -0.0138 & -1.0529 & 4.0640 \\ 0.0000 & 0.9983 & 3.4864 & -13.5081 \\ 0.0000 & 0.0000 & 0.8358 & 0.7626 \end{bmatrix}.$$

(4) $7.2077, -0.1039 \pm 1.5769i$. **(5)** $-11, -22, 17$. **(6)** $2, 3, 9$.

(7) Method fails. $\mathbf{A}_0 - 7\mathbf{I}$ does not have linearly independent columns, so no **QR**-decomposition is possible.

(8) $2, 2, 16$. **(9)** $1, 3, 3$. **(10)** $2, 3 \pm i$ **(11)** $1, \pm i$.

(12) $\pm i, 2 \pm 3i$. **(13)** $3.1265 \pm 1.2638i, -2.6265 \pm 0.7590i$.

(14) $0.0102, 0.8431, 3.8581, 30.2887$.

Section 6.5

(1) $x = 2.225, y = 1.464$. **(2)** $x = 3.171, y = 2.286$.

(3) $x = 9.879, y = 18.398$. **(4)** $x = -1.174, y = 8.105$.

(5) $x = 1.512, y = 0.639, z = 0.945$. **(6)** $x = 7.845, y = 1.548, z = 5.190$.

(7) $x = 81.003, y = 50.870, z = 38.801$.

(8) $x = 2.818, y = -0.364, z = -1.364$. **(9)** 2 and 4.

(10) (b) $y = 2.3x + 8.1$, (c) 21.9.

(11) (b) $y = -2.6x + 54.4$, (c) 31 in week 9, 28 in week 10.

(12) (b) $y = 0.27x + 10.24$, (c) 12.4.

(13) $$m = \frac{N\sum_{i=1}^{N} x_i y_i - \sum_{i=1}^{N} x_i \sum_{i=1}^{N} y_i}{N\sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i\right)^2}, \quad c = \frac{\sum_{i=1}^{N} y_i \sum_{i=1}^{N} x_i^2 - \sum_{i=1}^{N} x_i \sum_{i=1}^{N} x_i y_i}{N\sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i\right)^2}.$$

If $N\sum_{i=1}^{N} x_i^2$ is near $(\sum_{i=1}^{N} x_i)^2$, then the denominator is near zero.

(14) $\sum_{i=1}^{N} x_i' = 0$, so the denominator for m and c as suggested in Problem 13 is simply $N\sum_{i=1}^{N}(x_i')^2$.

(15) $y = 2.3x' + 15$. **(16)** $y = -2.6x' + 42.9$.

(17) (a) $y = -0.198x' + 21.18$, (b) Year 2000 is coded as $x' = 30$; $y(30) = 15.2$.

(23) $\mathbf{E} = \begin{bmatrix} 0.841 \\ 0.210 \\ -2.312 \end{bmatrix}$. **(24)** $\mathbf{E} = \begin{bmatrix} 0.160 \\ 0.069 \\ -0.042 \\ -0.173 \end{bmatrix}$.

Chapter 7

Section 7.1

(1) (a) $\begin{bmatrix} 0 & -4 & 8 \\ 0 & 4 & -8 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 8 & -16 \\ 0 & -8 & 16 \\ 0 & 0 & 0 \end{bmatrix}$; (b) $\begin{bmatrix} 57 & 78 \\ 117 & 174 \end{bmatrix}, \begin{bmatrix} 234 & 348 \\ 522 & 756 \end{bmatrix}$.

(2) $p_k(\mathbf{A}) = \begin{bmatrix} p_k(\lambda_1) & 0 & 0 \\ 0 & p_k(\lambda_2) & 0 \\ 0 & 0 & p_k(\lambda_3) \end{bmatrix}$. **(4)** In general, $\mathbf{AB} \neq \mathbf{BA}$.

(5) Yes. **(6)** $\begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}$. **(7)** $\begin{bmatrix} 2 \\ 0 \\ 3/2 \end{bmatrix}$.

(8) 2–2 element tends to ∞, so limit diverges. **(9)** a, b, d, and f.

(10) f. **(11)** All except c. **(13)** $\begin{bmatrix} e & 0 \\ 0 & e^2 \end{bmatrix}$. **(14)** $\begin{bmatrix} e^{-1} & 0 \\ 0 & e^{28} \end{bmatrix}$.

(15) $\begin{bmatrix} e^2 & 0 & 0 \\ 0 & e^{-2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. **(16)** $\sin(\mathbf{A}) = \begin{bmatrix} \sin(\lambda_1) & 0 & \cdots & 0 \\ 0 & \sin(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sin(\lambda_n) \end{bmatrix}$.

(17) $\begin{bmatrix} \sin(1) & 0 \\ 0 & \sin(2) \end{bmatrix}$. **(18)** $\begin{bmatrix} \sin(-1) & 0 \\ 0 & \sin(28) \end{bmatrix}$.

(19) $\cos \mathbf{A} = \sum_{k=0}^{\infty} \dfrac{(-1)^k \mathbf{A}^{2k}}{(2k)!}$, $\cos \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \cos(1) & 0 \\ 0 & \cos(2) \end{bmatrix}$.

(20) $\begin{bmatrix} \cos(2) & 0 & 0 \\ 0 & \cos(-2) & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Section 7.2

(1) $\mathbf{A}^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$. **(2)** Since $a_0 = 0$, the inverse does not exist.

(3) Since $a_0 = 0$, the inverse does not exist.

(4) $\mathbf{A}^{-1} = \begin{bmatrix} -1/3 & -1/3 & 2/3 \\ -1/3 & 1/6 & 1/6 \\ 1/2 & 1/4 & -1/4 \end{bmatrix}$. **(5)** $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Section 7.3

(1) $\begin{aligned} 1 &= \alpha_1 + \alpha_0, \\ -1 &= -\alpha_1 + \alpha_0; \end{aligned}$ $\begin{bmatrix} -2 & 3 \\ -1 & 2 \end{bmatrix}$. **(2)** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(3) $\begin{aligned} 0 &= \alpha_0, \\ -1 &= -\alpha_1 + \alpha_0; \end{aligned}$ $\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$.

(4) $\begin{aligned} 0 &= \alpha_0 \\ 1 &= -\alpha_1 + \alpha_0; \end{aligned}$ $\begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}$. **(5)** $\begin{bmatrix} -3 & 6 \\ -1 & 2 \end{bmatrix}$. **(6)** $\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$.

(7) $\begin{aligned} 3^{78} &= 3\alpha_1 + \alpha_0, \\ 4^{78} &= 4^{78} = 4\alpha_1 + \alpha_0; \end{aligned}$ $\begin{bmatrix} -4^{78} + 2(3^{78}) & -4^{78} + 3^{78} \\ 2(4^{78}) - 2(3^{78}) & 2(4^{78}) - 3^{78} \end{bmatrix}$.

(8) $\begin{bmatrix} -4^{41} + 2(3^{41}) & -4^{41} + 3^{41} \\ 2(4^{41}) - 2(3^{41}) & 2(4^{41}) - 3^{41} \end{bmatrix}$.

(9) $\begin{aligned} 1 &= \alpha_2 + \alpha_1 + \alpha_0, \\ 1 &= \alpha_2 - \alpha_1 + \alpha_0, \\ 2^{222} &= 4\alpha_2 + 2\alpha_1 + \alpha_0; \end{aligned}$ $\begin{bmatrix} 1 & 0 & (-4 + 4(2^{222}))/3 \\ 0 & 1 & (-2 + 2(2^{222}))/3 \\ 0 & 0 & 2^{222} \end{bmatrix}$.

(10) $\begin{aligned} 3^{17} &= 9\alpha_2 + 3\alpha_1 + \alpha_0, \\ 5^{17} &= 25\alpha_2 + 5\alpha_1 + \alpha_0, \\ 10^{17} &= 100\alpha_2 + 10\alpha_1 + \alpha_0. \end{aligned}$

(11) $\begin{aligned} 2^{25} &= 8\alpha_3 + 4\alpha_2 + 2\alpha_1 + \alpha_0, \\ (-2)^{25} &= -8\alpha_3 + 4\alpha_2 - 2\alpha_1 + \alpha_0, \\ 3^{25} &= 27\alpha_3 + 9\alpha_2 + 3\alpha_1 + \alpha_0, \\ 4^{25} &= 64\alpha_3 + 16\alpha_2 + 4\alpha_1 + \alpha_0. \end{aligned}$

(12) $\begin{aligned} 1 &= \alpha_3 + \alpha_2 + \alpha_1 + \alpha_0, \\ (-2)^{25} &= -8\alpha_3 + 4\alpha_2 - 2\alpha_1 + \alpha_0, \\ 3^{25} &= 27\alpha_3 + 9\alpha_2 + 3\alpha_1 + \alpha_0, \\ (-4^{25}) &= -64\alpha_3 + 16\alpha_2 - 4\alpha_1 + \alpha_0. \end{aligned}$

(13) $\begin{aligned} 1 &= \alpha_4 + \alpha_3 + \alpha_2 + \alpha_1 + \alpha_0, \\ 1 &= \alpha_4 - \alpha_3 + \alpha_2 - \alpha_1 + \alpha_0, \\ 256 &= 16\alpha_4 + 8\alpha_3 + 4\alpha_2 + 2\alpha_1 + \alpha_0, \\ 256 &= 16\alpha_4 - 8\alpha_3 + 4\alpha_2 - 2\alpha_1 + \alpha_0, \\ 6{,}561 &= 81\alpha_4 + 27\alpha_3 + 9\alpha_2 + 3\alpha_1 + \alpha_0. \end{aligned}$

(14) $\begin{aligned} 5{,}837 &= 9\alpha_2 + 3\alpha_1 + \alpha_0, \\ 381{,}255 &= 25\alpha_2 + 5\alpha_1 + \alpha_0, \\ 10^8 - 3(10)^5 + 5 &= 100\alpha_2 + 10\alpha_1 + \alpha_0. \end{aligned}$

(15)
$$\begin{aligned} 165 &= 8\alpha_3 + 4\alpha_2 + 2\alpha_1 + \alpha_0, \\ 357 &= -8\alpha_3 + 4\alpha_2 - 2\alpha_1 + \alpha_0, \\ 5{,}837 &= 27\alpha_3 + 9\alpha_2 + 3\alpha_1 + \alpha_0, \\ 62{,}469 &= 64\alpha_3 + 16\alpha_2 + 4\alpha_1 + \alpha_0. \end{aligned}$$

(16)
$$\begin{aligned} 3 &= \alpha_3 + \alpha_2 + \alpha_1 + \alpha_0, \\ 357 &= -8\alpha_3 + 4\alpha_2 - 2\alpha_1 + \alpha_0, \\ 5{,}837 &= 27\alpha_3 + 9\alpha_2 + 3\alpha_1 + \alpha_0, \\ 68{,}613 &= -64\alpha_3 + 16\alpha_2 - 4\alpha_1 + \alpha_0. \end{aligned}$$

(17)
$$\begin{aligned} 15 &= \alpha_3 + \alpha_2 + \alpha_1 + \alpha_0, \\ 960 &= -8\alpha_3 + 4\alpha_2 - 2\alpha_1 + \alpha_0, \\ 59{,}235 &= 27\alpha_3 + 9\alpha_2 + 3\alpha_1 + \alpha_0, \\ 1{,}048{,}160 &= -64\alpha_3 + 16\alpha_2 - 4\alpha_1 + \alpha_0. \end{aligned}$$

(18)
$$\begin{aligned} 15 &= \alpha_4 + \alpha_3 + \alpha_2 + \alpha_1 + \alpha_0, \\ -13 &= \alpha_4 - \alpha_3 + \alpha_2 - \alpha_1 + \alpha_0, \\ 1{,}088 &= 16\alpha_4 + 8\alpha_3 + 4\alpha_2 + 2\alpha_1 + \alpha_0, \\ 960 &= 16\alpha_4 - 8\alpha_3 + 4\alpha_2 - 2\alpha_1 + \alpha_0, \\ 59{,}235 &= 81\alpha_4 + 27\alpha_3 + 9\alpha_2 + 3\alpha_1 + \alpha_0. \end{aligned}$$

(19) $\begin{bmatrix} 9 & -9 \\ 3 & -3 \end{bmatrix}$. **(20)** $\begin{bmatrix} 6 & -9 \\ 3 & -6 \end{bmatrix}$. **(21)** $\begin{bmatrix} -50{,}801 & -56{,}632 \\ 113{,}264 & 119{,}095 \end{bmatrix}$.

(22) $\begin{bmatrix} 3{,}077 & -5{,}120 \\ 1{,}024 & -3{,}067 \end{bmatrix}$. **(23)** $\begin{bmatrix} 938 & 160 \\ -32 & 1130 \end{bmatrix}$. **(24)** $\begin{bmatrix} 2 & -4 & -3 \\ 0 & 0 & 0 \\ 1 & -5 & -2 \end{bmatrix}$.

(25)
$$\begin{aligned} 2{,}569 &= 4\alpha_2 + 2\alpha_1 + \alpha_0, \\ 5{,}633 &= 4\alpha_2 - 2\alpha_1 + \alpha_0, \\ 5 &= \alpha_2 + \alpha_1 + \alpha_0. \end{aligned} \qquad \begin{bmatrix} -339 & -766 & 1110 \\ -4440 & 4101 & 344 \\ -1376 & -3064 & 4445 \end{bmatrix}.$$

(26)
$$\begin{aligned} 0.814453 &= 0.25\alpha_2 + 0.5\alpha_1 + \alpha_0, \\ 0.810547 &= 0.25\alpha_2 - 0.5\alpha_1 + \alpha_0, \\ 0.988285 &= 0.0625\alpha_2 + 0.25\alpha_1 + \alpha_0. \end{aligned} \qquad \begin{bmatrix} 1.045578 & 0.003906 & -0.932312 \\ 0.058270 & 0.812500 & -0.229172 \\ 0.014323 & 0.000977 & 0.755207 \end{bmatrix}.$$

Section 7.4

(1)
$$\begin{aligned} 128 &= 2\alpha_1 + \alpha_0, \\ 448 &= \alpha_1. \end{aligned}$$

(2)
$$\begin{aligned} 128 &= 4\alpha_2 + 2\alpha_1 + \alpha_0, \\ 448 &= 4\alpha_2 + \alpha_1, \\ 1{,}344 &= 2\alpha_2. \end{aligned}$$

(3)
$$\begin{aligned} 128 &= 4\alpha_2 + 2\alpha_1 + \alpha_0, \\ 448 &= 4\alpha_2 + \alpha_1, \\ 1 &= \alpha_2 + \alpha_1 + \alpha_0. \end{aligned}$$

(4)
$$\begin{aligned} 59{,}049 &= 3\alpha_1 + \alpha_0, \\ 196{,}830 &= \alpha_1. \end{aligned}$$

(5) $59{,}049 = 9\alpha_2 + 3\alpha_1 + \alpha_0,$
$196{,}830 = 6\alpha_2 + \alpha_1,$
$590{,}490 = 2\alpha_2.$

(6) $59{,}049 = 27\alpha_3 + 9\alpha_2 + 3\alpha_1 + \alpha_0,$
$196{,}830 = 27\alpha_3 + 6\alpha_2 + \alpha_1,$
$590{,}490 = 18\alpha_3 + 2\alpha_2,$
$1{,}574{,}640 = 6\alpha_3.$

(7) $512 = 8\alpha_3 + 4\alpha_2 + 2\alpha_1 + \alpha_0,$
$2{,}304 = 12\alpha_3 + 4\alpha_2 + \alpha_1,$
$9{,}216 = 12\alpha_3 + 2\alpha_2,$
$32{,}256 = 6\alpha_3.$

(8) $512 = 8\alpha_3 + 4\alpha_2 + 2\alpha_1 + \alpha_0,$
$2{,}304 = 12\alpha_3 + 4\alpha_2 + \alpha_1,$
$9{,}216 = 12\alpha_3 + 2\alpha_2$
$1 = \alpha_3 + \alpha_2 + \alpha_1 + \alpha_0$

(9) $512 = 8\alpha_3 + 4\alpha_2 + 2\alpha_1 + \alpha_0,$
$2{,}304 = 12\alpha_3 + 4\alpha_2 + \alpha_1,$
$1 = \alpha_3 + \alpha_2 + \alpha_1 + \alpha_0.$
$9 = 3\alpha_3 + 2\alpha_2 + \alpha_1.$

(10)
$$(5)^{10} - 3(5)^5 = \alpha_5(5)^5 + \alpha_4(5)^4 + \alpha_3(5)^3 + \alpha_2(5)^2 + \alpha_1(5) + \alpha_0,$$
$$10(5)^9 - 15(5)^4 = 5\alpha_5(5)^4 + 4\alpha_4(5)^3 + 3\alpha_3(5)^2 + 2\alpha_2(5) + \alpha_1,$$
$$90(5)^8 - 60(5)^3 = 20\alpha_5(5)^3 + 12\alpha_4(5)^2 + 6\alpha_3(5) + 2\alpha_2,$$
$$720(5)^7 - 180(5)^2 = 60\alpha_5(5)^2 + 24\alpha_4(5) + 6\alpha_3,$$
$$(2)^{10} - 3(2)^5 = \alpha_5(2)^5 + \alpha_4(2)^4 + \alpha_3(2)^3 + \alpha_2(2)^2 + \alpha_1(2) + \alpha_0,$$
$$10(2)^9 - 15(2)^4 = 5\alpha_5(2)^4 + 4\alpha_4(2)^3 + 3\alpha_3(2)^2 + 2\alpha_2(2) + \alpha_1.$$

(11) $\begin{bmatrix} 729 & 0 \\ 0 & 729 \end{bmatrix}.$ **(12)** $\begin{bmatrix} 4 & 1 & -3 \\ 0 & -1 & 0 \\ 5 & 1 & -4 \end{bmatrix}.$ **(13)** $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

Section 7.5

(1) $e = \alpha_1 + \alpha_0,$
$e^2 = 2\alpha_1 + \alpha_0.$

(2) $e^2 = 2\alpha_1 + \alpha_0,$
$e^2 = \alpha_1.$

(3) $e^2 = 4\alpha_2 + 2\alpha_1 + \alpha_0,$
$e^2 = 4\alpha_2 + \alpha_1,$
$e^2 = 2\alpha_2.$

(4) $e^1 = \alpha_2 + \alpha_1 + \alpha_0,$
$e^{-2} = 4\alpha_2 - 2\alpha_1 + \alpha_0,$
$e^3 = 9\alpha_2 + 3\alpha_1 + \alpha_0.$

(5) $e^{-2} = 4\alpha_2 - 2\alpha_1 + \alpha_0,$
$e^{-2} = -4\alpha_2 + \alpha_1,$
$e^1 = \alpha_2 + \alpha_1 + \alpha_0.$

(6) $\sin(1) = \alpha_2 + \alpha_1 + \alpha_0,$
$\sin(2) = 4\alpha_2 + 2\alpha_1 + \alpha_0,$
$\sin(3) = 9\alpha_2 + 3\alpha_1 + \alpha_0.$

(7) $\sin(-2) = 4\alpha_2 - 2\alpha_1 + \alpha_0,$
$\cos(-2) = -4\alpha_2 + \alpha_1,$
$\sin(1) = \alpha_2 + \alpha_1 + \alpha_0.$

(8) $e^2 = 8\alpha_3 + 4\alpha_2 + 2\alpha_1 + \alpha_0,$
$e^2 = 12\alpha_3 + 4\alpha_2 + \alpha_1,$
$e^2 = 12\alpha_3 + 2\alpha_2,$
$e^2 = 6\alpha_3.$

(9) $e^2 = 8\alpha_3 + 4\alpha_2 + 2\alpha_1 + \alpha_0,$
$e^2 = 12\alpha_3 + 4\alpha_2 + \alpha_1,$
$e^{-2} = -8\alpha_3 + 4\alpha_2 - 2\alpha_1 + \alpha_0,$
$e^{-2} = 12\alpha_3 - 4\alpha_2 + \alpha_1.$

(10)
$$\begin{aligned}\sin(2) &= 8\alpha_3 + 4\alpha_2 + 2\alpha_1 + \alpha_0,\\ \cos(2) &= 12\alpha_3 + 4\alpha_2 + \alpha_1,\\ \sin(-2) &= -8\alpha_3 + 4\alpha_2 - 2\alpha_1 + \alpha_0,\\ \cos(-2) &= 12\alpha_3 - 4\alpha_2 + \alpha_1.\end{aligned}$$

(11)
$$\begin{aligned}e^3 &= 27\alpha_3 + 9\alpha_2 + 3\alpha_1 + \alpha_0,\\ e^3 &= 27\alpha_3 + 6\alpha_2 + \alpha_1,\\ e^3 &= 18\alpha_3 + 2\alpha_2,\\ e^{-1} &= -\alpha_3 + \alpha_2 - \alpha_1 + \alpha_0.\end{aligned}$$

(12)
$$\begin{aligned}\cos(3) &= 27\alpha_3 + 9\alpha_2 + 3\alpha_1 + \alpha_0,\\ -\sin(3) &= 27\alpha_3 + 6\alpha_2 + \alpha_1,\\ -\cos(3) &= 18\alpha_3 + 2\alpha_2,\\ \cos(-1) &= -\alpha_3 + \alpha_2 - \alpha_1 + \alpha_0.\end{aligned}$$

(13) $\dfrac{1}{7}\begin{bmatrix} 3e^5 + 4e^{-2} & 3e^5 - 3e^{-2} \\ 4e^5 - 4e^{-2} & 4e^5 + 3e^{-2} \end{bmatrix}$. **(14)** $e^3\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$.

(15) $e^2\begin{bmatrix} 0 & 1 & 3 \\ -1 & 2 & 5 \\ 0 & 0 & 1 \end{bmatrix}$. **(16)** $\dfrac{1}{16}\begin{bmatrix} 12e^2 + 4e^{-2} & 4e^2 - 4e^{-2} & 38e^2 + 2e^{-2} \\ 12e^2 - 12e^{-2} & 4e^2 + 12e^{-2} & 46e^2 - 6e^{-2} \\ 0 & 0 & 16e^2 \end{bmatrix}$.

(17) $\dfrac{1}{5}\begin{bmatrix} -1 & 6 \\ 4 & 1 \end{bmatrix}$.

(18) (a) $\begin{bmatrix} \log(3/2) & \log(3/2) - \log(1/2) \\ 0 & \log(1/2) \end{bmatrix}$.

(b) and (c) are not defined since they possess eigenvalues having absolute value greater than 1.

(d) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Section 7.6

(1) $1/7\begin{bmatrix} 3e^{8t} + 4e^{t} & 4e^{8t} - 4e^{t} \\ 3e^{8t} - 3e^{t} & 4e^{8t} + 3e^{t} \end{bmatrix}$.

(2) $\begin{bmatrix} (2/\sqrt{3})\sinh\sqrt{3}t + \cosh\sqrt{3}t & (1/\sqrt{3})\sinh\sqrt{3}t \\ (-1/\sqrt{3})\sinh\sqrt{3}t & (-2/\sqrt{3})\sinh\sqrt{3}t + \cosh\sqrt{3}t \end{bmatrix}$.

Note:

$$\sinh\sqrt{3}t = \frac{e^{\sqrt{3}t} - e^{-\sqrt{3}t}}{2} \quad \text{and} \quad \cosh\sqrt{3}t = \frac{e^{\sqrt{3}t} + e^{-\sqrt{3}t}}{2}.$$

(3) $e^{3t}\begin{bmatrix} 1+t & t \\ -t & 1-t \end{bmatrix}$. **(4)** $\begin{bmatrix} 1.4e^{-2t} - 0.4e^{-7t} & 0.2e^{-2t} - 0.2e^{-7t} \\ -2.8e^{-2t} + 2.8e^{-7t} & -0.4e^{-2t} + 1.4e^{-7t} \end{bmatrix}$.

(5) $\begin{bmatrix} 0.8e^{-2t} + 0.2e^{-7t} & 0.4e^{-2t} - 0.4e^{-7t} \\ 0.4e^{-2t} - 0.4e^{-7t} & 0.2e^{-2t} + 0.8e^{-7t} \end{bmatrix}$.

(6) $\begin{bmatrix} 0.5e^{-4t} + 0.5e^{-16t} & 0.5e^{-4t} - 0.5e^{-16t} \\ 0.5e^{-4t} - 0.5e^{-16t} & 0.5e^{-4t} + 0.5e^{-16t} \end{bmatrix}$. **(7)** $\begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$.

(8) $\dfrac{1}{12}\begin{bmatrix} 12e^{t} & 0 & 0 \\ -9e^{t} + 14e^{3t} - 5e^{-3t} & 8e^{3t} + 4e^{-3t} & 4e^{3t} - 4e^{-3t} \\ -24e^{t} + 14e^{3t} + 10e^{-3t} & 8e^{3t} - 8e^{-3t} & 4e^{3t} + 8e^{-3t} \end{bmatrix}$.

Section 7.7

(1) $\begin{bmatrix} (1/2)\sin 2t + \cos 2t & (-1/2)\sin 2t \\ (5/2)\sin 2t & (-1/2)\sin 2t + \cos 2t \end{bmatrix}$.

(2) $\begin{bmatrix} \sqrt{2}\sin\sqrt{2}t + \cos\sqrt{2}t & -\sqrt{2}\sin\sqrt{2}t \\ (3/\sqrt{2})\sin\sqrt{2}t & -\sqrt{2}\sin\sqrt{2}t + \cos\sqrt{2}t \end{bmatrix}$.

(3) $\begin{bmatrix} \cos(8t) & \dfrac{1}{8}\sin(8t) \\ -8\sin(8t) & \cos(8t) \end{bmatrix}$.

(4) $\dfrac{1}{4}\begin{bmatrix} 2\sin(8t) + 4\cos(8t) & -4\sin(8t) \\ 5\sin(8t) & -2\sin(8t) + 4\cos(8t) \end{bmatrix}$.

(5) $\begin{bmatrix} 2\sin(t) + \cos(t) & 5\sin(t) \\ -\sin(t) & -2\sin(t) + \cos(t) \end{bmatrix}$.

(6) $\dfrac{1}{3}e^{-4t}\begin{bmatrix} 4\sin(3t) + 3\cos(3t) & \sin(3t) \\ -25\sin(3t) & -4\sin(3t) + 3\cos(3t) \end{bmatrix}$.

(7) $e^{4t}\begin{bmatrix} -\sin t + \cos t & \sin t \\ -2\sin t & \sin t + \cos t \end{bmatrix}$.

(8) $\begin{bmatrix} 1 & -2 + 2\cos(t) + \sin(t) & -5 + 5\cos(t) \\ 0 & \cos(t) - 2\sin(t) & -5\sin(t) \\ 0 & \sin(t) & \cos(t) + 2\sin(t) \end{bmatrix}$.

Section 7.8

(3) $\mathbf{A}$ does not have an inverse.

(8) $e^{\mathbf{A}} = \begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix}$, $e^{\mathbf{B}} = \begin{bmatrix} 1 & e-1 \\ 0 & e \end{bmatrix}$, $e^{\mathbf{A}}e^{\mathbf{B}} = \begin{bmatrix} e & 2e^2-2e \\ 0 & e \end{bmatrix}$,

$e^{\mathbf{B}}e^{\mathbf{A}} = \begin{bmatrix} e & 2e-2 \\ 0 & e \end{bmatrix}$, $e^{\mathbf{A}+\mathbf{B}} = \begin{bmatrix} e & 2e \\ 0 & e \end{bmatrix}$.

(9) $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$. Also see Problem 10.

(11) First show that for any integer n, $(\mathbf{P}^{-1}\mathbf{B}\mathbf{P})^n = \mathbf{P}^{-1}\mathbf{B}^n\mathbf{P}$, and then use Eq. (6) directly.

Section 7.9

(1) (a) $\begin{bmatrix} -\sin t & 2t \\ 2 & e^{(t-1)} \end{bmatrix}$. (b) $\begin{bmatrix} 6t^2e^{t^3} & 2t-1 & 0 \\ 2t+3 & 2\cos 2t & 1 \\ -18t\cos^2(3t^2)\sin(3t^2) & 0 & 1/t \end{bmatrix}$.

(4) $\begin{bmatrix} \sin t + c_1 & \frac{1}{3}t^3 - t + c_2 \\ t^2 + c_3 & e^{(t-1)} + c_4 \end{bmatrix}$.

Chapter 8

Section 8.1

(1) $\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, $\mathbf{A}(t) = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$, $t_0 = 0$.

(2) $\mathbf{x}(t) = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}$, $\mathbf{A}(t) = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$; $\mathbf{c} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $t_0 = 0$.

(3) $\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, $\mathbf{A}(t) = \begin{bmatrix} -3 & 4 \\ 4 & -4 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $t_0 = 0$.

(4) $\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, $\mathbf{A}(t) = \begin{bmatrix} 3 & 0 \\ 2 & 0 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} t \\ t+1 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $t_0 = 0$.

(5) $\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, $\mathbf{A}(t) = \begin{bmatrix} 3t^2 & 7 \\ 1 & t \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} 2 \\ 2t \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, $t_0 = 1$.

(6) $\mathbf{x}(t) = \begin{bmatrix} u(t) \\ v(t) \\ w(t) \end{bmatrix}$, $\mathbf{A}(t) = \begin{bmatrix} e^t & t & 1 \\ t^2 & -3 & t+1 \\ 0 & 1 & e^{t^2} \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $t_0 = 4$.

(7) $\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$, $\mathbf{A}(t) = \begin{bmatrix} 0 & 6 & 1 \\ 1 & 0 & -3 \\ 0 & -2 & 0 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 10 \\ 10 \\ 20 \end{bmatrix}$, $t_0 = 0$.

(8) $\mathbf{x}(t) = \begin{bmatrix} r(t) \\ s(t) \\ u(t) \end{bmatrix}$, $\mathbf{A}(t) = \begin{bmatrix} t^2 & -3 & -\sin t \\ 1 & -1 & 0 \\ 2 & e^t & t^2-1 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} \sin t \\ t^2-1 \\ \cos t \end{bmatrix}$,

$\mathbf{c} = \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix}$, $t_0 = 1$.

(9) Only (c). **(10)** Only (c). **(11)** Only (b).

Section 8.2

(1) $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $\mathbf{A}(t) = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, $t_0 = 0$.

(2) $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $\mathbf{A}(t) = \begin{bmatrix} 0 & 1 \\ t & -e^t \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $t_0 = 1$.

(3) $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $\mathbf{A}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} 0 \\ t^2 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$, $t_0 = 0$.

(4) $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $\mathbf{A}(t) = \begin{bmatrix} 0 & 1 \\ 3 & 2e^t \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} 0 \\ 2e^{-t} \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $t_0 = 0$.

(5) $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $\mathbf{A}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $t_0 = 1$.

(6) $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$, $\mathbf{A}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/4 & 0 & -t/4 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 2 \\ 1 \\ -205 \end{bmatrix}$, $t_0 = -1$.

(7) $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$, $\mathbf{A}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & e^{-t} & -te^{-t} & 0 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ \pi \\ e^3 \end{bmatrix}$, $t_0 = 0$.

(8) $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \end{bmatrix}$, $\mathbf{A}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -4 & 0 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ t^2 - t \end{bmatrix}$,

$\mathbf{c} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$, $t_0 = \pi$.

Section 8.3

(1) $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ y_1(t) \end{bmatrix}$, $\mathbf{A}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 2 & 4 \\ 5 & 0 & -6 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$, $t_0 = 0$.

(2) $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ y_1(t) \\ y_2(t) \end{bmatrix}$, $\mathbf{A}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 4 \end{bmatrix}$, $t_0 = 0$.

(3) $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ y_1(t) \\ y_2(t) \end{bmatrix}$, $\mathbf{A}(t) = \begin{bmatrix} -4 & 0 & t^2 \\ 0 & 0 & 1 \\ t^2 & t & 0 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$, $t_0 = 2$.

(4) $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ y_1(t) \\ y_2(t) \end{bmatrix}$, $\mathbf{A}(t) = \begin{bmatrix} -4 & 0 & 2 \\ 0 & 0 & 1 \\ 3 & t & 0 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} t \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $t_0 = 3$.

$$\textbf{(5)}\quad \mathbf{x}(t)=\begin{bmatrix} x_1(t)\\ x_2(t)\\ y_1(t)\\ y_2(t)\\ y_3(t)\\ y_4(t)\end{bmatrix},\ \mathbf{A}(t)=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 2 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ t & 0 & -t & 0 & 1 & 0\end{bmatrix},\ \mathbf{f}(t)=\begin{bmatrix} 0\\ -t\\ 0\\ 0\\ 0\\ -e^t\end{bmatrix},$$

$$\mathbf{c}=\begin{bmatrix} 2\\ 0\\ 0\\ 3\\ 9\\ 4\end{bmatrix},\ t_0=-1.$$

$$\textbf{(6)}\quad \mathbf{x}(t)=\begin{bmatrix} x_1(t)\\ x_2(t)\\ x_3(t)\\ y_1(t)\\ y_2(t)\end{bmatrix},\ \mathbf{A}(t)=\begin{bmatrix} 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 1 & 0 & 0 & -1 & 1\\ 0 & 0 & 0 & 0 & 1\\ -1 & 0 & 1 & 0 & 2\end{bmatrix},\ \mathbf{f}(t)=\begin{bmatrix} 0\\ 0\\ 0\\ 0\\ 0\end{bmatrix},\ \mathbf{c}=\begin{bmatrix} 21\\ 4\\ -5\\ 5\\ 7\end{bmatrix},\ t_0=0.$$

$$\textbf{(7)}\quad \mathbf{x}(t)=\begin{bmatrix} x_1(t)\\ y_1(t)\\ y_2(t)\\ z_1(t)\end{bmatrix},\ \mathbf{A}(t)=\begin{bmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 1 & 0 & 0\end{bmatrix},\ \mathbf{f}(t)=\begin{bmatrix} -2\\ 0\\ -2\\ 0\end{bmatrix},\ \mathbf{c}=\begin{bmatrix} 1\\ 2\\ 17\\ 0\end{bmatrix},\ t_0=\pi.$$

$$\textbf{(8)}\quad \mathbf{x}(t)=\begin{bmatrix} x_1(t)\\ x_2(t)\\ y_1(t)\\ y_2(t)\\ z_1(t)\\ z_2(t)\end{bmatrix},\ \mathbf{A}(t)=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 1 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0 & -1 & 0\end{bmatrix},\ \mathbf{f}(t)=\begin{bmatrix} 0\\ 2\\ 0\\ -1\\ 0\\ 1\end{bmatrix},$$

$$\mathbf{c}=\begin{bmatrix} 4\\ -4\\ 5\\ -5\\ 9\\ -9\end{bmatrix},\ t_0=20.$$

Section 8.4

(3) (a) $e^{-3t}\begin{bmatrix} 1 & -t & t^2/2 \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{bmatrix}$, (b) $e^{3(t-2)}\begin{bmatrix} 1 & (t-2) & (t-2)^2/2 \\ 0 & 1 & (t-2) \\ 0 & 0 & 1 \end{bmatrix}$,

(c) $e^{3(t-s)}\begin{bmatrix} 1 & (t-s) & (t-s)^2/2 \\ 0 & 1 & (t-s) \\ 0 & 0 & 1 \end{bmatrix}$,

(d) $e^{-3(t-2)}\begin{bmatrix} 1 & -(t-2) & (t-2)^2/2 \\ 0 & 1 & -(t-s) \\ 0 & 0 & 1 \end{bmatrix}$.

(5) (a) $\frac{1}{6}\begin{bmatrix} 2e^{-5t}+4e^{t} & 2e^{-5t}-2e^{t} \\ 4e^{-5t}-4e^{t} & 4e^{-5t}+2e^{t} \end{bmatrix}$,

(b) $\frac{1}{6}\begin{bmatrix} 2e^{-5s}+4e^{s} & 2e^{-5s}-2e^{s} \\ 4e^{-5s}-4e^{s} & 4e^{-5s}+2e^{s} \end{bmatrix}$,

(c) $\frac{1}{6}\begin{bmatrix} 2e^{5(t-3)}+4e^{-(t-3)} & 2e^{5(t-3)}-2e^{-(t-3)} \\ 4e^{5(t-3)}-4e^{-(t-3)} & 4e^{5(t-3)}+2e^{-(t-3)} \end{bmatrix}$.

(6) (a) $\frac{1}{3}\begin{bmatrix} \sin 3t + 3\cos 3t & -5\sin 3t \\ 2\sin 3t & -\sin 3t + 3\cos 3t \end{bmatrix}$,

(b) $\frac{1}{3}\begin{bmatrix} \sin 3s + 3\cos 3s & -5\sin 3s \\ 2\sin 3s & -\sin 3s + 3\cos 3s \end{bmatrix}$,

(c) $\frac{1}{3}\begin{bmatrix} \sin 3(t-s) + 3\cos 3(t-s) & -5\sin 3(t-s) \\ 2\sin 3(t-s) & -\sin 3(t-s) + 3\cos 3(t-s)t \end{bmatrix}$.

(7) $x(t) = 5e^{(t-2)} - 3e^{-(t-2)}, y(t) = 5e^{(t-2)} - e^{-(t-2)}$.

(8) $x(t) = 2e^{(t-1)} - 1, y(t) = 2e^{(t-1)} - 1$.

(9) $x(t) = k_3e^{t} + 3k_4e^{-t}, y(t) = k_3e^{t} + k_4e^{-t}$.

(10) $x(t) = k_3e^{t} + 3k_4e^{-t} - 1, y(t) = k_3e^{t} + k_4e^{-t} - 1$.

(11) $x(t) = \cos 2t - (1/6)\sin 2t + (1/3)\sin t$.

(12) $x(t) = t^4/24 + (5/4)t^2 - (2/3)t + 3/8$.

(13) $x(t) = (4/9)e^{2t} + (5/9)e^{-t} - (1/3)te^{-t}$.

(14) $x(t) = -8\cos t - 6\sin t + 8 + 6t$,

$y(t) = 4\cos t - 2\sin t - 3$.

Section 8.5

(4) First show that

$$\mathbf{\Phi}^{\mathsf{T}}(t_1,t_0)\left[\int_{t_0}^{t_1}\mathbf{\Phi}(t_1,s)\mathbf{\Phi}^{\mathsf{T}}(t_1,s)\,ds\right]^{-1}\mathbf{\Phi}(t_1,t_0)$$
$$=\left[\mathbf{\Phi}(t_0,t_1)\int_{t_0}^{t_1}\mathbf{\Phi}(t_1,s)\mathbf{\Phi}'(t_1,s)\,ds\mathbf{\Phi}^{\mathsf{T}}(t_0,t_1)\right]^{-1}$$
$$=\left[\int_{t_0}^{t_1}\mathbf{\Phi}(t_0,t_1)\mathbf{\Phi}(t_1,s)[\mathbf{\Phi}(t_0,t_1)\mathbf{\Phi}(t_1,s)]^{\mathsf{T}}\,ds\right]^{-1}.$$

Chapter 9

Section 9.1

(2) If $\mathbf{PA} = \mathbf{BP}$, then $\mathbf{P} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ which is singular.

(3) (a) Yes, (b) No, (c) No, (d) No. **(4)** 2, 2, and 4, same as $\mathbf{D}$.

(5) Combine Theorem 1 with Property 1 of Section 5.4.

(6) Different traces, so matrices are not similar.

(8) $\mathbf{A}^2 = \mathbf{AA} = (\mathbf{P}^{-1}\mathbf{BP})(\mathbf{P}^{-1}\mathbf{BP}) = \mathbf{P}^{-1}\mathbf{B}(\mathbf{PP}^{-1})\mathbf{BP} = \mathbf{P}^{-1}\mathbf{BIBP}$
$= \mathbf{P}^{-1}\mathbf{B}^2\mathbf{P}$.

(10) Show that $\mathbf{A}^{\mathsf{T}} = \mathbf{P}^{\mathsf{T}}\mathbf{B}^{\mathsf{T}}(\mathbf{P}^{\mathsf{T}})^{-1}$ and set $\mathbf{S} = (\mathbf{P}^{\mathsf{T}})^{-1}$.

(11) Take $\mathbf{P} = \mathbf{I}$.

(14) $\mathbf{By} = (\mathbf{PAP}^{-1})(\mathbf{Px}) = \mathbf{PAx} = \mathbf{P}(\lambda\mathbf{x}) = \lambda(\mathbf{Px}) = \lambda\mathbf{y}$.

(15) $\mathbf{A}_k\mathbf{R}_{k-1} = \mathbf{R}_{k-1}\mathbf{Q}_{k-1}\mathbf{R}_{k-1} = \mathbf{R}_{k-1}\mathbf{A}_{k-1}$. Show that $\mathbf{R}_{k-1}$ is invertible, and set $\mathbf{P} = (\mathbf{R}_{k-1})^{-1}$.

Section 9.2

(1) Yes; $\mathbf{M} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

(2) Yes; $\mathbf{M} = \begin{bmatrix} 2+i & 2-i \\ 1 & 1 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$. **(3)** No. **(4)** No.

(5) Yes; $\mathbf{M} = \begin{bmatrix} -10 & 0 & 0 \\ 1 & 1 & -3 \\ 8 & 2 & 1 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{bmatrix}$.

(6) Yes; $\mathbf{M} = \begin{bmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}$.

(7) Yes; $\mathbf{M} = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 5 & 2 \\ -1 & -3 & 3 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 14 \end{bmatrix}$.

(8) Yes; $\mathbf{M} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ -1 & 1 & 1 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

(9) No. **(10)** No. **(11)** No.

Section 9.3

(1) $\begin{bmatrix} 2 - 2^{27} & -1 + 2^{27} \\ 2 - 2^{28} & -1 + 2^{28} \end{bmatrix}$. **(2)** $\begin{bmatrix} 89 - 2^{17} & -88 + 2^{17} \\ 176 - 2^{18} & -175 + 2^{18} \end{bmatrix}$.

(3) $\begin{bmatrix} 2e - e^2 & -e + e^2 \\ 2e - 2e^2 & -e + 2e^2 \end{bmatrix}$. **(4)** $e^2 \begin{bmatrix} \cos\sqrt{5} & (1/\sqrt{5})\sin\sqrt{5} \\ -\sqrt{5}\sin\sqrt{5} & \cos\sqrt{5} \end{bmatrix}$.

(5) $(1/70)\begin{bmatrix} 70e & 0 & 0 \\ -7e + 25e^3 - 18e^{-4} & 10e^3 + 60e^{-4} & 30e^3 - 30e^{-4} \\ -56e + 50e^3 + 6e^{-4} & 20e^3 - 20e^{-4} & 60e^3 + 10e^{-4} \end{bmatrix}$.

(6) $(-1/4)\begin{bmatrix} -2e^3 - 2e^7 & e^3 - e^7 & 2e^3 - 2e^7 \\ 0 & -4e^3 & 0 \\ 2e^3 - 2e^7 & e^3 - e^7 & -2e^3 - 2e^7 \end{bmatrix}$.

(9) $(-1/4)\begin{bmatrix} -2\sin 3 - 2\sin 7 & \sin 3 - \sin 7 & 2\sin 3 - 2\sin 7 \\ 0 & -4\sin 3 & 0 \\ 2\sin 3 - 2\sin 7 & \sin 3 - \sin 7 & -2\sin 3 - 2\sin 7 \end{bmatrix}$.

Section 9.4

(1) (a) Yes, (b) No, (c) Yes, (d) Yes, (e) No, (f) No.

(2) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. **(3)** $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, **(4)** $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. **(5)** $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. **(6)** $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

(7) For $\lambda = 3$, $\mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$, and for $\lambda = 4$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Section 9.5

(1) $\mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$

(2) $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$

(3) $\mathbf{x}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$

(4) $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$

(5) $\mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 0 \\ 0 \end{bmatrix}.$ **(6)** $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$

(7) $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$ **(8)** $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$

(9) $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}.$

(10) $\mathbf{x}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} -1 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 7 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$

(11) $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$

Section 9.6

(1) (a) One chain of length 3; (b) two chains of length 3;
(c) one chain of length 3, and one chain of length 2;
(d) one chain of length 3, one chain of length 2, and one chain of length 1; (e) one chain of length 3 and two chains of length 1;
(f) cannot be done, the numbers as given are not compatible;
(g) two chains of length 2, and two chains of length 1;
(h) cannot be done, the numbers as given are not compatible;
(i) two chains of length 2 and one chain of length 1;
(j) two chains of length 2.

(2) $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$

(3) $\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ corresponds to $\lambda = 1$ and $\mathbf{y}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$,

$\mathbf{y}_1 = \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}$ correspond to $\lambda = 4$.

(4) $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.$

(5) $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$, $\mathbf{y}_1 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$ both correspond to $\lambda = 3$ and

$\mathbf{z}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ corresponds to $\lambda = 7$.

(6) $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{y}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$.

(7) $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

correspond to $\lambda = 3$ and

$\mathbf{y}_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{y}_1 = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 0 \end{bmatrix}$ correspond to $\lambda = 4$.

(8) $\mathbf{x}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 3 \\ 4 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{x}_1 = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

correspond to $\lambda = 4$, and

$\mathbf{y}_2 = \begin{bmatrix} -5 \\ -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{y}_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ correspond to $\lambda = 5$.

Section 9.7

(1) $\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$. **(2)** $\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$. **(3)** $\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$.

(4) $\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$. **(5)** $\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$.

(6) $\begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$. **(7)** $\begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$.

(8) $\begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$. **(9)** $\begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$.

(10) $\begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$. **(11)** $\begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$.

(12) $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$. **(13)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix}$. **(14)** $\begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}$.

(15) $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$. **(16)** $\begin{bmatrix} 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$.

Section 9.8

(1) $\begin{bmatrix} 16 & 32 & 24 \\ 0 & 16 & 32 \\ 0 & 0 & 16 \end{bmatrix}$. **(2)** $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -10 & 45 & -120 \\ 0 & 0 & 0 & 1 & -10 & 45 \\ 0 & 0 & 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

(3) $e^4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. **(4)** $e^2\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. **(5)** $\begin{bmatrix} -1 & 0 & \pi^2/12 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(6) $e^{\pi}\begin{bmatrix} 1 & \pi/3 & (\pi^2/12) - \pi \\ 0 & 1 & \pi/2 \\ 0 & 0 & 1 \end{bmatrix}$. **(7)** $\begin{bmatrix} e & 0 & 0 \\ -e + 2e^2 & 2e^2 & -e^2 \\ e^2 & e^2 & 0 \end{bmatrix}$.

(8) $$\begin{bmatrix} e^2 & e^2 & 0 & 0 \\ 0 & e^2 & 0 & 0 \\ 0 & 0 & (e^{3/2}/\sqrt{27})\left(\sin\frac{\sqrt{27}}{2} + \sqrt{27}\cos\frac{\sqrt{27}}{2}\right) & (e^{3/2}/\sqrt{27})\left(14\sin\frac{\sqrt{27}}{2}\right) \\ 0 & 0 & (e^{3/2}/\sqrt{27})\left(-2\sin\frac{\sqrt{27}}{2}\right) & (e^{3/2}/\sqrt{27})\left(-\sin\frac{\sqrt{27}}{2} + \sqrt{27}\cos\frac{\sqrt{27}}{2}\right) \end{bmatrix}.$$

Section 9.9

(2) $e^{2t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$. **(3)** $e^{-t}\begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$. **(4)** $e^{4t}\begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(5) $\begin{bmatrix} e^{2t} & te^{2t} & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$. **(6)** $(1/2)\begin{bmatrix} -e^{-t}+3e^{t} & -3e^{-t}+3e^{t} & 0 \\ e^{-t}-e^{t} & 3e^{-t}-e^{t} & 0 \\ 2te^{t} & 2te^{t} & 2e^{t} \end{bmatrix}$.

(7) $e^{2t}\begin{bmatrix} 1+t & t & 0 \\ -t & 1-t & 0 \\ t-\frac{1}{2}t^2 & 2t-\frac{1}{2}t^2 & 1 \end{bmatrix}$.

(8) $e^{t}\begin{bmatrix} 1+4t-t^2 & -2t+2t^2 & 2t \\ 2t-t^2/2 & 1-t+t^2 & t \\ -7t+\frac{3}{2}t^2 & 5t-3t^2 & 1-3t \end{bmatrix}$.

Chapter 10

Section 10.1

(1) $\begin{bmatrix} 2-3i & 1 \\ 0 & 2-i \end{bmatrix}$. **(2)** $\begin{bmatrix} 2+i & 1-3i \\ -2i & 4 \end{bmatrix}$. **(3)** $\begin{bmatrix} 3 & 5-6i \\ 5+6i & 4 \end{bmatrix}$.

(4) $\begin{bmatrix} 3 & 5+6i \\ 5-6i & 4 \end{bmatrix}$. **(5)** $\begin{bmatrix} 1+2i & -3+i \\ -2 & 4i \end{bmatrix}$. **(6)** $\begin{bmatrix} 8-i & 1-2i \\ 0 & 4-3i \end{bmatrix}$.

(7) $\begin{bmatrix} 4+2i & 2+3i \\ 2i & 6+i \end{bmatrix}$. **(8)** $\begin{bmatrix} -1+3i & -4-6i \\ -5+6i & -2+i \end{bmatrix}$.

(9) $\begin{bmatrix} 13-2i & 17+24i \\ 15-14i & 20 \end{bmatrix}$. **(10)** $\begin{bmatrix} 6+12i & -3+5i \\ 6+5i & 6+7i \end{bmatrix}$.

(11) $\begin{bmatrix} -4+12i & 2+i \\ 2+i & 3+4i \end{bmatrix}$. **(12)** $\begin{bmatrix} 15 & 2-8i \\ 2+8i & 20 \end{bmatrix}$.

(13) $\begin{bmatrix} 15 & 2+8i \\ 2-8i & 20 \end{bmatrix}$. **(14)** $\begin{bmatrix} e^{(2+3i)t} & \frac{i}{2}[e^{(2+i)t}-e^{(2+3i)t}] \\ 0 & e^{(2+i)t} \end{bmatrix}$.

(15) (a) $2i, -2i$; (b) $0, 0$; (c) $1+6i, 1-6i$.

(19) (a) $\begin{bmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix}$, (b) $\begin{bmatrix} 1/\sqrt{7} \\ (1-i)/\sqrt{7} \\ 2i/\sqrt{7} \end{bmatrix}$, (c) $\begin{bmatrix} -i/\sqrt{5} \\ (1+i)/\sqrt{5} \\ (1-i)/\sqrt{5} \end{bmatrix}$, (d) $\begin{bmatrix} 1/\sqrt{8} \\ 2/\sqrt{8} \\ i/\sqrt{8} \\ (1-i)/\sqrt{8} \end{bmatrix}$.

(21) $x = -(1 + \frac{3}{2}i)$. **(22)** $x = -3 - 15i, y = 3 - 3i$. **(23)** $\begin{bmatrix} \pm 1/\sqrt{3} \\ \mp 1/\sqrt{3} \\ \pm 1/\sqrt{3} \end{bmatrix}$.

(24) $\mathbf{q}_1 = \begin{bmatrix} i/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$, $\mathbf{q}_2 = \begin{bmatrix} 0 \\ i/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$, $\mathbf{q}_3 = \begin{bmatrix} 0 \\ 1/2 + i/2 \\ 0 \\ 1/2 - i/2 \end{bmatrix}$, $\mathbf{q}_4 = \begin{bmatrix} -1/2 - i/2 \\ 0 \\ 1/2 - i/2 \\ 0 \end{bmatrix}$.

(25) $\mathbf{q}_1 = \begin{bmatrix} 2/3 \\ (1 + 2i)/3 \end{bmatrix}$, $\mathbf{q}_2 = \begin{bmatrix} (-6 + 7i)/\sqrt{153} \\ (8 + 2i)/\sqrt{153} \end{bmatrix}$.

(26) $\mathbf{q}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ i/\sqrt{2} \end{bmatrix}$, $\mathbf{q}_2 = \begin{bmatrix} (1 + 2i)/\sqrt{14} \\ 2/\sqrt{14} \\ (2 - i)/\sqrt{14} \end{bmatrix}$, $\mathbf{q}_3 = \begin{bmatrix} (-2 - i)/\sqrt{35} \\ (4 - 3i)/\sqrt{35} \\ (-1 + 2i)/\sqrt{35} \end{bmatrix}$.

Section 10.2

(1) Take transposes. **(2)** (c), (f), and (g). **(3)** No. **(4)** No.
(5) Yes.

Section 10.3

(1) (a) $\mathbf{q}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$, $\mathbf{q}_2 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$, $\mathbf{q}_3 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$.

(b) $\mathbf{q}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \\ 0 \end{bmatrix}$, $\mathbf{q}_2 = \begin{bmatrix} 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$, $\mathbf{q}_3 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \end{bmatrix}$, $\mathbf{q}_4 = \begin{bmatrix} 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}$.

(c) $\mathbf{q}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$, $\mathbf{q}_2 = \begin{bmatrix} 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}$, $\mathbf{q}_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \\ 0 \end{bmatrix}$, $\mathbf{q}_4 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \end{bmatrix}$.

(d) $\mathbf{q}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{q}_2 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$, $\mathbf{q}_3 = \begin{bmatrix} 0 \\ 1/\sqrt{6} \\ -2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$, $\mathbf{q}_4 = \begin{bmatrix} 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}$.

(2) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$; the matrix is not real. **(3)** Follow the proof of Theorem 1.

(9) $(\mathbf{A} + \mathbf{B})^{\mathsf{T}} = \mathbf{A}^{\mathsf{T}} + \mathbf{B}^{\mathsf{T}} = \mathbf{A} + \mathbf{B}$.

(10) $(\mathbf{A} - k\mathbf{I})^{\mathsf{T}} = \mathbf{A}^{\mathsf{T}} - (k\mathbf{I})^{\mathsf{T}} = \mathbf{A} - k\mathbf{I}$.

(11) The two matrices must commute.

(15) Show that $\mathbf{B} = 0.5(\mathbf{A} + \mathbf{A}^{\mathsf{T}})$ is symmetric, that $\mathbf{C} = 0.5(\mathbf{A} - \mathbf{A}^{\mathsf{T}})$ is skew-symmetric, and that $\mathbf{A} = \mathbf{B} + \mathbf{C}$.

(16) Must be zero, because each diagonal element must equal its own negative.

Section 10.4

(1) (a), (c), and (d) are orthogonal, (b) is not.

(2) $\mathbf{P} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix}$.

(3) Show that $(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^{\mathsf{T}} = \mathbf{P}^{-1}\mathbf{A}^{\mathsf{T}}\mathbf{P}$.

(5) $\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 0 & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{3} & 1/\sqrt{3} & 0 & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix}$.

(6) Using Eq. (5), $\|\mathbf{P}\mathbf{x}\|^2 = \langle \mathbf{P}\mathbf{x}, \mathbf{P}\mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle = \|\mathbf{x}\|^2$.

(7) Use Eq. (5) and the results of Problem 6.

(8) $(\mathbf{P}\mathbf{Q})^{\mathsf{T}} = \mathbf{Q}^{\mathsf{T}}\mathbf{P}^{\mathsf{T}} = \mathbf{Q}^{-1}\mathbf{P}^{-1} = (\mathbf{P}\mathbf{Q})^{-1}$. **(9)** $\begin{bmatrix} 3/5 & -4/5 \\ -4/5 & -3/5 \end{bmatrix}$.

(10) $\begin{bmatrix} 7/25 & 24/25 \\ 24/25 & -7/25 \end{bmatrix}$. **(11)** $\dfrac{1}{11}\begin{bmatrix} 9 & 2 & 6 \\ 2 & 9 & -6 \\ 6 & -6 & -7 \end{bmatrix}$. **(12)** $\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$.

(13)
$$\mathbf{R}^{\mathsf{T}} = \left[\mathbf{I} - 2\left(\frac{\mathbf{x}\mathbf{x}^{\mathsf{T}}}{\|\mathbf{x}\|^2}\right)\right]^{\mathsf{T}} = \mathbf{I}^{\mathsf{T}} - \left(2\frac{\mathbf{x}\mathbf{x}^{\mathsf{T}}}{\|\mathbf{x}\|^2}\right)^{\mathsf{T}}$$
$$= \mathbf{I} - 2\frac{(\mathbf{x}\mathbf{x}^{\mathsf{T}})^{\mathsf{T}}}{\|\mathbf{x}\|^2} = \mathbf{I} - 2\frac{\mathbf{x}\mathbf{x}^{\mathsf{T}}}{\|\mathbf{x}\|^2} = \mathbf{R}.$$

(14) $$\mathbf{RR} = \left[\mathbf{I} - 2\left(\frac{\mathbf{x}\mathbf{x}^{\mathrm{T}}}{\|\mathbf{x}\|^2}\right)\right]\left[\mathbf{I} - 2\left(\frac{\mathbf{x}\mathbf{x}^{\mathrm{T}}}{\|\mathbf{x}\|^2}\right)\right]$$
$$= \mathbf{I} - 4\frac{\mathbf{x}\mathbf{x}^{\mathrm{T}}}{\|\mathbf{x}\|^2} + 4\frac{\mathbf{x}(\mathbf{x}^{\mathrm{T}}\mathbf{x})\mathbf{x}^{\mathrm{T}}}{\|\mathbf{x}\|^4}$$
$$= \mathbf{I} - 4\frac{\mathbf{x}\mathbf{x}^{\mathrm{T}}}{\|\mathbf{x}\|^2} + 4\frac{\mathbf{x}\|\mathbf{x}\|^2\mathbf{x}^{\mathrm{T}}}{\|\mathbf{x}\|^4} = \mathbf{I}.$$

(15) $\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{RR} = \mathbf{I}$ using Problems 13 and 14

Section 10.5

(1) (a) Hermitian;
$$\mathbf{q}_1 = \begin{bmatrix} i/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix};\ \mathbf{q}_2 = \begin{bmatrix} -1/2 \\ i/2 \end{bmatrix}.$$
(b) Not Hermitian. (c) Not Hermitian.
(d) Hermitian;
$$\mathbf{q}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{q}_2 = \begin{bmatrix} -3/\sqrt{10} \\ 0 \\ i/\sqrt{10} \end{bmatrix},\ \mathbf{q}_3 = \begin{bmatrix} i/\sqrt{10} \\ 0 \\ -3/\sqrt{10} \end{bmatrix}.$$
(e) Hermitian;
$$\mathbf{q}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 0 \\ i/\sqrt{3} \\ -i/\sqrt{3} \end{bmatrix},\ \mathbf{q}_2 = \begin{bmatrix} -i/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \\ 0 \end{bmatrix},\ \mathbf{q}_3 = \begin{bmatrix} 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix},\ \mathbf{q}_4 = \begin{bmatrix} i/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \\ -1/\sqrt{3} \end{bmatrix}.$$
(f) Not Hermitian.

(4) Use the properties of the inner product.

(5) Note that each of the four terms on the right-hand side of the equality in Problem 4 is of the form $\langle \mathbf{Az}, \mathbf{z} \rangle$, hence, is real. Thus, $\langle \mathbf{Az}, \mathbf{z} \rangle = \overline{\langle \mathbf{Az}, \mathbf{z} \rangle} = \langle \mathbf{z}, \mathbf{Az} \rangle = \langle \mathbf{A}^*\mathbf{z}, \mathbf{z} \rangle$. Use this result to show that $\langle \mathbf{Ax}, \mathbf{y} \rangle = \langle \mathbf{A}^*\mathbf{x}, \mathbf{y} \rangle$ and then conclude that $\mathbf{A} = \mathbf{A}^*$.

(6) $(\mathbf{A} + \mathbf{B})^{\mathrm{H}} = (\overline{\mathbf{A} + \mathbf{B}})^{\mathrm{T}} = (\bar{\mathbf{A}} + \bar{\mathbf{B}})^{\mathrm{T}} = \bar{\mathbf{A}}^{\mathrm{T}} + \bar{\mathbf{B}}^{\mathrm{T}} = \mathbf{A}^{\mathrm{H}} + \mathbf{B}^{\mathrm{H}} = \mathbf{A} + \mathbf{B}$.

(7) $(\mathbf{A} - k\mathbf{I})^{\mathrm{H}} = (\overline{\mathbf{A} - k\mathbf{I}})^{\mathrm{T}} = (\bar{\mathbf{A}} - \overline{k\mathbf{I}})^{\mathrm{T}} = \bar{\mathbf{A}}^{\mathrm{T}} - k\bar{\mathbf{I}}^{\mathrm{T}} = \mathbf{A}^{\mathrm{H}} - k\mathbf{I} = \mathbf{A} - k\mathbf{I}$.

(8) When the matrices commute.

(10) $$(e^{\mathbf{A}})^{\mathrm{H}} = \left(\sum_{n=0}^{\infty} \frac{\mathbf{A}^n}{n!}\right)^{\mathrm{H}} = \sum_{n=0}^{\infty} \frac{(\mathbf{A}^n)^{\mathrm{H}}}{n!} = \sum_{n=0}^{\infty} \frac{(\mathbf{A}^{\mathrm{H}})^n}{n!} = \sum_{n=0}^{\infty} \frac{\mathbf{A}^n}{n!} = e^{\mathbf{A}}.$$

(12) Show that $\mathbf{B} = 0.5(\mathbf{A} + \mathbf{A}^{\mathrm{H}})$ is Hermitian, that $\mathbf{C} = 0.5(\mathbf{A} - \mathbf{A}^{\mathrm{H}})$ is skew-Hermetian, and that $\mathbf{A} = \mathbf{B} + \mathbf{C}$.

(13) They are zero.

(14) $(i\mathbf{A})^{\mathrm{H}} = (\overline{i\mathbf{A}})^{\mathrm{T}} = (-i\bar{\mathbf{A}})^{\mathrm{T}} = -i\bar{\mathbf{A}}^{\mathrm{T}} = i(-\mathbf{A}^{\mathrm{H}}) = i\mathbf{A}$.

Section 10.6

(1) (a), (c), and (d) are unitary, (b) is not.

(2) $\mathbf{U} = 1/\sqrt{3}\begin{bmatrix} 1 & 0 & -i & i \\ 0 & 1 & 1 & 1 \\ i & 1 & -1 & 0 \\ -i & 1 & 0 & -1 \end{bmatrix}$.

(4) Show that $(\mathbf{U}^{-1}\mathbf{AU})^* = \mathbf{U}^{-1}\mathbf{A}^*\mathbf{U}$. **(6)** $\begin{bmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ i/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$.

(7) They are the same because the matrices are similar.

(8) $\|\mathbf{Ux}\|^2 = \langle \mathbf{Ux}, \mathbf{Ux} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle = \|\mathbf{x}\|^2$.

(9) $\langle \mathbf{Ux}, \mathbf{Uy} \rangle = \langle \mathbf{x}, \mathbf{U}^*\mathbf{Uy} \rangle = \langle \mathbf{x}, \mathbf{U}^{\mathsf{H}}\mathbf{Uy} \rangle = \langle \mathbf{x}, \mathbf{U}^{-1}\mathbf{Uy} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$

(10) $(\mathbf{UV})^{\mathsf{H}} = \mathbf{V}^{\mathsf{H}}\mathbf{U}^{\mathsf{H}} = \mathbf{V}^{-1}\mathbf{U}^{-1} = (\mathbf{UV})^{-1}$.

(11) If $\mathbf{Ax} = \lambda\mathbf{x}$, then $\lambda\langle \mathbf{x}, \mathbf{x} \rangle = \langle \lambda\mathbf{x}, \mathbf{x} \rangle = \langle \mathbf{Ax}, \mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{A}^{\mathsf{H}}\mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{A}^{-1}\mathbf{x} \rangle =$ $\langle \mathbf{x}, (1/\lambda)\mathbf{x} \rangle = (1/\bar{\lambda})\langle \mathbf{x}, \mathbf{x} \rangle$. Then $\lambda\bar{\lambda} = 1$, or $|\lambda|^2 = 1$.

(12) The determinant is the product of the eigenvalues.

(13)
$$\mathbf{U}^{\mathsf{H}} = \frac{1-i}{\sqrt{2}}\mathbf{A}^{\mathsf{T}} = \frac{1-i}{\sqrt{2}}\mathbf{A}^{-1},$$
$$\text{so } \mathbf{UU}^{\mathsf{H}} = \left(\frac{1+i}{\sqrt{2}}\mathbf{A}\right)\left(\frac{1-i}{\sqrt{2}}\mathbf{A}^{-1}\right) = \mathbf{AA}^{-1} = \mathbf{I}.$$

Section 10.8

(1) (a), (c), (e), and (f) are not positive definite. Vectors are
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

(2) (a) Positive definite. (b) Neither.
(c) Neither. (d) Nonnegative definite.

(7) $\langle (\mathbf{A} + \mathbf{B})\mathbf{x}, \mathbf{x} \rangle = \langle \mathbf{Ax}, \mathbf{x} \rangle + \langle \mathbf{Bx}, \mathbf{x} \rangle > 0$ if $\mathbf{x} \neq \mathbf{0}$.

(8) $\langle \mathbf{A}^{\mathsf{H}}\mathbf{x}, \mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{Ax} \rangle = \overline{\langle \mathbf{Ax}, \mathbf{x} \rangle} > 0$.

(9) The determinant is the product of the eigenvalues.

(10) The inverse has reciprocal eigenvalues, which are all positive if the eigenvalues themselves are. The inverse exists and is also positive definite.

(11) $\langle \mathbf{T}^{\mathsf{H}}\mathbf{ATx}, \mathbf{x} \rangle = \langle \mathbf{ATx}, \mathbf{Tx} \rangle = \langle \mathbf{Ay}, \mathbf{y} \rangle > 0$ if $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y} = \mathbf{Tx}$.

(13) $\begin{bmatrix} 1.577670 & 0.714813i \\ -0.714813i & 1.220263 \end{bmatrix}$. **(14)** $\begin{bmatrix} 3/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$.

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