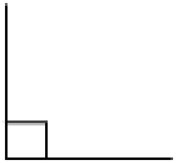
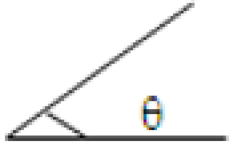
Angles

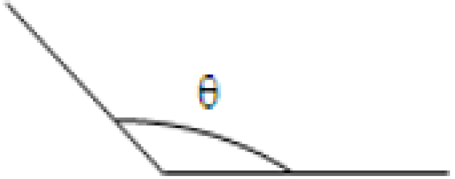
1. Right angle: Equal to 90°



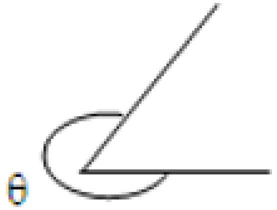
2. Acute angle: Less than 90°



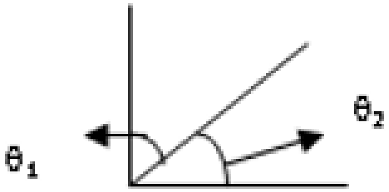
3. Obtuse angle: Angle between 90° and 180°



4. Reflex angle: Angle between 180° and 360°

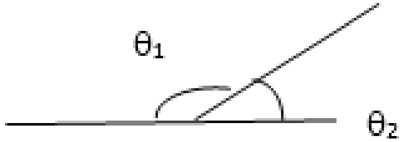


5. Complementary angle: Two angles whose sum is 90°



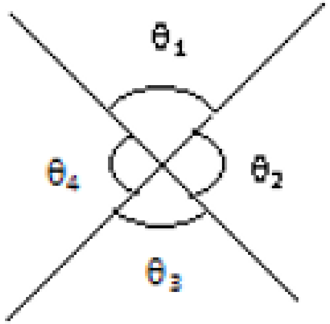
$\theta_1 + \theta_2 = 90°$

6. Supplementary angle: Two angles whose sum is 180°



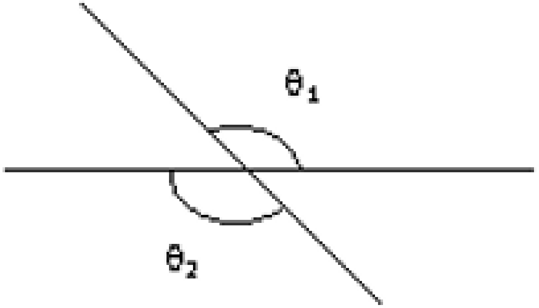
$\theta_1 + \theta_2 = 180°$

7. Sum of all the angles around a point is 360°

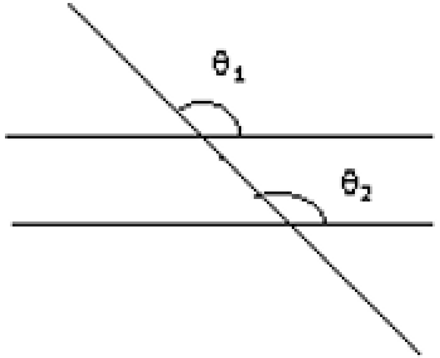


$\theta_1 + \theta_2 + \theta_3 + \theta_4 = 360°$

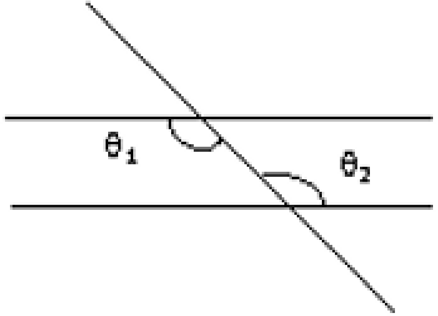
8. Vertically opposite angles are equal ($\theta_1 = \theta_2$)



9. Corresponding angles are equal when two parallel lines are cut by a traversal as shown in figure. Then, $\theta_1 = \theta_2$



10. Alternate angles are equal. [Alternate angles are formed when the figure is in the shape of 'z']

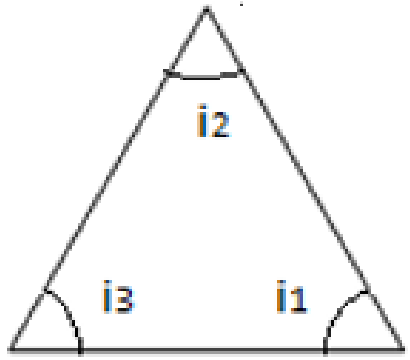


External and Internal angles:

Polygon: Any closed figure is called a polygon. Minimum three sides are required to form a polygon.

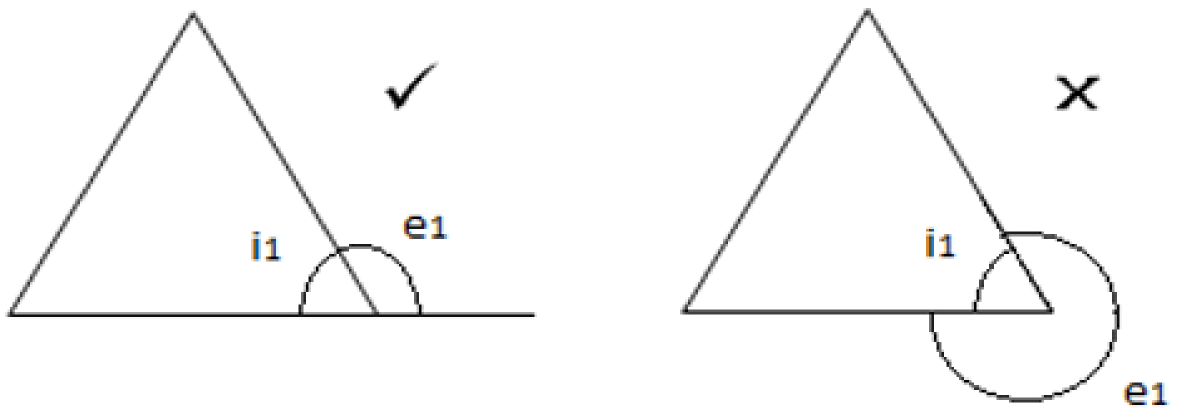
Regular polygon: All the sides and angles are equal. For any polygon, Number of vertices is equal to the number of sides.

Interior angle: In any regular polygon, the sum of interior angle is 180°.
Here i1,i2 and i3 are the interior angles of a triangle.



Exterior angle: Let us compare two figures a & b.

Figure (a) **Figure (b)**

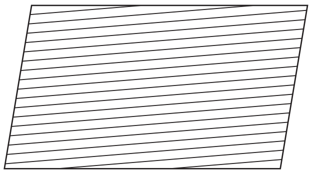


Exterior angle is supplementary but not reflexive. Hence the angle formed by projecting a side to another side
of the polygon is called exterior angle. So, **Figure b is incorrect**.

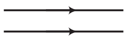
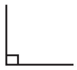
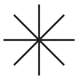
From exterior angle, we can find the interior angle. Because,

$$\text{Interior angle} + \text{Exterior angle} = 180°.$$

Plane : It is a flat surface, having length and breadth both, but no thickness. It is a two dimensional figure.

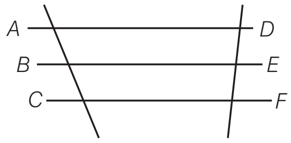


(Plane)

Type of Lines	Definition	Diagram
Parallel lines	Two lines, lying in a plane and having no common intersecting point are called parallel lines. The distance between two parallel lines is constant.	 Symbol : ||
Perpendicular lines	Two lines, which lie in a single plane and intersect eachother at right angle, are called perpendicular lines.	 Symbol : ⊥
Concurrent lines	More than two straight lines intersecting at the same point.	

Points to Remember

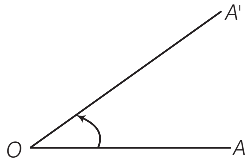
1. A line contains infinitely many points.
2. The intersection of two different lines is a point.
3. Through a given point, there pass an infinite number of lines and these lines are called **concurrent lines.**
4. Only one line can pass through any two particular points.
5. When more than two points lie on a line, they are called as **collinear points** else they are called as **non-collinear** points.
6. Two lines can intersect maximum at one point. This point is called as **point of intersection** and these lines are called as **intersecting lines**.
7. There are an infinite number of planes which pass through a single (particular) point.
8. When more than three points lie in the same plane, they are called as **coplanar,** else they are called as **non-coplanar.**
9. When more than one line lie in the same plane, then these lines are called as coplanar else they are called as non-coplanar.
10. When two planes intersect each other, they form a line *i.e.,* intersecting region is a line.
11. Two different lines which are perpendicular to the same (a third line) line, are necessarily parallel to each other, if all of them are lying on the same plane.
12. When two or more parallel lines are intercepted by some other intercepting lines, then the ratio of corresponding intercepts are equal. An intercepting line is generally called as a transversal.



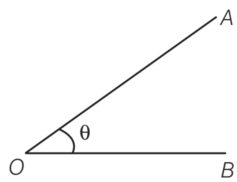
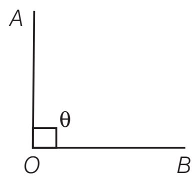
i.e., $\dfrac{AB}{BC} = \dfrac{DE}{EF}$

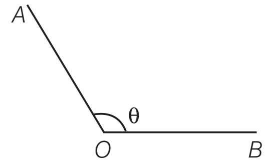
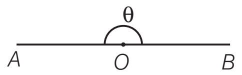
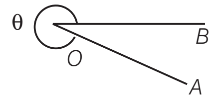
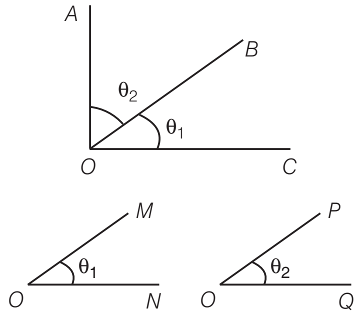
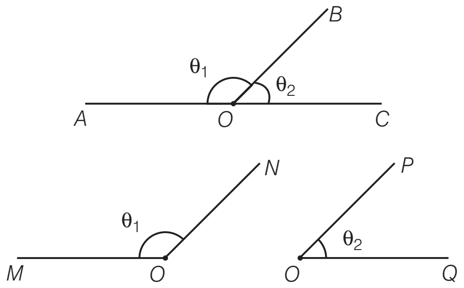
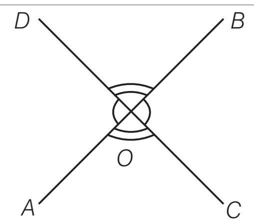
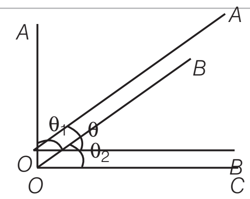
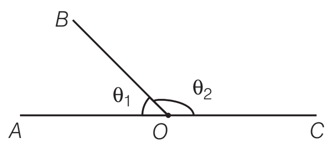
Angles

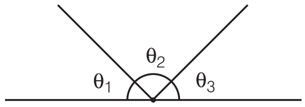
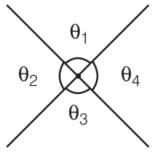
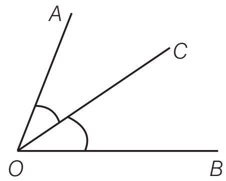
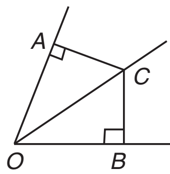
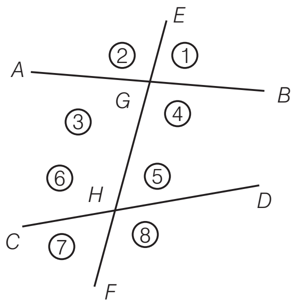
The amount of rotation about O, the vertex of the angle AOA', is called the magnitude of the angle.



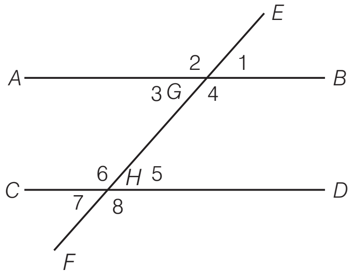
$m\angle AOA'$ denotes the measure of $\angle AOA'$. Angles are said to be congruent when their measure is same.
(symbol : ≅)

Types of Angles	Property	Diagram
Acute	$0° < \theta < 90°$ ($\angle AOB$ is an acute angle)	
Right	$\theta = 90°$ ($\angle AOB$ is a right angle)	

Types of Angles	Property	Diagram
Obtuse	$90° < \theta < 180°$ (∠AOB is an obtuse angle)	
Straight	$\theta = 180°$ (∠AOB is a straight angle)	
Reflex	$180° < \theta < 360°$ (∠AOB is a reflex angle)	
Complementary	$\theta_1 + \theta_2 = 90°$ Two angles whose sum is 90°, are complementary to each other	
Supplementary	$\theta_1 + \theta_2 = 180°$ Two angles whose sum is 180°, are supplementary to each other.	
Vertically opposite	∠DOA = ∠BOC and ∠DOB = ∠AOC	
Adjacent angles	∠AOB and ∠BOC are adjacent angles Adjacent angles must have a common side. (*e.g., OB*)	
Linear pair	∠AOB and ∠BOC are linear pair angles. One side must be common (*e.g., OB*) and these two angles must be supplementary.	

Types of Angles	Property	Diagram
Angles on the one side of a line	$\theta_1 + \theta_2 + \theta_3 = 180°$	
Angles round the point	$\theta_1 + \theta_2 + \theta_3 + \theta_4 = 360°$	
Angle bisector	OC is the angle bisector of $\angle AOB$. *i.e.*, $\angle AOC = \angle BOC = \frac{1}{2}(\angle AOB)$ When a line segment divides an angle equally into two parts, then it is said to be the angle bisector (OC)	 (Angle bisector is equidistant from the two sides of the angle) *i.e.*, $AC = BC$ 
Corresponding angles	When two lines are intersected by a transversal, then they form four pairs of corresponding angles (a) $\angle AGE, \angle CHG \Rightarrow (\angle 2, \angle 6)$ (b) $\angle AGH, \angle CHF \Rightarrow (\angle 3, \angle 7)$ (c) $\angle EGB, \angle GHD \Rightarrow (\angle 1, \angle 5)$ (d) $\angle BGH, \angle DHF \Rightarrow (\angle 4, \angle 8)$	
Exterior angles	These are following four angles (i) $\angle AGE \Rightarrow \angle 2$ (ii) $\angle CHF \Rightarrow \angle 7$ (iii) $\angle EGB \Rightarrow \angle 1$ (iv) $\angle DHF \Rightarrow \angle 8$	
Interior angles	These are following four angles (i) $\angle AGH \Rightarrow \angle 3$ (ii) $\angle GHC \Rightarrow \angle 6$ (iii) $\angle BGH \Rightarrow \angle 4$ (iv) $\angle DHF \Rightarrow \angle 5$	
Alternate angles	These are two pairs angles as following: (i) $\angle AGH, \angle GHD \ (\angle 3, \angle 5)$ (ii) $\angle GHC, \angle BGH \ (\angle 6, \angle 4)$	

When two parallel lines are intersected by a transversal, then



1. The pairs of corresponding angles so formed are congruent.

   i.e., $\angle 2 = \angle 6$

   $\angle 3 = \angle 7$

   $\angle 1 = \angle 5$

   $\angle 4 = \angle 8$

NOTE 1. If one pair of corresponding angles is congruent, then the rest pairs of corresponding angles are also congruent.

2. The pairs of alternate angles so formed are congruent

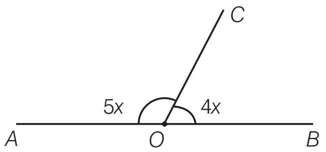
   i.e., $\angle 3 = \angle 5$ and $\angle 4 = \angle 6$

3. The pair of interior angles (*i.e.,* the interior angles on the same side of a transversal) are supplementary.

NOTE The converse of all the 3 rules is also true *i.e.,* if the pair of corresponding angles, or alternate angles or interior angles is equal (*i.e.,* congruent), then the two lines are parallel which are intersected by a transversal.

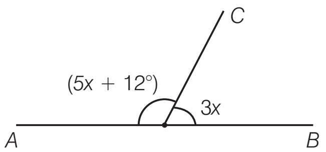
Introductory Exercise 12.1

1. What is the value of x in the following figure?



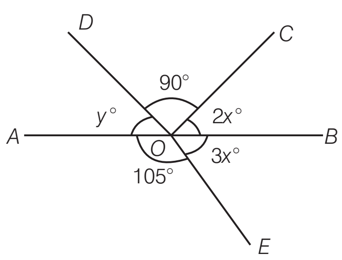
(a) 80° (b) 40°
(c) 20° (d) 25°

2. What is the value of x in the following figure?



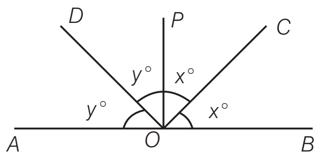
(a) 18° (b) 20°
(c) 21° (d) 24°

3. In the following figure AB is a straight line. Find $(x + y)$.



(a) 55° (b) 65°
(c) 75° (d) 80°

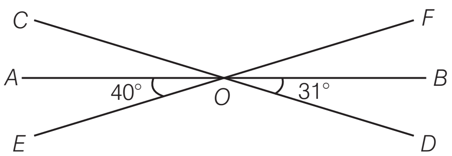
4. In the following figure $\angle BOP = 2x°$, $\angle AOP = 2y°$, OC and OD are angle bisectors of $\angle BOP$ and $\angle AOP$ respectively.



Find the value of $\angle COD$.

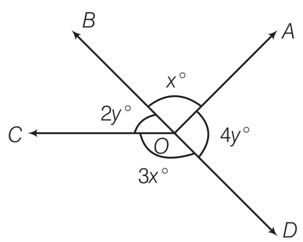
(a) 75° (b) 90° (c) 100° (d) 120°

5. In the following figure find the value of $\angle BOC$.



(a) 101° (b) 149° (c) 71° (d) 140°

6. Find y, if $x° = 36°$, as per the given diagram.



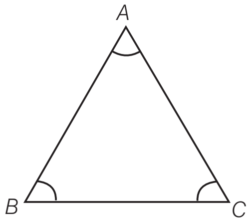
(a) 36° (b) 16° (c) 12° (d) 42°

7. If $(2x + 17)°$, $(x + 4)°$ are complementary, find x.

(a) 63° (b) 53° (c) 35° (d) 23°

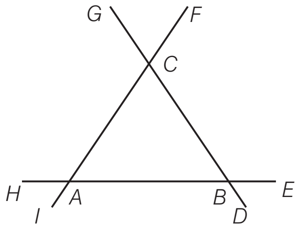
12.3 Triangles

Triangle : A three sided closed plane figure, which is formed by joining the three non-collinear points, is called as a triangle. It is denoted by the symbol Δ.



In the above Δ (triangle) ABC, A, B and C are three **vertices**, line segments AB, BC and AC are the three **sides** of the triangle. $\angle A, \angle B$ and $\angle C$ are the three **interior angles** of a triangle ABC.

In the adjoining figure $\angle FCB, \angle CBE, \angle ABD, \angle IAB, \angle HAC, \angle GCA$ are the **exterior angles** of the ΔABC.



Sum of the three interior angles of a triangle is always 180°.

Exterior angle = Sum of two interior opposite angles

e.g., $\angle CBE = \angle CAB + \angle BCA$

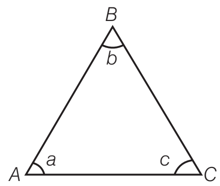
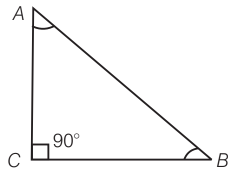
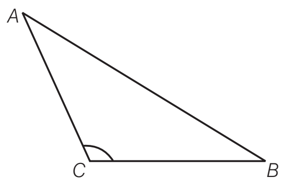
Perimeter of triangle is equal to sum of all the three sides *i.e.,* $a + b + c$

Semiperimeter of a triangle is half of the perimeter

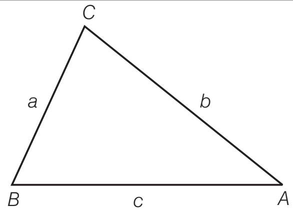
i.e., $s = \dfrac{a+b+c}{2}$, a, b, c are the length of three sides of a triangle.

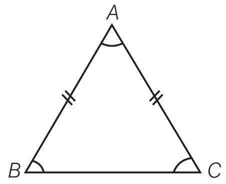
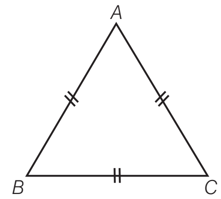
Types of Triangles

(A) According to interior angles

Types of Triangles	Property/Definition	Diagram
Acute angle triangle	Each of the angle of a triangle is less than 90° *i.e.,* $a < 90°, b < 90°, c < 90°$	 $\{\angle a, \angle b, \angle c\} < 90°$
Right angled triangle	One of the angle is equal to 90°, then it is called as right angled triangle. Rest two angles are complementary to each other.	 $\angle C = 90°$
Obtuse angle triangle	One of the angle is obtuse (*i.e.,* greater than 90°), then it is called as obtuse angle triangle.	 $\angle C > 90°$

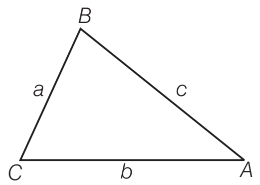
(B) According to the length of sides.

Types of Triangles	Property/Definition	Diagram
Scalene triangle	A triangle in which none of the three sides is equal is called a scalene triangle (all the three angles are also different).	 $a \neq b \neq c$

Isosceles triangle	A triangles in which at least two sides are equal is called an isosceles triangle. In this triangle, the angles opposite to the congruent sides are also equal.	 $AB = AC,\ \angle B = \angle C$
Equilateral triangle	A triangle in which all the three sides are equal called an equilateral triangle. In this triangle each angle is congruent and equal to 60°.	 $AB = BC = AC$ $\angle A = \angle B = \angle C = 60°$

Fundamental Properties of Triangles

1. Sum of any two sides is always greater than the third side.
2. The difference of any two sides is always less than the third side.
3. Greater angle has a greater side opposite to it and smaller angle has a smaller side opposite to it *i.e.,* if two sides of a triangle are not congruent then the angle opposite to the greater side is greater.
4. Let a, b and c be the three sides of a ΔABC and c is the largest side. Then



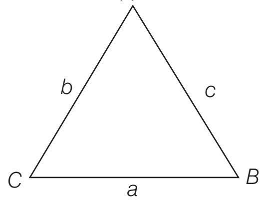
(i) if $c^2 < a^2 + b^2$, the triangle is **acute angle triangle**

(ii) if $c^2 = a^2 + b^2$, the triangle is **right angled triangle**

(iii) if $c^2 > a^2 + b^2$, the triangle is **obtuse angle triangle**

5. **Sine rule :** In a ΔABC, if a, b, c be the three sides opposite to the angles A, B, C respectively, then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



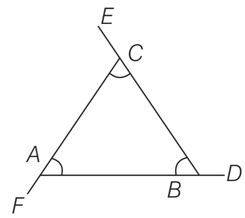
6. **Cosine rule :** In a ΔABC, if a, b, c be the sides opposite to angle A, B and C respectively, then

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \cos B = \frac{c^2 + a^2 - b^2}{2ca},$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

(These rules have been discussed already in trigonometry.)

7. The sum of all the three interior angles is always 180°



i.e., $\angle CAB + \angle ABC + \angle BCA = 180°$

8. The sum of three (ordered) exterior angles of a triangle is 360°

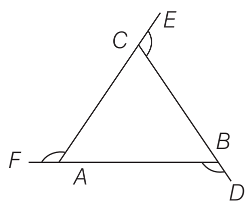


Fig. (i)

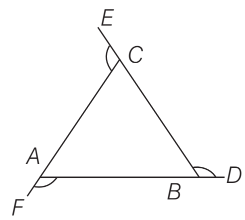


Fig. (ii)

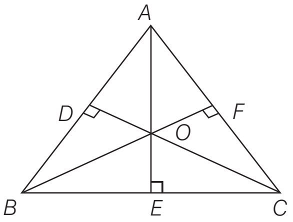
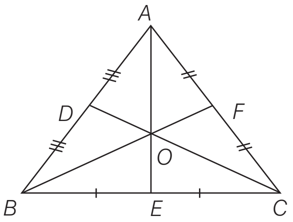
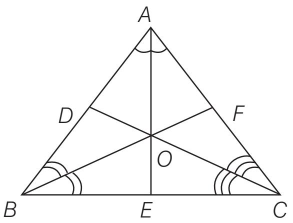
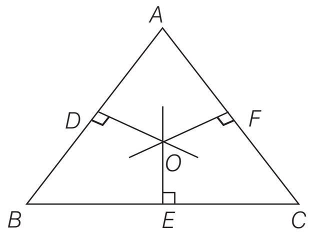
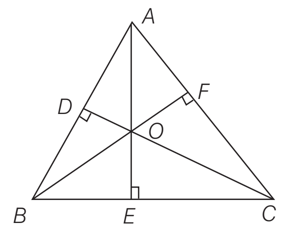
In fig. (i) : $(\angle FAC + \angle ECB + \angle DBA) = 360°$

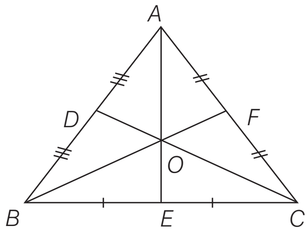
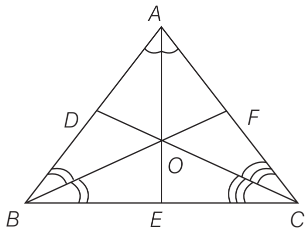
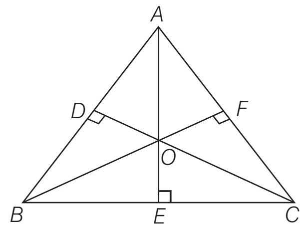
In fig. (ii) : $(\angle FAB + \angle DBC + \angle ECA) = 360°$

9. The sum of an interior angle and its adjacent exterior angle is 180°.

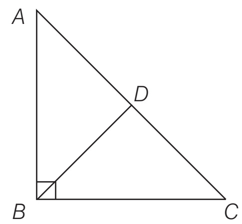
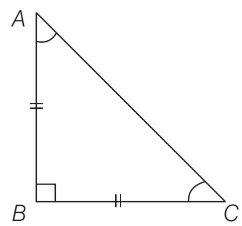
10. A triangle must has at least two acute angles.
11. In a triangle, the measure of an exterior angle equals the sum of the measures of the interior opposite angles.
12. The measure of an exterior angle of a triangle is greater than the measure of each of the opposite interior angles.

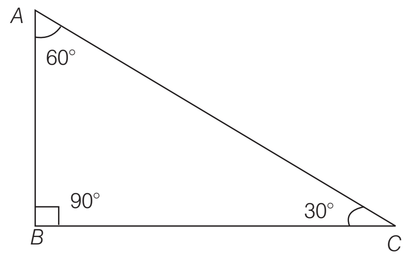
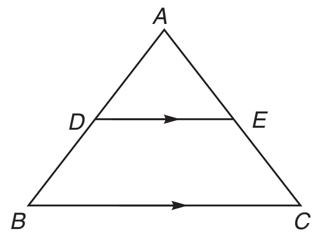
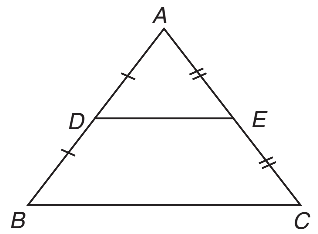
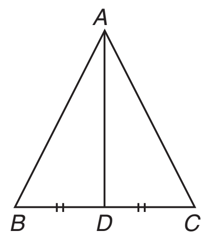
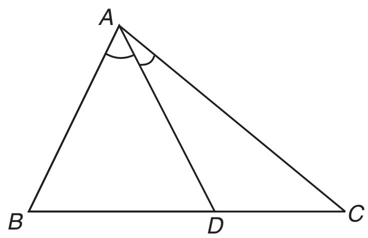
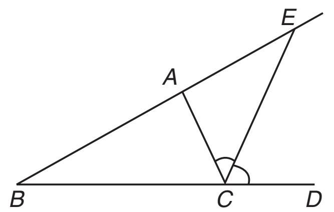
Important Definitions

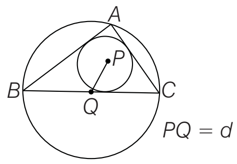
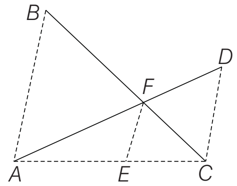
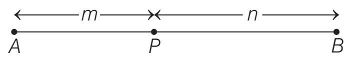
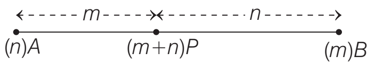
Nomenclature	Property/Definition	Diagram
Altitude (or height)	The perpendicular drawn from the opposite vertex of a side in a triangle is called an altitude of the triangle. • There are three altitudes in a triangle.	 *AE*, *CD* and *BF* are the altitudes
Median	The line segment joining the mid-point of a side to the vertex opposite to the side is called a median. • There are three medians in a triangle. • A median bisects the area of the triangle *i.e.,* $A(\Delta ABE) = A(\Delta AEC) = \frac{1}{2} A(\Delta ABC)$ etc.	 *AE*, *CD* and *BF* are the medians ($BE = CE$, $AD = BD$, $AF = CF$)
Angle bisector	A line segment which originates from a vertex and bisects the same angle is called an angle bisector. $\left(\angle BAE = \angle CAE = \frac{1}{2}\angle BAC\right)$ etc.	 *AE*, *CD* and *BF* are the angle bisectors.
Perpendicular bisector	A line segment which bisects a side perpendicularly (i.e, at right angle) is called a perpendicular bisector of a side of triangle. • All points on the perpendicular bisector of a line are equidistant from the ends of the line.	 *DO*, *EO* and *FO* are the perpendicular bisectors.
Orthocentre	The point of intersection of the three altitudes of the triangle is called the orthocentre. $\angle BOC = 180 - \angle A$ $\angle COA = 180 - \angle B$ $\angle AOB = 180 - \angle C$	 '*O*' is the orthocentre

Types of Triangles	Property/Definition	Diagram
Centroid	The point of intersection of the three medians of a triangle is called the centroid. A centroid divides each median in the ratio 2 : 1 (vertex : base) $$\frac{AO}{OE}=\frac{CO}{OD}=\frac{BO}{OF}=\frac{2}{1}$$	 'O' is the centroid.
Incentre	The point of intersection of the angle bisectors of a triangle is called the incentre. Incentre O is the always equidistant from all three sides *i.e.,* the perpendicular distance between the sides and incentre is always same for all the three sides.	 'O' is the incentre.
Circumcentre	The point of intersection of the perpendicular bisectors of the sides of a triangle is called the circumcentre. $OA = OB = OC$ = (circum radius) Circumcentre O is always equidistant from all the three vertices A, B and C.	 'O' is the circumcentre.

Important Theorems on Triangles

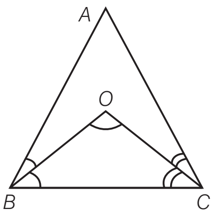
Theorem	Statement/Explanation	Diagram
Pythagoras theorem	The square of the hypotenuse of a right angled triangle is equal to the sum of the squares of the other two sides. *i.e.,* $(AC)^2 = (AB)^2 + (BC)^2$ • The converse of this theorem is also true. • The numbers which satisfy this relation, are called Pythagorean triplets. *e.g.,* (3, 4, 5), (5, 12, 13), (7, 24, 25), (8, 15, 17), (9, 40, 41), (11, 60, 61), (12, 35, 37), (16, 63, 65), (20, 21, 29), (28, 45, 53), (33, 56, 65) **Note :** All the multiples (or submultiples) of Pythagorean triplets also satisfy the relation. *e.g.,* (6, 8,10), (15,36, 39), (1.5, 2, 2.5) etc	 $\angle B = 90°$ $\overline{AC} \rightarrow$ Hypotenuse $AD = CD = BD$ (D is the mid-point of AC)
45° – 45° – 90° triangle theorem	If the angles of a triangle are 45°, 45° and 90°, then the hypotenuse (*i.e.,* longest side) is $\sqrt{2}$ times of any smaller side. Excluding hypotenuse rest two sides are equal. *i.e.,* $AB = BC$ and $AC = \sqrt{2}AB = \sqrt{2}BC$	 $\angle A = 45°$, $\angle B = 90°$, $\angle C = 45°$

Theorem	Statement/Explanation	Diagram
$30° - 60° - 90°$ triangle theorem	If the angles of a triangle are 30°, 60° and 90°,then the sides opposite to 30° angle is half of the hypotenuse and the side opposite to 60° is $\frac{\sqrt{3}}{2}$ times the hypotenuse . e.g., $AB = \frac{AC}{2}$ and $BC = \frac{\sqrt{3}}{2}AC$ $\therefore \quad AB : BC : AC = 1 : \sqrt{3} : 2$	
Basic proportionality theorem (BPT) **or Thales theorem**	Any line parallel to one side of a triangle divides the other two sides proportionally. So if *DE* is drawn parallel to *BC*, it would divide sides *AB* and *AC* proportionally *i.e.,* $\frac{AD}{DB} = \frac{AE}{EC}$ or $\frac{AD}{AB} = \frac{AE}{AC}$ $\frac{AD}{DE} = \frac{AB}{BC} = \frac{AE}{DE} = \frac{AC}{BC}$	
Mid-point theorem	If the mid-points of two adjacent sides of a triangle are joined by a line segment, then this segment is parallel to the third side. *i.e.,* if $AD = BD$ and $AE = CE$, then $DE \parallel BC$	
Apollonius theorem	In a triangle, the sum of the squares of any two sides of a triangle is equal to twice the sum of the square of the median to the third side and square of half the third side. *i.e.,* $AB^2 + AC^2 = 2(AD^2 + BD^2)$	 $BD = CD$ *AD* is the median
Interior angle bisector theorem	In a triangle the angle bisector of an angle divides the opposite side to the angle in the ratio of the remaining two sides. *i.e.,* $\frac{BD}{CD} = \frac{AB}{AC}$ and $BD \times AC - CD \times AB = AD^2$	
Exterior angle bisector theorem	In a triangle the angle bisector of any exterior angle of a triangle divides the side opposite to the external angle in the ratio of the remaining two sides *i.e.,* $\frac{BE}{AE} = \frac{BC}{AC}$	

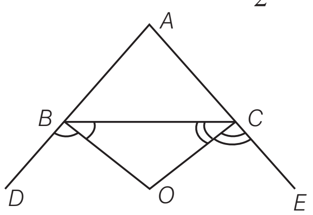
Euler's Theorem for a triangle	Let ΔABC have circumradius R and inradius r. Let d be the distance between the circumcentre and the incenter. Then we have $d^2 = R(R - 2r)$	
Crossed Ladder Theorem	Let the two line segments BC and AD intersect at a point F, such that the point E lies on AC and $AB \parallel CD \parallel EF$. Then, we have $\dfrac{1}{AB} + \dfrac{1}{CD} = \dfrac{1}{EF}$	
Mass Point Geometry	Let us consider a line AB which is intersected by a point P, such that $AP = m$ and $BP = n$, we have $\dfrac{\text{wt }(A)}{\text{wt }(B)} = \dfrac{n}{m}$ Therefore, the weight at A will be n, the weight at B will be m and the total weight at P will be $(m + n)$.	 

Useful Results

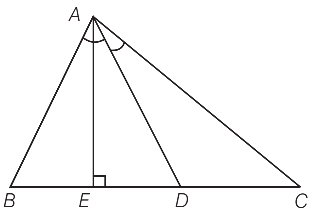
1. In a ΔABC, if the bisectors of $\angle B$ and $\angle C$ meet at O, then $\angle BOC = 90° + \frac{1}{2}\angle A$



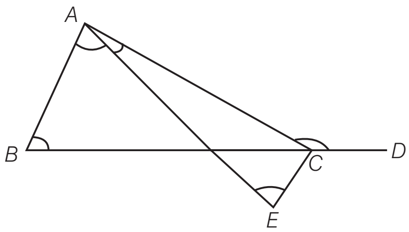
2. In a ΔABC, if sides AB and AC are produced to D and E respectively and the bisectors of $\angle DBC$ and $\angle ECB$ intersect at O, then $\angle BOC = 90° - \frac{1}{2}\angle A$



3. In a ΔABC, if AD is the angle bisector of $\angle BAC$ and $AE \perp BC$, then $\angle DAE = \frac{1}{2}(\angle ABC - \angle ACB)$

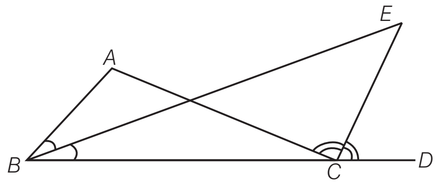


4. In a ΔABC, if BC is produced to D and AE is the angle bisector of $\angle A$, then



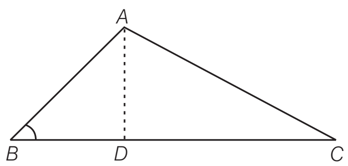
$$\angle ABC + \angle ACD = 2\angle AEC.$$

5. In a ΔABC, if side BC is produced to D and bisectors of $\angle ABC$ and $\angle ACD$ meet at E, then



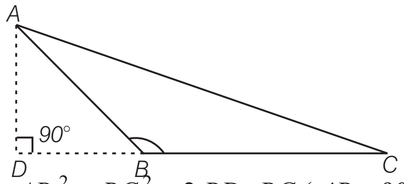
$$\angle BEC = \frac{1}{2}\angle BAC$$

6. In an acute angle ΔABC, AD is a perpendicular dropped on the opposite side of $\angle A$, then



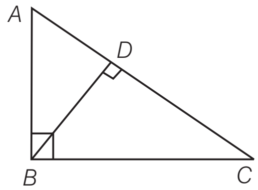
$$AC^2 = AB^2 + BC^2 - 2BD \cdot BC \ (\angle B < 90°)$$

7. In an obtuse angle ΔABC, AD is perpendicular dropped on BC. BC is produced to D to meet AD, then



$$AC^2 = AB^2 + BC^2 + 2\,BD \cdot BC \;(\angle B > 90°)$$

8. In a right angle ΔABC, $\angle B = 90°$ and AC is hypotenuse. The perpendicular BD is dropped on hypotenuse AC from right angle vertex B, then



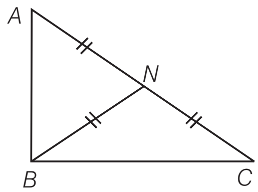
(i) $BD = \dfrac{AB \times BC}{AC}$ (ii) $AD = \dfrac{AB^2}{AC}$

(iii) $CD = \dfrac{BC^2}{AC}$

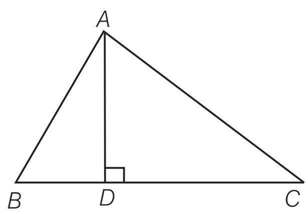
(iv) $\dfrac{1}{BD^2} = \dfrac{1}{AB^2} + \dfrac{1}{BC^2}$

In a right angled triangle, the median to the hypotenuse $= \dfrac{1}{2} \times$ hypotenuse

i.e., $BN = \dfrac{AC}{2}$ (as per the fig.)



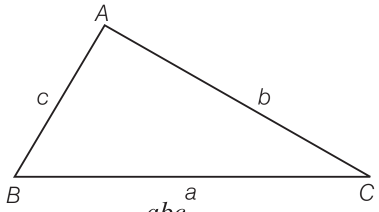
9. **Area of a triangle (General formula)**



$A\,(\Delta) = \dfrac{1}{2} \times$ base $\times$ height

$A\,(\Delta) = \dfrac{1}{2} \times BC \times AD$ (as per the figure.)

10. **Area of scalene triangle** $= \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$



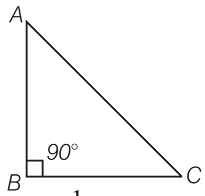
Also, $A\,(\Delta) = r \times s = \dfrac{abc}{4R}$

where a, b and c are the sides of the triangle.

$s \to$ semiperimeter $= \dfrac{a+b+c}{2}$, $r \to$ inradius

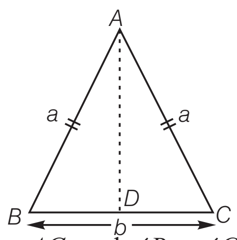
$R \to$ circumradius

11. **Area of right angled triangle**



$= \dfrac{1}{2} \times$ base $\times$ height $= \dfrac{1}{2} \times BC \times AB$ (as per the figure)

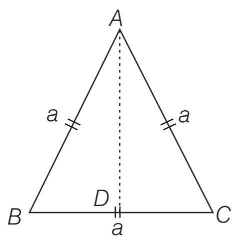
12. **Area of an isosceles triangle** $= \dfrac{b}{4}\sqrt{4a^2 - b^2}$



$AB = AC$ and $\angle B = \angle C$

$\Delta ABD \cong \Delta ACD$ ($AD \to$ Angle bisector, median, altitude and perpendicular bisector)

13. **Area of an equilateral triangle** $= \dfrac{\sqrt{3}}{4} a^2$



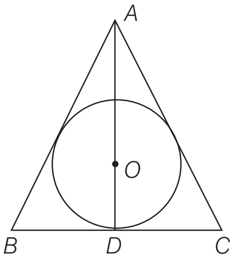
$$\left[A\,(\Delta) = \frac{1}{2} BC \times AD = \frac{1}{2} \times a \times \frac{\sqrt{3}}{2} a = \frac{\sqrt{3}}{4} a^2 \right]$$

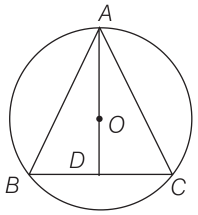
($a \to$ each side of the triangle)

$AD \rightarrow$ Altitude, median, angle bisector and perpendicular bisector also.

Inradius : $\dfrac{1}{3} \times \text{height} = \dfrac{\text{side}}{2\sqrt{3}}$,

$OD \rightarrow$ Inradius





Circumradius $= \dfrac{2}{3} \times \text{height} = \dfrac{\text{side}}{\sqrt{3}}$

$OA \rightarrow$ Circumradius

NOTE In equilateral triangle orthocentre centroid, incentre and circumcentre coincide at the same point.

Circumradius = 2 × inradius

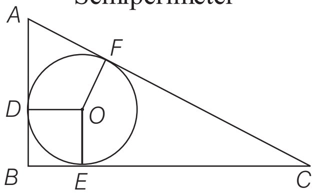
For the given perimeter of a triangle, the area of equilateral triangle is maximum.

For the given area of a triangle, the perimeter of equilateral triangle is minimum.

14. In a right angled triangle

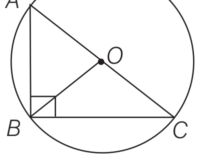
(i) Inradius $(r) = \dfrac{AB + BC - AC}{2}$

(ii) Inradius $(r) = \dfrac{\text{Area}}{\text{Semiperimeter}}$



$$DO = EO = FO\ (r)$$

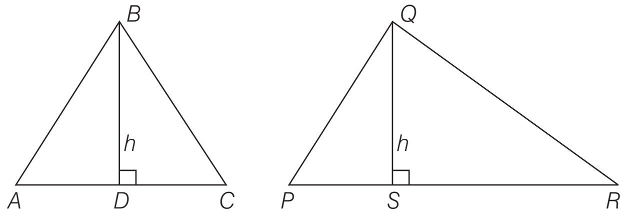
(iii) Circumradius $(R) = \dfrac{AC}{2} = \left(\dfrac{\text{hypotenuse}}{2}\right)$



$$AO = CO = BO = (R)$$

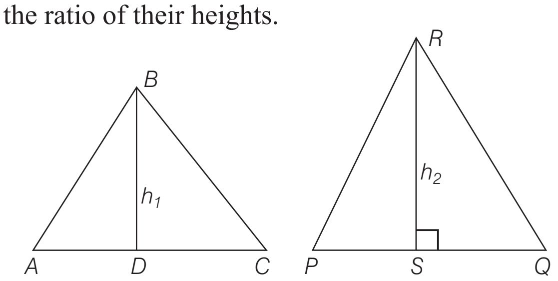
AC is the diameter.

15. The ratio of areas of two triangles of equal heights is equal to the ratio of their corresponding bases. i.e.,



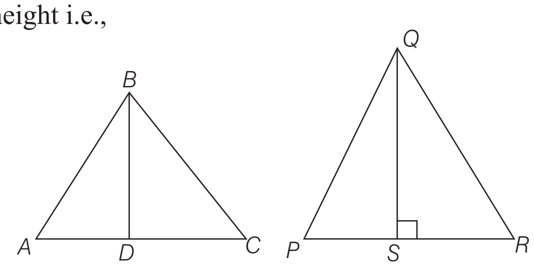
$$\frac{A\,(\Delta ABC)}{A\,(\Delta PQR)} = \frac{AC}{PR}$$

16. The ratio of areas of triangles of equal bases is equal to the ratio of their heights.



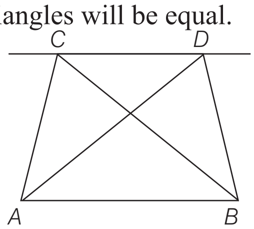
i.e., $$\frac{A\,(\Delta ABC)}{A\,(\Delta PQR)} = \frac{BD}{RS}$$

17. The ratio of the areas of two triangles is equal to the ratio of the products of base and its corresponding height i.e.,



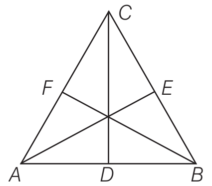
$$\frac{A\,(\Delta ABC)}{A\,(\Delta PQR)} = \frac{AC \times BD}{PR \times QS}$$

18. If the two triangles have the same base and lie between the same parallel lines (as shown in figure), then the area of two triangles will be equal.



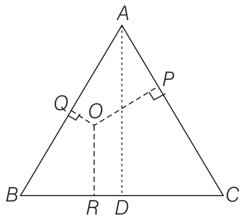
i.e., $A\,(\Delta ABC) = A\,(\Delta ADB)$

19. In a triangle AE, CD and BF are the medians, then

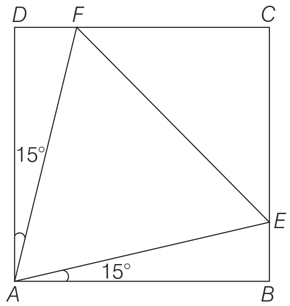


$$3\,(AB^2 + BC^2 + AC^2) = 4\,(CD^2 + BF^2 + AE^2)$$

20. In an equilateral triangle, if O is a point anywhere inside the equilateral triangle ABC, the sum of its distances from three sides is equal to the length of the altitude of the triangle. That is, as per the given diagram, $OP + OQ + OR = AD$.



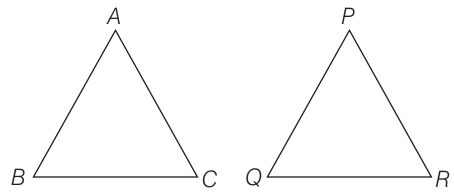
21. The largest possible area of an equilateral triangle inscribed in a unit square is $(2\sqrt{3} - 3)$.



- The maximum area can be enclosed only when one of the vertices of the triangle coincides with one of the vertices of the square and angle between the side of the triangle and the side of the rectangle at the point of coincidence is 15°.
- Also, the side of the equilateral triangle is $\sqrt{6} - \sqrt{2}$.
- Also, the area of $\Delta ADF = \Delta ABE = \frac{1}{2}(\Delta ECF)$.

Congruency of triangles

Two triangles are said to be congruent if they are equal in all respects. i.e.,

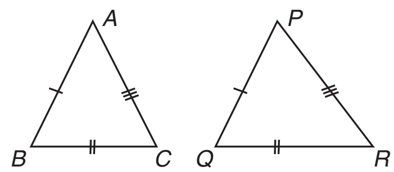
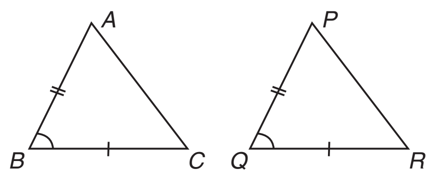


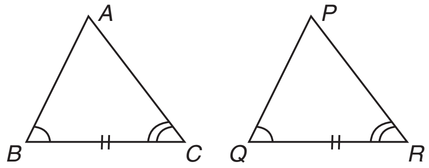
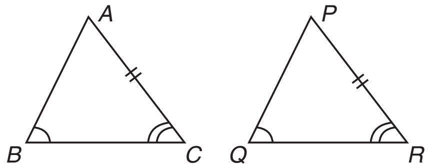
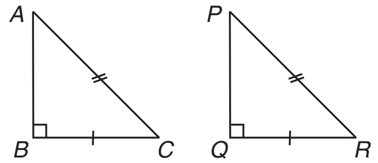
1. Each of the three sides of one triangle must be equal to the three respective sides of the other.
2. Each of the three angles of the one triangle must be equal to the three respective angles of the other.

i.e., $\left.\begin{array}{l} AB = PQ \\ AC = PR \\ BC = QR \end{array}\right\}$ and $\left.\begin{array}{l} \angle A = \angle P \\ \angle B = \angle Q \\ \angle C = \angle R \end{array}\right\}$

Tests for congruency

With the help of the following given tests, we can deduce without having detailed information about triangles that whether the given two triangles are congruent or not.

Test	Property	Diagram
$S-S-S$	(Side–Side–Side) If the three sides of one triangle are equal to the corresponding three sides of the other triangle, then the two triangles are congruent. $AB \cong PQ$, $AC \cong PR$, $BC \cong QR$ $\therefore$ $\Delta ABC \cong \Delta PQR$	
$S-A-S$	(Side–Angle–Side) If two sides and the angle included between them are congruent to the corresponding sides and the angle included between them, of the other triangle, then the two triangles are congruent. $AB \cong PQ$, $\angle ABC \cong \angle PQR$, $BC \cong QR$ $\therefore$ $\Delta ABC \cong \Delta PQR$	

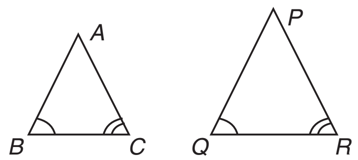
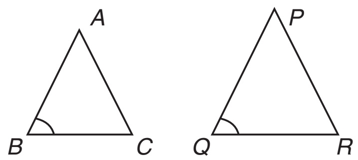
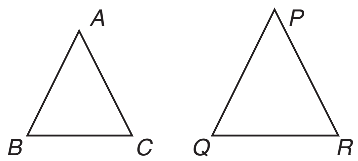
Test	Property	Diagram
$A-S-A$	(Angle–Side–Angle) If two angles and the included side of a triangle are congruent to the corresponding angles and the included side of the other triangle, then the two triangles are congruent. $\angle ABC \cong \angle PQR$, $BC \cong QR$, $\angle ACB \cong \angle PRQ$ $\therefore$ $\Delta ABC \cong \Delta PQR$	
$A-A-S$	(Angle–Angle–Side) If two angles and a side other than the included side of a one triangle are congruent to the corresponding angles and a corresponding side other than the included side of the other triangle, then the two triangles are congruent. $\angle ABC \cong \angle PQR$, $\angle ACB \cong \angle PRQ$ and $AC \cong PR$ (or $AB \cong PQ$)	
$R—H—S$	(Right angle–Hypotenuse–Side) If the hypotenuse and one side of the right angled triangle are congruent to the hypotenuse and a corresponding side of the other right angled triangle, then the two given triangles are congruent. $AC \cong PR$, $\angle B = \angle Q$ and $BC \cong QR$ $\therefore$ $\Delta ABC \cong \Delta PQR$	

Similarity of triangles

Two triangles are said to be similar if the corresponding **angles** are **congruent** and their corresponding **sides** are in **proportion**. The symbol for similarity is '~'. If $\Delta ABC \sim \Delta PQR$, then $\angle ABC \cong \angle PQR$, $\angle BCA \cong \angle QRP$, $\angle BAC \cong \angle QPR$

Tests for Similarity

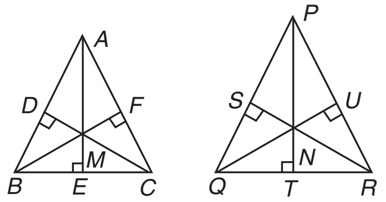
Through the tests for similarity we can deduce the similarity of triangles with minimum required information.

Test	Property/Definition	Diagram
$A-A$	Angle-Angle If the two angles of one triangle are congruent to the corresponding two angles of the other triangle, then the two triangles are said to be similar. $\angle ABC \cong \angle PQR$ $\angle ACB \cong \angle PRQ$ $\therefore$ $\Delta ABC \sim \Delta PQR$	
S-A-S	Side-Angle-Side If the two sides of one triangle are proportional to the corresponding two sides of the other triangle and the angle included by them are congruent, then the two triangles are similar. *i.e.,* $\frac{AB}{PQ} = \frac{BC}{QR}$ and $\angle ABC = \angle PQR$ $\therefore$ $\Delta ABC \sim \Delta PQR$.	 $\frac{AB}{PQ} = \frac{BC}{QR} = K$ (K is any constant)
S-S-S	Side-Side-Side If the three sides of one triangle are proportional to the corresponding three sides of the other triangle, then the two triangles are similar. *i.e.,* $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$ $\therefore$ $\Delta ABC \sim \Delta PQR$	 $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR} = K$

NOTE When the corresponding sides are in proportion , then the corresponding angles are in proportion.

Properties of Similar Triangles

If the two triangles are similar, then for the proportional/corresponding sides we have the following results.



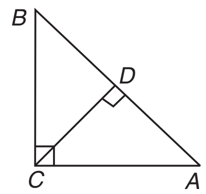
1. Ratio of sides = Ratio of heights (altitudes)
   = Ratio of medians
   = Ratio of angle bisectors
   = Ratio of inradii
   = Ratio of circumradii

2. Ratio of areas = Ratio of squares of corresponding sides.
   i.e., if $\Delta ABC \sim \Delta PQR$, then

$$\frac{A\,(\Delta ABC)}{A\,(\Delta PQR)} = \frac{(AB)^2}{(PQ)^2} = \frac{(BC)^2}{(QR)^2} = \frac{(AC)^2}{(PR)^2}$$

NOTE Rule 1 can also apply with rule 2.

3. In a right angled triangle, the triangles on each side of the altitude drawn from the vertex of the right angle to the hypotenuse are similar to the original triangle and to each other too. *i.e.,* $\Delta BCA \sim \Delta BDC \sim \Delta CDA$.



Introductory Exercise 12.2

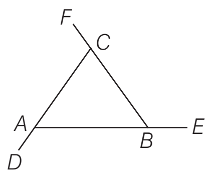
1. In a triangle ABC, if AB, BC and AC are the three sides of the triangle, then which of the statements is necessarily true?

(a) $AB + BC < AC$ (b) $AB + BC > AC$
(c) $AB + BC = AC$ (d) $AB^2 + BC^2 = AC^2$

2. The sides of a triangle are 12 cm, 8 cm and 6 cm respectively, the triangle is :

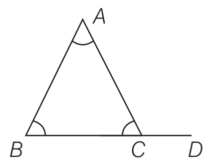
(a) acute (b) obtuse
(c) right (d) can't be determined

3. If the sides of a triangle are produced then the sum of the exterior angles *i.e,* $\angle DAB + \angle EBC + \angle FCA$ is equal to :



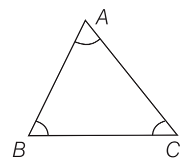
(a) 180° (b) 270° (c) 360° (d) 240°

4. In the given figure BC is produced to D and $\angle BAC = 40°$ and $\angle ABC = 70°$. Find the value of $\angle ACD$.



(a) 30° (b) 40°
(c) 70° (d) 110°

5. In a ΔABC, $\angle BAC > 90°$, then $\angle ABC$ and $\angle ACB$ must be :

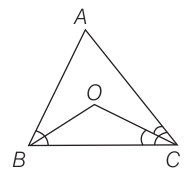


(a) acute
(b) obtuse
(c) one acute and one obtuse
(d) can't be determined

6. If the angles of a triangle are in the ratio 1 : 4 : 7, then the value of the largest angle is :

(a) 135°
(b) 84°
(c) 105°
(d) none of the above

7. In the adjoining figure $\angle B = 70°$ and $\angle C = 30°$. BO and CO are the angle bisectors of $\angle ABC$ and $\angle ACB$. Find the value of $\angle BOC$.



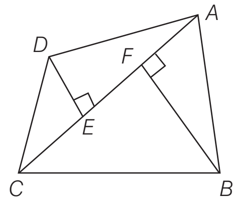
(a) 30° (b) 40°
(c) 120° (d) 130°

12.4 **Quadrilaterals**

A four sided closed figure is called a quadrilateral. It is denoted by symbol ‘ ’.

Properties

1. Sum of four interior angles is 360°.
2. The figure formed by joining the mid-points of a quadrilateral is a parallelogram.



3. The sum of opposite sides of a quadrilateral circumscribed about a circle, is always equal.
4. Area of quadrilateral $= \frac{1}{2} \times$ one of the diagonals × sum of the perpendiculars drawn to the diagonals from the opposite vertices. *i.e.,* $A\,(ABCD) = \frac{1}{2} \times AC \times (DE + BF)$

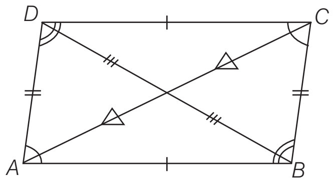
Types of Quadrilaterals

Parallelogram

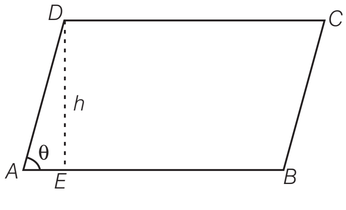
A quadrilateral whose opposite sides are parallel.

Properties

1. The opposite sides are parallel and equal.



2. Opposite angles are equal.
3. Sum of any two adjacent angles is 180°.
4. Diagonals bisect each other.



5. Diagonals need not be equal in length.
6. Diagonals need not bisect at right angle.
7. Diagonals need not bisect angles at the vertices.
8. Each diagonal divides a parallelogram into two congruent triangles.
9. Lines joining the mid-points of the adjacent sides of a **quadrilateral** form a **parallelogram.**
10. Lines joining the mid-points of of the adjacent sides of a **parallelogram** is a **parallelogram.**
11. The parallelogram that is inscribed in a circle is a rectangle.
12. The parallelogram that is circumscribed about a circle is a rhombus.
13. (a) Area of a parallelogram = base × height

    (b) Area of parallelogram

    = product of any two adjacent sides × sine of the included angles.

    $= AB \times AD \times \sin\theta$
14. Perimeter of a parallelogram $= 2$ (sum of any two adjacent sides)
15. $(AC)^2 + (BD)^2 = (AB)^2 + (BC)^2 + (CD)^2 + (AD)^2$

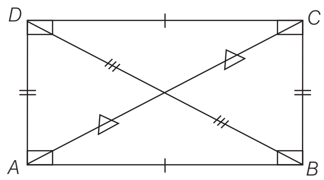
    $= 2\,(AB^2 + BC^2)$
16. Parallelogram that lie on the same base and between the same parallel lines are equal in area.
17. Area of a triangle is half of the area of a parallelogram which lie on the same base and between the same parallel lines.
18. A parallelogram is a rectangle if its diagonals are equal.

Rectangle

A parallelogram in which all the four angles at vertices are right (*i.e.,* 90°), is called a rectangle.

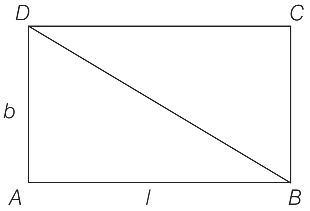
Properties

1. Opposite sides are parallel and equal.
2. Opposite angles are equal and of 90°.



3. Diagonals are equal and bisect each other, but not necessarily at right angles.
4. When a rectangle is inscribed in a circle, the diameter of the circle is equal to the diagonal of the rectangle.
5. For the given perimeter of rectangles, a square has maximum area.
6. The figure formed by joining the mid-points of the adjacent sides of a rectangle is a rhombus.

7. The quadrilateral formed by joining the mid-points of intersection of the angle bisectors of a parallelogram is a rectangle.
8. Every rectangle is a parallelogram.
9. Area of a rectangle = length × breadth $(= l \times b)$



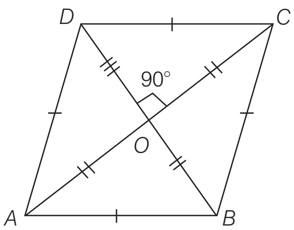
10. Diagonals of a rectangle $= \sqrt{l^2 + b^2}$
11. Perimeter of a rectangle $= 2\,(l + b)$
    $l \rightarrow$ length and $b \rightarrow$ breadth

Rhombus

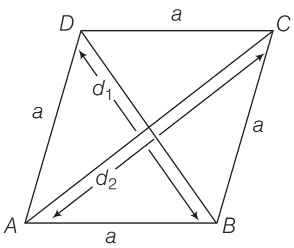
A parallelogram in which all sides are equal, is called a rhombus.

Properties

1. Opposite sides are parallel and equal.
2. Opposite angles are equal.
3. Diagonals bisect each other at right angle, but they are not necessarily equal.



4. Diagonals bisect the vertex angles.
5. Sum of any two adjacent angles is 180°.



6. Figure formed by joining the mid-points of the adjacent sides of a rhombus is a rectangle.
7. A parallelogram is a rhombus if its diagonals are perpendicular to each other.
8. (a) Area of a rhombus $= \frac{1}{2} \times$ product of diagonals

$$= \frac{1}{2} \times d_1 \times d_2$$

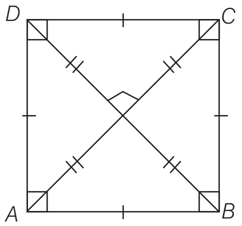
(b) Area of a rhombus = Product of adjacent sides × sine of the included angle.

Square

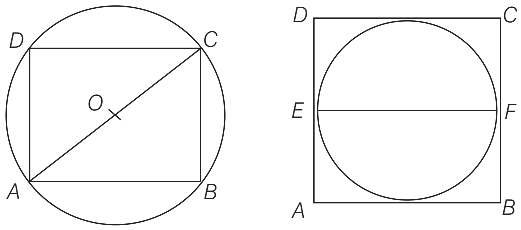
A rectangle whose all sides are equal or a rhombus whose all angles are equal is called a square. Thus each square is a parallelogram, a rectangle and a rhombus.

Properties

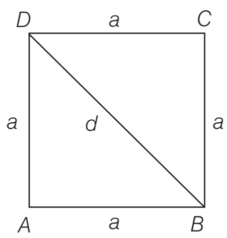
1. All side are equal and parallel.
2. All angles are right angles.
3. Diagonals are equal and bisect each other at right angle.



4. Diagonal of an inscribed square is equal to diameter of the inscribing circle.
5. Side of a circumscribed square is equal to the diameter of the inscribed circle.



6. The figure formed by joining the mid-points of the adjacent sides of a square is a square.
7. Area $= (\text{side})^2 = a^2 = \frac{(\text{diagonal})^2}{2} = \frac{d^2}{2}$



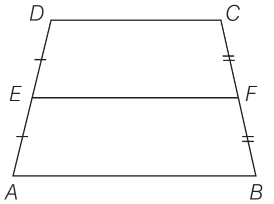
8. Diagonal = side $\sqrt{2} = a\sqrt{2}$
9. Perimeter $= 4 \times$ side $= 4a$

Trapezium

A quadrilateral whose only one pair of sides is parallel and other two sides are not parallel.

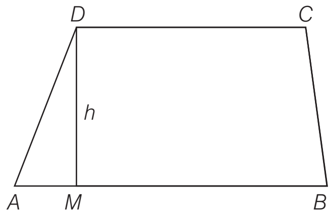
Properties

1. The line joining the mid-points of the oblique (non-parallel) sides is half the sum of the parallel sides and is called the median.

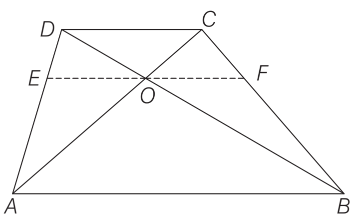


$$\left(i.e., \text{ Median} = \frac{1}{2} \times \text{sum of parallel sides} = \frac{1}{2} \times (AB + DC) = EF\right)$$

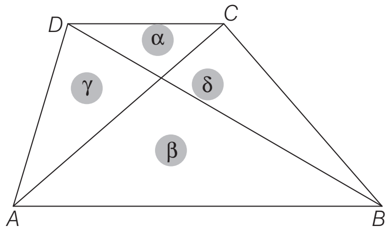
2. If the non-parallel sides are equal then the diagonals will also be equal to each other.
3. Diagonals intersect each other proportionally in the ratio of lengths of parallel sides.
4. By joining the mid-points of adjacent sides of a trapezium four similar triangles are obtained.
5. If a trapezium is inscribed in a circle, then it is an isosceles trapezium with equal oblique sides.
6. Area of trapezium $= \frac{1}{2} \times$ (sum of parallel sides × height)
   $= \frac{1}{2} \times (AB + CD) \times h$
7. $AC^2 + BD^2 = BC^2 + AD^2 + 2AB \cdot CD$



8. If a line segment EF, which is parallel to the two parallel sides of a trapezium $ABCD$, passes through a point O, which is the point of intersection of the two diagonals AC and BD, where point E lies on AD and point F lies on BC, we have
   (i) Point O bisects the line EF. (ii) $EF = \dfrac{2(AB)(CD)}{AB + CD}$

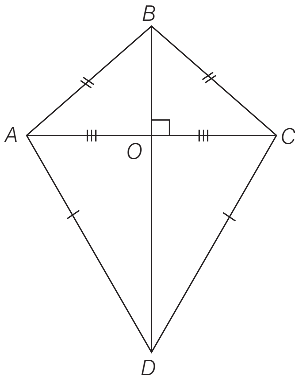


9. In a trapezium $ABCD$, as shown in the diagram, where AB is parallel to CD and the diagonals AC and BD intersect each other at O, and if the total area of the trapezium is A, we have
   (i) $\sqrt{A} = \sqrt{\alpha} + \sqrt{\beta}$
   (ii) $\gamma = \delta = \sqrt{\alpha \cdot \beta}$



Kite

In a kite two pairs of adjacent sides are equal.

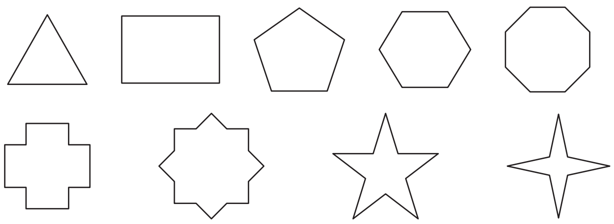


Properties

1. $AB = BC$ and $AD = CD$
2. Diagonals, AC and BD, are the perpendicular bisectors.
3. Longer diagonal (BD) bisects the shorter diagonal (AC).
4. Longer diagonal (BD) is the angle bisector of the pair of opposite angles.
5. Area of kite $= \frac{1}{2} \times$ product of diagonals
   $=$ inradius $\times$ semi-perimeter
6. A kite is symmetrical about the longer diagonal BD.
7. The longer diagonal (BD) divides the kite into two congruent triangles ABD and CBD.
8. The shorter diagonal divides the kite into two isosceles triangles ABC and ADC.
9. The kites are exactly the quadrilaterals that are both tangential and orthodiagonal.
10. A kite can always inscribe an incircle and the incentre is the intersection of the angular bisectors of kite.

12.5 **Polygons**

When three or more than three line segments are joined end to end on the same plane they form the closed area in various shapes. These shapes are called the polygons. Essentially, polygons are 2-dimensional plane figures. Look at the following figures to get the better picture of the shapes of polygons.

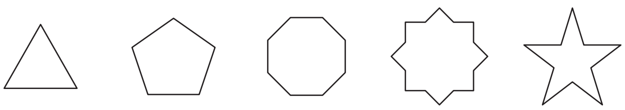


A polygon gets its name on the basis of the number of sides that it has. For example, a triangle has three sides, a quadrilateral has 4 sides, a pentagon has 5 sides, and so on.

Types of Polygon

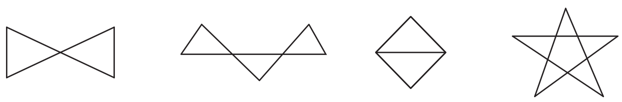
Simple Polygon

A polygon in which no more than two line segments meet at any vertex. In other words, the sides of the polygon do not self-intersect each other.



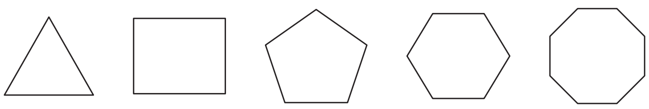
Complex Polygon

A polygon in which more than two line segments meet at any vertex. In other words the sides of the polygon self-intersect each other. A complex polygon is neither convex nor concave.



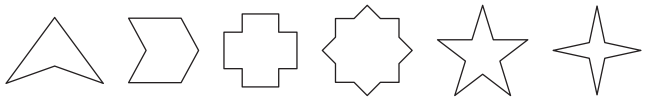
Convex Polygon

A polygon in which each of its interior angle is less than 180° is known as a convex polygon. A polygon is convex if and only if any line containing a side of the polygon doesn't contain a point in the interior of the polygon.



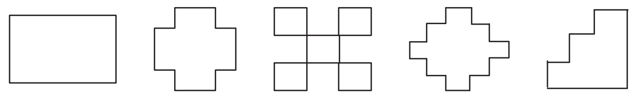
Concave Polygon

A polygon in which even if one interior angle is greater than 180° is known as a concave polygon.



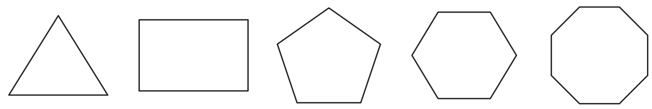
Rectilinear Polygon

A polygon whose sides meet at right angles, i.e., all its interior angles are 90° or 270°.



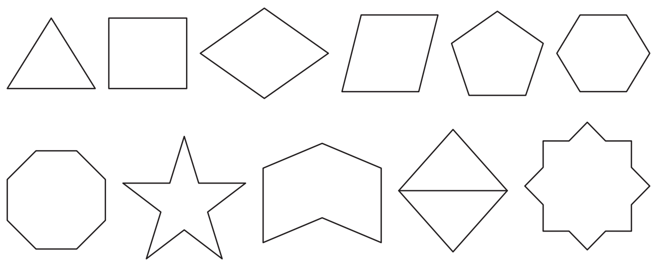
Equiangular Polygon

A polygon in which all its interior angles are equal is known as equiangular polygon.



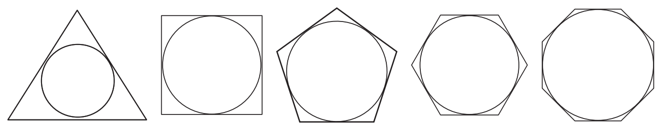
Equilateral Polygon

A polygon in which all sides
(or edges) are of the same length.



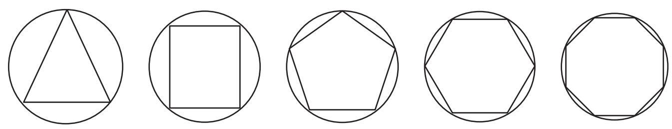
Tangential Polygon

A polygon whose all sides are tangent to an inscribed circle.



Cyclic Polygon

A polygon whose all the vertices lie on the circle.

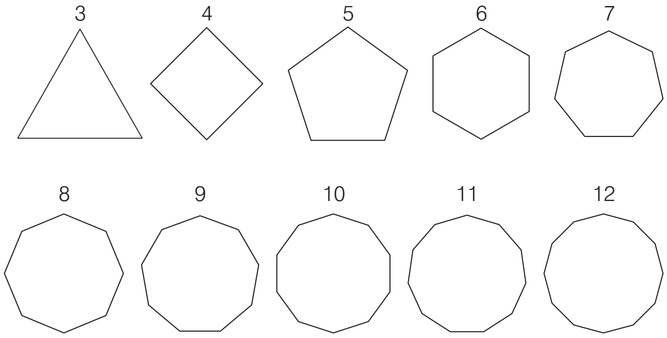


Regular Polygon

A polygon is regular if it is both cyclic and equilateral.

- A regular polygon has all its sides equal and all its interior angles equal.

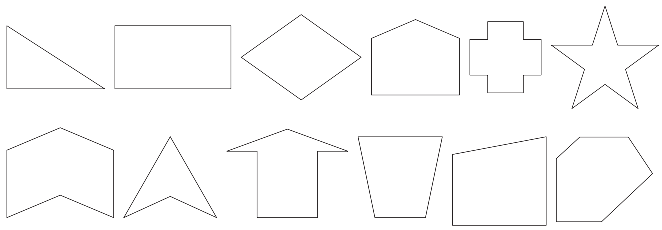
- A regular polygon has an incircle and a circumcircle. That is every regular polygon is tangential and cyclic polygon.



Irregular Polygon

A polygon which is not a regular one is known as an irregular polygon.

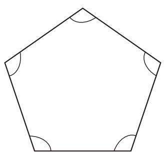
- If all the sides of a polygon are not equal, it's an irregular polygon.
- If all the interior angles of a polygon are not equal, it's an irregular polygon.



Basics of a Polygon

Interior Angle

Angle formed by two adjacent sides inside the polygon is known as interior angle.



Sum of all the interior angles of a regular polygon $= (n-2)\times 180°$

Each interior angle of a regular polygon

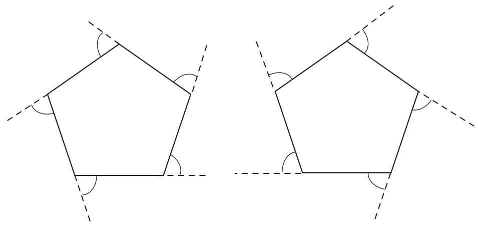
$$= \frac{(n-2)\times 180°}{n}$$

Each interior angle of a regular polygon

$$= (180° - \text{exterior angle})$$

Exterior Angle

Angle formed by two adjacent sides outside the polygon.



Sum of all exterior angles of a regular polygon $= 360°$ (always constant)

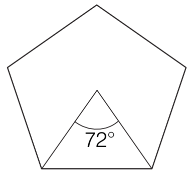
Each exterior angle of a regular polygon $= \dfrac{360°}{n}$

Each exterior angle of a regular polygon

$$= (180° - \text{ interior angle})$$

Central Angle

The angle at the center of the polygon formed by two radii, connected independently to the two adjacent vertices, is known as the central angle.

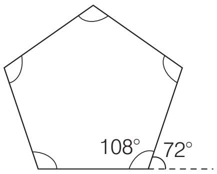


Each Central Angle $= \dfrac{2\pi}{n} = \dfrac{360°}{n}$

The following table shows some angles of the various regular polygons.

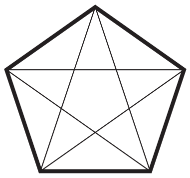
Number of Sides	Polygon	Sum of all the Angles	Each Interior Angle	Each Exterior Angle	Each Central Angle
3	Triangle	180°	60°	120°	120°
4	Quadrilateral	360°	90°	90°	90°
5	Pentagon	540°	108°	72°	72°
6	Hexagon	720°	120°	60°	60°
7	Heptagon	900°	$\left(128\frac{4}{7}\right)^\circ$	$\left(51\frac{3}{7}\right)^\circ$	$\left(51\frac{3}{7}\right)^\circ$
8	Octagon	1080°	135°	45°	45°
9	Nonagon	1260°	140°	40°	40°
10	Decagon	1440°	144°	36°	36°
…	…	…	…	…	…
…	…	…	…	…	…
n	n-gon	$(n-2)180°$	$\frac{(n-2)\times 180°}{n}$	$\frac{360°}{n}$	$\frac{360°}{n}$

In a polygon, Interior angle + Exterior angle $= 180°$



Diagonal

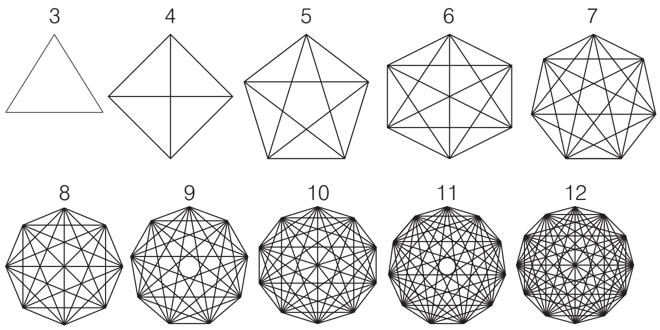
The line segments joining any two non-adjacent vertices are known as diagonals.



The number of diagonals in a polygon

$$= {}^{n}C_2 - n = \frac{n(n-3)}{2};\ \text{for } n \geq 3$$

Number of Sides	Polygon	Number of diagonals
3	Triangle	0
4	Quadrilateral	2
5	Pentagon	5
6	Hexagon	9
7	Heptagon	14
8	Octagon	20
9	Nonagon	27
10	Decagon	35
…	…	…
…	…	…
n	n-gon	$\frac{n(n-3)}{2}$



Longest Diagonal of a Regular Polygon

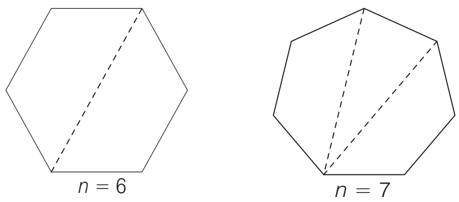
There are two different cases.

One when there is even number of sides in a polygon and the other one when there is odd number of sides in a polygon.

When n is even : The longest diagonal $= 2 \times$ radius of the polygon

When n is odd : The longest diagonal $= \dfrac{s}{2\sin\left(\dfrac{90}{n}\right)^{\circ}}$

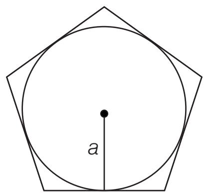
Where s denotes the length of each side and n denotes the number of sides in the regular polygon. And radius means circumradius.



Apothem (or Inradius)

It is the radius of the incircle of the regular polygon.

Apothem is perpendicular bisector of the side of the polygon.

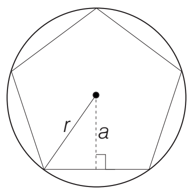


$$\text{Apothem of a regular polygon} = \frac{1}{2} \times s \times \cot\left(\frac{\pi}{n}\right)$$
$$= r\cos\left(\frac{\pi}{n}\right)$$

(Where, s and r are the side and radius of the regular polygon.)

Radius (or Circumradius)

It is the radius of the circumcircle of the regular polygon.



$$\text{Radius of a polygon} = \frac{1}{2} \times s \times \operatorname{cosec}\left(\frac{\pi}{n}\right) = a\sec\left(\frac{\pi}{n}\right)$$

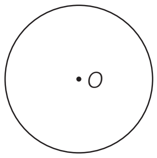
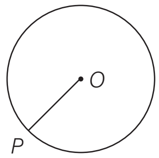
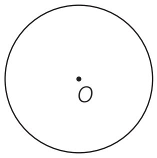
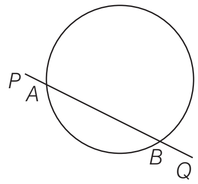
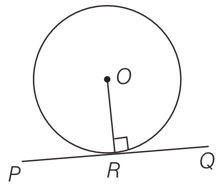
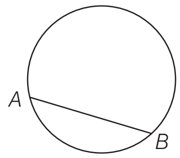
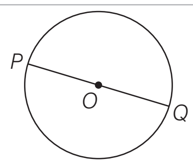
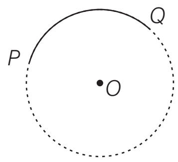
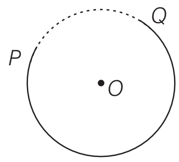
Where, s and a are the side and apothem of a regular polygon.

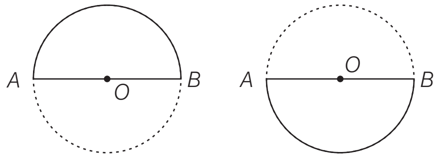
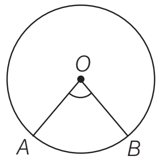
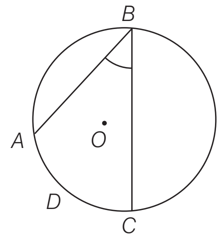
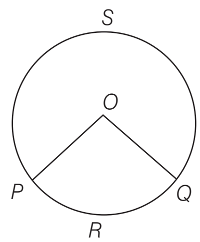
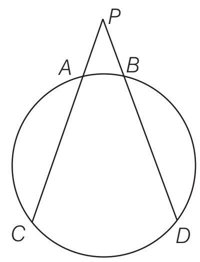
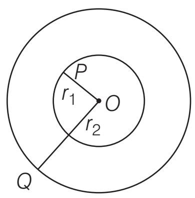
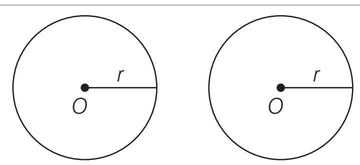
Sagitta

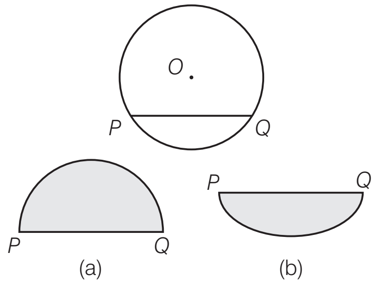
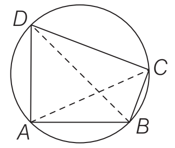
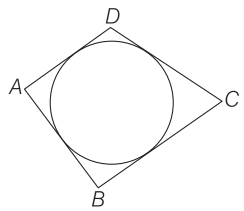
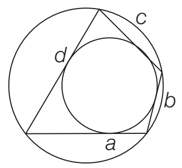
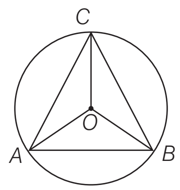
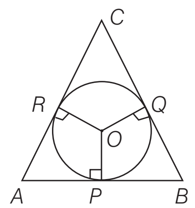
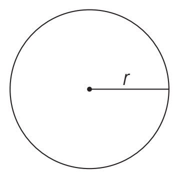
The perpendicular distance h from an arc's midpoint to the chord across it, equal to the radius r minus the apothem a.

12.6 Circles

Definition : A circle is a set of points on a plane which lie at a fixed distance from a fixed point.

Nomenclature	Definition	Diagram
Centre	The fixed point is called the centre. In the given diagram 'O' is the centre of the circle.	
Radius	The fixed distance is called a radius. In the given diagram OP is the radius of the circle. (Point P lies on the circumference)	
Circumference	The circumference of a circle is the distance around a circle, which is equal to $2\pi r$. ($r \rightarrow$ radius of the circle)	
Secant	A line segment which intersects the circle at two distinct points, is called as secant. In the given diagram secant PQ intersects circle at two points A and B.	
Tangent	A line segment which has one common point with the circumference of a circle *i.e.,* it touches only at one point is called as tangent of circle. The common point is called as point of contact. In the given diagram PQ is a tangent which touches the circle at a point R.	 (R is the point of contact) **NOTE** Radius is always perpendicular to tangent.
Chord	A line segment whose end points lie on the circle. In the given diagram AB is a chord.	
Diameter	A chord which passes through the centre of the circle is called the diameter of the circle. The length of the diameter is twice the length of the radius. In the given diagram PQ is the diameter of the circle. ($O \rightarrow$ is the centre of the circle)	
Arc	Any two points on the circle divides the circle into two parts the smaller part is called as minor arc and the larger part is called as major arc. It is denoted as '$\frown$'. In the given diagram $\overset{\frown}{PQ}$ is arc.	 $PQ \rightarrow$ minor arc  $PQ \rightarrow$ major arc

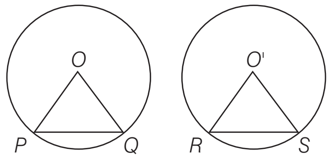
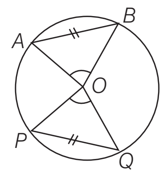
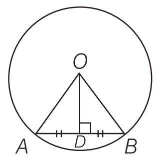
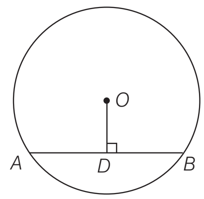
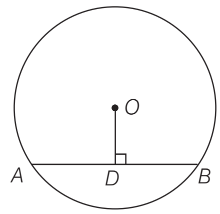
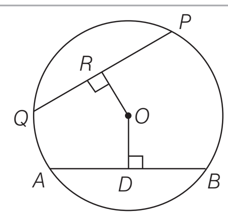
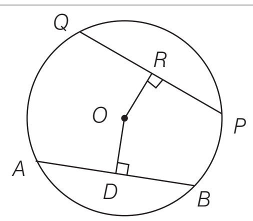
Nomenclature	Definition	Diagram
Semicircle	A diameter of the circle divides the circle into two equal parts. Each part is called as semicircle.	
Central angle	An angle formed at the centre of the circle, is called the central angle. In the given diagram $\angle AOB$ is the central angle.	
Inscribed angle	When two chords have one common end point, then the angle included between these two chords at the common point is called the inscribed angle. $\angle ABC$ is the inscribed angle by the arc $\overparen{ADC}$.	
Measure of an arc	Basically it is the central angle formed by an arc. *e.g.,* (a) measure of a circle = 360° (b) measure of a semicircle = 180° (c) measure of a minor arc = $\angle POQ$ (d) measure of a major arc = 360 – $\angle POQ$	 m (arc PRQ) = $m\,\angle POQ$ m (arc PSQ) = 360° – m (arc PRQ)
Intercepted arc	In the given diagram $\overparen{AB}$ and $\overparen{CD}$ are the two intercepted arcs, intercepted by $\angle CPD$. The end points of the arc must touch the arms of $\angle CPD$ *i.e., CP* and *DP*.	
Concentric circles	Circles having the same centre at a plane are called the concentric circles. In the given diagram there are two circles with radii r_1 and r_2 having the common (or same) centre. These are called as concentric circles.	
Congruent circles	Circles with equal radii are called as congruent circles.	

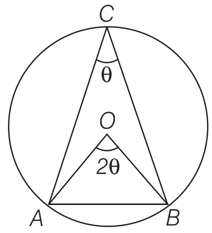
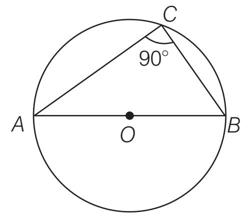
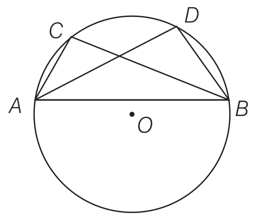
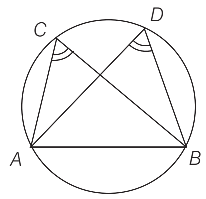
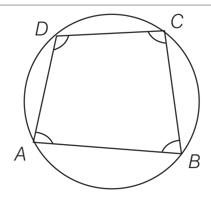
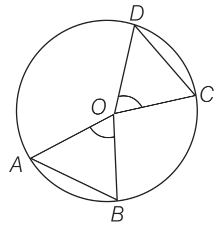
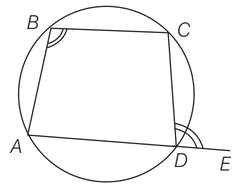
Nomenclature	Definition	Diagram
Segment of a circle	A chord divides a circle into two regions. These two regions are called the segments of a circle. (a) major segment (b) minor segment	
Cyclic Quadrilateral	A quadrilateral whose all the four vertices (A, B, C, D) lie on the circle. For a cyclic quadrilateral $ABCD$, we must have $(AB \times CD) + (BC \times AD) = (AC \times BD)$	
Tangential Quadrilateral	A tangential quadrilateral or circumscribed quadrilateral is a convex quadrilateral whose sides are all tangent to a single circle within the quadrilateral . For a tangential quadrilateral, we must have $AB + CD = BC + AD$	
Bicentric Quadrilateral	A quadrilateral is said to be bicentric if it is cyclic and tangential.	
Circumcircle	A circle which passes through all the three vertices of a triangle. Thus the circumcentre is always equidistant from the vertices of the triangle. $OA = OB = OC$ (circumradius)	
Incircle	A circle which touches all the three sides of a triangle *i.e.,* all the three sides of a triangle are tangents to the circle is called an incircle. Incircle is always equidistant from the sides of a triangle. $OP = OQ = OR$ (inradius of the circle)	
Curvature	Curvatures is a measure of how quickly a tangent line turns on a curve. We would expect the curvature to be 0 for a straight line, to be very small for curves which bend very little and to be large for curves which bend sharply. Larger the circle, smaller the curvature. Curvature of a circle $= \dfrac{1}{\text{radius}}$	 Curvature = $1/r$

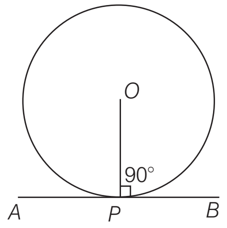
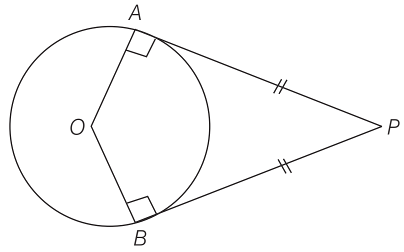
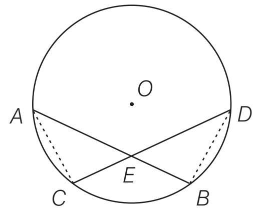
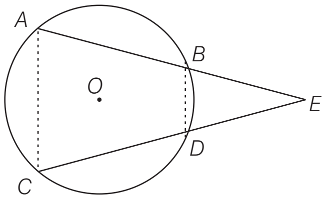
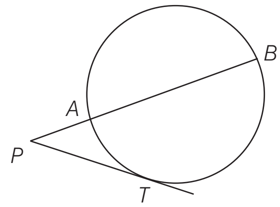
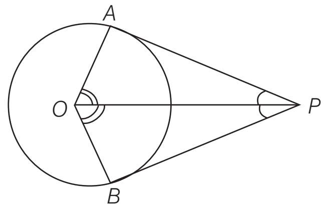
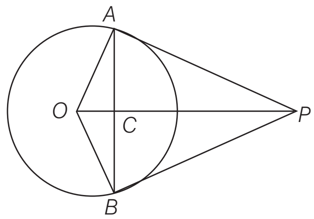
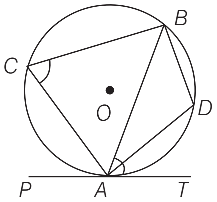
Theorems

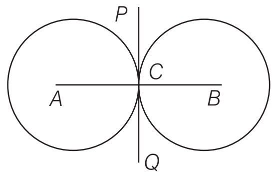
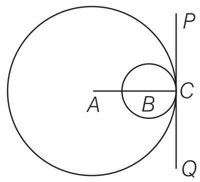
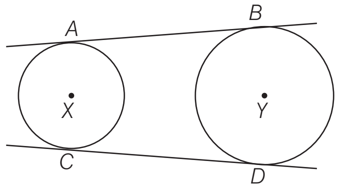
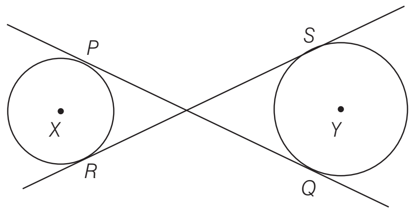
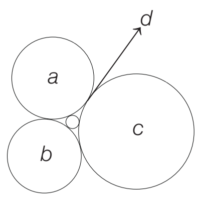
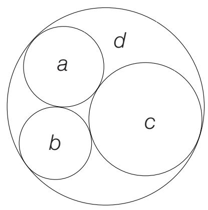
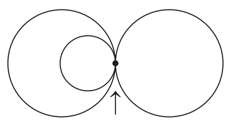
NOTE

- Two arcs of a circle (or of congruent circles) are congruent if their degree measures are equal.
- There is one and only one circle passes through three non-collinear points.

S. No.	Theorem	Diagram
1.	In a circle (or in congruent circles) equal chords are made by equal arcs. $\{OP = OQ\} = \{O'R = O'S\}$ and $PQ = RS$ $\therefore$ $PQ = RS$	
2.	Equal arcs (or chords) subtend equal angles at the centre $\because$ $PQ = AB$ $\therefore$ $\angle POQ = \angle AOB$	
3.	The perpendicular from the centre of a circle to a chord bisects the chord *i.e.,* if $OD \perp AB$ $\therefore$ $AB = 2AD = 2BD$	
4.	The line joining the centre of a circle to the mid-point of a chord is perpendicular to the chord. $\because$ $AD = DB$ $\therefore$ $OD \perp AB$	
5.	Perpendicular bisector of a chord passes through the centre. *i.e.,* if $OD \perp AB$ and $AD = DB$ $\therefore$ O is the centre of the circle.	
6.	Equal chords of a circle (or of congruent circles) are equidistant from the centre. $\because$ $AB = PQ$ $\therefore$ $OD = OR$	
7.	Chords which are equidistant from the centre in a circle (or in congruent, circles) are equal. $\because$ $OD = OR$ $\therefore$ $AB = PQ$	

S. No.	Theorem	Diagram
8.	The angle subtended by an arc (the degree measure of the arc) at the centre of a circle is twice the angle subtended by the arc at any point on the remaining part of the circle. $m\angle AOB = 2m\ \angle ACB.$	
9.	Angle in a semicircle is a right angle.	
10.	Angles in the same segment of a circle are equal. *i.e.,* $\angle ACB = \angle ADB$	
11.	If a line segment joining two points subtends equal angle at two other points lying on the same side of the line containing the segment, then the four points lie on the same circle. $\angle ACB = \angle ADB$ $\therefore$ Points A, C, D, B are concyclic *i.e.,* lie on the circle.	
12.	The sum of pair of opposite angles of a cyclic quadrilateral is 180°. $\angle DAB + \angle BCD = 180°$ and $\angle ABC + \angle CDA = 180°$ (Inverse of this theorem is also true)	
13.	Equal chords (or equal arcs) of a circle (or congruent circles) subtended equal angles at the centre. $AB = CD$ (or $AB = CD$) $\therefore$ $\angle AOB = \angle COD$ (Inverse of this theorem is also true)	
14.	If a side of a cyclic quadrilateral is produced, then the exterior angle is equal to the interior opposite angle. $m\ \angle CDE = m\ \angle ABC$	

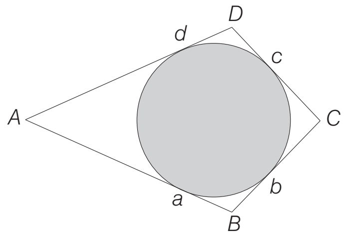
S. No.	Theorem	Diagram
15.	A tangent at any point of a circle is perpendicular to the radius through the point of contact. (Inverse of this theorem is also true)	
16.	The lengths of two tangents drawn from an external point to a circle are equal. *i.e.,* $AP = BP$	
17.	If two chords AB and CD of a circle, intersect inside a circle (outside the circle when produced at a point E), then $AE \times BE = CE \times DE$	
18.	If PB be a secant which intersects the circle at A and B and PT be a tangent at T, then $PA \cdot PB = (PT)^2$	
19.	From an external point from which the tangents are drawn to the circle with centre O, then (a) they subtend equal angles at the centre. (b) they are equally inclined to the line segment joining the centre of that point. $\angle AOP = \angle BOP$ and $\angle APO = \angle BPO$	
20.	If P is an external point from which the tangents to the circle with centre O touch it at A and B, then OP is the perpendicular bisector of AB. $OP \perp AB$ and $AC = BC$	
21.	**Alternate segment theorem :** If from the point of contact of a tangent, a chord is drawn, then the angles which the chord makes with the tangent line are equal respectively to the angles formed in the corresponding alternate segments. In the adjoining diagram. $\angle BAT = \angle BCA$ and $\angle BAP = \angle BDA$	

S. No.	Theorem	Diagram
22.	The point of contact of two tangents lies on the straight line joining the two centres. (a) When two circles touch externally, then the distance between their centres is equal to sum of their radii. *i.e.*, $AB = AC + BC$ (b) When two circles touch internally, the distance between their centres is equal to the difference between their radii. *i.e.*, $AB = AC - BC$	 
23.	For the two circles with centre X and Y and radii r_1 and r_2. AB and CD are two Direct Common Tangents (DCT), then the length of DCT $= \sqrt{(\text{distance between centres})^2 - (r_1 - r_2)^2}$	
24.	For the two circles with centre X and Y and radii r_1 and r_2. PQ and RS are two transverse common tangent, then length of TCT $= \sqrt{(\text{distance between centres})^2 - (r_1 + r_2)^2}$	
25.	**Descartes' circle equation theorem :** Given four mutually tangent circles with curvatures a, b, c and d, then $(a + b + c + d)^2 = 2(a^2 + b^2 + c^2 + d^2)$	
	If any one of the circles is circumscribing, say a circle of curvature d is circumscribing the circles of curvatures a, b and c, then $(a + b + c - d)^2 = 2(a^2 + b^2 + c^2 + (-d^2))$	
	In lay terms, if all the points of tangency are external, the curvatures are considered positive, but if one circle encompasses the others, that circle has negative curvature. Here, as all three circles are tangent to each other at the same point, Descartes' theorem does not apply.	

12.7 **Tangential Quadrilateral**

A tangential quadrilateral or circumscribed quadrilateral is a convex quadrilateral whose sides are all tangent to a single circle within the quadrilateral. This circle is called the incircle of the quadrilateral; its center is the incenter and its radius is called the inradius.

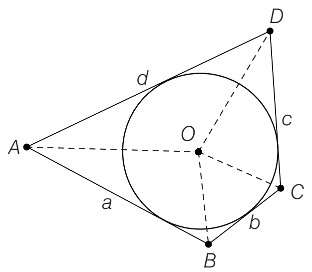
All triangles have an incircle, but not all quadrilaterals do. An example of a quadrilateral that cannot be tangential is a non-square rectangle.



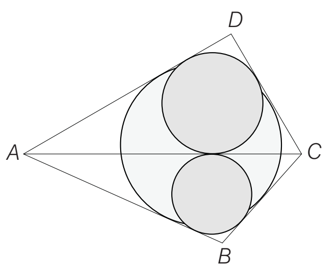
If a quadrilateral is both tangential and cyclic, it is called a bicentric quadrilateral. Examples of tangential quadrilaterals are squares, rhombi, and kites.

Properties

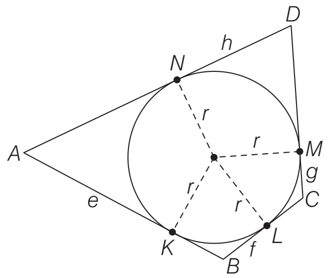
1. In a tangential quadrilateral, the four angle bisectors meet at the center of the incircle.
2. Conversely, a convex quadrilateral in which the four angle bisectors meet at a point must be tangential and the common point is the incenter.



3. The two pairs of opposite sides in a tangential quadrilateral add up to the same total length, which equals the semi perimeter s of the quadrilateral : $a + c = b + d = s$.
4. Conversely, a convex quadrilateral in which $a + c = b + d$ must be tangential.
5. A convex quadrilateral $ABCD$ is tangential, if and only if the incircles in the two triangles ABC and ADC are tangent to each other.



6. Area of the tangential quadrilateral —



(i) $A = r \times s$; where r is the inradius and s is the semi-perimeter

(ii) $A = \sqrt{(e+f+g+h)(efg+fgh+ghe+hef)}$;

where, e, f, g, h are the lengths of the tangents.

(iii) $A = \dfrac{1}{2}\sqrt{p^2q^2-(ac-bd)^2}$; where p, q are the diagonals and a, b, c, d are the sides of tangential quadrilateral.

(iv) $A = \sqrt{abcd-(eg-fh)^2}$; where a, b, c, d are the sides and e, f, g, h are the successive tangent lengths.

(v) When the tangential quadrilateral is cyclic, that is bi-centric, then $eg = fh$ and so the tangential quadrilateral will have the maximum area $\sqrt{abcd}$.

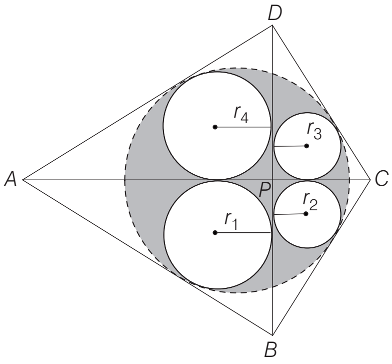
NOTE In the above diagram, it is considered as $AB = a, BC = b, CD = c, DA = d$ and $AK = e, BL = f, CM = g, DN = h$.

7. The maximum area of a tangential quadrilateral $A \le \sqrt{abcd}$
8. The minimum area of a tangential quadrilateral $A \ge 4r^2$
9. The minimum value of the semi-perimeter $s \ge 4r$
10. The inradius of a tangential quadrilateral $r = A/s$; where A is the area and s is the semi-perimeter of the tangential quadrilateral.
11. If e, f, g and h are the tangent lengths from A, B, C and D, respectively, to the points where the incircle is tangent to the sides of a tangential quadrilateral $ABCD$, then the lengths of the diagonals $p = AC$ and $q = BD$ are

$$p = \sqrt{\left(\frac{e+g}{f+h}\right)[(e+g)(f+h)+4fh]}$$

and $$q = \sqrt{\left(\frac{f+h}{e+g}\right)[(e+g)(f+h)+4eg]}$$

12. In the non-overlapping triangles APB, BPC, CPD, DPA formed by the diagonals in a convex quadrilateral $ABCD$, where the diagonals intersect at P, there are the following characterizations of tangential quadrilaterals.



(i) Let r_1, r_2, r_3 and r_4 denote the radii of the incircles in the four triangles APB, BPC, CPD and DPA respectively, then the quadrilateral is tangential if and only if $\frac{1}{r_1}+\frac{1}{r_3}=\frac{1}{r_2}+\frac{1}{r_4}$

(ii) If h_1, h_2, h_3 and h_4 denote the altitudes in the four triangles APB, BPC, CPD and DPA respectively, (from the diagonal intersection to the sides of the quadrilateral), then quadrilateral is tangential if and only if

$$\frac{1}{h_1}+\frac{1}{h_3}=\frac{1}{h_2}+\frac{1}{h_4}$$

(iii) If the exradii r_a, r_b, r_c and r_d in the four triangles APB, BPC, CPD and DPA respectively, (the four excircles are each tangent to one side of the quadrilateral and the extensions of its diagonals). A quadrilateral is tangential if and only if

$$\frac{1}{r_a}+\frac{1}{r_c}=\frac{1}{r_b}+\frac{1}{r_d}$$

(iv) If R_1, R_2, R_3 and R_4 denote the radii in the circumcircles of triangles APB, BPC, CPD and DPA, respectively, then the quadrilateral $ABCD$ is tangential if and only if

$$R_1+R_3=R_2+R_4$$

(v) A convex quadrilateral $ABCD$, with diagonals intersecting at P, is tangential if and only if the four excenters in triangles APB, BPC, CPD and DPA opposite the vertices B and D are concyclic. If R_a, R_b, R_c and R_d are the exradii in the triangles APB, BPC, CPD and DPA, respectively, opposite the vertices B and D, then another condition is that the quadrilateral is tangential if and only if

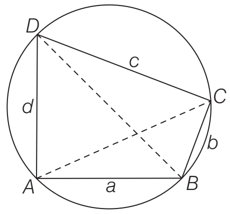
$$\frac{1}{R_a}+\frac{1}{R_c}=\frac{1}{R_b}=\frac{1}{R_d}$$

(vi) A convex quadrilateral $ABCD$ with diagonals intersecting at P is tangential if and only if

$$\frac{a}{\Delta(APB)}+\frac{c}{\Delta(CPD)}=\frac{b}{\Delta(BPC)}+\frac{d}{\Delta(DPA)}$$

Cyclic Quadrilateral

A cyclic quadrilateral is a quadrilateral for which a circle can be circumscribed so that it touches each vertex of the quadrilateral.



(i) Semi perimeter $s=\frac{a+b+c+d}{2}$

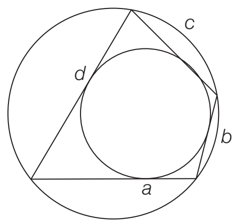
(ii) Area $A=\sqrt{(s-a)(s-b)(s-c)(s-d)}$

(iii) Area $A=\frac{\sqrt{(ab+cd)(ac+bd)(ad+bc)}}{4R}$

(iv) Circumradius $R=\frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}}$

Bicentric Quadrilateral

A quadrilateral is said to be bicentric if it is cyclic and tangential.



(i) If the sides be a, b, c, d the area of a bicentric quadrilateral is $A=\sqrt{abcd}$

(ii) Inradius, $r=\frac{2\sqrt{abcd}}{a+b+c+d}=\frac{\sqrt{abcd}}{s}$

(iii) Circumradius, $R=\frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{abcd}}$

(iv) In a bicyclic quadrilateral, the distance x between the incentre and circumcentre is given by

$$\frac{1}{(R-x)^2}+\frac{1}{(R+x)^2}=\frac{1}{(r)^2}$$