Using FORM in the context of Compton scattering

A. Context

Figure (1) gives the Feynman diagrams corresponding to the two lowest-order processes contributing to Compton scattering. As per Griffiths' (2008, Example 7.4) the probability amplitude $\mathcal{M}_1$ associated with the first of those Feynman diagrams is:

$$\frac{\mathcal{M}_1}{i} = \int \frac{d^4q}{(2\pi)^4} \underbrace{\epsilon_{\mu(2)}}_{\gamma \text{ in}} \left[\underbrace{\bar{u}_4}_{e^- \text{ out}} \underbrace{ig_e\gamma^\mu}_{\text{vertex}} \underbrace{\frac{i(\not{q} + mc)}{q^2 - m^2c^2}}_{\text{propogator}} \underbrace{ig_e\gamma^\kappa}_{\text{vertex}} \underbrace{u_1}_{e^- \text{ in}} \right] \underbrace{\epsilon^*_{\kappa(3)}}_{\gamma \text{ out}} \times$$

$$(2\pi)^8 \, \delta^4(p_1 - p_3 - q) \, \delta^4(p_2 + q - p_4) \, , \tag{1}$$

where $\bar{u}_4$ is the (adjoint) spinor associated with the outgoing electron, u_1 is the spinor representing the incoming electron, $\epsilon_{\mu(2)}$ are the components of the polarization 4-vector for the incoming photon (etc.), and the 'slash' notation $\not{q} \equiv q^\lambda \gamma_\lambda$. Spin and momentum labels have been dropped, i.e. more completely, for example, $\bar{u}_4 = \bar{u}^{(s_4)}(p_4)$ with two possible spin states ($s_4 = 1, 2$). (Implicitly, the mc factor is multiplied by the identity matrix.)

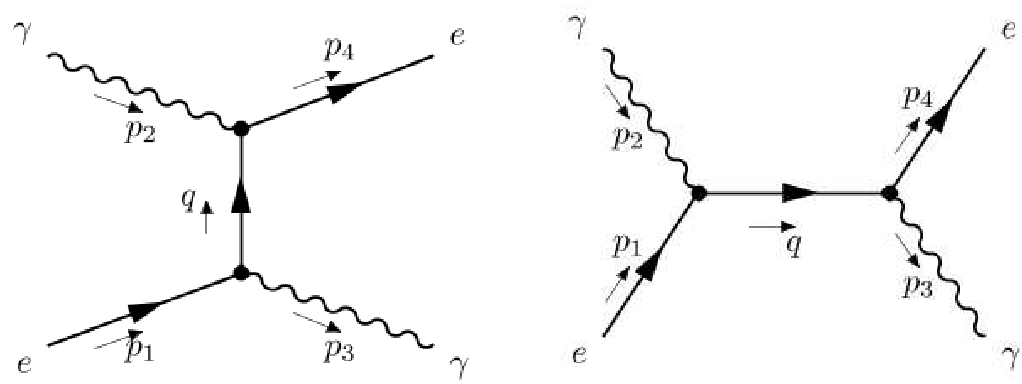

Figure 1: Feynman diagram of the two lowest-order (i.e. 2-vertex) processes contributing to Compton scattering (Griffths 2008, p246). Associated with the diagram on the left is an amplitude $\mathcal{M}_1$.

Integration sends $q \to (p_1 - p_3)$, and we drop the factor $(2\pi)^4 \, \delta^4(p_1 + p_2 - p_3 - p_4)$. Then noting that for fixed index μ the quantity $\epsilon_{\mu(2)}$ is just a scalar (and ditto $\epsilon^*_{\kappa(3)}$) so that $\epsilon_{\mu(2)}\bar{u}_4 = \bar{u}_4 \, \epsilon_{\mu(2)}$ (etc.), the amplitude associated with diagram 1 can be written

$$\mathcal{M}_1 = \frac{g_e^2}{(p_1 - p_3)^2 - m^2c^2} \, \left[\bar{u}_4 \not{\epsilon}_2 \, (\not{p}_1 - \not{p}_3 + mc) \, u_1 \not{\epsilon}^*_3 \right] \, , \tag{2}$$

where $\not\epsilon_2 \equiv \epsilon_{\mu(2)}\gamma^\mu$. Analogous steps for the second diagram give amplitude

$$\mathcal{M}_2 = \frac{g_e^2}{(p_1+p_2)^2 - m^2c^2} \left[\bar{u}_4 \not\epsilon_3^* \,(\not p_1 + \not p_2 + mc)\; u_1 \not\epsilon_2\right] . \tag{3}$$

To evaluate the cross section for Compton scattering the quantity

$$|\mathcal{M}|^2 = \mathcal{M}_1\mathcal{M}_1^* + \mathcal{M}_2\mathcal{M}_2^* + \mathcal{M}_1\mathcal{M}_2^* + \mathcal{M}_2\mathcal{M}_1^* \tag{4}$$

is needed. After some manipulation one obtains

$$\mathcal{M}_1\mathcal{M}_1^* = A^2\, \bar{u}_4 \not\epsilon_2 \,[\not p_1 - \not p_3 + mc]\, \not\epsilon_3^* u_1\, \bar{u}_1 \not\epsilon_3\, [\not p_1 - \not p_3 + mc]\, \not\epsilon_2^* u_4 , \tag{5}$$

$$\mathcal{M}_2\mathcal{M}_2^* = B^2\, \bar{u}_4 \not\epsilon_3^* \,[\not p_1 + \not p_2 + mc]\, \not\epsilon_2 u_1\, \bar{u}_1 \not\epsilon_2^*\, [\not p_1 + \not p_2 + mc]\, \not\epsilon_3 u_4 , \tag{6}$$

$$\mathcal{M}_1\mathcal{M}_2^* = AB\, \bar{u}_4 \not\epsilon_2 \,[\not p_1 - \not p_3 + mc]\, \not\epsilon_3^* u_1\, \bar{u}_1 \not\epsilon_2^*\, [\not p_1 + \not p_2 + mc]\, \not\epsilon_3 u_4 , \tag{7}$$

$$\mathcal{M}_2\mathcal{M}_1^* = AB\, \bar{u}_4 \not\epsilon_3^* \,[\not p_1 + \not p_2 + mc]\, \not\epsilon_2 u_1\, \bar{u}_1 \not\epsilon_3\, [\not p_1 - \not p_3 + mc]\, \not\epsilon_2^* u_4 , \tag{8}$$

where

$$A = \frac{g_e^2}{(p_1-p_3)^2 - m^2c^2} = \frac{g_e^2}{-2p_1\cdot p_3} , \tag{9}$$

$$B = \frac{g_e^2}{(p_1+p_2)^2 - m^2c^2} = \frac{g_e^2}{2p_1\cdot p_2} \tag{10}$$

(because $p_1^2 = m^2c^2$, $p_2^2 = p_3^2 = 0$).

Then averaging over the initial spin states and summing over final spin states, one obtains the four contributions

$$\langle|\mathcal{M}_1|^2\rangle = \frac{A^2}{4} Q_{\mu\lambda}Q_{\nu\kappa}\ \mathrm{Tr}\left[\gamma^\mu\,(\not p_1 - \not p_3 + mc)\,\gamma^\nu\,(\not p_1 + mc)\,\gamma^\kappa\,(\not p_1 - \not p_3 + mc)\,\gamma^\lambda\,(\not p_4 + mc)\right] , \tag{11}$$

$$\langle|\mathcal{M}_2|^2\rangle = \frac{B^2}{4} Q_{\mu\lambda}Q_{\nu\kappa}\ \mathrm{Tr}\left[\gamma^\mu\,(\not p_1 + \not p_2 + mc)\,\gamma^\nu\,(\not p_1 + mc)\,\gamma^\kappa\,(\not p_1 + \not p_2 + mc)\,\gamma^\lambda\,(\not p_4 + mc)\right] , \tag{12}$$

$$\langle\mathcal{M}_1\mathcal{M}_2^*\rangle = \frac{AB}{4} Q_{\mu\lambda}Q_{\nu\kappa}\ \mathrm{Tr}\left[\gamma^\mu\,(\not p_1 - \not p_3 + mc)\,\gamma^\nu\,(\not p_1 + mc)\,\gamma^\lambda\,(\not p_1 + \not p_2 + mc)\,\gamma^\kappa\,(\not p_4 + mc)\right] , \tag{13}$$

$$\langle\mathcal{M}_2\mathcal{M}_1^*\rangle = \frac{AB}{4} Q_{\mu\lambda}Q_{\nu\kappa}\ \mathrm{Tr}\left[\gamma^\mu\,(\not p_1 + \not p_2 + mc)\,\gamma^\nu\,(\not p_1 + mc)\,\gamma^\lambda\,(\not p_1 - \not p_3 + mc)\,\gamma^\kappa\,(\not p_4 + mc)\right] , \tag{14}$$

to

$$\langle\mathcal{M}^2\rangle = \langle|\mathcal{M}_1|^2\rangle + \langle|\mathcal{M}_2|^2\rangle + \langle\mathcal{M}_1\mathcal{M}_2^*\rangle + \langle\mathcal{M}_2\mathcal{M}_1^*\rangle . \tag{15}$$

In Eqs. (11-14) the tensor $Q_{\mu\lambda}$ is defined

$$Q_{\mu\lambda} = \sum_{s=1,2} \epsilon_\mu^{(s)}\,\epsilon_\lambda^{(s)*} ,$$

where the photon polarization vector $\epsilon^\mu = (\epsilon^0, \epsilon)$ with $\epsilon^0 = 0$. The completeness relation (for the photon polarization three vectors) is

$$\sum_{s=1,2} \epsilon_i^{(s)} \epsilon_j^{(s)*} = \delta_{ij} - \hat{p}_i \hat{p}_j \tag{16}$$

where $\hat{p}_i$ is the unit vector $\hat{p}_i = p_i/|p_i|$ (no summation). Hence

$$Q_{\mu\lambda} = \begin{cases} 0 \text{ , if } \mu = 0 \text{ or } \lambda = 0 \\ \delta_{\mu\lambda} - \hat{p}_\mu \hat{p}_\lambda \text{ , otherwise .} \end{cases} \tag{17}$$

Eqns. (11–17) concur with the solution manual written by D. Griffiths (available at www.academia.edu and elsewhere).

Evaluation of $\langle|\mathcal{M}|^2\rangle$ in terms of Mandelstam variables, using FORM

It can be seen from Eqs. (11, 12) that $\langle|\mathcal{M}_1|^2\rangle$ and $\langle|\mathcal{M}_2|^2\rangle$ are equivalent under the substitution $p_2 \leftrightarrow -p_3$ (or $s \leftrightarrow u$ in terms of the Mandelstam variables $s \equiv (p_1+p_2)^2$ and $u \equiv (p_2-p_4)^2$ used below), and the same symmetry links $\langle\mathcal{M}_1\mathcal{M}_2^*\rangle$ and $\langle\mathcal{M}_2\mathcal{M}_1^*\rangle$. Puzzlingly, that symmetry can be seen at one point in the output of the following FORM program, but, in the ultimate expression, the expected symmetry is not upheld. The final result for $\langle|\mathcal{M}|^2\rangle$, named "sum" in the FORM program below, is incorrect.

```
***  compton_scattering.frm  JDW 23 Oct. 2019 ***

Vectors p1, p2, p3, p4;
Symbols m,s,u,[ge^4],[2ge^4];
Symbols [1/(s-m^2)],[1/(s-m^2)^2],[1/(u-m^2)],[1/(u-m^2)^2],[1/(s-m^2)/(u-m^2)];
Indices alpha,beta,gamma,delta,rho,kappa,lambda,mu,nu;
Off Statistics;

Local traceM1M1 = d_(mu,lambda)*d_(nu,kappa)
                  *g_(1,mu)*(g_(1,p1)-g_(1,p3)+m)
                  *g_(1,nu)*(g_(1,p1)+m)
                  *g_(1,kappa)*(g_(1,p1)-g_(1,p3)+m)
                  *g_(1,lambda)*(g_(1,p4)+m);

Local traceM2M2 = d_(mu,lambda)*d_(nu,kappa)
                  *g_(2,mu)*(g_(2,p1)+g_(2,p2)+m)
                  *g_(2,nu)*(g_(2,p1)+m)
                  *g_(2,kappa)*(g_(2,p1)+g_(2,p2)+m)
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                  *g_(2,lambda)*(g_(2,p4)+m);

Local traceM1M2str = d_(mu,lambda)*d_(nu,kappa)
                    *g_(3,mu)*(g_(3,p1)-g_(3,p3)+m)
                    *g_(3,nu)*(g_(3,p1)+m)
                    *g_(3,lambda)*(g_(3,p1)+g_(3,p2)+m)
                    *g_(3,kappa)*(g_(3,p4)+m);

Local traceM2M1str = d_(mu,lambda)*d_(nu,kappa)
                    *g_(4,mu)*(g_(4,p1)+g_(4,p2)+m)
                    *g_(4,nu)*(g_(4,p1)+m)
                    *g_(4,lambda)*(g_(4,p1)-g_(4,p3)+m)
                    *g_(4,kappa)*(g_(4,p4)+m);


Local M1M1 =    div_(1,2)*[2ge^4]*traceM1M1*div_(1,4)*(-1/(2*p1.p3))^2;

Local M2M2 =    div_(1,2)*[2ge^4]*traceM2M2*div_(1,4)*( 1/(2*p1.p2))^2;

Local M1M2str = div_(1,2)*[2ge^4]*traceM1M2str*div_(1,4)*(-1/(2*p1.p3))*(1/(2*p1.p2));

Local M2M1str = div_(1,2)*[2ge^4]*traceM2M1str*div_(1,4)*(-1/(2*p1.p3))*(1/(2*p1.p2));

Local sum=M1M1+M2M2+M1M2str+M2M1str;

Trace4,1;
Trace4,2;
Trace4,3;
Trace4,4;
.sort;

repeat;
id p2.p2=0;
id p3.p3=0;
id p1.p1=m^2;
id p4.p4=m^2;
endrepeat;
.sort;
```

```
*** Introduce Mandelstam variables.

repeat;
id 1/(p1.p2)=2/(s-m^2);
id 1/(p1.p3)=-2/(u-m^2);
id p1.p4 = (u+s)/2;
id p2.p3=(u+s)/2-m^2;
id p2.p4 =  (m^2-u)/2;
id p3.p4 =  (s-m^2)/2;
endrepeat;
.sort;

repeat;
id s/(s-m^2)=1+m^2/(s-m^2);
endrepeat;
.sort;

repeat;
id 1/(s - m^2)/(s + m^2)=1/(s^2-m^4);
endrepeat;
.sort;

Bracket [2ge^4],m;
Print;
.sort;
.end;
```

Here are some selected outputs from the program, that present a puzzle. Prior to the complete conversion into Mandelstam variables

```
    traceM1M1 =
       + m^2 * (  - 16*u - 80*p1.p3 )

       + m^4 * ( 48 )

       + 16*p1.p3*s;
```

```
traceM2M2 =
    + m^2 * (  - 16*s + 80*p1.p2 )

    + m^4 * ( 48 )

    - 16*p1.p2*u;
```

showing the expected symmetry of $|\mathcal{M}_1|^2$ and $|\mathcal{M}_2|^2$. However upon completion

```
M1M1 =
    + [2ge^4] * (  - 1/(u - m^2)*s )

    + m^2*[2ge^4] * ( 5/(u - m^2) - 2/(u - m^2)/(u - m^2)*u )

    + m^4*[2ge^4] * ( 6/(u - m^2)/(u - m^2) );

M2M2 =
    + [2ge^4] * (  - 1/(s - m^2)*u )

    + m^2*[2ge^4] * ( 3/(s - m^2) )

    + m^4*[2ge^4] * ( 4/(s - m^2)/(s - m^2) );
```

such that the expected symmetry is not evident. The balance of the output

```
M1M2str =
    + [2ge^4] * (  - 1 + 1/(u - m^2)*u - 1/(s - m^2)*u + 1/(s - m^2)/(u -
      m^2)*u^2 )

    + m^2*[2ge^4] * ( 1/(s - m^2) - 1/(s - m^2)/(u - m^2)*u )

    + m^4*[2ge^4] * ( 4/(s - m^2)/(u - m^2) );

M2M1str =
    + [2ge^4] * (  - 1 + 1/(u - m^2)*u - 1/(s - m^2)*u + 1/(s - m^2)/(u -
      m^2)*u^2 )

    + m^2*[2ge^4] * ( 1/(s - m^2) - 1/(s - m^2)/(u - m^2)*u )

    + m^4*[2ge^4] * ( 4/(s - m^2)/(u - m^2) );
```

```
sum =
    + [2ge^4] * (  - 2 + 2/(u - m^2)*u - 1/(u - m^2)*s - 3/(s - m^2)*u + 2
      /(s - m^2)/(u - m^2)*u^2 )

    + m^2*[2ge^4] * ( 5/(u - m^2) - 2/(u - m^2)/(u - m^2)*u + 5/(s - m^2)
       - 2/(s - m^2)/(u - m^2)*u )

    + m^4*[2ge^4] * ( 6/(u - m^2)/(u - m^2) + 8/(s - m^2)/(u - m^2) + 4/(s
       - m^2)/(s - m^2) );
```

does show the expected symmetry of $\langle \mathcal{M}_1 \mathcal{M}_2^* \rangle$, $\langle \mathcal{M}_2 \mathcal{M}_1^* \rangle$, but, judged against Millar's (2014) result

$$\overline{|\mathcal{M}|^2} = 2e^4 \left[4m^4 \left(\frac{1}{s-m^2} + \frac{1}{u-m^2} \right)^2 + 4m^2 \left(\frac{1}{s-m^2} + \frac{1}{u-m^2} \right) - \frac{u-m^2}{s-m^2} - \frac{s-m^2}{u-m^2} \right]$$

the present result (from FORM) for $\langle |\mathcal{M}|^2 \rangle$ (i.e. "sum"), while promising, is not quite correct.

References

Griffiths, D. 2008. *Introduction to Elementary Particles*. Second edn. Wiley-VCH.